UNIVERSITY OF OXFORD
DEPARTMENT OF COMPUTER SCIENCE

Tutorial:

Probabilistic Model Checking

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"Scalable Analysis of Probabilistic Models and Programs"
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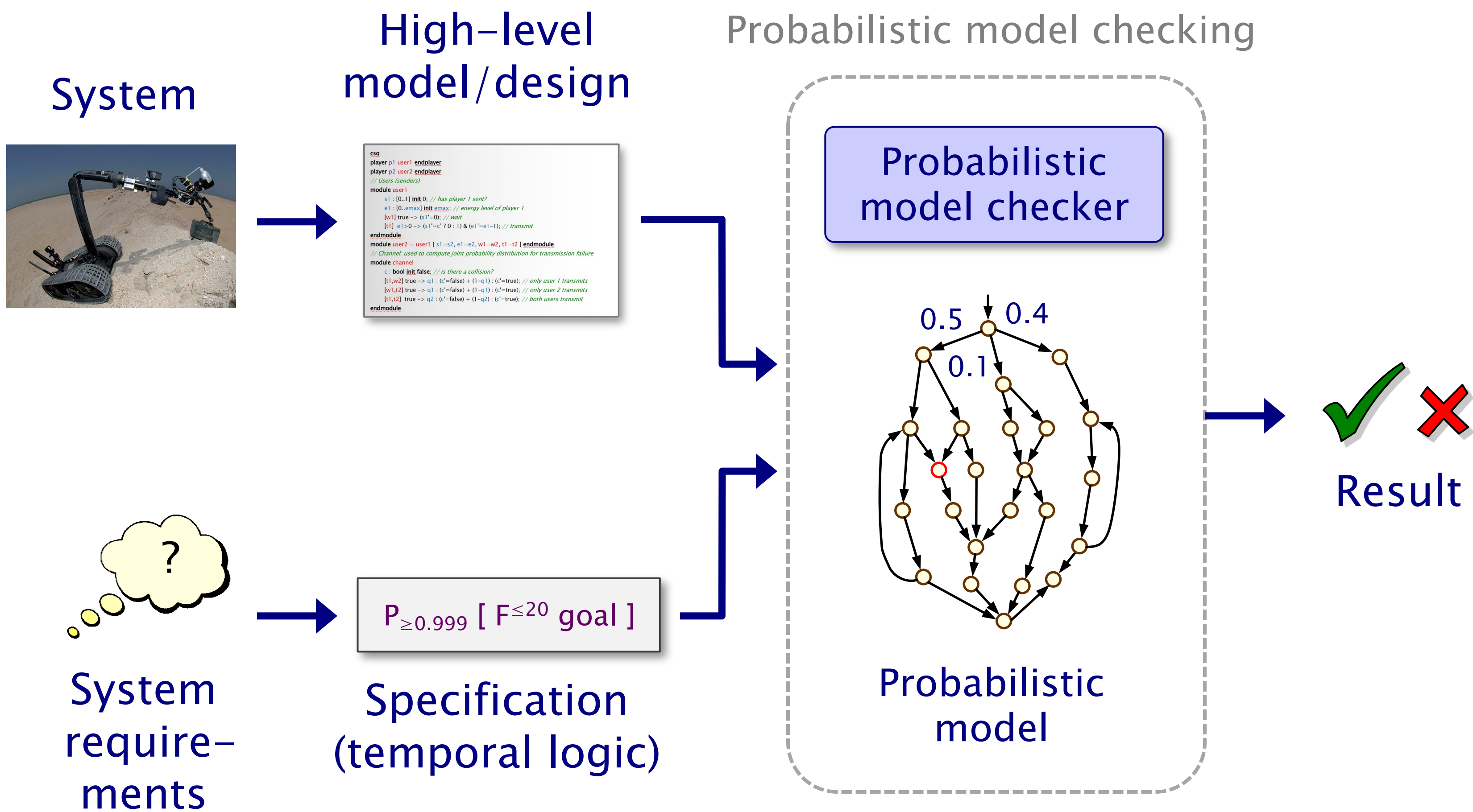
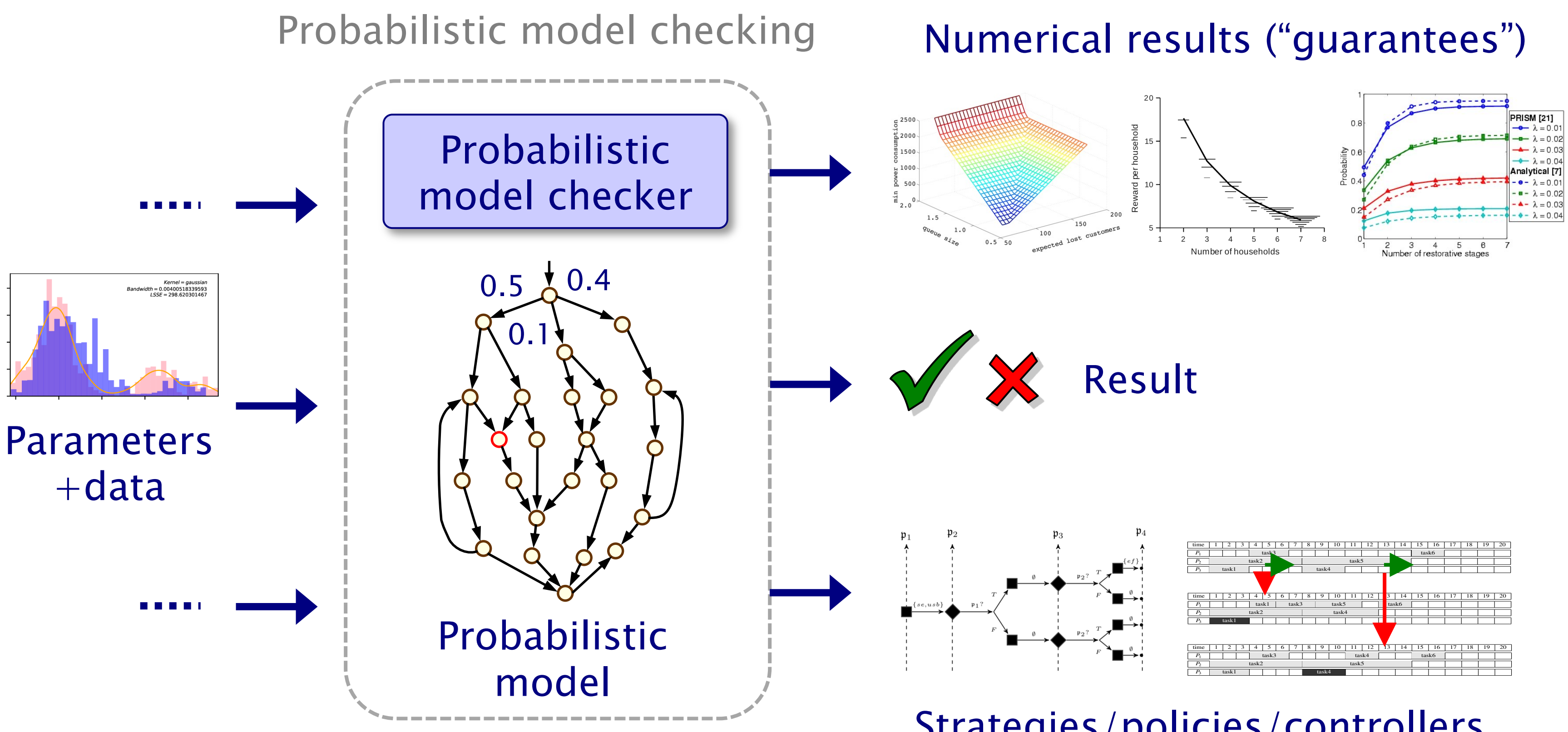
Probabilistic model checking (PMC)



