A Graphical Calculus for Quantum Observables

Ross Duncan and Bob Coecke
Oxford University Computing Laboratory

Motivation

- Quantum observables may be incompatible: position/momentum, polarisation, spin ...
- In traditional quantum logic approaches these observables are simply *incomparable* in the lattice.
- However if one wants to *compute* with quantum mechanics we need know how these observables relate to each other.

Classical Objects

In a $\dagger$-category $\mathcal{C}$, a triple (A, δ, ϵ) is called a *classical object* if :

- $\delta : A \to A \otimes A$ and $\epsilon : A \to I$ for a cocommutative comonoid;
- $\delta^\dagger : A \otimes A \to A$ and $\epsilon^\dagger : I \to A$ for a commutative monoid;
- they jointly satisfy the special frobenius condition.

The canonical example is **FDHilb** with $A = \mathbb{C}^2$,

$$\delta : |i\rangle \mapsto |ii\rangle$$

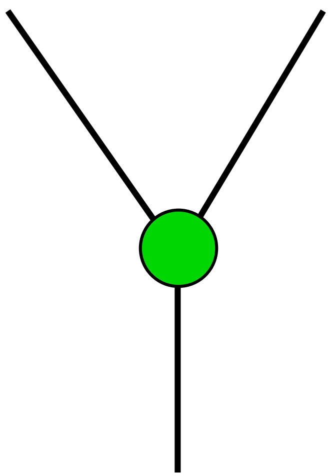
and

$$\epsilon : \sum |i\rangle \mapsto 1$$

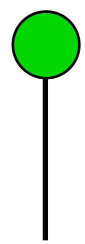
Classical Objects

Represent the free term model generated by (A, δ, ϵ) as graphs built up from:

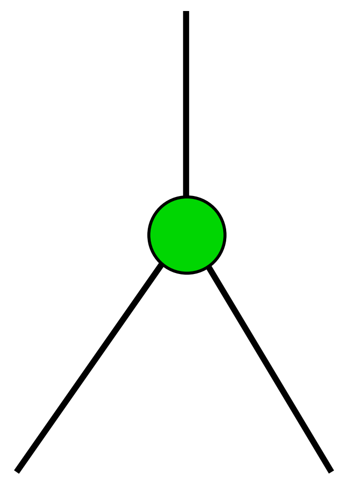
$\delta =$



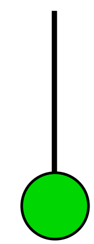
$\epsilon =$



$\delta^\dagger =$

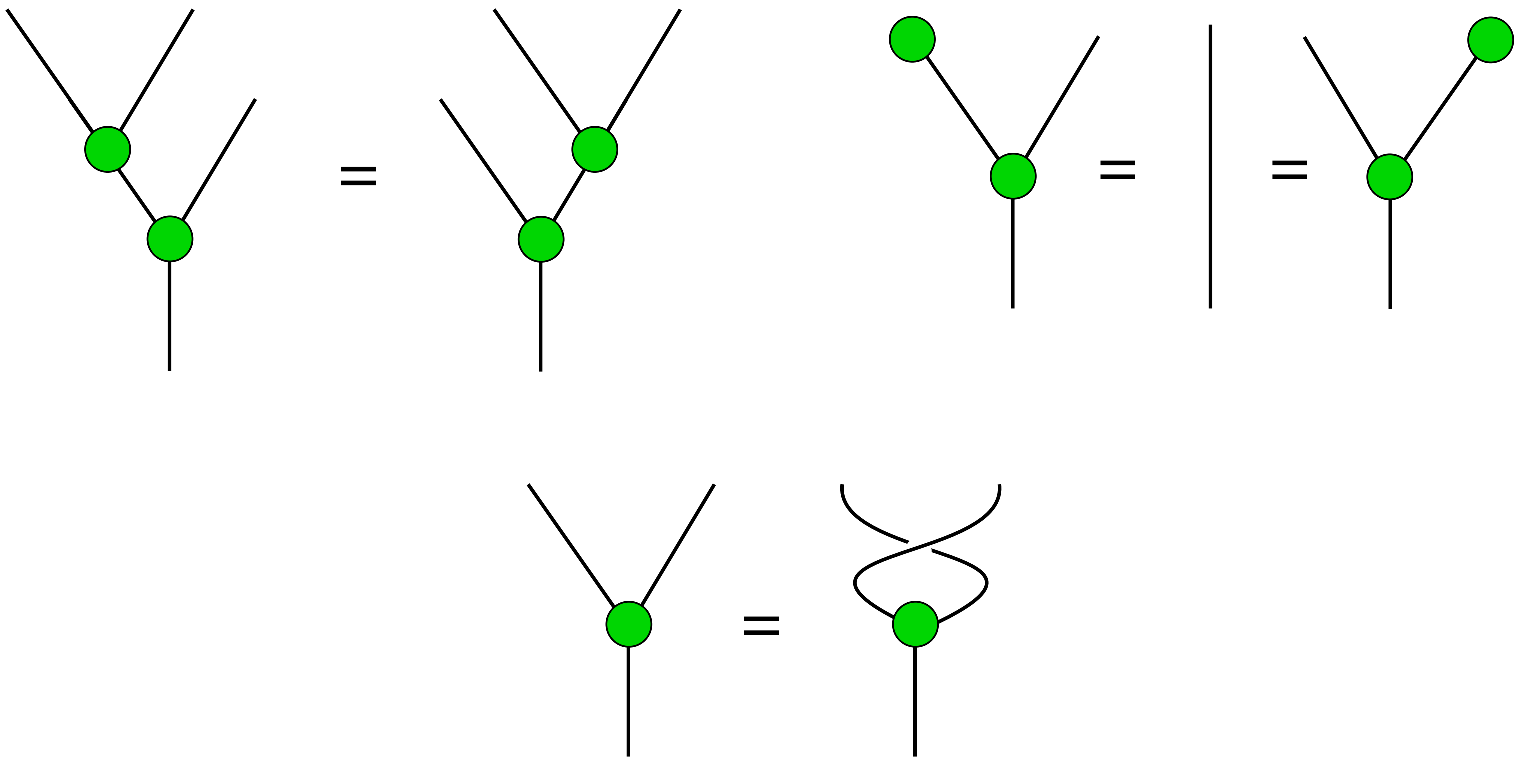


$\epsilon^\dagger =$



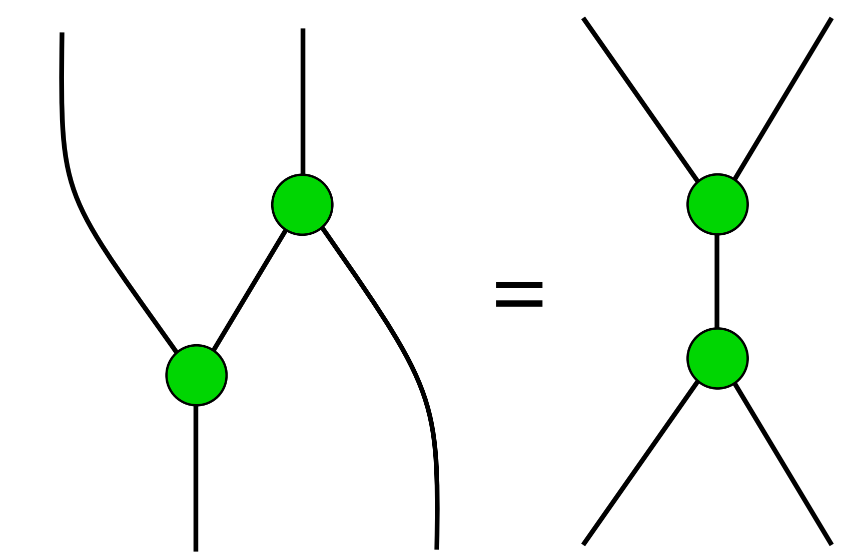
Classical Objects

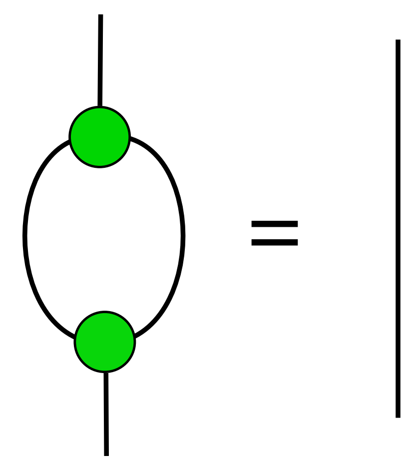
Comonoid laws:



Classical Objects

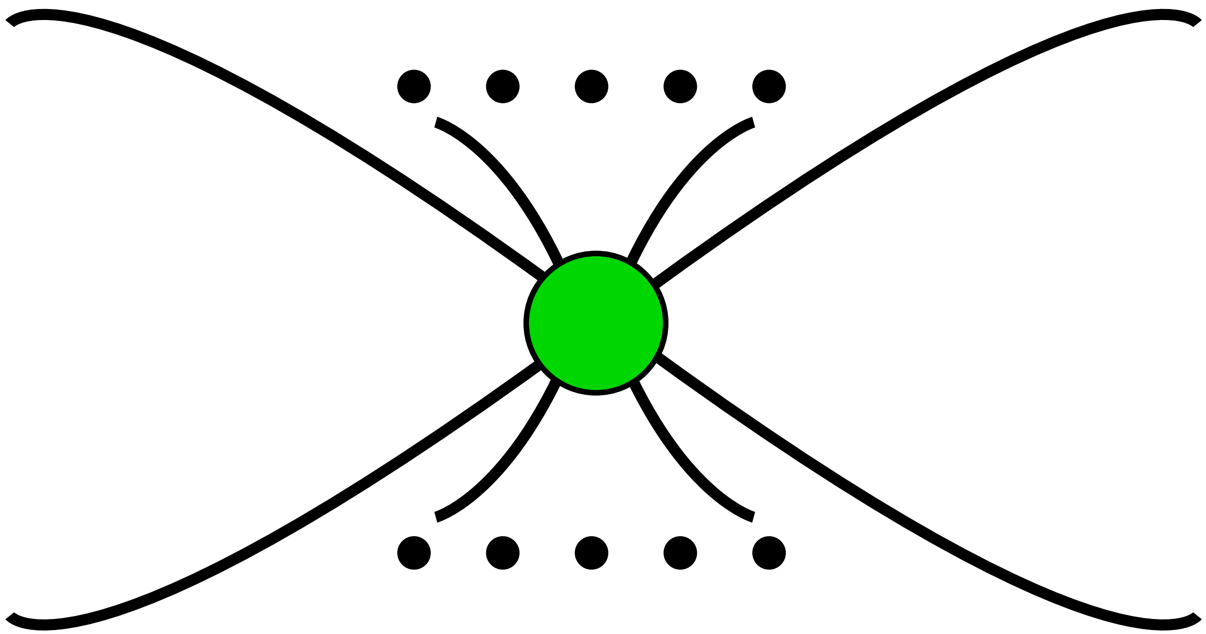
Special Frobenius laws:





Spider Theorem

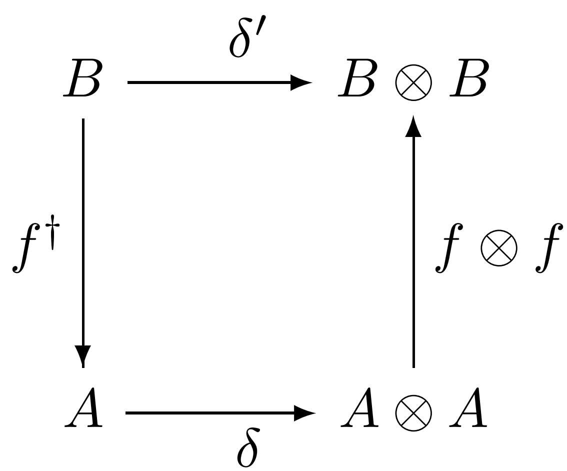
Theorem 1. *Any diagram constructed from a classical object is uniquely determined by the number of inputs and outputs.*

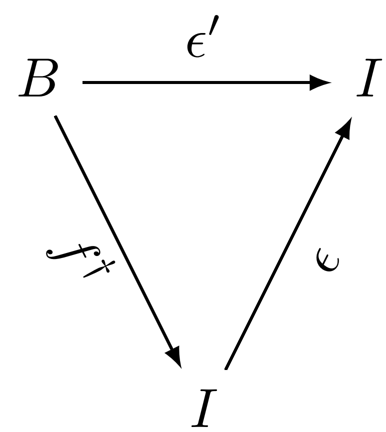


Therefore the graphical calculus for one classical object is rather uninteresting.

New classical structures from old

Given a classical object (A, δ, ϵ) and a unitary map $f : A \rightarrow B$ we can define a new classical object (B, δ', ϵ') by:

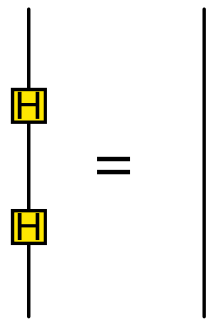




The Hadamard Map

The *Hadamard map* $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ enjoys a number of useful properties:

- Self adjointness: $H = H^\dagger$; and unitarity: $HH = \mathrm{id}$;



- The Hadamard exchanges the X and Z bases.

