Unchain My $\mathcal{EL}$ Reasoner

Yevgeny Kazakov, Markus Krötzsch, and František Simančík

Department of Computer Science, University of Oxford, UK

Abstract. We study a restriction of the classification procedure for $\mathcal{EL}^{++}$ where the inference rule for complex role inclusion axioms (RIAs) is applied in a "left-linear" way in analogy with the well-known procedure for computing the transitive closure of a binary relation. We introduce a notion of left-admissibility for a set of RIAs, which specifies when a subset of RIAs can be used in a left-linear way without loosing consequences, prove a criterion which can be used to effectively check this property, and describe some preliminary experimental results analyzing when the restricted procedure can give practical improvements.

1 Introduction

The description logic (DL) $\mathcal{EL}$ and its extension $\mathcal{EL}^{++}$ [1] provide the bases of the OWL EL profile of the Web Ontology Language [6] and are distinguished by having tractable worst-case complexity for the standard DL reasoning problems. The nice computational properties of $\mathcal{EL}$-style reasoning procedures such as optimal (polynomial) worst-case complexity and "pay-as-you-go" behavior, are commonly mentioned as main reasons for the improved practical performance of reasoners based on such procedures for large ontologies such as SNOMED CT [2,5,3,8].

Although $\mathcal{EL}^{++}$ admits a polynomial reasoning procedure, different features of $\mathcal{EL}^{++}$ contribute differently to the degree of this polynomial [4]. In particular, the $\mathcal{EL}^{++}$ rule for dealing with complex role inclusion axioms (RIAs) has $O(n^4)$ time complexity, which is higher than for other rules. Even for a single transitivity axiom, the rule can result in $O(n^3)$ inferences. Although complex role inclusion axioms are not used as commonly as other constructors in existing ontologies such as SNOMED CT, this might change in the future as more OWL EL ontologies emerge.

Inspired by an $O(n^2)$ algorithm for computing the transitive closure of a binary relation, in this paper we propose a refinement of the $\mathcal{EL}^{++}$ rule for dealing with complex RIAs. Our main idea is to restrict the rule so that inferences are applied in a *left-linear way*, that is, only a restricted number of the "initial" axioms can be used in all premises of the rule except for the left-most. To this end, we (i) formulate a notion of *left-admissibility* describing subsets of complex RIAs that can be used in a left-linear way without losing consequences, (ii) prove a criterion for left-admissibility that can be checked in polynomial time, and (iii) provide an experimental evaluation measuring the proportion of left-admissible RIAs and reduction in the number of inferences for a selection of commonly-used ontologies.

Table 1. The inference rules for $\mathcal{ELR}$, with N_C the set of all concept names

$$\mathsf{C_0}\ \frac{}{C \sqsubseteq C} : C \in N_C$$

$$\mathsf{C_1}\ \frac{C \sqsubseteq C'}{C \sqsubseteq D} : C' \sqsubseteq D \in \mathrm{KB}$$

$$\mathsf{C_2}\ \frac{(C \sqsubseteq C_i)_{i=1}^n}{C \sqsubseteq D} : \begin{array}{l} n > 1 \\ \bigsqcap_{i=1}^n C_i \sqsubseteq D \in \mathrm{KB} \end{array}$$

$$\mathsf{C_3}\ \frac{C \sqsubseteq C'}{C \sqsubseteq \exists R.D} : C' \sqsubseteq \exists R.D \in \mathrm{KB}$$

$$\mathsf{C_4}\ \frac{C \sqsubseteq \exists R.D \quad D \sqsubseteq D'}{C \sqsubseteq E} : \exists R.D' \sqsubseteq E \in \mathrm{KB}$$

$$\mathsf{C_5}\ \frac{C \sqsubseteq \exists S.D}{C \sqsubseteq \exists R.D} : S \sqsubseteq R \in \mathrm{KB}$$

$$\mathsf{C_6}\ \frac{(C_{i-1} \sqsubseteq \exists R_i.C_i)_{i=1}^n}{C_0 \sqsubseteq \exists R.C_n} : \begin{array}{l} n > 1 \\ R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathrm{KB} \end{array}$$

2 Reasoning in $\mathcal{ELR}$

We first introduce a basic classification calculus for the description logic $\mathcal{ELR}$ that will serve as a baseline for our study. $\mathcal{ELR}$ is the DL that supports only conjunction, existential role restrictions, role hierarchies, and role inclusion axioms, each of which can be used in arbitrary general concept inclusions and role inclusion axioms. We do not require regularity of RBoxes, and $\mathcal{ELR}$ can thus be viewed as a fragment of $\mathcal{EL}^{++}$ without top, bottom, nominals, and concrete domains. We use the following notation for role inclusion axioms.

Definition 1. *A* role chain *ρ is an expression of the form $R_1 \cdot \ldots \cdot R_n$, $n \geq 0$; when $n = 0$ then $\rho = \epsilon$ is the* empty role chain *and when $n \geq 2$ then ρ is a* complex role chain*. We denote by $\rho_1 \cdot \rho_2$ the* concatenation *of two role chains ρ_1 and ρ_2. A* (complex) role inclusion axiom *(short RIA) is an expression of the form $\rho \sqsubseteq R$ where ρ is a non-empty (complex) role chain and R a role. An RBox $\mathcal{R}$ is a finite set of RIAs.*

Table 1 shows the rules of a classification calculus for $\mathcal{ELR}$, obtained by restricting the calculus for $\mathcal{EL}^{++}$ [1]. The input to the rules are axioms from an $\mathcal{ELR}$ knowledge base that have been normalized as in [1]. The main difference is that we treat n-ary conjunctions/role chains in a single application of $\mathsf{C_2}$/$\mathsf{C_6}$, corresponding to the implementation we used for experiments. Each rule of inference consists of a premise, a conclusion, and possible side conditions. The calculus derives axioms of the form $C \sqsubseteq D$ and $C \sqsubseteq \exists R.D$ based on an input knowledge base KB, and it is sound and complete for classification in the sense that an axiom $C \sqsubseteq D$ is entailed by KB if and only if the exhaustive application of the inference rules can be used to derive $C \sqsubseteq D$. This follows immediately from the according result in [1] since it is easy to see that our rules correspond to the inference rules in that paper: $\mathsf{C_0}$ corresponds to the initialization, $\mathsf{C_1}$ to $\mathsf{CR_1}$, $\mathsf{C_2}$ to $\mathsf{CR_2}$, $\mathsf{C_3}$ to $\mathsf{CR_3}$, $\mathsf{C_4}$ to $\mathsf{CR_4}$, $\mathsf{C_5}$ to $\mathsf{CR_{10}}$, and $\mathsf{C_6}$ to $\mathsf{CR_{11}}$.

