Tractable Extensions of the Description Logic $\mathcal{EL}$ with Numerical Datatypes

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Abstract We consider extensions of the lightweight description logic (DL) $\mathcal{EL}$ with numerical datatypes such as naturals, integers, rationals and reals equipped with relations such as equality and inequalities. It is well-known that the main reasoning problems for such DLs are decidable in polynomial time provided that the datatypes enjoy the so-called convexity property. Unfortunately many combinations of the numerical relations violate convexity, which makes the usage of these datatypes rather limited in practice. In this paper, we make a more fine-grained complexity analysis of these DLs by considering restrictions not only on the kinds of relations that can be used in ontologies but also on their occurrences, such as allowing certain relations to appear only on the left-hand side of the axioms. To this end, we introduce a notion of safety for a numerical datatype with restrictions (NDR) which guarantees tractability, extend the $\mathcal{EL}$ reasoning algorithm to these cases, and provide a complete classification of safe NDRs for natural numbers, integers, rationals and reals.

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1 Introduction and Motivation

Description logics (DLs) [1] provide a logical foundation for modern ontology languages such as OWL[1] and OWL 2 [2]. $\mathcal{EL}^{++}$ [3] is a lightweight DL for which reasoning is tractable (i.e., can be performed in time that is polynomial w.r.t. the size of the input), and that offers sufficient expressivity for a number of life-sciences ontologies, such as SNOMED CT [4] or the Gene Ontology [5]. Additionally, $\mathcal{EL}^{++}$ underpins the EL Profile of OWL 2 [6], which is a sublanguage of OWL 2, particularly useful in applications involving large ontologies with many classes and/or properties. Among other constructors, $\mathcal{EL}^{++}$ supports limited usage of datatypes. In DL, datatypes (also called concrete domains) can be used to define new concepts by referring to particular values, such as strings or integers. For example, the concept $\mathsf{Human} \sqcap \exists \mathsf{hasAge}.(<, 18) \sqcap \exists \mathsf{hasName}.(=, \text{“}\mathsf{Alice}\text{”})$ describes humans whose age is less than 18 and whose name is “Alice”. Datatypes are characterised first by the domain their values can come from and also by the relations that can be used to constrain possible values. In our example, $(<, 18)$ refers to the domain of natural numbers and uses the relation “<” to constrain possible values to those less than 18, while $(=, \text{“}\mathsf{Alice}\text{”})$ refers to the domain of strings and uses the relation “=” to constrain the value to “Alice”.

In order to ensure that reasoning remains polynomial, $\mathcal{EL}^{++}$ allows only for datatypes which satisfy a condition called p-admissibility [3]. In a nutshell, this condition ensures that the satisfiability of datatype constraints can be solved in polynomial time, and that concept disjunction cannot be expressed using datatype concepts. For example, if we were to allow both $\leq$ and $\geq$ for integers, then we could express $A \sqsubseteq B \sqcup C$ by formulating the axioms $A \sqsubseteq \exists R.(\leq, 5)$, $\exists R.(\leq, 2) \sqsubseteq B$ and $\exists R.(\geq, 2) \sqsubseteq C$ for some fresh symbol R. Thus, allowing both $\leq$ and $\geq$ has the same effect as extending $\mathcal{EL}^{++}$ with disjunction, which is well known to cause intractability [3]. Similarly, we can show that p-admissibility prevents from having both $\leq$ and $=$ or both $\geq$ and $=$ in the language. For this reason, the EL Profile of OWL 2, which is based on $\mathcal{EL}^{++}$, admits only equality ($=$) in datatype expressions.

In this paper, we demonstrate how these restrictions can be significantly relaxed without loosing tractability. As a motivating example, consider the following axioms which might be used, e.g., in a pharmacy-related ontology:

$$\mathsf{Panadol} \sqsubseteq \exists \mathsf{contains}.(\mathsf{Paracetamol} \sqcap \exists \mathsf{mgPerTablet}.(=, 500)) \tag{1}$$

$$\begin{aligned}&\mathsf{Patient} \sqcap \exists \mathsf{hasAge}.(<, 6) \sqcap \\ &\quad \exists \mathsf{hasPrescription}.\exists \mathsf{contains}.(\mathsf{Paracetamol} \sqcap \exists \mathsf{mgPerTablet}.(>, 250)) \sqsubseteq \bot\end{aligned} \tag{2}$$

Axiom (1) states that the drug Panadol contains 500 mg of paracetamol per tablet, while axiom (2) states that a drug that contains more than 250 mg of

[1] http://www.w3.org/2004/OWL

paracetamol per tablet must not be prescribed to a patient younger than 6 years old. The ontology could be used, for example, to support clinical staff who want to check whether Panadol can be prescribed to a 3-year-old patient. This can easily be achieved by checking whether the following concept is satisfiable w.r.t. the ontology:

$$\mathsf{Patient} \sqcap \exists \mathsf{hasAge}.(=, 3) \sqcap \exists \mathsf{hasPrescription}.\mathsf{Panadol} \tag{3}$$

Unfortunately, this is not possible using $\mathcal{EL}^{++}$, because axioms (1) and (2) involve both equality (=) and inequalities (<, >), and this violates the *p*-admissibility restriction. In this paper we demonstrate that it is, however, possible to express axioms (1) and (2) and concept (3) in a tractable extension of $\mathcal{EL}$. A polynomial classification procedure can then be used to determine the satisfiability of (3) w.r.t. the ontology by checking if adding an axiom

$$\mathsf{X} \sqsubseteq \mathsf{Patient} \sqcap \exists \mathsf{hasAge}.(=, 3) \sqcap \exists \mathsf{hasPrescription}.\mathsf{Panadol}$$

for some new concept name X would entail $\mathsf{X} \sqsubseteq \bot$.

Our idea is based on the intuition that equality in (1) and (3) serves a different purpose than inequalities do in (2). Equality in (1) and (3) is used to state a *fact* (the content of a drug and the age of a patient) whereas inequalities in (2) are used to trigger a *rule* (what happens if a certain quantity of drug is prescribed to a patient of a certain age). In other words, equality is used *positively* and inequalities are used *negatively*. It seems reasonable to assume that positive usages of datatypes will typically involve equality since a fact can usually be precisely stated. On the other hand, it seems reasonable to assume that negative occurrences of datatypes can involve equality as well as inequalities since a rule usually applies to a range of situations. In this paper, we make a fine-grained study of datatypes in $\mathcal{EL}$ by considering restrictions not only on the kinds of relations included in a datatype, but also on whether the relations can be used positively or negatively.

The main contributions of this paper can be summarised as follows:

1. We introduce the notion of a *Numerical Datatype with Restrictions* (*NDR*) that specifies the domain of the datatype, the datatype relations that can be used positively and the datatype relations that can be used negatively.
2. We extend the $\mathcal{EL}$ reasoning algorithm [3] to provide a polynomial reasoning procedure for an extension of $\mathcal{EL}$ with NDRs, and we prove that this procedure is sound for any NDR.
3. We introduce the notion of a *safe NDR*, show that every extension of $\mathcal{EL}$ with a safe NDR is tractable, and prove that our reasoning procedure is complete for any safe NDR.
4. Finally, we provide a complete classification of safe NDRs for the cases of natural numbers, integers, rationals and reals. Notably, we demonstrate that the numerical datatype restrictions can be significantly relaxed by allowing arbitrary numerical relations to occur negatively—not only equality as currently specified in the OWL 2 EL Profile. As argued earlier, this combination is of particular interest to ontology engineering, and is thus a strong candidate for the next extension of the EL Profile in OWL 2.

This work is an extended version of a conference paper [7], and provides the full proofs for all obtained results. The paper is organized as follows. After providing the necessary technical background in Section 2, in Section 3 we formally define the extensions of $\mathcal{EL}$ with NDRs, characterise when such extensions are tractable using the notion of "safety", which is closely related to the notion of convexity [3], and describe a polynomial and sound classification procedure that is complete for safe NDRs. The remainder of the paper is concerned with identifying all safe NDRs for the domains of natural numbers (Section 4), integers (Section 5), rationals, and reals (Section 6). In Section 7 we discuss the related work. In Section 8 we summarise our results and outline the directions for future research.

2 Preliminaries

In this section we introduce an extension of $\mathcal{EL}^\perp$ [3] with numerical datatypes which we denote by $\mathcal{EL}^\perp(\mathcal{D})$. In the DL literature datatypes are best known under the name "concrete domains" [8]; in this paper we use the term "datatypes" to be more consistent with the terminology of OWL and OWL 2 [2]. The syntax of $\mathcal{EL}^\perp(\mathcal{D})$ uses a set of *concept names* N_C, a set of *role names* N_R and a set of *feature names* N_F. $\mathcal{EL}^\perp(\mathcal{D})$ is parametrised with a *numerical domain* $\mathcal{D}$, such that $\mathcal{D} \subseteq \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. N_C, N_R and N_F are countably infinite sets and, additionally, pairwise disjoint.

Definition 1 ($\mathcal{D}$-datatype restriction) We call (s, y), where $y \in \mathcal{D}$ and $s \in \{<, \leq, >, \geq, =\}$, a $\mathcal{D}$-*datatype restriction* (or simply a *datatype restriction* if the domain $\mathcal{D}$ is clear from the context). Given a domain $\mathcal{D}$, a $\mathcal{D}$-datatype restriction $r = (s, y)$ and an $x \in \mathcal{D}$, we say that x satisfies r and we write $r(x)$ iff $(x, y) \in s$, where s is interpreted in the usual way as the corresponding binary relation on real numbers.

Intuitively, datatype restrictions specify subsets of elements from the numerical domain using the (in)equality relations. For example, the restriction $(<, 5)$ over $\mathcal{D} = \mathbb{N}$ corresponds to the set $\{1, 2, 3, 4\} \subseteq \mathbb{N}$. In DLs datatype restrictions are used to define concepts by referring to elements in such subsets of numerical domains using the features from N_F. The set of concepts is recursively defined using the constructors listed in the middle column of Table 1, where C and D are concepts, $R \in N_R$, $F \in N_F$ and r is a $\mathcal{D}$-datatype restriction. We typically use the capital letters A, B to refer to concept names and the capital letters C, D or E to refer to concepts. We also set the abbreviations $N_C^\top = N_C \cup \{\top\}$ and $N_C^{\top,\perp} = N_C \cup \{\top, \perp\}$.

An *axiom* α *in* $\mathcal{EL}^\perp(\mathcal{D})$ or simply *an axiom* α is an expression of the form $C \sqsubseteq D$, where C and D are concepts. An $\mathcal{EL}^\perp(\mathcal{D})$-*ontology* $\mathcal{O}$ or simply *an ontology* $\mathcal{O}$ is a set of axioms. We say that a concept E occurs in a concept C iff E is used in the construction of C. E *occurs positively (negatively)* in an axiom $C \sqsubseteq D$ iff it occurs in the concept D (respectively C); alternatively we say that the axiom $C \sqsubseteq D$ has *positive (negative) occurrence* of E.

Table 1 Concept descriptions in $\mathcal{EL}^{\bot}(\mathcal{D})$

Name	Syntax	Semantics
Concept name	C	$C^{\mathcal{I}}$
Top	$\top$	$\Delta^{\mathcal{I}}$
Bottom	$\bot$	$\emptyset$
Conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
Existential restriction	$\exists R.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} :\ (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
Datatype restriction	$\exists F.r$	$\{x \in \Delta^{\mathcal{I}} \mid \exists v \in \mathcal{D} :\ (x,v) \in F^{\mathcal{I}} \wedge r(v)\}$

Table 2 Normal form of axioms and normalization rules for $\mathcal{EL}^{\bot}(\mathcal{D})$

Normal forms		Normalization rules	
NF1	$A' \sqsubseteq B'$	$C \sqcap H \sqsubseteq E$	$\rightarrow \{H \sqsubseteq A_f, C \sqcap A_f \sqsubseteq E\}$
NF2	$A_1 \sqcap A_2 \sqsubseteq B$	$\exists R.G \sqsubseteq D$	$\rightarrow \{G \sqsubseteq A_f, \exists R.A_f \sqsubseteq D\}$
NF3	$A \sqsubseteq \exists R.B$	$G \sqsubseteq H$	$\rightarrow \{G \sqsubseteq A_f, A_f \sqsubseteq H\}$
NF4	$\exists R.B \sqsubseteq A$	$C \sqsubseteq \exists R.H$	$\rightarrow \{C \sqsubseteq \exists R.A_f, A_f \sqsubseteq H\}$
NF5	$A \sqsubseteq \exists F.r$	$B \sqsubseteq C \sqcap D$	$\rightarrow \{B \sqsubseteq C, B \sqsubseteq D\}$
NF6	$\exists F.r \sqsubseteq A$	$\bot \sqsubseteq C$	$\rightarrow \emptyset$

An interpretation of $\mathcal{EL}^{\bot}(\mathcal{D})$ is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set which we call the *domain of the interpretation* and $\cdot^{\mathcal{I}}$ is the *interpretation function.* The interpretation function maps each concept name A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, each role name $R \in N_R$ to a relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and each feature name $F \in N_F$ to a relation $F^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \mathcal{D}$. Note that we do not require the interpretation of features to be functional or serial. In this respect, they correspond to the data properties in OWL 2 [2]. The constructors of $\mathcal{EL}^{\bot}(\mathcal{D})$ are interpreted as indicated in the right column of Table 1. For an axiom α, where $\alpha = C \sqsubseteq D$, we write $\mathcal{I} \models \alpha$ and we say that *an interpretation $\mathcal{I}$ satisfies an axiom* α, iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. If $\mathcal{I} \models \alpha$ for every $\alpha \in \mathcal{O}$, then $\mathcal{I}$ *is a model of* $\mathcal{O}$ and we write $\mathcal{I} \models \mathcal{O}$. Additionally, if every model $\mathcal{I}$ of $\mathcal{O}$ satisfies the axiom α then we say that $\mathcal{O}$ *entails* α and we write $\mathcal{O} \models \alpha$. We define the *signature of an ontology* $\mathcal{O}$ as the set $sig(\mathcal{O})$ of concept, role and feature names that occur in $\mathcal{O}$. We say that a concept, role or feature name X is *fresh w.r.t. an ontology* $\mathcal{O}$ iff $X \notin sig(\mathcal{O})$.

