A categorical framework for the quantum harmonic oscillator

arXiv:0706.0711

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Categorical Quantum Logic, Oxford
12 August 2007

Overview

- What is categorical quantum mechanics?
- What is the quantum harmonic oscillator?
- Constructing the state space categorically
- Graphical representation
- Raising and lowering operators
- Coherent states
- Exponentials
- A category of Hilbert spaces?

Categorical quantum mechanics

Traditionally	**Categorically**
Hilbert spaces and linear maps	Objects and morphisms in a category $\mathbf{C}$
Inner products	Contravariant functor $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$, identity on objects, $\dagger^2 = \mathrm{id}_{\mathbf{C}}$
Tensor product	Symmetric monoidal $\otimes$ on $\mathbf{C}$
Linearity	$\dagger$-biproducts $\oplus$ in $\mathbf{C}$
States	Morphisms $\phi : I \longrightarrow A$
Amplitudes	$\mathrm{Hom}_{\mathbf{C}}(I, I)$, always commutative

Symmetric monoidal †-category C with †-biproducts

Duals?

The quantum harmonic oscillator

Particle in an n-dimensional quadratic potential

State space is symmetric Fock space:

$$F(A) := \mathbb{C} \oplus A \oplus (A \otimes_s A) \oplus (A \otimes_s A \otimes_s A) \oplus \dots$$

Manipulated with raising and lowering operators, for $\phi : I \longrightarrow A$:

$$a_\phi : F(A) \longrightarrow F(A) \qquad a_\phi^\dagger : F(A) \longrightarrow F(A)$$

Canonical commutation relations:

$$a_\phi \circ a_\psi = a_\psi \circ a_\phi, \quad a_\phi^\dagger \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi^\dagger, \quad a_\phi \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi + (\psi^\dagger \circ \phi) \cdot \mathrm{id}_{F(A)}$$

Carries a natural commutative monoid structure
[Blute, Panangaden and Seely, 1994]

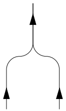
Natural isomorphisms: $F(A \oplus B) \simeq F(A) \otimes F(B)$

Internal commutative monoids

Category of internal commutative monoids $\mathbf{C}_+$

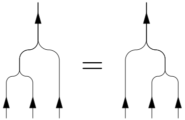
Multiplication and unit morphisms

Denoted $(A, g, u)_+$

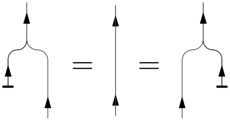


$g : A \otimes A \longrightarrow A$

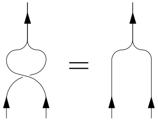
$u : I \longrightarrow A$



Associativity law



Unit laws



Commutativity law

Initial object	$I_+ := (I, \lambda_I, \mathrm{id}_I)_+$	Coproducts
Terminal object	$0_+ := (0, 0_{0\otimes 0, 0}, \mathrm{id}_0)_+$	Products

Categorical formulation

Problem: how to define our quantum system?
Solution: algebraically!

$$\mathbf{C} \underset{R \text{ forgetful}}{\overset{Q \text{ free}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathbf{C}_\times$$

$F := RQ$ is Fock space comonad
$\epsilon : RQ \stackrel{\cdot}{\longrightarrow} \mathrm{id}_{\mathbf{C}}$, $\eta : \mathrm{id}_{\mathbf{C}_\times} \stackrel{\cdot}{\longrightarrow} QR$
$(F(A), d_A, e_A)_\times := Q(A)$

$\epsilon_A : F(A) \longrightarrow A$ projects on to *single-particle* space
$e_A : F(A) \longrightarrow I$ projects onto *zero-particle* space

What to do with $\dagger : \mathbf{C} \longrightarrow \mathbf{C}$?

Introduce compatibility conditions:

- $F \circ \dagger = \dagger \circ F$;
- $\epsilon \circ \epsilon^\dagger = \mathrm{id}_{\mathrm{id}_{\mathbf{C}}}$ ... that is, $\epsilon_A \circ \epsilon_A^\dagger = \mathrm{id}_A$ for all $A \in \mathrm{Ob}(\mathbf{C})$;
- Products preserved unitarily.