3 Linear Use of Role Inclusion Axioms

One of the simplest examples of complex RIAs is a transitivity axiom:

$$R \cdot R \sqsubseteq R. \tag{1}$$

Transitivity axioms occur in many ontologies where they are used to express hierarchical relations between concepts, such as "part-of" or "child-of" hierarchies. Let us consider an ontology containing axioms expressing a simple R-hierarchy:

$$A_i \sqsubseteq \exists R.A_{i+1}, \quad 1 \leq i < n. \tag{2}$$

If we apply rule $\mathsf{C_6}$ to these axioms, we derive exactly axioms of the form:

$$A_i \sqsubseteq \exists R.A_j, \quad 1 \leq i < j \leq n, \tag{3}$$

using the following instances of rule $\mathsf{C_6}$:

$$\frac{A_i \sqsubseteq \exists R.A_j \quad A_j \sqsubseteq \exists R.A_k}{A_i \sqsubseteq \exists R.A_k} : \quad 1 \leq i < j < k \leq n. \tag{4}$$

There are exactly $n \cdot (n-1)/2$ possible axioms of the form (3) and there are exactly $n \cdot (n-1) \cdot (n-2)/6$ rule applications in (4). In particular, every axiom in (3) is derived $(n-2)/3$ times in average. Clearly, this demonstrates that rule $\mathsf{C_6}$ can be a source of inefficiency, especially for large n.

The inferences (4) look like the computation of the transitive closure for a binary relation, if we read $C \sqsubseteq \exists R.D$ as $\langle C, D\rangle \in R$. Using this correspondence, we can apply a more efficient algorithm for computing the transitive closure by restricting the second premise in $\mathsf{C_6}$ to the initial axioms only. Specifically, let us use $\sqsubseteq^0$ to distinguish the initial (told) axioms $C \sqsubseteq^0 \exists R.D$ from the axioms $C \sqsubseteq \exists R.D$ that are derived using inference rules. Then one can restrict rule $\mathsf{C_6}$ for transitivity axioms as follows:

$$\frac{C_1 \sqsubseteq \exists R.C_2 \quad C_2 \sqsubseteq^0 \exists R.C_3}{C_1 \sqsubseteq \exists R.C_3} : R \cdot R \sqsubseteq R \in \mathrm{KB}. \tag{5}$$

We will call the rule (5) a *left-linear rule* in analogy with left-linear production rules in context-free grammars because the conclusions of other inferences can be used here only in the left premise. By applying (5) to the input axioms (2) (written using $\sqsubseteq^0$), we obtain inferences of the following form:

$$\frac{A_i \sqsubseteq \exists R.A_j \quad A_j \sqsubseteq^0 \exists R.A_{j+1}}{A_i \sqsubseteq \exists R.A_{j+1}} : \quad 1 \leq i < j < n. \tag{6}$$

Table 2. Left-linear inference rules for $\mathcal{ELR}$

$$\mathsf{L_0}\quad \frac{}{C \sqsubseteq_{\mathcal{L}} C} : C \in N_C$$

$$\mathsf{L_1}\quad \frac{C \sqsubseteq_{\mathcal{L}} C'}{C \sqsubseteq_{\mathcal{L}} D} : C' \sqsubseteq D \in \mathrm{KB}$$

$$\mathsf{L_2}\quad \frac{(C \sqsubseteq_{\mathcal{L}} C_i)_{i=1}^n}{C \sqsubseteq_{\mathcal{L}} D} : \begin{array}{l} n > 1 \\ \prod_{i=1}^n C_i \sqsubseteq D \in \mathrm{KB} \end{array}$$

$$\mathsf{L_3}\quad \frac{C \sqsubseteq_{\mathcal{L}} C'}{C \sqsubseteq_{\mathcal{L}}^0 \exists R.D} : C' \sqsubseteq \exists R.D \in \mathrm{KB}$$

$$\mathsf{L_4}\quad \frac{C \sqsubseteq_{\mathcal{L}} \exists R.D \quad D \sqsubseteq_{\mathcal{L}} D'}{C \sqsubseteq_{\mathcal{L}} E} : \exists R.D' \sqsubseteq E \in \mathrm{KB}$$

$$\mathsf{L_5}\quad \frac{C \sqsubseteq_{\mathcal{L}} \exists S.D}{C \sqsubseteq_{\mathcal{L}} \exists R.D} : S \sqsubseteq R \in \mathcal{R}$$

$$\mathsf{L_5'}\quad \frac{C \sqsubseteq_{\mathcal{L}}^0 \exists S.D}{C \sqsubseteq_{\mathcal{L}}^0 \exists R.D} : S \sqsubseteq R \in \mathcal{R}$$

$$\mathsf{L_6}\quad \frac{(C_{i-1} \sqsubseteq_{\mathcal{L}} \exists R_i.C_i)_{i=1}^n}{C_0 \sqsubseteq_{\mathcal{L}} \exists R.C_n} : \begin{array}{l} n > 1 \\ R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R} \setminus \mathcal{L} \end{array}$$

$$\mathsf{L_6'}\quad \frac{C_0 \sqsubseteq_{\mathcal{L}} \exists R_1.C_1 \quad (C_{i-1} \sqsubseteq_{\mathcal{L}}^0 \exists R_i.C_i)_{i=2}^n}{C_0 \sqsubseteq_{\mathcal{L}} \exists R.C_n} : \begin{array}{l} n > 1 \\ R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{L} \end{array}$$

It is easy to see that there are exactly $(n-1)\cdot(n-2)/2$ rule applications in (6) producing exactly those axioms in (3) that are not in (2), and that every such axiom is derived exactly once. Clearly, this strategy represents an improvement over the application of the (unrestricted) rule $\mathsf{C_6}$.