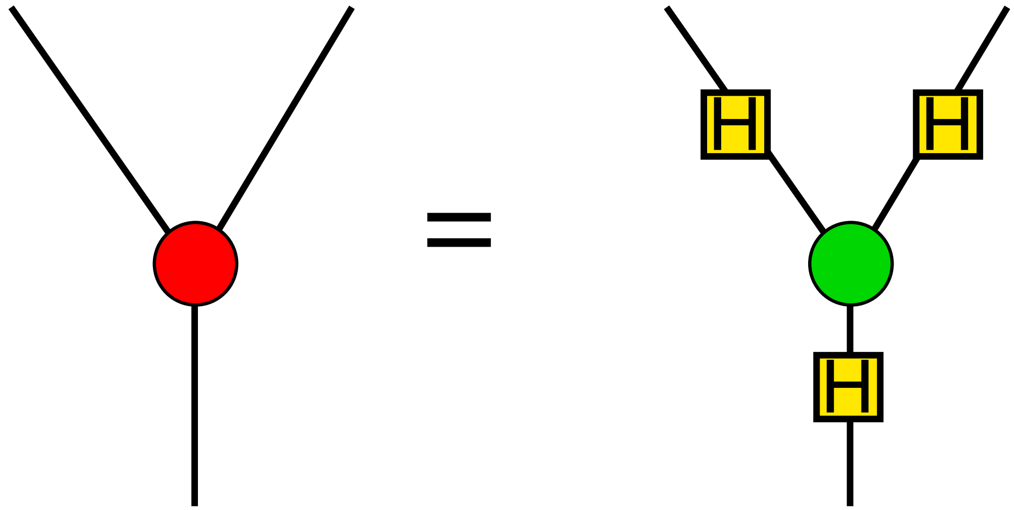
Two Classical Objects

Given (Q, δ, ϵ) and $(Q, \delta_H, \epsilon_H)$ we have the following:

- $\sqrt{2}\,|0\rangle = \epsilon_H^\dagger$;
- $\delta\epsilon_H^\dagger = \delta\,|0\rangle = |00\rangle = \epsilon_H^\dagger \otimes \epsilon_H^\dagger$;
- $|+\rangle = \epsilon^\dagger$
- $\delta_H\epsilon^\dagger = \delta_H\,|+\rangle = |{+}{+}\rangle = \epsilon^\dagger \otimes \epsilon^\dagger$

A 2nd Classical Structure

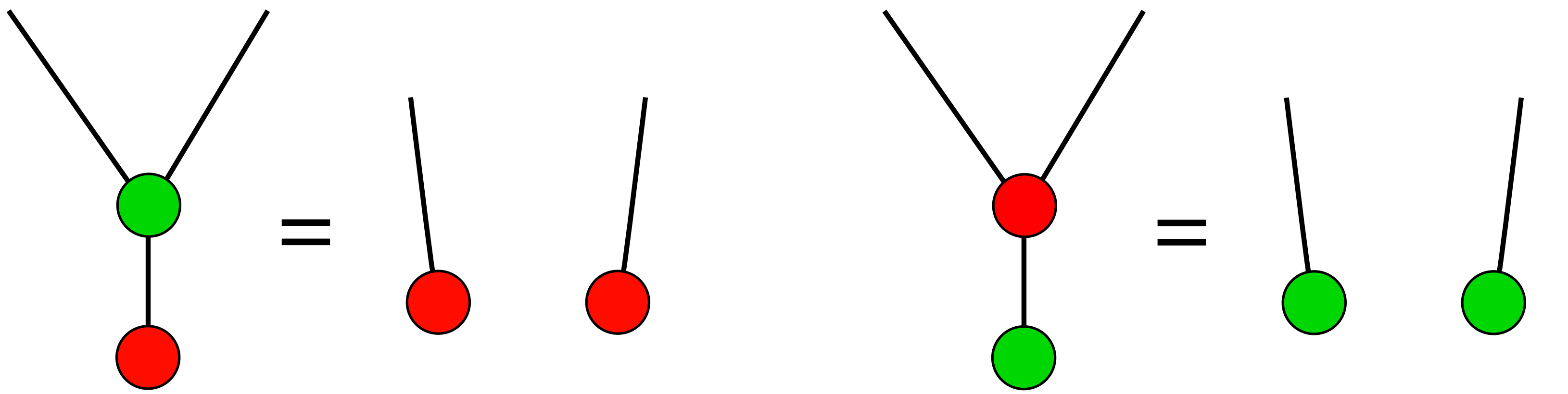
Represent the classical structure induced by H as a red dot:



We can immediately derive a law for changing the colour of dots by introducing H boxes. What other laws hold?

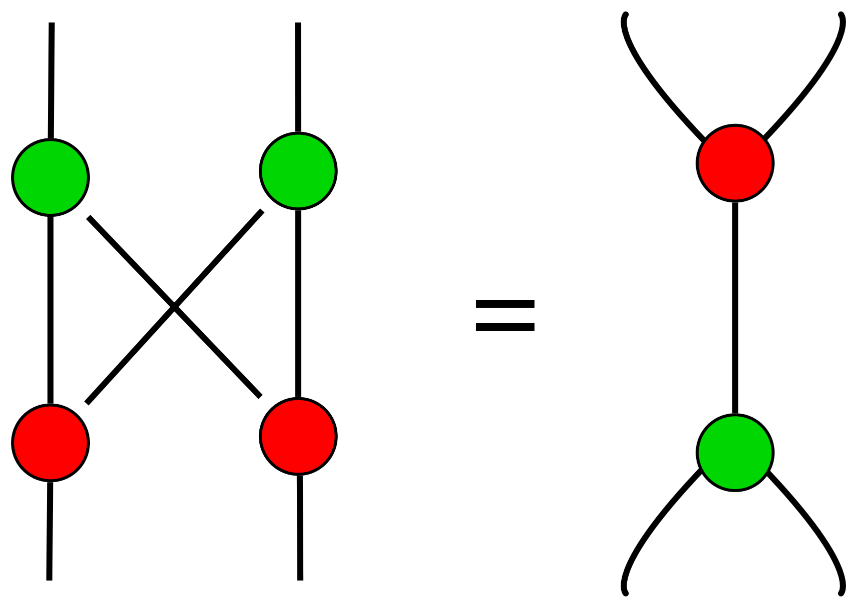
Bialgebraic Laws for Non-commuting observables

Cloning Laws:



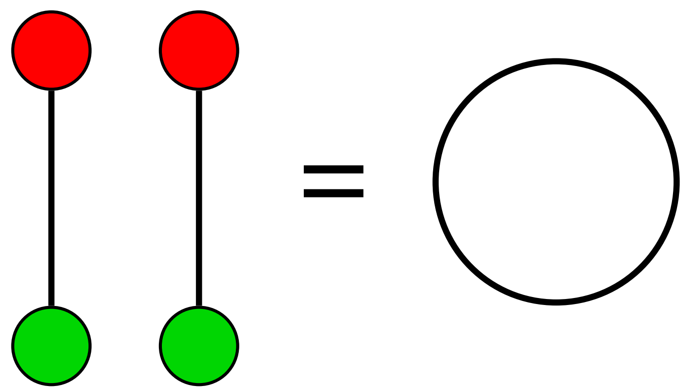
Bialgebraic Laws for Non-commuting observables

Bialgebra Law:



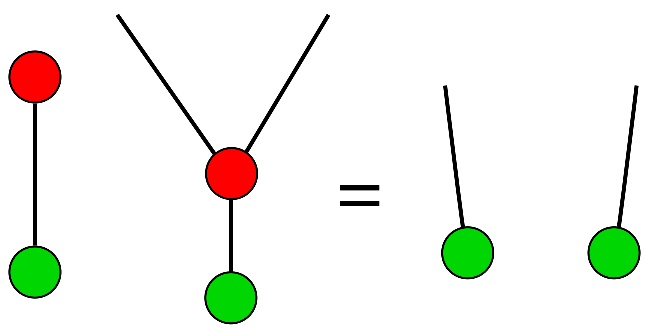
Bialgebraic Laws for Non-commuting observables

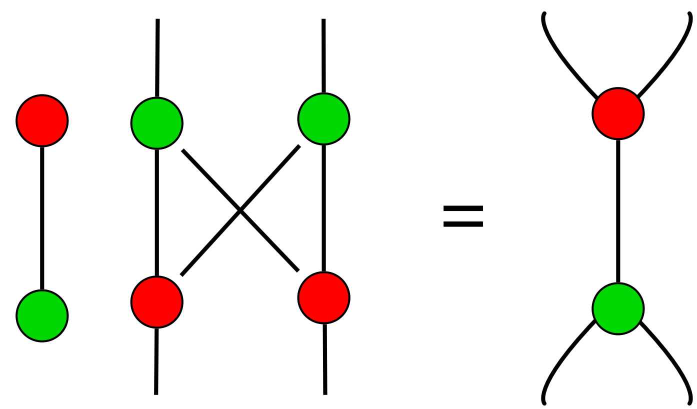
Dimension Law:

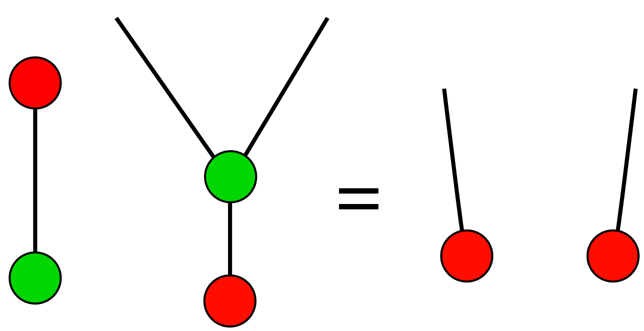


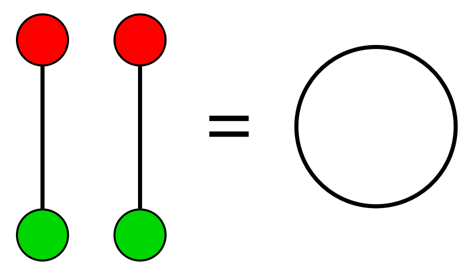
The pair of non-commuting observables fails to be a true bialgebra: every equation has a (hidden) scalar factor. Call this structure a *scaled bialgebra*.

Scaled Bialgebra Laws

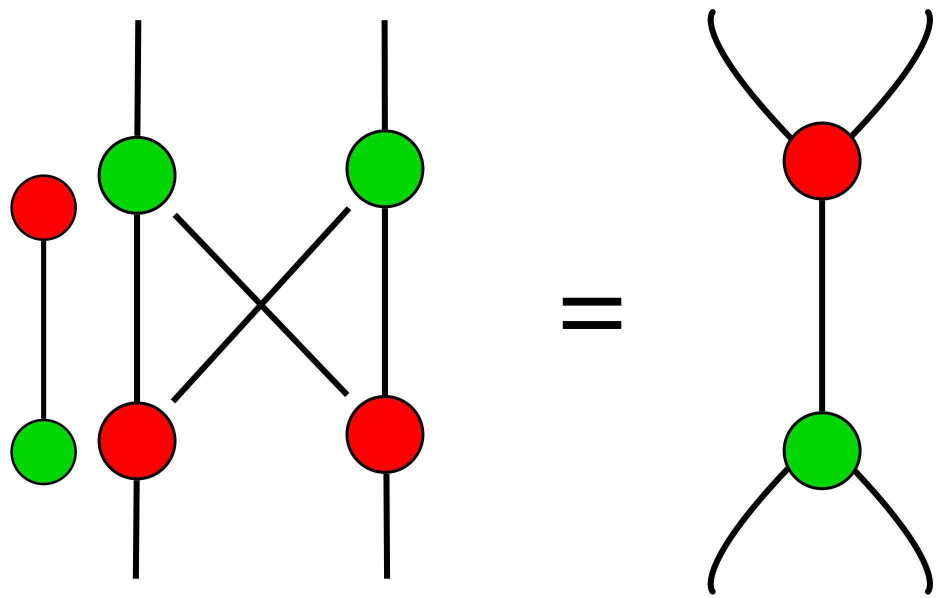








A Useful Lemma



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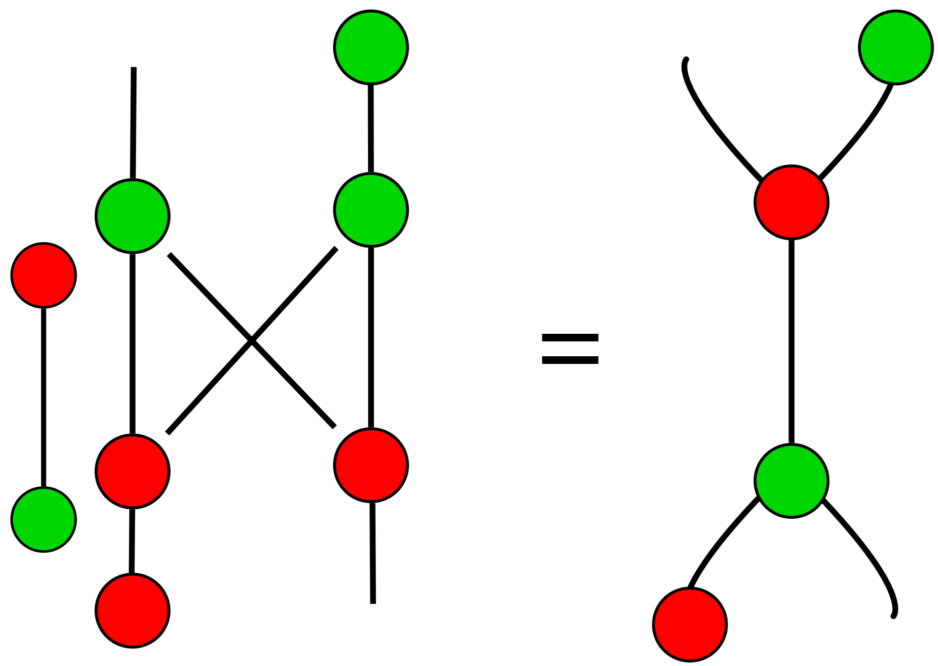

