Event Domains, Stable Functions and Proof-Nets

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Abstract

We pursue the program of exposing the intrinsic mathematical structure of the "space of proofs" of a logical system [AJ94b]. We study the case of Multiplicative-Additive Linear Logic (MALL). We use tools from Domain theory to develop a semantic notion of proof net for MALL, and prove a Sequentialization Theorem. This work forms part of a continuation of previous joint work with Radha Jagadeesan [AJ94b] and Paul-André Melliès [AM99].

Keywords: Linear Logic, Proof Nets, Domain Theory, Event Structures, Stable Functions.

Dedication

Gordon Plotkin was a major formative influence on me as a researcher, as he has been on the entire field of semantics of computation. He has also been a true friend, over these many years. It is a great pleasure to dedicate this paper to him on the occasion of his sixtieth birthday. I hope he will be pleased by the new uses it finds for some of the fundamental tools of semantic investigation which he has done so much to create and develop.

1 Introduction

One can distinguish two views on how Logic relates to Structure:

(i) **The Descriptive View.** Logic is used to *talk about* structure. This is the view taken in Model Theory, and in most of the uses of Logic (Temporal logics, MSO etc.) in Verification. It is by far the more prevalent and widely-understood view.

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(ii) **The Intrinsic View.** Logic is taken to *embody* structure. This is, implicitly or explicitly, the view taken in the Curry-Howard isomorphism, and more generally in Structural Proof Theory, and in (much of) Categorical Logic. For example, in the Curry-Howard isomorphism, one is not using logic to *talk about* functional programming; rather, logic (in this aspect) *is* functional programming.

If we are to find structure in the proof theory of a logic, we face a challenge. Proof systems are subject to many minor "design decisions", which does not impart confidence that the objects being described — formal proofs — have a robust intrinsic structure. It is perhaps useful to make an analogy with Geometry. A major concern of modern Geometry has been to find *instrinsic*, typically *coordinate-free*, descriptions of the geometric objects of study. We may view the rôle of *syntax* in Proof Theory as analogous to coordinates in Geometry; invaluable for computation, but an obstacle to finding the underlying invariant structure.

Some particularly promising progress in finding more intrinsic descriptions of proofs, their geometric structure, and their dynamics under Cut-elimination, has taken place in the study of proof-nets in Linear Logic [Gir87], and the associated study of Geometry of Interaction [Gir89]. On the semantic side, the development of Game Semantics and Full Completeness results [AJ94b] (and subsequently [Loa94,BS98,AM99,DHPP99,BHS05]) has greatly enriched and deepened the structural perspective.

In the present paper, we build on previous joint work with Radha Jagadeesan [AJ94a] and Paul-André Melliès [AM99]. We study Multiplicative-Additive Linear Logic (MALL). We use tools from Domain theory to develop a semantic notion of proof net for MALL, and prove a Sequentialization Theorem for this notion.

1.1   *Related Work*

We build on the previous work on proof-nets and semantics for MALL. In particular, our "semantic" approach to proof nets for MALL can be seen as an (in our opinion, more elegant) alternative to the development of weighted nets in [Gir96]. Recent work by Faggian and Curien on Ludics nets [FM05,CF05] is also clearly related — although it should be emphasized that Ludics deals with polarized (and hence "sequentialized") Linear Logic, whereas we are dealing with full classical MALL. See [Abr03] for further discussion of this issue.

An important recent contribution to the proof theory of MALL is the work of Hughes and van Glabeek [HvG05]. They give what can be considered an optimal notion of proof net for MALL, in the sense that it contains the minimal information necessary to reconstruct a sequent proof. We hope ultimately to extend our approach to give an analysis of their notion of proof net, and to relate it to our semantic ideas. However, this is left to future work.

Finally, as already mentioned, the present paper builds on our own previous joint work with Radha Jagadeesan [AJ94a] and Paul-André Melliès [AM99]. In particular, the underlying model which gives rise to our "semantic" notion of proof

nets is essentially derived from the construction given in [AJ94a] (which can be seen as the precursor of the Int or $\mathcal{G}$ construction [JSV96,Abr96]); even the idea of a domain-theoretic process for the additive part, which builds a tree for each additive resolution, to which is glued a permutation on the leaves giving the multiplicative structure, can be found in embryonic form in Section 7 of that paper.

The main novel feature of the present paper is the full semantic development of proof-nets. We give a detailed proof of the Sequentialization Theorem for our notion. This follows the lines of the proof in [Gir96] quite closely, but given the very different form of our proof-nets — no explicit links or weights, the principal ingredients in [Gir96] — this seems worthwhile. Moreover, we obtain a stronger result than that in [Gir96]. The relation of "sequentializability" used there is a many-many relation between proof nets and sequent proofs; while we define a canonical mapping from sequent proofs to proof nets, such that the sequent proof obtained from a proof net by sequentialization always denotes a proof net *which approximates the one we started with.* Indeed, this result can only meaningfully be stated in our domain-theoretic setting, where there is a natural order on proof-nets. This in turn opens up an interesting structure of "degrees of parallelism" within each equivalence class of proof nets under "extensional equivalence".

The present paper in fact forms a part of a larger work [Abr07], in which we revisit the work in [AM99], and prove Full Completeness of a concurrent game semantics for MALL. Much effort in [AM99] is expended on mapping strategies to proof-structures in the sense of [Gir96], in order to use the sequentialization result in that paper. The present treatment, in which we use a notion of proof net which is close to the semantic notion of strategy employed in [AM99], and prove sequentialization directly for that notion, seems more self-contained and illuminating.

1.2   Outline

We briefly outline the contents of the paper. In Section 2 we review proof nets and sequentialization for MLL, as a warm-up and template for the subsequent treatment of MALL. In Section 3, we review the basic syntax of MALL. Some notions of domain theory which we will use are reviewed in Section 4, to make the paper reasonably self-contained. In Section 5, the semantic notion of proof structure is introduced. Comparisons with other notions, and the domain-theoretic fine structure of semantic proof structures, are discussed in Section 6. The corresponding notion of proof net is defined in Section 7, and Sequentialization is proved in Section 8. Section 9 concludes.

2   MLL

The Multiplicative fragment of Linear Logic, minus the units — henceforth MLL — is a kind of logical paradise. Everything works beautifully smoothly and naturally. The ideas are simple and compelling, and yet non-trivial. Thus we will use it as a template for our subsequent discussion of Multiplicative-Additive Linear Logic (MALL).

2.1   Syntax of MLL

The formulas of the system are built from *literals*, *i.e.* propositional atoms α, β, ..., (*positive literals*), and their negations $\alpha^\perp$, $\beta^\perp$, ..., (*negative literals*), by the grammar

$$A \quad ::= \quad \alpha \mid \alpha^\perp \mid A \otimes A \mid A \,\invamp\, A.$$

Here $\otimes$ (Times) and $\invamp$ (Par) are the *multiplicative* connectives.
Negation is definitionally extended to general formulas by the equations

$$(A \otimes B)^\perp = A^\perp \invamp B^\perp \qquad (A \invamp B)^\perp = A^\perp \otimes B^\perp \qquad A^{\perp\perp} = A.$$

We also define *linear implication* $A \multimap B$ by:

$$A \multimap B \quad = \quad A^\perp \invamp B.$$

A sequent in MLL is an expression $\vdash \Gamma$, where Γ is a finite sequence of formulas.

2.2   Sequent Calculus for MLL

Axiom/Cut

$$\frac{}{\vdash A, A^\perp} \; \mathsf{Id} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} \; \mathsf{Cut}$$

Structural Rule

$$\frac{\vdash \Gamma}{\vdash \sigma\Gamma} \; \mathsf{Exchange}$$

Multiplicatives

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \; \mathsf{Times} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \invamp B} \; \mathsf{Par}$$

2.3   Proof Structures

We now turn to a "geometrization" of syntax. We shall introduce (Cut-free) proof-structures in a streamlined form [Gir88,AJ94b]. We consider firstly the version of the sequent calculus where the Axiom is restricted to atomic instances:

$$\frac{}{\vdash \alpha, \alpha^\perp} \; \mathsf{Id}$$

Note that any Cut-free proof of a sequent Γ will necessarily reproduce the structure of the formulas in Γ, in some order of application of the rules which is of no intrinsic significance, *except* insofar as it indicates how the occurrences of literals are introduced in pairs by the Axiom. Thus we take the bold step of saying that the essential content of a proof is this information, which can be represented by a listing of the matched pairs of literal occurrences $\{l_i, l_j\}$, where $l_j = l_i^{\perp}$. More conveniently, we can take a proof structure to be a *literal-respecting involution* on the set $\mathcal{L}(\Gamma)$ of literal occurrences in Γ: *i.e.* a permutation

$$\sigma : \mathcal{L}(\Gamma) \longrightarrow \mathcal{L}(\Gamma)$$

such that $\sigma = \sigma^{-1}$, and if $\sigma(a) = b$, then $\lambda(a) = \lambda(b)^{\perp}$, where $\lambda(a)$ is the literal of which a is an occurrence. Note that such a function is necessarily *fixpoint-free*, *i.e.* $\sigma(a) \neq a$ for all $a \in \mathcal{L}(\Gamma)$.

Example

Consider the sequent $\vdash \alpha^{\perp} \,\,\invamp\,\, \alpha^{\perp}, \alpha \otimes \alpha$. There are in fact only two Cut-free proofs of this sequent (corresponding to the identity and the twist map). They correspond to the following proof structures:

2.4 Interpreting Sequent Proofs as Proof Structures

We now show how every sequent proof can be interpreted as a proof structure.

Axiom

$$\frac{}{\vdash \alpha, \alpha^{\perp}} \;\; \mathsf{Id}$$

We assign the transposition $\alpha \leftrightarrow \alpha^{\perp}$.

Tensor

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \;\; \mathsf{Times}$$

Suppose we have assigned the permutation σ to the proof of $\vdash \Gamma, A$, and τ to the proof of $\vdash B, \Delta$. Then we assign the disjoint union $\sigma + \tau$ to the proof of $\Gamma, A \otimes B, \Delta$. This makes sense since $\mathcal{L}(\Gamma, A \otimes B, \Delta)$ is the disjoint union of $\mathcal{L}(\Gamma, A)$ and $\mathcal{L}(B, \Delta)$. Thus

$$\sigma + \tau(a) = \sigma(a), \quad a \in \mathcal{L}(\Gamma, A), \qquad \sigma + \tau(b) = \tau(b), \quad b \in \mathcal{L}(B, \Delta).$$

Par

$$\frac{\Gamma, A, B}{\vdash \Gamma, A \,\invamp\, B} \;\; \mathsf{Par}$$

Since (essentially) $\mathcal{L}(\Gamma, A \mathbin{\rotatebox[origin=c]{180}{\&}} B) = \mathcal{L}(\Gamma, A, B)$, we can assign the same permutation to the conclusion as to the premise!

2.5   Proof Nets

This raises the question: *how can we characterize which permutations arise as the interpretations of proofs?* If we can do this, we have the right to regard such permutations as being the intrinsic representations of proofs, laying bare their essential structure and content. A first approach is via a geometric criterion: this is the notion of *proof net*.

Switching Graphs

A *switching* S of Γ assigns L or R to each occurrence of $\mathbin{\rotatebox[origin=c]{180}{\&}}$. Given a sequent Γ, a proof structure σ, and a switching S, the *switching graph* $\mathcal{G}_\Gamma(\sigma, S)$ has:

- subformula occurrences in Γ as vertices;
- an edge connecting A to $A \otimes B$ and an edge connecting B to $A \otimes B$ for each occurrence of $A \otimes B$;
- an edge connecting A to $A \mathbin{\rotatebox[origin=c]{180}{\&}} B$ if S assigns L to $A \mathbin{\rotatebox[origin=c]{180}{\&}} B$, and an edge connecting B to $A \mathbin{\rotatebox[origin=c]{180}{\&}} B$ if S assigns R to $A \mathbin{\rotatebox[origin=c]{180}{\&}} B$;
- an edge connecting literal occurrences a and b if $\sigma(a) = b$.

