Reconsidering MacLane

Coherence for associativity in infinitary and untyped settings

Peter M. Hines

Oxford – March 2013

Topic of the talk:

Pure category theory . . .

for its own sake.

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Applications do exist:

Some applications:

1. *Logic & theoretical computing,*
2. *Quantum computation & foundations,*
3. *Linguistics & models of meaning,*
4. *Modular arithmetic / cryptography,*
5. *Decision procedures in group theory*

— these will not be discussed today!

The general area

We will be looking at

coherence theorems, and 'strictification',

for **associativity** and **related properties**.

These things we hold self-evident

A **category** $\mathcal{C}$ consists of

- A **proper class** of objects, $Ob(\mathcal{C})$.
- For all objects $A, B \in Ob(\mathcal{C})$, a **set** of arrows $\mathcal{C}(A, B)$.

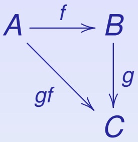
We will work diagrammatically:

An arrow $f \in \mathcal{C}(A, B)$ is drawn as

$$A \xrightarrow{f} B$$

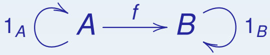
Axioms for category theory ...

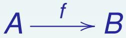
1. We may compose arrows:



   Composition is associative: $h(gf) = (hg)f$.

2. There is an identity arrow at every object:





That's all, folks!

Mapping between categories

A **functor** $\Gamma : \mathcal{C} \to \mathcal{D}$

$$X \xrightarrow{f} Y \qquad \text{in } \mathcal{C}$$

$$\Gamma(X) \xrightarrow{\Gamma(f)} \Gamma(Y) \qquad \text{in } \mathcal{D}$$

A simple property:

$\Gamma : \mathcal{C} \to \mathcal{D}$ preserves commuting diagrams:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{gf} & \downarrow g \\ & & Z \end{array} \qquad \text{Commutes in } \mathcal{C}$$

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Commutes in $\mathcal{C}$

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Commutes in $\mathcal{D}$

Categories with tensors

A **monoidal** category has a **monoidal tensor**:

A functor $\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

satisfying:

- *Associativity* $A \otimes (B \otimes C) \ \cong \ (A \otimes B) \otimes C$
- Existence of a *unit object* $A \otimes I \ \cong \ A \ \cong \ I \otimes A$

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Defining associativity:

We need a natural family of **associativity isomorphisms**

$$X \otimes (Y \otimes Z) \underset{\tau^{-1}_{X,Y,Z}}{\overset{\tau_{X,Y,Z}}{\rightleftarrows}} (X \otimes Y) \otimes Z$$

satisfying one very simple condition.

Yes, there are two paths you can go by, but ...

MacLane's **coherence** condition:

The two 'distinct' ways of re-arranging

$$A \otimes (B \otimes (C \otimes D))$$

into

$$((A \otimes B) \otimes C) \otimes D$$

must be equal.

Coherence is a simple form of **confluence**.

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The Pentagon condition

$$A \otimes (B \otimes (C \otimes D))$$

Two Steps *Three Steps*

$$((A \otimes B) \otimes C) \otimes D$$

We get a five-sided commuting diagram:

MacLane's Pentagon.

A simple special case:

When all *associativity isomorphisms* are **identities**,

$$\tau_{A,B,C} = 1_X \quad \text{for some object } X$$

$(\mathcal{C}, \otimes)$ is called strictly associative.

Important

This is *not* implied by

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C.$$

Equality of objects is not strict associativity.

(**Claim 1**) *Concrete example coming soon* ...

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Why is associativity generally ignored?

MacLane's coherence theorem

This provides a notion of 'confluence' for canonical diagrams.

