The spectrum problem for noncommutative rings and algebras

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What is a spectrum?

For our purposes, a *spectrum* is an assignment

$$\{\text{commutative algebras}\} \to \{\text{topological spaces}\}$$

of which there are several examples.

- Commutative rings: $\mathsf{Spec}(R) = \{\text{prime ideals of } R\}$
- C*-algebras: $\mathsf{Spec}(A) = \{\text{max. ideals of } A\} = \mathsf{Hom}(A, \mathbb{C})$
- Boolean algebra: $\mathsf{Spec}(B) = \{\text{ultrafilters of } B\} = \mathsf{Hom}(B, \{0, 1\})$

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Why are these so nice? Each Spec is a *(contravariant) functor:*

A homorphism $f : A \to B$ on the "algebra side" yields a continuous function $\mathrm{Spec}(f) \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

Dualities between categories

All of the examples mentioned actually provide *dualities* between a category of algebras and a category of spaces (after imposing the appropriate topological or geometric structure on the spectrum).

Examples of comm. algebra-geometry correspondence:

- Classical alg. geom.: reduced, f.g. comm. algebras $\leftrightsquigarrow$ affine varieties
- Algebraic geometry: commutative rings $\leftrightsquigarrow$ affine schemes
- Functional analysis: comm. C*-algebras $\leftrightsquigarrow$ compact Hausdorff spaces
- Logic: Boolean algebras $\leftrightsquigarrow$ Stone spaces

From commutative to noncommutative

The usual explanation of *noncommutative geometry:*

$$\{\text{commutative algebras}\} \leftrightsquigarrow \{\text{spaces}\}$$
$$\{\text{noncommutative algebras}\} \leftrightsquigarrow \{\text{“noncommutative” spaces}\}$$

(Various strands of noncommutative geometry come from different geometric theories: differential geometry, algebraic geometry, point-set topology, etc.)

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The acutal mathematics is done without any use of an underlying “noncommutative space.” The above is just metaphor and motivation.

(Common substitutes: module categories, homological conditions.)

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Why would we care?

- *Dimension theory* for noncommutative rings is hard: competing definitions, (when) are they equal?
- Which noncommutative rings are "geometrically nice"? (When do we expect them to be noetherian?)
- Quantum modeling: given an algebra of observables, how should we "visualize" the underlying "phase space" of the system?

These and other related questions could benefit if we had an actual "spatial" object to refer to when thinking geometrically about rings.

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This provides us with:

- Obstructions proving that certain constructions are impossible;
- Better ideas on how to make progress.

Taking commutative subalgebras seriously

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Suppose $F\colon \mathbf{Ring}^{\mathrm{op}} \to \{\text{“spaces”}\}$ is a “spectrum functor.”

(A) means we know what the $F(C)$ look like.

(B) gives us maps $F(A) \to F(C)$, compatible on intersections.

This also fits well with standard pictures of quantum physics:

- $A \leftrightsquigarrow$ algebra of *observables* for quantum system
- $C \leftrightsquigarrow$ “classical contexts” of the quantum system

Can we begin with a set of points?

Algebraic geometry provides clever ways to add structure to a topological space, and the structure can even be noncommutative.

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Question: Can we extend the functor $\operatorname{Spec}\colon \mathbf{cRing}^{\mathrm{op}} \to \mathbf{Set}$ to a functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ such that $R \neq 0 \implies F(R) \neq \varnothing$?

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Answer: No!

Obstruction to point sets

Theorem (R., 2012)

Any functor $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ *whose restriction to the full subcategory* $\mathbf{cRing}$ *equals* Spec *must assign the empty set to* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$.

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The proof proceeds in roughly three steps:

1. Find a "universal example" of $F\colon \mathbf{Ring} \to \mathbf{Set}$ extending Spec.
2. Show that this universal example assigns the empty set to $\mathbb{M}_n(\mathbb{C})$.
3. By universality, conclude that *every* such functor does the same!

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A surprise: Step (2) uses the Kochen-Specker theorem of quantum mechanics!

Obstruction to point sets, take 2

The analogue of this result also holds in the context of operator algebras.