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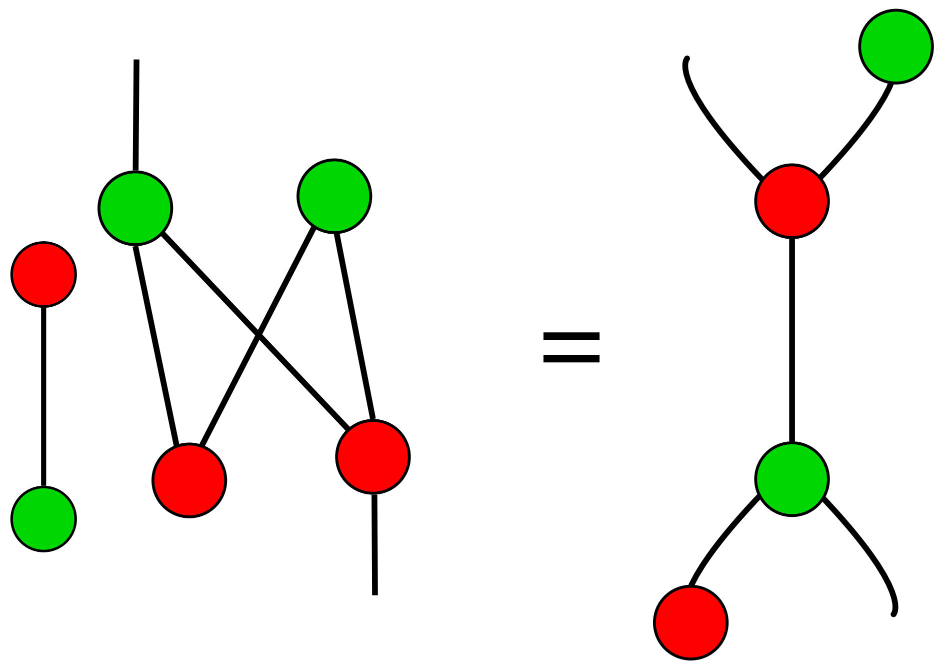

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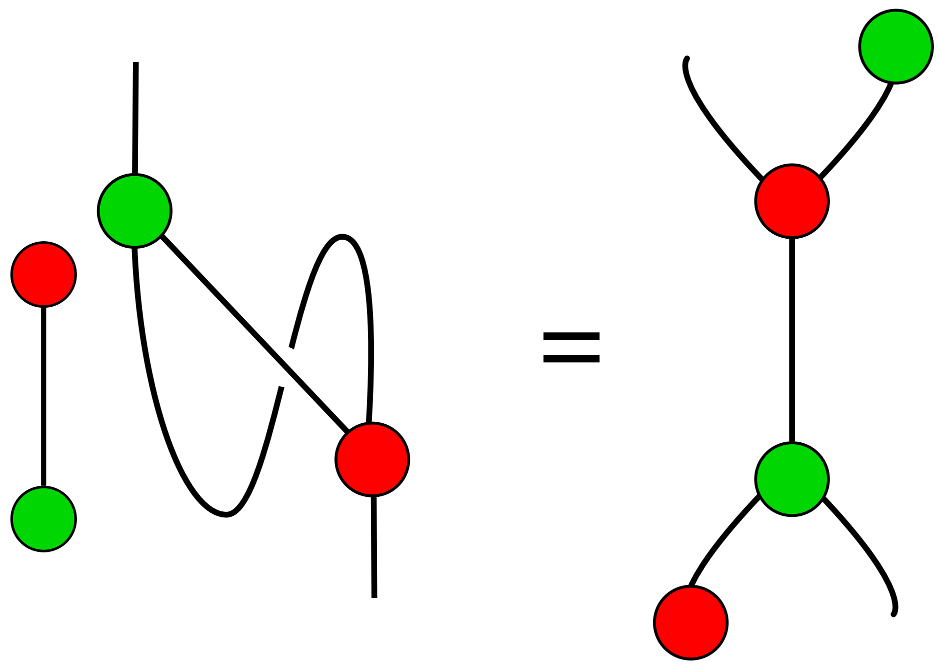

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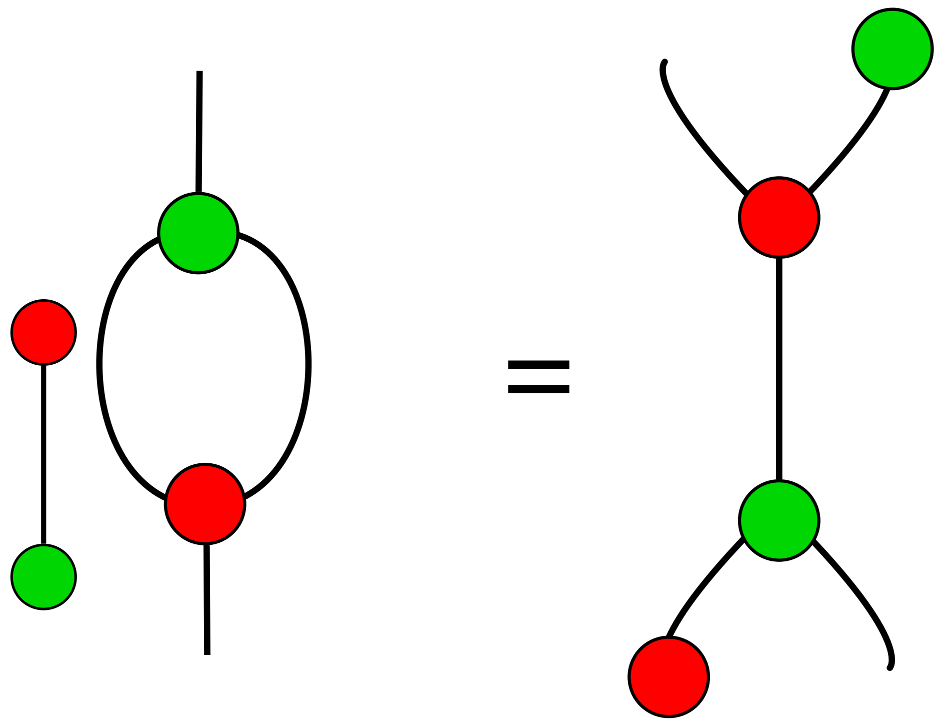

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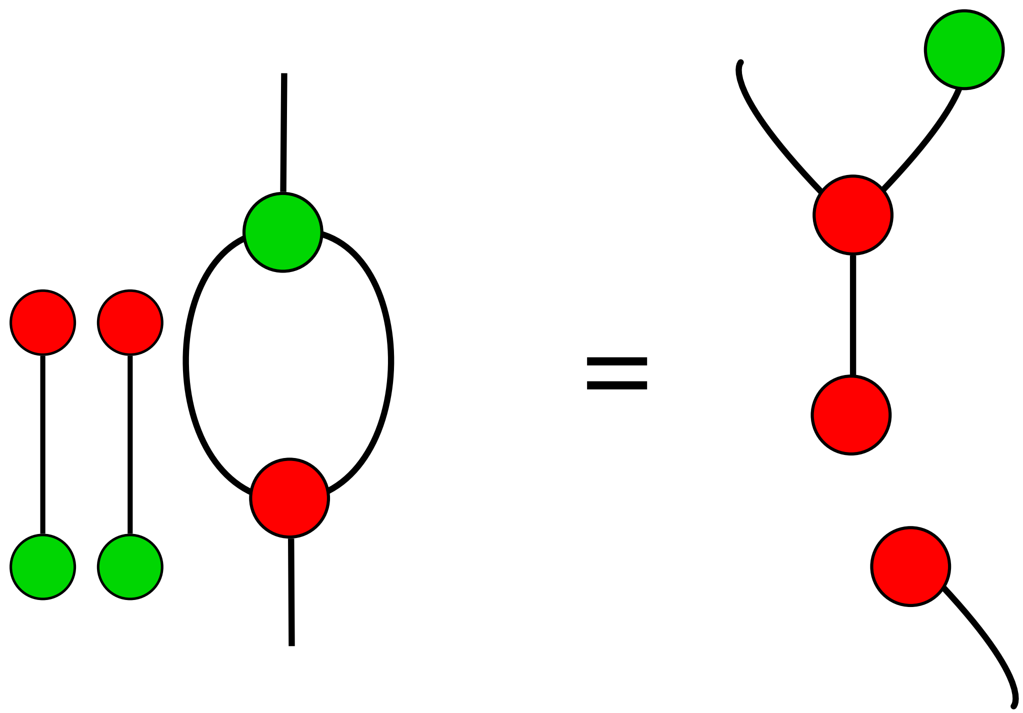

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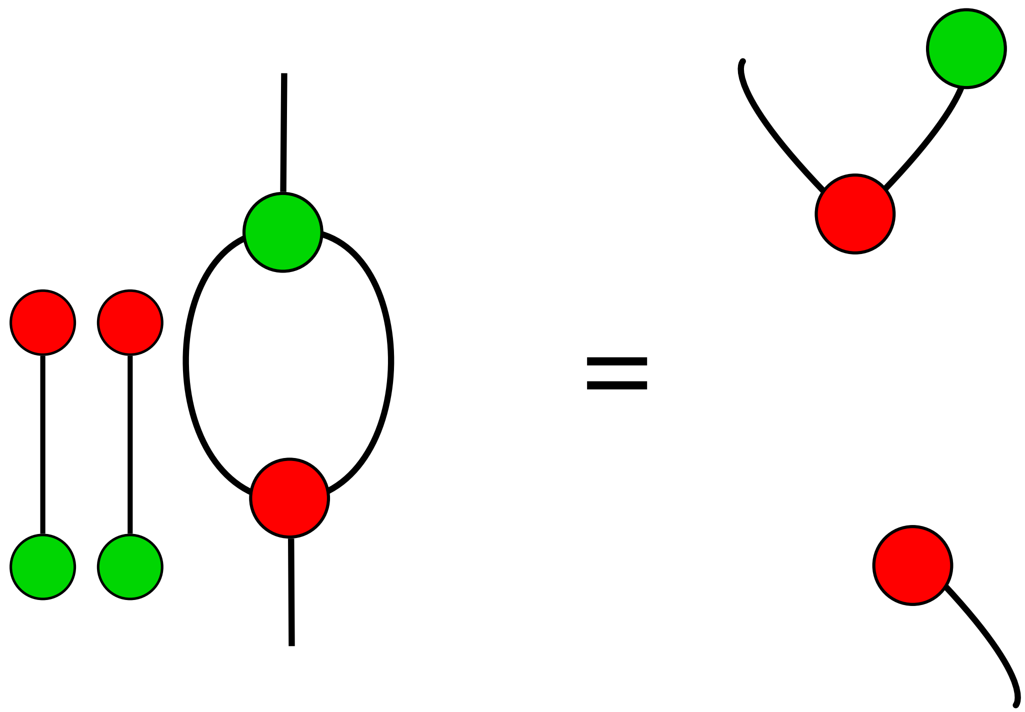

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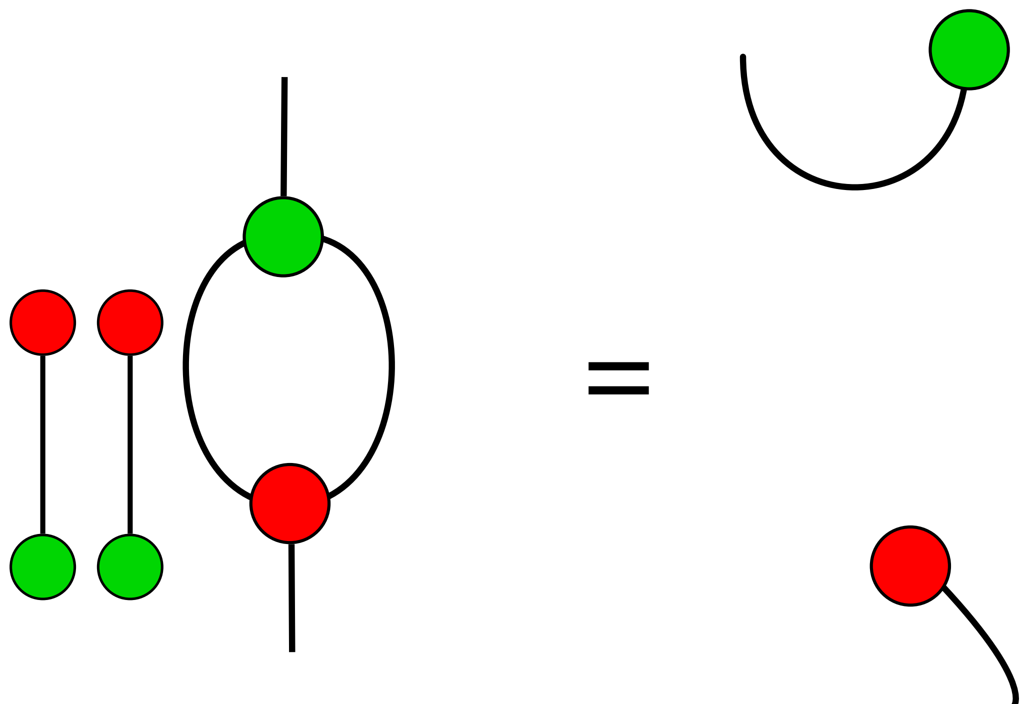

