

# Complementarity in categorical quantum mechanics

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## Abstract

We relate notions of complementarity in three layers of quantum mechanics: (i) von Neumann algebras, (ii) Hilbert spaces, and (iii) orthomodular lattices. Taking a more general categorical perspective of which the above are instances, we consider dagger monoidal kernel categories for (ii), so that (i) become (sub)endohomsets and (iii) become subobject lattices. By developing a ‘point-free’ definition of copyability we link (i) commutative von Neumann subalgebras, (ii) classical structures, and (iii) Boolean subalgebras.

## 1 Introduction

Complementarity is a supporting pillar of the Copenhagen interpretation of quantum mechanics. Unfortunately, Bohr’s own formulation of the principle remained imprecise and flexible [17]. Here, we understand it, roughly, to mean that complete knowledge of a quantum system can only be attained through examining all of its possible classical subsystems [7]. Notice that, perhaps unlike Bohr’s own, this interpretation concerns *all* classical contexts. To emphasize the distinction between talking about *two* totally incompatible classical contexts (as Bohr typically did), and mentioning all of them, we will speak of *complementary* classical contexts only when considering *two* of them. Only taken all together, (pairwise complementary) classical contexts give complete information, and we call them *completely complementary*. This paper considers instances of this interpretation of complementarity with regard to three aspects of quantum mechanics.

- (i) The observables of a quantum system form a von Neumann algebra. In this setting, complete complementarity is customarily taken to mean that one has to look at all commutative von Neumann subalgebras [13, 18].

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- (ii) The states of a quantum system are unit vectors in a Hilbert space, which can be coordinatized by choosing any orthonormal basis. Here, complete complementarity may be interpreted as saying that it takes measurements in all possible orthonormal bases (of many identical copies of a system) to determine its state (perfectly) [20].
- (iii) The measurable properties of a quantum system form an orthomodular lattice. Complete complementarity translates to this perspective as stating that the lattice structure is determined by all Boolean sublattices [15, 10].

In fact, we will take a more general perspective, as all three layers, separately, have recently been studied categorically.

- (i) The set of commutative von Neumann subalgebras of a von Neumann algebra gives rise to a topos of set-valued functors, whose intuitionistic internal logic sheds light on the the original noncommutative algebra in so far as (complete) complementarity is concerned [9, 6].
- (ii) The category of Hilbert spaces can be abstracted to a dagger monoidal category, in which much of quantum mechanics can still be formulated [1]. In this framework, orthonormal bases are characterized as so-called classical structures [4, 5, 2].
- (iii) Orthomodular lattices can be obtained as kernel subobjects in a so-called dagger kernel category [8]. This paper considers Boolean sublattices systematically, in the tradition of *e.g.* [11].

We will take the view that of these three layers, (ii) is the primitive one, which the others derive from. Indeed, our main results are in categories that are simultaneously dagger monoidal categories and dagger kernel categories. We give definitions of complementarity for (i) commutative von Neumann subalgebras, (ii) classical structures, and (iii) Boolean sublattices of the orthomodular lattice of kernels. By developing a notion of copyability, we obtain a bijective correspondence between complementary classical structures and complementary Boolean sublattices. Also, we characterize categorically what complementary commutative von Neumann subalgebras correspond to in terms of classical structures in the category of Hilbert spaces. The plan of the paper is as follows: Sections 2, 3 and 4 study layers (ii), (iii) and (i) respectively. Conclusions are then drawn in Section 5. The author is grateful to Samson Abramsky, Ross Duncan, Klaas Landsman, and Jamie Vicary for useful pointers and discussions.

## 2 Classical structures

**Definition 1** A *classical structure* in a dagger symmetric monoidal category  $\mathbf{D}$  is a commutative semigroup  $\delta: X \rightarrow X \otimes X$  that satisfies  $\delta^\dagger \circ \delta = \text{id}$  and the following so-called H\*-axiom: there is an involution  $*$ :  $\mathbf{D}(I, X)^{\text{op}} \rightarrow \mathbf{D}(I, X)$  such that  $\delta^\dagger \circ (x^* \otimes \text{id}) = (x^\dagger \otimes \text{id}) \circ \delta$ .

This terse definition suffices for this paper, because we will not explicitly use much of a classical structure except its type and the fact that it is dagger monic; for more information we refer to a forthcoming article [2].

The goal of this section is to find out when kernels  $k: K \rightarrow X$  are ‘compatible’ with a given classical structure. To do so, we develop a notion of copyability that has to be ‘point-free’ because  $K$  is typically not the monoidal unit  $I$ .

## 2.1 Kernels and tensor products

Fix a category  $\mathbf{D}$ , and assume it to be a dagger symmetric monoidal category [1] and a dagger kernel category [8] simultaneously, which additionally satisfies

$$\ker(f) \otimes \ker(g) = \ker(f \otimes \text{id}) \wedge \ker(\text{id} \otimes g)$$

for all morphisms  $f$  and  $g$ .<sup>1</sup> The categories **Hilb** and **Rel** both satisfy the above relationship between tensor products and kernels. Some coherence properties follow easily from the assumptions:

$$\begin{aligned} \ker(f) \otimes 0 &= 0, & 0 \otimes \ker(g) &= 0, \\ \ker(f) \otimes \text{id} &= \ker(f \otimes \text{id}), & \text{id} \otimes \ker(g) &= \ker(\text{id} \otimes g), \\ \ker(f) \otimes \text{id} = 0 &\Leftrightarrow \ker(f) = 0, & \text{id} \otimes \ker(g) = 0 &\Leftrightarrow \ker(g) = 0. \end{aligned}$$

Notice that requiring  $\ker(f \otimes g) = \ker(f) \otimes \ker(g)$  would have been too strong, for then  $\ker(f) \otimes \text{id} = \ker(f) \otimes \ker(0) = \ker(f \otimes 0) = \ker(0) = \text{id}$  for any  $f$ . Nevertheless, the following lemma shows that this property does hold when  $f = g$ .

