OXFORD UNIVERSITY

MATHEMATICS ADMISSIONS TEST

Wednesday 7 November 2012

Time Allowed: 2½ hours

For candidates applying for Mathematics, Computer Science or one of their joint degrees.

Write your name, UCAS Personal ID, Oxford college (to which you applied or were assigned) and your proposed degree course in BLOCK CAPITALS below.

NAME:

UCAS PERSONAL ID:

OXFORD COLLEGE (if known):

DEGREE COURSE:

DATE OF BIRTH:

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

- **Mathematics, Maths & Philosophy, Maths & Statistics applicants should attempt 1,2,3,4,5.**
- **Mathematics & Computer Science applicants should attempt 1,2,3,5,6.**
- **Computer Science, Computer Science & Philosophy applicants should attempt 1,2,5,6,7.**

Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

For Test Supervisors Use Only:

[ ] Tick here if special arrangements were made for the test. Please either include details below or securely attach to the test script a letter with the details.

Signature of Invigilator: ______________________________

FOR OFFICE USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. Which of the following lines is a tangent to the circle with equation

$$x^2 + y^2 = 4\,?$$

(a) $x + y = 2$; (b) $y = x - 2\sqrt{2}$; (c) $x = \sqrt{2}$; (d) $y = \sqrt{2} - x$.

B. Let $N = 2^k \times 4^m \times 8^n$ where k, m, n are positive whole numbers. Then N will definitely be a square number whenever

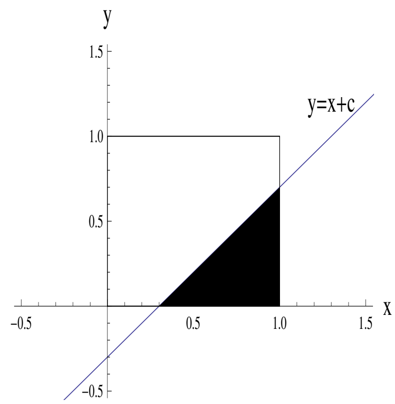
(a) k is even;
(b) $k + n$ is odd;
(c) k is odd but $m + n$ is even;
(d) $k + n$ is even.

Turn over

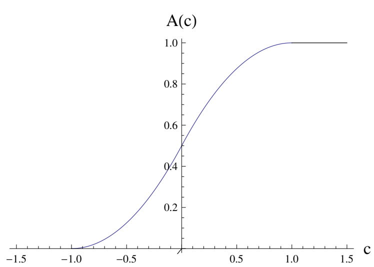
C. Which is the *smallest* of the following numbers?

(a) $\left(\sqrt{3}\right)^3$, (b) $\log_3\left(9^2\right)$, (c) $\left(3\sin\frac{\pi}{3}\right)^2$, (d) $\log_2\left(\log_2\left(8^5\right)\right)$.

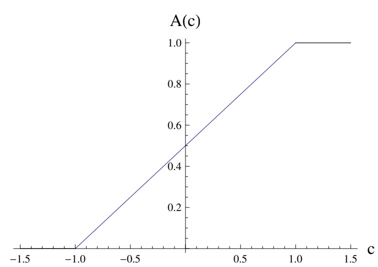
D. Shown below is a diagram of the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$ and the line $y = x + c$. The shaded region is the region of the square which lies below the line; this shaded region has area $A(c)$.



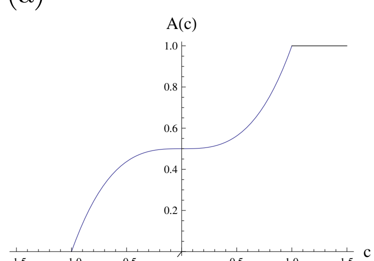
Which of the following graphs shows $A(c)$ as c varies?