Probabilistic model checking



$P_{\geq 0.999}$ [$F^{\leq 20}$ goal]

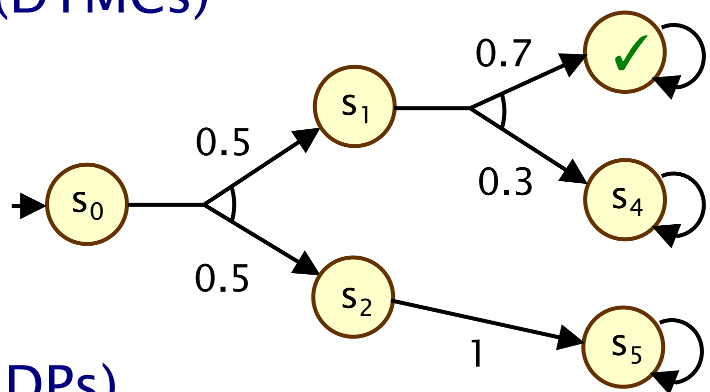
Overview

- Probabilistic models
- Temporal logic
  - a language for quantitative guarantees
- Techniques, tools & languages
- Multi-agent verification
  - stochastic multi-player games

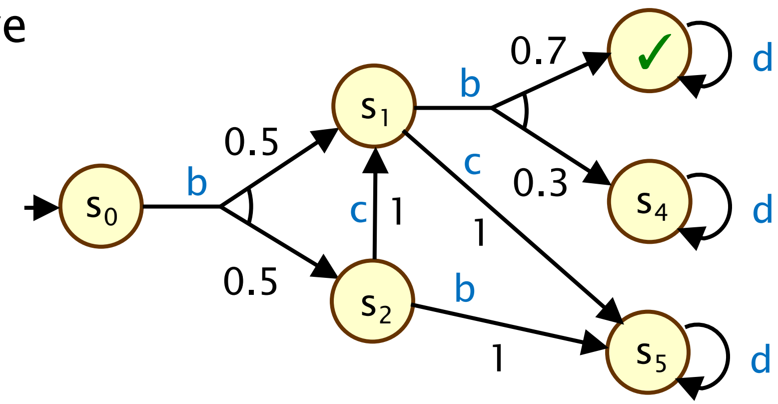
Probabilistic models

Probabilistic models

- Discrete-time Markov chains (DTMCs)
  - finite state space + discrete probabilities



- Markov decision processes (MDPs)
  - DTMCs + nondeterminism
  - policies (or strategies) resolve actions based on history



- Models for PMC:
  - mostly finite-state
  - mostly known in full

Models, models, models…

- Wide range of probabilistic models

  discrete states & probabilities: Markov chains
  + nondeterminism: Markov decision processes (MDPs)
  + real-time clocks: probabilistic timed automata (PTAs)
  + uncertainty: interval MDPs (IMDPs)
  + partial observability: partially observable MDPs (POMDPs)
  + multiple players: (turn-based) stochastic games
  + concurrency: concurrent stochastic games

- And many others
  - continuous-time Markov chains
  - Markov automata
  - stochastic timed/hybrid automata
  - …

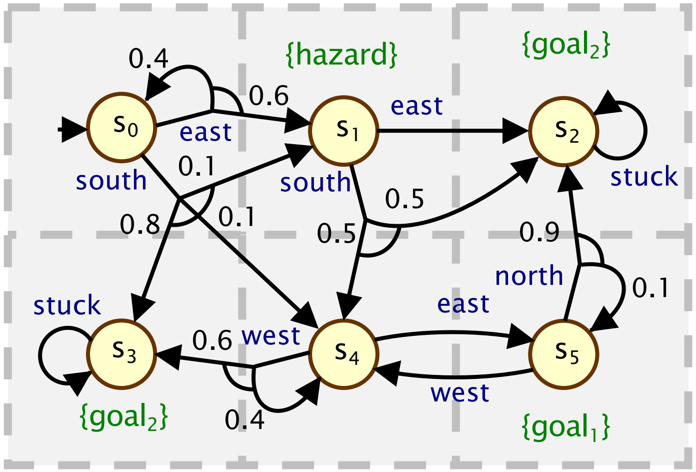
Temporal logic

Temporal logic

- Formal specification of desired/required behaviour
  - formal language for quantitative guarantees

