Supported quantales: an interface between modal logic and geometry

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0. PRELIMINARIES ON MODAL LOGIC

PROPOSITIONAL (NORMAL) MODAL LOGIC

Formulas:

$$\varphi ::= \text{propositional symbol} \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi$$

Kripke models:

$$(W, R, V)$$

- W is the set of *worlds*
- $R \subseteq W \times W$ is the *accessibility relation*
- V : Formulas $\to 2^W$ is the *valuation map*, which satisfies

$$\begin{aligned} V(\varphi \wedge \psi) &= V(\varphi) \cap V(\psi) \\ V(\neg\varphi) &= W \setminus V(\varphi) \\ V(\Diamond\varphi) &= \{x \in W \mid xRy \text{ for some } y \in V(\varphi)\} \end{aligned}$$

ACCESSIBILITY RELATIONS AS UNARY OPERATORS

Given a complete lattice L we denote by $\mathcal{Q}(L)$ the set of endomaps of L that preserve arbitrary joins. This is an example of a quantale.

There is a bijection (in fact an isomorphism of quantales)

$$\begin{array}{rcl} 2^{W\times W} & \xrightarrow{\cong} & \mathcal{Q}(2^W) \\ R & \mapsto & \Diamond_R \end{array}$$

given by, for all $Y \subseteq W$,

$$\Diamond_R(Y) = \{x \in W \mid xRy \text{ for some } y \in Y\}$$

Hence,

- accessibility relations are equivalent to union preserving operators on 2^W
- the third condition on valuation maps is just

$$V(\Diamond\varphi) = \Diamond_R(V(\varphi))$$

ALGEBRAIC SEMANTICS

By a *modal (Boolean) algebra* will be meant a Boolean algebra B equipped with a unary operation

$$\Diamond : B \to B$$

that satisfies the conditions

$$\begin{aligned} \Diamond 0 &= 0 \\ \Diamond(a \vee b) &= \Diamond a \vee \Diamond b \end{aligned}$$

An *algebraic model* $(B, \Diamond, V)$ consists of a modal algebra $(B, \Diamond)$ equipped with a *valuation map*

$$V : \text{Formulas} \to B$$

that satisfies

$$\begin{aligned} V(\varphi \wedge \psi) &= V(\varphi) \wedge V(\psi) \\ V(\neg\varphi) &= \neg V(\varphi) \\ V(\Diamond\varphi) &= \Diamond(V(\varphi)) \end{aligned}$$

LINDENBAUM ALGEBRAS AND EQUIVALENCE

There is an algebraic model $(\mathcal{L}, \Diamond_{\mathcal{L}}, V_{\mathcal{L}})$ which is universal in the sense that for any other model $(B, \Diamond, V)$ there is a unique homomorphism of modal algebras $\overline{V}$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{Formulas} & \xrightarrow{V_{\mathcal{L}}} & \mathcal{L} \\ & {\scriptstyle V} \searrow & \downarrow {\scriptstyle \overline{V}} \\ & & B \end{array}$$

The modal algebra $(\mathcal{L}, \Diamond_{\mathcal{L}})$ is the *Lindenbaum algebra* (for propositional normal modal logic — system K), and models can be *identified* with homomorphisms

$$\mathcal{L} \to B \ .$$

Two formulas φ and ψ are *equivalent*, and we write $\varphi \equiv \psi$, if and only if $V_{\mathcal{L}}(\varphi) = V_{\mathcal{L}}(\psi)$ — equivalently, if and only if $V(\varphi) = V(\psi)$ for every algebraic model $(B, \Diamond, V)$.

Completeness theorem (for system K). *$\varphi \equiv \psi$ if and only if $V(\varphi) = V(\psi)$ for every Kripke model (W, R, V).*

(This can be regarded as an extension of Stone's representation theorem.)

OTHER SYSTEMS OF PROPOSITIONAL MODAL LOGIC

There are similar completeness theorems for the following systems:

System T: The accessibility relations are *reflexive*, and the modal algebras satisfy

$$a \leq \Diamond a$$

System S4: The accessibility relations are *preorders*, and the modal algebras satisfy

$$\begin{aligned} a &\leq \Diamond a \\ \Diamond\Diamond a &\leq \Diamond a \end{aligned}$$

System S5: The accessibility relations are *equivalence relations*, and the modal algebras satisfy

$$\begin{aligned} a &\leq \Diamond a \\ \Diamond\Diamond a &\leq \Diamond a \\ a &\leq \neg\Diamond\neg\Diamond a \end{aligned}$$

1. WHY QUANTALES?

Definition. A *unital involutive quantale* Q is a complete lattice equipped with an additional structure of involutive monoid,

$$(ab)c = a(bc) \qquad ae = a = ea \qquad a^{**} = a \qquad (ab)^* = b^*a^* \ ,$$

which is compatible with arbitrary joins:

$$(\bigvee_i a_i)b = \bigvee_i a_i b \qquad b(\bigvee_i a_i) = \bigvee_i b a_i \qquad (\bigvee_i a_i)^* = \bigvee_i a_i^* \ .$$

(In other words, an involutive monoid in the monoidal category of sup-lattices.)