The Danos-Regnier criterion

A proof-structure σ for Γ is an *MLL proof-net* if for every switching S, $\mathcal{G}_\Gamma(\sigma, S)$ is *acyclic and connected*.

Proposition 2.1 (Soundness) *The proof structures arising as interpretations of sequent proofs are proof nets.*

The major result on MLL proof nets is the following [Gir87,DR89,Gir91,Tro92]:

Theorem 2.2 (Sequentialization Theorem) *Every proof net arises from a sequent proof.*

The key case in the proof is when all the non-literal conclusions in the sequent are tensors; we need to find a *splitting tensor* $A \otimes B$ such that we can split $\Gamma, A \otimes B$ into Γ_1, A and Γ_2, B in such a way that our proof-net decomposes into two sub-proof nets with these conclusions. This is done via the notion of *empire*. We will see all these ideas developed in detail in the more complex setting of MALL.

Discussion

The step involved in representing proof structures by permutations on literal occurrences — which is not the standard formulation of proof-nets [Gir87,Gir91,Tro92] — is already a significant step towards a semantic view of proofs. It leads directly to the Geometry of Interaction [Gir88,Gir89], and to Full Completeness results [AJ94b]. These in turn provide an elegant compositional account of the dynamics

of Cut-Elimination. Our approach can be seen as a continuation of these ideas in the richer setting of MALL, where new ideas are needed.

3 Syntax of MALL

The formulas of the system are built from *literals*, *i.e.* propositional atoms α, β, ..., and their negations $\alpha^\perp$, $\beta^\perp$, ..., by the grammar

$$A \quad ::= \quad \alpha \mid \alpha^\perp \mid A \otimes A \mid A \,\rotatebox[origin=c]{180}{\&}\, A \mid A \oplus A \mid A\&A.$$

Here $\otimes$ and $\rotatebox[origin=c]{180}{\&}$ are the *multiplicative* connectives, while $\oplus$, $\&$ are the *additive* connectives.

Negation is definitionally extended to general formulas by the equations

$$(A \otimes B)^\perp = A^\perp \,\rotatebox[origin=c]{180}{\&}\, B^\perp \qquad (A \,\rotatebox[origin=c]{180}{\&}\, B)^\perp = A^\perp \otimes B^\perp$$

$$(A \oplus B)^\perp = A^\perp \& B^\perp \qquad (A\&B)^\perp = A^\perp \oplus B^\perp$$

$$A^{\perp\perp} = A.$$

A sequent in MALL is an expression $\vdash \Gamma$, where Γ is a finite sequence of formulas.

3.1 Sequent Calculus for MALL

Axiom/Cut

$$\frac{}{\vdash A, A^\perp} \ \mathsf{Id} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta} \ \mathsf{Cut}$$

Structural Rule

$$\frac{\vdash \Gamma}{\vdash \sigma\Gamma} \ \mathsf{Exchange}$$

Multiplicatives

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \ \mathsf{Times} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \,\rotatebox[origin=c]{180}{\&}\, B} \ \mathsf{Par}$$

Additives

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \ \mathsf{PlusL} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \ \mathsf{PlusR} \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A\&B} \ \mathsf{With}$$

3.2 Generalized Axioms

In order to carry out the proof of the Sequentialization Theorem, it is useful to introduce *generalized axioms*, following [Gir91,Gir96]. In our setting, it is most convenient to proceed in the following manner. We introduce a set of *parameters* $\xi, \zeta, \ldots$, distinct from the propositional atoms $\alpha, \beta, \ldots$. Each parameter ξ has an arity k, and parameter instances $\xi_1, \ldots, \xi_k$. Formulas can be built from parameter instances as well as propositional atoms. We extend the definition of negation by

$$\xi_i^\perp = \xi_i.$$

For each parameter ξ of arity k there is a sequent rule (a "proper axiom"):

$$\frac{}{\vdash \xi_1, \ldots, \xi_k} \; \mathsf{Ax}$$

The idea is that $\xi_1, \ldots, \xi_k$ indicate the conclusions of a 'box'. Given a Cut-free sequent proof Π_1 of $\Gamma[\xi_1, \ldots, \xi_k]$, and a proof Π_2 of $\Delta = B_1, \ldots, B_k$, we can form a sequent proof $\Pi_1[\Pi_2]$ of $\Gamma[\Delta/\xi_1, \ldots, \xi_k]$ by replacing the use of Ax to derive $\vdash \xi_1, \ldots, \xi_k$ by Π_2.

3.3 Occurrences and Linear Contexts

It is necessary to speak of *occurrences* of formulas in a given formula or sequent. This often leads to awkwardness, imprecision, or both. A convenient way to handle occurrences is via *linear contexts*. These are built up with the same syntax as formulas or sequents, but with a single use of a "hole" $[\cdot]$.

Example The context $A \otimes ([\cdot] \, \& \, C)$ corresponds to the occurrence of B in the formula $A \otimes (B \, \& \, C)$.

Linear contexts are in evident biunique correspondence with occurrences, and permit convenient inductive definitions. We shall pass freely between an occurrence O of a formula A in a formula B (or a sequent Γ), and the corresponding context $C[\cdot]$ such that $C[A] = B$ (or $C[A] = \Gamma$). We shall use the letters O, P, Q, R for occurrences, V and W for With occurrences and L and M for literal occurrences. We write $\mathcal{O}(\Gamma)$ for the set of occurrences in a sequent Γ; and $\mathcal{L}(\Gamma)$ for the set of occurrences of literals.

4 Background on Domains

In order to make the paper reasonably self-contained, we shall briefly review some background material on domain theory. A useful reference is [Win86]. although we will work in a much more restricted setting. The seminal references for these ideas are [KP93,NPW81]

We make a global assumption, that *all domains considered in this paper are finite*. This means that we can disregard all considerations of completeness and continuity. All the definitions in this section are made under the assumption that the underlying poset is finite.

We shall work exclusively with *bounded-complete* posets; that is, partially ordered sets in which every bounded subset (*i.e.* subset having an upper bound in the poset) has a least upper bound. Such posets also have non-empty meets. Note in particular that bounded complete posets have least elements, denoted $\bot$.

A *prime* in a poset is an element p such that $p \sqsubseteq x \sqcup y$ implies $p \sqsubseteq x$ or $p \sqsubseteq y$. We write $\mathsf{Pr}(P)$ for the set of prime elements of P. An *event domain* is a bounded complete poset D in which every element is the least upper bound of the primes below it, which we write

$$x = \bigsqcup \downarrow_{\mathsf{Pr}}(x),$$

where $\downarrow_{\mathsf{Pr}}(x) = \{p \in \mathsf{Pr}(D) \mid p \sqsubseteq x\}$.

Proposition 4.1 *An event domain is distributive: that is*

$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$$

whenever y and z are bounded. Otherwise put, every principal lower set $\downarrow(x)$ is a distributive lattice.

The *covering relation* in a poset is defined by:

$$x \prec y \;\;\equiv\;\; x \sqsubset y \;\wedge\; (x \sqsubseteq z \sqsubseteq y \;\Rightarrow\; (x = z) \vee (y = z)).$$

An *atom* in an event domain is an element a such that $\bot \prec a$. An event domain is *atomic* if all primes other than $\bot$ are atoms.

Another class of event domains we will refer to are the (distributive) *concrete domains*, which satisfy an additional axiom. We omit the definition, which can be found in [KP93,Win86].

Constructions on Domains

We will use a few constructions on domains:

- The one-point domain, written **1**.
- The cartesian product $D \times E$, ordered pointwise.
- The lift $D_\bot$ obtained by adjoining a new bottom element to D.
- The separated sum $(D + E)_\bot$ obtained by forming the disjoint union of D and E, and adjoining a bottom element.
- Flat domains $X_\bot$ obtained by adjoining a bottom element to a set X, with the order relation: $x \sqsubseteq y$ iff $x = \bot$ or $x = y$.
- The set of partial bijections on a set X, ordered by inclusion.
- We use the notation $\mathbb{O} = \mathbf{1}_\bot$, for the *Sierpinski domain*, *i.e.* the 2-element lattice $\bot \sqsubseteq \top$.

Proposition 4.2 *Event domains and concrete domains are closed under all the above constructions. Atomic domains are closed under all but lifting and separated sum.*

Notation We write $\mathsf{Max}(D)$ for the set of maximal elements of an event domain D, and $\mathsf{MaxPr}(D)$ for the set of maximal primes in D, *i.e.* $\mathsf{MaxPr}(D) = \mathsf{Max}(\mathsf{Pr}(D))$. Thus a "maximal prime" is maximal in $\mathsf{Pr}(D)$, as a sub-poset of D.

4.1 Functions on Domains

All functions between domains will be assumed to be *monotone*:

$$x \sqsubseteq y \;\;\Rightarrow\;\; f(x) \sqsubseteq f(y).$$

We will be concerned with an additional property, of *stability* [Ber78]. A function $f : D \longrightarrow E$ is stable (first version) if whenever x and y are bounded, $f(x \sqcap y) = f(x) \sqcap f(y)$. There is an equivalent definition, which is more enlightening, and will prove more useful to us. Suppose we have an input $x \in D$, and $y \in E$ such that $y \sqsubseteq f(x)$. We define the *modulus of stability* to be the least $x' \sqsubseteq x$ such that $y \sqsubseteq f(x')$. Such an element may not exist in general. If it always does, we say that f is stable (second version), and denote this modulus by $\mathsf{M}(f, x, y)$. If f is stable (first version) we can define this modulus by

$$\mathsf{M}(f, x, y) = \bigsqcap \{x' \sqsubseteq x \mid y \sqsubseteq f(x')\}.$$

Conversely, one can show that if D and E are event domains (and actually much more generally), the second version of stability implies the first.

A further property we shall refer to will be sequentiality. We will not define this here; see [Cur93].

Proposition 4.3 *Both the stable and the sequential functions are closed under all the following operations associated with the constructions on domains described in the previous sub-section; constant functions, composition, identities, projections, pairing, injections, and the usual conditionals.*

We will also make some use of embedding-projections, which express how one domain fits as a sub-domain inside another. An embedding-projection is written

$$e : D \lhd E : p$$

where D and E are event domains, and $e : D \longrightarrow E$ and $p : E \longrightarrow D$ are monotone maps, satisfying:

$$p \circ e = 1_D, \qquad e \circ p \sqsubseteq 1_E.$$

The ordering on functions we are using here is the pointwise order:

$$f \sqsubseteq g \iff \forall x. f(x) \sqsubseteq g(x).$$

This is the only order on functions we will use, even for stable functions; we will never need to consider function spaces and higher-order functions.

4.2 Decompositions of Domains

We develop some technical notions which will prove useful.

We introduce a notion of *restriction* on event domains D. If d is an element of D, and P is a set of primes in D, we define:

$$d{\restriction}P = \bigsqcup \{p \in {\downarrow}_{\mathsf{Pr}}(d) \mid p \in P\}.$$

We can define a sub-domain D_P of D in either of the following two equivalent ways:

- D_P is generated by P, as the set of joins of bounded subsets of P.

- D_P is the image of the deflation $r_P : d \mapsto d{\upharpoonright}P$.

We can define an embedding-projection

$$e : D_P \vartriangleleft D : p$$

where $e : D_P \hookrightarrow D$ is the inclusion, and $p(d) = d{\upharpoonright}P$.
Now suppose that we have a monotone function

$$f : V_1 \times V_2 \longrightarrow D.$$

We assume given sets of primes $P_1, P_2 \subseteq D$, and corresponding embedding-projections

$$e_i : D_i \vartriangleleft D : p_i, \qquad i = 1, 2.$$

We assume that the sub-domains D_i are themselves event domains. Note that there are stable embedding-projections

$$\phi_i : V_i \vartriangleleft V : \pi_i, \qquad i = 1, 2$$

$$\phi_1(v_1) = (v_1, \bot), \qquad \phi_2(v_2) = (\bot, v_1),$$

where $V = V_1 \times V_2$. Thus we can define functions

$$f_i = p_i \circ f \circ \phi_i : V_i \longrightarrow D_i \qquad i = 1, 2.$$

Note that if f is stable, so are the f_i, since ϕ_i is stable, while p_i preserves all meets since it is a projection.