- Simple examples (PCTL)
  - Probabilistic reachability
    $P_{\geq 0.7}$ [F $goal_1$]
    $P_{\geq 0.6}$ [$F^{\leq 10}$ $goal_1$]
  - Probabilistic safety/invariance
    $P_{\geq 0.99}$ [G$\neg$hazard]
  - Numerical queries
    $P_{=?}$ [F $goal_1$]
    $P_{max=?}$ [F $goal_1$]

Example MDP (robot navigation)



- Extensions
  - richer temporal specs (LTL), costs/rewards, multi-objective, …

Linear temporal logic (LTL)

- LTL (linear temporal logic) syntax:
  - $\psi ::= \text{true} \mid a \mid \psi \wedge \psi \mid \neg\psi \mid X\,\psi \mid \psi\, U\, \psi \mid F\,\psi \mid G\,\psi$
- Propositional logic + temporal operators:
  - a is an atomic proposition (labelling a state)
  - $X\,\psi$ means "ψ is true in the next state"
  - $F\,\psi$ means "ψ is eventually true"
  - $G\,\psi$ means "ψ always remains true"
  - $\psi_1\, U\, \psi_2$ means "ψ_2 is true eventually and ψ_1 is true until then"
- Common alternative notation:
  - $\bigcirc$ (next), $\Diamond$ (eventually), $\Box$ (always) , U (until)

Linear temporal logic (LTL)

- LTL (linear temporal logic) syntax:
  - $\psi ::= \text{true} \mid a \mid \psi \wedge \psi \mid \neg\psi \mid X\,\psi \mid \psi\, U\, \psi \mid F\,\psi \mid G\,\psi$
- Commonly used LTL formulae:
  - $G\,(a \rightarrow F\,b)$ – "b always eventually follows a"
  - $G\,(a \rightarrow X\,b)$ – "b always immediately follows a”
  - $G\,F\,a$ – "a is true infinitely often"
  - $F\,G\,a$ – "a becomes true and remains true forever"
- Example: robot task specifications in LTL
  - e.g. $P_{>0.7}\,[\,(G\neg hazard) \wedge (GF\, goal_1)\,]$ – ”the probability of avoiding hazard and visiting $goal_1$ infinitely often is > 0.7"
  - e.g. $P_{max=?}\,[\,\neg zone_3\; U\; (zone_1 \wedge (F\, zone_4))\,]$ – "max. probability of patrolling zone 1 (whilst avoiding zone 3) then zone 4?”

Temporal logic

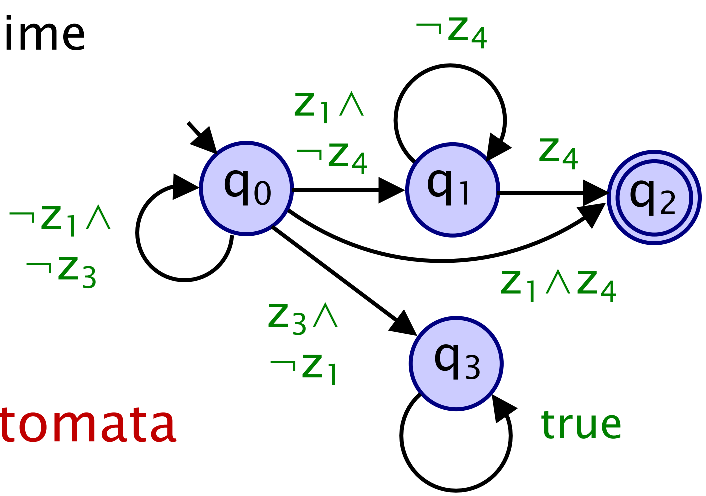
- Benefits of temporal logic
  - unambiguous, flexible, tractable behavioural specification
    - broad range of quantitative properties expressible
  - (probabilistic) guarantees on safety, performance, etc.
    - meaningful properties: event probabilities, time, energy,…

