Anyons & ribbon categories

Éric Oliver Paquette

Université de Montréal

Categorical Quantum Logic

-Oxford-

August 12, 2007

Anyons & ribbon categories

1. Anyons, ‘any’what?
2. Categorical language for anyons: Ribbon categories
3. Fusion rules and fusion spaces

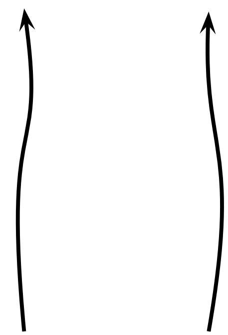
1. Anyons, ‘any’what?

Anyons, ‘any’what?

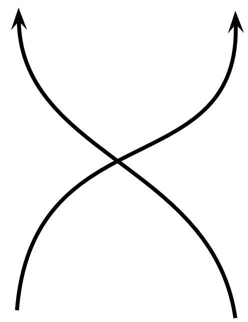
‘Anyon’ is a generic term coined in by Frank Wilczek to describe some particles that can acquire “any” phase when two ore more of them are interchanged, in that sense, they can be seen as a generalization of Bose and Fermi statistics. In fact, such an exchange of two such anyons can be expressed via representations of the braid group and hence, it permits us to encode information in topological features of a system composed of many anyons.

Anyons, ‘any’what?

Why braids? Because anyons live in a 2D world! Let’s give an idea why... Consider two indistinguishable particles moving in a 3D space starting and ending in the same configuration, then there are 2 different classes of paths:



and



i.e., direct and exchange.

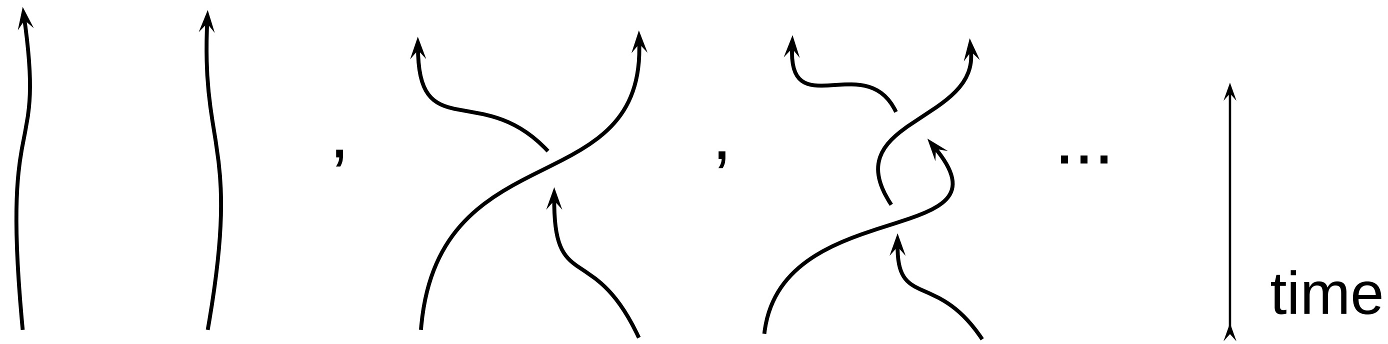
Anyons, 'any'what?

Passing to the relative coordinate space, it is clear that any path where the particles start and finishes in the same position can be contracted to a point which says that the space is, at least, simply connected. In fact, it is doubly connected since any two exchanges taken one after the other reduce to the trivial path. Thus, we really have two classes of path in the end.

Solving this with path integral, one sees that the amplitudes of direct and exchange either add up or contribute with a minus sign thus yielding Bose and Fermi statistics respectively.

Anyons, 'any'what?

Now, in two dimension, the game is a bit more complex – The relative coordinate space is no longer simply connected. We have an infinite number of classes of paths which is specified by the interchange of particles i.e.,



Again, would we solve this system with path, we would find that the factor of the amplitude for n swaps is multiplied by a phase $e^{in\phi}$.

Anyons, ‘any’what?

Of course, if ϕ is 0 or π, this collapses to Bose or Fermi statistics however, the more general type of particles involved in 2D is called *Anyons* as mentioned in the introduction.

Anyons, ‘any’what?

We will describe:

1. A labelling system for our anyons (objects of a category called ‘charge’ of the anyons),
2. Compound systems of anyons (monoidal category where I, the tensor unit, is the trivial charge),
3. Charge conjugation (Rigid structure)
4. A system of morphisms encapsulating the movement of the anyons (braids and ribbons from the categorical structure)
5. A way to calculate the charge of a system of n anyons (fusion rules from a semi-simple structure)

2. Categorical language for anyons: Ribbon categories

Ribbon categories

We start from a monoidal category. The first structure we add is the braid map:

Definition: A *braided monoidal category* $\langle \mathbf{C}, \otimes, I, \sigma \rangle$ is a monoidal category equipped with a natural isomorphism

$$\sigma : A \otimes B \xrightarrow{\sim} B \otimes A.$$

Moreover, it is required that the natural isomorphisms from the braided monoidal signature are such that for all $A, B, C, D \in |\mathbf{C}|$, the following diagrams commute in $\mathbf{C}$:

Ribbon categories

(i) *Pentagon axiom.*

$$
\begin{array}{ccc}
 & ((A \otimes B) \otimes C) \otimes D & \\
 \swarrow^{\alpha \otimes D} & & \searrow^{\alpha} \\
(A \otimes (B \otimes C)) \otimes D & & (A \otimes B) \otimes (C \otimes D) \\
\alpha \downarrow & & \downarrow \alpha \\
A \otimes ((B \otimes C) \otimes D) & \xrightarrow[A \otimes \alpha]{} & A \otimes (B \otimes (C \otimes D))
\end{array}
$$

(ii) *Triangle axiom.*

$$
\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{\alpha} & A \otimes (I \otimes B) \\
 & {}_{\rho \otimes B}\searrow \quad \swarrow_{A \otimes \lambda} & \\
 & A \otimes B &
\end{array}
$$

(iii) *Hexagon axiom.*

$$
\begin{array}{ccccccc}
 & & A\otimes(B\otimes C) & \xrightarrow{\sigma} & (B\otimes C)\otimes A & & \\
 & \overset{\alpha}{\nearrow} & & & & \overset{\alpha}{\searrow} & \\
(A\otimes B)\otimes C & & & & & & B\otimes(C\otimes A) \\
 & \underset{\sigma\otimes C}{\searrow} & & & & \underset{A\otimes\sigma}{\nearrow} & \\
 & & (B\otimes A)\otimes C & \xrightarrow[\alpha]{} & B\otimes(A\otimes C) & &
\end{array}
$$

and the same diagram commuting of σ^{-1} instead of σ.

Ribbon categories

The objects of such category as described above will be labels for our anyons called *charge*. Therefore, we need a notion to express the conjugation of charge and this is given via the notion of duals within our braided monoidal category.

Ribbon categories

Definition: Let **C** be a monoidal category and $A \in |\mathbf{C}|$. A *dual* of A is an object A^* together with the morphisms $i_A : I \to A \otimes A^*$ and $e_A : A^* \otimes A \to I$ that are such that

$$\begin{array}{ccc} A^* & \xrightarrow{A^* \otimes i} & A^* \otimes A \otimes A^* \\ & \searrow^{A^*} & \downarrow{\scriptstyle e \otimes A^*} \\ & & A^* \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{i \otimes A} & A \otimes A^* \otimes A \\ & \searrow^{A} & \downarrow{\scriptstyle A \otimes e} \\ & & A \end{array}$$

commute hence defining a *compact structure* for A.

Ribbon categories

Definition: Let $\mathbf{C}$ be a braided monoidal category then, $\mathbf{C}$ is said to be *rigid* if every $A \in |\mathbf{C}|$ has a dual.

Ribbon categories

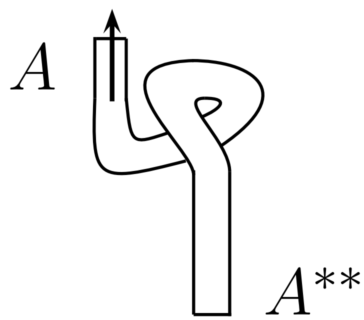
Now, rotating an anyon on itself is not the identity on the system, we therefore need to add yet another rule to our framework. Conceptually, if I rotate an anyon labelled by (say) A on itself by 2π it will be the same as if I rotate it around another anyon with a trivial charge (labelled with the tensor unit I) thus, we can think of the trajectory of this anyon as a belt turning around some centre; straightening the belt induces a ‘twist’ of 2π in it.

Ribbon categories

First, note that in *any* rigid braided monoidal category $\mathbf{C}$, one can define

$\gamma_A : A^{**} \to A$ as $\gamma_A := (A \otimes e_{A^*}) \circ (A \otimes \sigma^{-1}) \circ (i_A \otimes A^{**})$

for any $A \in |\mathbf{C}|$.



Ribbon categories

Definition: A rigid braided monoidal $\mathbf{C}$ is a *ribbon category*[a] if, for all $A \in |\mathbf{C}|$, it comes equipped with a natural isomorphisms

$$\delta_A : A \to A^{**}$$

subject to the following conditions:

i) $\delta_{A \otimes B} = \delta_A \otimes \delta_B$;

ii) $\delta_{A^*} = (\delta_A^*)^{-1}$ and,

iii) $\delta_I = 1$.

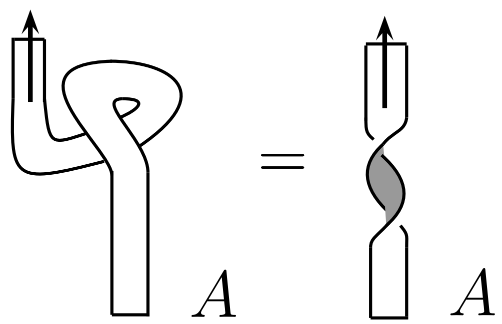
[a] Sometimes called a *tortile* category.