**Lemma 2** *We have  $\ker(f \otimes f) = \ker(f) \otimes \ker(f)$  for any morphism  $f$ .*

**PROOF** By definition of meet,  $\ker(f \otimes f) = \ker(f \otimes \text{id}) \wedge \ker(\text{id} \otimes f)$  is the pullback of  $\ker(f) \otimes \text{id}$  and  $\text{id} \otimes \ker(f)$ . Hence it suffices to prove that  $\ker(f) \otimes \ker(f)$  is a pullback of  $\ker(f) \otimes \text{id}$  and  $\text{id} \otimes \ker(f)$ , too. So suppose that  $(\text{id} \otimes \ker(f)) \circ p = (\ker(f) \otimes \text{id}) \circ q$ .

$$\begin{array}{c} \cdot \\ \begin{array}{ccccc} & & \xrightarrow{p} & & \\ \downarrow \varphi & & & & \\ K \otimes K & \xrightarrow{\ker(f) \otimes \text{id}} & X \otimes K & \xrightarrow{f \otimes \text{id}} & Y \otimes K \\ \downarrow \text{id} \otimes \ker(f) & & \downarrow \text{id} \otimes \ker(f) & & \downarrow \text{id} \otimes \ker(f) \\ K \otimes X & \xrightarrow{\ker(f) \otimes \text{id}} & X \otimes X & \xrightarrow{f \otimes \text{id}} & Y \otimes X \end{array} \\ \downarrow q \end{array}$$

<sup>1</sup>One might consider additional coherence requirements such as  $\ker(f \otimes g) = \ker(f \otimes \text{id}) \vee \ker(\text{id} \otimes g)$ , but these are not necessary for our present purposes.

Then

$$\begin{aligned}
(\text{id} \otimes \ker(f)) \circ (f \otimes \text{id}) \circ p &= (f \otimes \text{id}) \circ (\text{id} \otimes \ker(f)) \circ p \\
&= (f \otimes \text{id}) \circ (\ker(f) \otimes \text{id}) \circ q \\
&= (0 \otimes \text{id}) \circ q \\
&= 0.
\end{aligned}$$

Since  $\ker(f)$  is dagger monic, we find  $(f \otimes \text{id}) \circ p = 0$ . Therefore there exists  $\varphi$  such that  $p = \ker(f \otimes \text{id}) \circ \varphi$ . A symmetric argument shows that  $q = (\text{id} \otimes \ker(f)) \circ \varphi$ ; as  $\ker(f)$  is dagger monic,  $\varphi = (\ker(f)^\dagger \otimes \text{id}) \circ p = (\text{id} \otimes \ker(f)^\dagger) \circ q$  is the unique such morphism. Hence  $\ker(f) \otimes \ker(f)$  is indeed the pullback we were looking for.  $\square$

## 2.2 Copyability

Throughout this section we fix a classical structure  $\delta: X \rightarrow X \otimes X$ .

**Definition 3** A morphism  $k: K \rightarrow X$  is called *copyable* (along  $\delta$ ) when

$$\delta \circ P(k) = P(k \otimes k) \circ \delta,$$

where we write  $P(k) = k \circ k^\dagger$ .

We start by relating our definition of copyable to copyability of vectors as used in [3].

**Lemma 4** *The following are equivalent for a unit vector  $x$  in  $H \in \mathbf{Hilb}$ :*

- (a) *the morphism  $\mathbb{C} \rightarrow H$  defined by  $1 \mapsto x$  is copyable;*
- (b) *there is a phase  $z \in \mathbb{C}$  with  $|z| = 1$  such that  $\delta(x) = z \cdot (x \otimes x)$ ;*
- (c) *there is a unit vector  $x' \in H$  with  $P(x) = P(x')$  and  $\delta(x') = x' \otimes x'$ .*

PROOF For (a) $\Rightarrow$ (b):

$$\delta(x) = (\delta \circ P(x))(x) = (P(x) \otimes P(x)) \circ \delta(x) = \langle x \otimes x | \delta(x) \rangle \cdot (x \otimes x).$$

Taking  $z = \langle x \otimes x | \delta(x) \rangle$  gives  $|z| = \|\langle x \otimes x | \delta(x) \rangle \cdot (x \otimes x)\| = \|\delta(x)\| = \|x\| = 1$ .

Conversely, to see (b) $\Rightarrow$ (a):

$$\begin{aligned}
(\delta \circ P(x))(y) &= \delta \circ x \circ x^\dagger(y) \\
&= \langle x | y \rangle \cdot \delta(x) \\
&= \langle x | y \rangle \cdot z \cdot (x \otimes x) \\
&= \langle \delta(x) | \delta(y) \rangle \cdot z \cdot (x \otimes x) \\
&= |z|^2 \cdot \langle x \otimes x | \delta(y) \rangle \cdot (x \otimes x) \\
&= \langle x \otimes x | \delta(y) \rangle \cdot (x \otimes x) \\
&= (x \otimes x) \circ (x^\dagger \otimes x^\dagger) \circ \delta(y) \\
&= P(x \otimes x) \circ \delta(y).
\end{aligned}$$

Finally, the equivalence of (b) and (c) is established by the equality  $x' = z \cdot x$ .  $\square$

**Example 5** In any dagger kernel category with tensor products satisfying the coherence set out in Section 2.1, zero morphisms and identity morphisms are always copyable:

$$\begin{aligned}\delta \circ P(0) &= \delta \circ 0 = 0 \circ \delta = (0 \otimes 0) \circ \delta = P(0 \otimes 0) \circ \delta, \\ \delta \circ P(\text{id}) &= \delta \circ \text{id} = \delta = (\text{id} \otimes \text{id}) \circ \delta = P(\text{id} \otimes \text{id}) \circ \delta.\end{aligned}$$

These two kernels are called the *trivial* kernels.