One of the most common reasoning tasks w.r.t. an ontology $\mathcal{O}$ is the classification of an ontology $\mathcal{O}$, that is computing all axioms of the form $A \sqsubseteq B$, where A, $B \in N_C^{\top,\bot}$ and $\mathcal{O} \models A \sqsubseteq B$. The (transitively reduced) set of these *subsumption relations* is called the *taxonomy* of the ontology $\mathcal{O}$.

We say that an axiom in $\mathcal{EL}^{\bot}(\mathcal{D})$ is in normal form if it has one of the forms NF1-NF6 in the left part of Table 2, where $A' \in N_C^{\top}$, $B' \in N_C^{\top,\bot}$, $A_{(i)}, B \in N_C$, $R \in N_R$, $F \in N_F$, and r is a $\mathcal{D}$-datatype restriction. It holds that for each $\mathcal{EL}^{\bot}(\mathcal{D})$-ontology, if the normalization rules of the right part of Table 2 are applied, we obtain an ontology which contains only axioms in normal form [3]. For the rules of Table 2, we have that $B \in N_C$, $G, H \notin N_C$, $R \in N_R$, C, D, E, G and H are concepts and A_f is a fresh concept name w.r.t. the so far transformed ontology. For example, by successively applying the fourth and the fifth rule of Table 2 to the axiom (1), we replace

(1) by the normalised axioms $\mathsf{Panadol} \sqsubseteq \exists \mathsf{contains}.Y$, $Y \sqsubseteq \mathsf{Paracetamol}$ and $Y \sqsubseteq \exists \mathsf{mgPerTablet}.(=, 500)$, where Y is a fresh concept. It can be shown that normalisation of an ontology $\mathcal{O}$ can be performed in polynomial time and that the size of the produced ontology is linear in the size of the input ontology.

3 Numerical Datatypes with Restrictions

In this section we introduce the notion of a Numerical Datatype with Restrictions (NDR) which specifies which datatype relations can be used positively and negatively in ontologies. We then present a polynomial consequence-based classification procedure for $\mathcal{EL}^{\bot}$ extended with NDRs and prove its soundness. Finally we prove that the procedure is complete provided that the NDR satisfies special safety requirements.

Definition 2 (Numerical Datatype with Restrictions) A *numerical datatype with restrictions (NDR)* is a triple $(\mathcal{D}, O_+, O_-)$, where $\mathcal{D} \subseteq \mathbb{R}$ is a numerical domain, and $O_+, O_- \subseteq \{<, \leq, >, \geq, =\}$ are the sets of *positive* and, respectively, *negative relations*. An $\mathcal{EL}^{\bot}(\mathcal{D})$-axiom is an $\mathcal{EL}^{\bot}(\mathcal{D}, O_+, O_-)$-axiom if for every positive (negative) occurrence of a concept $\exists F.(s, y)$ in the axiom, $s \in O_+$ (respectively $s \in O_-$). An $\mathcal{EL}^{\bot}(\mathcal{D}, O_+, O_-)$-*ontology* is a set of $\mathcal{EL}^{\bot}(\mathcal{D}, O_+, O_-)$-axioms.

For example, $\exists F.(<, 5) \sqsubseteq \exists R.\exists F.(=, 3)$ is an $\mathcal{EL}^{\bot}(\mathbb{N}, \{=\}, \{<, \leq, >, \geq, =\})$-axiom, whereas the axioms $A \sqsubseteq \exists F.(\leq, 3)$ and $\exists F.(>, 1.5) \sqsubseteq B$ are not.

3.1 The Classification Procedure and Soundness

The classification procedure for $\mathcal{EL}^{\bot}(\mathcal{D}, O_+, O_-)$ that we are going to describe is closely related to the procedure for $\mathcal{EL}^{++}$ [3]. In order to formulate inference rules for datatypes we introduce notation for satisfiability of a datatype restriction and implication between datatype restrictions.

Definition 3 Let r_+ and r_- be $\mathcal{D}$-datatype restrictions. We write $r_+ \rightarrow_{\mathcal{D}} \bot$ if there is no $x \in \mathcal{D}$ such that $r_+(x)$ holds. Otherwise, we write $r_+ \nrightarrow_{\mathcal{D}} \bot$. We write $r_+ \rightarrow_{\mathcal{D}} r_-$ if $r_+(x)$ implies $r_-(x)$, for every $x \in \mathcal{D}$. Otherwise, we write $r_+ \nrightarrow_{\mathcal{D}} r_-$.

For example, $(<, 0) \rightarrow_{\mathbb{N}} \bot$ and $(<, 5) \rightarrow_{\mathbb{N}} (\leq, 4)$, but $(<, 5) \nrightarrow_{\mathbb{R}} (\leq, 4)$. Note that $r_+ \rightarrow_{\mathcal{D}} \bot$ implies that $r_+ \rightarrow_{\mathcal{D}} r_-$ for every restriction r_-. We assume that given r_+ and r_-, it is possible to decide in polynomial time whether $r_+ \rightarrow_{\mathcal{D}} \bot$ and $r_+ \rightarrow_{\mathcal{D}} r_-$. Table 3 shows that this is the case for $\mathcal{D} = \mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$.

The classification procedure for $\mathcal{EL}^{\bot}(\mathcal{D})$ takes as input an $\mathcal{EL}^{\bot}(\mathcal{D})$-ontology $\mathcal{O}$ whose axioms are in normal form and applies the inference rules in Table 4 to derive new axioms of the form $\mathsf{NF1}$, $\mathsf{NF3}$ and $\mathsf{NF5}$ in Table 2. The rules are applied to already derived axioms and use axioms in $\mathcal{O}$ and the properties

Table 3 Deciding $r_+ \rightarrow_\mathcal{D} \bot$ and $r_+ \rightarrow_\mathcal{D} r_-$ for $\mathcal{D} = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$

$r_+ \rightarrow_\mathcal{D} \bot$	when holds
$(<, n) \rightarrow_\mathcal{D} \bot$	$n = 0$, $\mathcal{D} = \mathbb{N}$
$(\leq, n) \rightarrow_\mathcal{D} \bot$	never
$(=, n) \rightarrow_\mathcal{D} \bot$	never
$(\geq, n) \rightarrow_\mathcal{D} \bot$	never
$(\geq, n) \rightarrow_\mathcal{D} \bot$	never
$r_+ \rightarrow_\mathcal{D} r_-$	when holds
$(<, n) \rightarrow_\mathcal{D} (<, m)$	$n \leq m$
$(<, n) \rightarrow_\mathcal{D} (\leq, m)$	$n \leq m + 1$, $\mathcal{D} = \mathbb{N}$, $\mathbb{Z}$, or $n \leq m$, $\mathcal{D} = \mathbb{R}$
$(<, n) \rightarrow_\mathcal{D} (=, m)$	$n = 0$, $\mathcal{D} = \mathbb{N}$ or $n = 1$, $m = 0$, $\mathcal{D} = \mathbb{N}$
$(<, n) \rightarrow_\mathcal{D} (\geq, m)$	$n = 0$, $\mathcal{D} = \mathbb{N}$
$(<, n) \rightarrow_\mathcal{D} (>, m)$	$n = 0$, $\mathcal{D} = \mathbb{N}$
$(\leq, n) \rightarrow_\mathcal{D} (<, m)$	$n < m$
$(\leq, n) \rightarrow_\mathcal{D} (\leq, m)$	$n \leq m$
$(\leq, n) \rightarrow_\mathcal{D} (=, m)$	$n = m = 0$, $\mathcal{D} = \mathbb{N}$
$(\leq, n) \rightarrow_\mathcal{D} (\geq, m)$	never
$(\leq, n) \rightarrow_\mathcal{D} (>, m)$	never
$(=, n) \rightarrow_\mathcal{D} (<, m)$	$n < m$
$(=, n) \rightarrow_\mathcal{D} (\leq, m)$	$n \leq m$
$(=, n) \rightarrow_\mathcal{D} (=, m)$	$n = m$
$(=, n) \rightarrow_\mathcal{D} (\geq, m)$	$n \geq m$
$(=, n) \rightarrow_\mathcal{D} (>, m)$	$n > m$
$(\geq, n) \rightarrow_\mathcal{D} (<, m)$	never
$(\geq, n) \rightarrow_\mathcal{D} (\leq, m)$	never
$(\geq, n) \rightarrow_\mathcal{D} (=, m)$	never
$(\geq, n) \rightarrow_\mathcal{D} (\geq, m)$	$n \geq m$
$(\geq, n) \rightarrow_\mathcal{D} (>, m)$	$n > m$
$(>, n) \rightarrow_\mathcal{D} (<, m)$	never
$(>, n) \rightarrow_\mathcal{D} (\leq, m)$	never
$(>, n) \rightarrow_\mathcal{D} (=, m)$	never
$(>, n) \rightarrow_\mathcal{D} (\geq, m)$	$n \geq m - 1$, $\mathcal{D} = \mathbb{N}$, $\mathbb{Z}$, or $n \geq m$, $\mathcal{D} = \mathbb{R}$
$(>, n) \rightarrow_\mathcal{D} (>, m)$	$n \geq m$

$r_+ \rightarrow_\mathcal{D} \bot$ and $r_+ \rightarrow_\mathcal{D} r_-$ as side-conditions. E.g., if $A \sqsubseteq \exists F.(<, 5)$ has been derived by previous application of the rules and $\exists F.(\leq, 4) \sqsubseteq B$ is in the input ontology, using **CD1** we can derive $A \sqsubseteq B$, since $(<, 5) \rightarrow_\mathbb{N} (\leq, 4)$.

The procedure terminates when no new axiom can be derived. It is easy to see that the procedure runs in polynomial time because no new datatype restrictions are created, and there are only polynomially many axioms of the form NF1, NF3 and NF5 possible over the symbols in $\mathcal{O}$. It can be easily shown that the procedure is sound because the rules derive logical consequences of the axioms:

Theorem 1 (Soundness) *Let $\mathcal{O}$ be an $\mathcal{EL}^\bot(\mathcal{D})$-ontology consisting of axioms in normal form and $\mathcal{O}'$ be the set of all axioms that are derivable using the rules of Table 4 for $\mathcal{O}$. Then every model $\mathcal{I}$ of $\mathcal{O}$ is a model of $\mathcal{O}'$ as well.*

Proof For every axiom $\alpha \in \mathcal{O}'$, we prove that $\mathcal{I} \models \alpha$ by induction on the length of the derivation of α.

Induction base: If α is obtained using rules **IR1** and **IR2** then clearly $\mathcal{I} \models \alpha$. If $\alpha = A \sqsubseteq \bot$ is obtained using rule **ID1**, then $A \sqsubseteq \exists F.r_+ \in \mathcal{O}$. Since $\mathcal{I} \models \mathcal{O}$,

Table 4 Reasoning rules in $\mathcal{EL}^{\bot}(\mathcal{D})$

IR1	$\frac{}{A \sqsubseteq A}$		IR2	$\frac{}{A \sqsubseteq \top}$	
			CR1	$\frac{A \sqsubseteq B}{A \sqsubseteq C'}$	$B \sqsubseteq C' \in \mathcal{O}$
CR2	$\frac{A \sqsubseteq B \quad A \sqsubseteq C}{A \sqsubseteq D}$	$B \sqcap C \sqsubseteq D \in \mathcal{O}$	CR3	$\frac{A \sqsubseteq B}{A \sqsubseteq \exists R.C}$	$B \sqsubseteq \exists R.C \in \mathcal{O}$
CR4	$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq C}{A \sqsubseteq D}$	$\exists R.C \sqsubseteq D \in \mathcal{O}$	CR5	$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq \bot}{A \sqsubseteq \bot}$	
ID1	$\frac{}{A \sqsubseteq \bot}$	$A \sqsubseteq \exists F.r_+ \in \mathcal{O}$, $r_+ \rightarrow_{\mathcal{D}} \bot$			
CD1	$\frac{A \sqsubseteq \exists F.r_+}{A \sqsubseteq B}$	$\exists F.r_- \sqsubseteq B \in \mathcal{O}$, $r_+ \rightarrow_{\mathcal{D}} r_-$			
CD2	$\frac{A \sqsubseteq B}{A \sqsubseteq \exists F.r_+}$	$B \sqsubseteq \exists F.r_+ \in \mathcal{O}$			

where A, B, C, $D \in N_C^{\top}$, $C' \in N_C^{\top,\bot}$, $R \in N_R, F \in N_F$ for all rules

$A^{\mathcal{I}} \subseteq (\exists F.r_+)^{\mathcal{I}}$. Since $r_+ \rightarrow_{\mathcal{D}} \bot$, $(\exists F.r_+)^{\mathcal{I}} = \emptyset$. Therefore, $A^{\mathcal{I}} \subseteq \emptyset$ and so $\mathcal{I} \models A \sqsubseteq \bot$.