Notation. $1 = \bigvee Q \qquad 0 = \bigvee \emptyset$

Example. $2^{W\times W}$ is a unital involutive quantale:

- Multiplication of binary relations is given by (forward) composition:

$$RS = R;S = S\circ R \ .$$

- The multiplicative unit e is the identity relation $\Delta_W = \{(x,x) \mid x \in W\}$.
- The involution is reversal: $R^* = \{(y,x) \mid xRy\}$.

Let W be a set, and $R \subseteq W \times W$ a binary relation on W. The *domain* of R is

$$\text{dom}\, R = \{x \in W \mid xRy \text{ for some } y \in W\} \ ,$$

but we may equivalently define it to be the (subdiagonal) relation

$$\varsigma R = \{(x, x) \in W \times W \mid x \in \text{dom}\, R\} \ ,$$

thus turning dom into an operation

$$\varsigma : 2^{W \times W} \longrightarrow 2^{W \times W}$$

We call ςR the *support* of R.

The image of ς consists of the set 2^{Δ_W} of subdiagonal relations, which of course is isomorphic to 2^W. We can thus define $\Diamond_R$ as an operation on 2^{Δ_W}, leading to the following simple formula for all $X \subseteq \Delta_W$:

$$\Diamond_R(X) = \varsigma(R; X) \ .$$

Hence, a Kripke model can be equivalently defined to be a triple (W, R, V), where the valuation map

$$V : \text{Formulas} \to 2^{\Delta_W}$$

satisfies

$$\begin{aligned} V(\varphi \wedge \psi) &= V(\varphi); V(\psi) \qquad \text{(note use of ; instead of } \cap\text{)} \\ V(\neg\varphi) &= \Delta_W \setminus V(\varphi) \\ V(\Diamond\varphi) &= \varsigma(R; V(\varphi)) \end{aligned}$$

If we further replace the second condition by the following two

$$\begin{aligned} V(\neg\varphi); V(\varphi) &= \emptyset \\ V(\neg\varphi) \cup V(\varphi) &= \Delta_W \end{aligned}$$

we see that the properties of V are entirely defined in terms of the *structure of unital quantale of* $2^{W \times W}$ *together with the additional operation* ς, which we shall now study.

EXAMPLE

An advantage of this redefinition of the semantics lies in its convenience, for instance when dealing with action logics such as *propositional dynamic logic*. There are now programs,

$$\alpha ::= \text{atomic programs} \mid \alpha;\beta \mid \alpha^* \mid \alpha \cup \beta \mid \varphi?$$

(here φ is a formula) and each program determines a modality:

$$\varphi ::= \text{atomic formulas} \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\alpha\rangle\varphi \; .$$

Then a *model* is a triple

$$(W, \Pi, V)$$

where W is the set of worlds (now called states), the map

$$\Pi : \text{Programs} \longrightarrow 2^{W \times W}$$

assigns meanings to programs, and V is the valuation map as before:

$$V : \text{Formulas} \longrightarrow 2^{\Delta_W} \; .$$

The conditions that the maps

$$V : \text{Formulas} \longrightarrow 2^{\Delta_W}$$
$$\Pi : \text{Programs} \longrightarrow 2^{W \times W}$$

must satisfy are easily stated. For Π we have

$$\begin{aligned}\Pi(\alpha;\beta) &= \Pi(\alpha);\Pi(\beta)\\ \Pi(\alpha^*) &= \bigcup_{n\in\mathbb{N}} \Pi(\alpha)^n\\ \Pi(\alpha\cup\beta) &= \Pi(\alpha)\cup\Pi(\beta)\\ \Pi(\varphi?) &= V(\varphi)\end{aligned}$$

and for V we have the usual propositional conditions plus the modal ones:

$$V(\langle\alpha\rangle\varphi) = \varsigma(\Pi(\alpha);V(\varphi))\ .$$

3. QUANTALE SEMANTICS OF MODAL LOGIC

SUPPORTED QUANTALES

Definition. Let Q be a unital involutive quantale. A *support* on Q is a join preserving map

$$\varsigma : Q \to Q$$

satisfying, for all $a \in Q$,

$$\varsigma a \leq e \qquad (1)$$

$$\varsigma a \leq aa^* \qquad (2)$$

$$a \leq \varsigma a a \qquad (3)$$

A *supported quantale* is a unital involutive quantale equipped with a specified support.