**Example 6** In the category **Hilb** of Hilbert spaces, a classical structure  $\delta$  corresponds to the choice of an orthonormal basis  $(e_i)$  [2], whereas a kernel corresponds to a (closed) linear subspace [8]. A kernel is copyable if and only if it is the linear span of a subset of the orthonormal basis.

**Example 7** In the category **Rel** of sets and relations, a classical structure  $\delta$  on  $X$  corresponds to (a disjoint union of) Abelian group structure(s) on  $X$  [14], and a kernel corresponds to a subset  $K \subseteq X$  [8]. A kernel is copyable iff

$$\begin{aligned}\{(x \cdot y, (x, y)) \mid x, y \in X, x \cdot y \in K\} &= \delta \circ P(k) \\ &= P(k \otimes k) \circ \delta \\ &= \{(x \cdot y, (x, y)) \mid x \in K, y \in K\},\end{aligned}$$

*i.e.* if and only if  $x \in K \wedge y \in K \Leftrightarrow x \cdot y \in K$ . One direction of this equivalence implies that  $K$  is a subsemigroup. Fixing  $k \in K$ , we see that for any  $x \in X$  there is  $y = x^{-1} \cdot k$  such that  $x \cdot y \in K$ . Therefore, the other direction implies that  $x \in K$ . That is,  $K = X$ . We conclude that the only copyable kernels in **Rel** are the trivial ones.

**Lemma 8** A dagger monic  $k$  is copyable if and only if there is a (unique) morphism  $\delta_k$  making the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{k^\dagger} & K & \xrightarrow{k} & X \\ \delta \downarrow & & \downarrow \delta_k & & \downarrow \delta \\ X \otimes X & \xrightarrow{k^\dagger \otimes k^\dagger} & K \otimes K & \xrightarrow{k \otimes k} & X \otimes X \end{array}$$

PROOF

$k$  is copyable

$$\iff (P(k) \otimes P(k)) \circ \delta = \delta \circ P(k)$$

$$\iff (P(k) \otimes P(k)) \circ \delta = (P(k) \otimes P(k)) \circ \delta \circ P(k) = \delta \circ P(k)$$

$$\iff \exists \delta_k. \delta \circ k = (k \otimes k) \circ \delta_k, \delta_k \circ k^\dagger = (k^\dagger \otimes k^\dagger) \circ \delta. \quad \square$$

**Proposition 9** If  $k$  is a copyable dagger monic,  $\delta_k$  is a classical structure.

PROOF We say that  $f$  is a dagger retract of  $g$  if there are dagger monics  $a$  and  $b$  making the following diagram commute:

$$\begin{array}{ccccc} \cdot & \xrightarrow{a} & \cdot & \xrightarrow{a^\dagger} & \cdot \\ f \downarrow & & \downarrow g & & \downarrow f \\ \cdot & \xrightarrow{b} & \cdot & \xrightarrow{b^\dagger} & \cdot \end{array}$$

Notice that if  $f$  and  $f'$  are both dagger retracts of  $g$  (along the same  $a$  and  $b$ ), then  $f = b^\dagger \circ b \circ f = b^\dagger \circ g \circ a = f' \circ a^\dagger \circ a = f'$ .

Now, if  $k$  is a copyable dagger monic, then it follows from Lemma 8 that  $\delta_k$  is a dagger retract of  $\delta$ , and  $\delta_k^\dagger$  is a dagger retract of  $\delta^\dagger$ . Therefore,  $\delta_k$  is associative, commutative, and is dagger monic. For example, to verify commutativity, notice that  $\gamma_k: K \otimes K \rightarrow K \otimes K$  is a dagger retract of  $\gamma: X \otimes X \rightarrow X \otimes X$ . Since dagger retracts compose, this means that  $\gamma_k \circ \delta_k$  and  $\delta_k$  are both dagger retracts (along the same morphisms) of  $\gamma \circ \delta = \delta$ . Hence  $\gamma_k \circ \delta_k = \delta_k$ . The other algebraic properties are verified similarly (including the Frobenius equation).

We are left to check the H\*-axiom. Let  $x: I \rightarrow K$ . Since  $\delta$  satisfies the H\*-axiom, there is  $(k \circ x)^*: I \rightarrow X$  such that

$$\delta^\dagger \circ ((k \circ x)^* \otimes \text{id}) = ((k \circ x)^\dagger \otimes \text{id}) \circ \delta.$$

Now put  $x^* = k^\dagger \circ (k \circ x)^*: I \rightarrow K$ . Then:

$$\begin{aligned} \delta_k^\dagger \circ (x^* \otimes \text{id}) &= \delta_k^\dagger \circ (k^\dagger \otimes \text{id}) \circ ((k \circ x)^* \otimes \text{id}) \\ &= \delta_k^\dagger \circ (k^\dagger \otimes k^\dagger) \circ ((k \circ x)^* \otimes \text{id}) \circ k \\ &= k^\dagger \circ \delta^\dagger \circ ((k \circ x)^* \otimes \text{id}) \circ k \\ &= k^\dagger \circ ((k \circ x)^\dagger \otimes \text{id}) \circ \delta \circ k \\ &= (x^\dagger \otimes \text{id}) \circ (k^\dagger \otimes k^\dagger) \circ \delta \circ k \\ &= (x^\dagger \otimes \text{id}) \circ \delta_k \circ k^\dagger \circ k \\ &= (x^\dagger \otimes \text{id}) \circ \delta_k. \end{aligned}$$

Hence  $\delta_k$  satisfies the H\*-axiom, too.  $\square$

**Corollary 10** *A dagger monic  $k$  is copyable if and only if its domain carries a classical structure  $\delta_k$  and  $k$  is simultaneously a (non-unital) monoid homomorphism and a (non-unital) comonoid homomorphism.*  $\square$

It stands to reason to define categories of classical structures to have morphisms as in the previous corollary; this also matches [12, 2.4.4].