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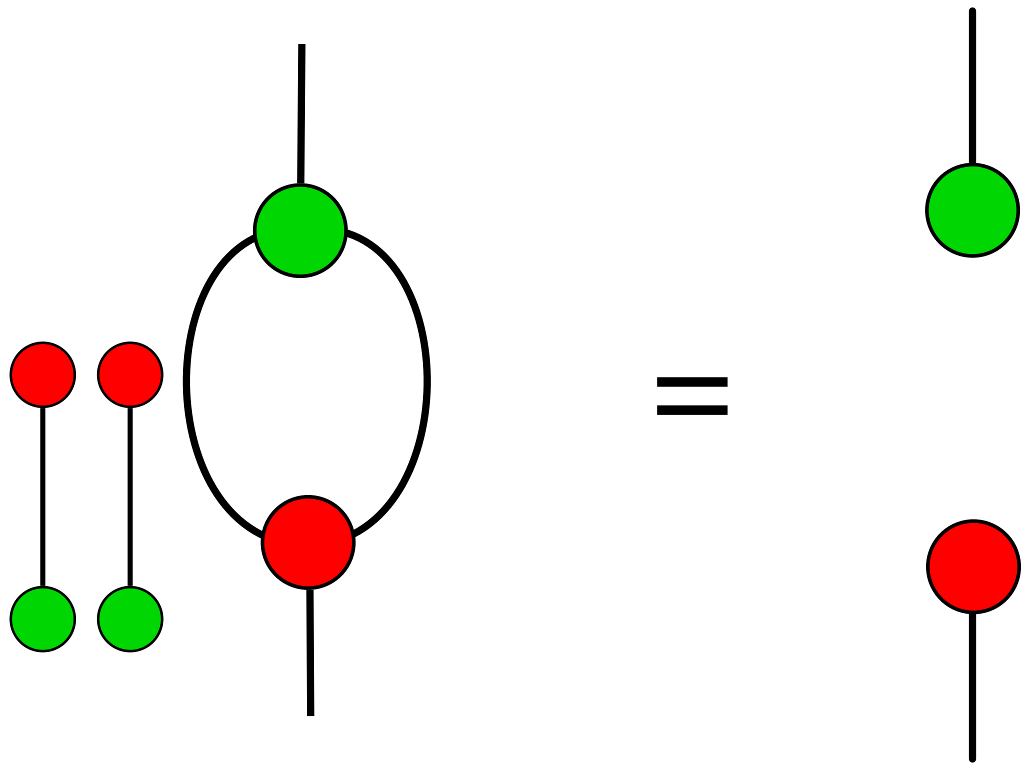

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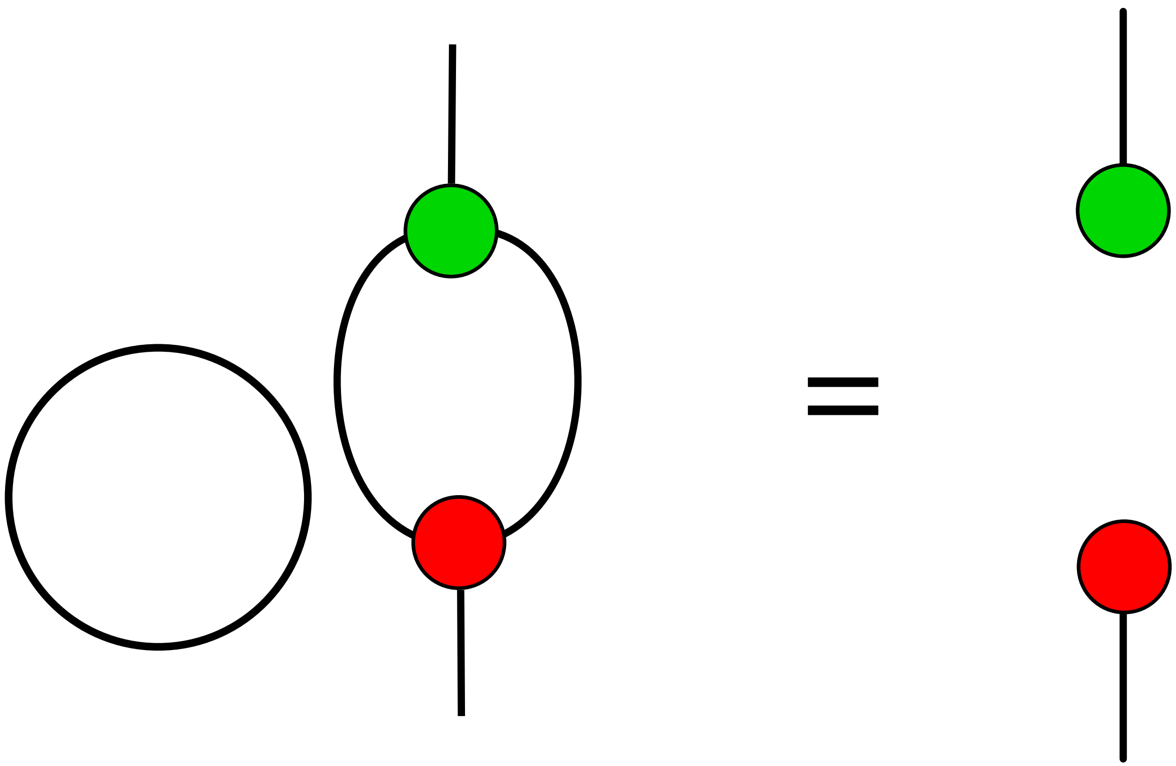
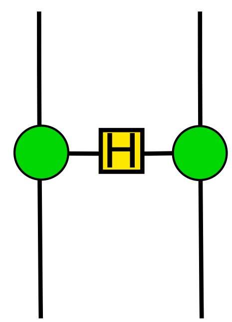


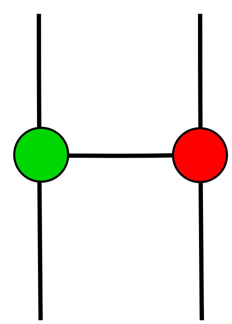
Therefore, the scaled bialgebra is in fact a *scaled Hopf algebra*, whose antipode is the identity times the dimension of the underlying space.

Representing Quantum Logic Gates

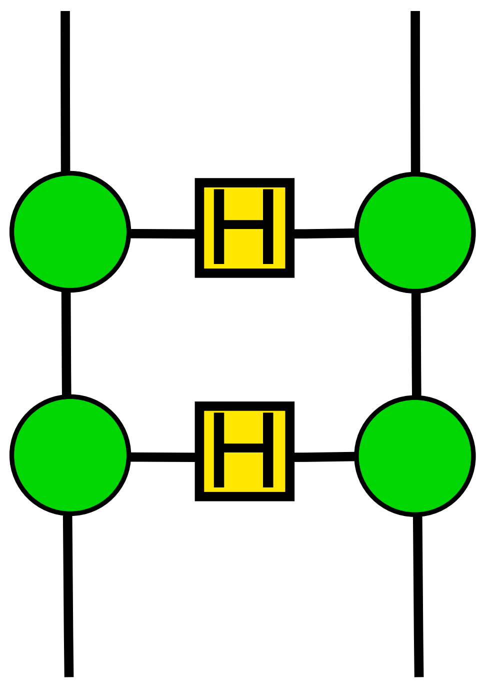
$$\wedge Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$$



$$\wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$



Example: $\wedge Z \circ \wedge Z = \mathrm{id}$



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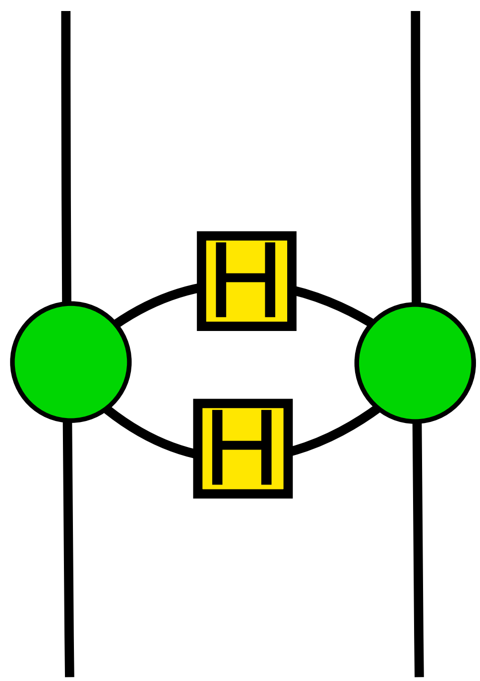

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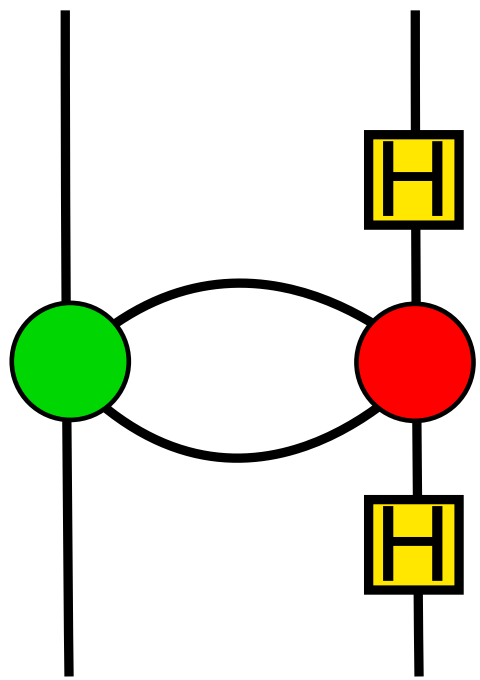

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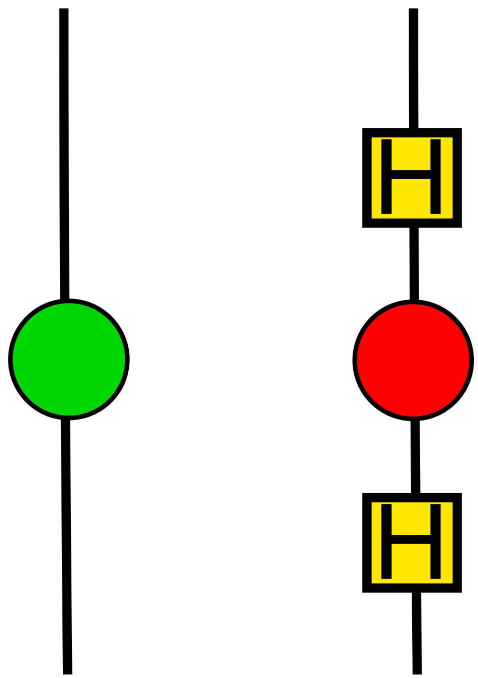

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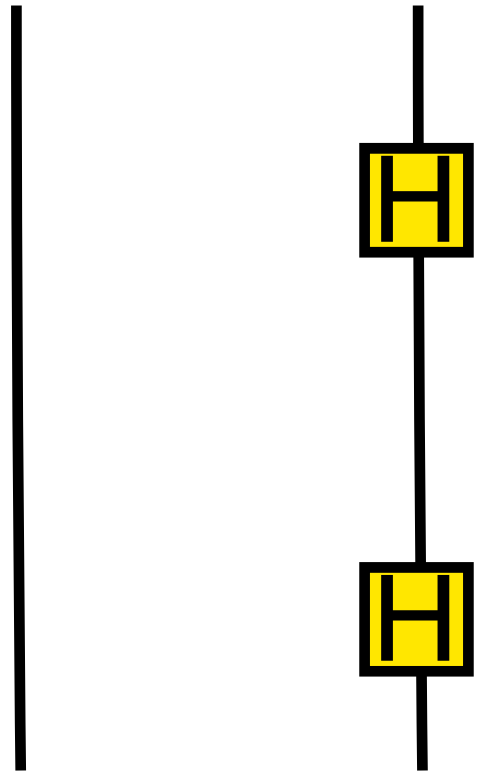

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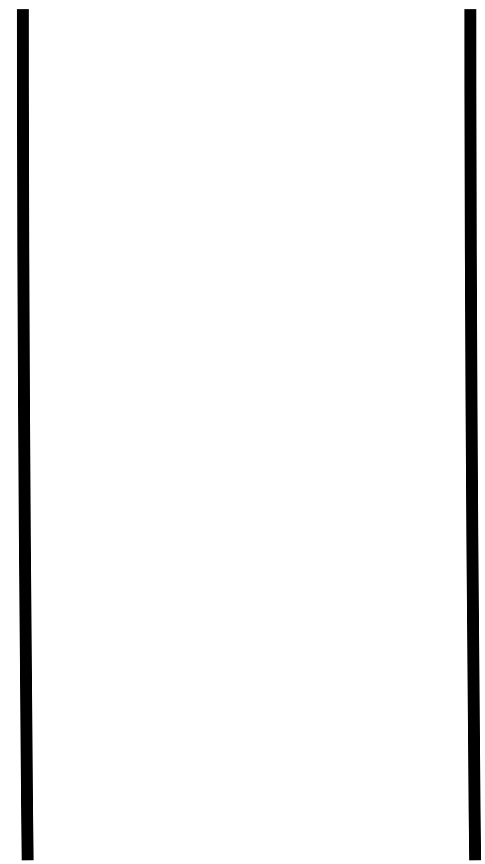
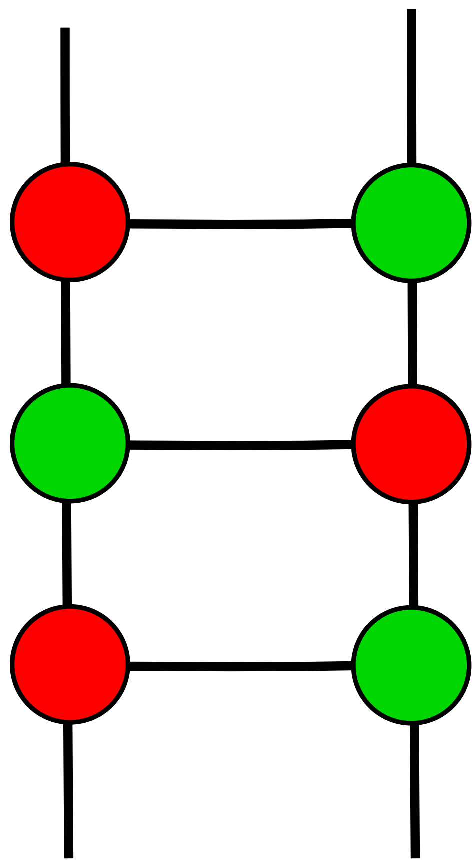