We use the idea above to formulate a calculus for $\mathcal{ELR}$ with a restricted version of rule $\mathsf{C_6}$. In order to do that, we need to specify where the "initial" axioms $C \sqsubseteq^0 \exists R.D$ come from. Clearly, we cannot take such axioms just from the knowledge base, since otherwise, e.g., we would not be able to derive $A \sqsubseteq \exists R.D$ for KB consisting of $A \sqsubseteq \exists R.B$, $B \sqsubseteq C$, $C \sqsubseteq \exists R.D$, and $R \cdot R \sqsubseteq R$, as we cannot avoid using $B \sqsubseteq \exists R.D$ in the second premise of $\mathsf{C_6}$. Similarly, we need to allow initial axioms to be produced by $\mathsf{C_5}$, since otherwise $A \sqsubseteq \exists R.C$ cannot be derived for KB consisting of $A \sqsubseteq \exists R.B$, $B \sqsubseteq \exists S.C$, $S \sqsubseteq R$, and $R \cdot R \sqsubseteq R$.

The new calculus for $\mathcal{ELR}$ is formulated in Table 2. The calculus is parametrized with a distinguished subset $\mathcal{L} \subseteq \mathcal{R}$ of complex RIAs. The RIAs in $\mathcal{L}$ can, similar to the transitivity axiom in the example above, only be used in a left-linear version $\mathsf{L_6'}$ of rule $\mathsf{C_6}$. The remaining axioms from $\mathcal{R} \setminus \mathcal{L}$ can be used without restrictions in rule $\mathsf{L_6}$. The *initial axioms* of the form $C \sqsubseteq_{\mathcal{L}}^0 \exists R.D$ are produced by rules $\mathsf{L_3}$ and $\mathsf{L_5'}$. We use $\mathcal{L}$ in the subscripts of $\sqsubseteq_{\mathcal{L}}$ and $\sqsubseteq_{\mathcal{L}}^0$ to emphasize that these relations depend on $\mathcal{L}$. We implicitly assume that $\sqsubseteq_{\mathcal{L}}^0 \subseteq \sqsubseteq_{\mathcal{L}}$; in particular, axioms of the form $C \sqsubseteq_{\mathcal{L}}^0 \exists R.D$ can also be used as premises of rules $\mathsf{L_4}$ and $\mathsf{L_6}$ and as the first premise of rule $\mathsf{L_6'}$. Note that if $\mathcal{L} = \emptyset$, our new calculus coincides with the original calculus for $\mathcal{ELR}$ (ignoring the distinction

Table 3. Left-linear composition of roles

E₀	$\dfrac{}{R \sqsubseteq^0_{\mathcal{L}} R}$	$: R \in N_R$
E₁	$\dfrac{\rho \sqsubseteq_{\mathcal{L}} S}{\rho \sqsubseteq_{\mathcal{L}} R}$	$: S \sqsubseteq R \in \mathcal{R}$
E′₁	$\dfrac{T \sqsubseteq^0_{\mathcal{L}} S}{T \sqsubseteq^0_{\mathcal{L}} R}$	$: S \sqsubseteq R \in \mathcal{R}$
E₂	$\dfrac{(\rho_i \sqsubseteq_{\mathcal{L}} R_i)_{i=1}^n}{\rho_1 \cdot \ldots \cdot \rho_n \sqsubseteq_{\mathcal{L}} R}$	$: \begin{array}{l} n > 1 \\ R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R} \setminus \mathcal{L} \end{array}$
E′₂	$\dfrac{\rho_1 \sqsubseteq_{\mathcal{L}} R \quad (T_i \sqsubseteq^0_{\mathcal{L}} R_i)_{i=2}^n}{\rho_1 \cdot T_2 \cdot \ldots \cdot T_n \sqsubseteq_{\mathcal{L}} R}$	$: \begin{array}{l} n > 1 \\ R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{L} \end{array}$

between $\sqsubseteq^0_{\mathcal{L}}$ and $\sqsubseteq_{\mathcal{L}}$). Clearly, the larger $\mathcal{L}$ is, the more restricted our rules are, and so the less inferences are possible. Thus, in the remainder of the paper we are concerned with the problem of finding subsets $\mathcal{L}$ of a given $\mathcal{R}$ which do not result in lost consequences relative to the original calculus in Table 1.

4 Left-Admissible Role Inclusion Axioms

In this section we are concerned with the problem of how to determine, given a set of complex RIAs $\mathcal{L} \subseteq \mathcal{R}$, whether the calculus in Table 2 produces the same consequences as the original calculus in Table 1. In order to study the properties of the calculus in Table 2 for different subsets $\mathcal{L}$ of $\mathcal{R}$, consider the smallest relations $\sqsubseteq^0_{\mathcal{L}} \subseteq \sqsubseteq_{\mathcal{L}}$ on role chains satisfying the properties in Table 3. Note the similarities between the rules in Table 3 and rules in Table 2. Also note that unlike the derivation relation $\sqsubseteq^0_{\mathcal{L}}$ in Table 2, the relation $\sqsubseteq^0_{\mathcal{L}}$ in Table 3 does not depend on $\mathcal{L}$ and coincides with the closure of the role hierarchy. The following lemma can be easily proved using the correspondence between the rules **L₃**, **L₅**, **L′₅**, **L₆** and **L′₆** and the rules **E₀**, **E₁**, **E′₁**, **E₂** and **E′₂**.