## 2.3 Complementarity and mutual unbiasedness

**Definition 11** Two classical structures are *complementary* if no nontrivial kernel is simultaneously copyable along both.

The above definition clashes with complementarity of classical structures as defined in [3]. Let us spend some time developing a notion that does correspond to complementarity in the sense of [3] directly.

**Definition 12** A morphism  $x: U \rightarrow X$  is called *unbiased* (relative to  $\delta$ ) if and only if  $P(x^\dagger \circ k) = P(x^\dagger \circ l)$  for all copyable kernels  $k$  and  $l$ .

Recall that if the ambient category is simply well-pointed, then  $\text{KSub}(X)$  is atomic, and its atoms are precisely the kernels with domain  $I$  [8]. If such ‘points’ are unbiased in the above sense, then they are unbiased vectors in the sense of [3].

Another advantage of the previous definition in this point-based setting is that it does not need to specify what the scalars  $\langle k | x \rangle$  are. In the traditional point-based setting this scalar involves the dimension of the carrier Hilbert space, and is therefore limited to finite-dimensional spaces. The above definition can be interpreted regardless of dimensional aspects.

**Lemma 13** *If a nonzero kernel is copyable (along  $\delta$ ) then it is not unbiased (relative to  $\delta$ ).*

PROOF Let  $x$  be a nonzero copyable kernel. Then  $x^\perp$  is copyable, too, by Lemma 17. The trivial kernels are always copyable by Example 5. Hence  $k = \text{id}$  and  $l = x^\perp$  are both copyable kernels. But

$$P(x^\dagger \circ k) = P(x^\dagger) = x^\dagger \circ x \neq 0 = P(0) = P(x^\dagger \circ \ker(x^\dagger)) = P(x^\dagger \circ l),$$

and therefore  $x$  cannot be unbiased. □

**Definition 14** Two classical structures are *mutually unbiased* if a nontrivial kernel is unbiased relative to one whenever it is copyable along the other.

**Proposition 15** *Mutually unbiased classical structures are complementary.*

PROOF Suppose that  $\delta$  and  $\delta'$  are mutually unbiased, and let  $k$  be a kernel in the intersection. That is,  $k$  is copyable along both  $\delta$  and  $\delta'$ . By Lemma 13,  $k$  cannot be unbiased relative to  $\delta'$ . This contradicts mutual unbiasedness unless  $k$  were a trivial kernel. □

There is no converse to the previous proposition. For example, consider the object  $\mathbb{C}^2$  in the category **Hilb**. A classical structure corresponds to an orthonormal basis of  $\mathbb{C}^2$ , and hence corresponds (up to sign) to a single ray. The collections of copyables of classical structures induced by two different rays always have trivial intersection. But certainly not every pair of different orthonormal bases is mutually unbiased. We conclude that Definition 14 is too strong for our purposes.

### 3 Boolean subalgebras of orthomodular lattices

This section concerns level (iii) of the Introduction. We will prove that kernels that are copyable along  $\delta$  form a Boolean subalgebra of the orthomodular lattice of all kernel subobjects of  $X$ .

**Lemma 16** *The copyable kernels form a sub-meetsemilattice of  $\text{KSub}(X)$ .*

PROOF The bottom element 0 is always copyable by Example 5. So we have to prove that if  $k$  and  $l$  are copyable kernels, then so is  $k \wedge l$ . Recall that  $k \wedge l$  is defined as the pullback.

$$\begin{array}{ccc} K \wedge L & \xrightarrow{p} & L \\ q \downarrow & \lrcorner & \downarrow l \\ K & \xrightarrow{k} & X \end{array}$$

Together with the assumption that  $k$  and  $l$  are copyable, this means that the top, back, right and bottom face of the following cube commute:

$$\begin{array}{ccccc} & & K & \xrightarrow{k} & X \\ & \nearrow q & \uparrow & & \nearrow l \\ K \wedge L & \xrightarrow{p} & L & & \\ \uparrow \varphi & & \delta_k^\dagger \downarrow & & \uparrow \delta_l^\dagger \\ & \nearrow q \otimes q & K \otimes K & \xrightarrow{k \otimes k} & X \otimes X \\ & & \uparrow & & \uparrow l \otimes l \\ (K \wedge L)^{\otimes 2} & \xrightarrow{p \otimes p} & L \otimes L & & \end{array}$$

Hence  $l \circ \delta_l^\dagger \circ (p \otimes p) = k \circ \delta_k^\dagger \circ (q \otimes q)$ . Therefore, by the universal property of pullbacks, there exists a dashed morphism  $\varphi$  making the left and front sides of the above cube commute. Using the fact that  $p$  and  $q$  are dagger monic, we deduce  $\varphi = (k \wedge l)^\dagger \circ \delta^\dagger \circ ((k \wedge l) \otimes (k \wedge l))$ . This means that the left square in the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{(k \wedge l)^\dagger} & K \wedge L & \xrightarrow{k \wedge l} & X \\ \delta \downarrow & & \downarrow \varphi^\dagger & & \downarrow \delta \\ X \otimes X & \xrightarrow{(k \wedge l)^\dagger \otimes (k \wedge l)^\dagger} & (K \wedge L) \otimes (K \wedge L) & \xrightarrow{(k \wedge l) \otimes (k \wedge l)} & X \otimes X \end{array}$$