A diagram is **canonical** if its arrows are built up from

$$\{ \tau_{\_,\_,\_} \ , \ \tau^{-1}_{\_,\_,\_} \ , \ 1_{\_} \ , \ \_ \otimes \_ \}$$

Two common descriptions of MacLane's theorem:

1. Every canonical diagram commutes.

2. We can treat

$$A \otimes (B \otimes C) \underset{\tau^{-1}_{A,B,C}}{\overset{\tau_{A,B,C}}{\rightleftarrows}} (A \otimes B) \otimes C$$

as a strict identity

$$A \otimes B \otimes C \underset{1_{A \otimes B \otimes C}}{\overset{1_{A \otimes B \otimes C}}{\rightleftarrows}} A \otimes B \otimes C$$

with no 'harmful side-effects'.

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Two inaccurate descriptions of MacLane's theorem:

1. ~~Every canonical diagram commutes.~~

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~~with no 'harmful side-effects'.~~

Two more claims:

- Not every canonical diagram commutes.

  (Claim 2)

- Treating associativity **isomorphisms** as **strict identities** can have major consequences.

  (Claim 3)

A simple example:

The **Cantor monoid** $\mathcal{U}$ (single-object category).

- Single object $\mathbb{N}$.
- Arrows: all bijections on $\mathbb{N}$.

The monoidal structure

We have a tensor $(\_ \star \_) : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$.

$$(f \star g)(n) = \begin{cases} 2.f\left(\frac{n}{2}\right) & n \text{ even,} \\ 2.g\left(\frac{n-1}{2}\right) + 1 & n \text{ odd.} \end{cases}$$

Properties of the Cantor monoid (I)

The Cantor monoid has only one object —

$$\mathbb{N} \star (\mathbb{N} \star \mathbb{N}) = \mathbb{N} = (\mathbb{N} \star \mathbb{N}) \star \mathbb{N}$$

$(\_ \star \_) : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is associative *up to a natural isomorphism*

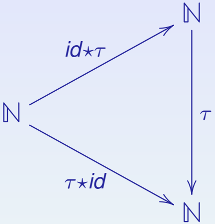
$$\tau(n) = \begin{cases} 2n & n \ (mod\ 2) = 0, \\ n+1 & n \ (mod\ 4) = 1, \\ \frac{n-1}{2} & n \ (mod\ 4) = 3. \end{cases}$$

that satisfies MacLane's pentagon condition.

This is not the identity map!

Properties of the Cantor monoid (II)

Not all canonical diagrams commute:



This diagram does *not* commute.

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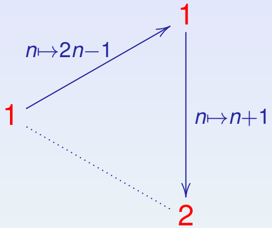
Properties of the Cantor monoid (II)

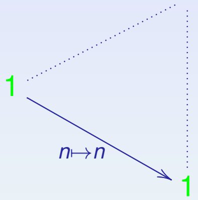
Using an actual number:



On the upper path, $1 \mapsto 2$.

Properties of the Cantor monoid (II)

Taking the left hand path:



$1 \neq 2$, so this diagram does *not* commute.

Properties of the Cantor monoid (III)

Forcing *strict associativity* by taking a quotient

$$\tau \sim id$$

collapses $\mathcal{U}(\mathbb{N}, \mathbb{N})$ to a single element.

The algebraic proof ...

The canonical isomorphisms of the Cantor monoid generate a representation of Thompson's group $\mathcal{F}$, and so have a representation in terms of an embedding of P_2, the two-generator polycyclic monoid. However, polycyclic monoids are Hilbert-Post complete, and so any non-trivial congruence (i.e. composition-preserving equivalence relation) on P_2 that identifies τ and *id* must force a collapse to the trivial monoid $\{1\}$.

— A categorical proof is *simpler* and *more general*.

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What does MacLane's thm. actually say?

If in doubt ...

. . . ask the experts:

http://en.wikipedia.org/wiki/Monoidal_category

WIKIPEDIA
The Free Encyclopedia

*"It follows that **any diagram** whose morphisms are built using [canonical isomorphisms], identities and tensor product commutes."*

What does the man himself say?