Lemma 1. *For every subset $\mathcal{L} \subseteq \mathcal{R}$ of complex RIAs, all concepts A and B, and every role R, the following two conditions are equivalent:*

(i) $A \sqsubseteq_{\mathcal{L}} \exists R.B$.
(ii) *There exist $C_0, \ldots, C_n$ and $R_1 \cdot \ldots \cdot R_n \sqsubseteq_{\mathcal{L}} R$ such that $A = C_0$, $B = C_n$, and $C_{i-1} \sqsubseteq^0_{\mathcal{L}} \exists R_i.C_i$ $(1 \le i \le n)$.*

The following properties can be proved by induction on $S \sqsubseteq^0_{\mathcal{L}} T$:

$$\text{if } R \sqsubseteq^0_{\mathcal{L}} S \text{ and } S \sqsubseteq^0_{\mathcal{L}} T, \text{ then } R \sqsubseteq^0_{\mathcal{L}} T, \tag{7}$$
$$\text{if } \rho \sqsubseteq_{\mathcal{L}} S \text{ and } S \sqsubseteq^0_{\mathcal{L}} T, \text{ then } \rho \sqsubseteq_{\mathcal{L}} T. \tag{8}$$

The necessary and sufficient condition on $\mathcal{L} \subseteq \mathcal{R}$ that guarantees that our new calculus for $\mathcal{ELR}$ derives the same consequences as the original calculus, can now be defined as follows:

Definition 2. *A set of complex RIAs $\mathcal{L} \subseteq \mathcal{R}$ is* left-admissible *for $\mathcal{R}$ if the following condition holds:*

$$\text{if } (\rho_i \sqsubseteq_{\mathcal{L}} R_i)_{i=1}^n \text{ and } R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}, \text{ then } \rho_1 \cdot \ldots \cdot \rho_n \sqsubseteq_{\mathcal{L}} R. \quad (9)$$

Intuitively, $\mathcal{L} \subseteq \mathcal{R}$ is *left-admissible* if the relation $\sqsubseteq_{\mathcal{L}}$ is closed under the unrestricted version of the rule $\mathbf{E_2}$ when RIAs $R_1 \cdot \ldots \cdot R_n \sqsubseteq R$ can be taken not only from $\mathcal{R} \setminus \mathcal{L}$, but from the whole $\mathcal{R}$ (note the similarity of (9) and $\mathbf{E_2}$). Left-admissibility thus ensures that the relation $\sqsubseteq_{\mathcal{L}}$ coincides with the unrestricted relation $\sqsubseteq_{\emptyset}$, i.e., the relation for $\mathcal{L} = \emptyset$.

Example 1. Consider $\mathcal{R}$ consisting of the axiom

$$\mathsf{isPartOf} \cdot \mathsf{isProperPartOf} \sqsubseteq \mathsf{isProperPartOf}. \quad (10)$$

It is easy to show that the following relations hold for any (of the two) $\mathcal{L} \subseteq \mathcal{R}$:

$$\mathsf{isPartOf} \sqsubseteq^{o}_{\mathcal{L}} \mathsf{isPartOf} \qquad (\text{by } \mathbf{E_0}), \quad (11)$$

$$\mathsf{isProperPartOf} \sqsubseteq^{o}_{\mathcal{L}} \mathsf{isProperPartOf} \qquad (\text{by } \mathbf{E_0}), \quad (12)$$

$$\mathsf{isPartOf} \cdot \mathsf{isProperPartOf} \sqsubseteq_{\mathcal{L}} \mathsf{isProperPartOf} \qquad (\text{by } \mathbf{E_2} \text{ or } \mathbf{E_2'}). \quad (13)$$

The following relation, however, holds only for $\mathcal{L} = \emptyset$:

$$\mathsf{isPartOf} \cdot \mathsf{isPartOf} \cdot \mathsf{isProperPartOf} \sqsubseteq_{\mathcal{L}} \mathsf{isProperPartOf} \qquad (\text{by } \mathbf{E_2}). \quad (14)$$

When $\mathcal{L} = \mathcal{R}$, one cannot use $\mathbf{E_2'}$ to produce (14) from (11) and (13) using (10). Therefore $\mathcal{L} = \mathcal{R}$ is not left-admissible for $\mathcal{R}$ according to Definition 2.

Now suppose $\mathcal{R}$ is extended with the transitivity axiom

$$\mathsf{isPartOf} \cdot \mathsf{isPartOf} \sqsubseteq \mathsf{isPartOf}. \quad (15)$$

Then, similarly, for any $\mathcal{L} \subseteq \mathcal{R}$ we have

$$\mathsf{isPartOf} \cdot \mathsf{isPartOf} \sqsubseteq_{\mathcal{L}} \mathsf{isPartOf} \qquad (\text{by } \mathbf{E_2} \text{ or } \mathbf{E_2'}). \quad (16)$$

And now (14) can be produced from (16) and (12) using (10) for any $\mathcal{L} \subseteq \mathcal{R}$. In fact, one can show that any $\mathcal{L} \subseteq \mathcal{R}$ will be left-admissible for the extended $\mathcal{R}$.

Theorem 1. *Let* KB *be a knowledge base with RBox $\mathcal{R}$, and $\mathcal{L} \subseteq \mathcal{R}$ be left-admissible for $\mathcal{R}$. Then $C \sqsubseteq D$ is derivable by rules in Table 1 using* KB *iff $C \sqsubseteq_{\mathcal{L}} D$ is derivable by rules in Table 2 using* KB.

Proof. The "if" direction of the theorem is straightforward since each rule in Table 2 is a restriction of a corresponding rule in Table 1.

To prove the "only if" direction, assume to the contrary that there exists $C \sqsubseteq D$ derivable by rules in Table 1 such that $C \not\sqsubseteq_{\mathcal{L}} D$. Without loss of generality, $C \sqsubseteq D$ is produced by some rule in Table 1 from some premises $C_i \sqsubseteq D_i$, $0 \le i < n$, $n \ge 0$, such that $C_i \sqsubseteq_{\mathcal{L}} D_i$. We obtain contradiction by considering all possible cases for such a rule.