The right square is seen to commute analogously—take daggers of all the vertical morphisms in the cube. Therefore the whole rectangle commutes. In other words,  $\delta \circ P(k \wedge l) = (P(k \wedge l) \otimes P(k \wedge l)) \circ \delta$ , that is,  $k \wedge l$  is copyable.  $\square$

**Lemma 17** *The copyable kernels form an orthocomplemented sublattice of the orthomodular lattice  $\text{KSub}(X)$ .*

PROOF We have to prove that if  $k$  is a copyable kernel, then so is  $k^\perp = \ker(k^\dagger)$ .

$$\begin{array}{ccccccccc}
K & \xrightarrow{k} & X & \xrightarrow{(k^\perp)^\dagger} & K^\perp & \xrightarrow{k^\perp} & X & \xrightarrow{k^\dagger} & K \\
\uparrow \delta_k^\dagger & & \uparrow \delta^\dagger & & \uparrow \delta^\dagger & & \uparrow \delta^\dagger & & \uparrow \delta_k^\dagger \\
K \otimes K & \xrightarrow{k \otimes k} & X \otimes X & \xrightarrow{(k^\perp)^\dagger \otimes (k^\perp)^\dagger} & K^\perp \otimes K^\perp & \xrightarrow{k^\perp \otimes k^\perp} & X \otimes X & \xrightarrow{k^\dagger \otimes k^\dagger} & K \otimes K \\
& & & & \uparrow f & & & & \\
& & & & g & & & & 
\end{array}$$

Since  $k$  is copyable, we have  $k^\dagger \circ \delta^\dagger \circ (k^\perp \otimes k^\perp) = \delta_k^\dagger \circ (k^\dagger \otimes k^\dagger) \circ (k^\perp \otimes k^\perp) = \delta_k^\dagger \circ (0 \otimes 0) = 0$ , so that the dashed arrow  $f$  in the above diagram exists, making the square to its right commute. Since  $k^\perp$  is dagger monic,  $f$  must equal  $\text{coker}(k) \circ \delta^\dagger \circ (\ker(k^\dagger) \otimes \ker(k^\dagger))$ .

Similarly, it follows from copyability of  $k$  that  $(k^\perp)^\dagger \circ \delta_k^\dagger \circ (k \otimes k) = 0$ , so that the dashed arrow  $g$  exists. Since  $g$  must be  $\text{coker}(k) \circ \delta^\dagger \circ (\text{coker}(k)^\dagger \otimes \text{coker}(k)^\dagger)$ , we see that  $f$  and  $g$  coincide. Hence the rectangle composed of the middle two squares commutes. Taking its dagger yields the following commutative diagram:

$$\begin{array}{ccccc}
& & P(k^\perp) & & \\
& \curvearrowright & & \curvearrowleft & \\
X & \xrightarrow{(k^\perp)^\dagger} & K^\perp & \xrightarrow{k^\perp} & X \\
\downarrow \delta & & \downarrow f^\dagger & & \downarrow \delta \\
X \otimes X & \xrightarrow{(k^\perp)^\dagger \otimes (k^\perp)^\dagger} & K^\perp \otimes K^\perp & \xrightarrow{k^\perp \otimes k^\perp} & X \otimes X \\
& \curvearrowleft & & \curvearrowright & \\
& & P(k^\perp) \otimes P(k^\perp) & & 
\end{array}$$

That is,  $k^\perp$  is copyable.  $\square$

Notice that if the classical structure had a unit  $\varepsilon$ , the previous result would have been impossible if we had additionally demanded  $\varepsilon \circ P(k) = \varepsilon$  for  $k$  to be copyable, since then  $\varepsilon = \varepsilon \circ P(k^\perp) = \varepsilon \circ P(k) \circ P(k^\perp) = \varepsilon \circ P(k \wedge k^\perp) = \varepsilon \circ 0 = 0$ . Compare [2].

**Lemma 18** [8, Theorem 1] *An orthocomplemented sublattice  $L$  of  $\text{KSub}(X)$  is Boolean if and only if  $k \wedge l = 0$  implies  $l^\dagger \circ k = 0$  for all  $k, l \in L$ .  $\square$*

**Theorem 19** *The copyable kernels form a Boolean subalgebra of the orthomodular lattice  $\text{KSub}(X)$ .*

PROOF By the previous lemmas, it suffices to prove that if  $k \wedge l = 0$  for copyable kernels  $k$  and  $l$ , then  $l^\dagger \circ k = 0$ . So let  $k$  and  $l$  be copyable kernels and suppose  $k \wedge l = 0$ . Say  $k = \ker(f)$  and  $l = \ker(g)$ . Then

$$(f \otimes \text{id}) \circ (k \otimes l) = (f \circ k) \otimes l = 0 \otimes l = 0,$$

so that  $k \otimes l \leq \ker(f \otimes \text{id}) = k \otimes \text{id} \leq (k \otimes \text{id}) \wedge (\text{id} \otimes k) = k \otimes k$ . Similarly,  $k \otimes l \leq l \otimes l$ . Therefore the bottom, top, back and right faces of the following cube commute:

$$\begin{array}{ccccc}
& & K & \xrightarrow{k} & X \\
& \nearrow 0 & \uparrow & & \nearrow l \\
0 & \xrightarrow{0} & L & & X \\
& \uparrow \delta_k^\dagger & \uparrow \delta_l^\dagger & & \uparrow \delta^\dagger \\
\varphi \downarrow & & K \otimes K & \xrightarrow{k \otimes k} & X \otimes X \\
& \nearrow \text{id} \otimes (k^\dagger \circ l) & \nearrow k \otimes l & & \nearrow l \otimes l \\
K \otimes L & \xrightarrow{(l^\dagger \circ k) \otimes \text{id}} & L \otimes L & & X \otimes X
\end{array}$$