Ribbon categories

This is enough to define the ‘twist’ that I spoke of above:

Definition: Let $\mathbf{C}$ be a ribbon category and $A \in |\mathbf{C}|$. The *twist* map is given by the composite natural isomorphism

$$\theta_A := \gamma_A \circ \delta_A : A \to A.$$



Ribbon categories

[Freyd] Recall that a category is:

a. A category is *preadditive* if its homsets are (additive) abelian groups and the composition of morphism is bilinear.

b. It is *additive* if every finite set of objects has a biproduct.

c. It is *preabelian* if every morphism has both a kernel and a cokernel.

d. It is *abelian* if every monomorphism and epimorphism is normal.

Ribbon categories

We also need:

Definition: Let $\mathbf{C}$ be an abelian category then, $S \in |\mathbf{C}|$ such that $S \not\simeq 0$ is said to be *simple* if for all $B \in |\mathbf{C}|$ any $f : B \hookrightarrow S$ is either the zero morphism or an isomorphism.

This is the same as saying that A has no other subobject than 0 and itself.

Ribbon categories

From this, we have:

Definition: An abelian category $\mathbf{C}$ is *semisimple* if any $A \in |\mathbf{C}|$ is such that

$$A \simeq \bigoplus_{j \in J} N_j S_j$$

with S a simple object, J is the set of isomorphism classes of simple objects and $N_j \in \mathbb{N}$ are such that only a finite number of them are non-zero.

Ribbon categories

This is enough to give our last definition:

Definition: A *semisimple ribbon category* **C** is a semisimple category endowed with a ribbon structure where the tensor unit $I \in |\mathbf{C}|$ is simple, the tensor product is bilinear and where for each simple object $S \in |\mathbf{C}|$, End$(S) \simeq \mathbb{K}$, where $\mathbb{K}$ is a field of characteristic 0.

Ribbon categories

Remark: The correct categorical structure that we need in order to speak of anyons is the one of *modular tensor categories* – ribbon categories satisfying some additional assumptions; I won't get into those details now.

3. Fusion rules and fusion spaces

Fusion

Suppose that two charges S and T can combine in order to give yet another charge U in N_{ST}^{U} ways we will write this as $S \otimes T \simeq N_{ST}^{U} U$; this makes sense since the category is semisimple. There, the lower labels of N_{ST}^{U} then express which labels fuse in order to yield the upper label. We get,

Fusion

Definition: Let S and T be simple object in **C** the *fusion rule* of S and T is

$$S \otimes T \simeq \bigoplus_U N_{ST}^U U.$$

The coefficient $N_{ST}^U = \text{Dim}(\text{Hom}(S \otimes T, U))$ are called the *fusion coefficients*.

Of course, in the above definition, the direct sum over U means that we sum over all classes of isomorphisms of simple objects. We notice also that

$$N_{ST}^U = N_{TS}^U = N_{TU^*}^{S^*} = N_{S^*T^*}^{U^*}$$

Fusion

Now, the computation (braidings, twists...) will occur within the space that encodes the various ways a set of anyons can fuse together according to the fusion rules (associativity of $\otimes$).

Definition: A *fusion space* is a Hilbert space $\mathcal{H}_{ST}^{U}$ of dimension equal N_{ST}^{U} and the states it contains are called *fusion states*.

Fusion

Remark: The way we pass from the configuration space of anyons to a Hilbert space is highly non-trivial; in fact, this is where the modular functor comes in play. We leave this for now in order to keep the exposition simple.

Fusion

Let us introduce the following notation: A fusion state where S fuses with T obtaining a total charge of U in the η^{th} possible way will be denoted by $|ST; U, \eta\rangle$ so that the set of basis vectors that spans $\mathcal{H}_{ST}^{U}$ is

$$\{|ST; U, \eta\rangle \mid \eta = 1, 2, ..., N_{ST}^{U}\}.$$

Note, however, that if we fuse together S and T as above, U may also vary yielding a set of Hilbert space. However, we will compute braiding the anyons S and T and the total charge of the fusion state is the measured charge U ultimately.

Fusion

In that sense, superposition of different final state is not allowed, the superposition really occurs in the various ways the anyons can fuse to yield the total charge U thus, what we actually get is a tuple of Hilbert spaces $\langle \mathcal{H}_{ST}^{U_1}, ..., \mathcal{H}_{ST}^{U_j} \rangle_{j \in J}$ each carrying a different branch of the computation. Each of these spaces are mutually orthogonal in a Hilbert space $\mathcal{H}_{ST}$ of dimension $\sum_{j \in J} N_{ST}^{U_j}$ spanned by the vectors

$$\{|ST; j, \eta\rangle \mid \eta = 1, 2, ..., N_{ST}^{U},\ j \in J\}.$$

where J is the (finite) set of isomorphisms classes of simple objects.

More...

Of course, there is a lot more. Here I just gave an outline of the structure needed for computing with anyons.

Documentation:

- (Physics) Preskill's notes on Topological quantum computation
- (Maths) My master's thesis: `www.inexistant.net`
- (Bridging the gap) Joint –introductory– paper with Prakash Panangaden (in progress)