Consequence-Based Reasoning for Description Logic Ontologies

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OVERVIEW

- Introduction to Description Logic
  - Reasoning problems
  - Hierarchy of DLs
  - Related formalisms
- Tableau-based reasoning procedures
  - Key reasoning phases
  - Practical limitations
- Consequence-based reasoning procedures
  - Reasoning in the DL $\mathcal{EL}$
  - Extension to Horn $\mathcal{SHIQ}$
  - Advantages
- Related methods
  - Hyper-resolution
  - Ordered resolution
  - Automata-based methods
- Conclusions

Outline

Syntax and Semantics of DLs

- The syntax

$$\text{Heart} \sqsubseteq \text{Organ} \sqcap \exists \text{isComponentOf}.\text{CirculatorySystem}$$

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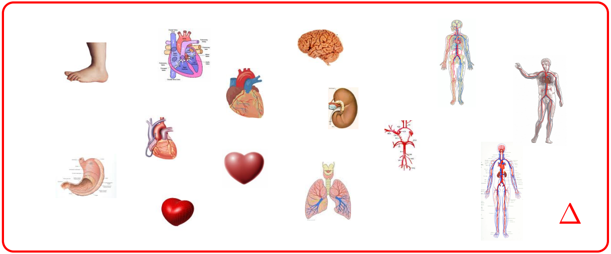

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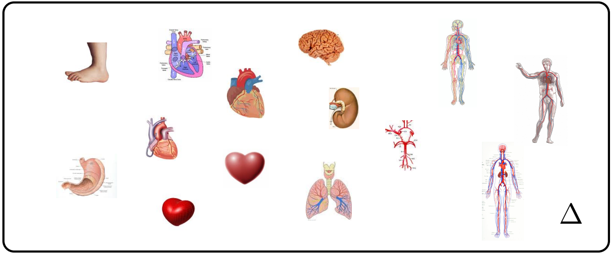

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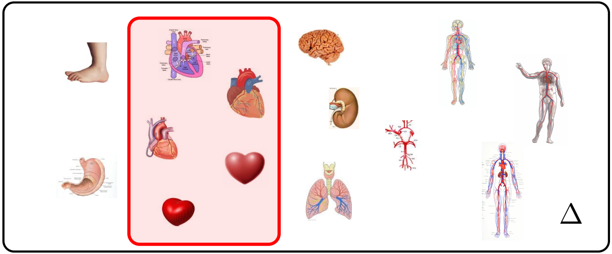

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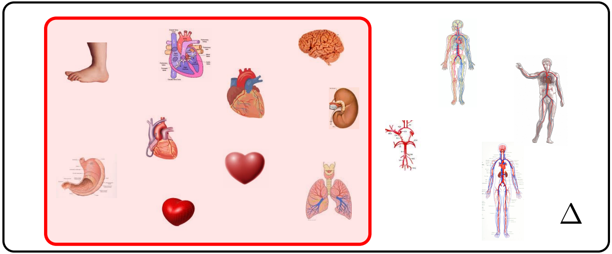

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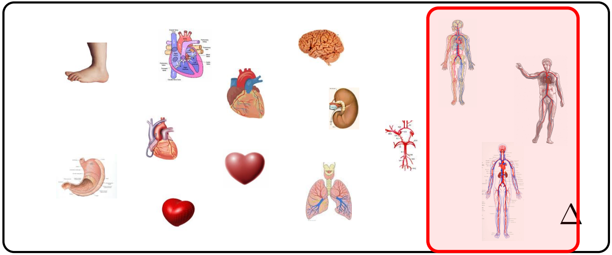

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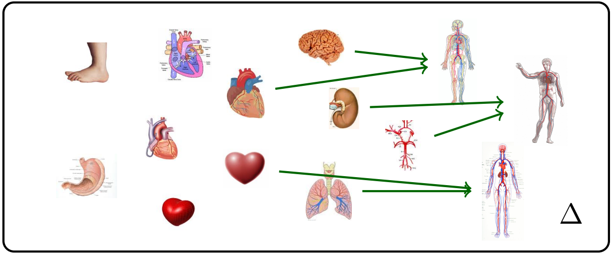

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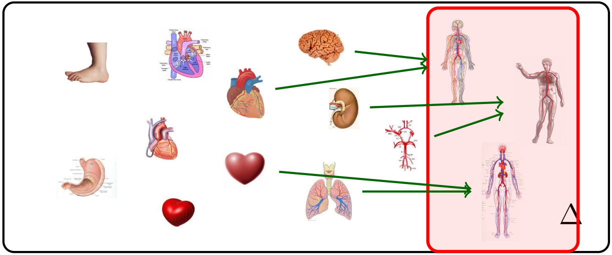

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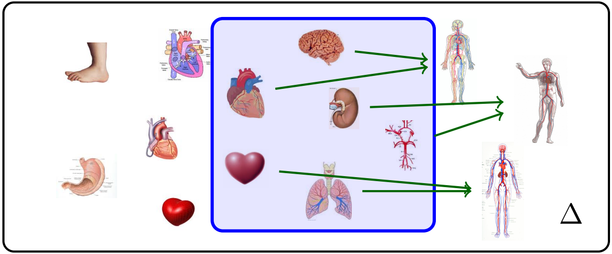

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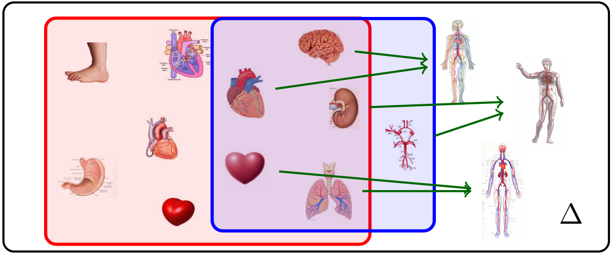

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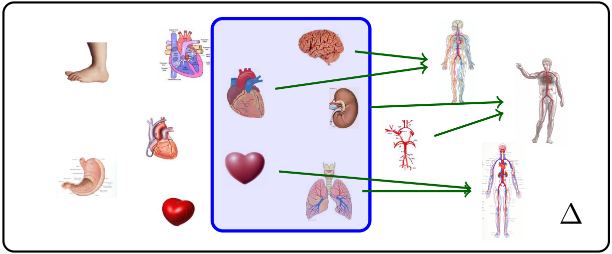

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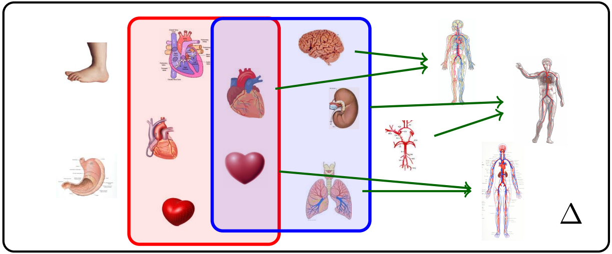

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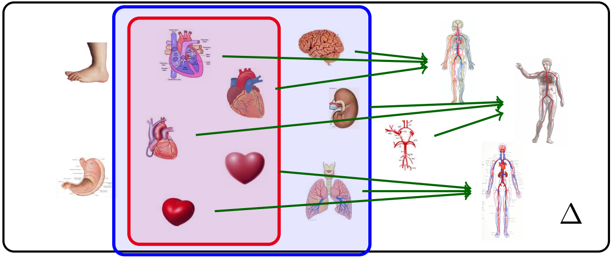

Hierarchy of DLs

Name	DL syntax	First-Order syntax	
intersection	$C_1 \sqcap C_2$	$C_1(x) \wedge C_2(x)$	
union	$C_1 \sqcup C_2$	$C_1(x) \vee C_2(x)$	$= \mathcal{A}$
complement	$\neg C$	$\neg C(x)$	$\mathcal{L}$
value restriction	$\forall r.C$	$\forall y.[r(x,y) \rightarrow C(y)]$	$\mathcal{C}$
existential restr.	$\exists r.C$	$\exists y.[r(x,y) \wedge C(y)]$	
concept inclusion	$C_1 \sqsubseteq C_2$	$\forall x.[C_1(x) \rightarrow C_2(x)]$	

- Basic DL $\mathcal{ALC}$ [Schmidt-Schauß, Smolka; 1991]:
  - is a syntactic variant of $\mathcal{K}_n$:
    - $\forall r.C \Rightarrow \Box_r C$
    - $\exists r.C \Rightarrow \Diamond_r C$
  - is a subset of $\mathcal{GF}^2$
  - has tree-model property
  - has finite model property
  - satisfiability problem is ExpTime-complete

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concept inclusion	$C_1 \sqsubseteq C_2$	$\forall x.[C_1(x) \rightarrow C_2(x)]$	
transitivity	$Tra(r)$	$\forall xyz.[r(x,y) \wedge r(y,z) \rightarrow r(x,z)]$	$= \mathcal{S}$
functionality	$Fun(r)$	$\forall xyz.[r(x,y) \wedge r(x,z) \rightarrow y \simeq z]$	$+\mathcal{F}$
role inclusion	$r_1 \sqsubseteq r_2$	$\forall xy.[r_1(x,y) \rightarrow r_2(x,y)]$	$+\mathcal{H}$
inverse roles	$[\ldots r^- \ldots]$	$[\ldots r(y,x) \ldots]$	$+\mathcal{I}$

- $\mathcal{SHIF}$:
  - has a generalized tree-model property (transitivity)
  - has no finite-model property (because of functionality)
  - satisfiability problem is ExpTime-complete

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number restriction	$\leqslant n\, r.C$	$\exists^{\leq n} y.[r(x,y) \wedge C(y)]$	$+\mathcal{Q}$
nominals	o	$x \simeq o$	$+\mathcal{O}$

- $\mathcal{SHOIQ}$:
  - no tree-model property (even generalized)
  - satisfiability is NExpTime-complete (can be translated to $\mathcal{C}^2$)