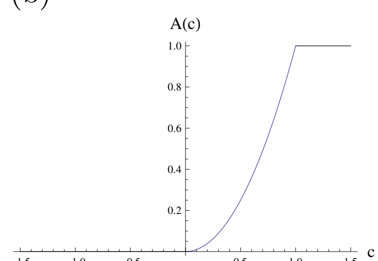
(a)



(b)



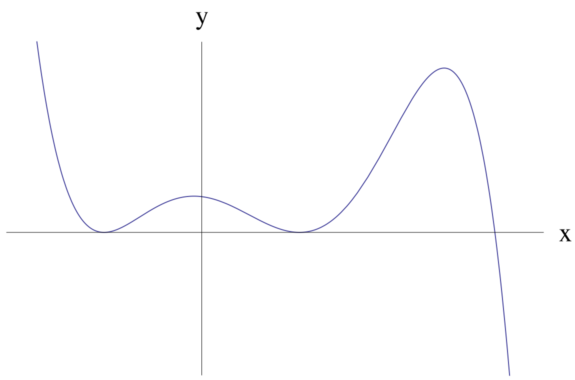
(c)



(d)

E. Which one of the following equations could possibly have the graph given below?

(a) $y = (3-x)^2\,(3+x)^2\,(1-x)$;
(b) $y = -x^2\,(x-9)\,(x^2-3)$;
(c) $y = (x-6)\,(x-2)^2\,(x+2)^2$;
(d) $y = (x^2-1)^2\,(3-x)$.



F. Let

$$T = \left(\int_{-\pi/2}^{\pi/2} \cos x \,\mathrm{d}x\right) \times \left(\int_{\pi}^{2\pi} \sin x \,\mathrm{d}x\right) \times \left(\int_{0}^{\pi/8} \frac{\mathrm{d}x}{\cos 3x}\right).$$

Which of the following is true?

(a) $T = 0$; (b) $T < 0$; (c) $T > 0$; (d) T is not defined.

Turn over

G. There are *positive* real numbers x and y which solve the equations

$$2x + ky = 4, \qquad\qquad x + y = k$$

for

(a) all values of k; (b) no values of k; (c) $k = 2$ only; (d) only $k > -2$.

H. In the region $0 < x \leqslant 2\pi$, the equation

$$\int_0^x \sin(\sin t)\,\mathrm{d}t = 0$$

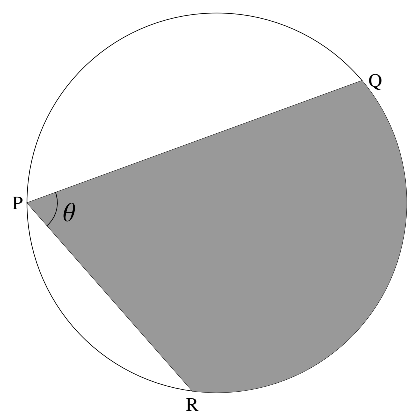
has

(a) no solution; (b) one solution; (c) two solutions; (d) three solutions.

I. The vertices of an equilateral triangle are labelled X, Y and Z. The points X, Y and Z lie on a circle of circumference 10 units. Let P and A be the numerical values of the triangle's perimeter and area, respectively. Which of the following is true?

(a) $\dfrac{A}{P}=\dfrac{5}{4\pi}$; (b) $P<A$; (c) $\dfrac{P}{A}=\dfrac{10}{3\pi}$; (d) P^2 is rational.

J. If two chords QP and RP on a circle of radius 1 meet in an angle θ at P, for example as drawn in the diagram below,



then the largest possible area of the shaded region RPQ is

(a) $\theta\left(1+\cos\left(\dfrac{\theta}{2}\right)\right)$; (b) $\theta+\sin\theta$; (c) $\dfrac{\pi}{2}\left(1-\cos\theta\right)$; (d) θ.