How can we (approximately) reconstruct f from f_1 and f_2? We assume that $f \circ \phi_i$ factors through the inclusion $D_i \hookrightarrow D$, $i = 1, 2$. Hence we can define stable functions

$$f'_i = e_i \circ f_i \circ \pi_i : V \longrightarrow D \qquad i = 1, 2.$$

Note that $f'_i \sqsubseteq f$, $i = 1, 2$. Hence we can define $f_1[f_2] : V \longrightarrow E$ by

$$f_1[f_2] : v \mapsto f'_1(v) \sqcup f'_2(v).$$

Proposition 4.4 (i) $f_1[f_2] \sqsubseteq f$.
(ii) *If* $f_i \sqsubseteq g_i$, $i = 1, 2$, *then* $f_1[f_2] \sqsubseteq g_1[g_2]$.

5 Semantic Proof Structures

We begin by reviewing the Hughes-van Glabeek (HvG) definition of proof structure and proof net [HvG05].

Preliminary definitions for HvG Proof nets

They define an *additive resolution* of a MALL sequent Γ to be the result of deleting one argument of each occurrence of an additive connective $\&/\oplus$. An *axiom link* is an edge between a pair of complementary occurrences of some literal. A *linking* λ on an additive resolution of Γ is a set of axiom links, each of which involves occurrences which remain in the additive resolution, and such that each literal occurrence remaining in the additive resolution is in exactly one link in λ. A $\&$-resolution of Γ is the result of deleting one argument of each occurrence of $\&$ in Γ. A linking is *on a $\&$-resolution* if every literal occurrence in the linking is in the $\&$-resolution. A $\otimes$-*switching* of an additive resolution of Γ is the result of deleting one argument of each occurrence of $\otimes$ in the additive resolution.

Definition 5.1 An HvG proof-net for Γ is a set Θ of linkings on additive resolutions of Γ such that:

(i) For each $\&$-resolution of Γ, there is exactly one linking $\lambda \in \Theta$ on that $\&$-resolution.

(ii) Every $\otimes$-switching of every $\lambda \in \Theta$ is connected and acyclic.

(iii) A further, rather subtle technical condition known as *Toggling*.

We shall not discuss Toggling further here. It remains a far from intuitive notion, and a goal for future research is to understand it better. We hope that the tools developed in the present paper will help towards this. We shall take a minor liberty with terminology, and refer to sets of linkings satisfying the first of the above conditions as *HvG proof-structures*.

5.1 A Domain-theoretic Formalization of HvG

The various notions used by Hughes and van Glabeek, such as additive resolution, $\&$-resolution etc., are quite intuitive. They also provide formal definitions of these concepts, in terms of labelled graphs. We shall pursue an alternative, more "semantic" formalization, in domain-theoretic terms. This exposes some mathematical structure inherent in these definitions, but not made explicit in [HvG05], and which will prove useful and enlightening. Our approach will build on the semantic insights from [AJ94a,AM99].

Given a MALL sequent Γ, we shall introduce a number of event domains associated with Γ. We will use these in our presentation of MALL proof-nets.

Firstly, we define a poset $\mathcal{D}(A)$ for each MALL formula A, inductively as follows:

$$
\begin{aligned}
\mathcal{D}(\alpha) \quad &= \quad \mathcal{D}(\alpha^\perp) \quad = \quad \mathbb{O} \\
\mathcal{D}(A \otimes B) \quad &= \quad \mathcal{D}(A \,\wp\, B) \quad = \quad \mathcal{D}(A) \times \mathcal{D}(B) \\
\mathcal{D}(A \,\&\, B) \quad &= \quad \mathcal{D}(A \oplus B) \quad = \quad (\mathcal{D}(A) + \mathcal{D}(B))_\perp \\
&\quad\;\; \mathcal{D}(\xi) \quad = \quad \mathbb{O}.
\end{aligned}
$$

We extend this assignment to sequents $\Gamma = A_1, \ldots, A_k$ by

$$
\mathcal{D}(\Gamma) = \mathcal{D}(A_1) \times \cdots \times \mathcal{D}(A_k).
$$

Note that this is consistent with treating the sequent as the Par of its formulas. Recall that $\mathbb{O} = \mathbf{1}_\perp$ is the two-element lattice $\perp \sqsubseteq \top$.

The intuitions behind these definitions follow those from [AJ94a,AM99], and are discussed extensively in [Abr03]. Briefly, the interpretation of multplicative connectives as products reflect their connection with *concurrency and causality*; while the interpretation of the additives as separated sum reflect their connection with *choice, conflict and moments of synchronization* (cf. [Gir87,Gir88]). The interpretation of the atoms by the Sierpinski domain $\mathbb{O}$ should be seen as a convenient *instance* at which to take what is really a *parametric* (in fact, *functorial*) definition, with the propositional atoms as the parameters.
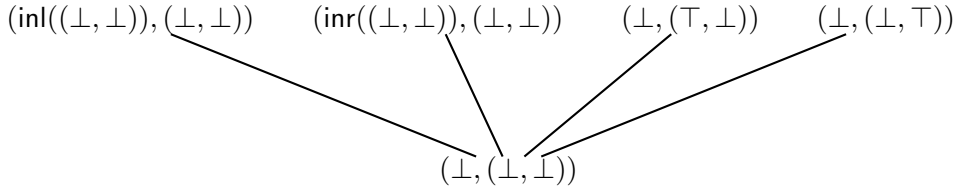
Notation We shall often write $\mathcal{D}_\Gamma$ rather than $\mathcal{D}(\Gamma)$ for convenience.

Example

Consider the sequent $\Gamma = (\alpha^\perp \wp \alpha^\perp) \,\&\, (\alpha^\perp \wp \alpha^\perp), \alpha \otimes \alpha$. The domain $\mathcal{D}_\Gamma$ is:

$$
(\mathbb{O}^2 + \mathbb{O}^2)_\perp \times \mathbb{O}^2.
$$

We illustrate the bottom element and atoms of this domain as follows:

The increase in the ordering in the first component to decide the separated sum corresponds to resolving the additive choice.

We now relate this formal structure to the HvG notions, with the following simple observation.

Proposition 5.2 *The maximal elements of $\mathcal{D}_\Gamma$ are in one-to-one correspondence with the additive resolutions of Γ; while the maximal primes of $\mathcal{D}_\Gamma$ are in one-to-one correspondence with the occurrences of literals and parameter instances in Γ.*

The latter part of this proposition shows the convenience of using $\mathbb{O}$ as the interpretation of the literals. Note, for example, that the interpretation of any purely multiplicative sequent Γ is the product $\mathbb{O}^{\mathcal{L}(\Gamma)}$ indexed over the literal occurrences in Γ. There is a unique maximal element, namely the tuple in which all components are $\top$. The maximal primes are those tuples in which exactly one component is $\top$, yielding the bijective correspondence. If we had used the one-point domain $\mathbf{1}$ instead of $\mathbb{O}$, then the corresponding product would still have just one element!

In the above example, note that there are two additive resolutions of this formula, corresponding to choosing the left or right argument of $\&$; the corresponding maximal elements are $(\mathsf{inl}((\top, \top)), (\top, \top))$ and $(\mathsf{inr}((\top, \top)), (\top, \top))$. The leftmost occurrence of α (positive literal) corresponds to the maximal prime $(\bot, (\top, \bot))$.

Notation

For each literal occurrence L in Γ, we write a_L for the corresponding maximal prime in $\mathcal{D}_\Gamma$. Given $d \in \mathcal{D}_\Gamma$, we define:

$$|d| = \{a_L \mid L \in \mathcal{L}(\Gamma), a_L \sqsubseteq d\}.$$

Fact 5.3 *If $d \sqsubseteq d'$, then $|d|_\Gamma \subseteq |d'|_\Gamma$.*

5.1.2 Formalizing Linkings

To specify a linking on an additive resolution of Γ explicitly, we must give two things:

- An additive resolution of Γ, which by Proposition 5.2 corresponds to a maximal element d of $\mathcal{D}_\Gamma$.

- A set of axiom links which partition the literal occurrences in the additive resolution into complementary pairings $\{L, L^\perp\}$. This amounts to specifying a *literal-respecting partial involution* (just as for MLL) on $|d|$.

This is naturally formalized as a *dependent sum*. We write $S^\partial(X)$ for the set of partial bijections on a set X, ordered by inclusion. Representing partial bijections by their graphs, we have:

$$X \subseteq Y \implies S^\partial(X) \subseteq S^\partial(Y).$$

We now define the *domain of pre-linkings*:

$$\mathcal{E}_\Gamma = (\Sigma d \in \mathcal{D}_\Gamma)S^\partial(|d|)$$
$$= \{(d, \pi) \mid d \in \mathcal{D}_\Gamma, \ \pi \in S^\partial(|d|)\}$$

with the pointwise ordering:

$$(d, \pi) \sqsubseteq (d', \pi') \iff d \sqsubseteq d' \wedge \pi \subseteq \pi'.$$

We define a *linking* to be a maximal element (d, π) of $\mathcal{E}_\Gamma$ such that π is a literal-respecting involution on $|d|$.
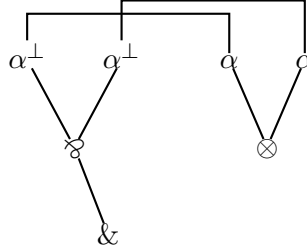
Example Continued

Continuing with the example from the previous sub-section, a maximal element of $\mathcal{D}_\Gamma$ is $d = (\mathsf{inl}((\top, \top)), (\top, \top))$. The corresponding set of literal occurrences is $|d| = \{a_i \mid i = 1, \ldots, 4\}$, where

$$a_1 = (\mathsf{inl}((\top, \bot)), (\bot, \bot)), \quad a_2 = (\mathsf{inl}((\bot, \top)), (\bot, \bot)), \quad a_3 = (\bot, (\top, \bot)), \quad a_4 = (\bot, (\bot, \top)).$$

The linkings on the additive resolution d are

$$(d, \{a_1 \leftrightarrow a_3, a_2 \leftrightarrow a_4\}), \quad (d, \{a_1 \leftrightarrow a_4, a_2 \leftrightarrow a_3\}).$$

We illustrate the first of these linkings as follows:

5.1.3 Formalizing &-Resolutions

Let $\mathcal{W}(\Gamma)$ be the set of occurrences of Withs (*i.e.* of subformulas of the form $A \& B$) in Γ. Let $\mathcal{V}_\Gamma$ be the poset of partial functions from $\mathcal{W}(\Gamma)$ into $\mathbb{B} = \{0, 1\}$, ordered by inclusion. Equivalently, $\mathcal{V}_\Gamma = \mathbb{B}_\bot^{\mathcal{W}(\Gamma)}$, a product of flat domains. We refer to elements of $\mathcal{V}_\Gamma$ as *(partial) valuations*; maximal elements are *total valuations*. These correspond to &-resolutions on Γ in an evident fashion.

If v is a valuation, and $W \in \mathcal{W}(\Gamma)$, we define a new valuation $v_{\neg W}$, with

$$v_{\neg W}(W') = v(W'), \qquad W' \neq W$$

$$v_{\neg W}(W) = \neg v(W)$$

where:

$$\neg 0 = 1, \quad \neg 1 = 0, \quad \neg \bot = \bot.$$

We refer to this as "toggling W".

We also define the valuation $v \backslash W$, such that $v \backslash W : W \mapsto \bot$, and $v \backslash W : W' \mapsto v(W')$, for $W' \neq W$. This extends to a set of With occurrences, $v \backslash \{W_1, \ldots, W_k\}$, in the obvious fashion.

5.1.4 HvG Proof Structures Formalized

We can now formalize HvG proof structures in our terms.