Categories for the working mathematician (1^{st} ed.)

- *Moreover all diagrams involving [canonical iso.s] must commute.* **(p. 158)**
- *These three [coherence] diagrams imply that "all" such diagrams commute.* **(p. 159)**
- *We can only prove that every "formal" diagram commutes.* **(p. 161)**

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What does his theorem say?

MacLane's coherence theorem for associativity

All diagrams within the image of a certain functor are guaranteed to commute.

- In some *ideal world*, this includes all canonical diagrams.
- In the *real world*, this might not be the case.

MacLane talks about unwanted identifications of objects.

Where does **T**hat come **F**rom? *Identification of objects is not a categorical concept!*

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Coherence for associativity

— a closer look

A technicality: It is standard to work with *monogenic categories*.

Objects are generated by:

- Some object S,
- The tensor $(\_ \otimes \_)$.

This is not a restriction — S is thought of as a ‘variable symbol’.

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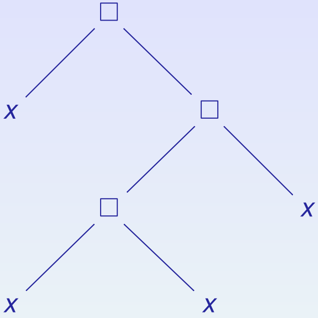
The source of the functor

This is based on (non-empty) *binary trees*.



- **Leaves** labelled by x,
- **Branchings** labelled by □.

The **rank** of a tree is the number of leaves.

The source of the functor (II)

MacLane's category $\mathcal{W}$.

- **(Objects)** All non-empty binary trees.
- **(Arrows)** A unique arrow between any two trees *of the same rank*.

  — write this as $(v \leftarrow u) \in \mathcal{W}(u, v)$.

Key points:

1. $(\_\square\_)$ is a monoidal tensor on $\mathcal{W}$.
2. $\mathcal{W}$ is **skeletal** — all diagrams over $\mathcal{W}$ commute.

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The functor itself

Given an object S of a monoidal category $(\mathcal{C}, \otimes)$,

MacLane's theorem simply gives a monoidal functor

$$\mathcal{W}Sub_S : (\mathcal{W}, \Box) \to (\mathcal{C}, \otimes)$$

Why this is interesting ...

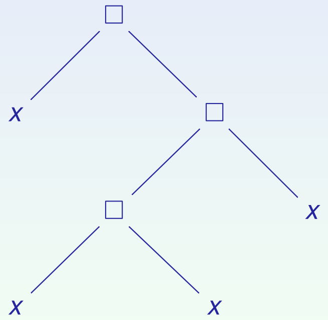
- *Every* diagram over $\mathcal{W}$ commutes.
- *Every* diagram in the image of this functor commutes.
- *Every* arrow in the image is a canonical isomorphism.

An inductively defined functor

On objects:

- $\mathcal{W}Sub_S(x) = S$,
- $\mathcal{W}Sub_S(u \Box v) = \mathcal{W}Sub_S(u) \otimes \mathcal{W}Sub_S(v)$.

An object of $\mathcal{W}$:



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An inductively defined functor (I)

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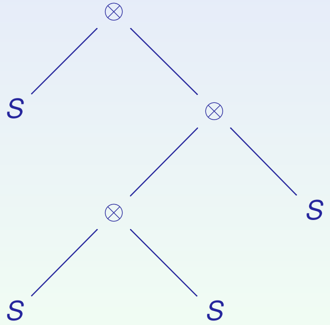
An object of $\mathcal{C}$:



An inductively defined functor (II)

On arrows:

- $\mathcal{W}Sub(u \leftarrow u) = 1_{\_}$.
- $\mathcal{W}Sub(a\Box v \leftarrow a\Box u) = 1_{\_} \otimes \mathcal{W}Sub(v \leftarrow u)$.
- $\mathcal{W}Sub(v\Box b \leftarrow u\Box b) = \mathcal{W}Sub(v \leftarrow u) \otimes 1_{\_}$.
- $\mathcal{W}Sub((a\Box b)\Box c \leftarrow a\Box(b\Box c)) = \tau_{\_,\_,\_}$.