Induction step: For the cases when axiom α is obtained using rules **IR1**-**CR5** (that do not involve datatypes) the proof is the same as for $\mathcal{EL}^{++}$ [3]. If $\alpha = A \sqsubseteq B$ is obtained using **CD1** from $A \sqsubseteq \exists F.r_+$, then by induction hypothesis, $A^{\mathcal{I}} \subseteq (\exists F.r_+)^{\mathcal{I}}$. Since $\mathcal{I} \models \mathcal{O}$, $(\exists F.r_-)^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ and by $r_+ \rightarrow_{\mathcal{D}} r_-$, we have that $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$. So, $\mathcal{I} \models A \sqsubseteq B$. If $\alpha = A \sqsubseteq \exists F.r_+$ is obtained using **CD2** from $A \sqsubseteq B$, then, by induction hypothesis, $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$. Since $\mathcal{I} \models \mathcal{O}$, $B^{\mathcal{I}} \subseteq (\exists F.r_+)^{\mathcal{I}}$ and, so, $A^{\mathcal{I}} \subseteq (\exists F.r_+)^{\mathcal{I}}$. So, $\mathcal{I} \models A \sqsubseteq \exists F.r_+$. □

3.2 Completeness and Safe NDRs

The completeness proof of the procedure presented in the current section is based on the canonical model construction similarly as for $\mathcal{EL}^{++}$ [3]. In order to deal with datatypes in the canonical model we introduce a notion of a *datatype constraint*.

Definition 4 (Constraint) A *constraint* over $(\mathcal{D}, O_+, O_-)$ is defined as a pair of sets (S_+, S_-), such that $S_+ = \{(s_+^1, y_1), \ldots, (s_+^n, y_n)\}$, $n \geq 0$, with $s_+^i \in O_+$, $y_i \in \mathcal{D}$ $(1 \leq i \leq n)$; $S_- = \{(s_-^1, z_1), \ldots, (s_-^m, z_m)\}$, $m \geq 0$, with $s_-^j \in O_-$, $z_j \in \mathcal{D}$ $(0 \leq j \leq m)$. (S_+, S_-) is *trivial* iff there exists an $r^+ \in S_+$, such that $r^+ \rightarrow_{\mathcal{D}} \bot$, or there exist $r^+ \in S_+$ and $r^- \in S_-$, such that $r^+ \rightarrow_{\mathcal{D}} r^-$.

Intuitively, a constraint specifies which datatype restrictions should hold in a model and which should not. Trivial constrains specify restrictions that are trivially unsatisfiable. For example, $(\{(\leq, 2)\}, \{(<, 5), (>, 3)\})$ is a trivial constraint over $(\mathbb{N}, \{\leq\}, \{<, >\})$ because $(\leq, 2) \rightarrow_{\mathbb{N}} (<, 5)$.

Definition 5 (Solution) A *solution* for (S_+, S_-) is a set $V \subseteq \mathcal{D}$ such that (i) for every $r_+ \in S_+$ there exists $x \in V$ such that $r_+(x)$ holds, and (ii) for every $r_- \in S_-$ and every $x \in V$, $r_-(x)$ does not hold. A constraint (S_+, S_-) is *satisfiable* if there exists a solution for (S_+, S_-).

For example $(\{(<, 5)\}, \{(=, 3)\})$ has a solution (e.g. $\{4\}$) and, hence, it is satisfiable. Note that the empty constraint $(\{\}, \{\})$ is also satisfiable and non-trivial. It is easy to see that every trivial constraint is not satisfiable: if $r^+ \rightarrow_{\mathcal{D}} \bot$, then condition (i) of Definition 5 is violated; if $r^+ \rightarrow_{\mathcal{D}} r^-$ then (i) and (ii) cannot hold together. If a constraint is non-trivial, it does not yet mean that it is satisfiable. For example the constraint $(\{(<, 2)\}, \{(=, 0), (=, 1)\})$ over $(\mathbb{N}, \{<\}, \{=\})$ is non-trivial because $(<, 2) \not\rightarrow_{\mathbb{N}} \bot$, $(<, 2) \not\rightarrow_{\mathbb{N}} (=, 0)$ and $(<, 2) \not\rightarrow_{\mathbb{N}} (=, 1)$, but it has no solution $V \subseteq \mathbb{N}$. We are particularly interested in "safe" NDRs for which this never happens, that is, all non-trivial constraints are satisfiable. We will demonstrate that our classification procedure is complete for such NDRs and therefore, extensions of $\mathcal{EL}$ with safe NDRs are tractable.

Definition 6 (NDR Safety) Let $(\mathcal{D}, O_+, O_-)$ be an NDR. $(\mathcal{D}, O_+, O_-)$ is *safe* iff every non-trivial constraint over $(\mathcal{D}, O_+, O_-)$ is satisfiable.

Our goal now is to give classification of all (of finitely many) NDRs over $\mathcal{D} = \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ that are safe. In the following we establish a link between the NDR safety property and the convexity property as defined by Baader et al. [3], which will be later used for proving safety. A datatype $\mathcal{D}$ is convex, if whenever a conjunction of $\mathcal{D}$-datatype restrictions implies a disjunction of such restrictions, then the conjunction also implies some of its disjuncts [3]. We will refer to this property as a strong convexity property, and demonstrate that the notion of safety for NDRs corresponds to a weaker version of it.

Definition 7 (Strong and Weak Convexity) The NDR $(\mathcal{D}, O_+, O_-)$ is *strongly convex* if whenever $(\bigwedge_{i=1}^n r_+^i) \rightarrow_{\mathcal{D}} (\bigvee_{j=1}^m r_-^j)$, for some $r_+^i = (s_+^i, y_i)$, $r_-^j = (s_-^j, z_j)$, $s_+^i \in O_+$, $s_-^j \in O_-$, and $y_i, z_j \in \mathcal{D}$ $(1 \leq i \leq n, 1 \leq j \leq m)$, then there exists j $(1 \leq j \leq m)$ such that $(\bigwedge_{i=1}^n r_+^i) \rightarrow_{\mathcal{D}} r_-^j$. The NDR $(\mathcal{D}, O_+, O_-)$ is *weakly convex* if this property holds for $n = 1$.

For example the NDR $(\mathbb{Z}, \{<, >\}, \{=\})$ is weakly convex since the implications $(<, y) \rightarrow_{\mathbb{Z}} (\bigvee_{j=1}^m (=, z_j))$ and $(>, y) \rightarrow_{\mathbb{Z}} (\bigvee_{j=1}^m (=, z_j))$ never hold. But it is not strongly convex since, e.g., $(>, 2) \wedge (<, 5) \rightarrow_{\mathbb{Z}} (=, 3) \vee (=, 4)$, but $(>, 2) \wedge (<, 5) \not\rightarrow_{\mathbb{Z}} (=, 3)$ and $(>, 2) \wedge (<, 5) \not\rightarrow_{\mathbb{Z}} (=, 4)$.

Lemma 1 $(\mathcal{D}, O_+, O_-)$ *is safe iff it is weakly convex.*

Proof Suppose that $(\mathcal{D}, O_+, O_-)$ is not weakly convex. We prove that it is not safe. Since it is not weakly convex, there exists some $r_+ \rightarrow_{\mathcal{D}} \bigvee_{j=1}^m r_-^j$ such that $r_+ \not\to_\mathcal{D} r_-^j$ for all j with $1 \leq j \leq m$. In order to prove non-safety, it is sufficient to define a non-trivial constraint (S_+, S_-) over $(\mathcal{D}, O_+, O_-)$ that is not satisfiable. Let (S_+, S_-) be such that $S_+ = \{r_+\}$ and $S_- = \{r_-^j\}_{j=1}^m$. (S_+, S_-) is non-trivial because $r_+ \not\to_\mathcal{D} r_-^j$ and $r_+ \not\to_\mathcal{D} \bot$ (otherwise, e.g., $r_+ \to_\mathcal{D} r_-^1$, which exists since $m \geq 1$). (S_+, S_-) is not satisfiable because, otherwise, there exists an $x \in V$ such that $r_+(x)$ and $\neg r_-^j(x)$ for all j with $1 \leq j \leq m$, and so, the implication $r_+ \to_\mathcal{D} (\bigvee_{j=1}^m r_-^j)$ cannot hold.

Conversely, we prove that if $(\mathcal{D}, O_+, O_-)$ is not safe, then it is not weakly convex. Since it is not safe, there exists a non-trivial constraint (S_+, S_-) that is not satisfiable, where $S_+ = \{r_+^i\}_{i=1}^n$ and $S_- = \{r_-^j\}_{j=1}^m$. First note that $S_- \neq \emptyset$, since, otherwise $V = \mathcal{D}$ is a solution for $(S_+, S_-) = (S_+, \emptyset)$ since $r_+^i \not\to_\mathcal{D} \bot$ $(1 \leq i \leq n)$. Similarly, $S_+ \neq \emptyset$ since, otherwise, $V = \emptyset$ is a solution for $(S_+, S_-) = (\emptyset, S_-)$. Since (S_+, S_-) is non-trivial, $r_+^i \not\to_\mathcal{D} r_-^j$ for all i with $1 \leq i \leq n$ and all j with $1 \leq j \leq m$. We claim that there exists some i with $1 \leq i \leq n$ such that $r_+^i \to_\mathcal{D} (\bigvee_{j=1}^m r_-^j)$. Indeed, otherwise for every i $(1 \leq i \leq n)$ there exists $e_i \in \mathcal{D}$ such that $r_+^i(e_i)$ and $\neg r_-^j(e_i)$ hold for all j with $1 \leq j \leq m$, and so $V = \{e_i\}_{i=1}^n$ is a solution for (S_+, S_-). Since $r_+^i \to_\mathcal{D} (\bigvee_{j=1}^m r_-^j)$ but $r_+^i \not\to_\mathcal{D} r_-^j$ $(1 \leq j \leq m)$, $(\mathcal{D}, O_+, O_-)$ is not weakly convex. □

We will now prove that safety for NDRs is a sufficient condition for completeness for the classification algorithm based on the rules in Table 4.

Theorem 2 (Completeness) *Let $(\mathcal{D}, O_+, O_-)$ be a safe NDR, $\mathcal{O}$ an ontology consisting of $\mathcal{EL}^\bot(\mathcal{D}, O_+, O_-)$-axioms in normal form, and $\mathcal{O}'$ the axioms derivable under the rules of Table 4 using $\mathcal{O}$. Then for every $A \in N_C^\top$ and $B \in N_C^{\top,\bot}$, $A, B \in sig(\mathcal{O})$, if $\mathcal{O} \models A \sqsubseteq B$, then $A \sqsubseteq B \in \mathcal{O}'$ or $A \sqsubseteq \bot \in \mathcal{O}'$.*

Proof The proof is analogous to the completeness proof of the subsumption algorithm for $\mathcal{EL}^{++}$ [3]: we will build a (canonical) model $\mathcal{I}$ for $\mathcal{O}$ using $\mathcal{O}'$ and show that for all A, B if $A \sqsubseteq B \notin \mathcal{O}'$ and $A \sqsubseteq \bot \notin \mathcal{O}'$, then $\mathcal{I} \not\models A \sqsubseteq B$. W.l.o.g. there exists at least one A, such that $A \sqsubseteq \bot \notin \mathcal{O}$ since otherwise this claim is trivial.

For every $A \in N_C$, $F \in N_F$, define the constraint $(S_+(A,F), S_-(A,F))$ over $(\mathcal{D}, O_+, O_-)$, as follows:

$$S_+(A,F) = \begin{cases} \emptyset & \text{if } A \sqsubseteq \bot \in \mathcal{O}', \\ \{r_+ \mid A \sqsubseteq \exists F.r_+ \in \mathcal{O}'\} & \text{otherwise.} \end{cases} \tag{3}$$

$$S_-(A,F) = \{r_- \mid \exists F.r_- \sqsubseteq B \in \mathcal{O}, A \sqsubseteq B \notin \mathcal{O}'\} \tag{4}$$

First, we show that $(S_+(A,F), S_-(A,F))$ is a non-trivial constraint. Indeed, if $r_+ \to_\mathcal{D} \bot$ for some $r_+ \in S_+(A,F)$, then by (3) we have $A \sqsubseteq \bot \notin \mathcal{O}'$ and $A \sqsubseteq \exists F.r_+ \in \mathcal{O}'$, which is not possible since $\mathcal{O}'$ is closed under the rule **ID1**. Similarly, there exists no $r_+ \in S_+(A,F)$ and $r_- \in S_-(A,F)$ such that $r_+ \to_\mathcal{D} r_-$ holds because, otherwise by (3) and (4) $A \sqsubseteq \exists F.r_+ \in \mathcal{O}'$, $\exists F.r_- \sqsubseteq B \in \mathcal{O}$, and $A \sqsubseteq B \notin \mathcal{O}'$, which is not possible since $\mathcal{O}'$ is closed under the rule **CD1**. Since

$(S_+(A,F), S_-(A,F))$ is a non-trivial constraint over the safe NDR $(\mathcal{D}, O_+, O_-)$, by Definition 4, it is satisfiable. Let us denote by $V(A,F)$ some fixed solution for $(S_+(A,F), S_-(A,F))$.