Definition of HvG proof structures: first attempt

An HvG proof structure on a MALL sequent Γ is a function

$$f : \mathsf{Max}(\mathcal{V}_\Gamma) \longrightarrow \mathsf{Max}(\mathcal{E}_\Gamma)$$

which assigns to each &-resolution (total valuation) a linking on an additive resolution of Γ (a maximal element (d, π) of $\mathcal{E}_\Gamma$ such that π is a literal-respecting involution on $|d|$).

This definition falls short of capturing the HvG definition in that we have not expressed the condition that the linking corresponding to a &-resolution must be *on* that &-resolution. Intuitively, this expresses the idea that the With resolution is being specified "externally" by the environment or context, and we must simply replicate this resolution in the corresponding "slice" of the proof structure.

We illustrate the necessity for this condition by the following (non-)example.

Non-Example

Consider the sequent $\Gamma = \alpha \oplus (\alpha \& \alpha), \alpha^\perp$, and the set of linkings $\{\lambda_1, \lambda_2\}$:
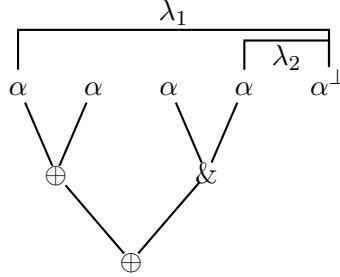
If we assign the first linking λ_1 to the valuation which sets the unique With-occurrence W to 0, and the second to the valuation setting $W = 1$, then we get a function satisfying the above definition; and yet this set of linkings evidently does not correspond to any sequent proof. However, both these linkings are on the &-valuation $[W = 1]$, and hence this is not an HvG proof structure.

In order to formulate the condition that a linking is on a &-resolution, we need to capture the idea that every literal occurrence L remaining in the additive resolution d, *i.e.* such that $a_L \sqsubseteq d$, induces a partial valuation on With occurrences which is consistent with the &-resolution.

To do this, we need to relate occurrences in the sequent Γ to elements of $\mathcal{D}_\Gamma$.

Semantic Occurrences

For each occurrence of a formula B in A, with corresponding linear context $C[\cdot]$, we define $(\!|C[\cdot]|\!)_A \in \mathsf{Pr}(\mathcal{D}(A))$:

$$(\!|[\cdot]|\!)_A = \bot_{\mathcal{D}(A)}$$

$$(\!|C[\cdot]\,\mathbf{m}\,B|\!)_{A\,\mathbf{m}\,B} = ((\!|C[\cdot]|\!)_A, \bot), \quad \mathbf{m} \in \{\otimes, \mathbin{\rotatebox[origin=c]{180}{\&}}\}$$

$$(\!|A\,\mathbf{m}\,C[\cdot]|\!)_{A\,\mathbf{m}\,B} = (\bot, (\!|C[\cdot]|\!)_B), \quad \mathbf{m} \in \{\otimes, \mathbin{\rotatebox[origin=c]{180}{\&}}\}$$

$$(\!|C[\cdot]\,\mathbf{a}\,B|\!)_{A\,\mathbf{a}\,B} = \mathsf{inl}((\!|C[\cdot]|\!)_A), \quad \mathbf{a} \in \{\&, \oplus\}$$

$$(\!|A\,\mathbf{a}\,C[\cdot]|\!)_{A\,\mathbf{a}\,B} = \mathsf{inr}((\!|C[\cdot]|\!)_B), \quad \mathbf{a} \in \{\&, \oplus\}$$

We extend this to occurrences in sequents $\Gamma = \Gamma_1, A, \Gamma_2$ by

$$(\!|\Gamma_1, C[\cdot], \Gamma_2|\!)_\Gamma = (\bot, \ldots, \bot, (\!|C[\cdot]|\!)_A, \bot, \ldots, \bot).$$

Example Continued

In our running example, the unique With occurrence W has the corresponding prime $(\!|W|\!) = (\bot, (\bot, \bot))$. Note that in general many distinct syntactic occurrences can be mapped to the same prime in $\mathcal{D}_\Gamma$.

We define a function

$$\mathsf{out} : \mathcal{D}_\Gamma \longrightarrow \mathcal{V}_\Gamma$$

which makes explicit how an additive resolution induces a valuation (in general partial) on With occurrences. For each $W \in \mathcal{W}(\Gamma)$, with corresponding context $C[\cdot]$ with $C[A \& B] = \Gamma$, we define:

$$\mathsf{out}(d)(W) = \begin{cases} 0, & (\!|C[[\cdot] \& B]|\!)_\Gamma \sqsubseteq d \\ 1, & (\!|C[A \& [\cdot]]|\!)_\Gamma \sqsubseteq d \\ \bot, & \text{otherwise} \end{cases}$$

An immediate consequence of the definitions is the following:

Proposition 5.4 *If $d \in \mathcal{D}_\Gamma$ and $W \in \mathcal{W}(\Gamma)$, then:*

$$(\!|W|\!) \not\sqsubseteq d \implies \mathsf{out}(d)(W) = \bot.$$

Example Continued

In our running example, the additive resolution $(\mathsf{inl}((\top, \top)), (\top, \top))$ is mapped by out to the valuation $[W = 0]$, while $(\mathsf{inr}((\top, \top)), (\top, \top))$ is mapped to $[W = 1]$. Note that $(\bot, (\top, \top))$ is mapped to $[W = \bot]$.

For convenience, we then lift this function to linkings:

$$\mathbf{p} : \mathcal{E}_\Gamma \longrightarrow \mathcal{V}_\Gamma :: (d, \pi) \mapsto \mathsf{out}(d).$$

Now given a total valuation v, and a linking (d, π), we can define (d, π) to be *on* v if for all literal occurrences L:

$$a_L \sqsubseteq d \implies \mathsf{out}(a_L) \sqsubseteq v.$$

Note that we are using the pointwise order on functions to compare the (in general partial) valuation $\mathsf{out}(a_L)$ with v.

There is a final subtlety. The HvG definition of a proof structure is simply a *set* of linkings. When we "uniformize" their definition into a function from valuations to linkings, this set will be the *image* of the function. Thus the condition that for every &-resolution, there is a unique linking in the set which is on that &-resolution, translates to uniqueness *in the image of the function* — a global rather than pointwise property.

Finally, we can provide our formal definition of HvG proof structures:

Definition 5.5 An HvG proof structure on a MALL sequent Γ is a function

$$f : \mathsf{Max}(\mathcal{V}_\Gamma) \longrightarrow \mathsf{Max}(\mathcal{E}_\Gamma)$$

which assigns to each &-resolution (total valuation) v a linking (d, π) which is on v. Moreover, it must satisfy the following *image-uniqueness* condition: for all valuations v', if (d, π) is on v', then $f(v') = (d, \pi) = f(v)$.

Proposition 5.6 *The image-uniqueness condition is equivalent to the following, more "local" condition:*
(Tog) *For all With occurrences W, if $(\!|W|\!) \not\sqsubseteq d$, then $f(v) = f(v_{\neg W})$.*

Proof. Assume image-uniqueness. If $(\!|W|\!) \not\sqsubseteq d$, then for all $a_L \sqsubseteq d$, $\mathsf{out}(a_L)(W) = \bot$. Hence d is on $v_{\neg W}$, and $f(v) = f(v_{\neg W})$.

For the converse, suppose that d is on v'. We can write $v' = v_{\neg W_1 \cdots \neg W_k}$. Suppose that for some W_i, we had $(\!|W_i|\!) \sqsubseteq d$. Then for some literal occurrence L above W_i, we would have $a_L \sqsubseteq d$, and $\mathsf{out}(a_L)(W_i) = v(W_i)$. This would contradict d on v', since clearly $\mathsf{out}(a_L) \not\sqsubseteq v'$. So we must have $(\!|W_i|\!) \not\sqsubseteq d$, $i = 1, \ldots, k$. Hence by k applications of **(Tog)**, $f(v) = f(v')$. $\square$

Non-Example Continued

We re-examine our counter-example to our first attempt at defining HvG. This translates into the function

$$[W = 0] \mapsto ((\mathsf{inl}(\mathsf{inl}(\top)), \top), \{a_1 \leftrightarrow a_5\}), \qquad [W = 1] \mapsto ((\mathsf{inr}(\mathsf{inr}(\top)), \top), \{a_4 \leftrightarrow a_5\})$$

where

$$a_1 = (\mathsf{inl}(\mathsf{inl}(\top)), \bot), \quad a_4 = (\mathsf{inr}(\mathsf{inr}(\top)), \bot), \quad a_5 = (\bot, \top).$$

Note that this function does assign a linking to each valuation, which *is* on that valuation; however, since the assignment to $[W = 0]$ is also on $[W = 1]$ (since $\mathsf{out}(a_1) = \bot$), it fails the image-uniqueness property.

5.2   Monotone Proof Structures

The HvG definitions are phrased in terms of complete additive and &-resolutions, which correrspond to maximal elements of our domains. Nevertheless, we have already found the domain structure useful, in defining occurrences as primes, and in formalizing the condition of a linking being on a &-resolution. Moreover, the HvG Toggling condition and related notions refer implicitly to partial valuations (via saturated sets of linkings) [HvG05].

We shall now make further and more essential use of the domain structure, to formulate a wider notion of proof structure, which we will subsequently use as the basis for our notion of proof net.

A *monotone proof structure* is simply a monotone function

$$f : \mathcal{V}_\Gamma \longrightarrow \mathcal{E}_\Gamma.$$

Note that such a function maps arbitrary valuations, not just total ones, into arbitrary pre-linkings.

What conditions should such a proof structure satisfy? An obvious one is that its restriction to *total* valuations should give rise to an HvG proof structure:

(PS1) The monotone function f cuts down to a map

$$f^{\mathsf{m}} : \mathsf{Max}(\mathcal{V}_\Gamma) \longrightarrow \mathsf{Max}(\mathcal{E}_\Gamma)$$

which is an HvG proof structure.

Next, we shall generalize the condition of unique linkings for each &-resolution to cover partial valuations. The HvG definitions are phrased in a "top-down" style in terms of linkings. In our setting, it is preferable to work upwards in the ordering, in a more constructive fashion. It then becomes more natural to formulate conditions in terms of the With occurrences themselves. In fact, we have already seen an example of such a reformulation in Proposition 5.6.

A first condition is that "relevant" With occurrences — those reachable in the output — should be decided as they are in the input. Formally:

(PS2) For $W \in \mathcal{W}(\Gamma)$, and $f(v) = (d, \pi)$:

$$(\!(W)\!) \sqsubseteq d \implies \mathsf{out}(d)(W) = v(W).$$

Proposition 5.7 *The condition **(PS2)** implies that* $\mathbf{p} \circ f \sqsubseteq \mathsf{id}$, *and that* (d, π) *is on* v.

Proof. For a With occurrence $W \in \mathcal{W}(\Gamma)$, if $\mathsf{out}(d)(W) \neq \perp$, then $(\!(W)\!) \sqsubseteq d$, and hence by **(PS2)**, $\mathsf{out}(d)(W) \sqsubseteq v(W)$. Thus $\mathbf{p} \circ f \sqsubseteq \mathsf{id}$. Similarly, if $a_L \sqsubseteq d$, then $\mathsf{out}(a_L) \sqsubseteq \mathsf{out}(d) \sqsubseteq v$. □

Example For a simple example, consider the formula $\alpha \& \alpha$. Any monotone function f which maps $[W = 0]$ either to $\perp$ or to $\mathsf{inr}(\top)$ violates **(PS2)**; in fact, the only permissible choice is of the form $\mathsf{inl}(d)$.