Bio-Medical Ontologies

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- Simple inclusions:

$$\mathsf{Heart} \sqsubseteq \mathsf{Organ} \sqcap \exists \mathsf{isPartOf}.\mathsf{Chest}$$
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- Concept definitions:

$$\begin{aligned} \text{MuscularOrgan} &\equiv \text{Organ} \sqcap \exists \text{hasPart}.\text{Muscle} \\ \text{HeartDisease} &\equiv \text{Disorder} \sqcap \exists \text{affects}.\exists \text{isPartOf}.\text{Heart} \\ \text{KidneyExamination} &\equiv \text{ClinicalAct} \sqcap \\ &\quad \exists \text{hasSubprocess}.(\text{Examination} \sqcap \exists \text{involves}.\text{Kidney}) \end{aligned}$$

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- General concept inclusions:

$$\begin{aligned} \text{Structure} \sqcap \exists \text{isPartOf}.\text{Heart} &\sqsubseteq \\ &\exists \text{isComponentOf}.\text{CardiovascularSystem} \end{aligned}$$

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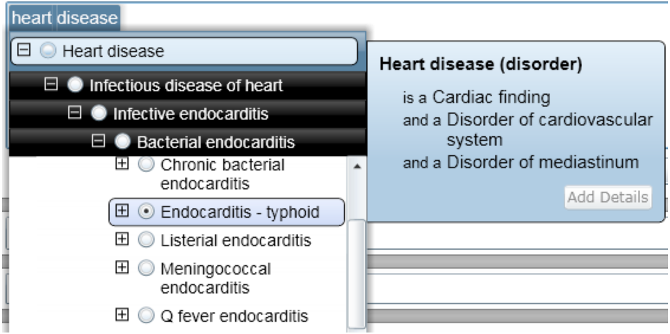

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4. Related Methods
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  3. To check $\mathcal{O} \models A \sqsubseteq B$, build a model for $A \sqcap \neg B$.

Example

$$\text{Myocarditis} \sqsubseteq \text{Disorder} \sqcap \exists \text{affects}.\text{Myocardium}$$
$$\text{Myocardium} \sqsubseteq \text{Muscle} \sqcap \exists \text{isPartOf}.\text{Heart}$$
$$\text{HeartDisease} \equiv \text{Disorder} \sqcap \exists \text{affects}.\exists \text{isPartOf}.\text{Heart}$$

$$?-\ \text{Myocarditis} \sqsubseteq \text{HeartDisease}$$

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✔ Myocarditis $\sqsubseteq$ Disorder $\sqcap$ $\exists$affects.Myocardium

✔ Myocardium $\sqsubseteq$ Muscle $\sqcap$ $\exists$isPartOf.Heart

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✔ HeartDisease $\sqsubseteq$ Disorder $\sqcap$ $\exists$affects.$\exists$isPartOf.Heart

✘ Disorder $\sqsubseteq$ $\neg\exists$affects.$\exists$isPartOf.Heart $\sqcup$ HeartDisease

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✔ $\text{Disorder} \sqsubseteq \forall\text{affects}.\forall\text{isPartOf}.\neg\text{Heart} \sqcup \text{HeartDisease}$

?− $\text{Myocarditis} \sqsubseteq \text{HeartDisease}$

1. Normalization

Example

$$\begin{aligned}
\text{Myocarditis} &\sqsubseteq \text{Disorder} \sqcap \exists \text{affects}.\text{Myocardium} \\
\text{Myocardium} &\sqsubseteq \text{Muscle} \sqcap \exists \text{isPartOf}.\text{Heart} \\
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\text{Disorder} &\sqsubseteq \forall \text{affects}.\forall \text{isPartOf}.\neg \text{Heart} \sqcup \text{HeartDisease}
\end{aligned}$$

$$?- \text{Myocarditis} \sqsubseteq \text{HeartDisease}$$

1. Normalization
2. Initialization

Example

$$\begin{aligned}
\text{Myocarditis} &\sqsubseteq \text{Disorder} \sqcap \exists\text{affects}.\text{Myocardium}\\
\text{Myocardium} &\sqsubseteq \text{Muscle} \sqcap \exists\text{isPartOf}.\text{Heart}\\
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\end{aligned}$$

$?-$ Myocarditis $\sqsubseteq$ HeartDisease

1. Normalization
2. Initialization

Myocarditis, $\neg$HeartDisease

EXAMPLE

$$\text{Myocarditis} \sqsubseteq \text{Disorder} \sqcap \exists \text{affects}.\text{Myocardium}$$
$$\text{Myocardium} \sqsubseteq \text{Muscle} \sqcap \exists \text{isPartOf}.\text{Heart}$$
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$$?- \text{Myocarditis} \sqsubseteq \text{HeartDisease}$$

1. Normalization
2. Initialization
3. Expansion

$\text{Myocarditis}, \neg\text{HeartDisease}$

Example

$$\begin{aligned}
\text{Myocarditis} &\sqsubseteq \text{Disorder} \sqcap \exists \text{affects}.\text{Myocardium} \\
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\end{aligned}$$

$$?\!-\ \text{Myocarditis} \sqsubseteq \text{HeartDisease}$$

1. Normalization
2. Initialization
3. Expansion

Myocarditis, ¬HeartDisease, Disorder

Example

$$\begin{aligned}
\blacktriangleright\ \mathsf{Myocarditis} &\sqsubseteq \mathsf{Disorder} \sqcap \exists \mathsf{affects}.\mathsf{Myocardium} \\
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?-\ \mathsf{Myocarditis} &\sqsubseteq \mathsf{HeartDisease}
\end{aligned}$$

1. Normalization
2. Initialization
3. **Expansion**

Myocarditis, ¬HeartDisease, Disorder,
• ∃affects.Myocardium

Example

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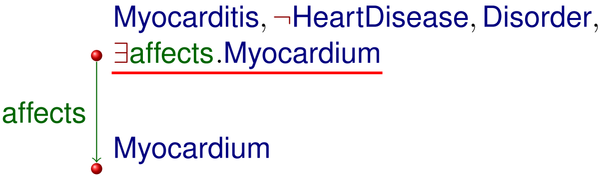
$$?- \ \text{Myocarditis} \sqsubseteq \text{HeartDisease}$$

1. Normalization
2. Initialization
3. Expansion



Example

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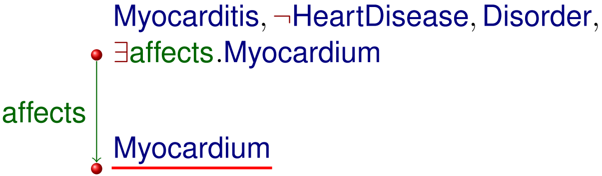
1. Normalization
2. Initialization
3. Expansion



Example

$$\mathsf{Myocarditis} \sqsubseteq \mathsf{Disorder} \sqcap \exists \mathsf{affects}.\mathsf{Myocardium}$$

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1. Normalization
2. Initialization
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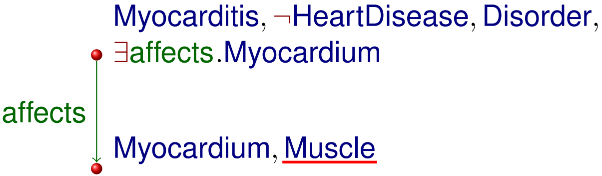

Example

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$$?-\ \mathsf{Myocarditis} \sqsubseteq \mathsf{HeartDisease}$$

1. Normalization
2. Initialization
3. **Expansion**

Myocarditis, $\neg$HeartDisease, Disorder, $\exists$affects.Myocardium

affects ↓

Myocardium, Muscle, $\underline{\exists \mathsf{isPartOf}.\mathsf{Heart}}$

Example

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1. Normalization
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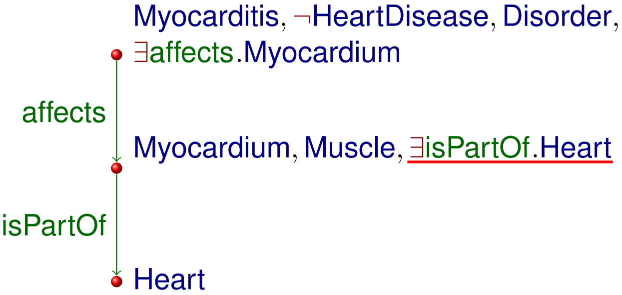

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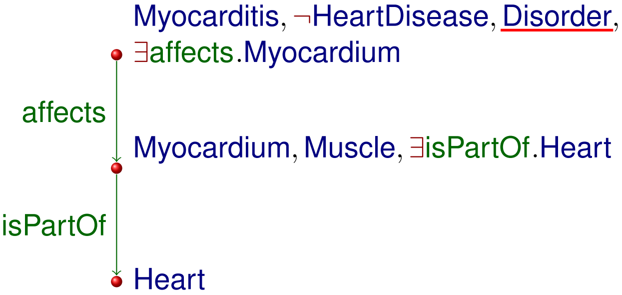
1. Normalization
2. Initialization
3. Expansion



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1. Normalization
2. Initialization
3. Expansion

Myocarditis, ¬HeartDisease, Disorder, ∃affects.Myocardium, <u>∀affects.∀isPartOf.¬Heart</u>

↓ affects

Myocardium, Muscle, ∃isPartOf.Heart

↓ isPartOf

Heart

Example

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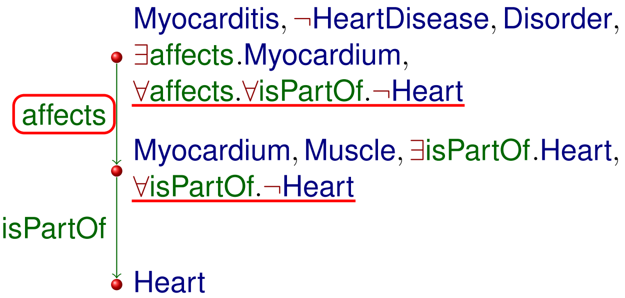
1. Normalization
2. Initialization
3. Expansion



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1. Normalization
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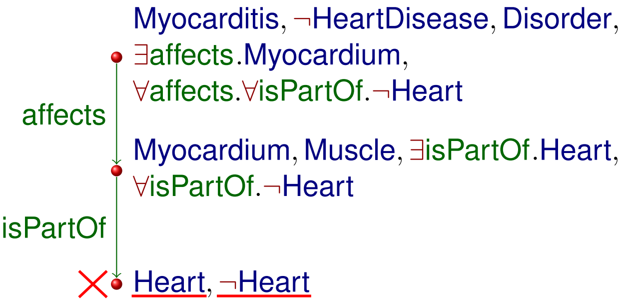