Roughly: A C*-algebra is a $\mathbb{C}$-algebra with an antilinear involution $x \mapsto x^*$ and a norm $\| \bullet \|$, which is norm-complete, has $\|xy\| \leq \|x\|\|y\|$, and $\|x^*x\| = \|x\|^2$.
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Theorem (R., 2012)

Any functor $\mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Set}$ *whose restriction to the full subcategory* **cCstar** *is isomorphic to the Gelfand spectrum must assign the empty set to* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$.

Prime partial ideals

How do we define the "universal extension" of Spec?

(**Idea**: Think of a ring only in terms of its commutative subrings.)

Definitions: Let R be a ring.

- A subset $I \subseteq R$ is a partial ideal if, for all commutative subrings $C \subseteq R$, $I \cap C$ is an ideal of C.
- A partial ideal $\mathfrak{p} \neq R$ is prime if $\mathfrak{p} \cap C$ is a prime ideal of C for all commutative subrings $C \subseteq R$.
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Example: $\{\text{nilpotent elements}\} \subseteq R$ is always a partial ideal, even though it's not generally an ideal for noncommutative R.

Functoriality

This "spectrum" gives a functor $p\text{-}\mathrm{Spec}\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$.

Lemma: If $f\colon R \to S$ is a ring homorphism and $\mathfrak{p} \subseteq S$ is a prime partial ideal, then so is $f^{-1}(\mathfrak{p})$.

In fact, it is the "universal extension" of Spec.

Theorem

Let $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ be any functor such that $F|_{\mathbf{cRing}} \cong \mathrm{Spec}$. Then there exists a unique morphism of functors $F \to p\text{-}\mathrm{Spec}$ preserving the isomorphism with Spec.

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Idea: In fact, $p\text{-}\mathrm{Spec}(R) = \varprojlim \mathrm{Spec}(C)$ where C ranges over the commutative subrings of R. The claim follows rather immediately from the universal property of the limit.

Partial spectrum to Kochen-Specker

Any $\mathfrak{p} \subseteq \mathbb{M}_3(\mathbb{C})$ induces a coloring on the set of projections $\text{Proj}(\mathbb{M}_3(\mathbb{C}))$: say those in $\mathfrak{p}$ are "black" and those outside are "white."

Exercise: if $P_1 + P_2 + P_3 = I$ is a sum of orthogonal projections, then two P_i lie in $\mathfrak{p}$ (black), exactly one lies outside (white).

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But this is the kind of coloring that Kochen and Specker proved to be impossible! (It's a non-contextual assignment of "yes" and "no" values to the observables in $\mathrm{Proj}(\mathbb{M}_3(\mathbb{C}))$.)

Corollary

For $n \geq 3$, $p\text{-}\mathrm{Spec}(\mathbb{M}_n(\mathbb{C})) = \varnothing$.

Proof of the first obstruction

Now we have all the pieces for the proof.

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Any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ *whose restriction to* $\mathbf{cRing}$ *is isomorphic to* Spec *must assign* $F(\mathbb{M}_n(\mathbb{C})) = \varnothing$ *for* $n \geq 3$.

Proof: Let F be as above. Then there exists a morphism of functors $F \to p\text{-}\mathrm{Spec}$ by universality of the latter.

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Proof: Let F be as above. Then there exists a morphism of functors $F \to p\text{-}\mathrm{Spec}$ by universality of the latter. For $n \geq 3$, the Kochen-Specker Theorem implies that $p\text{-}\mathrm{Spec}(\mathbb{M}_n(\mathbb{C})) = \varnothing$.

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Corollary: The same obstruction holds for $\mathbb{M}_n(A)$ whenever $\mathbb{C} \subseteq A$.

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Corollary: The same obstruction holds for $\mathbb{M}_n(A)$ whenever $\mathbb{C} \subseteq A$.

Idea: $\mathbb{C} \subseteq A$ induces $\mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(A)$ and $F(\mathbb{M}_n(A)) \to F(\mathbb{M}_n(\mathbb{C})) = \varnothing$.

The case $n = 2$

Question: What happens for $\mathbb{M}_2(\mathbb{C})$?