Proposition 5.8 *Given a valuation v, and $d \in \mathcal{D}_\Gamma$ such that $\mathsf{out}(d) \sqsubseteq v$, there is a least $d' \sqsupseteq d$ satisfying* **(PS2)**: *i.e. such that, for all With occurrences W:*

$$(\!|W|\!) \sqsubseteq d' \implies \mathsf{out}(d')(W) = v(W).$$

Notation: *We write $d' = \gamma_v(d)$.*

Proof. We define $D_v = \{x \in \mathcal{D}_\Gamma \mid \mathsf{out}(x) \sqsubseteq v\}$. We define a function on D_v:

$$x \mapsto x \sqcup \{p \in \mathsf{Pr}(\mathcal{D}_\Gamma) \mid \exists W. (\!|W|\!) \sqsubseteq x \wedge (\!|W|\!) \prec p \wedge \mathsf{out}(x)(W) \sqsubset \mathsf{out}(p)(W) = v(W)\}.$$

This function is increasing, and hence has a least fixpoint d' above d. By the fixpoint property, d' satisfies **(PS2)**. It is easily checked that d' is the meet of all elements of $\mathcal{D}_\Gamma$ above d which satisfy **(PS2)**, and that γ_v is a closure operator on D_v. Intuitively, the process of constructing $\gamma_v(d) = d'$ "fills all (hereditarily) accessible With cells according to v". $\qquad\square$

Proposition 5.9 **(PS2)** *is equivalent to the following, more algebraic condition:*

$$\mathbf{p} \circ f \sqsubseteq \mathsf{id} \ \wedge \ (\gamma_v \times \mathsf{id}) \circ f(v) = f(v).$$

We shall generalize the "global" aspect of the unique linkings condition by the following equation:
(PS3) If v is a total valuation, then $f \circ \mathbf{p} \circ f(v) = f(v)$.
This encapsulates the idea that the only With choices which actually affect the output are those which are "relevant" or "reachable", and hence appear in the output. This equation can also be seen as replacing the rather obscure "technical condition" in [Gir96].

Proposition 5.10 *Let f be a monotone proof structure satisfying* **(PS2)**. *Then* **(PS3)** *is equivalent to the following, more "local" condition:*
(Loc): *If v is total and $f(v) = (d, \pi)$, then for any $W_1, \ldots, W_k \in \mathcal{W}(\Gamma)$:*

$$\forall i. (\!|W_i|\!) \not\sqsubseteq d \implies f(v) = f(v \backslash \{W_1, \ldots, W_k\}).$$

Proof. Firstly, assume **(PS3)**. If $(\!|W_i|\!) \not\sqsubseteq d$, then $\mathbf{p} \circ f(v)(W_i) = \bot$. Hence $f(v) = f \circ \mathbf{p} \circ f(v) \sqsubseteq f(v \backslash \{W_1, \ldots, W_k\})$, while $f(v \backslash \{W_1, \ldots, W_k\}) \sqsubseteq f(v)$ by monotonicity.

Conversely, assume **(Loc)**. By **(PS2)**, $v' = \mathbf{p} \circ f(v) \sqsubseteq v$. We can write $v' = v \backslash \{W_1, \ldots, W_k\}$, where $(\!|W_i|\!) \not\sqsubseteq d$, $i = 1, \ldots, k$. Hence by **(Loc)**, $f(v') = f(v)$. $\quad\square$

By monotonicity, the condition **(Loc)** evidently implies that if $f(v) = (d, \pi)$ with v total, then for any $W \in \mathcal{W}(\Gamma)$:

$$(\!|W|\!) \not\sqsubseteq d \implies f(v) = f(v_{\neg W})$$

which by Proposition 5.6 is equivalent to the image-uniqueness part of the linkings condition.

Hence **(PS1)** is over-specified; it is sufficient to ask that f cuts down to a map f^{m} carrying total valuations to linkings.

Discussion

It would have been more in the spirit of making definitions constructively, "from below", to have stated the condition **(PS3)** for *all* valuations, not just total ones. In fact, this stronger axiom would have led to a perfectly viable theory of stable proof nets; in particular, the stable proof nets we shall assign to sequent proofs in Section 7 do satisfy this stronger condition. Our reasons for preferring the weaker axiom are as follows:

- Firstly, we wish to allow a wider class of proof structures, beyond the stable ones, with an eye to future developments. The restricted condition we have given allows for this. For example, Proposition 6.1 in the next Section would no longer hold if we used the stronger axiom.
- The condition **(PS3)** as stated suffices to prove Sequentialization (see in particular the proof of Lemma 8.7), and hence we get a stronger version of the result by using it.

Non-Example Continued

We look again our counter-example to our first attempt at defining HvG. Any monotone extension of the function

$$[W = 0] \mapsto ((\mathsf{inl}(\mathsf{inl}(\top)), \top), \{a_1 \leftrightarrow a_5\}), \qquad [W = 1] \mapsto ((\mathsf{inr}(\mathsf{inr}(\top)), \top), \{a_4 \leftrightarrow a_5\})$$

must map $[W = \bot]$ to $(\bot, d)$, $d \in \mathbb{O}$. Such a function violates **(PS3)**, since

$$f \circ \mathbf{p} \circ f([W = 0]) = f([W = \bot]) = (\bot, d) \neq f([W = 0]).$$

Definition 5.11 We call a monotone function satisfying **(PS1)**–**(PS3)** a *semantic proof structure.*

Example Continued

In our running example, the map

$$[W = \bot] \mapsto ((\bot, (\bot, \bot)), \varnothing), \qquad [W = 0] \mapsto (d_1, \pi_1), \qquad [W = 1] \mapsto (d_2, \pi_2)$$

where we display the linkings as

$$(d_1, \pi_1) = ((\mathsf{inl}((\overset{a_1}{\bullet}, \overset{a_2}{\bullet})), (\overset{a_3}{\bullet}, \overset{a_4}{\bullet})), \{a_1 \leftrightarrow a_3, a_2 \leftrightarrow a_4\})$$

$$(d_2, \pi_2) = ((\mathsf{inr}((\overset{a_1'}{\bullet}, \overset{a_2'}{\bullet})), (\overset{a_3}{\bullet}, \overset{a_4}{\bullet})), \{a_1' \leftrightarrow a_4, a_2' \leftrightarrow a_3\})$$

is a semantic proof structure.

6 Proof Structures Compared

6.1 HvG vs. Semantic Proof Structures

Semantic proof structures contain more information than HvG structures; they describe a *process* of developing an additive resolution and a linking as a function of increasing partial information about the &-resolution, as provided by an external environment. This has both positive and negative aspects:

- On the plus side, this process view leads to an elegant, compositional approach to Cut-Elimination, as shown in [AJ94a,AM99].

- On the negative side, we can see the HvG proof structures as a "fully abstract" representation; by adding extra, "intensional" information we are making additional distinctions.

We can define an *extensional equivalence* on semantic proof structures:

$$f \approx g \iff f^{\mathsf{m}} = g^{\mathsf{m}}.$$

Thus semantic proof structures are extensionally equivalent if they determine the same HvG proof structures. Conversely, to each HvG proof structure f there is a corresponding extensional equivalence class $E[f]$ of semantic proof structures, comprising all *monotone extensions of f satisfying **(PS1)**–**(PS3)***. We can think of these extensions as *realizations* of f.

6.2 Structure of extensional equivalence classes

We make some basic observations.

Proposition 6.1 *Each HvG proof structure*

$$f : \mathsf{Max}(\mathcal{V}_\Gamma) \longrightarrow \mathsf{Max}(\mathcal{E}_\Gamma)$$

has a greatest monotone extension

$$\hat{f} : \mathcal{V}_\Gamma \longrightarrow \mathcal{E}_\Gamma :: v \mapsto \bigsqcap \{f(v') \mid v \sqsubseteq v' \in \mathsf{Max}(\mathcal{V}_\Gamma)\}$$

to a semantic proof structure. We refer to this as the canonical extension.

Proof. Since Hughes and van Glabbeek do not consider generalized axioms, we shall assume these are not present in the sequent. The condition **(PS1)** is immediate, since $(\hat{f})^{\mathsf{m}} = f$. Consider some valuation v, and a With occurrence W. Let $(d, \pi) = \hat{f}(v)$. Consider firstly the case when $(\!|W|\!) \sqsubseteq d$, and the sub-case when $v(W) \neq \bot$. For any maximal extension v' of v, there is a literal occurrence L above W, such that $(\!|W|\!) \sqsubseteq a_L \sqsubseteq d'$, where $f(v') = (d', \pi')$. Since d' must be on v', we have $\mathsf{out}(a_L)(W) = v'(W) = v(W)$. This shows that $\mathsf{out}(d)(W) = v(W)$. Now consider the sub-case when $v(W) = \bot$. Then there are maximal extensions v_1 and v_2 with $v_1(W) = 0$ and $v_2(W) = 1$. Let $(d_i, \pi_i) = f(v_i)$, $i = 1, 2$. By the same reasoning as

in the first sub-case, $\mathsf{out}(d_1)(W) = 0$ and $\mathsf{out}(d_2)(W) = 1$. Hence $\mathsf{out}(d)(W) = \bot$. Thus $\hat{f}$ satisfies **(PS2)**.

Now suppose that v is total, and $\langle\!\langle W_i \rangle\!\rangle \not\sqsubseteq d$, $i = 1, \ldots, k$. By Proposition 5.10, to show that $\hat{f}$ satisfies **(PS3)**, it suffices to show that $\hat{f}(v) = \hat{f}(v \backslash \{W_1, \ldots, W_k\})$. The maximal extensions of $v \backslash \{W_1, \ldots, W_k\}$ have the form $v' = v_{\neg W_{i_1} \cdots \neg W_{i_l}}$, where the W_{i_j} are a subset of the W_i. For each such v', we have $f(v) = f(v')$ by Proposition 5.6. Hence $\hat{f}(v) = \hat{f}(v \backslash \{W_1, \ldots, W_k\})$, as required. $\qquad\square$

Note that $(\hat{f})^{\mathsf{m}} = f$, so we can recover the HvG proof structure from its canonical extension. In effect, we can regard HvG proof structures as embedded in the larger space of semantic proof structures via their canonical extensions. Moreover, any extensional equivalence class can be written as $E[f^{\mathsf{m}}]$, where f is any representative of the class.

It is easily seen that semantic proof structures in the same extensional equivalence class are closed under pointwise sup. Hence we have the following.

Proposition 6.2 $E[f]$ *forms an upper semilattice under the pointwise ordering, with* $\hat{f}$ *as the greatest element.*

6.3 Sequential vs. parallel realizations

Proposition 6.3 $\mathcal{D}_\Gamma$ *and* $\mathcal{E}_\Gamma$ *are event domains.* $S^\partial(X)$ *and* $\mathcal{V}_\Gamma$ *are atomic domains.*

Since semantic proof structures are monotone functions between event domains, they can in particular be *stable* or *sequential*. We note that in general, the canonical extension $\hat{f}$ of an HvG proof structure will be a parallel function — *i.e.* neither sequential nor stable. The example given at the end of Section 1 of [HvG05] provides a suitable illustration of this, which we will not reproduce here. A fine structure of "degrees of parallelism" thus opens up in looking at the realizations $E[f]$ of an HvG proof structure.

6.4 Girard proof structures

It is also possible to relate semantic proof structures to Girard's notion of proof structures for MALL [Gir96], but we will not elaborate on this here. Note that Hughes and van Glabeek [HvG05] discuss how their proof structures can be converted into Girard proof structures (in general not satisfying the monomial condition [Gir96]). Also, in the extended version of [AM99], another route is given from a game semantics closely related to our present approach to Girard proof structures (in this case, which *do* satisfy the monomial condition). The purpose of this was to use the Sequentializability Theorem of [Gir96] in proving the Full Completeness of the concurrent games model in [AM99]. One the main motivations for our present approach, by contrast, is to prove Sequentialization directly for our "semantic proof nets", and hence enable a more self-contained, and in our view illuminating, route to Full Completeness.

7 Stable Proof Nets

We now come to the key definition of *stable proof nets*. These will be the class of semantic proof structures which correspond to MALL proofs.