Example. $2^{W \times W}$. Will see others later.

Some properties...

- $a = \varsigma a = a^* = a^2$, for all $a \leq e$
- $\downarrow e = \varsigma Q$ has trivial involution ($a^* = a$) and $ab = a \wedge b$ (it is a locale!)
- $a \leq aa^*a \quad (\Rightarrow Q$ is a Gelfand quantale)
- $\varsigma a1 = a1$
- the map $(-)1 : \varsigma Q \to Q1$ is a retraction, split by $\varsigma : Q1 \to \varsigma Q$

($Q1$ is the set $\mathrm{R}(Q)$ of *right-sided* elements of Q.)

Definition. An *abstract Kripke model* of propositional modal logic is a triple

$$(Q, r, v)$$

consisting of

- a supported quantale Q (the "quantale of worlds")
- an *accessibility element* $r \in Q$
- a *valuation map* $v : \text{Formulas} \to \varsigma Q$

satisfying:

$$\begin{array}{rcll} v(\varphi \wedge \psi) & = & v(\varphi)v(\psi) & \qquad [= v(\varphi) \wedge v(\psi)] \\ v(\neg\varphi)v(\varphi) & = & 0 & \\ v(\neg\varphi) \vee v(\varphi) & = & e & \\ v(\Diamond\varphi) & = & \varsigma(r\, v(\varphi)) & \end{array}$$

Or, for *intuitionistic logic*, replace the two middle conditions by a single one using the pseudo-complement in ςQ (this is a locale and therefore a Heyting algebra):

$$v(\neg\varphi) = v(\varphi) \to 0\ .$$

EXAMPLES

The following variations on the notion of model have an obvious justification:

T-models: The accessibility element r satisfies $e \leq r$.

S4-models: The accessibility element satisfies $e \leq r = r^2$.

S5-models: The accessibility element satisfies $e \leq r = r^2 = r^*$.

By a K-model will be meant one for which the accessibility element has no restriction.

EXAMPLES

Also, going back to propositional dynamic logic, a model is now a triple

$$(Q, \pi, v)$$

where Q is a supported quantale and π and v are maps

$$\begin{aligned} v &: \text{Formulas} \to \varsigma Q \\ \pi &: \text{Programs} \to Q \end{aligned}$$

that satisfy the conditions:

$$\begin{aligned} \pi(\alpha;\beta) &= \pi(\alpha)\pi(\beta) \\ \pi(\alpha^*) &= \bigvee_{n\in\mathbb{N}} \pi(\alpha)^n \\ \pi(\alpha\cup\beta) &= \pi(\alpha)\vee\pi(\beta) \\ \pi(\varphi?) &= v(\varphi) \\ v(\langle\alpha\rangle\varphi) &= \varsigma(\pi(\alpha)\,v(\varphi)) \\ \text{etc.} & \end{aligned}$$

Later we shall also look at temporal logic.

STABLY SUPPORTED QUANTALES

It is convenient to restrict to a particular class of supported quantales:

Definition. A support is *stable* if the following (equivalent) conditions hold:

1. $\varsigma(ab) = \varsigma(a\varsigma b)$
2. $\varsigma(a1) = \varsigma a$
3. the map $(-)1 : \varsigma Q \to Q1$ is an order isomorphism
4. many others...

A quantale equipped with a stable support is *stably supported*.

Example. $2^{W \times W}$

Theorem.

1. *If Q has a stable support ς then it has no other support, and we have*

$$\varsigma a = e \wedge aa^* = e \wedge a1 \ .$$

2. *The category of stably supported quantales is a reflective full subcategory of the category of unital involutive quantales.*

Hence,

- Being stably supported is a *property* of a unital involutive quantale, rather than additional structure.
- For a stably supported quantale Q the homomorphisms of unital involutive quantales $Q \to 2^{W \times W}$ (the relational representations) necessarily preserve the support.

PROPERTIES OF THE QUANTALE SEMANTICS

From now on we restrict to stably supported quantales.

Theorem. *The quantale semantics extends the modal algebra semantics (and therefore the Kripke semantics) while retaining the same notions of equivalence of formulas (and the same theorems, etc.):*

$$\varphi \equiv \psi$$

if and only if $v(\varphi) = v(\psi)$ *for all abstract Kripke models* (Q, r, v).