The only non-trivial case is an application of the rule $\mathbf{C_6}$ since all other rules have a direct counterpart in Table 2. In this case $D_{i-1} = \exists R_i.C_i$ $(1 \leq i \leq n)$, $C = C_0$, $D = \exists R.C_n$ and $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}$. By case $(i) \Rightarrow (ii)$ of Lemma 1 applied to each $C_{i-1} \sqsubseteq_{\mathcal{L}} \exists R_i.C_i$, there exist $C_i^{j-1} \sqsubseteq^0_{\mathcal{L}} \exists R_i^j.C_i^j$ $(1 \leq j \leq m_i)$ such that $C_i^0 = C_{i-1}$, $C_i^{m_i} = C_i$ and $R_i^1 \cdot \ldots \cdot R_i^{m_i} \sqsubseteq_{\mathcal{L}} R_i$. Define $\rho_i := R_i^1 \cdot \ldots \cdot R_i^{m_i}$. Since $\rho_i \sqsubseteq_{\mathcal{L}} R_i$ $(1 \leq i \leq n)$, $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}$, and $\mathcal{L}$ is left-admissible, we have $\rho_i \cdot \ldots \cdot \rho_n \sqsubseteq_{\mathcal{L}} R$. By case $(i) \Leftarrow (ii)$ of Lemma 1 for $((C_i^{j-1} \sqsubseteq^0_{\mathcal{L}} \exists R_i^j.C_i^j)_{j=1}^{m_i})_{i=1}^n$ using $\rho_i \cdot \ldots \cdot \rho_n \sqsubseteq_{\mathcal{L}} R$ we obtain $C = C_0 = C_1^0 \sqsubseteq_{\mathcal{L}} \exists R.C_n^{m_n} = \exists R.C_n = D$, which contradicts $C \not\sqsubseteq_{\mathcal{L}} D$. This proves the theorem. □

5 Recognizing Left-Admissibility

It is difficult in general to verify the conditions of left-admissibility formulated in Definition 2 since this requires checking property (9) for a potentially infinite number of role chains ρ_i. In this section we give an equivalent formulation for left-admissibility, which can be checked in polynomial time.

We start with the following sufficient condition for left-admissibility:

Lemma 2. *Let $\mathcal{L} \subseteq \mathcal{R}$ be sets of complex RIAs that satisfy the property:*

$$\textit{if } \rho \sqsubseteq S \in \mathcal{R} \textit{ and } \rho_1 \cdot S \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R, \textit{ then } \rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R. \tag{17}$$

Then $\mathcal{L}$ is left-admissible for $\mathcal{R}$.

Proof. We first show that (17) implies the following stronger property:

$$\text{if } \rho \sqsubseteq_{\mathcal{L}} S \text{ and } \rho_1 \cdot S \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R, \text{ then } \rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R. \tag{18}$$

The proof of (18) is by induction on the derivation of $\rho \sqsubseteq_{\mathcal{L}} S$. In the base case $\mathbf{E_0}$ we have $\rho = S$ and the claim $\rho_1 \cdot S \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$ is part of the precondition. In all other cases $\mathbf{E_1}$–$\mathbf{E_2'}$ there exist $(\sigma_i \sqsubseteq_{\mathcal{L}} S_i)_{i=1}^n$, $n \geq 1$ such that $\rho = \sigma_1 \cdot \ldots \cdot \sigma_n$ and $S_1 \cdot \ldots \cdot S_n \sqsubseteq S \in \mathcal{R}$. By (17) $\rho_1 \cdot S_1 \cdot \ldots \cdot S_n \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$. Now use the induction hypothesis (18) for each $\sigma_i \sqsubseteq_{\mathcal{L}} S_i$ to iteratively expand the left-hand side of $\rho_1 \cdot S_1 \cdot \ldots \cdot S_n \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$ to obtain the claim $\rho_1 \cdot \rho \cdot \rho_2 = \rho_1 \cdot \sigma_1 \cdot \ldots \cdot \sigma_n \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$.

To finish the proof of the lemma, we now show that (18) implies (9). To this end, consider any $(\rho_i \sqsubseteq_{\mathcal{L}} R_i)_{i=1}^n$ and $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}$. We must prove that $\rho_1 \cdot \ldots \cdot \rho_n \sqsubseteq_{\mathcal{L}} R$. For this, note that $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R}$ implies $R_1 \cdot \ldots \cdot R_n \sqsubseteq_{\mathcal{L}} R$, and use (18) for each $\rho_i \sqsubseteq_{\mathcal{L}} R_i$ to iteratively expand the left-hand side of $R_1 \cdot \ldots \cdot R_n \sqsubseteq_{\mathcal{L}} R$ to obtain the desired $\rho_1 \cdot \ldots \cdot \rho_n \sqsubseteq_{\mathcal{L}} R$. □

The following lemma formulates some useful closure properties of the relation $\sqsubseteq_{\mathcal{L}}$, which hold for arbitrary $\mathcal{L}$:

Lemma 3. *Let $\mathcal{L} \subseteq \mathcal{R}$ be sets of RIAs. If $\rho_1 \sqsubseteq_{\mathcal{L}} S_1$, $(T_i \sqsubseteq^0_{\mathcal{L}} S_i)_{i=2}^n$, $n \geq 1$, and $S_1 \cdot \ldots \cdot S_n \sqsubseteq_{\mathcal{L}} R$, then $\rho_1 \cdot T_2 \cdot \ldots \cdot T_n \sqsubseteq_{\mathcal{L}} R$.*

Proof. Let $\rho = \rho_1 \cdot T_2 \cdot \ldots \cdot T_n$. We will show $\rho \sqsubseteq_{\mathcal{L}} R$ by induction on the derivation of $S_1 \cdot \ldots \cdot S_n \sqsubseteq_{\mathcal{L}} R$.

E$_0$: $n = 1$ and $S_1 = R$. The claim $\rho_1 \sqsubseteq_{\mathcal{L}} R$ is part of the precondition.

E$_1$: $S_1 \cdot \ldots \cdot S_n \sqsubseteq_{\mathcal{L}} S$ and $S \sqsubseteq R \in \mathcal{R}$. By the induction hypothesis $\rho \sqsubseteq_{\mathcal{L}} S$ from which $\rho \sqsubseteq_{\mathcal{L}} R$ follows by **E$_1$**.

E$'_1$: Analogous to the case of **E$_1$**.

E$_2$: $S_1 \cdot \ldots \cdot S_n = \sigma_1 \cdot \ldots \cdot \sigma_m$, $m > 1$, $(\sigma_i \sqsubseteq_{\mathcal{L}} R_i)_{i=1}^m$ and $R_1 \cdot \ldots \cdot R_m \sqsubseteq R \in \mathcal{R} \setminus \mathcal{L}$. Let s_i and e_i be the start and the end indices of σ_i in $S_1 \cdot \ldots \cdot S_n$. By the induction hypothesis $\rho_1 \cdot T_2 \cdot \ldots \cdot T_{e_1} \sqsubseteq_{\mathcal{L}} R_1$ and $(T_{s_i} \cdot \ldots \cdot T_{e_i} \sqsubseteq_{\mathcal{L}} R_i)_{i=2}^m$, from which $\rho \sqsubseteq_{\mathcal{L}} R$ follows by **E$_2$**.