Turn over

2. For **ALL APPLICANTS.**

Let

$$f(x) = x + 1 \qquad \text{and} \qquad g(x) = 2x.$$

We will, for example, write fg to denote the function "perform g then perform f" so that

$$fg(x) = f(g(x)) = 2x + 1.$$

If $i \geqslant 0$ is an integer we will, for example, write f^i to denote the function which performs f i times, so that

$$f^i(x) = \underbrace{fff\cdots f}_{i \text{ times}}(x) = x + i.$$

(i) Show that

$$f^2g(x) = gf(x).$$

(ii) Note that

$$gf^2g(x) = 4x + 4.$$

Find all the other ways of combining f and g that result in the function $4x + 4$.

(iii) Let $i, j, k \geqslant 0$ be integers. Determine the function

$$f^i g f^j g f^k(x).$$

(iv) Let $m \geqslant 0$ be an integer. How many different ways of combining the functions f and g are there that result in the function $4x + 4m$?

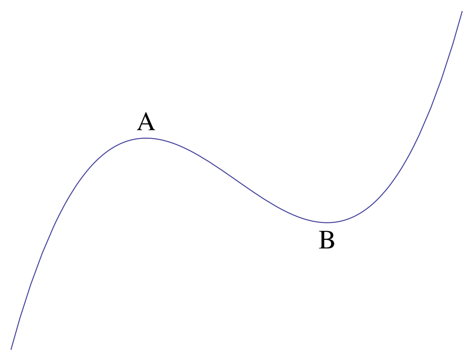
Turn over

3.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\end{array}\right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 14.**

Let $f(x) = x^3 + ax^2 + bx + c$, where the coefficients a, b and c are real numbers. The figure below shows a section of the graph of $y = f(x)$. The curve has two distinct turning points; these are located at A and B, as shown. (Note that the axes have been omitted deliberately.)



(i) Find a condition on the coefficients a, b, c such that the curve has two distinct turning points if, and only if, this condition is satisfied.

It may be assumed from now on that the condition on the coefficients in (i) is satisfied.

(ii) Let x_1 and x_2 denote the x coordinates of A and B, respectively. Show that

$$x_2 - x_1 = \frac{2}{3}\sqrt{a^2 - 3b}.$$

(iii) Suppose now that the graph of $y = f(x)$ is translated so that the turning point at A now lies at the origin. Let $g(x)$ be the cubic function such that $y = g(x)$ has the translated graph. Show that

$$g(x) = x^2\left(x - \sqrt{a^2 - 3b}\right).$$

(iv) Let R be the area of the region enclosed by the x-axis and the graph $y = g(x)$. Show that if a and b are rational then R is also rational.

(v) Is it possible for R to be a non-zero rational number when a and b are both irrational? Justify your answer.

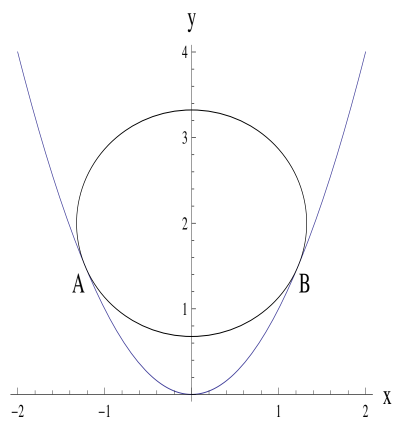
Turn over

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 14.**

The diagram below shows the parabola $y = x^2$ and a circle with centre $(0, 2)$ just 'resting' on the parabola. By 'resting' we mean that the circle and parabola are tangential to each other at the points A and B.



(i) Let (x, y) be a point on the parabola such that $x \neq 0$. Show that the gradient of the line joining this point to the centre of the circle is given by

$$\frac{x^2 - 2}{x}.$$

(ii) With the help of the result from part (i), or otherwise, show that the coordinates of B are given by

$$\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right).$$

(iii) Show that the area of the sector of the circle enclosed by the radius to A, the minor arc AB and the radius to B is equal to

$$\frac{7}{4}\cos^{-1}\left(\frac{1}{\sqrt{7}}\right).$$

(iv) Suppose now that a circle with centre $(0, a)$ is resting on the parabola, where $a > 0$. Find the range of values of a for which the circle and parabola touch at two distinct points.