Proposition

Let $\mathcal{I} \subseteq \mathbb{M}_2(\mathbb{C})$ be a set of idempotents such that the set of all idempotents of $\mathbb{M}_2(\mathbb{C})$ is partitioned as $\{0, 1\} \sqcup \mathcal{I} \sqcup \{1 - e : e \in \mathcal{I}\}$. There is a bijection between:

- *The set of prime partial ideals of $\mathbb{M}_2(\mathbb{C})$;*
- *The set of functions $\mathcal{I} \to \{0, 1\}$.*

So for the particular functor $F = p\text{-}\mathrm{Spec}$, the set $p\text{-}\mathrm{Spec}(\mathbb{M}_2(\mathbb{C}))$ has cardinality $2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$. (It's huge!)

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Theorem: Any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ whose restriction to **cRing** is isomorphic to Spec must assign $F(\mathbb{M}_n(\mathbb{C})) = \varnothing$ for $n \geq 3$.

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- As before, reduce to $F = p\text{-}\mathrm{Spec}$.
- As before, any $\mathfrak{p} \in p\text{-}\mathrm{Spec}(\mathbb{M}_3(\mathbb{Z}))$ yields a Kochen-Specker coloring of the idempotent ($E = E^2$) integer matrices.

Q': Is there an "integer-valued" Kochen-Specker theorem?

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Q': Is there an "integer-valued" Kochen-Specker theorem? Yes!

Kochen-Specker uncolorable vector configurations in the literature typically use vectors in $\mathbb{C}^3$ (even $\mathbb{R}^3$) with irrational entries. So there was real work to be done here.

Colorability of projections in various rings

First try: Look at orthogonal projections in $\mathbb{M}_3(\mathbb{R})$ whose entries happen to be integer or rational. In general, for *any* commutative ring R, can consider $\mathrm{Proj}(\mathbb{M}_3(R)) = \{\text{symmetric idempotents}\}$.

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Theorem: (M. Ben-Zvi, A. Ma, R., 2015?)

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Idea: J. Bub (1996) produced an uncolorable configuration of (non-unit) vectors with integer entries such that $\|v\|^2$ all divide 30. Analyze $\mathbb{F}_p$ for $p = 2, 3, 5$ as special cases. Use functoriality for the rest.

Colorability of idempotents in various rings

This means we really *need* to consider non-symmetric idempotent matrices.

- Counting argument $\implies$ idempotents of $\mathbb{M}_3(\mathbb{F}_p)$ cannot be colored whenever $p \equiv 2 \pmod 3$. (Communicated to us by A. Chirvasitu.)

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Theorem (M. Ben-Zvi, A. Ma, R., 2015?)

There is no Kochen-Specker coloring of the set of idempotents in $\mathbb{M}_n(\mathbb{Z})$ for any $n \geq 3$. (Same is true of $\mathbb{M}_n(R)$ for any ring R.)

Finishing the spectrum obstruction

As mentioned before, this directly implies

Theorem

Given any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ *extending* Spec *as before, we have* $F(\mathbb{M}_n(\mathbb{Z})) = \varnothing$ *for any ring* R *and any integer* $n \geq 3$.

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Given any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ *extending* Spec *as before, we have* $F(\mathbb{M}_n(\mathbb{Z})) = \varnothing$ *for any ring* R *and any integer* $n \geq 3$.

Corollary: We also get $F(\mathbb{M}_n(R)) = \varnothing$ for any ring R and $n \geq 3$.

Again: $\mathbb{Z} \to R$ gives $\mathbb{M}_n(\mathbb{Z}) \to \mathbb{M}_n(R)$ and $F(\mathbb{M}_n(R)) \to F(\mathbb{M}_n(\mathbb{Z})) = \varnothing$.

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Theorem (van den Berg & Heunen, 2014)

Any functor $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ *whose restriction to* $\mathbf{cRing}^{\mathrm{op}}$ *is isomorphic to* Spec *(considered as a locale) must assign the trivial locale to* $\mathbb{M}_n(R)$ *for any ring* R *with* $\mathbb{C} \subseteq R$ *and any* $n \geq 3$. *(The same holds for C*-algebras.)*

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Corollary (Ben-Zvi, Chirvasitu, Ma, R.)

The obstruction above still holds with any ring R *and any integer* $n \geq 3$*.*

A toy problem

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I'm not ready to belive so. *We need to be even more creative!*

A positive example: Jointly w/ Chris Heunen, we were able to extend a kind of spectrum to noncommutative algebras.