“LINDENBAUM QUANTALES”

For each modal algebra $(B, \Diamond)$ there is a stably supported quantale $\mathbf{Q}(B)$, which can be presented by generators and relations taking as set of generators $B \cup \{r\}$ with $r \notin B$, and with relations that make B a modal subalgebra of $\varsigma\mathbf{Q}(B)$ with respect to the modal operator defined on the latter by $\Diamond a = \varsigma(ra)$. This has a *universal property* analogous to that of a *Lindenbaum algebra*. Contrary to the situation with modal algebras, however, where the modal operators $\Diamond$ (or $\langle\alpha\rangle$, etc.) have to be specified in advance, and have to be preserved by the homomorphisms (that is, each modal logic gives rise to a particular kind of modal algebra), here *the algebra of unital involutive quantales is common to any of the modal logics we have seen so far*; that is, the type of modal logic under consideration is encoded as a *theory* in the language of unital involutive quantales, making the latter a kind of “meta modal logic”:

Theorem. *There is a bijective correspondence between abstract Kripke models* (Q, r, v) *(with stably supported* Q*) and homomorphisms of unital involutive quantales*

$$\mathbf{Q}(\mathcal{L}) \to Q \ .$$

In particular, the usual Kripke models can be identified with the relational representations of $\mathbf{Q}(\mathcal{L})$*, i.e., the homomorphisms of unital involutive quantales*

$$\mathbf{Q}(\mathcal{L}) \to 2^{W \times W} \ .$$

In order to obtain similar facts for other systems, such as T, S4, S5, propositional dynamic logic, etc., one must define appropriate "Lindenbaum quantales". For instance, the "Lindenbaum quantale" for S5 is obtained as before, but in addition the generator r is subject to the relations

$$e \leq r = r^2 = r^* \ .$$

EXAMPLE — PROPOSITIONAL DYNAMIC LOGIC

For propositional dynamic logic all the programs are generators, and the relations are the obvious ones:

- $\alpha;\beta = \alpha\beta$
- $\alpha \cup \beta = \alpha \vee \beta$
- $\varphi? = \varphi$
- the iteration α^* (not involution) equals $\bigvee_{n\in\mathbb{N}} \alpha^n$
- $\langle\alpha\rangle\varphi = \varsigma(\alpha\varphi)$
- etc.

EXAMPLE — RAMIFIED TEMPORAL LOGIC

$$\begin{aligned}\varphi \;::=\;& \text{propositional symbol} \mid \neg\varphi \mid \varphi \wedge \psi \\ & \mid EX\varphi \mid EF\varphi \mid EG\varphi \mid AX\varphi \mid AF\varphi \mid AG\varphi\end{aligned}$$

An *abstract model* is a triple (Q, r, v) where, as before,

- Q is a supported quantale
- $r \in Q$
- $v : \text{Formulas} \to \varsigma Q$

Now the accessibility element r must satisfy a condition that prevents time from stopping,

$$\varsigma r = e$$

and the valuation map v must satisfy the following conditions:

- The usual propositional conditions
- Additional propositional conditions making $AX\varphi$ the complement of $EX\neg\varphi$, $AG\varphi$ the complement of $EF\neg\varphi$, and $AF\varphi$ the complement of $EG\neg\varphi$.

  (All of this can easily be made intuitionistic.)
- $v(EX\varphi) = \varsigma(rv(\varphi))$
- $v(EF\varphi) = \varsigma(\bigvee_{n\in\mathbb{N}} r^n v(\varphi))$

  (Variants could include having a separate generator instead of $\bigvee_{n\in\mathbb{N}} r^n$, subject to appropriate relations.)
- $v(EG\varphi) = \bigvee\{a \leq v(\varphi) \mid a \leq \varsigma(ra)\}$

3. GROUPOIDS AND INVERSE SEMIGROUPS

The notions of *groupoid* and of *inverse semigroup* generalize the notion of *group*.

Groupoids and inverse semigroups allow one to describe *more general notions of symmetry* than groups, namely taking into account *local symmetries* in *differential topology and geometry*.

Definition. An *inverse semigroup* S is an involutive semigroup,

$$\begin{aligned}(ab)c &= a(bc)\\ a^{**} &= a\\ (ab)^* &= b^*a^* \ ,\end{aligned}$$

(a^* is the *inverse* of a) satisfying

$$aa^*a = a \ ,$$

and such that all idempotents commute. The set of idempotents is denoted by $E(S)$.