Preserving products unitarily

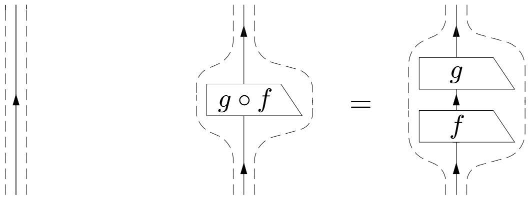
Unique natural isomorphisms induced in $\mathbf{C}_\times$:

$$k_{A,B} : Q(A \oplus B) \longrightarrow Q(A) \times Q(B)$$
$$k_0 : Q(0) \longrightarrow I_\times$$

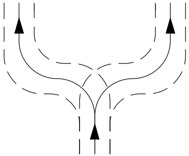
Require $Rk_{A,B}$, Rk_0 unitary in $\mathbf{C}$:

$$(Rk_{A,B})^\dagger = Rk_{A,B}^{-1} \qquad\qquad (Rk_0)^\dagger = Rk_0^{-1}$$

Graphical representations for F, e, d



$F(A)$ $\qquad F(g) \circ F(f) \;=\; F(g \circ f)$

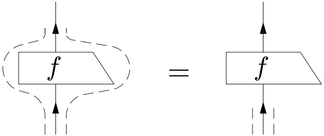


$d_A : F(A) \longrightarrow F(A) \otimes F(A)$ $\qquad e_A : F(A) \longrightarrow I$

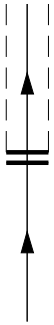
Graphical representation for ϵ, η

$\epsilon_A : F(A) \longrightarrow A$

Naturality



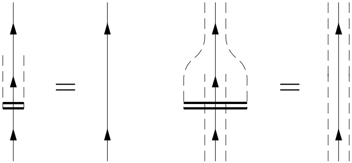
$$\epsilon_A \circ F(f) \quad = \quad f \circ \epsilon_A$$



$R\eta_{(A,g,u)_\times}$

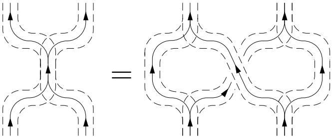
$R\eta_{(A,g,u)_\times}$:
$A \longrightarrow F(A)$

Adjunction equations

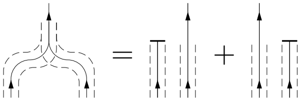


Some emergent properties...

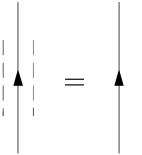
Bialgebra identity



$\epsilon \circ d^\dagger$ identity

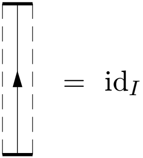


ϵ normalised

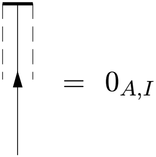


(cheated)

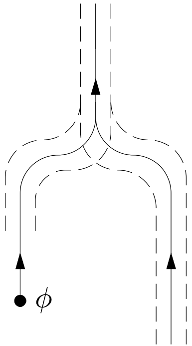
e normalised



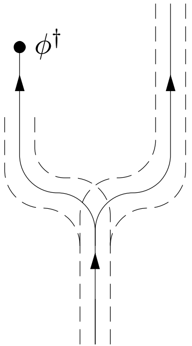
e, ϵ orthogonal



Raising and lowering operators



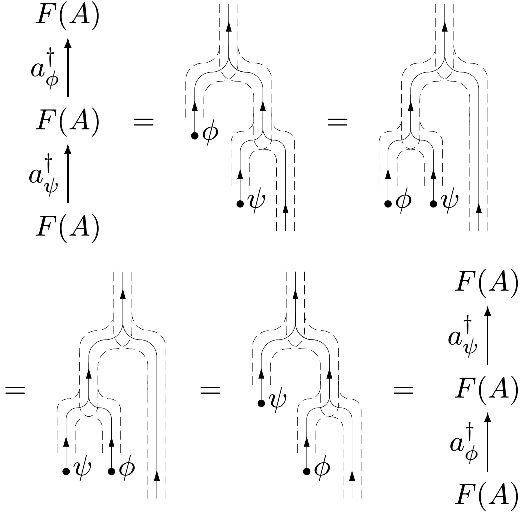
Raising morphism
$a_\phi^\dagger : F(A) \longrightarrow F(A)$



Lowering morphism
$a_\phi : F(A) \longrightarrow F(A)$

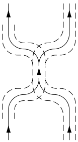
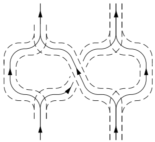
Canonical commutator

$$a^\dagger_\phi \circ a^\dagger_\psi = a^\dagger_\psi \circ a^\dagger_\phi$$

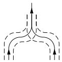


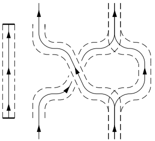
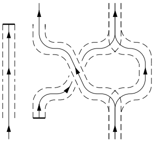
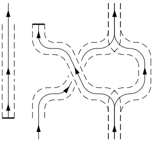
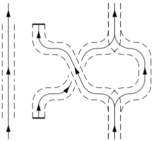
Canonical commutator