The universal property of the pullback formed by the top face yields the dashed morphism  $\varphi$  making the left and front faces commute. Hence  $\delta_l^\dagger \circ ((l^\dagger \circ k) \otimes \text{id}) = 0$ . But then, as  $k$  and  $l$  are copyable:

$$\begin{aligned}
l^\dagger \circ k &= l^\dagger \circ k \circ \delta_k^\dagger \circ \delta_k \\
&= l^\dagger \circ \delta^\dagger \circ (k \otimes k) \circ \delta_k \\
&= \delta_l^\dagger \circ (l^\dagger \otimes l^\dagger) \circ (k \otimes k) \circ \delta_k \\
&= \delta_l^\dagger \circ ((l^\dagger \circ k) \otimes \text{id}) \circ (\text{id} \otimes (l^\dagger \circ k)) \circ \delta_k \\
&= 0 \circ (\text{id} \otimes (l^\dagger \circ k)) \circ \delta_k \\
&= 0.
\end{aligned}$$

□

In the category **Hilb**, Example 6 shows that the copyable kernels in fact form a maximal Boolean subalgebra. Example 7 shows that this does not hold generally. For in **Rel**, one has  $\text{KSub}(X) = \mathcal{P}(X)$ , but the copyable kernels are always  $\{0, 1\}$ , which are only maximal if  $X$  has cardinality 1. Similarly, not every maximal Boolean subalgebra  $B$  of  $\text{KSub}(X)$  induces a classical structure on  $X$  of which  $B$  are the copyables, as does happen to be the case in **Hilb**.

The following definition expresses the standard view in (order-theoretic) quantum logic that Boolean subalgebras of orthomodular lattices are regarded as embodying complete complementarity. It is precisely what is needed to make Theorem 21 true.

**Definition 20** Two Boolean subalgebras of an orthomodular lattice are called *complementary* when they have trivial intersection.

**Theorem 21** *Two classical structures are complementary if and only if their collections of copyable kernels are complementary.* □

Hence we have linked, fully abstractly, complementarity in the order-theoretic sense to complementarity in the sense of classical structures.

## 4 Von Neumann algebras

Finally, this section advances to level (i) of the Introduction. We instantiate the dagger monoidal kernel category  $\mathbf{D}$  to be  $\mathbf{Hilb}$ . For any object  $H \in \mathbf{Hilb}$ , the endomset  $A = \mathbf{Hilb}(H, H)$  is then a type I von Neumann algebra. At this level, the notion of complementarity is formalized by considering all commutative von Neumann subalgebras  $C$  of  $A$ . We denote the collection of all such subalgebras of  $A$  by  $\mathcal{C}(A)$ . Let us recall some facts about this situation.

- (a) The set  $\text{Proj}(A) = \{p \in A \mid p^\dagger = p = p^2\}$  of projections is a complete, atomic, atomistic orthomodular lattice [16, p85].
- (b) There is an order isomorphism  $\text{Proj}(A) \cong \text{KSub}(H)$  [8, Proposition 12].
- (c) Any von Neumann algebra is generated by its projections [16, 6.3], so in particular  $C = \text{Proj}(C)''$ .
- (d) Since  $C$  is a subalgebra of  $A$ , also  $\text{Proj}(C)$  is a sublattice of  $\text{Proj}(A)$ .
- (e) Because  $C$  is commutative,  $\text{Proj}(C)$  is a Boolean algebra [16, 4.16].

The following lemma draws a conclusion of interest from these facts.

**Lemma 22** *Commutative von Neumann subalgebras  $C$  of  $A = \mathbf{Hilb}(H, H)$  are in bijective correspondence with Boolean subalgebras of  $\text{KSub}(H)$ .*

PROOF As in (b) above, we identify  $\text{Proj}(A)$  with  $\text{KSub}(H)$ . A commutative subalgebra  $C$  corresponds to the Boolean subalgebra  $\text{Proj}(C)$ . Conversely, a Boolean subalgebra  $B$  corresponds to the commutative subalgebra  $B''$  generated by it. These mappings are inverses because  $\text{Proj}(C)'' = C$  and  $\text{Proj}(B'') = B$ .  $\square$

We now set out to establish the relation between commutative subalgebras of  $A$  and classical structures on  $H$ .

**Lemma 23** *An orthocomplemented sublattice  $L$  of  $\text{KSub}(H)$  is Boolean if and only if the following equivalent conditions hold:*

- *there exists a classical structure on the greatest element of  $L$  along which every element of  $L$  is copyable;*
- *there exists a classical structure on  $H$  along which every element of  $L$  is copyable.*

PROOF Necessity is established by Theorem 19. For sufficiency, let  $L$  be a Boolean sublattice of  $\text{KSub}(H)$ . Since  $\text{KSub}(H)$  is complete by (a) above,  $\bigvee L$  exists. By atomicity (a),  $\bigvee L$  is completely determined by the set of atoms  $a_i$  below it. By definition of atoms,  $a_i \wedge a_j = 0$  when  $i \neq j$ . Because  $L$  is Boolean, it follows from Lemma 18 that  $a_i$  and  $a_j$  are orthogonal. Also, because  $\mathbf{Hilb}$  is simply well-pointed, the kernels  $a_i$  correspond to one-dimensional subspaces [8, Lemma 11]. That is, the  $a_i$  give an orthonormal basis for (the domain of) the greatest element of  $L$  (which can be extended to an orthonormal basis of  $H$ ). This, in turn, induces a classical structure  $\delta$  on  $L$  (or  $H$ ) [2]. Finally, Example 6 shows that the kernels  $a_i$ , and hence all  $l \in L$ , are copyable along  $\delta$ .  $\square$

**Theorem 24** For the von Neumann algebra  $A = \mathbf{Hilb}(H, H)$ :

$$\mathcal{C}(A) \cong \{L \subseteq \mathbf{KSub}(H) \mid L \text{ orthocomplemented sublattice,} \\ \exists \delta: 1_L \rightarrow 1_L \otimes 1_L \forall l \in L [l \text{ copyable along } \delta]\}.$$

PROOF This is just a combination of Lemma 22 and Lemma 23.  $\square$

The previous theorem implies that for any classical structure  $\delta$  on  $H$ , there is an induced commutative von Neumann subalgebra  $C \in \mathcal{C}(A)$  corresponding to the lattice  $L$  of all copyable kernels. The following definition and corollary finish the connections of complementarity across the three levels discussed in the Introduction.