Now, if for every $A \in (N_C^\top \cap sig(\mathcal{O}))$ we have $A \sqsubseteq \bot \in \mathcal{O}'$, then the theorem holds trivially. Otherwise, define the following interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$:

$$\Delta^\mathcal{I} = \{e_A \mid A \in (N_C^\top \cap sig(\mathcal{O})), A \sqsubseteq \bot \notin \mathcal{O}'\} \tag{5}$$
$$B^\mathcal{I} = \{e_A \mid e_A \in \Delta^\mathcal{I}, A \sqsubseteq B \in \mathcal{O}'\} \tag{6}$$
$$R^\mathcal{I} = \{(e_A, e_B) \mid A \sqsubseteq \exists R.B \in \mathcal{O}', e_A, e_B \in \Delta^\mathcal{I}\} \tag{7}$$
$$F^\mathcal{I} = \{(e_A, v) \mid v \in V(A,F)\} \tag{8}$$

Intuitively, the domain of the interpretation contains a distinguished element e_A for every concept name A, such that $A \sqsubseteq \bot \notin \mathcal{O}'$. Note that the domain $\Delta^\mathcal{I}$ is not empty since we have assumed that there exists at least one A such that $A \sqsubseteq \bot \notin \mathcal{O}'$. Hence, $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ is defined correctly. We prove that $\mathcal{I}$ is a model of $\mathcal{O}$ by considering all types of axioms $\alpha \in \mathcal{O}$ according to Table 2:

- NF1 : $\alpha = A' \sqsubseteq B'$: Take any $x \in A'^\mathcal{I}$. We need to prove that $x \in B'^\mathcal{I}$. By (5), $x = e_C$ for some $C \in (N_C^\top \cap sig(\mathcal{O}))$ such that $C \sqsubseteq \bot \notin \mathcal{O}'$. We first prove that $C \sqsubseteq A' \in \mathcal{O}'$. Indeed, if $A' \in N_C$, by (6), since $x = e_C \in A'^\mathcal{I}$, $C \sqsubseteq A' \in \mathcal{O}'$. Otherwise, $A' = \top$ and by **IR2**, we have that $C \sqsubseteq \top = C \sqsubseteq A' \in \mathcal{O}'$. Now, since $A' \sqsubseteq B' \in \mathcal{O}$ and $\mathcal{O}'$ is closed under **CR1**, $C \sqsubseteq B' \in \mathcal{O}'$. Since $C \sqsubseteq \bot \notin \mathcal{O}'$, we have either $B' = \top$, and so $x \in \Delta^\mathcal{I} = B'^\mathcal{I}$, or $B' \in N_C$ and by (6), $x = e_C \in B'^\mathcal{I}$.
- NF2 : $\alpha = A_1 \sqcap A_2 \sqsubseteq B$**:** Take any $x \in (A_1 \sqcap A_2)^\mathcal{I}$. We need to prove that $x \in B^\mathcal{I}$. By (5) $x = e_A$ for some $A \in (N_C^\top \cap sig(\mathcal{O}))$, and by (6), since $x = e_A \in (A_1 \sqcap A_2)^\mathcal{I} = A_1^\mathcal{I} \cap A_2^\mathcal{I}$, and $A_1, A_2 \in N_C$, $A \sqsubseteq A_1 \in \mathcal{O}'$ and $A \sqsubseteq A_2 \in \mathcal{O}'$. Since $A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{O}$ and $\mathcal{O}'$ is closed under **CR2**, $A \sqsubseteq B \in \mathcal{O}'$. Since $B \in N_C$, by (6) $x \in B^\mathcal{I}$.
- NF3 : $\alpha = A \sqsubseteq \exists R.B$**:** Take any $x \in A^\mathcal{I}$. We will prove that $x \in (\exists R.B)^\mathcal{I}$. By (5), $x = e_C$ for some $C \in (N_C^\top \cap sig(\mathcal{O}))$ with $C \sqsubseteq \bot \notin \mathcal{O}'$. Since $A \in N_C^\top$ and $x = e_C \in A^\mathcal{I}$, by (6), $C \sqsubseteq A \in \mathcal{O}'$. Since $A \sqsubseteq \exists R.B \in \mathcal{O}$ and $\mathcal{O}'$ is closed under **CR3**, $C \sqsubseteq \exists R.B \in \mathcal{O}'$. Since $C \sqsubseteq \bot \notin \mathcal{O}'$ and $\mathcal{O}'$ is closed under **CR5**, $B \sqsubseteq \bot \notin \mathcal{O}'$. Since additionally $B \in (N_C^\top \cap sig(\mathcal{O}))$, by (5), there exists $e_B \in \Delta^\mathcal{I}$. By (7), $(e_C, e_B) \in R^\mathcal{I}$. Since $\mathcal{O}'$ is closed under **IR1**, $B \sqsubseteq B \in \mathcal{O}'$, therefore, by (6), $e_B \in B^\mathcal{I}$. Thus, $x = e_C \in (\exists R.B)^\mathcal{I}$.
- NF4 : $\alpha = \exists R.B \sqsubseteq A$**:** Take any $x \in (\exists R.B)^\mathcal{I}$. We need to prove that $x \in A^\mathcal{I}$. By definition of the interpretation, there exists $y \in \Delta^\mathcal{I}$ such that $(x,y) \in R^\mathcal{I}$ and $y \in B^\mathcal{I}$. Since $(x,y) \in R^\mathcal{I}$, by (7) $x = e_C$ and $y = e_D$ such that $C \sqsubseteq \exists R.D \in \mathcal{O}'$, and since $y = e_D \in B^\mathcal{I}$ and $B \in N_C$, by (6), $D \sqsubseteq B \in \mathcal{O}'$. Since $\exists R.B \sqsubseteq A \in \mathcal{O}$, $C \sqsubseteq \exists R.D \in \mathcal{O}'$, $D \sqsubseteq B \in \mathcal{O}'$, and $\mathcal{O}'$ is closed under **CR4**, $C \sqsubseteq A \in \mathcal{O}'$. Hence, by (6), $x = e_C \in A^\mathcal{I}$.
- NF5 : $\alpha = A \sqsubseteq \exists F.r_+$**:** Take any $x \in A^\mathcal{I}$. We will prove that $x \in (\exists F.r_+)^\mathcal{I}$. By (5), $x = e_C$ for some $C \in (N_C^\top \cap sig(\mathcal{O}))$ such that $C \sqsubseteq \bot \notin \mathcal{O}'$. By (5), since $x = e_C \in A^\mathcal{I}$, $C \sqsubseteq A \in \mathcal{O}'$. Since $A \sqsubseteq \exists F.r_+ \in \mathcal{O}$ and $\mathcal{O}'$ is closed under **CD2**, $C \sqsubseteq \exists F.r_+ \in \mathcal{O}'$. Let $(S_+(C,F), S_-(C,F))$ be the

constraint defined according to (3) and (4) and $V(C, F)$ its solution. Since $C \sqsubseteq \exists F.r_+ \in \mathcal{O}'$ and $C \sqsubseteq \bot \notin \mathcal{O}'$, by (3), $r_+ \in S_+(C, F)$. Then there exists $v \in V(C, F)$ such that v satisfies r_+. By (8), we have $(e_C, v) \in F^{\mathcal{I}}$, hence $x = e_C \in (\exists F.r_+)^{\mathcal{I}}$ by the definition of the interpretation.

– $\mathsf{NF6}$: $\alpha = \exists F.r_- \sqsubseteq B$**:** Take any $x \in (\exists F.r_-)^{\mathcal{I}}$. We need to prove that $x \in B^{\mathcal{I}}$. By (5), $x = e_C$ for some $C \in (N_C^{\top} \cap sig(\mathcal{O}))$ such that $C \sqsubseteq \bot \notin \mathcal{O}'$. Let $(S_+(C, F), S_-(C, F))$ be the constraint defined according to (3) and (4) and $V(C, F)$ its solution. Since $x \in (\exists F.r_-)^{\mathcal{I}}$, by the definition of the interpretation, there exists $v \in \mathcal{D}$ such that $(x, v) \in F^{\mathcal{I}}$ and v satisfies r_-. By (8), $v \in V(C, F)$. Since $V(C, F)$ is a solution for $(S_+(C, F), S_-(C, F))$ and v satisfies r_-, we have $r_- \notin S_-(C, F)$. Hence, by (4) and since $\exists F.r_- \sqsubseteq B \in \mathcal{O}$, $C \sqsubseteq B \in \mathcal{O}'$. Since $B \in N_C$, by (6), $x = e_C \in B^{\mathcal{I}}$.

To conclude the proof of the theorem, suppose to the contrary that there exist $A \in N_C^{\top}$ and $B \in N_C^{\top,\bot}$, $A, B \in sig(\mathcal{O})$, such that $\mathcal{O} \models A \sqsubseteq B$, but $A \sqsubseteq B \notin \mathcal{O}'$ and $A \sqsubseteq \bot \notin \mathcal{O}'$. Let $\mathcal{I}$ be the model defined by (5)–(8). As shown above, $\mathcal{I} \models \mathcal{O}$. Since $A \in (N_C^{\top} \cap sig(\mathcal{O}))$ and $A \sqsubseteq \bot \notin \mathcal{O}'$, by (5), $e_A \in \Delta^{\mathcal{I}}$. Since $A \in (N_C^{\top} \cap sig(\mathcal{O}))$ and $\mathcal{O}'$ is closed under **IR1**, $A \sqsubseteq A \in \mathcal{O}'$. Therefore, by (6), $e_A \in A^{\mathcal{I}}$. Since $\mathcal{I} \models \mathcal{O}$ and $\mathcal{O} \models A \sqsubseteq B$, $e_A \in B^{\mathcal{I}}$. Since $A \in (N_C^{\top} \cap sig(\mathcal{O}))$ and $\mathcal{O}'$ is closed under **IR2**, $A \sqsubseteq \top \in \mathcal{O}'$. Therefore, since $A \sqsubseteq B \notin \mathcal{O}'$, $B \neq \top$. Also, since $e_A \in B^{\mathcal{I}}$, $B \neq \bot$. Hence $B \in N_C$ and by (6), since $e_A \in B^{\mathcal{I}}$, $A \sqsubseteq B \in \mathcal{O}'$. This contradicts our assumption that $A \sqsubseteq B \notin \mathcal{O}'$. Thus, the proof by contradiction implies the statement of the theorem. □

Corollary 1 *The classification of $\mathcal{EL}^{\bot}(\mathcal{D}, O_+, O_-)$-ontologies for safe NDRs $(\mathcal{D}, O_+, O_-)$ can be computed in polynomial time in the size of the ontology.*

Proof Given an $\mathcal{EL}^{\bot}(\mathcal{D}, O_+, O_-)$-ontology $\mathcal{O}$, let $\mathcal{O}_N$ be the result of applying the normalisation rules from Table 2 to $\mathcal{O}$, and $\mathcal{O}'$ be the axioms derivable by rules in Table 4 using $\mathcal{O}_N$. As has been pointed out, both of these computations can be performed in polynomial time. By Theorem 1 and Theorem 2, the result of classification can be computed by taking all $A \sqsubseteq B$ for A, $B \in (N_C^{\top,\bot} \cap sig(\mathcal{O}))$, where either $A = \bot$, or $A \sqsubseteq B \in \mathcal{O}'$ or $A \sqsubseteq \bot \in \mathcal{O}'$. □

4 Maximal Safe NDRs for $\mathbb{N}$

In this section we present a full classification of safe NDRs for natural numbers; within this section we assume that every constraint is over the domain $\mathcal{D} = \mathbb{N}$. The main result of this section is presented in Table 5, which lists all maximal safe NDRs for $\mathbb{N}$. We prove that: (*i*) every NDR in Table 5 is safe, (*ii*) extending any of these NDRs with a new relation leads to non-safety and (*iii*) every safe NDR for $\mathbb{N}$ is contained in some NDR in Table 5.

In order to prove that the NDRs in Table 5 are safe, by Definition 6 we need to demonstrate that every non-trivial constraint over each of these NDRs is satisfiable. In the next Lemma we show that w.l.o.g. we can focus our attention only on constraints of a reduced form.