E$'_2$: $S_1 \cdot \ldots \cdot S_k \sqsubseteq_{\mathcal{L}} R_1$, $k \geq 1$, $(S_{i+k-1} \sqsubseteq^0_{\mathcal{L}} R_i)_{i=2}^m$, $m > 1$, $k + m = n + 1$ and $R_1 \cdot \ldots \cdot R_m \sqsubseteq R \in \mathcal{L}$. By the induction hypothesis $\rho_1 \cdot T_2 \cdot \ldots \cdot T_k \sqsubseteq_{\mathcal{L}} R_1$. By (7) we have $(T_{i+k-1} \sqsubseteq^0_{\mathcal{L}} R_i)_{i=2}^m$. Then $\rho \sqsubseteq_{\mathcal{L}} R$ follows by **E$'_2$**. □

We are now ready to formulate our main criterion for left-admissibility:

Theorem 2. *A subset $\mathcal{L} \subseteq \mathcal{R}$ of complex RIAs is left-admissible for an RBox $\mathcal{R}$ if and only if the following property holds:*

$$\text{if } \rho \sqsubseteq S_1 \in \mathcal{R},\ S_1 \sqsubseteq^0_{\mathcal{L}} S_2 \text{ and } \rho_1 \cdot S_2 \cdot \rho_2 \sqsubseteq R \in \mathcal{L}, \text{ then } \rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R. \quad (19)$$

Proof. The "only if" direction of the theorem can be easily shown using Definition 2 since $\rho \sqsubseteq S_1 \in \mathcal{R}$ and $S_1 \sqsubseteq^0_{\mathcal{L}} S_2$ imply $\rho \sqsubseteq_{\mathcal{L}} S_2$.

To show the "if" direction, we first prove the following strengthening of (19):

$$\text{if } \rho \sqsubseteq S_1 \in \mathcal{R},\ S_1 \sqsubseteq^0_{\mathcal{L}} S_2 \text{ and } \rho_1 \cdot S_2 \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R, \text{ then } \rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R. \quad (20)$$

The proof of (20) is by induction on the derivation of $\rho_1 \cdot S_2 \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$.

E$_0$: $S_2 = R$ and $\rho_1 = \rho_2 = \epsilon$. Then (20) follows from (8).

E$_1$: $\rho_1 \cdot S_2 \cdot \rho_2 \sqsubseteq_{\mathcal{L}} S$ and $S \sqsubseteq R \in \mathcal{R}$. By the induction hypothesis $\rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} S$, from which $\rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$ follows by **E$_1$**.

E$'_1$: Analogous to the case of **E$_1$**.

E$_2$: $\rho_1 \cdot S_2 \cdot \rho_2 = \sigma_1 \cdot \ldots \cdot \sigma_n$, $(\sigma_i \sqsubseteq_{\mathcal{L}} R_i)_{i=1}^n$ and $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{R} \setminus \mathcal{L}$. Let k be such that S_2 occurs in σ_k, that is $\sigma_1 \cdot \ldots \cdot \sigma_{k-1} = \rho'_1$, $\sigma_k = \rho''_1 \cdot S_2 \cdot \rho''_2$, $\sigma_{k+1} \cdot \ldots \cdot \sigma_n = \rho'_2$ and $\rho_1 = \rho'_1 \cdot \rho''_1$, $\rho_2 = \rho''_2 \cdot \rho'_2$. By the induction hypothesis $\rho''_1 \cdot \rho \cdot \rho''_2 \sqsubseteq_{\mathcal{L}} R_k$, from which $\rho_1 \cdot \rho \cdot \rho_2 = \sigma_1 \cdot \ldots \cdot \sigma_{k-1} \cdot \rho''_1 \cdot \rho \cdot \rho''_2 \cdot \sigma_{k+1} \cdot \ldots \cdot \sigma_n \sqsubseteq_{\mathcal{L}} R$ follows by **E$_2$**.

E$'_2$: $\rho_1 \cdot S_2 \cdot \rho_2 = \sigma_1 \cdot T_2 \cdot \ldots \cdot T_n$, $\sigma_1 \sqsubseteq_{\mathcal{L}} R_1$, $(T_i \sqsubseteq^0_{\mathcal{L}} R_i)_{i=2}^n$ and $R_1 \cdot \ldots \cdot R_n \sqsubseteq R \in \mathcal{L}$. If S_2 occurs in σ_1, then this is analogous to the case of **E$_2$**. Otherwise, let k be such that $\sigma_1 \cdot T_2 \cdot \ldots \cdot T_{k-1} = \rho_1$, $T_k = S_2$ and $T_{k+1} \cdot \ldots \cdot T_n = \rho_2$. By (7) $S_1 \sqsubseteq^0_{\mathcal{L}} R_k$. By (19) applied to $\rho \sqsubseteq S_1 \in \mathcal{R}$, $S_1 \sqsubseteq^0_{\mathcal{L}} R_k$ and $R_1 \cdot \ldots \cdot R_n \sqsubseteq_{\mathcal{L}} R$ we obtain $R_1 \cdot \ldots \cdot R_{k-1} \cdot \rho \cdot R_{k+1} \cdot \ldots \cdot R_n \sqsubseteq_{\mathcal{L}} R$, from which $\rho_1 \cdot \rho \cdot \rho_2 = \sigma_1 \cdot T_2 \cdot \ldots \cdot T_{k-1} \cdot \rho \cdot T_{k+1} \cdot \ldots \cdot T_n \sqsubseteq_{\mathcal{L}} R$ follows by Lemma 3.