Example

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$$?-\ \mathsf{Myocarditis} \sqsubseteq \mathsf{HeartDisease}$$

1. Normalization
2. Initialization
3. Expansion
4. **Backtracking**

Myocarditis, ¬HeartDisease, Disorder, ∃affects.Myocardium, ~~∀affects.∀isPartOf.¬Heart~~

affects ↓

Myocardium, Muscle, ∃isPartOf.Heart, ~~∀isPartOf.¬Heart~~

isPartOf ↓

Heart

Example

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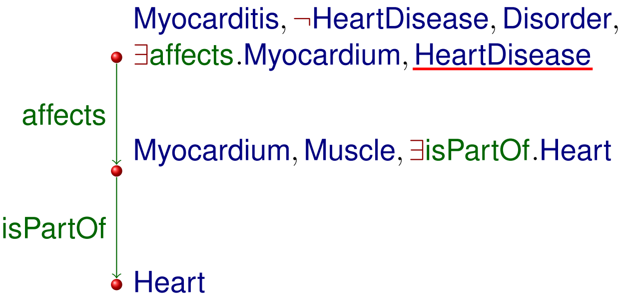
$$

1. Normalization
2. Initialization
3. Expansion
4. **Backtracking**



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1. Normalization
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3. Expansion
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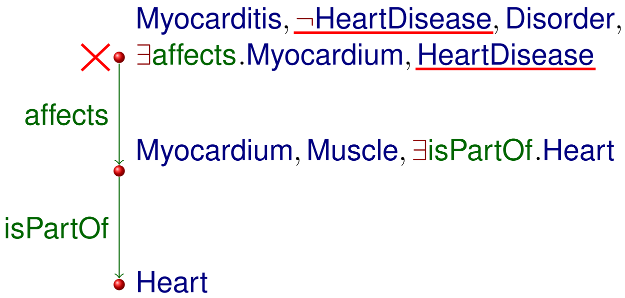

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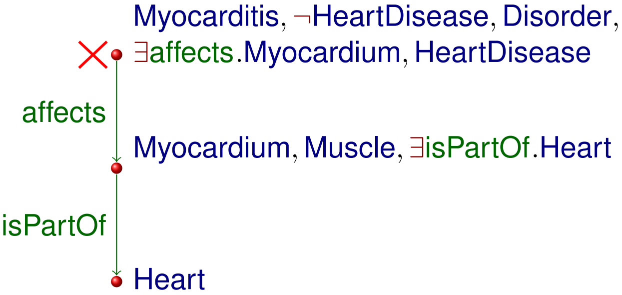
\end{aligned}$$

$?-\ \text{Myocarditis} \sqsubseteq \text{HeartDisease}$ – *Yes*!

1. Normalization
2. Initialization
3. Expansion
4. Backtracking



Observations

1. Classification requires enumeration:
   - Every subsumption $A \sqsubseteq B$ has to be checked separately
   - E.g., 300,000 atomic concepts (SNOMED CT) result in 90,000,000,000 subsumption tests
   - Over 99.99% of subsumptions do not hold

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2. Excessive non-determinism:
   - Concept definitions $A \equiv B \sqcap \exists R.C$ are very common
   - Normalization produces disjunctions: $B \sqsubseteq A \sqcup \forall R.\neg C$
   - Often B is a generic commonly-occuring concept:

     HeartDisease ≡ Disorder ⊓ ∃affects.∃isPartOf.Heart

   - And so, the rules with Disorder ⊑ . . . apply very often

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     HeartDisease $\equiv$ Disorder $\sqcap$ $\exists$affects.$\exists$isPartOf.Heart
   - And so, the rules with Disorder $\sqsubseteq \ldots$ apply very often
3. The models can be very very very large…
   - which makes every subsumption test very expensive

Reciprocal Links

Example

$$\begin{aligned} \text{Heart} &\sqsubseteq \text{Organ} \\ \text{MuscularOrgan} &\equiv \text{Organ} \sqcap \exists \text{hasPart}.\text{Muscle} \\ \text{Myocardium} &\sqsubseteq \text{Muscle} \sqcap \exists \text{isPartOf}.\text{Heart} \\ \\ \text{isPartOf} &\sqsubseteq \text{hasPart}^{-} \end{aligned}$$

$$\not\models \text{Heart} \sqsubseteq \text{MuscularOrgan}$$

Reciprocal Links

Example

$$\mathsf{Heart} \sqsubseteq \mathsf{Organ}$$
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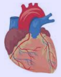
$$\mathsf{isPartOf} \sqsubseteq \mathsf{hasPart}^{-}$$

$$\not\models \mathsf{Heart} \sqsubseteq \mathsf{MuscularOrgan}$$



Heart

Organ

Reciprocal Links

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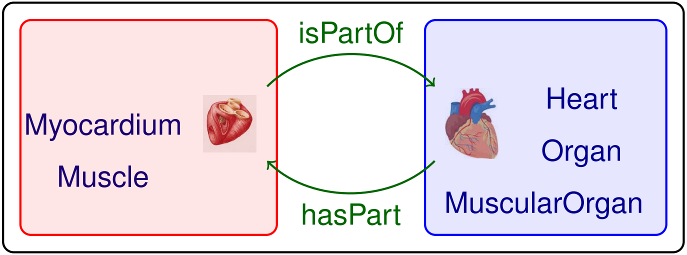
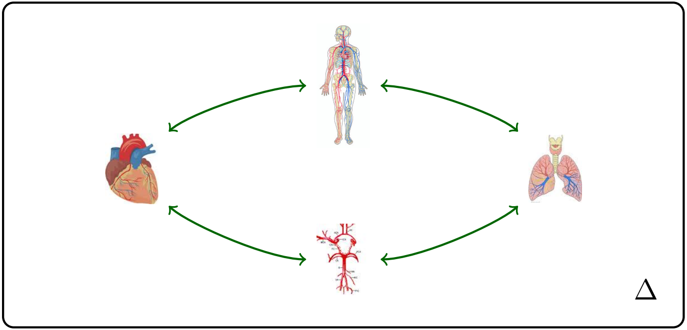


Cycles in Ontologies

Example

$$\text{Heart} \sqsubseteq \exists \text{isComponentOf}.\text{CirculatorySystem}$$
$$\text{CirculatorySystem} \sqsubseteq \exists \text{hasComponent}.\text{Lungs}$$
$$\text{Lungs} \sqsubseteq \exists \text{isServedBy}.\text{PulmonaryArtery}$$
$$\text{PulmonaryArtery} \sqsubseteq \exists \text{serves}.\text{Heart}$$



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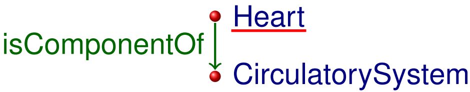

CYCLES IN ONTOLOGIES

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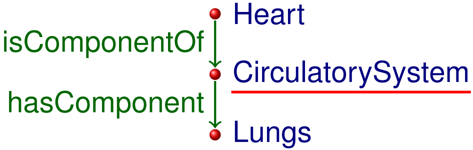

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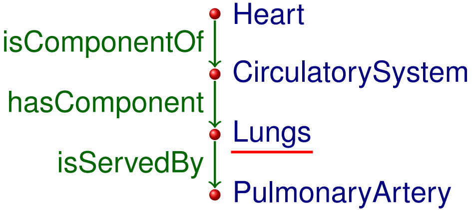

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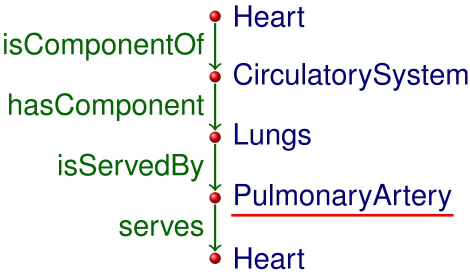

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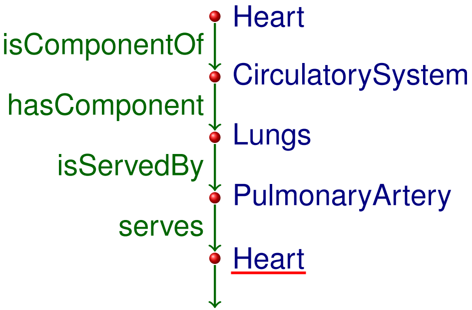

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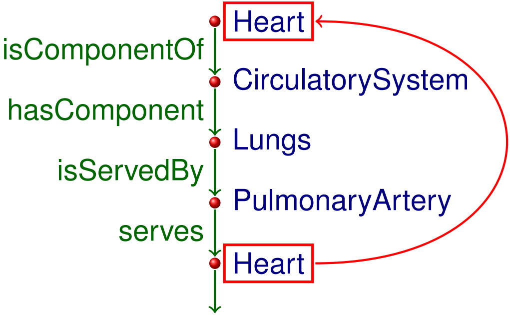
\end{aligned}$$



Blocking in Practice

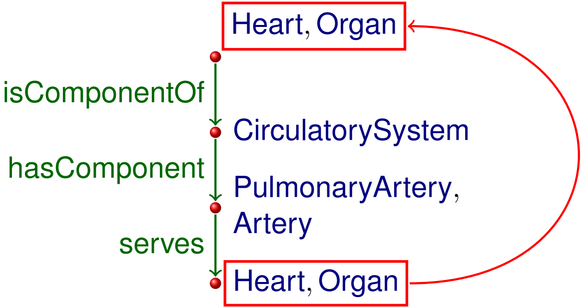
Example

Heart–component – CirculatorySystem

PulmonaryArtery–component – CirculatorySystem

PulmonaryArtery–serve – Heart

ArterialOrgan $\equiv$ Organ $\sqcap$ $\exists$isServedBy.Artery



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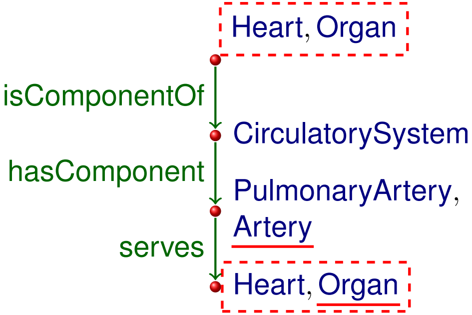