One might ask: why should "semantic" objects such as our semantic proof structures be seen as reasonable representations of proofs? But: why not! We are seeking a "geometric", intrinsic representation of proofs. As we have seen, MLL proof nets can be represented as certain permutations on literals, also a "geometric" and "semantic" idea. The important point is that this representation has sufficient structure to allow us to prove a Sequentialization Theorem, which enables us to find a sequent proof for each such permutation. More particularly, given the permutation, we can define a set of switching graphs and formulate the Danos-Regnier criterion to identify which permutations arise from proofs.

We shall follow an entirely analogous procedure here. We shall use the domain-theoretic structure, and in particular the property of *stability* of the proof structure, to define a dependency relation in a semantic style, as an analogue to the syntactic notion in [Gir96]. This will then allow us to define the set of switching graphs for a stable semantic proof structure, and to formulate a Danos-Regnier criterion. Hence we can define our notion of stable proof net, for which we can prove an appropriate form of Sequentialization Theorem (in fact, a stronger form than in [Gir96]).

7.1 The dependency relation

Let $f : \mathcal{V}_\Gamma \longrightarrow \mathcal{E}_\Gamma$ be a stable proof structure, and $v \in \mathsf{Max}(\mathcal{V}_\Gamma)$. Given a With occurrence $W \in \mathcal{W}(\Gamma)$, and an occurrence $O \in \mathcal{O}(\Gamma)$, we say that O *depends on W in v* if $v'(W)\!\downarrow$, where $v' = \mathsf{M}(f, v, e)$, and one of the following cases holds:

(i) $e = (\langle\!\langle O \rangle\!\rangle_\Gamma, \varnothing)$.

(ii) $O = L$ is an occurrence of a literal l, and there is an occurrence M of $l^\perp$ such that
$$e = (a_L \sqcup a_M, \{(a_L, a_M), (a_M, a_L)\}).$$

(iii) O is an occurrence ξ_i, and $e = (\langle\!\langle \xi_1 \rangle\!\rangle_\Gamma \sqcup \cdots \sqcup \langle\!\langle \xi_k \rangle\!\rangle_\Gamma, \varnothing)$.

.

7.2 Switchings and Switching Graphs

Now given $v \in \mathsf{Max}(\mathcal{V}_\Gamma)$ with $f(v) = (d, \pi)$, define a switching S to be an assignment of L or R to every occurrence of $\invamp$ in d, and a choice of a *jump* for every occurrence W of a With in d, where a jump is an occurrence O depending on W in v. We say that a jump is *normal* if it is the premise of W specified by $v(W)$, and *proper* otherwise.

We can then define a switching graph $\mathcal{G}_\Gamma(f, v, S)$ with:

• vertices given by the subformula occurrences in d, *i.e.* occurrences $O \in \mathcal{O}(\Gamma)$ with $\langle\!\langle O \rangle\!\rangle_\Gamma \sqsubseteq d$;

- an edge connecting A to $A \otimes B$ and an edge connecting B to $A \otimes B$ for each occurrence of $A \otimes B$;

- an edge connecting A to $A \,\invamp\, B$ if S assigns L to $A \,\invamp\, B$, and an edge connecting B to $A \,\invamp\, B$ if S assigns R to $A \,\invamp\, B$;

- an edge connecting literal occurrences L and M if $\pi(a_L) = a_M$;

- if $\xi_1, \ldots, \xi_k$ are the occurrences of a parameter ξ, there are edges connecting ξ_i and ξ_{i+1}, for $1 \le i < k$;

- an edge connecting each $\oplus$ to its unique premise in d;

- an edge connecting each With occurrence to its jump as specified by S.

We say that f is a *stable MALL proof net* if for every $v \in \mathsf{Max}(\mathcal{V}_\Gamma)$ and switching S, $\mathcal{G}_\Gamma(f, v, S)$ is connected and acyclic.

Example Continued

We show one of the switching graphs for the proof structure considered above:
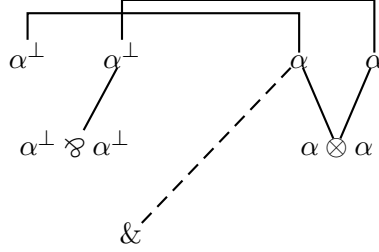
Here we are showing the graph for the valuation $v = [W = 0]$, and the switching S which sets the Par occurrence to R, and chooses the proper jump from the With occurrence W to the leftmost occurrence of α. Note that this occurrence depends on W because the axiom link it is connected to in this valuation does.

This switching graph is a tree; and in fact, the proof structure is a proof-net.

7.3 Assignment of Proof Structures to Cut-Free Sequent Proofs

Axiom

$$\frac{}{\vdash \alpha, \alpha^\perp} \ \mathsf{Id}$$

We assign $\perp \mapsto ((\top, \top), \{(a_L, a_M), (a_M, a_L)\})$, where L and M are the occurrences of α and $\alpha^\perp$.

Generalized Axiom

$$\frac{}{\vdash \xi_1, \ldots, \xi_k} \ \mathsf{Ax}$$

Since $|\xi_1, \ldots \xi_k|_\Gamma = \varnothing$, there is a unique proof structure for this sequent, which we assign.

Multiplicatives

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \; \mathsf{Times}$$

Suppose we have assigned f_1 to the proof of $\vdash \Gamma, A$, and f_2 to the proof of $\vdash B, \Delta$. The set of With occurrences in $\vdash \Gamma, A \otimes B, \Delta$ is the disjoint union of the occurrences in the two premises, which in turn induces a decomposition of valuations as $v = (v_1, v_2)$. Suppose that $f_i(v_i) = (d_i, \pi_i)$, $i = 1, 2$. We assign the proof structure f, defined by

$$f(v) = ((d_1, d_2), \pi_1 + \pi_2)$$

to the conclusion of the rule.

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \,\invamp\, B} \; \mathsf{Par}$$

In this case, up to associativity of the cartesian product and disjoint union, the *same* proof structure is assigned to the conclusion as to the premise of the Par rule.

Additives

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \; \mathsf{PlusL} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B} \; \mathsf{PlusR}$$

Suppose the proof structure assigned to the premise of the PlusL rule is f. Then the proof structure assigned to the conclusion is given by

$$\mathsf{L}(f) :: v \mapsto ((\boldsymbol{d}, \mathsf{inl}(d)), \pi) \qquad \text{where} \quad f(v) = ((\boldsymbol{d}, d), \pi).$$

The assignment $g \mapsto \mathsf{R}(g)$ for the PlusR rule is similar.

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B} \; \mathsf{With}$$

Suppose the proof structures assigned to the two premises of the rule are f and g. We write a valuation in $\mathcal{V}(\Gamma, A \,\&\, B)$ as (v, b), where $b \in \mathbb{B}_\perp$ is the value assigned to the With occurrence $A \,\&\, B$ appearing in the conclusion of the rule. Then we assign the proof structure h to the conclusion, where:

$$
\begin{aligned}
h(v, 0) &= \mathsf{L}(f)(v) \\
h(v, 1) &= \mathsf{R}(g)(v) \\
h(v, \perp) &= \perp_v.
\end{aligned}
$$

Here, using the notation of Proposition 5.8, $\perp_v = (\gamma_{v,\perp}(\perp), \varnothing)$.

Note that the last equation in this definition is *the only place where any latitude appears* in the definition of the assignment of proof structures to sequent proofs. The above definition can be written using a *sequential conditional*:

$$h(v, b) = \perp_v \ \sqcup \ \textbf{if } b \textbf{ then } \mathsf{L}(f)(v) \textbf{ else } \mathsf{R}(g)(v).$$

This is the *least* (most conservative, most lazy) possible definition.

Note that we can use the *same assignment* to define the HvG proof structure

$$f : \mathsf{Max}(\mathcal{V}_\Gamma) \longrightarrow \mathsf{Max}(\mathcal{E}_\Gamma)$$

assigned to a sequent proof, except that we can simply omit the last case in the definition of the function assigned to the With rule, since we are only concerned with total valuations.

Proposition 7.1 *For every sequent proof, the corresponding proof structure given by the above assignment is sequential.*

Proof. We argue by induction on the structure of the sequent proof. All cases except for the With rule are straightforward, by virtue of Proposition 4.3. For the With rule, we informally exhibit a sequential algorithm (in the sense of [Cur93]) for computing the function assigned to the rule. Recalling the demand-driven nature of sequential algorithms, we distinguish two cases in the output cell which may be requested:

- In the first case, the requested cell may correspond to a With occurrence which is hereditarily accessible from the initial state $\perp$, and hence should be filled according to v. The important point is that whether this case applies can be *statically determined*, independent of the input (v, b). A With occurrence W is hereditarily accessible if the path from W to the root of its formula tree contains only multiplicatives and Withs. In this case, taking the With occurrences on this path in reverse order, starting from the root (conclusion), we form a sequence of queries to the input v. If all of these queries are answered successfully, i.e. v is defined on all of them, and gives the answers corresponding to the path to W, then we ask v for W itself, and fill the cell for W accordingly. It is clear that the cells we can fill in this way correspond exactly to $\perp_v$.

- In case any other cell is requested, we firstly request the cell corresponding to b; if we receive an answer 0, we apply the sequential algorithm corresponding to f, while if we receive 1, we apply the sequential algorithm corresponding to g.

Note that we can assume inductively that f and g, since they both satisfy **(PS2)**, would agree on the cells covered by the first case, and would fill them as in $\perp_v$. $\square$

Theorem 7.2 (Soundness) *For every sequent proof, its denotation as a proof structure is a proof net.*

We could also use a *parallel conditional* in defining the proof net assignment for

the With rule: the final case $h(v, \perp) = \perp_v$ is then replaced by

$$h(v, \perp) = \mathsf{L}(f)(v) \sqcap \mathsf{R}(g)(v).$$

This is the *greatest* (most eager) definition compatible with the monotonicity of h. This still yields a well-defined proof structure, which is indeed extensionally equivalent to the sequential one given by the above assignment. However, we would no longer have stability, and hence could not define the dependency relation and jumps as above. This would correspond to doing without the monomial condition in [Gir96]. Hughes and van Glabbeek have shown that it is possible to work without the monomial condition; it remains a challenge to conceptualize their approach within our framework.

8 Sequentialization

The major result on proof nets is the Sequentialization Theorem, due to Girard, and incorporating a significant refinement due to Danos and Regnier [Gir87,DR89,Gir91,Gir96].

Theorem 8.1 (Sequentialization) *For every stable proof net f, there is a sequent proof Π such that $g \sqsubseteq f$ (in the pointwise order), where $g = \llbracket \Pi \rrbracket$ is the sequential proof net assigned to Π.*

Corollary 8.2 *With notation as in the Theorem: $f^{\mathsf{m}} = g^{\mathsf{m}}$.*

Proof. If $g \sqsubseteq f$, then for any $v \in \mathsf{Max}(\mathcal{V}_\Gamma)$, $g(v) \sqsubseteq f(v)$. But $g(v) \in \mathsf{Max}(\mathcal{E}_\Gamma)$, hence $g(v) = f(v)$. □

Since our notions of proof structure and proof net are formulated very differently to those in [Gir96], and since our result is stronger, we shall give a detailed proof of this Theorem. We shall follow the proof in [Gir96] quite closely, indicating where differences arise.

8.1 Empires

In this subsection, we fix a sequent Γ, stable proof net f, and total valuation $v \in \mathsf{Max}(\mathcal{V}_\Gamma)$. All switchings S will be relative to v.

We say than an occurrence P of a formula A is a *premise* of an occurrence O of a formula B in $\mathcal{G}_\Gamma(f, v, S)$ if there is an edge from P to O in $\mathcal{G}_\Gamma(f, v, S)$, and A is an immediate subformula of B. Note that P can be the premise of at most one occurrence.

Let O be an occurrence in $\mathcal{G}_\Gamma(f, v, S)$. We consider the sub-graph $\mathcal{G}_\Gamma^{O-}(f, v, S)$ obtained by erasing the edge, if any, connecting O as a premise to an occurrence P. Since $\mathcal{G}_\Gamma(f, v, S)$ is connected by hypothesis, $\mathcal{G}_\Gamma^{O-}(f, v, S)$ contains at most two connected components. We define $\mathcal{G}_\Gamma^O(f, v, S)$ to be the component containing O.