$$P_{>0.7} [(G \neg hazard) \wedge (GF\ goal_1)]$$

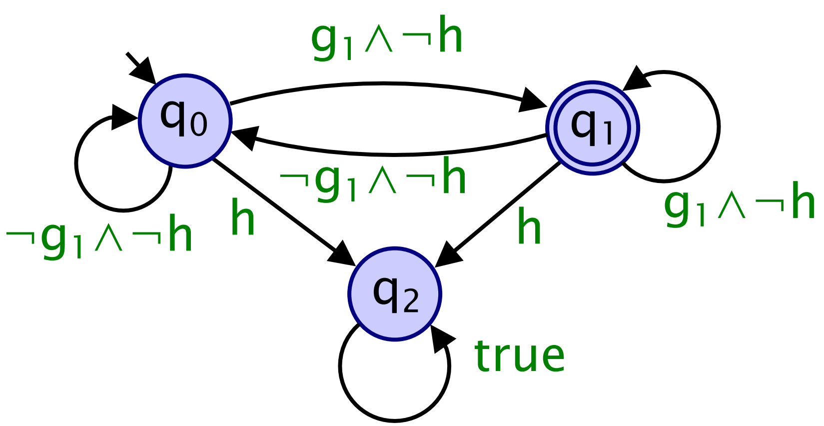
    - (c.f. ad-hoc reward structures, e.g. with discounting)
    - caveat: accuracy of model (and its solution)
  - efficient LTL-to-automata translation
    - optimal (finite-memory) policy synthesis (via product MDP)
    - correctness monitoring / shielding
    - task progress metrics

LTL & automata

- Safe/co-safe LTL: (deterministic) finite automata
  - (non-)satisfaction occurs in finite time
  - $\neg zone_3$ U ($zone_1 \wedge$ (F $zone_4$))