The goal was to fill in the blank below:

$$\begin{array}{ccc} \{\text{commutative AW*-algebras}\} & \xrightarrow{\sim} & \{\text{complete Boolean algebras}\} \\ \big\downarrow & & \big\downarrow \\ \{\text{AW*-algebras}\} & \longrightarrow & \{???\} \end{array}$$

AW*-algebras: C*-algebras with many projections

Projections: $p = p^2 = p^*$ in a C*-algebra; think *orthogonal projection*

Definition: (Kaplansky 1951) An *AW*-algebra* is a C*-algebra A that satisfies the following equivalent conditions:

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Kaplansky's motivation: isolate the "algebraic" part of the theory of *W*-algebras* (i.e., *von Neumann algebras*, key players in "noncommutative measure theory"). The term is meant to stand for "abstract W*-algebra."

Proj(A) is complete *orthomodular* lattice (with orthogonal complement $p^\perp = 1 - p$), and a complete Boolean algebra when A is commutative

Examples of AW*-algebras

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Fact: Every maximal abelian $*$-subalgebra (MASA) of an AW*-algebra is again an AW*-algebra.

So noncommutative AW*-algebras contain many complete Boolean algebras!

Stone duality

Boolean algebra: $(B, 0, 1, \vee, \wedge, \neg)$

$$\mathsf{Spec}(B) = \{\text{prime ideals}\} \cong \{\text{ultrafilters}\} = \mathsf{Hom}(B, \{0, 1\})$$

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B complete $\Leftrightarrow$ Spec(B) is Stonean $\Leftrightarrow$ C(Spec(B)) is an AW*-algebra.

If we "skip the space," can we find "quantum complete Boolean algebras" to act as a spectrum for noncommutative AW*-algebras?

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If we "skip the space," can we find "quantum complete Boolean algebras" to act as a spectrum for noncommutative AW*-algebras? Yes!!

$$\begin{array}{ccccc}
\textbf{cAWstar} & \xrightarrow{\sim} & \textbf{Stonean} & \xrightarrow{\sim} & \textbf{CBoolean} \\
\downarrow & & & & \downarrow \\
\textbf{AWstar} & \longrightarrow & & \longrightarrow & \textbf{ActiveLat}
\end{array}$$

How can we "quantize" Boolean algebras?

Say $B = \mathrm{Proj}(A)$ for a commutative AW*-algebra A with $p, q \in B$:

- $p \wedge q = pq$;
- $p \vee q = p + q - pq$;
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Idea: Think of the noncommutative product $(1 - 2p)(1 - 2q)$ as a "quantum symmetric difference," even though the latter need not have the form $1 - 2p'$ for any projection $p' \in A$.

Active lattices

Definition (roughly): An active lattice consists of the following data:

- A complete *orthomodular lattice* P
- A *group* G with an injection $P \hookrightarrow G$ onto a generating set of "reflections"
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For each AW*-algebra A, we get an active lattice $\mathrm{AProj}(A)$ with:

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This gives us a functor $\mathrm{AProj}\colon \mathbf{AWstar} \to \mathbf{ActiveLat}$ (with appropriate morphisms on each side).

Active lattices determine AW*-algebras

Theorem (Heunen and R., 2014)

The functor $\mathrm{AProj}\colon \textbf{AWstar} \to \textbf{ActiveLat}$ *is a full and faithful embedding, i.e., there is a bijection between morphisms* $A \to B$ *between AW*-algebras and* $\mathrm{AProj}(A) \to \mathrm{AProj}(B)$ *of active lattices.*

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It seems likely that there are active lattices that do not arise from an AW*-algebra.

Question

Which active lattices L *satisfy* $L \cong \mathrm{AProj}(A)$ *for some AW*-algebra* A*? (That is, what is the essential image of* AProj*?)*

This is a kind of "coordinatization problem" similar to some that have appeared in lattice theory before. (Most notably, von Neumann's continuous geometries.)

What's next?

The obstructions suggest to me that we should make creative attempts to fill in the "???" below:

$$\begin{array}{ccc}
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<u>We know:</u> **Alg** can't be **AWstar** [Heunen & R., 2015], but not much else!

Thank you! (Plus some references)

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