We define the *empire* of O, $\mathbf{e}O = \bigcap_S \mathcal{G}_\Gamma^O(f, v, S)$, to be the intersection of the graphs $\mathcal{G}_\Gamma^O(f, v, S)$ as S ranges over all switchings relative to v.

We verify the basic properties of empires.

Lemma 8.3 (Closure Properties) *Let O be an occurrence in $\mathcal{G}_\Gamma(f, v, S)$.*

(i) *If O is the premise of an occurrence O' of a Par or With, then O' is not in $\mathbf{e}O$.*

(ii) *If $\alpha \rule{1cm}{0.4pt} \alpha^\perp$ is an axiom link in $\mathcal{G}_\Gamma(f, v, S)$, then $\alpha \in \mathbf{e}O$ iff $\alpha^\perp \in \mathbf{e}O$.*

(iii) *If $\xi_1, \ldots, \xi_k$ are the occurrences of ξ in $\mathcal{G}_\Gamma(f, v, S)$, then $\xi_i \in \mathbf{e}O$ iff $\xi_j \in \mathbf{e}O$, for $1 \leq i, j \leq k$.*

(iv) *An occurrence of $A \otimes B$, both of whose premises are distinct from O, is in $\mathbf{e}O$ iff the corresponding occurrence of A is in $\mathbf{e}O$, iff the corresponding occurrence of B is in $\mathbf{e}O$.*

(v) *An occurrence of $A \oplus B$, both of whose premises are distinct from O, is in $\mathbf{e}O$ iff the corresponding occurrence of C is in $\mathbf{e}O$, where C is whichever of A or B is present in $\mathcal{G}_\Gamma(f, v, S)$.*

(vi) *An occurrence of $A \bindnasrepma B$,, both of whose premises are distinct from O, is in $\mathbf{e}O$ iff the corresponding occurrences of both A and B are in $\mathbf{e}O$.*

(vii) *An occurrence W of $A \& B$, both of whose premises are distinct from O, is in $\mathbf{e}O$ iff for every occurrence O' depending on W in v, O' is in $\mathbf{e}O$.*

Proof. Parts (i)–(v) are straightforward from the definitions. Parts (vi) and (vii) are similar; we prove (vii). Suppose firstly that $w \in \mathbf{e}O$, and O' depends on W in v. Assume for a contradiction that for some switching S, $O' \notin \mathcal{G}_\Gamma^O(f, v, S)$. It must be the case that O is the premise of a link P, and S connects O' and P. Moreover, after removing the edge $O \rule{1cm}{0.4pt} P$, O' is in the same component as P. The jump from W specified by S links it to an occurrence Q. This edge cannot lie on the path from P to O' in $\mathcal{G}_\Gamma^{O-}(f, v, S)$. If we modify S by replacing this jump by one to O', resulting in a switching S', then W is still in $\mathcal{G}_\Gamma^O(f, v, S')$, since W is in $\mathbf{e}O$. Moreover, P and O' are still connected in $\mathcal{G}_\Gamma^O(f, v, S')$, while W is now connected to O'. Hence we get a cycle

$$O \cdots W O' \cdots P O$$

in $\mathcal{G}_\Gamma(f, v, S')$, yielding the required contradiction.

Now assume that all occurrences O' depending on W in v are in $\mathbf{e}O$. (Note that there is at least one such occurrence, namely the normal jump of W). Given any switching S, $\mathcal{G}_\Gamma(f, v, S)$ has an edge between W and some occurrence O', which by hypothesis belongs to $\mathbf{e}O$ and hence to $\mathcal{G}_\Gamma^O(f, v, S)$. Hence W belongs to $\mathcal{G}_\Gamma^O(f, v, S)$. It follows that W is in $\mathbf{e}O$. $\square$

Lemma 8.4 (Principal Switchings) *For each occurrence O, there is a switching S such that $\mathbf{e}O = \mathcal{G}_\Gamma^O(f, v, S)$. S is called a* principal switching *for O.*

Proof. S is defined as follows. If O is the premise of a Par or With occurrence O', set S to connect O to O'. Otherwise:

- If O' is an occurrence of a Par formula which is not in $\mathbf{e}O$, then by Lemma 8.3, at least one of the premises of O', say P, is not in $\mathbf{e}O$. Set S to connect P and O'.

- If W is an occurrence of a With formula which is not in $\mathbf{e}O$, then by Lemma 8.3, at least one occurrence depending on W in v, say P, is not in $\mathbf{e}O$. Set S to connect P and W.

The remaining With and Par occurrences can be set arbitrarily by S. $\qquad\square$

Lemma 8.5 (Nesting Property) *Let O and O' be distinct occurrences in $\mathcal{G}_\Gamma(f, v, S)$, and assume that $O' \notin \mathbf{e}O$. Then*

(i) *If $O \in \mathbf{e}O'$, then $\mathbf{e}O \subset \mathbf{e}O'$.*

(ii) *If $O \notin \mathbf{e}O'$, then $\mathbf{e}O \cap \mathbf{e}O' = \varnothing$.*

Proof. The construction of a principal switching S for $\mathbf{e}O'$ as in Lemma 8.4 is further specified as follows:

- If P is a Par or With occurrence which is in $\mathbf{e}O'$ but not in $\mathbf{e}O$, then we set S as we would for a principal switching for $\mathbf{e}O$.

- If O is the premise of a Par or With occurrence Q in $\mathbf{e}O'$, then we set S so as to connect O and Q.

Thus S is still a principal switching for $\mathbf{e}O'$, so that $\mathbf{e}O' = \mathcal{G}_\Gamma^{O'}(f, v, S)$, which moreover does not contain any edges connecting $\mathbf{e}O \cap \mathbf{e}O'$ with $\mathbf{e}O^c \cap \mathbf{e}O'$, except possibly for an edge between O and Q. We argue by cases:

- If $O \in \mathbf{e}O'$, since by assumption $O' \notin \mathbf{e}O$, while $O' \in \mathbf{e}O'$, there is an edge between $\mathbf{e}O$ and $\mathbf{e}O^c$ in $\mathbf{e}O'$, which must be between O and Q. This implies that any path from O to O' in $\mathcal{G}_\Gamma^{O'}(f, v, S)$ must go via the edge from O to Q, and hence there is no path from O to O' in $\mathcal{G}_\Gamma^O(f, v, S) \cap \mathcal{G}_\Gamma^{O'}(f, v, S)$. Since the only edge in $\mathcal{G}_\Gamma^O(f, v, S)$ which is not in $\mathcal{G}_\Gamma^{O'}(f, v, S)$, if any, is that connecting O' as premise to its conclusion, we conclude that we must have $\mathcal{G}_\Gamma^O(f, v, S) \subseteq \mathcal{G}_\Gamma^{O'}(f, v, S)$. But then $\mathbf{e}O \subseteq \mathcal{G}_\Gamma^O(f, v, S) \subseteq \mathcal{G}_\Gamma^{O'}(f, v, S) = \mathbf{e}O'$. Also, $O' \in \mathbf{e}O' \setminus \mathbf{e}O$.

- If $O \notin \mathbf{e}O'$, there is no edge between $\mathbf{e}O$ and $\mathbf{e}O^c$ in $\mathbf{e}O'$, and since $O' \in \mathbf{e}O^c$, no occurrence in $\mathbf{e}O$ is in $\mathbf{e}O'$. Since in this case the conditions on O and O' are symmetric, we conclude that $\mathbf{e}O \cap \mathbf{e}O' = \varnothing$.

$\qquad\square$

An occurrence Q in $\mathbf{e}O$ is said to be a *door* of $\mathbf{e}O$ if it is either a premise of an occurrence R which is not in $\mathbf{e}O$, or a conclusion in Γ. The occurrence O itself is a door of $\mathbf{e}O$, the *main door*; the other doors are the *auxiliary doors*. The set of doors of $\mathbf{e}O$ is the *border* of $\mathbf{e}O$.

The following is an immediate consequence of Lemma 8.3.

Lemma 8.6 *Let Q be an auxiliary door of $\mathbf{e}O$ which is not a conclusion. Then Q is the premise of a Par or With occurrence.*

8.2 Maximal Empires

In this sub-section, we keep Γ, f and v fixed as before, with $f(v) = (d, \pi)$. We additionally assume that O is a With occurrence, and that $\mathbf{e}O$ is maximal among

empires of this form.

Lemma 8.7 *Let W be any With occurrence, and consider $P, Q \in \mathcal{G}_\Gamma(f, v, S)$ such that both P and Q depend on W in v. Then $P \in \mathbf{e}O$ iff $Q \in \mathbf{e}O$.*

Proof. Suppose that $P \in \mathbf{e}O$. Then there is $e \sqsubseteq f(v)$ such that $v_0(W){\downarrow}$, where $v_0 = \mathsf{M}(f, v, e)$. Hence $f(v) \neq f(v \backslash W)$. Since $f(v) = f \circ \mathbf{p} \circ f(v)$, this implies that $\mathbf{p} \circ f(v)(W){\downarrow}$, and hence that $(\!|W|\!)_\Gamma \sqsubseteq d$. By Lemma 8.3, $P \in \mathbf{e}W$, hence $\mathbf{e}O \cap \mathbf{e}W \neq \varnothing$. By maximality of $\mathbf{e}O$, $\mathbf{e}O \subset \mathbf{e}W$ is impossible. Hence by Lemma 8.5, $W \in \mathbf{e}O$. By Lemma 8.3, this implies $Q \in \mathbf{e}O$. $\qquad\square$

The proof of this Lemma shows how the equation $f = f \circ \mathbf{p} \circ f$ takes the place of Girard's "technical condition" on his proof structures.

Lemma 8.8 *If P is a premise of a With or Par occurrence Q, and P depends on a With occurrence $W \neq Q$ in v, then Q also depends on W in v.*

Proof. If Q is a Par occurrence, then $(\!|P|\!)_\Gamma = (\!|Q|\!)_\Gamma$. If Q is a With occurrence, let $v_0 = \mathsf{M}(f, v, (\!|Q|\!)_\Gamma)$. By property **(PS2)** of proof structures, $\mathsf{M}(f, v, (\!|P|\!)_\Gamma) = v_0 \cup \{Q \mapsto v(Q)\}$, so if P depends on W, so does Q. $\qquad\square$

Lemma 8.9 *If P is a border occurrence of $\mathbf{e}O$, and W is any With occurrence, with $f(v_{\neg W}) = (d', \pi')$, then $(\!|P|\!)_\Gamma \sqsubseteq d'$, i.e. the border occurrences are still present in $f(v_{\neg W})$.*

Proof. Either P is a conclusion, in which case $(\!|P|\!)_\Gamma = \bot$, or P is a premise of some occurrence Q, which by Lemma 8.6 must be a With or Par. Suppose for a contradiction that $(\!|P|\!)_\Gamma \not\sqsubseteq d'$. Then P depends on W in v, and hence also by Lemma 8.8 so does Q, but $P \in \mathbf{e}O$ while $Q \notin \mathbf{e}O$. Lemma 8.7 yields the required contradiction. $\qquad\square$

8.3 Stability of Maximal Empires

In this section we show that if O is a With occurrence in a total valuation v with a maximal empire $\mathbf{e}O$ relative to v, then its empire remains maximal in any total valuation v'. It is sufficient to show that $\mathbf{e}O$ remains maximal in $v_{\neg W}$ for any With occurrence W, since any total valuation v' can be reached from v by successively "toggling" With occurrences.