Example.

1. The partial bijections on a set X (the symmetric inverse semigroup of X).
2. The locally defined homeomorphisms of a topological space (pseudo-group).
3. The locally defined diffeomorphisms of a smooth manifold.
4. Any semigroup of partial isometries on a Hilbert space, or, more generally, of a C*-algebra.

The *natural order* of an inverse semigroup S is defined by

$$s \leq t \iff s = ft \text{ for some } f \in E(S)$$

Theorem. *Let S be an inverse semigroup. The set $\mathcal{L}(S)$ of order ideals of S is a stably supported quantale. (In fact this yields a functor from inverse semigroups to stably supported quantales that has a right adjoint.)*

Hence, modal logics can be interpreted in any of the structures mentioned before that yield inverse semigroups!

Furthermore this is not the only construction of a stably supported quantale from an inverse semigroup:

Example. Consider the pseudo-group $\Gamma(X)$ of local homeomorphisms of a topological space X with topology $\Omega(X)$. The idempotents are the identity maps on open sets, and therefore we have

$$\Omega(X) \cong E(\Gamma(X))$$

This inverse semigroup is *complete* in the sense that its natural order has all possible joins, and if we take the least quotient Q of $\mathcal{L}(\Gamma(X))$ that forces those joins to be preserved by the map

$$\Gamma(X) \to \mathcal{L}(\Gamma(X)) \to Q$$

we obtain a stably supported quantale such that

$$\varsigma Q \cong \Omega X \ .$$

Therefore a model of modal logic on the space X can be naturally defined to consist of an abstract Kripke model whose quantale is Q. In particular, this provides a natural way in which to define an intuitionistic topological semantics for propositional modal logics.

GROUPOIDS

$$\begin{array}{ccc} G_1 \underset{G_0}{\times} G_1 & \xrightarrow{\pi_1} & G_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle r} \\ G_1 & \xrightarrow[d]{} & G_0 \end{array}$$

$$G \quad = \quad G_1 \underset{G_0}{\times} G_1 \xrightarrow{m} \overset{i}{\circlearrowleft} G_1 \; \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{u} \\ \xrightarrow[d]{} \end{array} \; G_0$$

In Set, Top, Loc, etc...

EXAMPLES

- Groupoids can be constructed from arbitrary inverse semigroups.
- In particular, groupoids of germs of local homeomorphisms, local diffeomorphisms, etc.
- The fundamental groupoid of a topological space.
- The monodromy groupoid of a foliation (a generalization of the previous example).
- The holonomy groupoid of a foliation.
- The dual groupoid of a C*-algebra.

All of these are topological groupoids.

SUPPORTED QUANTALES FROM GROUPOIDS

Example. Let G be a discrete groupoid:

$$G \quad = \quad G_1 \times_{G_0} G_1 \xrightarrow{\ m\ } G_1 \overset{i}{\circlearrowleft} \begin{array}{c} \xrightarrow{\ r\ } \\ \xleftarrow{\ u\ } \\ \xrightarrow[\ d\ ]{} \end{array} G_0$$

Then 2^{G_1} is a stably supported quantale:

$$\begin{aligned} UV &= \{xy \mid x \in U,\ y \in V,\ r(x) = d(y)\} \\ e &= G_0 \\ U^* &= U^{-1} \end{aligned}$$

As particular cases we have the powerset of a discrete group, and the quantale of binary relations on a set ($G_1 = W \times W$).

Example. More generally, any topological groupoid with open unit space G_0 and open multiplication map m has a topology $\Omega(G_1)$ that is a stably supported sub-quantale of 2^{G_1}. Such groupoids are precisely the étale groupoids.

(In fact *localic* étale groupoids are equivalent to their quantales.)

$\Rightarrow$ Semantics of (intuitionistic or not) propositional modal logic on any étale groupoid!

Not all the previous groupoids are étale, for instance the holonomy groupoids, which however are always weakly equivalent to étale ones.

Ultimately it may be useful to define direct interpretations of modal logic on the topologies of non-étale groupoids. That requires a generalization of the theory of supported quantales.

BIBLIOGRAPHY

The notion of *supported quantale* and its relation to *inverse semigroups* and *groupoids* can be found in the preprint

P. Resende, Étale groupoids and their quantales,

which can be downloaded from http://arxiv.org/abs/math/0412478.

The applications to *modal logic* will appear elsewhere and have first been addressed (in the context of non-involutive quantales) in the MSc thesis of Cátia Vaz (http://www.cc.isel.ipl.pt/pessoais/CatiaVaz/).