Example: $3 \times \wedge X =$ **swap**



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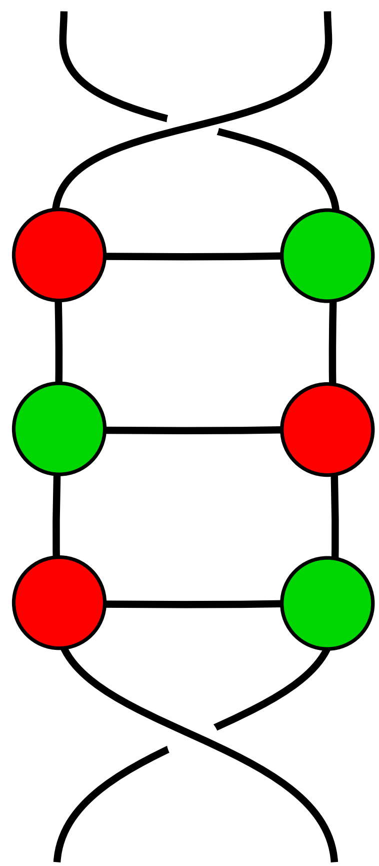

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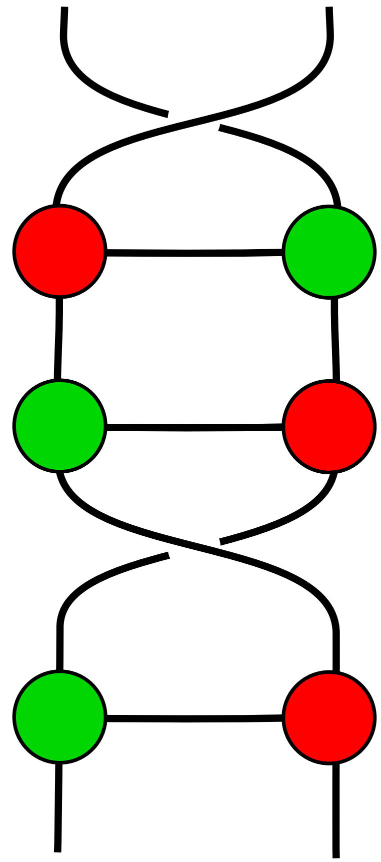

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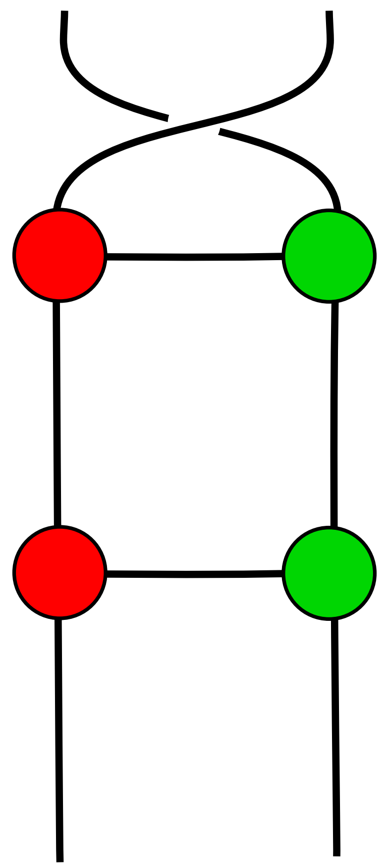

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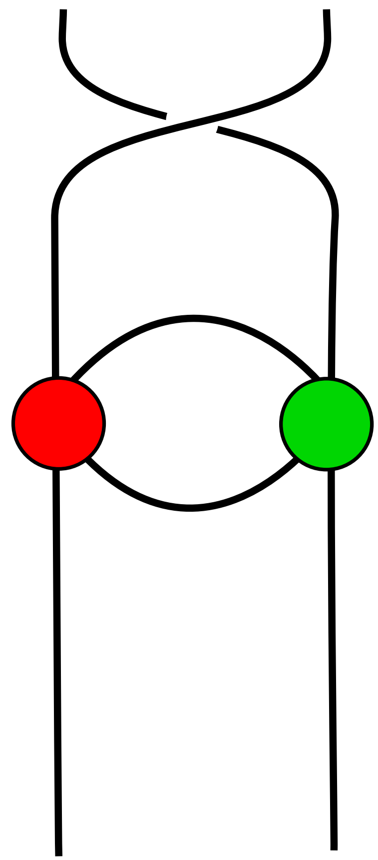

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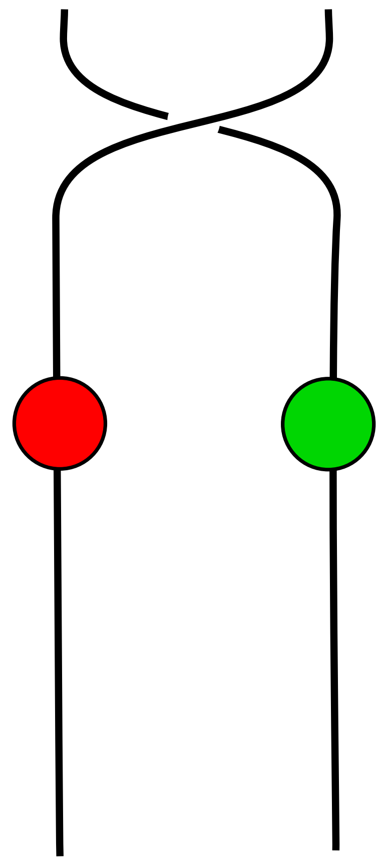

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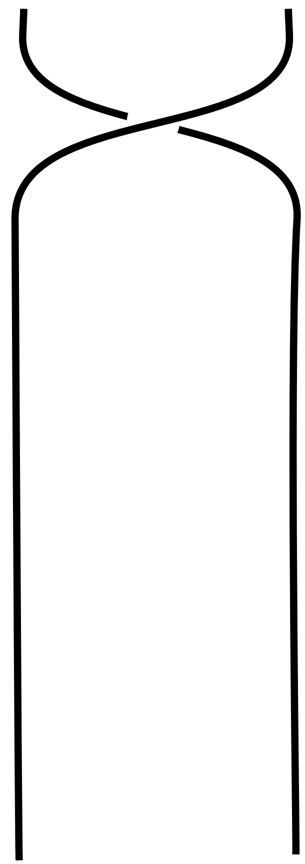
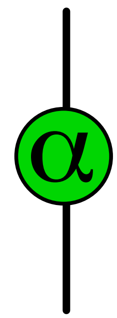
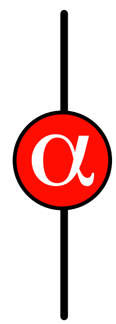


Incorporating Phases

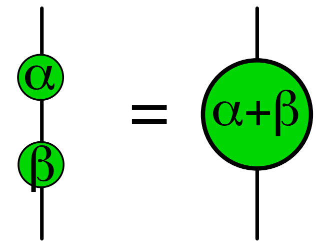
Let $\alpha \in (0, 2\pi)$; consider the maps:

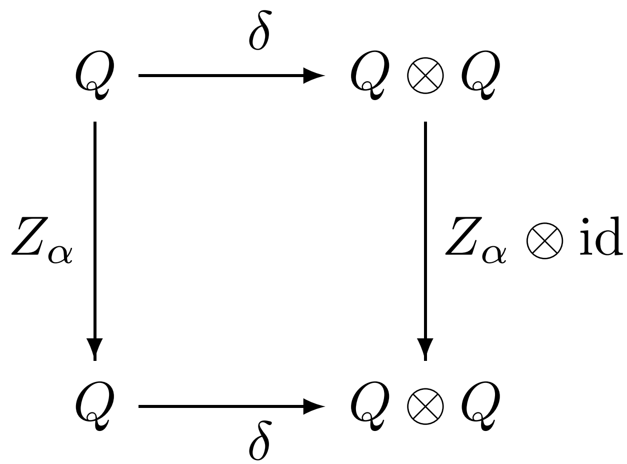
$$Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} =$$ 

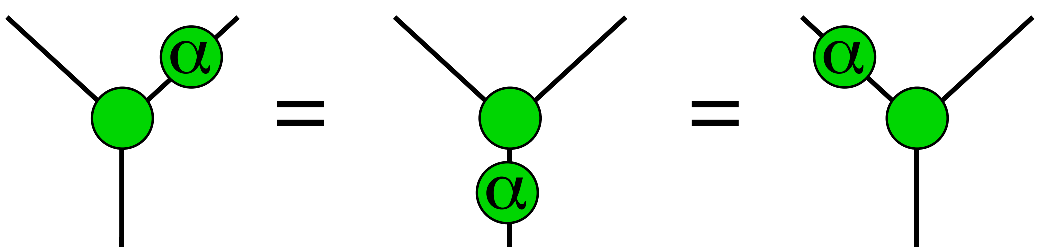
$$X_\alpha = HZ_\alpha H =$$ 

Incorporating Phases

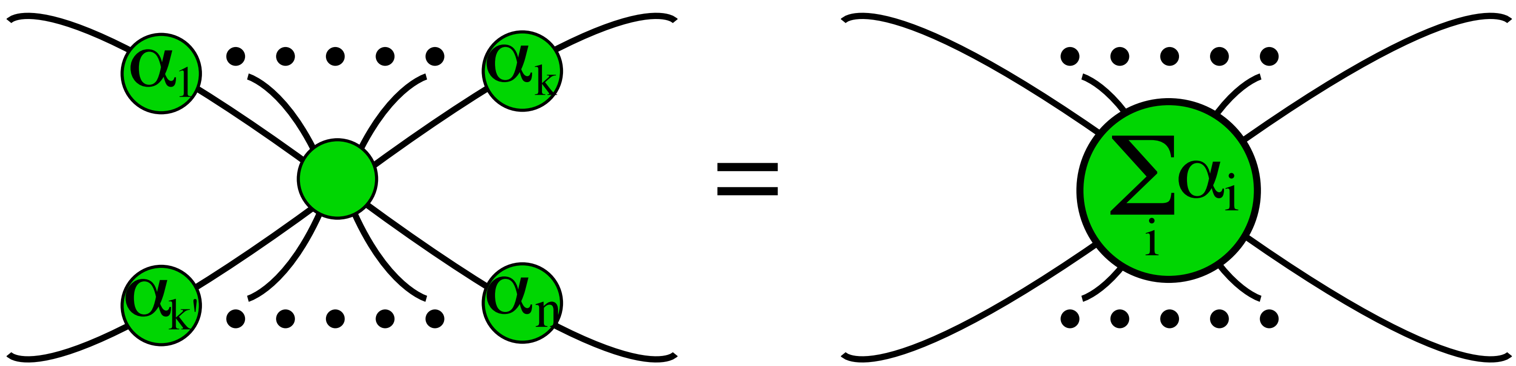
$$Z_\alpha \circ Z_\beta = Z_{\alpha+\beta}$$