(v) Let r be the radius of a circle with centre $(0, a)$ that is resting on the parabola. Express a as a function of r, distinguishing between the cases in which the circle is, and is not, in contact with the vertex of the parabola.

Turn over

5. For **ALL APPLICANTS.**

A particular robot has three commands:

F: Move forward a unit distance;
L: Turn left 90^o;
R: Turn right 90^o.

A *program* is a sequence of commands. We consider particular programs P_n (for $n \geqslant 0$) in this question. The basic program P_0 just instructs the robot to move forward:

$$P_0 = \mathbf{F}.$$

The program P_{n+1} (for $n \geqslant 0$) involves performing P_n, turning left, performing P_n again, then turning right:

$$P_{n+1} = P_n\,\mathbf{L}\,P_n\,\mathbf{R}.$$

So, for example, $P_1 = \mathbf{F\,L\,F\,R}$.

(i) Write down the program P_2.

(ii) How far does the robot travel during the program P_n? In other words, how many **F** commands does it perform?

(iii) Let l_n be the total number of commands in P_n; so, for example, $l_0 = 1$ and $l_1 = 4$.

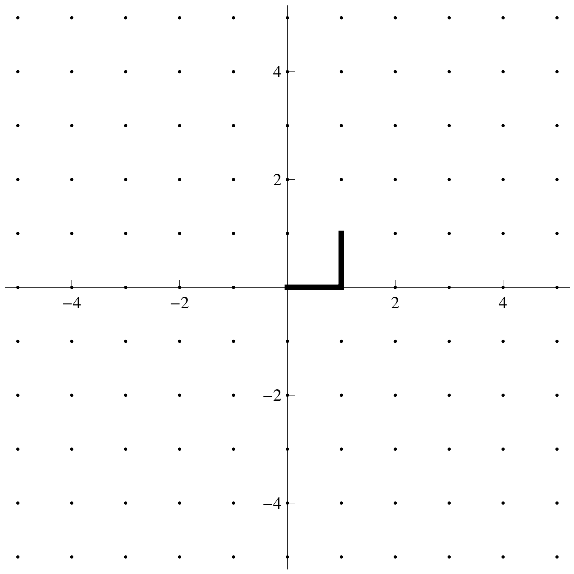
Write down an equation relating l_{n+1} to l_n. Hence write down a formula for l_n in terms of n. No proof is required. **Hint:** consider $l_n + 2$.

(iv) The robot starts at the origin, facing along the positive x-axis. What direction is the robot facing after performing the program P_n?

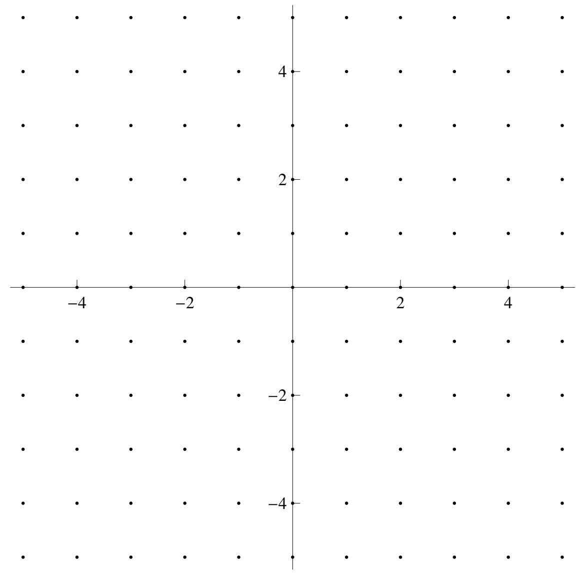
(v) The left-hand diagram on the opposite page shows the path the robot takes when it performs the program P_1. On the right-hand diagram opposite, draw the path it takes when it performs the program P_4.