By construction:

1. Every arrow in the image of $\mathcal{W}Sub$ is a canonical iso.
2. Every canonical isomorphism is in the image of $\mathcal{W}Sub$.

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When do canonical diagrams *not* commute?

If $\mathcal{W}Sub$ is an *embedding* of $(\mathcal{W}, \Box)$, there are no problems

...all canonical diagrams commute!

In general, this is not true.

"There are unwanted identifications of objects"

$$\equiv$$

The functor $\mathcal{W}Sub$ is not monic.

There does exist a monic-epic decomposition of $\mathcal{W}Sub$. (Claim 5)

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How to Rectify the Anomaly

Given a **badly-behaved** category $(\mathcal{C}, \otimes)$, we can

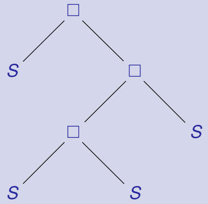
*build a **well-behaved** version*. (Claim 6)

Think of this as the **Platonic Ideal** of $(\mathcal{C}, \otimes)$.

We assume $\mathcal{C}$ is *monogenic*, with objects generated by $\{S, \_ \otimes \_\}$

Constructing $Plat_{\mathcal{C}}$

Objects are free binary trees



Leaves labelled by $S \in Ob(\mathcal{C})$,

Branchings labelled by $\Box$.

There is an **instantiation map** $Inst : Ob(Plat_{\mathcal{C}}) \rightarrow Ob(\mathcal{C})$

$$S\Box((S\Box S)\Box S) \mapsto S \otimes ((S \otimes S) \otimes S)$$

This is not just a matter of syntax!

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What about arrows?

Homsets are copies of homsets of $\mathcal{C}$

Given trees T_1, T_2,

$$Plat_{\mathcal{C}}(T_1, T_2) = \mathcal{C}(Inst(T_1), Inst(T_2))$$

Composition is inherited from $\mathcal{C}$ in the obvious way.

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The tensor $(\_\square\_) : Plat_{\mathcal{C}} \times Plat_{\mathcal{C}} \to Plat_{\mathcal{C}}$

$$\left. \begin{array}{c} A \xrightarrow{f} X \\ \\ B \xrightarrow{g} Y \end{array} \right\} \quad A\square X \xrightarrow{f\square g} B\square Y$$

The tensor of $Plat_{\mathcal{C}}$ is

- **(Objects)** A free formal pairing, $A\square B$,
- **(Arrows)** Inherited from $(\mathcal{C}, \otimes)$, so $f\square g \overset{def.}{=} f \otimes g$.

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Some properties of the platonic ideal ...

1. The functor

$$\mathcal{W}Subs_S : (\mathcal{W}, \Box) \to (Plat_{\mathcal{C}}, \Box)$$

is always **monic.**

2. As a corollary:

All canonical diagrams of $(Plat_{\mathcal{C}}, \Box)$ commute.

3. Instantiation defines a monoidal epimorphism

$$Inst : (Plat_{\mathcal{C}}, \Box) \to (\mathcal{C}, \otimes)$$

through which McL'.s substitution functor always factors.

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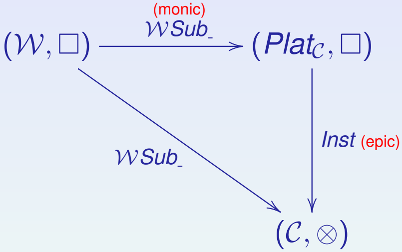
A monic / epic decomposition

MacLane's substitution functor always factors through the platonic ideal:



This gives a monic / epic decomposition of his functor.