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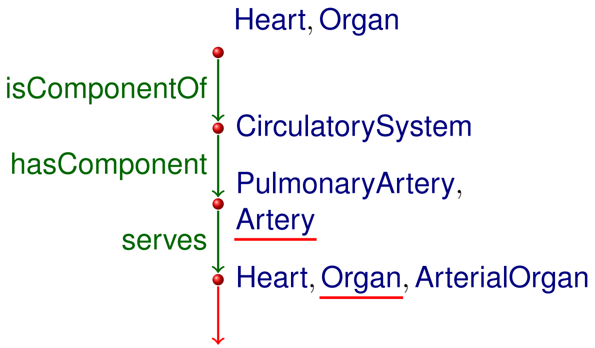

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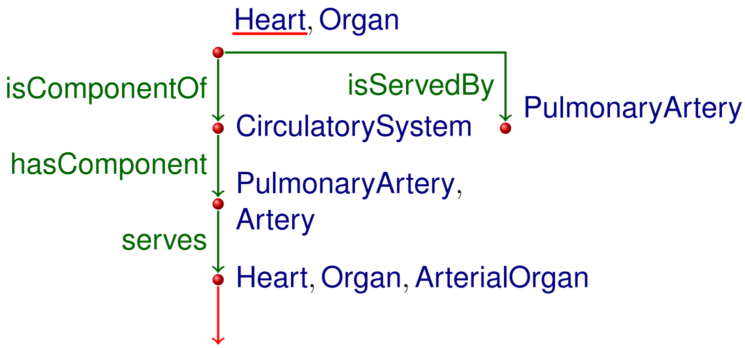

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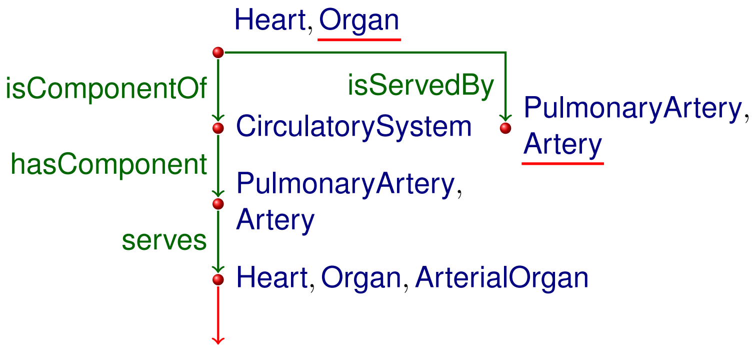

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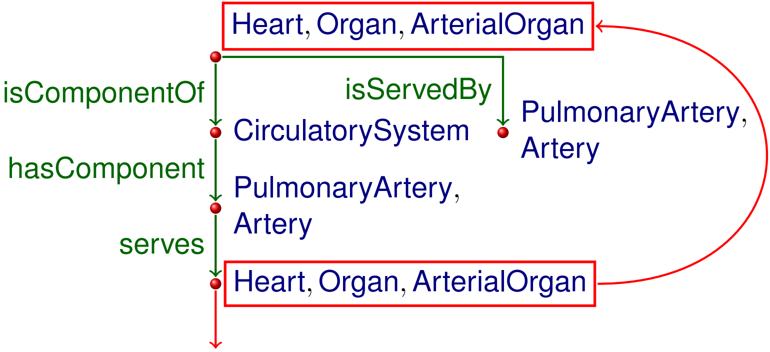

OBSERVATIONS

1. Blocking is not persistent:
   - Blocking of nodes also depend on predecessor nodes
   - The “pairwise blocking” strategy is commonly used
   - Nodes are frequently blocked and unblocked
   - Highly dependent on the order of rule applications

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2. Models can be very large:
   - Contain similar nodes at different stages of expansion
   - The parts below the blocked are not discarded
3. Blocking conditions are hard to check
   - Required after every rule application

OUTLINE

$\mathcal{EL}$ Family of DLs

- Introduced by [Baader, Brandt, Lutz; IJCAI 2003, 2005]

Name	DL syntax	First-Order syntax	
top	$\top$	$\top$	
intersection	$C_1 \sqcap C_2$	$C_1(x) \wedge C_2(x)$	$= \mathcal{E}$
existential restr.	$\exists r.C$	$\exists y.[r(x,y) \wedge C(y)]$	$\mathcal{L}$
concept inclusion	$C_1 \sqsubseteq C_2$	$\forall x.[C_1(x) \to C_2(x)]$	

- Redefines the basic DL: $\mathcal{EL} = \mathcal{ALC} \setminus \{\bot, \neg, \forall\}$
- Reasoning problems are PTime-complete

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- $\mathcal{EL}^{++}$:
  - has polynomial-model property
  - classification can be computed in polynomial time
  - basis of the **OWL 2 EL** profile

$\mathcal{ELH}$ Expressivity

- Surprisingly useful:

SNOMED CT	GO	NCI	Galen
✔	✔	✔	

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- Simple inclusions:

$$\text{Myocardium} \sqsubseteq \text{Muscle} \sqcap \exists \text{isPartOf}.\text{Heart}$$
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$$\text{MuscularOrgan} \equiv \text{Organ} \sqcap \exists \text{hasPart}.\text{Muscle}$$
$$\text{KidneyExamination} \equiv \text{ClinicalAct} \sqcap \exists \text{hasSubprocess}.(\text{Examination} \sqcap \exists \text{involves}.\text{Kidney})$$

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- General concept inclusions:

$$\text{Structure} \sqcap \exists \text{isPartOf}.\text{Heart} \sqsubseteq \exists \text{isComponentOf}.\text{CardiovascularSystem}$$

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Example (Galen)

✔	$\text{BasilarArtery} \sqsubseteq \exists \text{hasBranch}.\text{VertebralArtery}$
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✘	$\text{hasBranch} \sqsubseteq \text{isBranchOf}^{-}$
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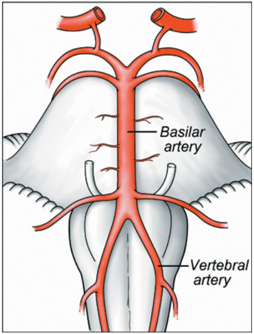

- Over 95% of axioms in Galen are in $\mathcal{ELH}$

$\mathcal{ELH}$ Classification Procedure

1. Normalization / structural transformation:

Example

$A \sqsubseteq \exists R.(B \sqcap C)$

$\mathcal{ELH}$ Classification Procedure

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Example

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$A \sqsubseteq B \quad A \sqcap B \sqsubseteq C \quad A \sqsubseteq \exists R.B \quad \exists R.B \sqsubseteq C \quad R \sqsubseteq S$

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CR5 $\dfrac{A \sqsubseteq \exists R.B \quad B \sqsubseteq C \quad \exists R.C \sqsubseteq D}{A \sqsubseteq D}$

Observations

1. Procedure is more goal-directed:
   - Derives only subsumptions of the form $A \sqsubseteq B$ or $A \sqsubseteq \exists r.B$
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   - Only consequences of the axioms are derived
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2. Useful computational properties:
   - Polynomial worst-case complexity
   - No non-determinism, no backtracking
   - Relatively easy to implement
   - Easy to track dependencies for explanations
   - Can be made incremental, distributed, and parallel

Reciprocal Links and Cycles

Example

$$
\begin{aligned}
\text{Heart} &\sqsubseteq \exists \text{isComponentOf}.\text{CirculatorySystem} \\
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- Inferences require matching existential restrictions:

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- No inference is made for just positive existential restrictions (FMA is trivially classified)

Beyond $\mathcal{ELH}$

- Galen uses two constructors that are outside of $\mathcal{ELH}$: inverse roles and role functionality:

Example (Galen)

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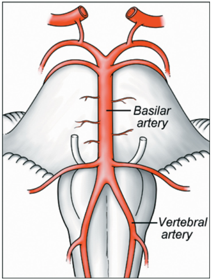

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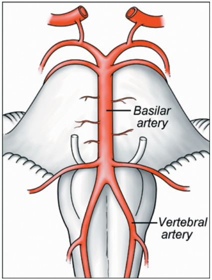

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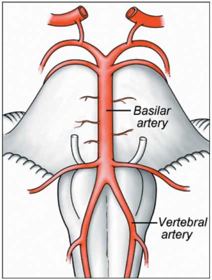

- Adding either results in complexity increase from PTime to ExpTime [Baader, Brandt, Lutz 2005; 2008]
- We are not scared of the high complexity!

$\mathcal{SHIQ}$

Name	DL syntax	First-Order syntax	
intersection	$C_1 \sqcap C_2$	$C_1(x) \wedge C_2(x)$	
union	$C_1 \sqcup C_2$	$C_1(x) \vee C_2(x)$	$= \mathcal{A}$
complement	$\neg C$	$\neg C(x)$	$\mathcal{L}$
value restriction	$\forall r.C$	$\forall y.[r(x,y) \rightarrow C(y)]$	$\mathcal{C}$
existential restr.	$\exists r.C$	$\exists y.[r(x,y) \wedge C(y)]$	
transitivity	$Tra(r)$	$\forall xyz.[r(x,y) \wedge r(y,z) \rightarrow r(x,z)]$	$= \mathcal{S}$
functionality	$Fun(r)$	$\forall xyz.[r(x,y) \wedge r(x,z) \rightarrow y \simeq z]$	$+\mathcal{F}$
role inclusion	$r_1 \sqsubseteq r_2$	$\forall xy.[r_1(x,y) \rightarrow r_2(x,y)]$	$+\mathcal{H}$
inverse roles	$[\dots r^- \dots]$	$[\dots r(y,x) \dots]$	$+\mathcal{I}$
number restriction	$\leqslant n\, r.C$	$\exists^{\leq n} y.[r(x,y) \wedge C(y)]$	$+\mathcal{Q}$

- $\mathcal{SHIQ}$:
  - has a generalized tree-model property (transitivity)
  - has no finite-model property (because of functionality)
  - satisfiability problem is ExpTime-complete