**Definition 25** Two commutative von Neumann subalgebras of  $\mathbf{Hilb}(H, H)$  are *complementary* when their intersection is the trivial subalgebra  $\{z \cdot \text{id} \mid z \in \mathbb{C}\}$ .

**Corollary 26** Two classical structures on an object  $H$  in  $\mathbf{Hilb}$  are *complementary* if and only if they induce complementary commutative von Neumann subalgebras of  $\mathbf{Hilb}(H, H)$ .  $\square$

Unlike  $\text{Proj}(A)$ , the sublattice  $\text{Proj}(C)$  is not atomic for general  $C \in \mathcal{C}(A)$ ; for a counterexample, take  $H = L^2([0, 1])$  and  $C = L^\infty([0, 1])$ . If this does happen to be the case, for example if we restrict the ambient category  $\mathbf{D}$  to that of finite-dimensional Hilbert spaces, we can strengthen the characterization of  $\mathcal{C}(A)$  in Theorem 24.

**Proposition 27** For a finite-dimensional Hilbert space  $H$  and the von Neumann algebra  $A = \mathbf{fdHilb}(H, H)$ :

$$\mathcal{C}(A) \cong \{(k_i)_{i \in I} \mid k_i \in \mathbf{KSub}(H) \setminus \{0\}, k_i \wedge k_j = 0 \text{ for } i \neq j, \\ \exists \delta: H \rightarrow H \otimes H [k_i \text{ copyable along } \delta]\}.$$

PROOF Since every  $C \in \mathcal{C}(A)$  is finite-dimensional and hence  $\text{Proj}(C)$  is atomic, an orthocomplemented sublattice  $L$  as in Theorem 24 is completely determined by its atoms  $(k_i)$ .  $\square$

Every finite-dimensional  $C^*$ -algebra is a von Neumann algebra, and hence in case  $H$  is finite-dimensional and  $A = \mathbf{Hilb}(H, H)$ , we find that  $\mathcal{C}(A)$  is the collection of all commutative  $C^*$ -subalgebras of  $A$ . Notice that the characterization of Proposition 27 above has no need for the cumbersome combinatorial symmetry considerations of [9, 1.4.5]. Corollary 10 gives another characterization of  $\mathcal{C}(A)$  in the finite-dimensional case, purely in terms of classical structures and their morphisms.

**Proposition 28** For a finite-dimensional Hilbert space  $H$  and the von Neumann algebra  $A = \mathbf{fdHilb}(H, H)$ :

$$\mathcal{C}(A) \cong \{(\delta_i)_{i \in I} \mid \delta_i, \delta_j \text{ classical structures, complementary when } i \neq j, \\ \exists \delta: H \rightarrow H \otimes H \forall_i \exists_{k_i}: \delta_i \rightarrow \delta [k_i \text{ morphism of classical structures}]\}.$$

Hence  $\mathcal{C}(A)$  is isomorphic to the collection of cocones in the category of classical substructures on  $H$  that are pairwise complementary.  $\square$

## 5 Concluding remarks

Observing the similarities across the three levels of quantum mechanics considered, we now propose the following precise formulation of complete complementarity.

A collection of classical structures is *completely complementary* when its members are pairwise complementary and jointly epic.

Compare also [19]. Notice that this formulation is almost information-theoretic.

The view on  $\mathcal{C}(A)$  provided by Section 4 holds several promises for the study of functors on  $\mathcal{C}(A)$  that we intend to explore further in future work:

- One can consider variations in the study of **Set**-valued functors on  $\mathcal{C}(A)$  by choosing different morphisms on  $\mathcal{C}(A)$ : *e.g.* inclusions [9], or reverse inclusions [6]. In the above perspective, the natural direction that suggests itself is that of morphisms between classical structures, *i.e.* inclusions. Moreover, a more interesting choice of morphisms based on classical structures (see *e.g.* [4]) could make  $\mathcal{C}(A)$  into a category that is not just a partially ordered set.
- The topos of functors on  $\mathcal{C}(A)$  can be abstracted away from **Hilb** to any dagger monoidal kernel category that satisfies a suitable ‘spectral assumption’ linking commutative submonoids of endohomsets to classical structures. For example, one could lift Theorem 24 or even Proposition 28 to a definition, and study **Set**-valued functors on these characterizations of  $\mathcal{C}(A)$  in any dagger monoidal kernel category.

In fact, in this generalized setting, there is no need for the base category to be **Set**. After all, the basic objects of study of *e.g.* [6] are really partial orders of subobjects in a functor category. This just happens to be a Heyting algebra because the functors take values in the topos **Set**, but in principle less structured partial orders of subobjects are just as interesting, and perhaps are also justifiable physically.

- One of the weak points of the study of functors on  $\mathcal{C}(A)$  to date is that there is no obvious way to study compound systems. That is, there is no obvious (satisfactory) relation between  $\mathcal{C}(A \otimes B)$  and  $\mathcal{C}(A)$  and  $\mathcal{C}(B)$ . Considering  $A$  as (a submonoid of) an endohomset opens the broader context of a fibred setting in which studying entanglement is possible.

All in all, the above considerations strongly suggest studying fibrations of all classical structures over all objects of a dagger (kernel) monoidal category. As a first step, we mention the following lemma, showing that copyable kernels are stable under pullback (along morphisms of classical structures).