Table 5 Maximal safe NDRs for $\mathbb{N}$: $\mathsf{NDR}_i^{\mathbb{N}} = (O_+^i, O_-^i)$, $1 \leq i \leq 4$

$\mathsf{NDR}_i^{\mathbb{N}}$	O_+^i	O_-^i
$\mathsf{NDR}_1^{\mathbb{N}}$	$\{=\}$	$\{<, \leq, >, \geq, =\}$
$\mathsf{NDR}_2^{\mathbb{N}}$	$\{<, \leq, >, \geq, =\}$	$\{<, \leq\}$
$\mathsf{NDR}_3^{\mathbb{N}}$	$\{<, \leq, >, \geq, =\}$	$\{>, \geq\}$
$\mathsf{NDR}_4^{\mathbb{N}}$	$\{>, \geq, =\}$	$\{<, \leq, =\}$

Lemma 2 *Let (S_+, S_-) be a non-trivial constraint over an NDR from Table 5. Then there exists a non-trivial constraint (S'_+, S'_-) over the same NDR such that (S_+, S_-) is satisfiable iff (S'_+, S'_-) is satisfiable and:*

$$S'_+ \subseteq \{(\leq, y_0), (=, y_1), \ldots, (=, y_n), (\geq, y_{n+1})\}, \tag{9}$$
$$S'_- \subseteq \{(\leq, z_0), (=, z_1), \ldots, (=, z_m), (\geq, z_{m+1})\}. \tag{10}$$

Proof Given a non-trivial constraint (S_+, S_-) let us apply the following transformation rules to (S_+, S_-):

$$(S_+ \cup \{(<, y)\}, S_-) \Rightarrow (S_+ \cup \{(\leq, y-1)\}, S_-), \text{ if } y \geq 1 \tag{11}$$
$$(S_+ \cup \{(>, y)\}, S_-) \Rightarrow (S_+ \cup \{(\geq, y+1)\}, S_-) \tag{12}$$
$$(S_+, S_- \cup \{(<, 0)\}) \Rightarrow (S_+, S_-) \tag{13}$$
$$(S_+, S_- \cup \{(<, z)\}) \Rightarrow (S_+, S_- \cup \{(\leq, z-1)\}), \text{ if } z \geq 1 \tag{14}$$
$$(S_+, S_- \cup \{(>, z)\}) \Rightarrow (S_+, S_- \cup \{(\geq, z+1)\}) \tag{15}$$
$$(S_+ \cup \{(\leq, y_1), (\leq, y_2)\}, S_-) \Rightarrow (S_+ \cup \{(\leq, \min\{y_1, y_2\})\}, S_-) \tag{16}$$
$$(S_+ \cup \{(\geq, y_1), (\geq, y_2)\}, S_-) \Rightarrow (S_+ \cup \{(\leq, \max\{y_1, y_2\})\}, S_-) \tag{17}$$
$$(S_+, S_- \cup \{(\leq, z_1), (\leq, z_2)\}) \Rightarrow (S_+, S_- \cup \{(\leq, \max\{z_1, z_2\})\}) \tag{18}$$
$$(S_+, S_- \cup \{(\geq, z_1), (\geq, z_2)\}) \Rightarrow (S_+, S_- \cup \{(\geq, \min\{z_1, z_2\})\}) \tag{19}$$

It is easy to see that that rules (11)-(19) can be applied only finitely many times. Indeed, every transformation either reduces the number of restrictions with relations $<$ and $>$ by rules (11)–(15), or leaves this number the same and reduces the number of restrictions with relations $\leq$, $\geq$ by rules (16)–(19). Let (S'_+, S'_-) be obtained from (S_+, S_-) after exhaustively applying the transformations (11)-(19). It is easy to see that (S'_+, S'_-) remains a non-trivial constraint over the same NDR from Table 5 as (S_+, S_-) and (S'_+, S'_-) is satisfiable iff (S_+, S_-) is satisfiable. Moreover, since (S_+, S_-) is non-trivial, $(<, 0) \notin S_+$, and therefore (S'_+, S'_-) does not contain restrictions of the form $(<, x)$ and $(>, x)$, because all of them are eliminated by rules (11)–(15). Similarly, each set S'_+ and S'_- contains at most one restriction of the form $(\leq, x)$ and at most one restriction of the form $(\geq, x)$ as a consequence of the transformation rules (16)–(19). Therefore (9) and (10) hold. □

Lemma 3 *Every NDR in Table 5 is safe.*

Proof According to Definition 6, in order to prove safety for an NDR we need to find a solution for every non-trivial constraint over the NDR. Recall that a solution V for a constraint (S_+, S_-) over an NDR is a set $V \subseteq \mathcal{D} = \mathbb{N}$ such that every $r_+ \in S_+$ is satisfied by at least one value $v \in V$ but no $r_- \in S_-$ is satisfied by any value $v \in V$. By Lemma 2, w.l.o.g. we can assume that S_+ and S_- are of the form (9) and (10), respectively. We construct the solution V by performing the following case analysis over the content of S_+ and S_-:

Case 1: $S_+ = \{(=, y_1), \ldots, (=, y_n)\}$, $n \geq 0$. Define $V := \{y_1, \ldots, y_n\}$. Clearly, every restriction in S_+ is satisfied by some $y_i \in V$, but no restriction in S_- is satisfied by V: if y_i, with $1 \leq i \leq n$, satisfies some restriction $r_- \in S_-$, then $(=, y_i) \rightarrow_{\mathbb{N}} r_-$, which contradicts the non-triviality of (S_+, S_-).

Case 2: $S_+ \cap \{(\leq, y_0), (\geq, y_{n+1})\} \neq \emptyset$. We further distinguish cases according to the content of S_-. Note that we do not examine the case where $\{(\leq, z_0), (\geq, z_{m+1})\} \subseteq S_-$, because this is not possible for NDRs in Table 5.

Case 2.1: $S_- = \{(=, z_1), \ldots, (=, z_m)\}$, $m \geq 0$. Define $V := \mathbb{N} \backslash \{z_1, \ldots, z_m\}$. It is easy to see that V satisfies all restrictions except for those in S_-. Since (S_+, S_-) is non-trivial, and thus $S_+ \cap S_- = \emptyset$, V is a solution for (S_+, S_-).

Case 2.2: $S_- = \{(\leq, z_0)\}$. Define $V := \{v \in \mathbb{N} \mid v > z_0\}$. It is easy to see that V satisfies all restrictions except for restrictions of the form $(\leq, y)$ and $(=, y)$ with $y \leq z_0$. Since such restrictions imply $(\leq, z_0)$ and (S_+, S_-) is non-trivial, S_+ cannot contain them. Therefore, V is a solution for (S_+, S_-).

Case 2.3: $S_- = \{(\geq, z_{m+1})\}$. Define $V := \{v \in \mathbb{N} \mid v < z_{m+1}\}$. It is easy to see that V satisfies all restrictions except for restrictions of the form $(\geq, y)$ and $(=, y)$ with $y \geq z_{m+1}$. Since such restrictions imply $(\geq, z_{m+1})$ and (S_+, S_-) is non-trivial, S_+ cannot contain them. Therefore, V is a solution for (S_+, S_-).

Case 2.4: $S_- = \{(\leq, z_0), (=, z_1), \ldots, (=, z_m)\}$, $m \geq 1$. Define $V := \{v \in \mathbb{N} \mid v > z_0 \backslash \{z_1, \ldots, z_m\}\}$. From Table 5 one can see that S_+ cannot contain restrictions of the form $(\leq, y)$. It is easy to see that from the remaining restrictions, V satisfies all restrictions except for those of the form $(=, y)$ with $y \leq z_0$ or $y = z_j$ $(1 \leq j \leq m)$. Since such restrictions imply restrictions in S_-, S_+ cannot contain them. Therefore, V is a solution for (S_+, S_-). □

We are now in a position to prove that the NDRs in Table 5 are maximal safe, that is, they contain all safe NDRs over $\mathcal{D} = \mathbb{N}$. In order to prove this property, we first list several cases of non-safe NDRs for $\mathbb{N}$, and then, show that by extending NDRs listed in Table 5, we fall into one of these cases.

Lemma 4 *Let* $(\mathbb{N}, O_+, O_-)$ *be an NDR. Then:*

(a) If $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$, $O_- \cap \{<, \leq\} \neq \emptyset$ *and* $O_- \cap \{>, \geq\} \neq \emptyset$*, then* $(\mathbb{N}, O_+, O_-)$ *is non-safe.*
(b) If $O_+ \cap \{>, \geq\} \neq \emptyset$, $O_- \cap \{>, \geq\} \neq \emptyset$ *and* $\{=\} \subseteq O_-$*, then* $(\mathbb{N}, O_+, O_-)$ *is non-safe.*
(c) If $O_+ \cap \{<, \leq\} \neq \emptyset$ *and* $\{=\} \subseteq O_-$*, then* $(\mathbb{N}, O_+, O_-)$ *is non-safe.*

Proof In order to prove that $(\mathbb{N}, O_+, O_-)$ is non-safe, by Lemma 1 it suffices to prove that it is not weakly convex. Recall that by Definition 7, $(\mathbb{N}, O_+, O_-)$ is

Table 6 Examples of non-safe NDRs for $\mathbb{N}$ where $(s_+, y) \rightarrow_{\mathbb{N}} (s^1_-, z_1) \vee (s^2_-, z_2)$, $(s_+, y) \nrightarrow_{\mathbb{N}} (s^1_-, z_1)$ and $(s_+, y) \nrightarrow_{\mathbb{N}} (s^2_-, z_2)$

$\{s_+\}$	$\{s^1_-, s^2_-\}$	y	z_1	z_2
$\{<\}, \{\leq\}$	$\{<, \geq\}, \{\leq, >\}, \{\leq, \geq\}$	3	1	1
$\{<\}, \{\leq\}$	$\{<, >\}$	3	2	1
$\{>\}, \{\geq\}$	$\{<, \geq\}, \{\leq, >\}, \{\leq, \geq\}$	1	3	3
$\{>\}, \{\geq\}$	$\{<, >\}$	1	3	2
$\{>\}$	$\{=, \geq\}$	1	2	3
$\{>\}$	$\{=, >\}$	1	2	2
$\{\geq\}$	$\{=, \geq\}$	1	1	2
$\{\geq\}$	$\{=, >\}$	1	1	1
$\{<\}$	$\{=\}$	3	1	2
$\{\leq\}$	$\{=\}$	2	1	2

weakly convex iff for every $r_+ = (s_+, y)$ and $r^i_- = (s^i_-, z_i)$, such that $s_+ \in O_+$, $s^i_- \in O_-$ and y, $z_i \in \mathbb{N}$ $(1 \leq i \leq n)$, if $r_+ \rightarrow_{\mathbb{N}} \bigvee_{i=1}^{n} r^i_-$, then there exists an i with $1 \leq i \leq n$ such that $r_+ \rightarrow_{\mathbb{N}} r^i_-$. For each of the cases (a), (b) and (c) of the lemma, we provide counterexamples that violate the weak convexity condition, namely a triple of restrictions (s_+, y), (s^1_-, z_1) and (s^2_-, z_2), such that $s_+ \in O_+$, s^1_-, $s^2_- \in O_-$, y, z_1, $z_2 \in \mathbb{N}$, $(s_+, y) \rightarrow_{\mathbb{N}} (s^1_-, z_1) \vee (s^2_-, z_2)$, $(s_+, y) \nrightarrow_{\mathbb{N}} (s^1_-, z_1)$ and $(s_+, y) \nrightarrow_{\mathbb{N}} (s^2_-, z_2)$. The counterexamples are listed in Table 6: the first four lines correspond to case (a), the next four lines to case (b) and the final two lines to case (c). □

Lemma 5 *Every NDR in Table 5 is maximal safe, that is if any new relation is added to O_+ or O_-, the NDR becomes non-safe.*

Proof We examine all cases of adding a new relation to NDRs in Table 5:
$\underline{\mathsf{NDR}^{\mathbb{N}}_1}$: By Lemma 4(a), if any of $<$, $\leq$, $>$, $\geq$ is added to O_+, then $\mathsf{NDR}^{\mathbb{N}}_1$ becomes non-safe.
$\underline{\mathsf{NDR}^{\mathbb{N}}_2}$: By Lemma 4(a), if $>$ or $\geq$ is added to O_-, then $\mathsf{NDR}^{\mathbb{N}}_2$ becomes non-safe. By Lemma 4(c), when $=$ is added to O_-, $\mathsf{NDR}^{\mathbb{N}}_2$ becomes non-safe.
$\underline{\mathsf{NDR}^{\mathbb{N}}_3}$: By Lemma 4(a), if $<$ or $\leq$ is added to O_-, then $\mathsf{NDR}^{\mathbb{N}}_3$ becomes non-safe. By Lemma 4(c), when $=$ is added to O_-, $\mathsf{NDR}^{\mathbb{N}}_3$ becomes non-safe.
$\underline{\mathsf{NDR}^{\mathbb{N}}_4}$: By Lemma 4(b), if $>$ or $\geq$ is added to O_-, then $\mathsf{NDR}^{\mathbb{N}}_4$ is non-safe. By Lemma 4(c), when $<$ or $\leq$ is added to O_+, $\mathsf{NDR}^{\mathbb{N}}_4$ becomes non-safe. □

It remains to prove that the list of safe NDRs in Table 5 subsumes every safe NDR for $\mathcal{D} = \mathbb{N}$.

Lemma 6 *If $(\mathbb{N}, O_+, O_-)$ is a safe NDR, then $O_+ \subseteq O^i_+$ and $O_- \subseteq O^i_-$ for some $\mathsf{NDR}^{\mathbb{N}}_i = (\mathbb{N}, O^i_+, O^i_-)$ in Table 5, $(1 \leq i \leq 4)$.*

Proof The proof is by case analysis of possible relations in O_+ and O_-.

Case 1: $O_+ \cap \{<, \leq, >, \geq\} = \emptyset$. In this case, $O_+ \subseteq O^1_+$ and $O_- \subseteq O^1_-$.

Case 2: $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$. By Lemma 4(a) and since the NDR is safe, $O_- \cap \{<, \leq\} = \emptyset$ or $O_- \cap \{>, \geq\} = \emptyset$. Therefore, we examine two cases: either $O_- \subseteq \{>, \geq, =\}$ or $O_- \subseteq \{<, \leq, =\}$.

