TRACTABLE EXTENSIONS OF THE DESCRIPTION LOGIC $\mathcal{EL}$ WITH NUMERICAL DATATYPES

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Outline

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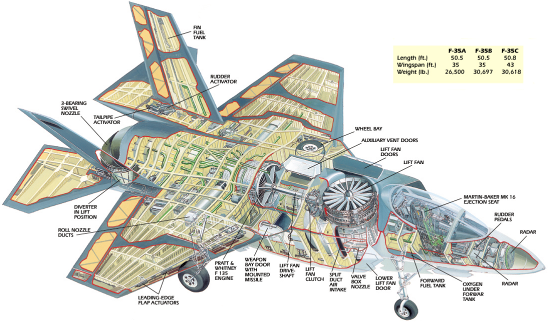

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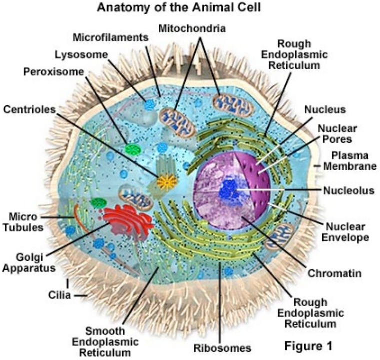

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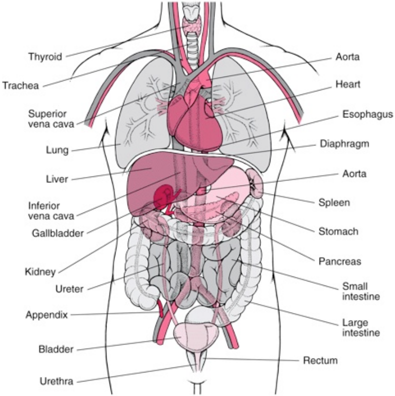

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$\text{Panadol} \sqsubseteq \text{Drug}$

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- Foundations of W3C ontology languages (OWL and OWL 2)

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$$\mathsf{X} \equiv \mathsf{Patient} \sqcap \exists \mathsf{hasAge}.[= 3] \sqcap \exists \mathsf{hasPrescription}.\mathsf{Panadol}$$

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- DL reasoning tasks:
  - Check satisfiability of a concept ($\mathcal{O} \models \mathsf{X} \sqsubseteq \bot$)
  - Check satisfiability of an ontology ($\mathcal{O} \models \bot$)
  - Check subsumption ($\mathcal{O} \models \mathsf{A} \sqsubseteq \mathsf{B}$)
  - Classification (compute all $\mathsf{A} \sqsubseteq \mathsf{B}$ such that $\mathcal{O} \models \mathsf{A} \sqsubseteq \mathsf{B}$) $\rightsquigarrow$ more general than all tasks above

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$$\begin{array}{rcl} A & \sqsubseteq & \exists F.[< 2] \\ \exists F.[= 1] & \sqsubseteq & B \\ \exists F.[= 0] & \sqsubseteq & C \\ \hline & A \sqsubseteq B \sqcup C & \end{array}$$

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- EL Profile of OWL 2 admits only equality
- Absence of inequalities reduces the utility of OWL 2 EL

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  - Full classification of cases where datatypes are used and tractability is preserved for $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$
  - Polynomial, sound and complete reasoning procedure for extensions of $\mathcal{EL}^{\perp}$ with restricted numerical datatypes

$\mathcal{EL}$ Family of Description Logics

- The $\mathcal{EL}$ language:

	Syntax	Semantics
Atomic concept	C	$C(x)$
Top	$\top$	$\top$
Conjunction	C $\sqcap$ D	$C(x) \wedge D(x)$
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Datatype restriction	$\exists$F.*range*	$\exists v : F(x, v) \wedge v \in range$

$$range = [< n] \mid [\leq n] \mid [> n] \mid [\geq n] \mid [= n] \subseteq \mathcal{D} = \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$$

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ALGORITHM STAGES

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Normal forms	
NF1	$A \sqsubseteq B$
NF2	$A_1 \sqcap A_2 \sqsubseteq B$
NF3	$A \sqsubseteq \exists R.B$
NF4	$\exists R.B \sqsubseteq A$
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2. Saturation of the axioms under inference rules, such as:

$$\frac{A \sqsubseteq B \quad A \sqsubseteq C}{A \sqsubseteq D} \quad B \sqcap C \sqsubseteq D \in \mathcal{O}$$