Having proved (20), condition (17) now follows by taking $S_1 = S_2 = S$ in (20) and using rule **E$_0$** to derive $S \sqsubseteq^0_{\mathcal{L}} S$. Therefore $\mathcal{L}$ is left-admissible for $\mathcal{R}$. □

Condition (19) in Theorem 2 can be checked in polynomial time in the size of $\mathcal{R}$. Indeed, there are only polynomially many possible instances of the precondition in (19). For every such precondition, the property $\rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$ can be checked in polynomial time by, e.g., applying Lemma 1: $\rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$ holds iff $C_0 \sqsubseteq_{\mathcal{L}} \exists R.C_n$ is derivable from $(C_{i-1} \sqsubseteq^0_{\mathcal{L}} \exists R_i.C_i)_{i=1}^n$, where $R_1 \cdot \ldots \cdot R_n = \rho_1 \cdot \rho \cdot \rho_2$.

Theorem 2 can help checking if a given set $\mathcal{L}$ is left-admissible, but does not explain how to find such a set without exhaustively checking al possible subsets of $\mathcal{R}$. The following sufficient condition will help us quickly find a suitable left-admissible set of RIAs in practice:

Theorem 3. *For a set of RIAs $\mathcal{R}$ let $\mathcal{L}(\mathcal{R})$ be the set of exactly those complex RIAs $\sigma \sqsubseteq R \in \mathcal{R}$ that satisfy the following condition for all ρ, ρ_1, ρ_2, S_1, S_2:*

$$\text{if } \rho \sqsubseteq S_1 \in \mathcal{R},\ S_1 \sqsubseteq^0_{\mathcal{R}} S_2 \text{ and } \rho_1 \cdot S_2 \cdot \rho_2 = \sigma, \text{ then } \rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{R}} R. \quad (21)$$

Then $\mathcal{L}(\mathcal{R})$ is left-admissible for $\mathcal{R}$.

Proof. Note that the relations $\sqsubseteq_{\mathcal{L}}$ are anti-monotonic in $\mathcal{L}$, that is for $\mathcal{L}_1 \subseteq \mathcal{L}_2$ we have $\sqsubseteq_{\mathcal{L}_1} \supseteq \sqsubseteq_{\mathcal{L}_2}$. Let $\mathcal{L} = \mathcal{L}(\mathcal{R})$. Since $\mathcal{L} \subseteq \mathcal{R}$, we have $\sqsubseteq_{\mathcal{L}} \supseteq \sqsubseteq_{\mathcal{R}}$, and, since $\sqsubseteq^0_{\mathcal{L}}$ does not depend on $\mathcal{L}$, we have $\sqsubseteq^0_{\mathcal{L}} = \sqsubseteq^0_{\mathcal{R}}$. Now it is easy to show that $\mathcal{L}$ satisfies (19): Suppose $\rho \sqsubseteq S_1 \in \mathcal{R}$, $S_1 \sqsubseteq^0_{\mathcal{L}} S_2$ and $\rho_1 \cdot S_2 \cdot \rho_2 \sqsubseteq R \in \mathcal{L}$. Then $\sqsubseteq^0_{\mathcal{L}} = \sqsubseteq^0_{\mathcal{R}}$ implies $S_1 \sqsubseteq^0_{\mathcal{R}} S_2$, so $\rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{R}} R$ by (21). Then $\sqsubseteq_{\mathcal{L}} \supseteq \sqsubseteq_{\mathcal{R}}$ implies $\rho_1 \cdot \rho \cdot \rho_2 \sqsubseteq_{\mathcal{L}} R$, so (19) holds. Therefore $\mathcal{L} = \mathcal{L}(\mathcal{R})$ is left-admissible for $\mathcal{R}$ by Theorem 2. $\square$

6 Experimental Evaluation

In this section we present the results of an experimental comparison of applying the calculi in Sections 2 and 3 to several commonly considered $\mathcal{EL}$ ontologies that contain complex RIAs, and discuss whether and to which extent our optimized treatment of RIAs can improve the performance of reasoning in practice.

To evaluate the proposed algorithms, we have implemented the calculi described in Sections 2 and 3 in a prototype Java-based reasoner ELK.[1] All experiments were conducted using Java 1.6 on a 2.5 GHz quad core CPU with 4GB RAM running Fedora 13 Linux.

Our test ontology suite includes GO,[2] FMA-lite,[3] and an OWL EL version of GALEN.[4] These ontologies contain only (left-admissible) RIAs of the form $R \sqsubseteq S$ and $R \cdot R \sqsubseteq R$. In order to test which proportion of complex RIAs in realistic ontologies is left-admissible, we additionally considered the two latest versions of GALEN,[5] namely GALEN7 and GALEN8, which contain RIAs of the form $R \sqsubseteq S$, $R \cdot S \sqsubseteq R$, and $S \cdot R \sqsubseteq R$. We reduced these ontologies to $\mathcal{ELR}$ by removing all axioms for role functionalities and role inverses and replacing all datatypes by fresh atomic concepts. It is worth noting that the RIAs in GALEN7 and GALEN8 do not satisfy the regularity restrictions of OWL 2 [7]

[1] http://code.google.com/p/elk-reasoner/
[2] obtained from http://lat.inf.tu-dresden.de/~meng/toyont.html
[3] obtained from http://www.bioontology.org/wiki/index.php/FMAInOwl
[4] obtained from http://condor-reasoner.googlecode.com/
[5] obtained from http://www.opengalen.org/sources/sources.html