Case 2.1: $O_- \subseteq \{>, \geq, =\}$. We further distinguish whether $O_- \subseteq \{>, \geq\}$ or $O_- \cap \{=\} \neq \emptyset$.

Case 2.1.1: $O_- \subseteq \{>, \geq\}$. In this case, $\{>, \geq\} = O^3_-$, so $O_- \subseteq O^3_-$; also $O_+ \subseteq O^3_+$.

Case 2.1.2: $O_- \cap \{=\} \neq \emptyset$. By Lemma 4(c) and since the NDR is safe, $O_+ \subseteq \{>, \geq, =\}$; also since $O^4_+ = \{>, \geq, =\}$, $O_+ \subseteq O^4_+$. By Lemma 4(b) and since the NDR is safe, $O_- \cap \{>, \geq\} = \emptyset$. Therefore, $O_- = \{=\}$ and since $\{=\} \subseteq O^4_-$, $O_- \subseteq O^4_-$

Case 2.2: $O_- \subseteq \{<, \leq, =\}$. We distinguish whether $O_+ \subseteq \{>, \geq, =\}$ or $O_+ \cap \{<, \leq\} \neq \emptyset$.

Case 2.2.1: $O_+ \subseteq \{>, \geq, =\}$. If $O_+ \subseteq \{>, \geq, =\}$, then by $\{>, \geq, =\} = O^4_+$, $O_+ \subseteq O^4_+$. Also since $O^4_- = \{<, \leq, =\}$, $O_- \subseteq O^4_-$.

Case 2.2.2: $O_+ \cap \{<, \leq\} \neq \emptyset$. By Lemma 4(c) and since the NDR is safe, $\{=\} \cap O_- = \emptyset$. Therefore, $O_- \subseteq \{<, \leq\}$. Since $\{<, \leq\} = O^2_-$, $O_- \subseteq O^2_-$; also $O_+ \subseteq O^2_+$. □

5 Maximal Safe NDRs for $\mathbb{Z}$

In this section, we identify the maximal safe NDRs for the domain of integers ($\mathbb{Z}$). Table 7 lists all maximal safe NDRs for $\mathbb{Z}$. Compared to the Table 5, we have two new maximal safe NDRs, namely $\mathsf{NDR}^{\mathbb{Z}}_2$ and $\mathsf{NDR}^{\mathbb{Z}}_6$. This is because integers do not have a minimal element as in the case of naturals. In particular positive occurrences of $<$ or $\leq$ together with negative occurrence of $=$ are no longer dangerous: e.g., $(\leq, 1) \rightarrow_{\mathbb{N}} (=, 1) \vee (=, 0)$, but $(\leq, 1) \not\rightarrow_{\mathbb{Z}} (=, 1) \vee (=, 0)$.

Lemma 7 *Let (S_+, S_-) be a non-trivial constraint over an NDR from Table 7. Then there exists a non-trivial constraint (S'_+, S'_-) over the same NDR such that (S_+, S_-) is satisfiable iff (S'_+, S'_-) is satisfiable and:*

$$S'_+ \subseteq \{(\leq, y_0), (=, y_1), \ldots, (=, y_n), (\geq, y_{n+1})\}, \quad (20)$$

$$S'_- \subseteq \{(\leq, z_0), (=, z_1), \ldots, (=, z_m), (\geq, z_{m+1})\}. \quad (21)$$

Proof The proof of this lemma is analogous to the proof of Lemma 2. The constraint (S'_+, S'_-) can be obtained using the following transformation rules:

$$(S_+ \cup \{(<, y)\}, S_-) \Rightarrow (S_+ \cup \{(\leq, y-1)\}, S_-) \quad (22)$$

$$(S_+ \cup \{(>, y)\}, S_-) \Rightarrow (S_+ \cup \{(\geq, y+1)\}, S_-) \quad (23)$$

$$(S_+, S_- \cup \{(<, z)\}) \Rightarrow (S_+, S_- \cup \{(\leq, z-1)\}) \quad (24)$$

$$(S_+, S_- \cup \{(>, z)\}) \Rightarrow (S_+, S_- \cup \{(\geq, z+1)\}) \quad (25)$$

$$(S_+ \cup \{(\leq, y_1), (\leq, y_2)\}, S_-) \Rightarrow (S_+ \cup \{(\leq, \min\{y_1, y_2\})\}, S_-) \quad (26)$$

$$(S_+ \cup \{(\geq, y_1), (\geq, y_2)\}, S_-) \Rightarrow (S_+ \cup \{(\leq, \max\{y_1, y_2\})\}, S_-) \quad (27)$$

$$(S_+, S_- \cup \{(\leq, z_1), (\leq, z_2)\}) \Rightarrow (S_+, S_- \cup \{(\leq, \max\{z_1, z_2\})\}) \quad (28)$$

$$(S_+, S_- \cup \{(\geq, z_1), (\geq, z_2)\}) \Rightarrow (S_+, S_- \cup \{(\geq, \min\{z_1, z_2\})\}) \quad (29)$$

Note that rule (13) is now omitted because integer numbers do not have a lower bound. □

Table 7 Maximal safe NDRs for $\mathbb{Z}$: $\mathsf{NDR}_i^{\mathbb{Z}} = (O_+^i, O_-^i)$, $1 \leq i \leq 6$

$\mathsf{NDR}_i^{\mathbb{Z}}$	O_+^i	O_-^i
$\mathsf{NDR}_1^{\mathbb{Z}}$	$\{=\}$	$\{<, \leq, >, \geq, =\}$
$\mathsf{NDR}_2^{\mathbb{Z}}$	$\{<, \leq, >, \geq, =\}$	$\{=\}$
$\mathsf{NDR}_3^{\mathbb{Z}}$	$\{<, \leq, >, \geq, =\}$	$\{<, \leq\}$
$\mathsf{NDR}_4^{\mathbb{Z}}$	$\{<, \leq, >, \geq, =\}$	$\{>, \geq\}$
$\mathsf{NDR}_5^{\mathbb{Z}}$	$\{>, \geq, =\}$	$\{<, \leq, =\}$
$\mathsf{NDR}_6^{\mathbb{Z}}$	$\{<, \leq, =\}$	$\{>, \geq, =\}$

Lemma 8 *Every NDR in Table 7 is safe.*

Proof Let (S_+, S_-) be a non-trivial constraint over an NDR in Table 7. By Lemma 7, w.l.o.g. we can assume that S_+ and S_- are of the form (20) and (21), respectively. We construct a solution V for (S_+, S_-) by performing the following case analysis over the content of S_+ and S_-:

Case 1: $S_+ = \{(=, y_1), \ldots, (=, y_n)\}$, $n \geq 0$. Define $V := \{y_1, \ldots, y_n\}$. Clearly, every restriction in S_+ is satisfied by some value in V. On the other hand, no restriction in S_- is satisfied by V: if y_i, with $1 \leq i \leq n$, satisfies some restriction $r_- \in S_-$, then $(=, y_i) \rightarrow_{\mathbb{N}} r_-$, which contradicts the non-triviality of (S_+, S_-).

Case 2: $S_+ \cap \{(\leq, y_0), (\geq, y_{n+1})\} \neq \emptyset$. We further distinguish cases according to the content of S_-. Note that we do not examine the case where $\{(\leq, z_0), (\geq, z_{m+1})\} \subseteq S_-$, because this is not possible for NDRs in Table 7.

Case 2.1: $S_- = \{(=, z_1), \ldots, (=, z_m)\}$, $m \geq 0$. Define $V := \mathbb{Z} \setminus \{z_1, \ldots, z_m\}$. It is easy to see that V satisfies all restrictions except for those in S_-. Since (S_+, S_-) is non-trivial, and thus $S_+ \cap S_- = \emptyset$, V is a solution for (S_+, S_-).

Case 2.2: $S_- = \{(\leq, z_0)\}$. Define $V := \{v \in \mathbb{Z} \mid v > z_0\}$. It is easy to see that V satisfies all restrictions except for restrictions of the form $(\leq, y)$ and $(=, y)$ with $y \leq z_0$. Since such restrictions imply $(\leq, z_0)$ and (S_+, S_-) is non-trivial, S_+ cannot contain them. Therefore, V is a solution for (S_+, S_-).

Case 2.3: $S_- = \{(\geq, z_m)\}$. This case is symmetrical to Case 2.2.

Case 2.4: $S_- = \{(\leq, z_0), (=, z_1), \ldots, (=, z_m)\}$, $m \geq 1$. Define solution $V := \{v \in \mathbb{Z} \mid v > z_0\} \setminus \{z_1, \ldots, z_m\}$. In this case, S_+ cannot contain restrictions of the form $(\leq, y)$ as it can be seen from Table 7. It is also easy to see that from the remaining restrictions, V satisfies all restrictions except for restrictions of the form $(=, y)$ with $y \leq z_0$ or $y = z_j$ $(1 \leq j \leq m)$. Since such restrictions imply restrictions in S_-, S_+ cannot contain them. Therefore, V is a solution for (S_+, S_-).

Case 2.5: $S_- = \{(=, z_1), \ldots, (=, z_m), (\geq, z_{m+1})\}$, $m \geq 1$. This case is symmetrical to Case 2.4 □

Lemma 9 *Let $(\mathbb{Z}, O_+, O_-)$ be an NDR. Then:*

(a) If $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$, $O_- \cap \{<, \leq\} \neq \emptyset$ and $O_- \cap \{>, \geq\} \neq \emptyset$, then $(\mathbb{Z}, O_+, O_-)$ is non-safe.

(b) If $O_+ \cap \{>, \geq\} \neq \emptyset$, $O_- \cap \{>, \geq\} \neq \emptyset$ and $\{=\} \subseteq O_-$, then $(\mathbb{Z}, O_+, O_-)$ is non-safe.

Table 8 Examples of non-safe NDRs for $\mathbb{Z}$ where $(s_+, y) \rightarrow_{\mathbb{Z}} (s_-^1, z_1) \vee (s_-^2, z_2)$, $(s_+, y) \not\rightarrow_{\mathbb{Z}} (s_-^1, z_1)$ and $(s_+, y) \not\rightarrow_{\mathbb{Z}} (s_-^2, z_2)$

$\{s_+\}$	$\{s_-^1, s_-^2\}$	y	z_1	z_2
$\{<\}$, $\{\leq\}$	$\{<, \geq\}$, $\{\leq, >\}$, $\{\leq, \geq\}$	3	1	1
$\{<\}$, $\{\leq\}$	$\{<, >\}$	3	2	1
$\{>\}$, $\{\geq\}$	$\{<, \geq\}$, $\{\leq, >\}$, $\{\leq, \geq\}$	1	3	3
$\{>\}$, $\{\geq\}$	$\{<, >\}$	1	3	2
$\{>\}$	$\{=, \geq\}$	1	2	3
$\{>\}$	$\{=, >\}$	1	2	2
$\{\geq\}$	$\{=, \geq\}$	1	1	2
$\{\geq\}$	$\{=, >\}$	1	1	1
$\{<\}$	$\{=, \leq\}$	3	2	1
$\{<\}$	$\{=, <\}$	3	2	2
$\{\leq\}$	$\{=, \leq\}$	2	2	1
$\{\leq\}$	$\{=, <\}$	2	2	2

(c) If $O_+ \cap \{<, \leq\} \neq \emptyset$, $O_- \cap \{<, \leq\} \neq \emptyset$ and $\{=\} \subseteq O_-$, then $(\mathbb{Z}, O_+, O_-)$ is non-safe.

Proof In order to prove that $(\mathbb{Z}, O_+, O_-)$ is non-safe, by Lemma 1 it suffices to prove that it is not weakly convex. For each of the cases (a), (b) and (c) of the lemma, we provide counterexamples that violate the weak convexity condition, namely a triple of restrictions (s_+, y), (s_-^1, z_1) and (s_-^2, z_2), such that $s_+ \in O_+$, s_-^1, $s_-^2 \in O_-$, y, z_1, $z_2 \in \mathbb{Z}$, $(s_+, y) \rightarrow_{\mathbb{N}} (s_-^1, z_1) \vee (s_-^2, z_2)$, but $(s_+, y) \not\rightarrow_{\mathbb{Z}} (s_-^1, z_1)$ and $(s_+, y) \not\rightarrow_{\mathbb{N}} (s_-^2, z_2)$. The counterexamples are listed in Table 8: the first four lines correspond to case (a), the next four lines to case (b) and the final four lines to case (c). □

Lemma 10 *Every NDR in Table 7 is maximal safe, that is if any new relation is added to O_+ or O_-, the NDR becomes non-safe.*

Proof We examine all cases of adding a new relation to NDRs in Table 7:
$\underline{\mathsf{NDR}_1^{\mathbb{Z}}}$: By Lemma 9(a), if any of $<$, $\leq$, $>$, $\geq$ is added to O_+, then $\mathsf{NDR}_1^{\mathbb{Z}}$ becomes non-safe.
$\underline{\mathsf{NDR}_2^{\mathbb{Z}}}$: By Lemma 9(b), if $>$ or $\geq$ is added to O_-, then $\mathsf{NDR}_2^{\mathbb{Z}}$ becomes non-safe. By Lemma 9(c), when $<$ or $\leq$ is added to O_-, then $\mathsf{NDR}_2^{\mathbb{Z}}$ becomes non-safe.
$\underline{\mathsf{NDR}_3^{\mathbb{Z}}}$: By Lemma 9(a), if $>$ or $\geq$ is added to O_-, then $\mathsf{NDR}_3^{\mathbb{Z}}$ becomes non-safe. By Lemma 9(c), when $=$ is added to O_-, then $\mathsf{NDR}_3^{\mathbb{Z}}$ becomes non-safe.
$\underline{\mathsf{NDR}_4^{\mathbb{Z}}}$: By Lemma 9(a), if $<$ or $\leq$ is added to O_-, then $\mathsf{NDR}_4^{\mathbb{Z}}$ becomes non-safe. By Lemma 9(b), if $=$ is added to O_-, then $\mathsf{NDR}_4^{\mathbb{Z}}$ becomes non-safe.
$\underline{\mathsf{NDR}_5^{\mathbb{Z}}}$: By Lemma 9(a), if $>$ or $\geq$ is added to O_-, then $\mathsf{NDR}_5^{\mathbb{Z}}$ becomes non-safe. By Lemma 9(c), when $<$ or $\leq$ is added to O_+, then $\mathsf{NDR}_5^{\mathbb{Z}}$ becomes non-safe.
$\underline{\mathsf{NDR}_6^{\mathbb{Z}}}$: By Lemma 9(a), if $<$ or $\leq$ is added to O_-, then $\mathsf{NDR}_6^{\mathbb{Z}}$ becomes non-safe. By Lemma 9(b), if $>$ or $\geq$ is added to O_+, then $\mathsf{NDR}_6^{\mathbb{Z}}$ becomes non-safe.
□

Lemma 11 *If $(\mathbb{Z}, O_+, O_-)$ is a safe NDR, then $O_+ \subseteq O_+^i$ and $O_- \subseteq O_-^i$ for some $\mathsf{NDR}_i^{\mathbb{Z}} = (\mathbb{Z}, O_+^i, O_-^i)$ in Table 7, $(1 \leq i \leq 6)$.*

Proof The proof is by case analysis of possible relations in O_+ and O_-.