Examples of the Platonic ideal (I)

A **strictly associative** category $(\mathcal{C}, \otimes)$.

Its Platonic ideal $(Plat_C, \Box)$ is associative **up to isomorphism**.

The objects

are distinct.

A question:

What are the associativity isomorphisms?

Examples of the Platonic ideal (II)

A particularly interesting case:

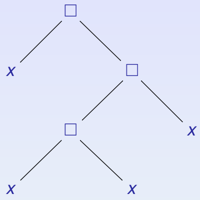
The trivial monoidal category $(\mathcal{I}, \otimes)$.

- **Objects:** $Ob(\mathcal{I}) = \{x\}$.
- **Arrows:** $\mathcal{I}(x, x) = \{1_x\}$.
- **Tensor:**

$$x \otimes x = x \ , \ 1_x \otimes 1_x = 1_x$$

What is the platonic ideal of $\mathcal{I}$?

(Objects) All non-empty binary trees:



(Arrows) For all trees T_1, T_2,

$$Plat_{\mathcal{I}}(T_1, T_2) \text{ is a single-element set.}$$

There is a unique arrow between any two objects!

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Can you tell what it is yet?

(P.H. 1998) The skeletal self-similar category $(\mathcal{X}, \Box)$

- **Objects:** *All non-empty binary trees.*
- **Arrows:** *A unique arrow between any two objects.*

This monoidal category:

1. was introduced to study **self-similarity** $S \cong S \otimes S$,
2. contains MacLane's $(\mathcal{W}, \Box)$ as a wide subcategory.

The categorical identity $S \cong S \otimes S$

Exhibited by two canonical isomorphisms:

- **(Code)** $\lhd : S \otimes S \to S$
- **(Decode)** $\rhd : S \to S \otimes S$

These are *unique* (up to *unique isomorphism*).

Examples

- *The natural numbers $\mathbb{N}$, Separable Hilbert spaces, Infinite matrices, Cantor set & other fractals, &c.*
- *C-monoids, and other untyped (single-object) monoidal categories*
- *Any unit object I of a monoidal category . . .*

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You are unique - just like everybody else

Unique *up to unique isomorphism*

is not the same as

actually unique.

Elementary remarks on units in monoidal categories – J. Kock

The theory of Saavedra units: **actual uniqueness of arrows**

$$S \underset{\triangleleft}{\overset{\triangleright}{\rightleftarrows}} S \otimes S$$

implies that S is the unit object.

(Claim 7) Coherence for self-similarity provides an alternative proof.

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Can we have **strict self-similarity**?

Can the code / decode maps

$$\lhd : S \otimes S \rightarrow S \quad , \quad \rhd : S \rightarrow S \otimes S$$

be **strict identities**?

In untyped monoidal categories:

We only have one object, $S = S \otimes S$.

The **code** / **decode maps** are both the identity.

Untyped $\equiv$ Strictly Self-Similar.

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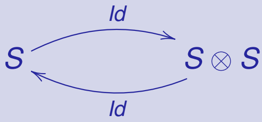

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Untyped $\equiv$ **Strictly Self-Similar**.

Strictifying self-similarity

(**Claim 8**) There exists a *strictification* procedure for self-similarity.

(**Claim 9**) One cannot simultaneously strictify self-similarity and associativity.

An essential preliminary

We need a coherence theorem for self-similarity.

Coherence for Self-Similarity

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A straightforward coherence theorem

We base this on the category $(\mathcal{X}, \Box)$

- **Objects** All non-empty binary trees.
- **Arrows** A unique arrow between any two trees.

This category is skeletal — all diagrams over $\mathcal{X}$ commute.