Table 4. Ontology metrics and experimental results

	GO	FMA-lite	OWL GALEN	GALEN7	GALEN8
Number of normalized input axioms					
$A \sqsubseteq B$	28,896	121,708	71,366	92,749	588,806
$\bigsqcap_{i=1}^{n} A_i \sqsubseteq B, n \geq 2$	0	0	11,561	12,097	122,527
$A \sqsubseteq \exists R.B$	1,796	12,355	14,115	15,105	106,065
$\exists R.A \sqsubseteq B$	0	0	7,549	7,973	93,241
$R \sqsubseteq S$	0	3	958	972	996
$R_1 \cdot R_2 \sqsubseteq R_3 \in \mathcal{R}$	1	1	58	385	385
$R_1 \cdot R_2 \sqsubseteq R_3 \in \mathcal{L}$	1	1	58	183	183
Number of derived axioms					
$A \sqsubseteq B$	206,205	1,035,527	1,119,636	1,770,895	11,462,383
$A \sqsubseteq \exists R.B$	33,985	867,209	2,282,471	3,299,376	24,998,147
Number of rule applications					
$\mathbf{C_0}/\mathbf{L_0}$	19,468	78,977	25,963	30,534	202,664
$\mathbf{C_1}/\mathbf{L_1}$	241,834	958,754	1,396,379	2,917,625	15,900,712
$\mathbf{C_2}/\mathbf{L_2}$	0	0	259,654	372,780	2,639,688
$\mathbf{C_3}/\mathbf{L_3}$	21,994	114,724	339,880	446,980	3,780,076
$\mathbf{C_4}/\mathbf{L_4}$	0	0	949,148	1,316,768	17,217,292
$\mathbf{C_5}/\mathbf{L_5}+\mathbf{L_5'}$	0	96,891	2,023,828	2,903,821	21,988,137
$\mathbf{C_6}$	34,753	5,807,992	275,248	1,264,208	11,867,857
$\mathbf{L_6}+\mathbf{L_6'}$	19,756	1,186,733	216,982	1,087,328	8,728,711

and, for this reason, no OWL reasoner can handle them in the unreduced form. We have excluded SNOMED CT from our experiments for the reason that the only complex RIA it contains is redundant for classification in the sense that rule $\mathbf{C_6}$ is never applied on this ontology.

In our experiments, we first normalized all test ontologies using structural transformation, and applied Theorem 3 to identify left-admissible sets of RIAs. Table 4 presents statistics on the number of axioms of each type and the number of left-admissible RIAs for each of the tested ontologies. For GALEN7 and GALEN8, which contain identical complex RIAs, we found a left-admissible subset containing 183 out of the total 385 complex RIAs. In this case, we additionally checked that adding any one of the remaining complex RIAs to the previously found 183 violates the conditions of Theorem 2, showing that the left-admissible subset of RIAs we found is maximal. For the remaining ontologies, the full set of RIAs is left-admissible since transitivity axioms are the only kind of complex RIA. In general, the computation of left-admissible RIAs had no relevant impact on overall performance, running in less than 0.5 seconds in all cases.

For each of the tested ontologies, we computed the saturation under the inference rules of Table 1 and Table 2. Table 4 presents the total number of different conclusions of each type, and the total number of inferences for each rule. In accordance with Theorem 1, both approaches produce the same conclusions. For this reason, the number of applications of each rule $\mathbf{C_0}$–$\mathbf{C_5}$ coincides with the corresponding number for rules $\mathsf{L_0}$–$\mathsf{L_5'}$. Differences between the two approaches are found in the number of applications of $\mathsf{C_6}$ on the one hand, and the combined number of applications of $\mathsf{L_6}$ and $\mathsf{L_6'}$ on the other hand. As can be seen from the results, the effect of our optimization strongly depends on the input ontology, with the largest relative reductions obtained for FMA-lite and GO, and less significant reductions for all versions of GALEN.

Although the reduction in the number of rule application is significant for FMA-lite and GO, this, surprisingly, did not translate to a significant reduction in the running time for our prototype implementation. For FMA-lite, for example, the running time is reduced just from 7.2 to 6.1 seconds (15.3%), which is less than expected for more than 65% reduction in the number of inferences. For other ontologies the reduction in the running time was even less measurable.

One possible explanation for this effect is that a rule application producing a new consequence costs more than a rule application producing a previously derived consequence because the first requires a (relatively expensive) memory allocation. Since our optimized procedure derives exactly the same conclusions, it reduces only the number of inferences of the second kind. Nevertheless, our optimization can give improvement in some cases and should not be difficult to implement (at least for transitivity) in any reasoner based on the original $\mathcal{EL}$ calculus [1], such as in CEL/jCEL [2] or Snorocket [5].

References

1. Baader, F., Brandt, S., Lutz, C.: Pushing the $\mathcal{EL}$ envelope. In: Kaelbling, L., Saffiotti, A. (eds.) Proc. 19th Int. Joint Conf. on Artificial Intelligence (IJCAI'05). pp. 364–369. Professional Book Center (2005)
2. Baader, F., Lutz, C., Suntisrivaraporn, B.: CEL—a polynomial-time reasoner for life science ontologies. In: Furbach, U., Shankar, N. (eds.) Proc. 3rd Int. Joint Conf. on Automated Reasoning (IJCAR'06). LNCS, vol. 4130, pp. 287–291. Springer (2006)
3. Delaitre, V., Kazakov, Y.: Classifying $\mathcal{ELH}$ ontologies in SQL databases. In: Patel-Schneider, P.F., Hoekstra, R. (eds.) Proc. OWLED 2009 Workshop on OWL: Experiences and Directions. CEUR Workshop Proceedings, vol. 529. CEUR-WS.org (2009)
4. Krötzsch, M.: Efficient rule-based inferencing for OWL EL. In: Walsh, T. (ed.) Proc. 22nd Int. Conf. on Artificial Intelligence (IJCAI'11). IJCAI (2011), to appear
5. Lawley, M.J., Bousquet, C.: Fast classification in Protégé: Snorocket as an OWL 2 EL reasoner. In: Taylor, K., Meyer, T., Orgun, M. (eds.) Proc. 6th Australasian Ontology Workshop (IAOA'10). Conferences in Research and Practice in Information Technology, vol. 122, pp. 45–49. Australian Computer Society Inc. (2010)
6. Motik, B., Cuenca Grau, B., Horrocks, I., Wu, Z., Fokoue, A., Lutz, C. (eds.): OWL 2 Web Ontology Language: Profiles. W3C Recommendation (27 October 2009), available at http://www.w3.org/TR/owl2-profiles/
7. Motik, B., Patel-Schneider, P.F., Parsia, B. (eds.): OWL 2 Web Ontology Language: Structural Specification and Functional-Style Syntax. W3C Recommendation (27 October 2009), available at http://www.w3.org/TR/owl2-syntax/
8. Simančík, F., Kazakov, Y., Horrocks, I.: Consequence-based reasoning beyond Horn ontologies. In: Walsh, T. (ed.) Proc. 22nd Int. Conf. on Artificial Intelligence (IJCAI'11). IJCAI (2011), to appear