Case 1: $O_+ \cap \{<, \leq, >, \geq\} = \emptyset$. In this case, $O_+ \subseteq O_+^1$ and $O_- \subseteq O_-^1$.

Case 2: $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$. By Lemma 9(a) and since the NDR is safe, $O_- \cap \{<, \leq\} = \emptyset$ or $O_- \cap \{>, \geq\} = \emptyset$. Therefore, we distinguish two cases: either $O_- \subseteq \{>, \geq, =\}$ or $O_- \subseteq \{<, \leq, =\}$.

Case 2.1: $O_- \subseteq \{>, \geq, =\}$. We further distinguish on whether $O_- \subseteq \{>, \geq\}$ or $O_- \cap \{=\} \neq \emptyset$.

Case 2.1.1: $O_- \subseteq \{>, \geq\}$. Then $O_- \subseteq O_-^4$ and $O_+ \subseteq O_+^4$.

Case 2.1.2: $O_- \cap \{=\} \neq \emptyset$. If $O_- = \{=\}$, then $O_- \subseteq O_-^2$ and $O_+ \subseteq O_+^2$. Otherwise, $O_- \cap \{>, \geq\} \neq \emptyset$. By Lemma 9(b) and since the NDR is safe, $O_+ \cap \{>, \geq\} = \emptyset$. Thus, $O_+ \subseteq \{<, \leq, =\} = O_+^6$ and $O_- \subseteq \{>, \geq, =\} = O_-^6$.

Case 2.2: $O_- \subseteq \{<, \leq, =\}$. We further distinguish on whether $O_- \subseteq \{<, \leq\}$ or $O_- \cap \{=\} \neq \emptyset$.

Case 2.2.1: $O_- \subseteq \{<, \leq\}$. Then $O_- \subseteq O_-^3$ and $O_+ \subseteq O_+^3$.

Case 2.2.2: $O_- \cap \{=\} \neq \emptyset$. If $O_- = \{=\}$, then $O_- \subseteq O_-^2$ and $O_+ \subseteq O_+^2$. Otherwise, $O_- \cap \{<, \leq\} \neq \emptyset$. By Lemma 9(c) and since the NDR is safe, $O_+ \cap \{<, \leq\} = \emptyset$. Thus, $O_+ \subseteq \{>, \geq, =\} = O_+^5$ and $O_- \subseteq \{<, \leq, =\} = O_-^5$.

□

6 Maximal Safe NDRs for $\mathbb{R}$ and $\mathbb{Q}$

We continue with the domain of real numbers ($\mathbb{R}$) and rational numbers ($\mathbb{Q}$). Table 9 lists all maximal safe NDRs for these domains. Reals and rationals are examples of dense domains: between every two different numbers there is always a third one. This property results in new safe NDRs. Specifically, either $\leq$ or $\geq$ can be added to O_- of $\mathsf{NDR}_2^{\mathbb{Z}}$ from Table 7 because they do not violate the weak convexity property: e.g., $(\leq, 5) \rightarrow_{\mathbb{Z}} (=, 5) \vee (\leq, 4)$, but $(\leq, 5) \nrightarrow_{\mathbb{R}} (=, 5) \vee (\leq, 4)$. For the same reason, O_+ of $\mathsf{NDR}_5^{\mathbb{Z}}$ and $\mathsf{NDR}_6^{\mathbb{Z}}$ from Table 7 can be extended with $<$ and $>$: e.g., $(<, 5) \rightarrow_{\mathbb{Z}} (=, 4) \vee (\leq, 3)$, but $(<, 5) \nrightarrow_{\mathbb{R}} (=, 4) \vee (\leq, 3)$.

Below we provide only proofs for $\mathbb{R}$. The proofs for $\mathbb{Q}$ are identical.

Lemma 12 *Let (S_+, S_-) be a non-trivial constraint over an NDR from Table 9. Then there exists a non-trivial constraint (S'_+, S'_-) over the same NDR such that (S_+, S_-) is satisfiable iff (S'_+, S'_-) is satisfiable and:*

$$S'_+ \subseteq \{(<, y_0^s), (\leq, y_0), (=, y_1), \ldots, (=, y_n), (\geq, y_{n+1}), (>, y_{n+1}^s)\}, \quad (30)$$
$$S'_- \subseteq \{(<, z_0^s), (\leq, z_0), (=, z_1), \ldots, (=, z_m), (\geq, z_{m+1}), (>, z_{m+1}^s)\}. \quad (31)$$

Proof The proof is similar to the proof of Lemma 2 for the case of natural numbers. Since, we are no longer able to eliminate strict inequalities (real

Table 9 Maximal safe NDRs: $\mathsf{NDR}_i^{\mathcal{D}} = (O_+^i, O_-^i)$, $1 \le i \le 7$, for $\mathcal{D} = \mathbb{R}$ and $\mathcal{D} = \mathbb{Q}$

$\mathsf{NDR}_i^{\mathbb{Q}}$	$\mathsf{NDR}_i^{\mathbb{R}}$	O_+^i	O_-^i
$\mathsf{NDR}_1^{\mathbb{Q}}$	$\mathsf{NDR}_1^{\mathbb{R}}$	$\{=\}$	$\{<, \le, >, \ge, =\}$
$\mathsf{NDR}_2^{\mathbb{Q}}$	$\mathsf{NDR}_2^{\mathbb{R}}$	$\{<, \le, >, \ge, =\}$	$\{\le, =\}$
$\mathsf{NDR}_3^{\mathbb{Q}}$	$\mathsf{NDR}_3^{\mathbb{R}}$	$\{<, \le, >, \ge, =\}$	$\{\ge, =\}$
$\mathsf{NDR}_4^{\mathbb{Q}}$	$\mathsf{NDR}_4^{\mathbb{R}}$	$\{<, \le, >, \ge, =\}$	$\{<, \le\}$
$\mathsf{NDR}_5^{\mathbb{Q}}$	$\mathsf{NDR}_5^{\mathbb{R}}$	$\{<, \le, >, \ge, =\}$	$\{>, \ge\}$
$\mathsf{NDR}_6^{\mathbb{Q}}$	$\mathsf{NDR}_6^{\mathbb{R}}$	$\{<, >, \ge, =\}$	$\{<, \le, =\}$
$\mathsf{NDR}_7^{\mathbb{Q}}$	$\mathsf{NDR}_7^{\mathbb{R}}$	$\{<, \le, >, =\}$	$\{>, \ge, =\}$

numbers is a dense domain) we only apply the rules that eliminate duplicate occurrences of inequalities:

$$(S_+ \cup \{(<, y_1), (<, y_2)\}, S_-) \Rightarrow (S_+ \cup \{(<, \min\{y_1, y_2\})\}, S_-) \tag{32}$$
$$(S_+ \cup \{(>, y_1), (>, y_2)\}, S_-) \Rightarrow (S_+ \cup \{(>, \max\{y_1, y_2\})\}, S_-) \tag{33}$$
$$(S_+ \cup \{(\le, y_1), (\le, y_2)\}, S_-) \Rightarrow (S_+ \cup \{(\le, \min\{y_1, y_2\})\}, S_-) \tag{34}$$
$$(S_+ \cup \{(\ge, y_1), (\ge, y_2)\}, S_-) \Rightarrow (S_+ \cup \{(\le, \max\{y_1, y_2\})\}, S_-) \tag{35}$$
$$(S_+, S_- \cup \{(<, z_1), (<, z_2)\}) \Rightarrow (S_+, S_- \cup \{(<, \max\{z_1, z_2\})\}) \tag{36}$$
$$(S_+, S_- \cup \{(>, z_1), (>, z_2)\}) \Rightarrow (S_+, S_- \cup \{(>, \min\{z_1, z_2\})\}) \tag{37}$$
$$(S_+, S_- \cup \{(\le, z_1), (\le, z_2)\}) \Rightarrow (S_+, S_- \cup \{(\le, \max\{z_1, z_2\})\}) \tag{38}$$
$$(S_+, S_- \cup \{(\ge, z_1), (\ge, z_2)\}) \Rightarrow (S_+, S_- \cup \{(\ge, \min\{z_1, z_2\})\}) \tag{39}$$

□

Lemma 13 *Every NDR in Table 9 is safe.*

Proof Let (S_+, S_-) be a non-trivial constraint over an NDR in Table 9. By Lemma 12, w.l.o.g. we can assume that S_+ and S_- are of the form (30) and (31), respectively. We construct a solution V for (S_+, S_-) by performing the following case analysis over the content of S_+ and S_-:

Case 1: $S_+ = \{(=, y_1), \ldots, (=, y_n)\}$. Then the solution $V := \{y_1, \ldots, y_n\}$.

Case 2: $S_+ \cap \{(<, y_0^s), (\le, y_0), (\ge, y_{n+1}), (>, y_{n+1}^s)\} \neq \emptyset$. We further distinguish cases according the content of S_-:

Case 2.1: $S_- = \{(=, z_1), \ldots, (=, z_m)\}$, $m \ge 0$. Then $V := \mathbb{R} \setminus \{z_1, \ldots, z_m\}$.

Case 2.2: $S_- = \{(\le, z_0), (=, z_1), \ldots, (=, z_m)\}$, $m \ge 0$. Then the solution $V := \{v \in \mathbb{R} \mid v > z_0\} \setminus \{z_1, \ldots, z_m\}$.

Case 2.3: $S_- = \{(=, z_1), \ldots, (=, z_m), (\ge, z_{m+1})\}$, $m \ge 0$. Then the solution $V := \{v \in \mathbb{R} \mid v < z_{m+1}\} \setminus \{z_1, \ldots, z_m\}$.

Case 2.4: $S_- = \{(<, z_0^s), (=, z_1), \ldots, (=, z_m)\}$, $m \ge 0$. Then the solution $V := \{v \in \mathbb{R} \mid v \ge z_0^s\} \setminus \{z_1, \ldots, z_m\}$.

Case 2.5: $S_- = \{(=, z_1), \ldots, (=, z_m), (>, z_{m+1}^s)\}$, $m \ge 0$. Then the solution $V := \{v \in \mathbb{R} \mid v \le z_{m+1}^s\} \setminus \{z_1, \ldots, z_m\}$.

Case 2.6: $S_- = \{(<, z_0^s), (\le, z_0), (=, z_1), \ldots, (=, z_m)\}$, $m \ge 0$. Then the solution $V := \{v \in \mathbb{R} \mid v \ge z_0^s \wedge v > z_0\} \setminus \{z_1, \ldots, z_m\}$.