Horn $\mathcal{SHIQ}$

Name	positive	negative	Horn-
intersection	$\cdot \sqsubseteq C_1 \sqcap C_2$	$C_1 \sqcap C_2 \sqsubseteq \cdot$	
union	$-$	$C_1 \sqcup C_2 \sqsubseteq \cdot$	$= \mathcal{A}$
complement	$\cdot \sqsubseteq \neg C$	$-$	$\mathcal{L}$
value restriction	$\cdot \sqsubseteq \forall r.C$	$-$	$\mathcal{C}$
existential restr.	$\cdot \sqsubseteq \exists r.C$	$\exists r.C \sqsubseteq \cdot$	
transitivity	$Tra(r)$		$= \mathcal{S}$
functionality	$Fun(r)$		$+\mathcal{F}$
role inclusion	$r_1 \sqsubseteq r_2$		$+\mathcal{H}$
inverse roles	$[\ldots r^- \ldots]$		$+\mathcal{I}$
number restriction	$\cdot \sqsubseteq \leqslant 1\, r.C$	$-$	$+\mathcal{Q}$

- Horn $\mathcal{SHIQ}$:
  - can be translated to the Horn fragment of first-order logic
  - the reasoning problems are ExpTime-complete
  - data complexity (quiering assertions) is PTime-complete [Hustadt, Motik, Saatler; JAR 2007]

NEW INFERENCE RULES

1

$$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \forall R.C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

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2

$$\frac{A \sqsubseteq \exists R.B \quad \exists R^{-}.A \sqsubseteq C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

$$[(\exists R^{-}.A \sqsubseteq C) \equiv (A \sqsubseteq \forall R.C)]$$

New Inference Rules

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$$\frac{A \sqsubseteq \exists R.B \quad \exists R^{-}.A \sqsubseteq C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

$[(\exists R^{-}.A \sqsubseteq C) \equiv (A \sqsubseteq \forall R.C)]$

3
$$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \exists R.C \quad Fun(R)}{A \sqsubseteq \exists R.(B \sqcap C)}$$

New Inference Rules

1. $$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \forall R.C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

2. $$\frac{A \sqsubseteq \exists R.B \quad \exists R^{-}.A \sqsubseteq C}{A \sqsubseteq \exists R.(B \sqcap C)}$$ $[(\exists R^{-}.A \sqsubseteq C) \equiv (A \sqsubseteq \forall R.C)]$

3. $$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \exists R.C \quad Fun(R)}{A \sqsubseteq \exists R.(B \sqcap C)}$$

4. $$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \exists R.C \quad B \sqsubseteq D \quad C \sqsubseteq D \quad A \sqsubseteq \leqslant 1\, R.D}{A \sqsubseteq \exists R.(B \sqcap C)}$$

New Inference Rules

1. $$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \forall R.C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

2. $$\frac{A \sqsubseteq \exists R.B \quad \exists R^{-}.A \sqsubseteq C}{A \sqsubseteq \exists R.(B \sqcap C)}$$ $[(\exists R^{-}.A \sqsubseteq C) \equiv (A \sqsubseteq \forall R.C)]$

3. $$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \exists R.C \quad Fun(R)}{A \sqsubseteq \exists R.(B \sqcap C)}$$

4. $$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \exists R.C \quad B \sqsubseteq D \quad C \sqsubseteq D \quad A \sqsubseteq \leqslant 1\, R.D}{A \sqsubseteq \exists R.(B \sqcap C)}$$

5. Old rules should be extended for new conjunctions:

CR5 $$\frac{A \sqsubseteq \exists R.(B \sqcap C) \quad B \sqcap C \sqsubseteq D \quad \exists R.D \sqsubseteq E}{A \sqsubseteq E}$$

New Inference Rules

1

$$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \forall R.C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

2

$$\frac{A \sqsubseteq \exists R.B \quad \exists R^{-}.A \sqsubseteq C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

$[(\exists R^{-}.A \sqsubseteq C) \equiv (A \sqsubseteq \forall R.C)]$

3

$$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \exists R.C \quad Fun(R)}{A \sqsubseteq \exists R.(B \sqcap C)}$$

4

$$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \exists R.C \quad B \sqsubseteq D \quad C \sqsubseteq D \quad A \sqsubseteq\, \leqslant 1\, R.D}{A \sqsubseteq \exists R.(B \sqcap C)}$$

5 Old rules should be extended for new conjunctions:

$$\text{CR5}\quad \frac{A \sqsubseteq \exists R.(B \sqcap C) \quad \boxed{B \sqcap C} \sqsubseteq D \quad \boxed{\exists R.D \sqsubseteq E}}{A \sqsubseteq E}$$

New Inference Rules

1
$$\frac{M \sqsubseteq \exists R.N \quad M \sqsubseteq \forall R.C}{M \sqsubseteq \exists R.(N \sqcap C)}$$

2
$$\frac{M \sqcap A \sqsubseteq \exists R.N \quad \exists R^{-}.A \sqsubseteq C}{M \sqcap A \sqsubseteq \exists R.(N \sqcap C)}$$

3
$$\frac{M \sqsubseteq \exists R.N_1 \quad M \sqsubseteq \exists R.N_2 \quad Fun(R)}{M \sqsubseteq \exists R.(N_1 \sqcap N_2)}$$

4
$$\frac{M \sqsubseteq \exists R.N_1 \quad M \sqsubseteq \exists R.N_2 \quad N_1 \sqsubseteq D \quad N_2 \sqsubseteq D \quad M \sqsubseteq \leqslant 1\, R.D}{M \sqsubseteq \exists R.(N_1 \sqcap N_2)}$$

5 Old rules should be extended for new conjunctions:

CR5
$$\frac{M \sqsubseteq \exists R.N \quad M \sqsubseteq D \quad \boxed{\exists R.D \sqsubseteq E}}{M \sqsubseteq E}$$

$$\boxed{M, N_* = \bigsqcap A_i}$$

all rules

Observations

1. Optimal complexity:
   - Derives only subsumptions of the form:

     $\bigsqcap A_i \sqsubseteq B$ or $\bigsqcap A_i \sqsubseteq \exists R.\bigsqcap B_j$
   - At most exponential number of inferences is possible

OBSERVATIONS

1. Optimal complexity:
   - Derives only subsumptions of the form:

     $\bigsqcap A_i \sqsubseteq B$ or $\bigsqcap A_i \sqsubseteq \exists R. \bigsqcap B_j$

   - At most exponential number of inferences is possible
2. "Pay as you go" behaviour:
   - Remains polynomial for $\mathcal{ELH}$
   - because the rules forming conjunctions never apply:

$$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \forall R.C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

Observations

1. Optimal complexity:
   - Derives only subsumptions of the form:

     $\bigsqcap A_i \sqsubseteq B \quad \text{or} \quad \bigsqcap A_i \sqsubseteq \exists R.\bigsqcap B_j$
   - At most exponential number of inferences is possible
2. "Pay as you go" behaviour:
   - Remains polynomial for $\mathcal{ELH}$
   - because the rules forming conjunctions never apply:

$$\frac{A \sqsubseteq \exists R.B \quad A \sqsubseteq \forall R.C}{A \sqsubseteq \exists R.(B \sqcap C)}$$

(the second premise and the conclusion are crossed out on the slide)

EXPERIMENTAL RESULTS

	GO	NCI	Galen v.0	Galen v.7	SNOMED CT
Concepts:	20465	27652	2748	23136	389472
FACT++	15.24	6.05	465.35	—	650.37
HERMIT	199.52	169.47	45.72	—	—
PELLET	72.02	26.47	—	—	—
CEL	1.84	5.76	—	—	1185.70
CB	1.17	3.57	0.32	9.58	49.44
Speed-Up:	1.57X	1.61X	143X	∞	13.15X

- The prototype reasoner CB implementing the procedure is available open source from:

`cb-reasoner.googlecode.com`

[Demo?]

Outline

Tableau vs. Hyper-Resolution

Example

$$A \sqsubseteq \exists R.B$$

$$B \sqsubseteq A$$

$$\exists R.A \sqsubseteq C$$

$$\overline{\quad ?- A \sqsubseteq C \quad}$$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

➤ $\exists R.A \sqsubseteq C$

$$\frac{}{?- A \sqsubseteq C}$$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

➤ $?- A \sqsubseteq C$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

$?- A \sqsubseteq C$

- $A, \neg C$

Tableau vs. Hyper-Resolution

Example

➤ $\underline{A} \sqsubseteq \exists R.B$

$$\frac{\begin{array}{l} B \sqsubseteq A \\ A \sqsubseteq \forall R^{-}.C \end{array}}{?- A \sqsubseteq C}$$

- $\underline{A}, \neg C$

TABLEAU VS. HYPER-RESOLUTION

EXAMPLE

- ➤ $A \sqsubseteq \exists R.B$

$$\begin{array}{l} B \sqsubseteq A \\ A \sqsubseteq \forall R^{-}.C \\ \hline ?- A \sqsubseteq C \end{array}$$

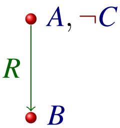


Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

➤ $B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

$?- A \sqsubseteq C$

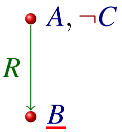


Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

➤ $B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

$?- A \sqsubseteq C$

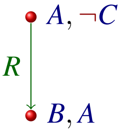


Tableau vs. Hyper-Resolution

Example

$$A \sqsubseteq \exists R.B$$

$$B \sqsubseteq A$$

$$➤ \underline{A} \sqsubseteq \forall R^{-}.C$$

$$?- A \sqsubseteq C$$

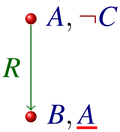


Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

➤ $A \sqsubseteq \forall R^{-}.C$

$?- A \sqsubseteq C$

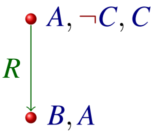


TABLEAU VS. HYPER-RESOLUTION

EXAMPLE

$$A \sqsubseteq \exists R.B$$

$$B \sqsubseteq A$$

$$A \sqsubseteq \forall R^{-}.C$$

$$?-\ A \sqsubseteq C$$

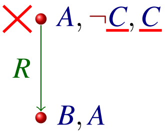


Tableau vs. Hyper-Resolution

Example

➤ $A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
➤ $B \sqsubseteq A$	$\neg B(x) \vee A(x)$
➤ $A \sqsubseteq \forall R^{-}.C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