We will define a monoidal substitution functor:

$$\mathcal{X}Sub : (\mathcal{X}, \Box) \to (\mathcal{C}, \otimes)$$

The self-similarity substitution functor

An inductive definition of $\mathcal{X}Sub : (\mathcal{X}, \Box) \to (\mathcal{C}, \otimes)$

On objects:

$$\begin{array}{rcl} x & \mapsto & S \\ u\Box v & \mapsto & \mathcal{X}Sub(u) \otimes \mathcal{X}Sub(v) \end{array}$$

On arrows:

$$\begin{array}{rcl} (x \leftarrow x) & \mapsto & 1_S \in \mathcal{C}(S,S) \\ & & \\ (x \leftarrow x\Box x) & \mapsto & \lhd \in \mathcal{C}(S \otimes S, S) \\ (x\Box x \leftarrow x) & \mapsto & \rhd \in \mathcal{C}(S, S \otimes S) \\ & & \\ (b\Box v \leftarrow a\Box u) & \mapsto & \mathcal{X}Sub(b \leftarrow a) \otimes \mathcal{X}Sub(v \leftarrow u) \end{array}$$

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Interesting properties:

1. $\mathcal{X}Sub : (\mathcal{X}, \Box) \rightarrow (\mathcal{C}, \otimes)$ is always functorial.
2. Every arrow built up from

$$\{\lhd \ , \ \rhd \ , \ 1_S \ , \ \_ \otimes \_\}$$

   is the image of an arrow in $\mathcal{X}$.
3. Every diagram in the image of $\mathcal{X}Sub$ commutes.

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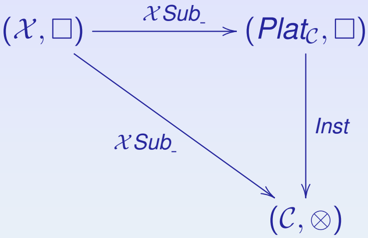
$\mathcal{X}$*Sub* factors through the Platonic ideal

There is a monic-epic decomposition of $\mathcal{X}$*Sub*.



Every canonical (for self-similarity) diagram in $(Plat_{\mathcal{C}}, \Box)$ commutes.

Relating associativity and self-similarity

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Comparing the *associativity* and *self-similarity* categories.

MacLane's $(\mathcal{W}, \Box)$

Objects: Binary trees.

Arrows: Unique arrow between two trees *of the same rank*.

The category $(\mathcal{X}, \Box)$

Objects: Binary trees.

Arrows: Unique arrow between *any two trees*.

There is an obvious inclusion $(\mathcal{W}, \Box) \hookrightarrow (\mathcal{X}, \Box)$

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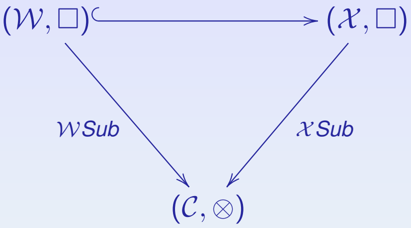
Is *associativity* a restriction of *self-similarity*?

Does the following diagram commute?



Does the **associativity** functor

factor through

the **self-similarity** functor?

Proof by contradiction:

Let's assume this is the case.

Special arrows of $(\mathcal{X}, \Box)$

For arbitrary trees u, e, v,

$$t_{uev} = ((u \Box e) \Box v \leftarrow u \Box (e \Box v))$$

$$l_v = (v \leftarrow e \Box v)$$

$$r_u = (u \leftarrow u \Box e)$$

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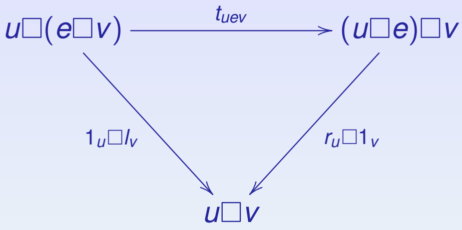
Since all diagrams over X commute:

The following diagram over $(\mathcal{X}, \Box)$ commutes:



Let's apply $\mathcal{X}Sub$ to this diagram.