(vi) Let (x_n, y_n) be the position of the robot after performing the program P_n, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$. Give an equation relating (x_{n+1}, y_{n+1}) to (x_n, y_n).

What is (x_8, y_8)? What is (x_{8k}, y_{8k})?



P_1

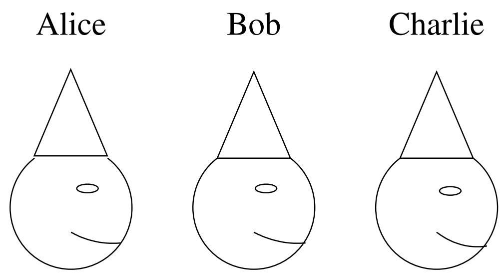


P_4

Turn over

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**



Alice, Bob and Charlie are well-known expert logicians; they always tell the truth.

They are sat in a row, as illustrated above. In each of the scenarios below, their father puts a red or blue hat on each of their heads. Alice can see Bob's and Charlie's hats, but not her own; Bob can see only Charlie's hat; Charlie can see none of the hats. All three of them are aware of this arrangement.

(i) Their father puts a hat on each of their heads and says: "Each of your hats is either red or blue. At least one of you has a red hat." Alice then says "I know the colour of my hat." What colour is each person's hat? Explain your answer.

(ii) Their father puts a new hat on each of their heads and again says: "Each of your hats is either red or blue. At least one of you has a red hat." Alice then says "I don't know the colour of my hat." Bob then says "I don't know the colour of my hat." What colour is Charlie's hat? Explain your answer.

(iii) Their father puts a new hat on each of their heads and says: "Each of your hats is either red or blue. At least one of you has a red hat, and at least one of you has a blue hat." Alice says "I know the colour of my hat." Bob then says "Mine is red." What colour is each person's hat? Explain your answer.

(iv) Their father puts a new hat on each of their heads and says: "Each of your hats is either red or blue. At least one of you has a red hat, and at least one of you has a blue hat." Alice then says "I don't know the colour of my hat." Bob then says "My hat is red". What colour is Charlie's hat? Explain your answer.

(v) Their father puts a new hat on each of their heads and says: "Each of your hats is either red or blue. Two of you who are seated adjacently both have red hats." Alice then says "I don't know the colour of my hat." What colour is Charlie's hat? Explain your answer.

Turn over

7.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

Amy and Brian play a game together, as follows. They take it in turns to write down a number from the set $\{0, 1, 2\}$, with Amy playing first. On each turn (except Amy's first turn), the player must not repeat the number just played by the previous player.

In their first version of the game, Brian wins if, after he plays, the sum of all the numbers played so far is a multiple of 3. For example, Brian will win after the sequence

2,0	1,2	1,0

(where we draw a box around each round) because the sum of the numbers is 6. Amy wins if Brian has not won within five rounds; for example, Amy wins after the sequence

2,0	1,2	1,2	0,2	1,2

(i) Show that if Amy starts by playing either 1 or 2, then Brian can immediately win.

(ii) Suppose, instead, Amy starts by playing 0. Show that Brian can always win within two rounds.

They now decide to change the rules so that Brian wins if, after he plays, the sum of all the numbers played so far is *one less than a multiple of 3*. Again, Amy wins if Brian has not won within five rounds. It is still the case that a player must not repeat the number just played previously.

(iii) Show that if Amy starts by playing either 0 or 2, then Brian can immediately win.

(iv) Suppose, instead, Amy starts by playing 1. Explain why it cannot benefit Brian to play 2, assuming Amy plays with the best strategy.

(v) So suppose Amy starts by playing 1, and Brian then plays 0. How should Amy play next?

(vi) Assuming both play with the best strategies, who will win the game? Explain your answer.

End of last question

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