**Lemma 29** *Suppose  $f: X \rightarrow Y$  is a morphism of classical structures from  $\delta_X: X \rightarrow X \otimes X$  to  $\delta_Y: Y \rightarrow Y \otimes Y$ . If a kernel  $k: K \rightarrow X$  is copyable along  $\delta_Y$ , then the kernel  $f^*(k)$  is copyable along  $\delta_X$ .*

PROOF By Lemma 2, we have

$$\begin{aligned}
(f \otimes f)^*(k \otimes k) &= \ker(\text{coker}(k \otimes k) \circ (f \otimes f)) \\
&= \ker((\text{coker}(k) \otimes \text{coker}(k)) \circ (f \otimes f)) \\
&= \ker((\text{coker}(k) \circ f) \otimes (\text{coker}(k) \circ f)) \\
&= \ker(\text{coker}(k) \circ f) \otimes \ker(\text{coker}(k) \circ f) \\
&= (f^*k) \otimes (f^*k).
\end{aligned}$$

This makes the pullback in the bottom face of the following cube commute.

$$\begin{array}{ccccc}
& & f^*K & \xrightarrow{\quad} & K \\
& f^*k \swarrow & \downarrow \varphi & \searrow f & \downarrow \delta_k \\
X & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & K \\
\delta_X \downarrow & & \downarrow \delta_Y & & \downarrow \delta_k \\
X \otimes X & \xrightarrow{\quad} & Y \otimes Y & \xrightarrow{\quad} & K \otimes K \\
& (f^*k)^{\otimes 2} \swarrow & \downarrow & \searrow k \otimes k & \\
& & (f^*K)^{\otimes 2} & \xrightarrow{\quad} & K \otimes K
\end{array}$$

The pullback in the top face commutes by definition. The front and right faces commute because  $f$  is a morphism of classical structures and  $k$  is copyable. Hence the dashed morphism  $\varphi$  exists making the entire cube commute.

Similarly, taking daggers of the solid vertical morphisms and using the universal property of the pullback in the top face yields a morphism  $\psi: (f^*K)^{\otimes 2} \rightarrow f^*K$  such that  $(f^*k) \circ \psi = \delta_X^\dagger \circ (f^*k)^{\otimes 2}$ . Using the fact that  $f^*k$  is dagger monic, one deduces

$$\psi = (f^*k)^\dagger \circ \delta_X^\dagger \circ ((f^*k) \otimes (f^*k)) = \varphi^\dagger.$$

Therefore  $f^*k$  is copyable by Lemma 8.  $\square$

Finally, we remark that we have not used the H\*-axiom (or the Frobenius equation) at all in this paper. Apparently, the combination of (copyable) kernels with the dagger monic type  $X \rightarrow X \otimes X$  of classical structures suffices for these purposes.

## References

- [1] Samson Abramsky and Bob Coecke. Categorical quantum mechanics. In *Handbook of quantum logic and quantum structures: quantum logic*, pages 261–324. Elsevier, 2009.
- [2] Samson Abramsky and Chris Heunen. Classical structures in infinite-dimensional categorical quantum mechanics. *preprint*, 2010.
- [3] Bob Coecke and Ross Duncan. Interacting quantum observables: Categorical algebra and diagrammatics. In *Automata, Languages and Programming, ICALP 2008*, number 5126 in Lecture Notes in Computer Science, pages 298–310. Springer, 2008.

- [4] Bob Coecke and Duško Pavlović. Quantum measurements without sums. In *Mathematics of Quantum Computing and Technology*. Taylor and Francis, 2007.
- [5] Bob Coecke, Duško Pavlović, and Jamie Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science*, 2009.
- [6] Andreas Döring and Christopher J. Isham. ‘What is a thing?’: Topos theory in the foundations of physics. In *New Structures for Physics*, Lecture Notes in Physics. Springer, 2009.
- [7] Carsten Held. The meaning of complementarity. *Studies in History and Philosophy of Science Part A*, 25:871–893, 1994.
- [8] Chris Heunen and Bart Jacobs. Quantum logic in dagger kernel categories. *Order*, 2010.
- [9] Chris Heunen, Nicolaas P. Landsman, and Bas Spitters. Bohrification. In *Deep Beauty*. Cambridge University Press, 2009.
- [10] Gudrun Kalmbach. *Orthomodular Lattices*. Academic Press, 1983.
- [11] Simon Kochen and Ernst Specker. The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics*, 17:59–87, 1967.
- [12] Joachim Kock. *Frobenius algebras and 2-D Topological Quantum Field Theories*. Number 59 in London Mathematical Society Student Texts. Cambridge University Press, 2003.
- [13] Nicolaas P. Landsman. *Handbook of the philosophy of science vol 2: Philosophy of Physics*, chapter Between classical and quantum, pages 417–554. North-Holland, 2007.
- [14] Duško Pavlović. Quantum and classical structures in nondeterministic computation. In P. Bruza et al., editor, *Third International symposium on Quantum Interaction*, volume 5494 of *Lecture Notes in Artificial Intelligence*, pages 143–157. Springer, 2009.
- [15] Constantin Piron. *Foundations of quantum physics*. Number 19 in Mathematical Physics Monographs. W.A. Benjamin, 1976.
- [16] Miklós Rédei. *Quantum Logic in Algebraic Approach*. Kluwer, 1998.
- [17] Erhard Scheibe. *The logical analysis of quantum mechanics*. Pergamon, 1973.
- [18] Franco Strocchi. *Elements of quantum mechanics of infinite systems*. World scientific, 1985.
- [19] Benno van den Berg and Chris Heunen. Noncommutativity as a colimit. *arXiv:1003.3618*, 2010.
- [20] John von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Springer, 1932.