By Assumption: $t_{uev} \mapsto \tau_{U,E,V}$ (assoc. iso.)

Notation: $u \mapsto U$, $v \mapsto V$, $e \mapsto E$, $l_v \mapsto \lambda_V$, $r_u \mapsto \rho_U$

Since all diagrams over X commute:

The following diagram over $(\mathcal{X}, \Box)$ commutes:

$$\begin{array}{ccc} u\Box(e\Box v) & \xrightarrow{t_{uev}} & (u\Box e)\Box v \\ & {\scriptstyle 1_u\Box l_v} \searrow \quad \swarrow {\scriptstyle r_u\Box 1_v} & \\ & u\Box v & \end{array}$$

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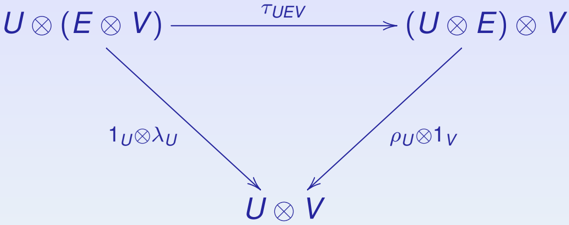

This is MacLane's **units triangle**
— E is the unit object for $(\mathcal{C}, \otimes)$.

The choice of e was *arbitrary* — every object is the unit object!

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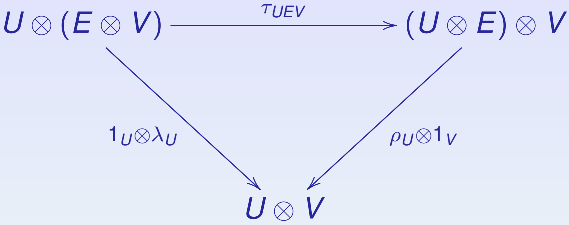

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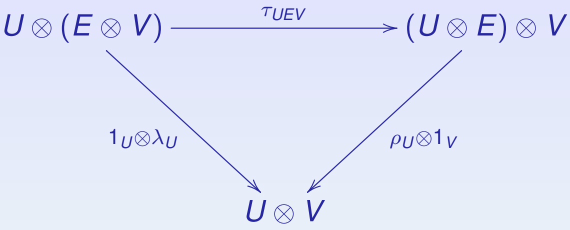

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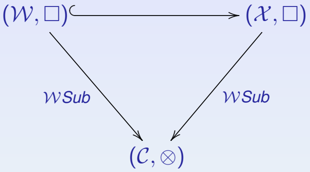
A general result

The following commutes



exactly when $(\mathcal{C}, \otimes)$ is **degenerate** —

i.e. all objects are isomorphic to the unit object.

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A special case:

1. **Strict associativity:** All arrows of $(\mathcal{W}, \Box)$ are mapped to identities of $(\mathcal{C}, \otimes)$
2. **Strict self-similarity:** All arrows of $(\mathcal{X}, \Box)$ are mapped to the identity of $(\mathcal{C}, \otimes)$.

$\mathcal{W}Sub$ trivially factors through $\mathcal{X}Sub$.

The conclusion

Strictly associative untyped monoidal categories are **degenerate**.

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What this means ...

Untyped categorical structures can

never be strictly associative.

A practical corollary:

LISP programmers will never get rid of all those parentheses.

```
(funcall ((lambda (f) #'(lambda (n) (funcall f f n)))
          #'(lambda (f n)
              (if (= n 0)
                  1
                (* n (funcall f f (- n 1))))))
         8)
```

Question: what about the **strictification** procedure?