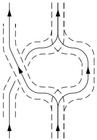
$$a_\phi \circ a_\psi^\dagger = a_\psi^\dagger \circ a_\phi + (\phi^\dagger \circ \psi) \cdot \mathrm{id}_{F(A)}$$

$a_A \circ a_A^\dagger$ =  = 

$\epsilon \circ d^\dagger$ identity

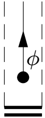
 =  + 

=  +  +  + 

=  + 0 + 0 +  $= a_A^\dagger \circ a_A + \mathrm{id}_A \otimes \mathrm{id}_{F(A)}$

Coherent states

single-particle state $\phi : I \longrightarrow A$ becomes multi-particle state $\mathrm{Coh}(\phi) : I \longrightarrow F(A)$



Employ $R\eta_{I_\times} : I \longrightarrow F(I)$ here

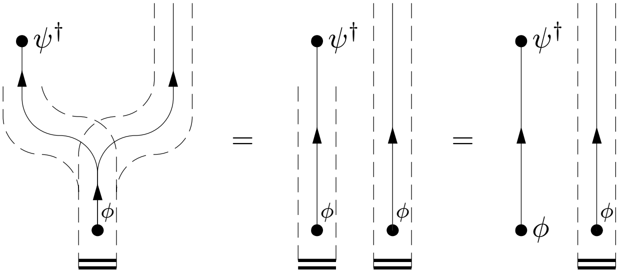
Correspond to *points* $\mathrm{Hom}_{\mathbf{C}_\times}\big(I_\times, Q(A)\big)$

Have classical properties:

- Can be copied:
  $d_A \circ \mathrm{Coh}(\phi) = \mathrm{Coh}(\phi) \otimes \mathrm{Coh}(\phi)$
- Can be deleted:
  $e_A \circ \mathrm{Coh}(\phi) = \mathrm{id}_I$
- Unchanged by lowering operator:
  $a_\psi \circ \mathrm{Coh}(\phi) = (\psi^\dagger \circ \phi) \cdot \mathrm{Coh}(\phi)$

Coherent state is eigenstate of a_ψ

$$a_\psi \circ \mathrm{Coh}(\phi) = (\psi^\dagger \circ \phi) \cdot \mathrm{Coh}(\phi)$$



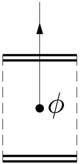
Morphism exponentials

Construct an exponential

$$\exp_{(A,g,u)_+}(\phi)$$

from a state $\phi : I \longrightarrow A$ and a commutative monoid $(A,g,u)_+$

Intuition: $\exp_{(A,g,u)_+}(\phi) = \frac{1}{0!} \cdot u + \frac{1}{1!} \cdot \phi + \frac{1}{2!} \cdot g \circ (\phi \otimes \phi) + \cdots$



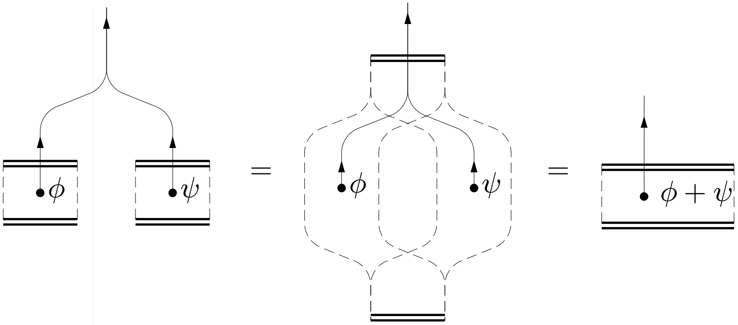
employing

$$(R\eta_{(A,g^\dagger,u^\dagger)_\times})^\dagger : F(A) \longrightarrow A$$
$$R\eta_{I_\times} : I \longrightarrow F(I)$$

Has the following familiar properties:

- Additivity: $g \circ \big(\exp(\phi) \otimes \exp(\psi)\big) = \exp(\phi + \psi)$
- Unit: $\exp_{(A,g,u)_+}(0_{I,A}) = u$

Proof of additivity



C = Hilb?

Need *unbounded* operators (e.g. stages of η, d)

Problem:
Unbounded operators don't always compose!

Suggested solution:
Use inner-product spaces, not Hilbert spaces
Allows a well-behaved set of unbounded operators

Duals? Needed for *operator exponentials*.

Inner	Lacks duals
Rel	Lacks interesting scalars
FdHilb	Lacks free commutative monoid functor

'Not enough room' for duals, interesting scalars and Fock space all at once!

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- A category of Hilbert spaces?