Reasoning Rules (part I)

Rules from $\mathcal{EL}^{\perp}$

IR1 $\dfrac{}{A \sqsubseteq A}$ **IR2** $\dfrac{}{A \sqsubseteq \top}$ **CR1** $\dfrac{A \sqsubseteq B}{A \sqsubseteq C}$ $B \sqsubseteq C \in \mathcal{O}$

CR2 $\dfrac{A \sqsubseteq B \quad A \sqsubseteq C}{A \sqsubseteq D}$ $B \sqcap C \sqsubseteq D \in \mathcal{O}$

CR3 $\dfrac{A \sqsubseteq B}{A \sqsubseteq \exists R.C}$ $B \sqsubseteq \exists R.C \in \mathcal{O}$

CR4 $\dfrac{A \sqsubseteq \exists R.B \quad B \sqsubseteq C}{A \sqsubseteq D}$ $\exists R.C \sqsubseteq D \in \mathcal{O}$

CR5 $\dfrac{A \sqsubseteq \exists R.B \quad B \sqsubseteq \bot}{A \sqsubseteq \bot}$

Reasoning Rules (part II)

New rules for datatypes

$$\frac{\mathsf{A} \sqsubseteq \exists \mathsf{F}.[< m]}{\mathsf{A} \sqsubseteq \mathsf{B}} \quad \exists \mathsf{F}.[< n] \sqsubseteq \mathsf{B} \in \mathcal{O},\; m \leq n$$

Reasoning Rules (part II)

New rules for datatypes

ID1 $\dfrac{}{A \sqsubseteq \perp}$ $A \sqsubseteq \exists F.[< 0] \in \mathcal{O}$

CD1 $\dfrac{A \sqsubseteq B}{A \sqsubseteq \exists F.range}$ $B \sqsubseteq \exists F.range \in \mathcal{O}$

CD2$_{(<,<)}$ $\dfrac{A \sqsubseteq \exists F.[< m]}{A \sqsubseteq B}$ $\exists F.[< n] \sqsubseteq B \in \mathcal{O},\ m \leq n$

CD2$_{(=,<)}$ $\dfrac{A \sqsubseteq \exists F.[= m]}{A \sqsubseteq B}$ $\exists F.[< n] \sqsubseteq B \in \mathcal{O},\ m < n$

CD2$_{(=,=)}$ $\dfrac{A \sqsubseteq \exists F.[= m]}{A \sqsubseteq B}$ $\exists F.[= n] \sqsubseteq B \in \mathcal{O},\ m = n$ ...

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  - complete under certain restrictions on datatypes
- What type of restrictions?

Restrictions for $\mathbb{N}$

Negative relations	Positive relations
$<, \leq, >, \geq, =$	$=$

Example

$\text{Panadol} \sqsubseteq \exists\text{contains}.(\text{Paracetamol} \sqcap \exists\text{mgPerTablet}.[= 500])$

$\text{Patient} \sqcap \exists\text{hasAge}.[< 6] \sqcap \exists\text{hasPrescription}.$
$\exists\text{contains}.(\text{Paracetamol} \sqcap \exists\text{mgPerTablet}.[> 250]) \sqsubseteq \perp$

$X \sqsubseteq \text{Patient} \sqcap \exists\text{hasAge}.[= 3] \sqcap \exists\text{hasPrescription}.\text{Panadol}$

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Example

$$\begin{array}{l} A \sqsubseteq \exists F.[< 2] \\ \exists F.[= 1] \sqsubseteq B \\ \exists F.[= 0] \sqsubseteq C \\ \hline A \sqsubseteq B \sqcup C \end{array}$$

Restrictions for $\mathbb{Z}$

Negative relations	Positive relations
$<, \leq, >, \geq, =$	$=$
$=$	$<, \leq, >, \geq, =$
$<, \leq$	$<, \leq, >, \geq, =$
$>, \geq$	$<, \leq, >, \geq, =$
$<, \leq, =$	$>, \geq, =$
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Restrictions for $\mathbb{Z}$

Negative relations	Positive relations
$<, \leq, >, \geq, =$	$=$
$=$	$<, \leq, >, \geq, =$
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$<, \leq$	$<, \leq, >, \geq, =$
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Outline

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- Candidate for an OWL 2 EL Profile extension

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