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Another way of looking at things:

One cannot simultaneously *strictify*

(I) Associativity $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$

(II) Self-Similarity $S \cong S \otimes S$

The no simultaneous strictification property

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How to strictify self-similarity (I)

- Start with a monogenic category $(\mathcal{C}, \otimes)$, generated by a self-similar object

$$S \underset{\lhd}{\overset{\rhd}{\rightleftarrows}} S \otimes S$$

- Construct its platonic ideal $(Plat_{\mathcal{C}}, \Box)$
- Use the (monic) self-similarity substitution functor

$$\mathcal{X}Sub : (\mathcal{X}, \Box) \to (Plat_{\mathcal{C}}, \Box)$$

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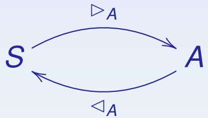
- The image of $\mathcal{X}Sub$ is a wide subcategory of $(\mathcal{P}lat_{\mathcal{C}}, \Box)$.

  It contains, for all objects A,

  a unique pair of inverse arrows



- Use these to define an **endofunctor** $\Phi : Plat_{\mathcal{C}} \rightarrow Plat_{\mathcal{C}}$.

How to strictify self-similarity (II)

- The image of $\mathcal{X}Sub$ is a wide subcategory of $(\mathcal{P}lat_{\mathcal{C}}, \Box)$.

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$$S \underset{\lhd_A}{\overset{\rhd_A}{\rightleftarrows}} A$$

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The **convolution endofunctor**

- **Objects**

$$\Phi(A) = S \ , \quad \text{for all objects } A$$

- **Arrows**

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \rhd_A} \uparrow & & \downarrow {\scriptstyle \lhd_B} \\ S & \xrightarrow[\quad \Phi(f) \quad]{} & S \end{array}$$

- **Functoriality** is trivial ... Φ it is also fully faithful.

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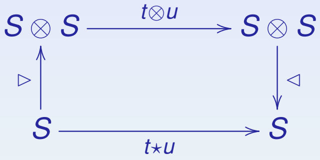
A tensor on $\mathcal{C}(S, S)$

As a final step, we define

$$(\_ \star \_) : C(S, S) \times \mathcal{C}(S, S) \to \mathcal{C}(S, S)$$

by



$(C(S, S), \star)$ is an untyped monoidal category!

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Convolution as a monoidal functor

- Recall, $Plat_{\mathcal{C}}(S,S) \cong \mathcal{C}(S,S)$.
- Up to this obvious isomorphism,

$$\Phi : (Plat_{\mathcal{C}}, \Box) \to (\mathcal{C}(S,S), \star)$$

is a monoidal functor.

What we have ...

A fully faithful monoidal functor from $Plat_{\mathcal{C}}$

to an untyped monoidal category.

— every canonical (for self-similarity) arrow is mapped to 1_S.

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To arrive where we started . . .

A monogenic category:

- **The generating object:** natural numbers $\mathbb{N}$.
- **The arrows** bijective functions.
- **The tensor** disjoint union $A \uplus B = A \times \{0\} \cup B \times \{1\}$.

The self-similar structure:

Based on the familiar **Cantor pairing** $c(n, i) = 2n + i$.

Let us strictify this self-similar structure.

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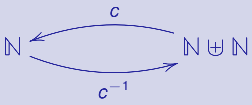

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The end is where we started from

The Cantor monoid:

The object	The natural numbers $\mathbb{N}$
The arrows	All bijections $\mathbb{N} \rightarrow \mathbb{N}$
The tensor	$(f \star g)(n) = \begin{cases} 2.f\left(\frac{n}{2}\right) & n \text{ even}, \\ 2.g\left(\frac{n-1}{2}\right)+1 & n \text{ odd}. \end{cases}$
The associativity isomorphism	$\tau(n) = \begin{cases} 2n & n\ (mod\ 2) = 0, \\ n+1 & n\ (mod\ 4) = 1, \\ \frac{n-3}{2} & n\ (mod\ 4) = 3. \end{cases}$
The symmetry isomorphism	$\sigma(n) = \begin{cases} n+1 & n \text{ even}, \\ n-1 & n \text{ odd}. \end{cases}$