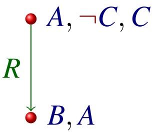
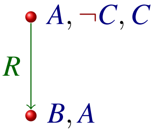


TABLEAU VS. HYPER-RESOLUTION

EXAMPLE

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
?– $A \sqsubseteq C$	



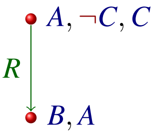
$A(c)$

$\neg C(c)$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg \underline{A(x)} \vee R(x, f(x))$
	$\neg \underline{A(x)} \vee B(f(x))$
$B \sqsubseteq A$	$\neg \underline{B(x)} \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg \underline{R(x, y)} \vee \neg \underline{A(y)} \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg \underline{C(c)}$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

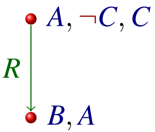
$?- A \sqsubseteq C$

$\neg A(x) \vee R(x, f(x))$

$\neg A(x) \vee B(f(x))$

$\neg B(x) \vee A(x)$

$\neg R(x,y) \vee \neg A(y) \vee C(x)$



$A(c)$

$\neg C(c)$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

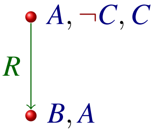
$?- A \sqsubseteq C$

➤ $\neg A(x) \vee R(x, f(x))$

$\neg A(x) \vee B(f(x))$

$\neg B(x) \vee A(x)$

$\neg R(x,y) \vee \neg A(y) \vee C(x)$



➤ $A(c)$

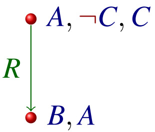
$\neg C(c)$

$R(c, f(c))$

TABLEAU VS. HYPER-RESOLUTION

EXAMPLE

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	➤ $\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



➤ $A(c)$

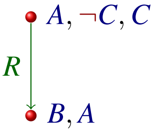
$\neg C(c)$

$R(c, f(c))$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	➤ $\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



➤ $A(c)$

$\neg C(c)$

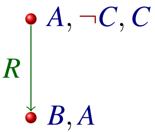
$R(c, f(c))$

$B(f(c))$

TABLEAU VS. HYPER-RESOLUTION

EXAMPLE

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	➤ $\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

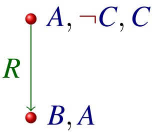
➤ $B(f(c))$

Tableau vs. Hyper-Resolution

Example

$$A \sqsubseteq \exists R.B \qquad \neg A(x) \vee R(x, f(x))$$
$$\neg A(x) \vee B(f(x))$$
$$B \sqsubseteq A \qquad \neg B(x) \vee A(x)$$
$$A \sqsubseteq \forall R^{-}.C \qquad \neg R(x,y) \vee \neg A(y) \vee C(x)$$

$$?- A \sqsubseteq C$$



$A(c)$

$\neg C(c)$

$R(c, f(c))$

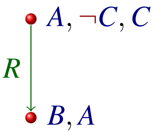
$B(f(c))$

$A(f(c))$

TABLEAU VS. HYPER-RESOLUTION

EXAMPLE

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	➤ $\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

➤ $R(c, f(c))$

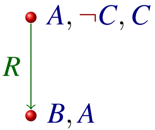
$B(f(c))$

➤ $A(f(c))$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	➤ $\neg R(x, y) \vee \neg A(y) \vee C(x)$
$?-\ A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

➤ $R(c, f(c))$

$B(f(c))$

➤ $A(f(c))$

$C(c)$

Tableau vs. Hyper-Resolution

Example

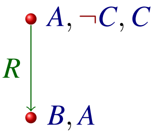
$A \sqsubseteq \exists R.B$ $\qquad$ $\neg A(x) \vee R(x, f(x))$

$\qquad\qquad\qquad$ $\neg A(x) \vee B(f(x))$

$B \sqsubseteq A$ $\qquad$ $\neg B(x) \vee A(x)$

$A \sqsubseteq \forall R^{-}.C$ $\qquad$ $\neg R(x, y) \vee \neg A(y) \vee C(x)$

$?- A \sqsubseteq C$



$A(c)$

➤ $\neg C(c)$

$R(c, f(c))$

$B(f(c))$

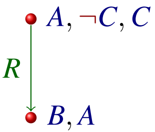
$A(f(c))$

➤ $C(c)$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

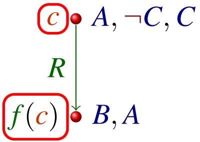
$C(c)$

$\bot$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

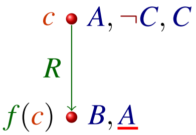
$C(c)$

$\bot$

Tableau vs. Hyper-Resolution

Example

$$A \sqsubseteq \exists R.B \qquad \neg A(x) \vee R(x, f(x))$$
$$\qquad \neg A(x) \vee B(f(x))$$
$$B \sqsubseteq A \qquad \neg B(x) \vee A(x)$$
$$A \sqsubseteq \forall R^{-}.C \qquad \neg R(x,y) \vee \neg A(y) \vee C(x)$$
$$?\!-\, A \sqsubseteq C$$

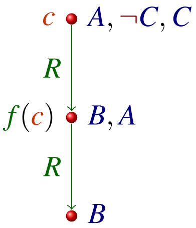


$$A(c)$$
$$\neg C(c)$$
$$R(c, f(c))$$
$$B(f(c))$$
$$A(f(c))$$
$$C(c)$$
$$\bot$$

Tableau vs. Hyper-Resolution

Example

➤ $A \sqsubseteq \exists R.B$	$\neg A(x) \lor R(x, f(x))$
	$\neg A(x) \lor B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \lor A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \lor \neg A(y) \lor C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

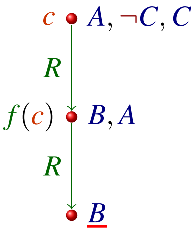
$C(c)$

$\bot$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
➤ $B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

$C(c)$

$\bot$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

➤ $B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

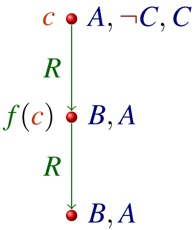
$?- A \sqsubseteq C$

$\neg A(x) \vee R(x, f(x))$

$\neg A(x) \vee B(f(x))$

$\neg B(x) \vee A(x)$

$\neg R(x, y) \vee \neg A(y) \vee C(x)$



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

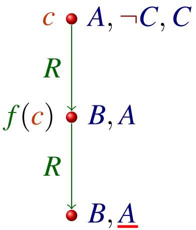
$C(c)$

$\perp$

Tableau vs. Hyper-Resolution

Example

➤ $A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

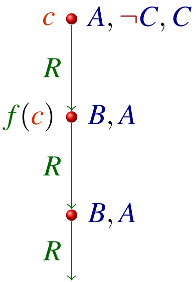
$C(c)$

$\bot$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

$C(c)$

$\bot$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

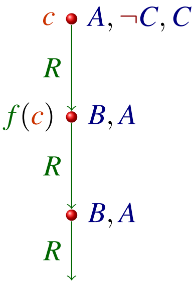
$?- A \sqsubseteq C$

$\neg A(x) \vee R(x, f(x))$

$\neg A(x) \vee B(f(x))$

$\neg B(x) \vee A(x)$

$\neg R(x,y) \vee \neg A(y) \vee C(x)$



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

$C(c)$

$\bot$

TABLEAU VS. HYPER-RESOLUTION

EXAMPLE

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

$A \sqsubseteq \forall R^{-}.C$

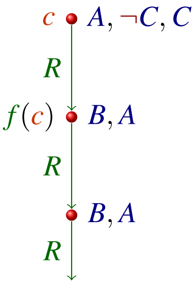
$?- A \sqsubseteq C$

$\neg A(x) \vee R(x, f(x))$

$\neg A(x) \vee B(f(x))$

$\neg B(x) \vee A(x)$

$\neg R(x,y) \vee \neg A(y) \vee C(x)$



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

$C(c)$

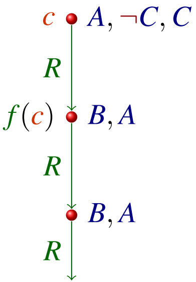
$\bot$

$R(f(c), f(f(c)))$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	➤ $\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$ $\quad$ $R(f(c), f(f(c)))$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

➤ $A(f(c))$

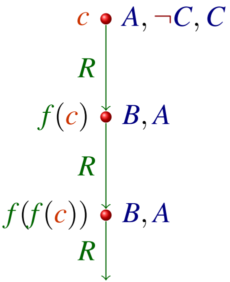
$C(c)$

$\bot$

Tableau vs. Hyper-Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	➤ $\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

➤ $A(f(c))$

$C(c)$

$\bot$

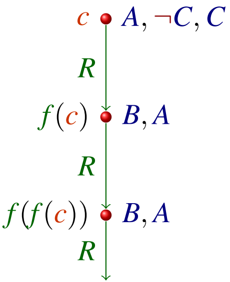
$R(f(c), f(f(c)))$

$B(f(f(c)))$

TABLEAU VS. HYPER-RESOLUTION

EXAMPLE

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$A \sqsubseteq \forall R^{-}.C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	



$A(c)$

$\neg C(c)$

$R(c, f(c))$

$B(f(c))$

$A(f(c))$

$C(c)$

$\bot$

$R(f(c), f(f(c)))$

$B(f(f(c)))$

$\ldots$

No termination!