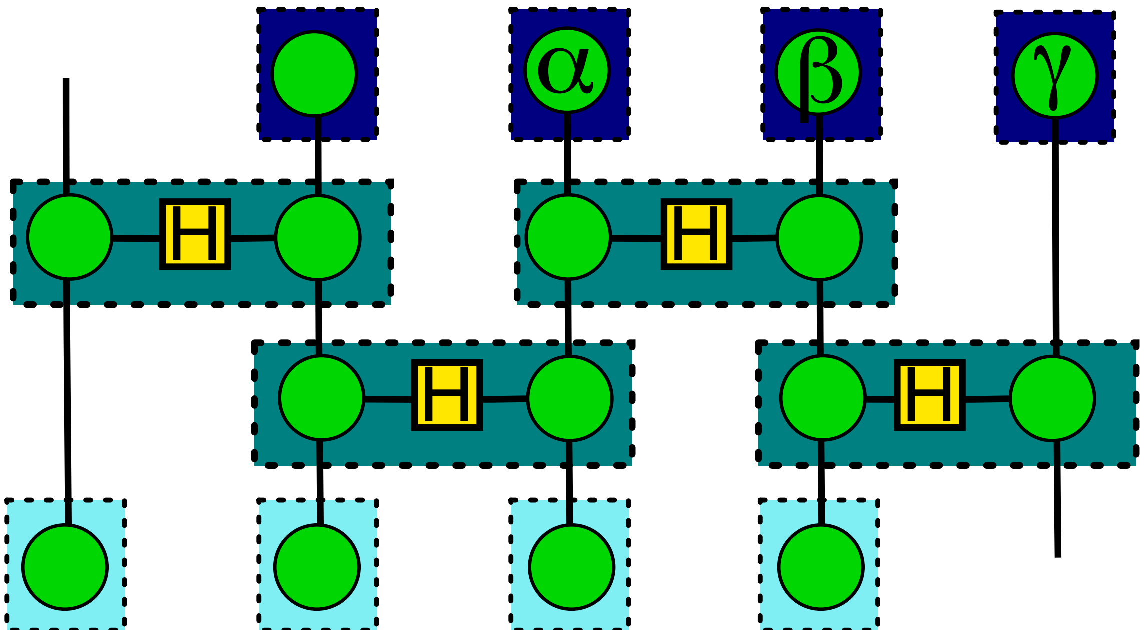
Generalised Spider Law



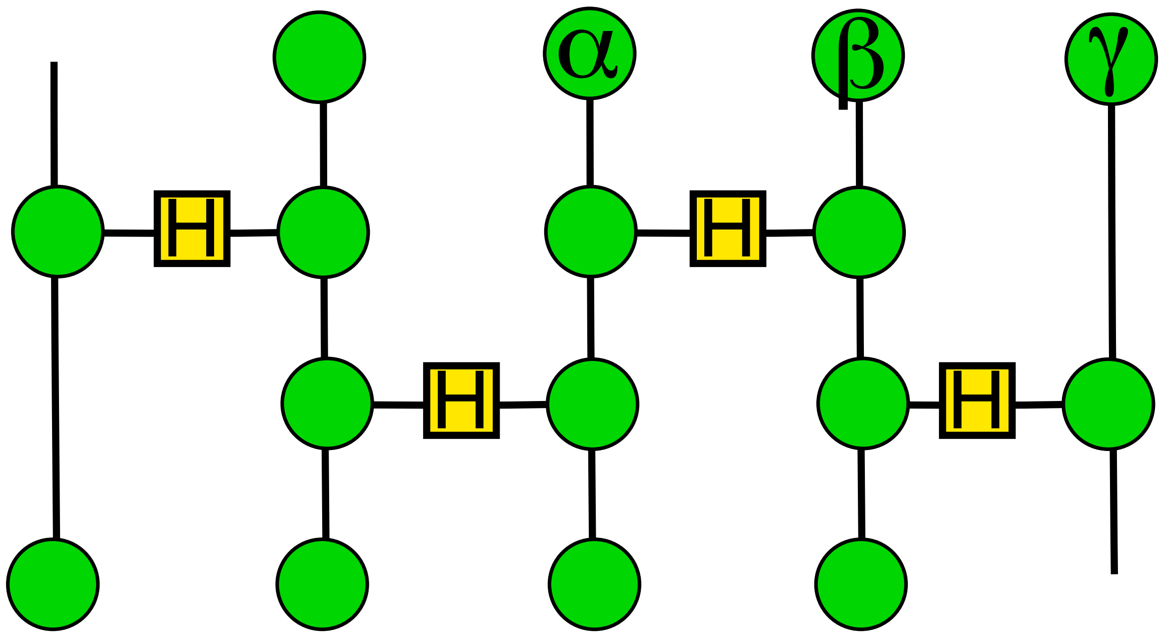
General unitary U

Proposition 2. *If U is a unitary on $\mathbb{C}^2$ there exist α, β, γ such that $U = Z_\alpha X_\beta Z_\gamma$.*

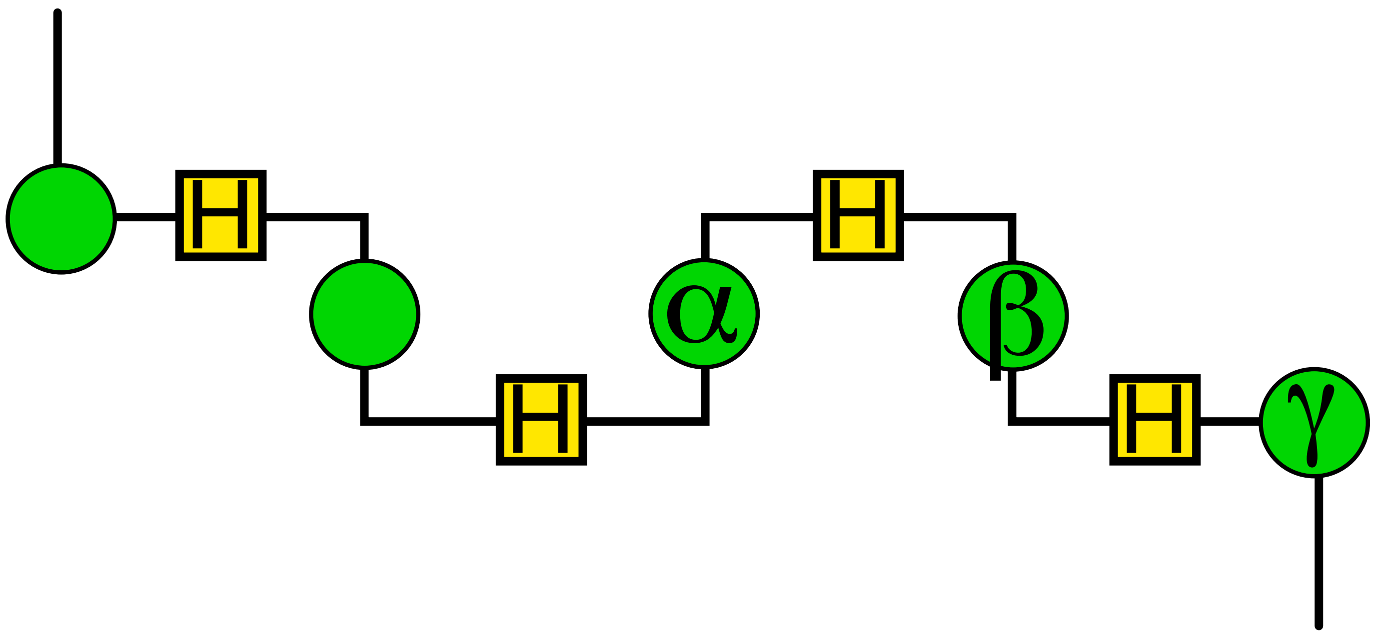
Here is (part of) a measurement based program to compute this:



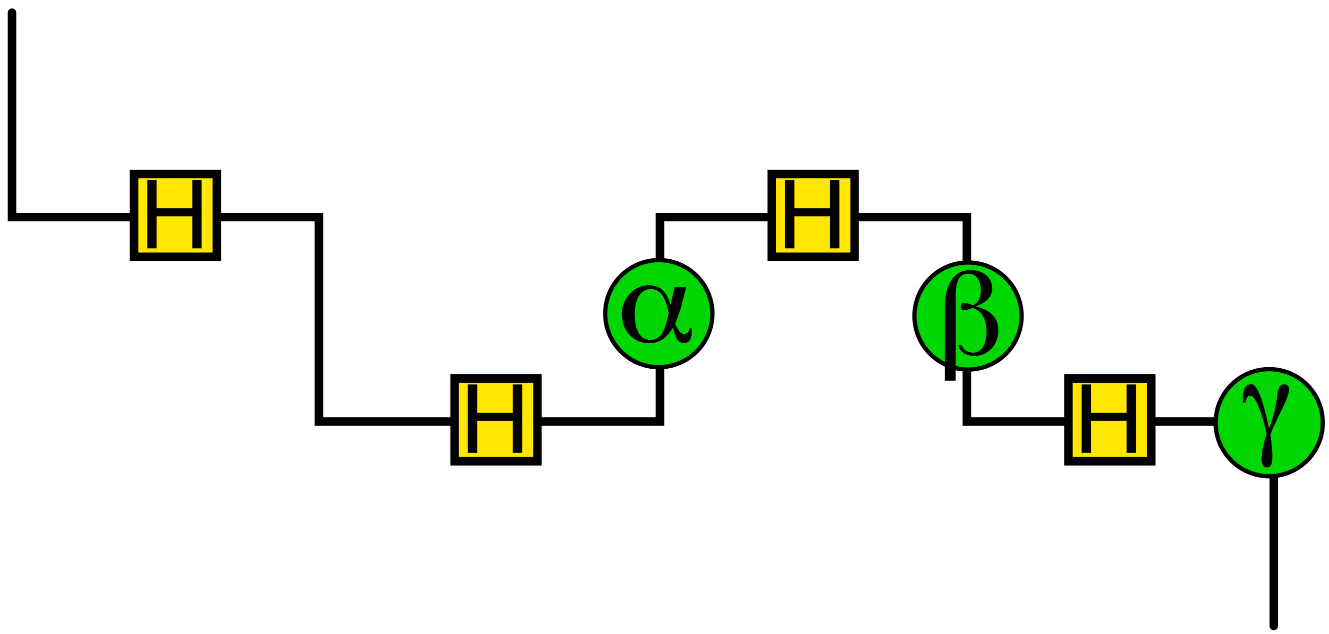
General unitary U



General unitary U



General unitary U



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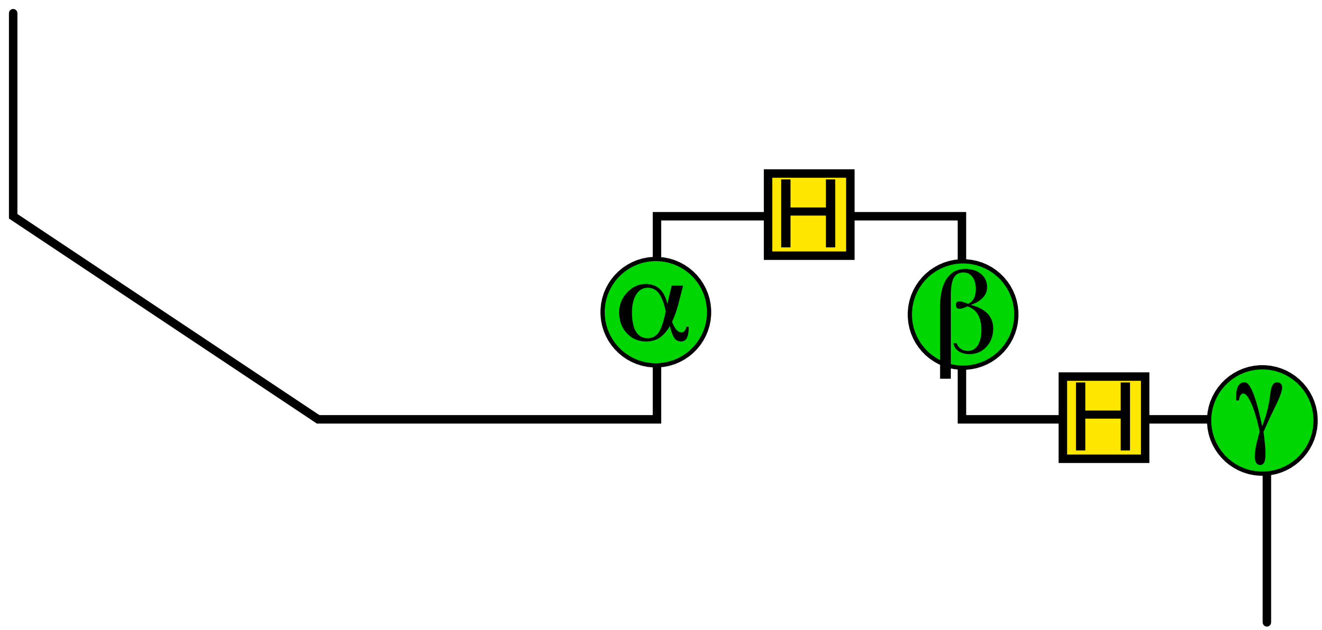
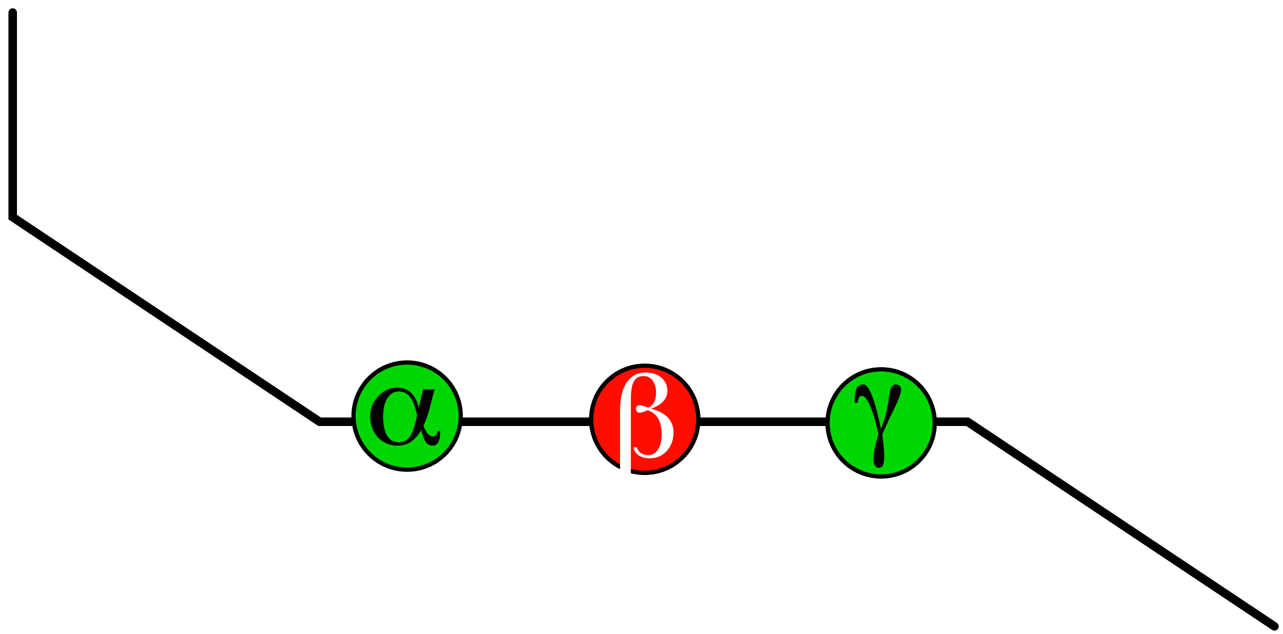