Table 10 Examples of non-safe NDRs for $\mathbb{R}$ where $(s_+, y) \rightarrow_{\mathbb{R}} (s^1_-, z_1) \vee (s^2_-, z_2)$, $(s_+, y) \not\rightarrow_{\mathbb{R}} (s^1_-, z_1)$ and $(s_+, y) \not\rightarrow_{\mathbb{R}} (s^2_-, z_2)$

$\{s_+\}$	$\{s^1_-, s^2_-\}$	y	z_1	z_2
$\{<\}$, $\{\leq\}$	$\{<, \geq\}$, $\{\leq, >\}$, $\{\leq, \geq\}$	3	1	1
$\{<\}$, $\{\leq\}$	$\{<, >\}$	3	2	1
$\{>\}$, $\{\geq\}$	$\{<, \geq\}$, $\{\leq, >\}$, $\{\leq, \geq\}$	1	3	3
$\{>\}$, $\{\geq\}$	$\{<, >\}$	1	3	2
$\{\geq\}$	$\{=, >\}$	1	1	1
$\{\leq\}$	$\{=, <\}$	1	1	1

Case 2.7: $S_- = \{(=, z_1), \ldots, (=, z_m), (\geq, z_{m+1}), (>, z^s_{m+1})\}$, $m \geq 0$. Then the solution $V := \{v \in \mathbb{R} \mid v < z_{m+1} \wedge v \leq z^s_{m+1}\} \setminus \{z_1, \ldots, z_m\}$. □

Lemma 14 *Let $(\mathbb{R}, O_+, O_-)$ be an NDR. Then:*

(a) If $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$, $O_- \cap \{<, \leq\} \neq \emptyset$ and $O_- \cap \{>, \geq\} \neq \emptyset$, then $(\mathbb{R}, O_+, O_-)$ is non-safe.
(b) If $\{\geq\} \in O_+$ and $O_- \cap \{>, =\} \neq \emptyset$, then $(\mathbb{R}, O_+, O_-)$ is non-safe.
(c) If $\{\leq\} \in O_+$ and $O_- \cap \{<, =\} \neq \emptyset$, then $(\mathbb{R}, O_+, O_-)$ is non-safe.

Proof In order to prove that $(\mathbb{R}, O_+, O_-)$ is non-safe, by Lemma 1 it suffices to prove that it is not weakly convex. For each of the cases (a), (b) and (c) of the lemma, we provide counterexamples that violate the weak convexity condition, namely a triple of restrictions (s_+, y), (s^1_-, z_1) and (s^2_-, z_2), such that $s_+ \in O_+$, s^1_-, $s^2_- \in O_-$, y, z_1, $z_2 \in \mathbb{R}$, $(s_+, y) \rightarrow_{\mathbb{R}} (s^1_-, z_1) \vee (s^2_-, z_2)$, $(s_+, y) \not\rightarrow_{\mathbb{R}} (s^1_-, z_1)$ and $(s_+, y) \not\rightarrow_{\mathbb{R}} (s^2_-, z_2)$. The counterexamples are listed in Table 10: the first four lines correspond to case (a), the penultimate line to case (b) and the last line to case (c). □

Lemma 15 *Every NDR in Table 9 is maximal safe, that is if any new relation is added to O_+ or O_-, the NDR becomes non-safe.*

Proof We examine all cases of adding a new relation to NDRs in Table 9:
$\mathsf{NDR}^{\mathbb{R}}_1$: By Lemma 14 (a), if any of $<$, $\leq$, $>$, $\geq$ is added to O_+, then $\mathsf{NDR}^{\mathbb{R}}_1$ becomes non-safe.
$\mathsf{NDR}^{\mathbb{R}}_2$: By Lemma 14 (a), if $\geq$ is added to O_-, then $\mathsf{NDR}^{\mathbb{R}}_2$ becomes non-safe. By Lemma 14 (b), when $>$ is added to O_-, $\mathsf{NDR}^{\mathbb{R}}_2$ becomes non-safe. Finally, by Lemma 14 (c), if $<$ is added to O_-, then $\mathsf{NDR}^{\mathbb{R}}_2$ becomes non-safe.
$\mathsf{NDR}^{\mathbb{R}}_3$: By Lemma 14 (a), if $\leq$ is added to O_-, then $\mathsf{NDR}^{\mathbb{R}}_3$ becomes non-safe. By Lemma 14 (b), when $>$ is added to O_-, $\mathsf{NDR}^{\mathbb{R}}_3$ becomes non-safe. Finally, by Lemma 14 (c), if $<$ is added to O_-, then $\mathsf{NDR}^{\mathbb{R}}_3$ becomes non-safe.
$\mathsf{NDR}^{\mathbb{R}}_4$: By Lemma 14 (a), if $>$ or $\geq$ is added to O_-, then $\mathsf{NDR}^{\mathbb{R}}_4$ becomes non-safe. By Lemma 14 (c), if $=$ is added to O_-, then $\mathsf{NDR}^{\mathbb{R}}_4$ becomes non-safe.
$\mathsf{NDR}^{\mathbb{R}}_5$: By Lemma 14 (a), if $<$ or $\leq$ is added to O_-, then $\mathsf{NDR}^{\mathbb{R}}_5$ becomes non-safe. By Lemma 14 (b), when $=$ is added to O_-, $\mathsf{NDR}^{\mathbb{R}}_5$ becomes non-safe.
$\mathsf{NDR}^{\mathbb{R}}_6$: By Lemma 14 (a), if $>$ or $\geq$ is added to O_-, then $\mathsf{NDR}^{\mathbb{R}}_6$ becomes non-safe. By Lemma 14 (c), if $\leq$ is added to O_+, then $\mathsf{NDR}^{\mathbb{R}}_6$ becomes non-safe.

$\underline{\mathsf{NDR}_7^{\mathbb{R}}}$: By Lemma 14 (a), if $<$ or $\leq$ is added to O_-, then $\mathsf{NDR}_7^{\mathbb{R}}$ becomes non-safe. Similarly, by Lemma 14 (b), if $\geq$ is added to O_+, then $\mathsf{NDR}_7^{\mathbb{R}}$ becomes non-safe. □

Lemma 16 *If $(\mathbb{R}, O_+, O_-)$ is a safe NDR, then $O_+ \subseteq O_+^i$ and $O_- \subseteq O_-^i$ for some $\mathsf{NDR}_i^{\mathbb{R}} = (\mathbb{R}, O_+^i, O_-^i)$ in Table 9, $(1 \leq i \leq 7)$.*

Proof The proof is by case analysis of possible relations in O_+ and O_-.

Case 1: $O_+ \cap \{<, \leq, >, \geq\} = \emptyset$. In this case, $O_+ \subseteq O_+^1$ and $O_- \subseteq O_-^1$.

Case 2: $O_+ \cap \{<, \leq, >, \geq\} \neq \emptyset$. By Lemma 14 (a) and since the NDR is safe, $O_- \cap \{<, \leq\} = \emptyset$ or $O_- \cap \{>, \geq\} = \emptyset$. Therefore, we examine two cases: either $O_- \subseteq \{>, \geq, =\}$ or $O_- \subseteq \{<, \leq, =\}$.

Case 2.1: $O_- \subseteq \{>, \geq, =\}$. We further examine whether $O_- \subseteq \{>, \geq\}$ or $\{=\} \subseteq O_-$.

Case 2.1.1: $O_- \subseteq \{>, \geq\}$. Then $O_- \subseteq O_-^5$ and $O_+ \subseteq O_+^5$.

Case 2.1.2: $\{=\} \subseteq O_-$. We distinguish two cases: either $O_- \subseteq \{\geq, =\}$ or $O_- \cap \{>, =\} \neq \emptyset$.

Case 2.1.2.1: $O_- \subseteq \{\geq, =\}$. In this case $O_+ \subseteq O_+^3$ and $O_- \subseteq O_-^3$.

Case 2.1.2.2: $O_- \cap \{>, =\} \neq \emptyset$. By Lemma 14 (b) and since the NDR is safe $O_+ \cap \{\geq\} = \emptyset$. So, $O_+ \subseteq \{<, \leq, >, =\} = O_+^7$ and $O_- \subseteq \{>, \geq, =\} = O_-^7$.

Case 2.2: $O_- \subseteq \{<, \leq, =\}$. We further examine whether $O_- \subseteq \{<, \leq\}$ or $\{=\} \subseteq O_-$.

Case 2.2.1: $O_- \subseteq \{<, \leq\}$. Then $O_- \subseteq O_-^4$ and $O_+ \subseteq O_+^4$.

Case 2.2.2: $\{=\} \subseteq O_-$. We distinguish two cases: either $O_- \subseteq \{\leq, =\}$ or $O_- \cap \{<, =\} \neq \emptyset$.

Case 2.2.2.1: $O_- \subseteq \{\leq, =\}$. In this case $O_+ \subseteq O_+^2$ and $O_- \subseteq O_-^2$.

Case 2.2.2.2: $O_- \cap \{<, =\} \neq \emptyset$. By Lemma 14 (c) and since the NDR is safe, $O_+ \cap \{\leq\} = \emptyset$. So, $O_+ \subseteq \{<, >, \geq, =\} = O_+^6$ and $O_- \subseteq \{<, \leq, =\} = O_-^6$. □

7 Related Work

Datatypes have been extensively studied in the context of DLs [3,8,9]. Extensions of expressive DLs with datatypes have been examined in depth [8] with the main focus on decidability. Baader, Brandt and Lutz [3] formulated tractable extensions of $\mathcal{EL}$ with datatypes using a p-admissibility restriction for datatypes. A datatype $\mathcal{D}$ is p-admissible if (i) satisfiability and implication of conjunctions of datatype restrictions can be decided in polynomial time, and (ii) $\mathcal{D}$ is convex: if a conjunction of datatype restrictions implies a disjunction of datatype restrictions, then it also implies one of its disjuncts [3]. In our work instead of condition (i) we require that implication and satisfiability of just datatype restrictions (not conjunctions since we do not consider functional features) is decidable in polynomial time. Condition (ii) is replaced with the requirement of safety for NDRs, where, in addition, we take into account the polarity for occurrences of datatype restrictions. The refined restrictions give more possibilities for the use of datatypes in tractable languages, as demonstrated by the example given in the introduction. Furthermore, Baader, Brandt and Lutz did not provide a classification of p-admissible datatypes; in our case we provide such a classification for natural numbers, integers, rationals and reals. The EL Profile of OWL 2 [2] is inspired by $\mathcal{EL}^{++}$ and restricts all OWL 2 datatypes to satisfy p-admissibility. In particular, only equality can be used in datatype restrictions. Our result can allow for a significant extension of datatypes in the OWL 2 EL Profile, where, in addition, inequalities can be used negatively. We believe that this result can be extended to many more datatype restrictions outside of OWL 2, such as intervals, or user-definable restrictions, such as predicates expressing that an integer is odd or prime.

Our work is not the only one where the convexity property for extensions of $\mathcal{EL}$ is relaxed without losing tractability. It has been shown [9] that the convexity requirement is not necessary provided that (i) the ontology contains only concept definitions of the form $A \equiv C$, where A is a concept name, and (ii) every concept name occurs at most once in the left-hand side of the definition. While this requirement seems natural since concepts in ontologies are typically defined only once, it disallows the usage of general concept inclusion axioms (GCIs), such as the axiom (2) given in the introduction, which do not cause any problem in our case.

Another related polynomial extension of $\mathcal{EL}$ has been considered by Viorica Sofronie-Stokkermans [10], where in addition to the standard concept constructors in $\mathcal{EL}$, one can use concept of the form $\downarrow m$, $\uparrow n$, and $[m; n]$, interpreted as (semi-)intervals over a partially ordered set $(\mathcal{P}, <)$. The difference with the results discussed before, is that the domain $\mathcal{P}$ and therefore the values of the end-point parameters m and n are not fixed but can be chosen arbitrarily for every interpretation.[2] This property guarantees that the extension is polynomial: in contrast to intervals over concrete datatypes such as integers or reals, intervals over abstract partially ordered sets have the convexity property. For example, the property $[n; n] \subseteq \downarrow m \cup \uparrow m$ holds for real numbers since $\{x \in \mathbb{R} \mid x \leq m\} \cup \{x \in \mathbb{R} \mid x \geq m\} = \mathbb{R}$ for every $m \in \mathbb{R}$, but does not hold for partially ordered sets in general, where elements can be incomparable.

8 Conclusions and Future Work

In this work we made a fine-grained analysis of extensions of $\mathcal{EL}$ with numerical datatypes, by distinguishing not only the types of relations but also the polarities of their occurrences in axioms. We made a full classification of cases where these restrictions result in a tractable extension for natural numbers, integers, rationals and reals. One practically relevant case for these datatypes is when positive occurrences of datatype expressions can only use equality and negative occurrences can use any of the numerical relations considered. This case was motivated by an example of a pharmacy-related ontology and can

[2] Personal communication with Viorica Sofronie-Stokkermans

be proposed as a candidate for a successor of the OWL 2 EL Profile [6]. For the cases where the extension is tractable, we provided a polynomial sound and complete consequence-based reasoning procedure, which can be seen as an extension of the completion-based procedure for $\mathcal{EL}$ [3]. We think that the procedure can be extended to accommodate other constructors in $\mathcal{EL}^{++}$ such as (complex) role inclusions, nominals, domain and range restrictions and assertions since these constructors do not interact with datatypes [11].

In future work we also plan to consider other OWL datatypes, such as strings, binary data or date and time, functional features, and to try to merge our procedure with the consequence-based procedure for Horn $\mathcal{SHIQ}$ [12]. For example, to extend the procedure with functional features, we probably need a notion of "functional safety" for an NDR that corresponds to the strong convexity property (see Definition 7). It might be possible to achieve even higher expressivity by combining different NDRs for features and datatypes that do not interact in the ontology. Currently, using two safe NDRs in a single ontology may result in intractability. For example, allowing the usage of both $\mathsf{NDR}_1^{\mathbb{Z}}$ and $\mathsf{NDR}_2^{\mathbb{Z}}$ in Table 7 is equivalent to not having any restrictions at all. One possible solution to this problem is to specify explicitly which features can be used with which NDRs in order to separate their usage in ontologies. As we mentioned in the previous section, it will be also interesting to look into more expressive datatype restrictions, such as intervals or user-definable predicates, and restrictions containing unknown parameters or variables.

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