C.B. vs. Ordered Resolution

Example

$$A \sqsubseteq \exists R.B$$

$$\frac{\begin{array}{r} B \sqsubseteq A \\ \exists R.A \sqsubseteq C \end{array}}{?- A \sqsubseteq C}$$

C.B. vs. Ordered Resolution

Example

$$A \sqsubseteq \exists R.B$$

$$\frac{\begin{array}{c} B \sqsubseteq A \\ \exists R.A \sqsubseteq C \end{array}}{?- A \sqsubseteq C}$$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

C.B. vs. Ordered Resolution

Example

- $A \sqsubseteq \exists R.B$ — $\neg A(x) \vee R(x, f(x))$
  — $\neg A(x) \vee B(f(x))$
- $B \sqsubseteq A$ — $\neg B(x) \vee A(x)$
- $\exists R.A \sqsubseteq C$ — $\neg R(x,y) \vee \neg A(y) \vee C(x)$

$?- A \sqsubseteq C$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$? - A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$ ➤ $\neg A(x) \vee R(x, f(x))$

$\neg A(x) \vee B(f(x))$

$B \sqsubseteq A$ $\neg B(x) \vee A(x)$

$\exists R.A \sqsubseteq C$ ➤ $\neg R(x, y) \vee \neg A(y) \vee C(x)$

$?- A \sqsubseteq C$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$

$\exists R.A \sqsubseteq C$

$?-A \sqsubseteq C$

➤ $\neg A(x) \vee \underline{R(x, f(x))}$

$\neg A(x) \vee \underline{B(f(x))}$

$\underline{\neg B(x)} \vee A(x)$

➤ $\underline{\neg R(x,y)} \vee \neg A(y) \vee C(x)$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$\underline{A(c)}$

$\underline{\neg C(c)}$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	➤ $\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	➤ $\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

C.B. vs. Ordered Resolution

Example

$$A \sqsubseteq \exists R.B \qquad \neg A(x) \vee R(x, f(x))$$

$$\neg A(x) \vee B(f(x))$$

$$B \sqsubseteq A \qquad \neg B(x) \vee A(x)$$

$$\exists R.A \sqsubseteq C \qquad \neg R(x, y) \vee \neg A(y) \vee C(x)$$

$$?- A \sqsubseteq C$$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$$A(c)$$

$$\neg C(c)$$

$$\neg A(x) \vee \neg A(f(x)) \vee C(x)$$

$$\neg A(x) \vee A(f(x))$$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

$\neg A(x) \vee A(f(x))$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$ | $\neg A(x) \vee \underline{R(x, f(x))}$

$\neg A(x) \vee \underline{B(f(x))}$

$B \sqsubseteq A$ | $\underline{\neg B(x)} \vee A(x)$

$\exists R.A \sqsubseteq C$ | $\underline{\neg R(x, y)} \vee \neg A(y) \vee C(x)$

$?- A \sqsubseteq C$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$\underline{A(c)}$

$\neg\underline{C(c)}$

➤ $\neg A(x) \vee \underline{\neg A(f(x))} \vee C(x)$

➤ $\neg A(x) \vee \underline{A(f(x))}$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee \underline{R(x, f(x))}$
	$\neg A(x) \vee \underline{B(f(x))}$
$B \sqsubseteq A$	$\underline{\neg B(x)} \vee A(x)$
$\exists R.A \sqsubseteq C$	$\underline{\neg R(x,y)} \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$\underline{A(c)}$

$\underline{\neg C(c)}$

➤ $\neg A(x) \vee \underline{\neg A(f(x))} \vee C(x)$

➤ $\neg A(x) \vee \underline{A(f(x))}$

$\neg A(x) \vee C(x)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

$\neg A(x) \vee A(f(x))$

$\neg A(x) \vee C(x)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee \underline{R(x, f(x))}$
	$\neg A(x) \vee \underline{B(f(x))}$
$B \sqsubseteq A$	$\neg \underline{B(x)} \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg \underline{R(x,y)} \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$\underline{A(c)}$

$\neg \underline{C(c)}$

$\neg A(x) \vee \neg \underline{A(f(x))} \vee C(x)$

➤ $\neg A(x) \vee \underline{A(f(x))}$

➤ $\neg \underline{A(x)} \vee C(x)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

➤ $\neg A(x) \vee A(f(x))$

➤ $\neg A(x) \vee C(x)$

$\neg A(x) \vee C(f(x))$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

$\neg A(x) \vee A(f(x))$

$\neg A(x) \vee C(x)$

$\neg A(x) \vee C(f(x))$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$? - A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

$\neg A(x) \vee A(f(x))$

$\neg A(x) \vee C(x)$

$\neg A(x) \vee C(f(x))$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$ $\neg A(x) \vee \underline{R(x, f(x))}$

$\neg A(x) \vee \underline{B(f(x))}$

$B \sqsubseteq A$ $\underline{\neg B(x)} \vee A(x)$

$\exists R.A \sqsubseteq C$ $\underline{\neg R(x, y)} \vee \neg A(y) \vee C(x)$

$?- A \sqsubseteq C$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

➤ $\underline{A(c)}$

$\underline{\neg C(c)}$

$\neg A(x) \vee \underline{\neg A(f(x))} \vee C(x)$

$\neg A(x) \vee \underline{A(f(x))}$

➤ $\underline{\neg A(x)} \vee C(x)$

$\neg A(x) \vee \underline{C(f(x))}$

$C(c)$

C.B. vs. Ordered Resolution

Example

$$A \sqsubseteq \exists R.B \qquad \neg A(x) \vee R(x, f(x))$$
$$\neg A(x) \vee B(f(x))$$
$$B \sqsubseteq A \qquad \neg B(x) \vee A(x)$$
$$\exists R.A \sqsubseteq C \qquad \neg R(x,y) \vee \neg A(y) \vee C(x)$$
$$?- A \sqsubseteq C$$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$$A(c)$$
$$\neg C(c)$$
$$\neg A(x) \vee \neg A(f(x)) \vee C(x)$$
$$\neg A(x) \vee A(f(x))$$
$$\neg A(x) \vee C(x)$$
$$\neg A(x) \vee C(f(x))$$
$$C(c)$$

C.B. vs. Ordered Resolution

Example

$$A \sqsubseteq \exists R.B$$
$$B \sqsubseteq A$$
$$\exists R.A \sqsubseteq C$$
$$?- A \sqsubseteq C$$

$$\neg A(x) \vee R(x, f(x))$$
$$\neg A(x) \vee B(f(x))$$
$$\neg B(x) \vee A(x)$$
$$\neg R(x, y) \vee \neg A(y) \vee C(x)$$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$$A(c)$$
$$\neg C(c)$$
$$\neg A(x) \vee \neg A(f(x)) \vee C(x)$$
$$\neg A(x) \vee A(f(x))$$
$$\neg A(x) \vee C(x)$$
$$\neg A(x) \vee C(f(x))$$
$$C(c)$$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x,y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

➤ $\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

$\neg A(x) \vee A(f(x))$

$\neg A(x) \vee C(x)$

$\neg A(x) \vee C(f(x))$

➤ $C(c)$

$\bot$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$	$\neg A(x) \vee R(x, f(x))$
	$\neg A(x) \vee B(f(x))$
$B \sqsubseteq A$	$\neg B(x) \vee A(x)$
$\exists R.A \sqsubseteq C$	$\neg R(x, y) \vee \neg A(y) \vee C(x)$
$?- A \sqsubseteq C$	

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$A(c)$

$\neg C(c)$

$\neg A(x) \vee \neg A(f(x)) \vee C(x)$

$\neg A(x) \vee A(f(x))$

$\neg A(x) \vee C(x)$

$\neg A(x) \vee C(f(x))$

$C(c)$

$\bot$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$

$B \sqsubseteq A$ (crossed out)

$\exists R.A \sqsubseteq C$

$?- A \sqsubseteq C$

$\neg A(x) \vee \underline{R(x, f(x))}$

$\neg A(x) \vee \underline{B(f(x))}$

$\underline{\neg B(x)} \vee A(x)$

$\underline{\neg R(x,y)} \vee \neg A(y) \vee C(x)$

$$\frac{A \sqsubseteq \exists R.B \quad B \sqsubseteq A \quad \exists R.A \sqsubseteq C}{A \sqsubseteq C}$$

$\underline{A(c)}$

$\underline{\neg C(c)}$

$\neg A(x) \vee \underline{\neg A(f(x))} \vee C(x)$

$\neg A(x) \vee \underline{A(f(x))}$

$\underline{\neg A(x)} \vee C(x)$

$\neg A(x) \vee \underline{C(f(x))}$

$\underline{C(c)}$

$\bot$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$

~~$B \sqsubseteq A$~~

$\exists R.A \sqsubseteq C$

$?- A \sqsubseteq C$

$\neg A(x) \vee \underline{R(x, f(x))}$

$\neg A(x) \vee \underline{B(f(x))}$

$\underline{\neg B(x)} \vee A(x)$

$\underline{\neg R(x, y)} \vee \neg A(y) \vee C(x)$

$A \sqsubseteq \exists R.B$ ~~$B \sqsubseteq A$~~ $\exists R.A \sqsubseteq C$

~~$A \sqsubseteq C$~~

$\underline{A(c)}$

$\underline{\neg C(c)}$

$\neg A(x) \vee \underline{\neg A(f(x))} \vee C(x)$

$\neg A(x) \vee \underline{A(f(x))}$

$\underline{\neg A(x)} \vee C(x)$

$\neg A(x) \vee \underline{C(f(x))}$

$\underline{C(c)}$

$\bot$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$

~~$B \sqsubseteq A$~~

$\exists R.A \sqsubseteq C$

$?- A \sqsubseteq C$

- $\neg A(x) \vee R(x, f(x))$
  $\neg A(x) \vee B(f(x))$
- $\neg R(x, y) \vee \neg A(y) \vee C(x)$

$$\frac{A \sqsubseteq \exists R.B \quad \cancel{B \sqsubseteq A} \quad \exists R.A \sqsubseteq C}{\cancel{A \sqsubseteq C}}$$

$A(c)$

$\neg C(c)$

- $\neg A(x) \vee \neg A(f(x)) \vee C(x)$

C.B. vs. Ordered Resolution

Example

$A \sqsubseteq \exists R.B$ ➤ $\neg A(x) \vee R(x, f(x))$

$\neg A(x) \vee B(f(x))$

~~$B \sqsubseteq A$~~

$\exists R.A \sqsubseteq C$ ➤ $\neg R(x,y) \vee \neg A(y) \vee C(x)$

$?- A \sqsubseteq C$

$$\frac{A \sqsubseteq \exists R.B \quad \cancel{B \sqsubseteq A} \quad \exists R.A \sqsubseteq C}{\cancel{A \sqsubseteq C}}$$

$A(c)$

$\neg C(c)$

➤ $\neg A(x) \vee \neg A(f(x)) \vee C(x)$

Every pair of (unrelated) axioms result in a resolution inference:

$A_1 \equiv B_1 \sqcap \exists R.C_1$

$A_2 \equiv B_2 \sqcap \exists R.C_2$

Automata-Based Procedures

Example

$$A \sqsubseteq \exists R.B$$
$$B \sqsubseteq A$$
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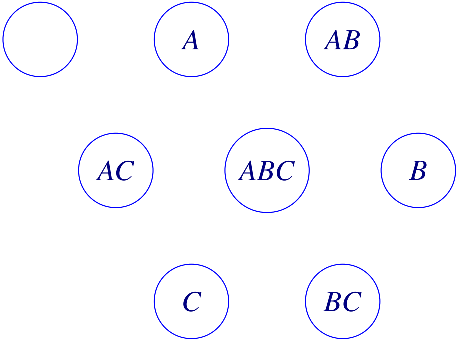

Automata-Based Procedures

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➤ $A \sqsubseteq \exists R.B$

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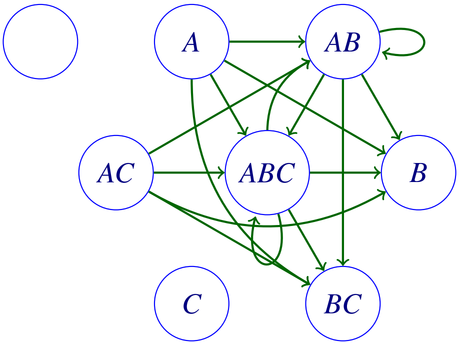

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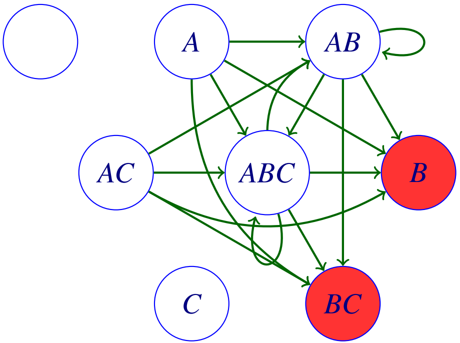

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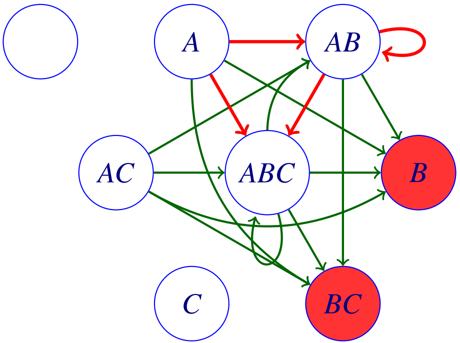

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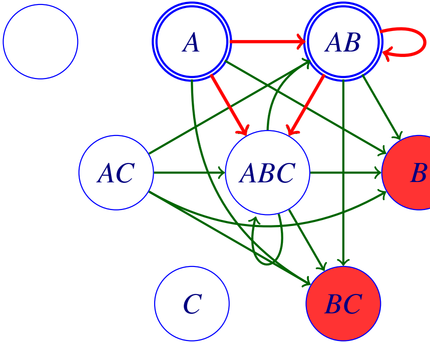

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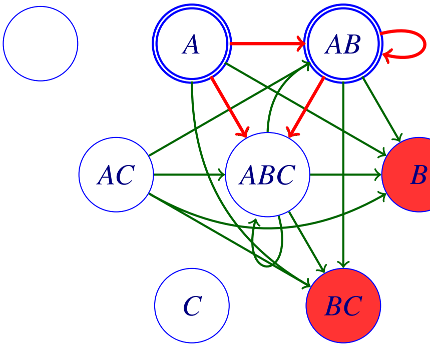

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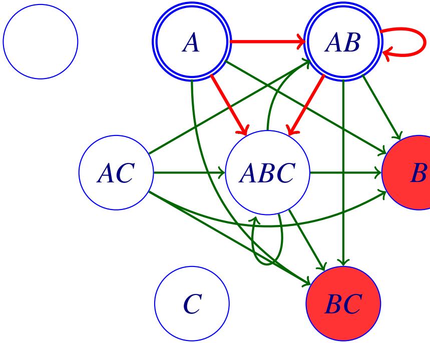

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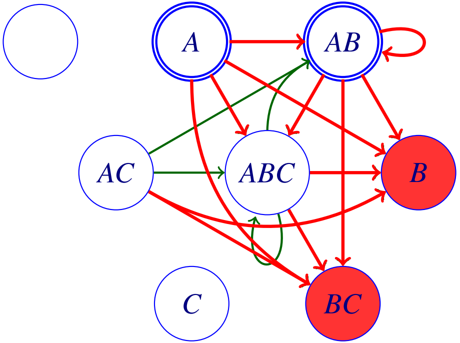

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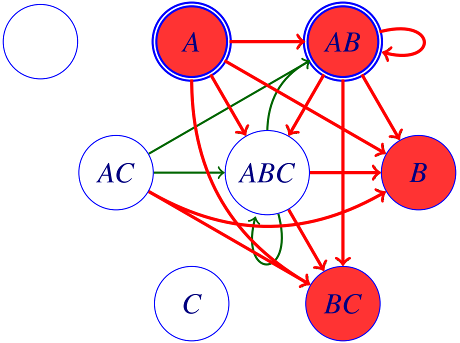

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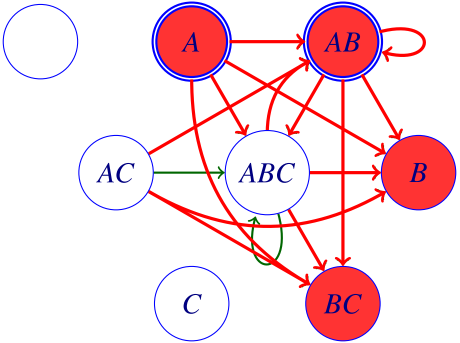

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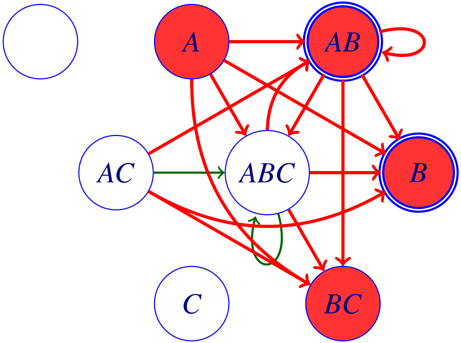

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- Solvable in polynomial time by propagating inconsistent states.

- Note that other subsumption relations can be also determined

Observations

1. Direct implementation is exponential even in the best case:
   - Builds exponentially-many states
   - Symbolic representation (BDDs, ZDDs) can be used to reduce the complexity [Pan, Sattler, Vadi; 2006]

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2. Efficinet implementations are already available:
   - Tableau and hyper-resolution can be seen as bottom-up procedures that search for a run
   - Consequence-based and ordered resolution can be seen as top-down procedures that propagate inconsistent states:

$$\frac{A \sqsubseteq \exists R.B \quad \boxed{B \sqsubseteq C} \quad \exists R.C \sqsubseteq D}{\boxed{A \sqsubseteq D}} \quad \rightsquigarrow \quad \frac{\{B, \neg C\} \text{ is inconsistent}}{\{A, \neg D\} \text{ is inconsistent}}$$

Outline

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- Tableau-based reasoners are catching up:
  - Hyper-tableau procedures reduce non-determinism
  - Smarter blocking: “core blocking”, “speculative blocking”
  - Reducing the number of subsumption tests by finding non-subsumptions from the models

Lessons learned

- What is important:
  - Knowing the input (kinds of constructors, their usage)
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  - Complying with standards:
    not a big deal if nominals are not supported
- Something to consider:
  - Things are not as easy as they may seem
  - Reductions (e.g., to general ATP) don't work well in the end
  - Implementation makes huge difference: profile a lot!

References

- Baader, F., Brandt, S., Lutz, C.: Pushing the EL Envelope. IJCAI 2005: 364-369
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- Glimm, B., Horrocks, I., Motik, B.: Optimized Description Logic Reasoning via Core Blocking. IJCAR 2010.

Thank you for your attension!

The Inference Rules for Horn $\mathcal{SHIQ}$

$$\frac{}{M \sqcap A \sqsubseteq A} \qquad \frac{}{M \sqsubseteq \top} \qquad \frac{M \sqsubseteq A_1 \;\ldots\; M \sqsubseteq A_n}{M \sqsubseteq C} : \prod_{i=1}^{n} A_i \sqsubseteq C \in \mathcal{O}$$

$$\frac{M \sqsubseteq \exists R.N \quad N \sqsubseteq \bot}{M \sqsubseteq \bot} \qquad \frac{M \sqsubseteq \exists R_1.N \quad M \sqsubseteq \forall R_2.A}{M \sqsubseteq \exists R_1.(N \sqcap A)} : R_1 \sqsubseteq_{\mathcal{O}} R_2$$

$$\frac{M \sqsubseteq \exists R_1.N \quad N \sqsubseteq \forall R_2.A}{M \sqsubseteq A} : R_1 \sqsubseteq_{\mathcal{O}} R_2^-$$

$$\frac{\begin{array}{ll} M \sqsubseteq \exists R_1.N_1 & N_1 \sqsubseteq B \\ M \sqsubseteq \exists R_2.N_2 & N_2 \sqsubseteq B \\ M \sqsubseteq \leqslant 1\, S.B & \end{array}}{M \sqsubseteq \exists R_1.(N_1 \sqcap N_2)} : \begin{array}{l} R_1 \sqsubseteq_{\mathcal{O}} S \\ R_2 \sqsubseteq_{\mathcal{O}} S \end{array} \qquad \frac{\begin{array}{ll} M \sqsubseteq \exists R_1.N_1 & M \sqsubseteq B \\ N_1 \sqsubseteq \exists R_2.(N_2 \sqcap A) & \\ N_1 \sqsubseteq \leqslant 1\, S.B & N_2 \sqcap A \sqsubseteq B \end{array}}{M \sqsubseteq A \quad M \sqsubseteq \exists R_2^-.N_1} : \begin{array}{l} R_1 \sqsubseteq_{\mathcal{O}} S^- \\ R_2 \sqsubseteq_{\mathcal{O}} S \end{array}$$

Where $M, N = \prod A_i$