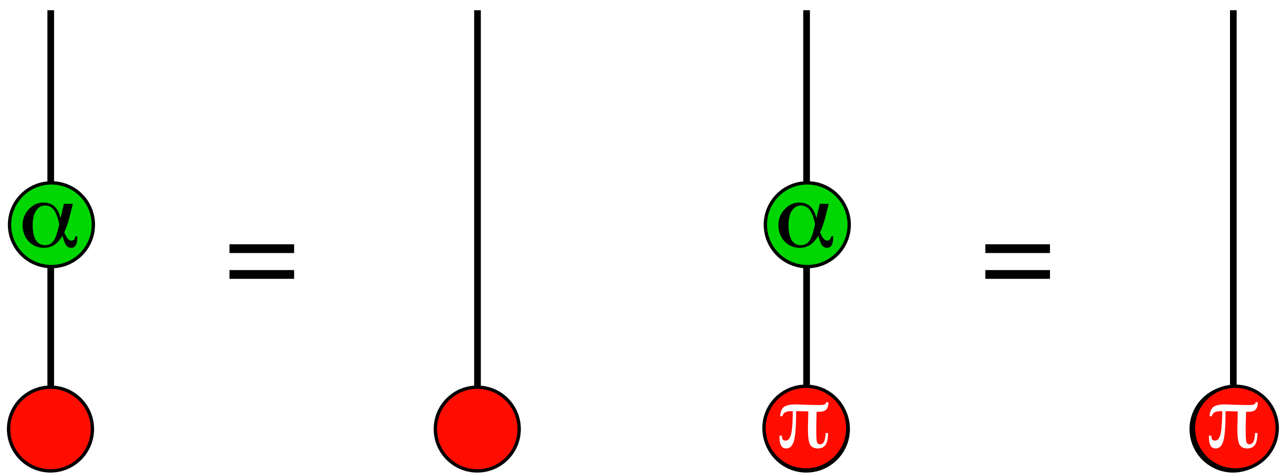
General unitary U



$$= Z_\alpha X_\beta Z_\gamma$$

How do phases interact?

$Z_\alpha |0\rangle = |0\rangle$ $\quad$ $Z_\alpha |1\rangle = e^{i\alpha} |1\rangle = |1\rangle$



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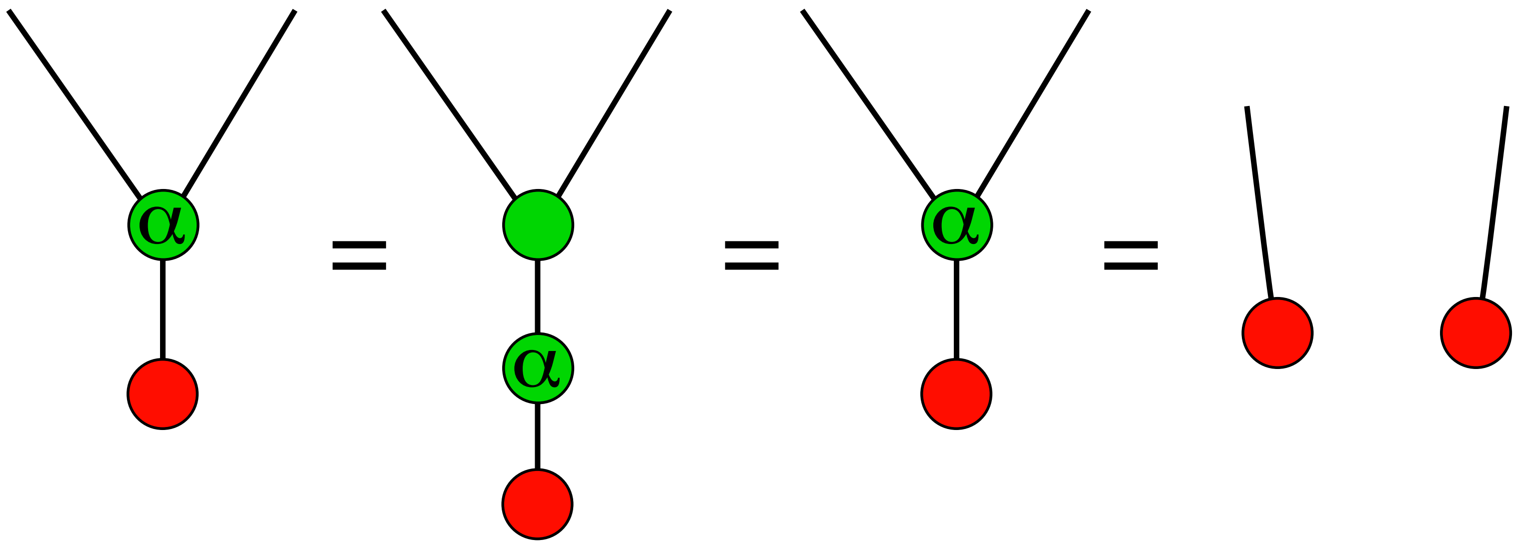

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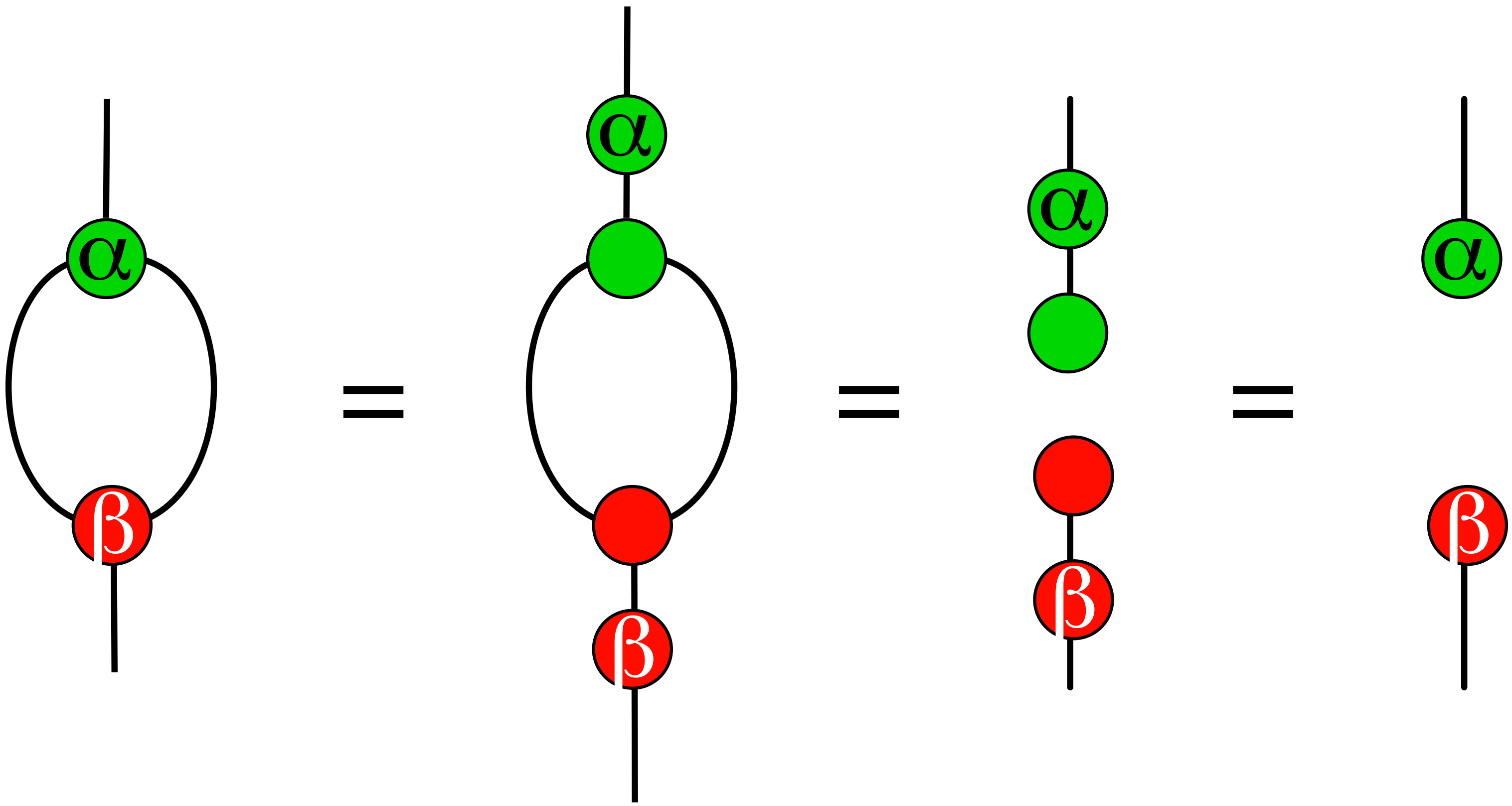
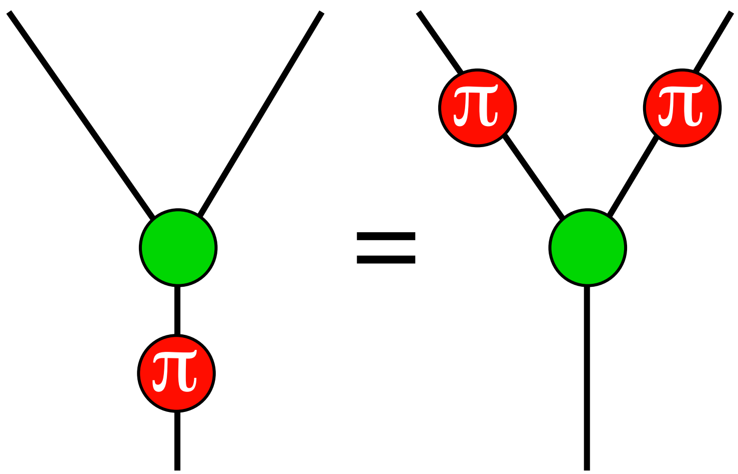


“Negation”

$$X_\pi = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad :: \begin{cases} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{cases}$$

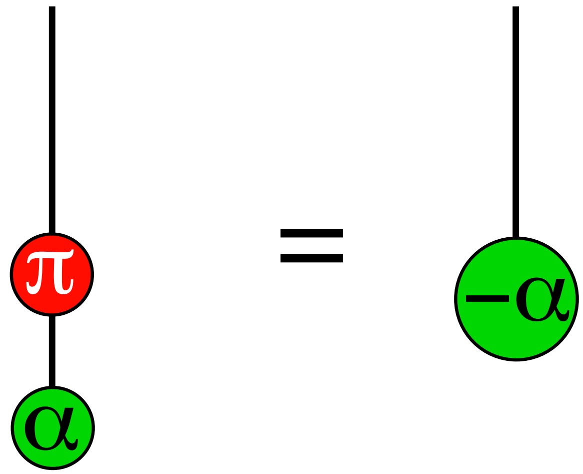
$$\begin{array}{ccc} Q & \xrightarrow{\ \delta\ } & Q \otimes Q \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle X \otimes X} \\ Q & \xrightarrow[\ \delta\ ]{} & Q \otimes Q \end{array}$$

“Negation”



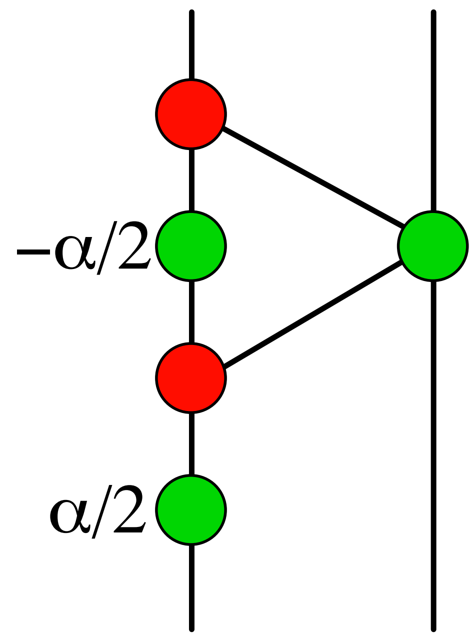
“Negation”

$$X :: |0\rangle + e^{i\alpha}|1\rangle \mapsto e^{i\alpha}|1\rangle + |0\rangle = |0\rangle + e^{-i\alpha}|1\rangle$$



Representing Controlled Phase

$$\wedge Z_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{pmatrix} =$$



Example: Quantum Fourier Transform

Among the most important quantum algorithms, the quantum fourier transform is a key stage of factoring.

$$|j_0 j_1 \cdots j_n\rangle \mapsto (|0\rangle + e^{2\pi i \alpha_0} |1\rangle)(|0\rangle + e^{2\pi i \alpha_1} |1\rangle) \cdots (|0\rangle + e^{2\pi i \alpha_n} |1\rangle)$$