Lemma 8.10 *Suppose that O depends neither on W nor on W' in v. Then this remains true in $v_{\neg W}$.*

Proof. If $v_0 = \mathsf{M}(f, v, (\!|O|\!)_\Gamma)$, then $v_0(W) = \bot = v_0(W')$. Hence $(\!|O|\!)_\Gamma \sqsubseteq f(v \backslash \{W, W'\})$, and $v_0 = \mathsf{M}(f, v_{\neg W}, (\!|O|\!)_\Gamma)$. $\qquad\square$

Lemma 8.11 *Suppose that $P \in \mathbf{e}O$ with respect to v, and that P does not depend on a With occurrence W. Then we still have $P \in \mathbf{e}O$ in $v_{\neg W}$.*

Proof. If O is a conclusion, $\mathbf{e}O$ contains all occurrences in $f(v)$ for any v, so the Lemma holds straightforwardly.

Otherwise, O is the premise of some occurrence Q, and $Q \notin \mathbf{e}O$. We argue by cases:

- If $W \in \mathbf{e}O$, then by Lemma 8.3, the occurrences outside $\mathbf{e}O$ do not depend on W in v, hence by Lemma 8.10, changing from v to $v_{\neg W}$ does not alter the dependency relation outside $\mathbf{e}O$. Assume for a contradiction that there is a switching S for $v_{\neg W}$, with $P \notin \mathcal{G}_\Gamma^O(f, v_{\neg W}, S)$. Since O is the premise of Q, $Q \notin \mathcal{G}_\Gamma^O(f, v_{\neg W}, S)$. Then we can define a switching S' for v which makes the same choices outside $\mathbf{e}O$ as S, which is possible since the dependencies are the same outside $\mathbf{e}O$. Then P and Q are still connected in $\mathcal{G}^{O-}(f, v, S')$, and so $P \notin \mathcal{G}^O(f, v, S')$, yielding the required contradiction.

- If $W \notin \mathbf{e}O$, then by Lemma 8.7, no occurrences inside $\mathbf{e}O$ depend on W, hence by Lemma 8.10, changing from v to $v_{\neg W}$ does not alter the dependency relation inside $\mathbf{e}O$. Assume for a contradiction that there is a switching S for $v_{\neg W}$, with $P \notin \mathcal{G}_\Gamma^O(f, v_{\neg W}, S)$. Then we can define a switching S' for v which makes the same choices inside $\mathbf{e}O$ as S, which is possible since the dependencies are the same inside $\mathbf{e}O$. Then P and O are still not connected in $\mathcal{G}^{O-}(f, v, S')$, and so $P \notin \mathcal{G}^O(f, v, S')$, yielding the required contradiction.

$\square$

Lemma 8.12 $\mathbf{e}O$ *is maximal with respect to* $v_{\neg W}$.

Proof. Assume for a contradiction that, with respect to $v_{\neg W}$, $\mathbf{e}O \subset \mathbf{e}P$ with $\mathbf{e}P$ maximal. By Lemma 8.5 (with respect to $v_{\neg W}$), $O \in \mathbf{e}P$, but $P \notin \mathbf{e}O$. By Lemma 8.9, O does not depend on W. Hence by the maximality of $\mathbf{e}P$ with respect to $v_{\neg W}$, and Lemma 8.11, we still have $O \in \mathbf{e}P$ in v. Similarly, using the maximality of $\mathbf{e}O$ with respect to v, and arguing contrapositively, we must still have $P \notin \mathbf{e}O$ with respect to v. This yields the required contradiction to the assumed maximality of $\mathbf{e}O$ with respect to v. $\square$

8.4 *Proof of the Main Theorem*

Proof. We argue by induction on the number n of With occurrences in Γ.

Base Case

The base case is $n = 0$. In this case, we can use a minor variation on the standard argument for MLL Sequentialization [Gir91]. We argue by induction on the size of Γ, and cases on the principal connectives of Γ.

- If all formulas in Γ are atomic, we can argue as usual that a proof-net must consist of a single (possibly generalized) axiom link.

- If one of the conclusions is a Par or Plus, then we can remove this outermost connective, obtain a new proof net, and argue inductively that this new proof net has a sequentialization Π. A single application of a Par or Plus rule to Π then yields a sequentialization of the original proof net.

- If all compound formulas in Γ are Times formulas $A_i \otimes B_i$, then among these we take one, say A_i, with a maximal empire.

  Claim The border of $\mathbf{e}A$ consists only of A_i and conclusions. We argue by contradiction. If some occurrence P in the border is not a conclusion, it must be the premise of a Par occurrence Q which is not in $\mathbf{e}A_i$. But Q is the hereditary premise of some A_j or B_j, say A_j. By Lemma 8.3, $Q \notin \mathbf{e}A_i$ implies $A_j \notin \mathbf{e}A_i$. It also follows from Lemma 8.3 that $P \in \mathbf{e}A_j$, whence $\mathbf{e}A_i \cap \mathbf{e}A_j \neq \varnothing$. By Lemma 8.5, $\mathbf{e}A_i \subset \mathbf{e}A_j$, contradicting the maximality of $\mathbf{e}A_i$.

  It follows from the Claim that $\mathbf{e}A_i$ is independent of the choice of switching; hence so also is $\mathbf{e}B_i$. It follows from this that the removal of the Times in $A_i \otimes B_i$ splits the proof net into two connected components, both of which are proof nets and to which the induction hypothesis can be applied, yielding sequentializations Π_1 and Π_2. We can then apply a Times rule to Π_1 and Π_2 to obtain a sequentialization of the original proof net.

Note that in this case, $\mathcal{V}_\Gamma = \mathbf{1}$, and the proof-net consists of an additive resolution of all the Plus nodes, together with a set of axiom links for this additive resolution. It is immediate that in this case $f = [\![\Pi]\!]$, where f is the proof net, and Π the sequentialization.

Inductive Step

If $n > 0$, we choose some total valuation v arbitrarily, and choose a With occurrence O such that $\mathbf{e}O$ is maximal with respect to v. By Lemma 8.9, $\mathbf{e}O$ has the same boundary with respect to any valuation v', while by Lemma 8.12 it remains maximal with respect to v'. Moreover, if W is any With occurrence which is in $\mathbf{e}O$ with respect to v, then it is in $\mathbf{e}O$ with respect to any v' in which W is present. This follows from Lemma 8.11, since v' can be obtained from v by successively toggling With occurrences.

It follows that $\mathcal{W}(\Gamma)$ can be written as a disjoint union $\mathcal{W}(\Gamma) = \mathcal{W}_1 \uplus \mathcal{W}_2$, where $\mathcal{W}_2$ comprises those With occurrences which, in any valuation where they are present, are in $\mathbf{e}O$, while $\mathcal{W}_1$ comprises those which, in any valuation where they are present, are not in $\mathbf{e}O$. Clearly $\mathcal{V}_\Gamma \cong \mathcal{V}_1 \times \mathcal{V}_2$, where $\mathcal{V}_i = \mathbb{B}_\perp^{\mathcal{W}_i}$, $i = 1, 2$.

Let Γ_2 be the sequent corresponding to the conclusions of $\mathbf{e}O$. Two cases arise at this point:

- O is a conclusion of Γ, in which case $\Gamma_2 = \Gamma$.
- O is not a conclusion, in which case Γ_2 comprises a sequence of non-nested occurrences of subformulas of Γ, not all of which are conclusions of Γ. In this case, we can define a sequent $\Gamma_1[\xi_1, \ldots, \xi_k]$, in which the formulas of Γ_2 are replaced by the instances of a parameter ξ which does not appear in Γ, such that $\Gamma = \Gamma_1[\Gamma_2/\boldsymbol{\xi}]$.

We shall describe how to proceed in the second case, which effectively subsumes the first. We define two sets of occurrences: $\mathcal{O}_2$ is the set of all occurrences in $\mathbf{e}O$, while $\mathcal{O}_1$ is the set of all occurrences either outside $\mathbf{e}O$ or on its border. There are

corresponding sets of primes

$$P_i = \{(\!(O)\!)_\Gamma \mid O \in \mathcal{O}_i\}, \qquad i = 1, 2.$$

We can now apply the constructions of Section 4.2, to obtain sub-domains D_i of $\mathcal{D}_\Gamma$, corresponding sub-domains E_i of $\mathcal{E}_\Gamma$, and embedding-projections $E_i \lhd \mathcal{E}_\Gamma$, $i = 1, 2$. Moreover,

$$E_1 \cong \mathcal{E}(\Gamma_1), \qquad E_2 \cong \mathcal{E}(\Gamma_2)_\perp.$$

Hence there are embedding-projections

$$e_1 : \mathcal{E}(\Gamma_1) \lhd \mathcal{E}_\Gamma : p_1, \qquad e_2 : \mathcal{E}(\Gamma_2)_\perp \lhd \mathcal{E}_\Gamma : p_2.$$

There are also stable embedding-projections

$$\phi_1 : \mathcal{V}_1 \lhd \mathcal{V}_\Gamma : \psi_1, \qquad \phi_2 : \mathcal{V}_2 \lhd \mathcal{V}_\Gamma : \psi_2.$$

Since $p_2 \circ f \circ \phi_2$ factors through the inclusion $\mathcal{E}(\Gamma_2) \hookrightarrow \mathcal{E}(\Gamma_2)_\perp$, we can define stable functions

$$f_i = p_i \circ f \circ \phi_i : \mathcal{V}_i \longrightarrow \mathcal{E}(\Gamma_i), \qquad i = 1, 2.$$

These functions are readily seen to be proof nets. For f_2, note that any switching of Γ_2 relative to a valuation v_2 can be extended to a *principal switching* S' of $\mathbf{e}O$ relative to $v = (v_1, v_2)$, which implies that $\mathcal{G}_{\Gamma_2}(f_2, v_2, S) = \mathcal{G}_\Gamma^O(f, v, S')$ is acyclic and connected. For f_1 we similarly extend a valuation v_1 and switching S to v and S', and note that $\mathcal{G}_\Gamma(f, v, S')$ induces paths, not involving any other conclusions of Γ_2, between (the conclusions corresponding to) $\xi_{\sigma(1)}$ and $\xi_{\sigma(2)}$, ..., $\xi_{\sigma(k-1)}$ and $\xi_{\sigma(k)}$ for some permutation σ on $\{1, \ldots, k\}$. This reordering of the conclusions of the generalized axiom link does not affect the acyclicity of $\mathcal{G}_{\Gamma_1}(f_1, v_1, S)$, nor — since the number of vertices and edges remains the same — its connectedness. Hence the fact that f is a proof net implies that f_1 is a proof net.

The induction hypothesis applies immediately to f_1, and yields a sequent proof Π_1, with $[\![\Pi_1]\!] \sqsubseteq f_1$. In the case of f_2, it has a With formula as a conclusion in Γ_2. By setting the corresponding occurrence W to 0 or 1 in the valuation, we obtain proof nets f_2', f_2'' to which the induction hypothesis applies, yielding sequent proofs Π' and Π'' with $[\![\Pi']\!] \sqsubseteq f_2'$, $[\![\Pi'']\!] \sqsubseteq f_2''$. These can be combined using the With rule to yield a sequent proof Π_2. Clearly $[\![\Pi_2]\!] \sqsubseteq f_2$, by a pointwise argument for any $v_2 \in \mathcal{V}_2$, and cases on $v_2(W)$. Now $\Pi_1[\Pi_2]$ is a sequent proof of Γ, with

$$[\![\Pi_1[\Pi_2]]\!] = [\![\Pi_1]\!][[\![\Pi_2]\!]] \sqsubseteq f_1[f_2] \sqsubseteq f.$$

$\square$

9   Further Directions

We simply list some of the many directions for future work.

- The results of the present paper form a building block for a proof of Full Completeness for a concurrent game semantics of MALL. The strategies in this semantics are certain closure operators on the domains $\mathcal{D}_\Gamma$, which can readily be related to semantic proof structures. A refined version of the Full Completeness theorem from [AM99] has been developed in the current setting in [Abr07].

- The fine structure of the extensional equivalence classes of proof-nets, including the order structure and issues of sequentiality and parallelism, as discussed in Section 6, should be developed further.

- Going beyond the stable case, and analyzing the HvG approach in a more conceptual manner in the present setting, is a major desideratum.

- It would also be interesting to look at Linear Logic beyond MALL, in particular the exponentials, including weak versions suitable for analyzing complexity classes.

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