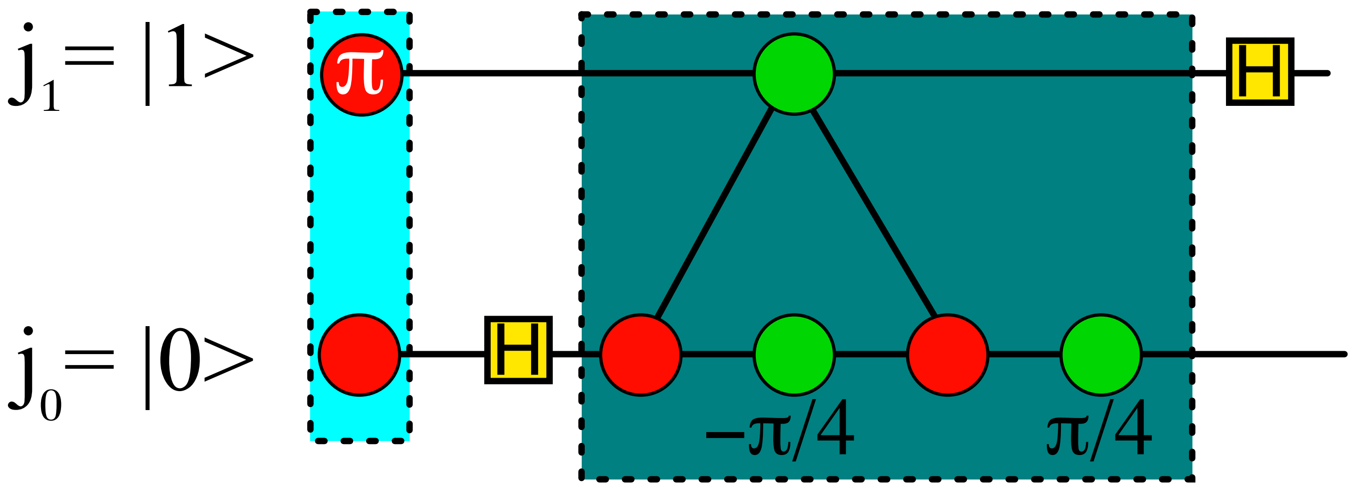
where $\alpha_k = 0.j_k \cdots j_n = \sum_{l=k}^{n} j_l / 2^k$

For 2 qubits:

$$|00\rangle \mapsto (|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \qquad |10\rangle \mapsto (|0\rangle + e^{i\pi} |1\rangle)(|0\rangle + |1\rangle)$$

$$|01\rangle \mapsto (|0\rangle + e^{i\pi/2} |1\rangle)(|0\rangle + e^{i\pi} |1\rangle) \qquad |11\rangle \mapsto (|0\rangle + e^{i3\pi/2} |1\rangle)(|0\rangle + e^{i\pi} |1\rangle)$$

Example: Quantum Fourier Transform



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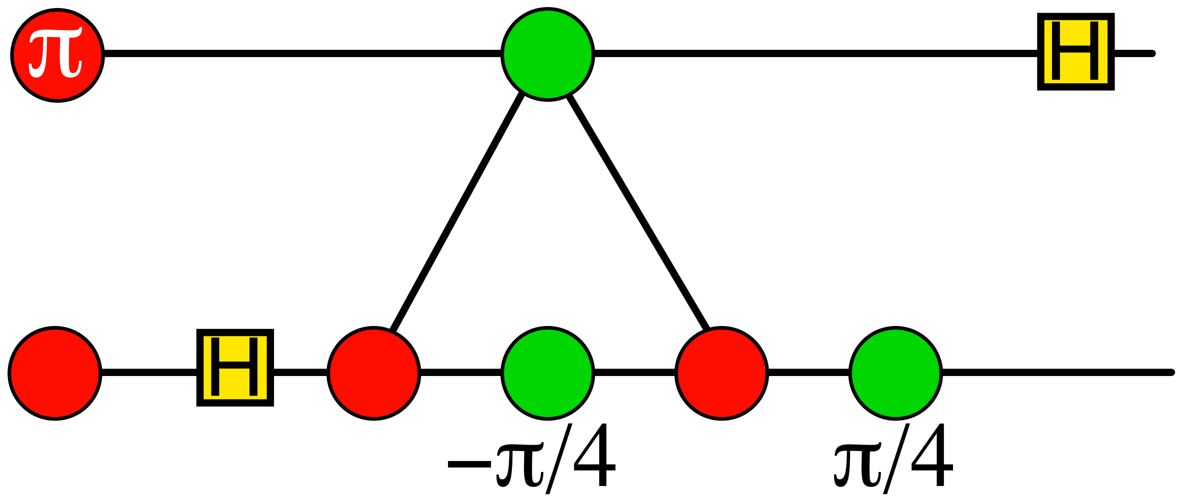

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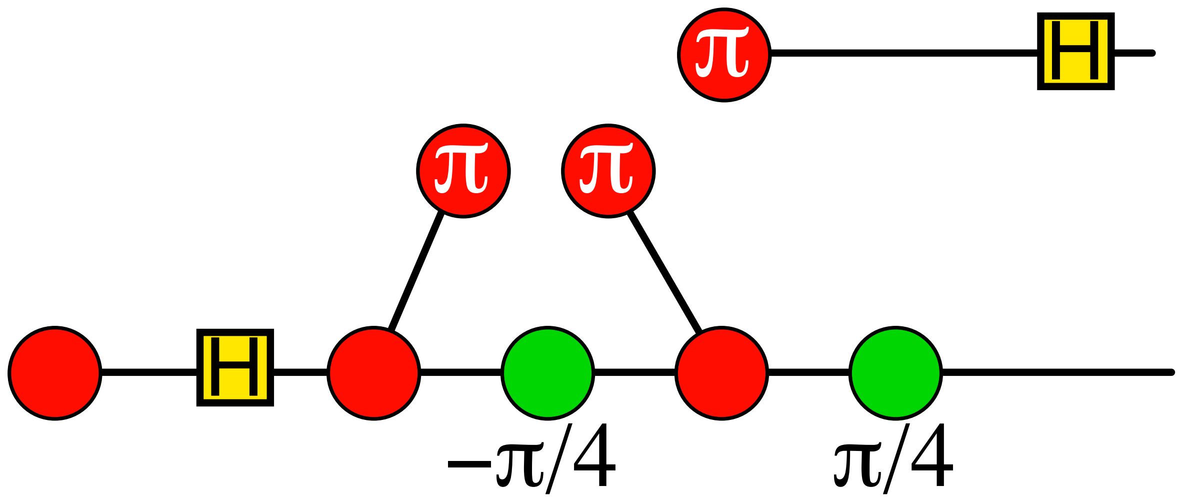
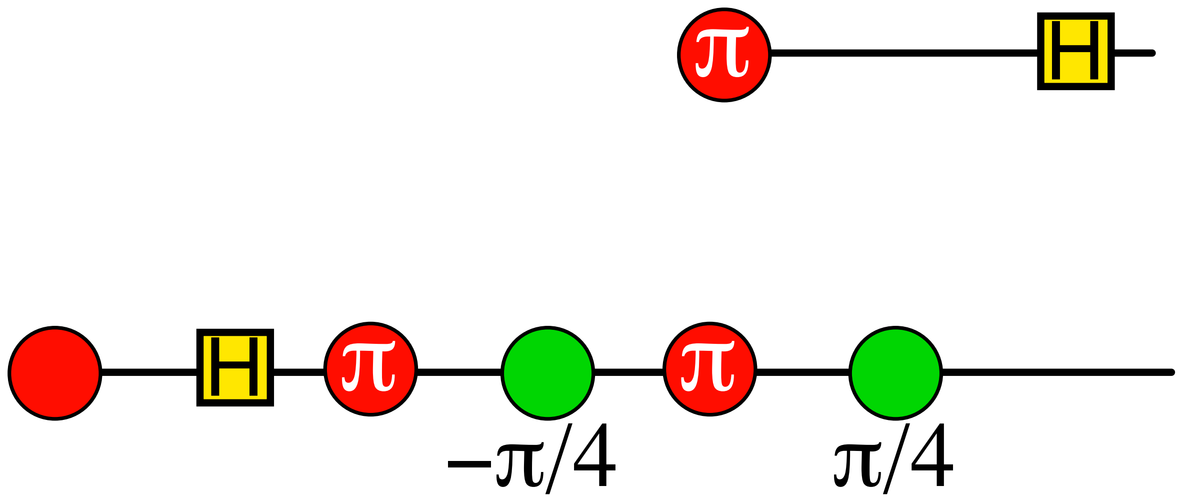


Example: Quantum Fourier Transform



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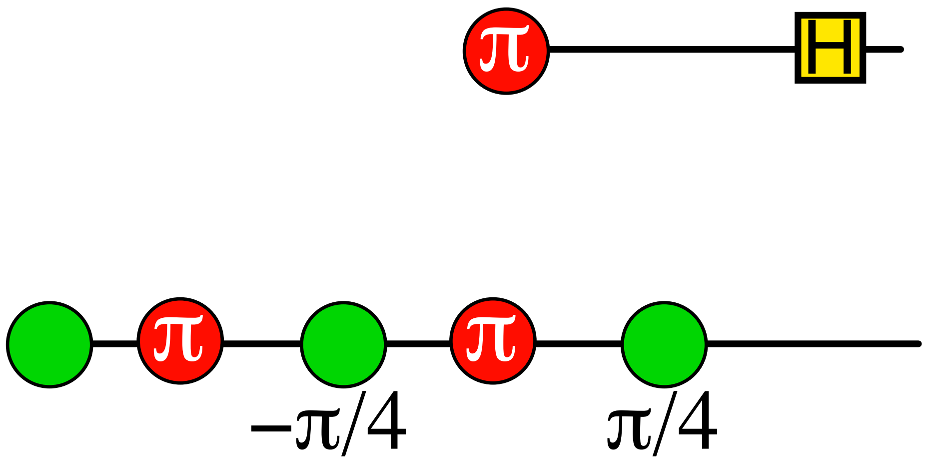

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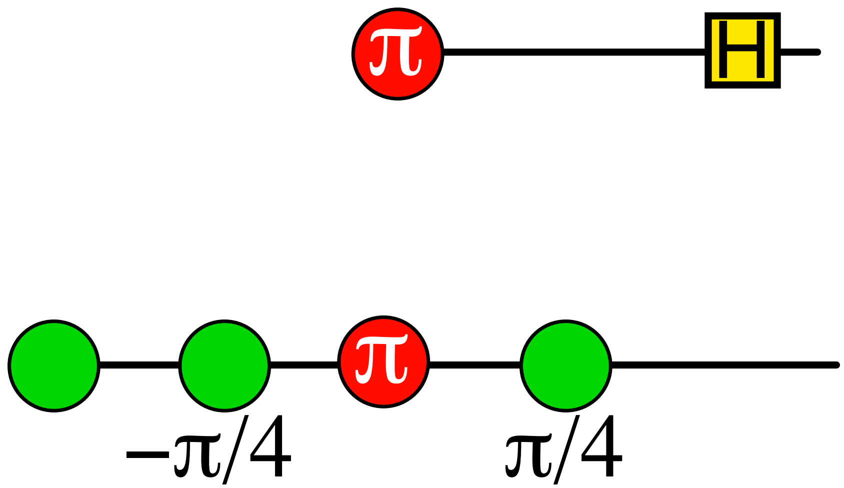
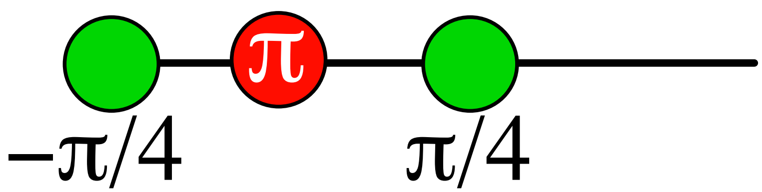
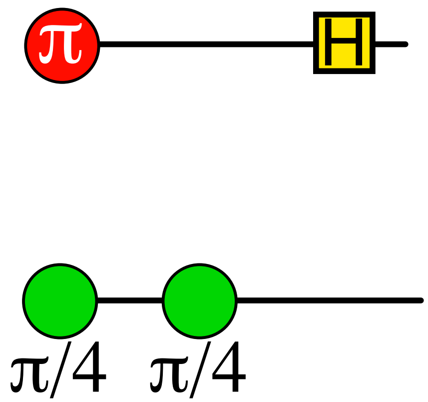


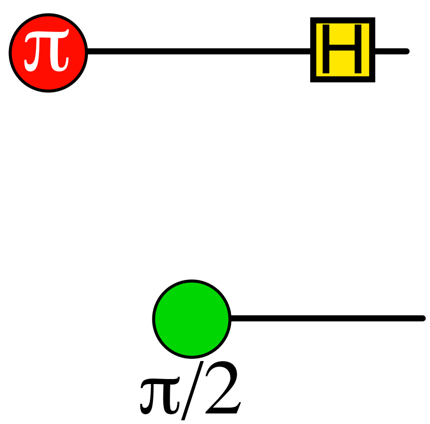
Example: Quantum Fourier Transform



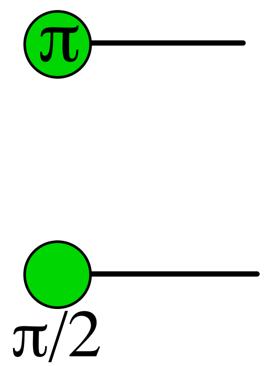
Example: Quantum Fourier Transform



Example: Quantum Fourier Transform



Example: Quantum Fourier Transform



which is the correct result!

Conclusions

- Pairs of incompatible observables form a Hopf algebra-like structure.
- This structure captures a fundamental aspect of quantum mechanics.
- The axioms are sufficiently strong to derive the properties of quantum logic gates and prove the correctness of important quantum algorithms.

Questions and Further Work

- What about completeness?
  - Are two observables sufficient?
  - Can we prove that there is another maximally unbiassed basis for the qubit?
  - What about other dimensionalities?
- How special is the choice of the H map?
- Formal properties:
  - Confluence? Termination?
  - Can this be mechanized?
  - Induction principals for reasoning about graphical rewriting?
- We simulated the QFT algorithm: what is the complexity of this simulation? Can complexity be studied in this setting?