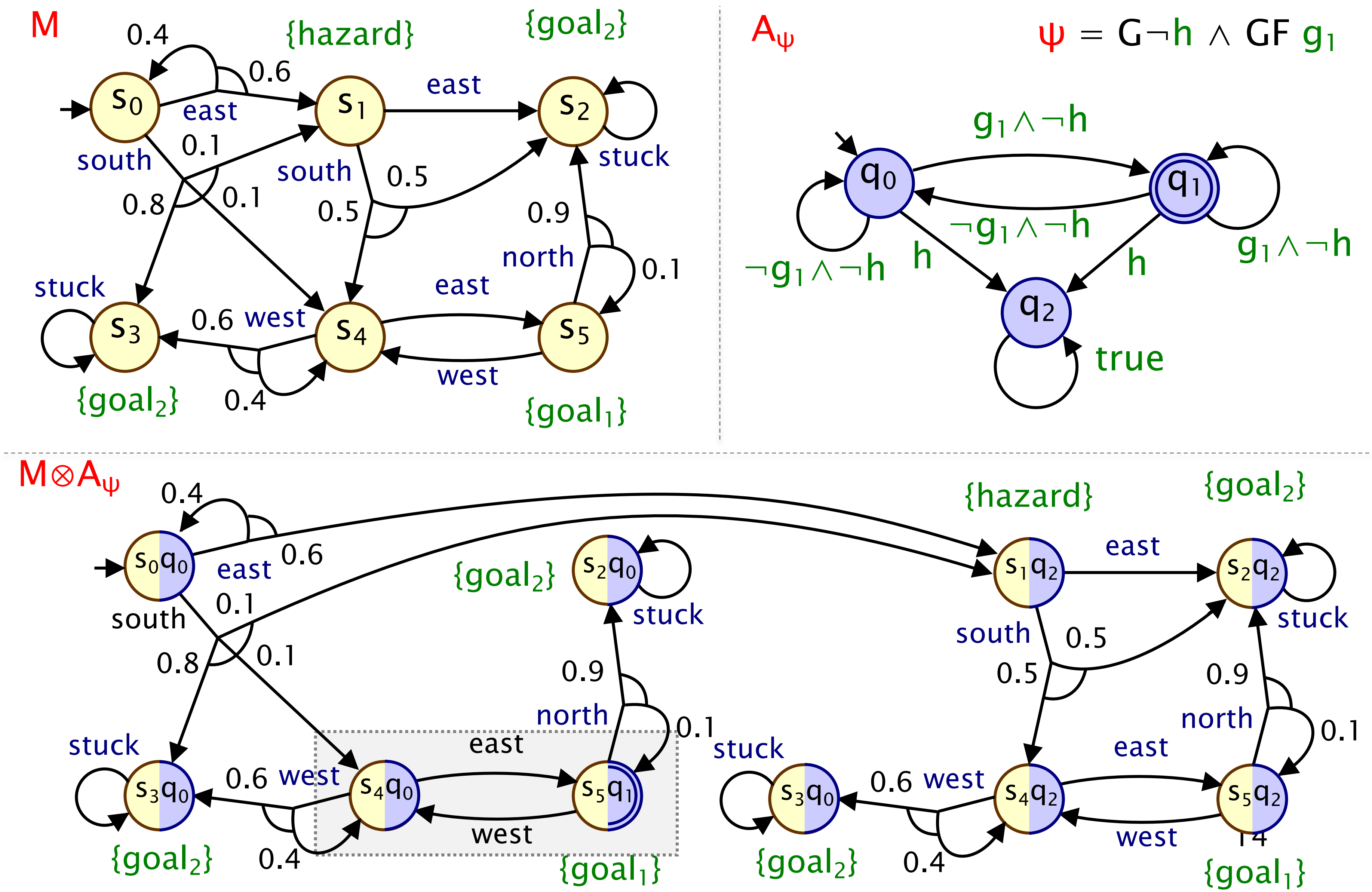


- Full LTL: e.g. (det.) Rabin/Buchi automata
  - G¬hazard $\wedge$ GF $goal_1$

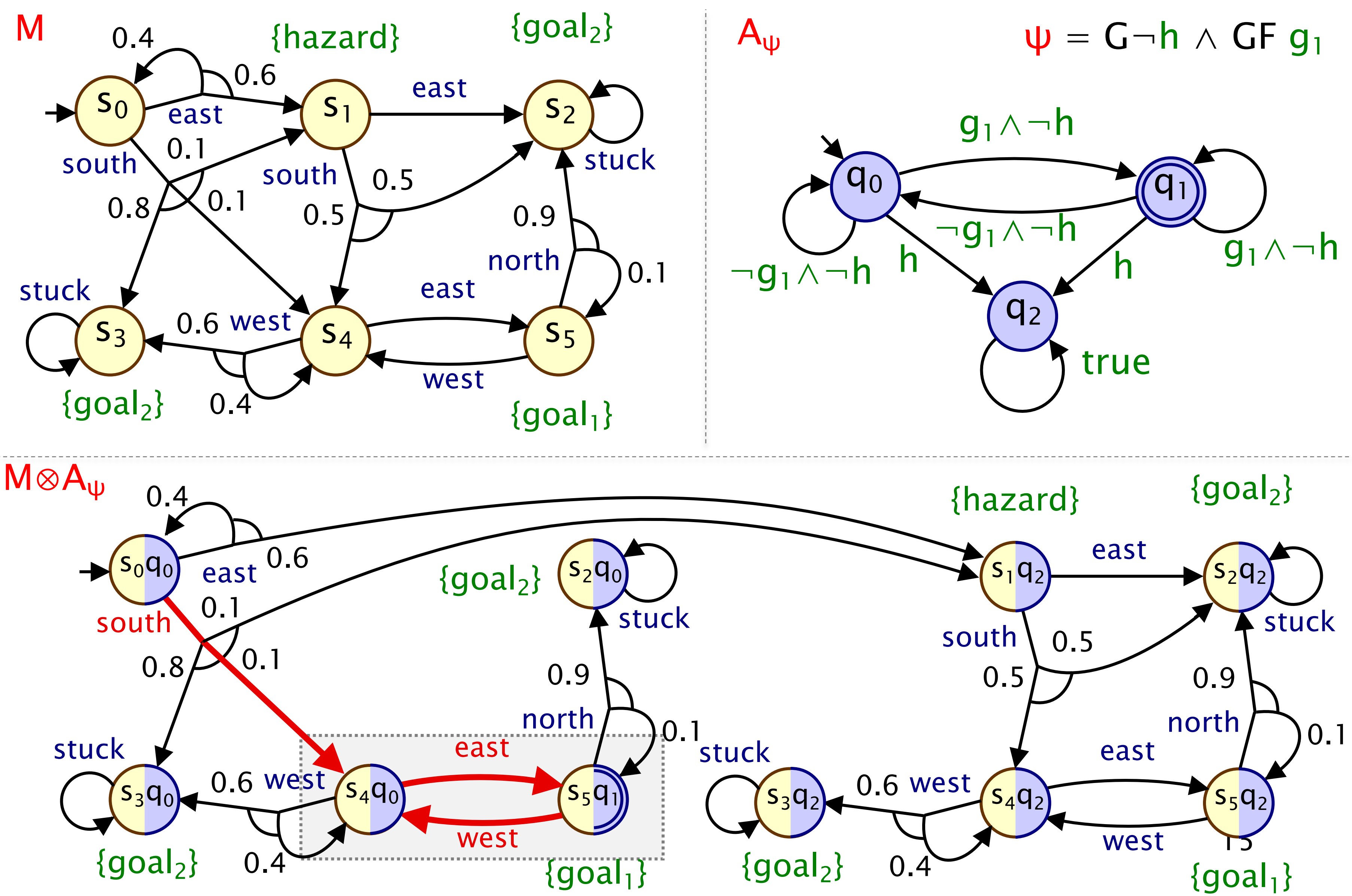


- Other useful LTL subclasses
  - GR(1), LTL\GU, …

LTL model checking via product MDP



LTL model checking via product MDP



Costs & Rewards

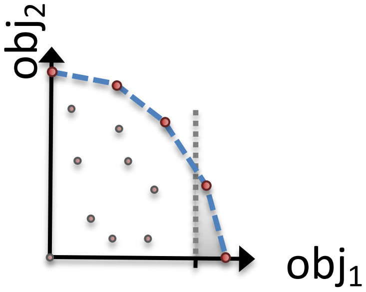
- Costs & rewards
  - i.e., values assigned to model states or transitions

- Temporal logic examples
  - $R^{energy}_{min=?}$ [F goal] – minimise the expected energy consumption until the the goal is reached
  - $R^{hazard}_{\leq 1.5}$[$C^{\leq 20}$] – the expected number of times that the robot enters the hazard location within 20 steps is at most 1.5
  - $R^{time}_{min=?}$ [$\neg zone_3$ U ($zone_1 \wedge$ (F $zone_4$))] – minimise expected time to patrol zones 1 then 4, without passing through 3

- Notes:
  1. mostly use the R (reward) operator, even for costs
  2. discounted rewards are more rarely used in this context

More temporal logic

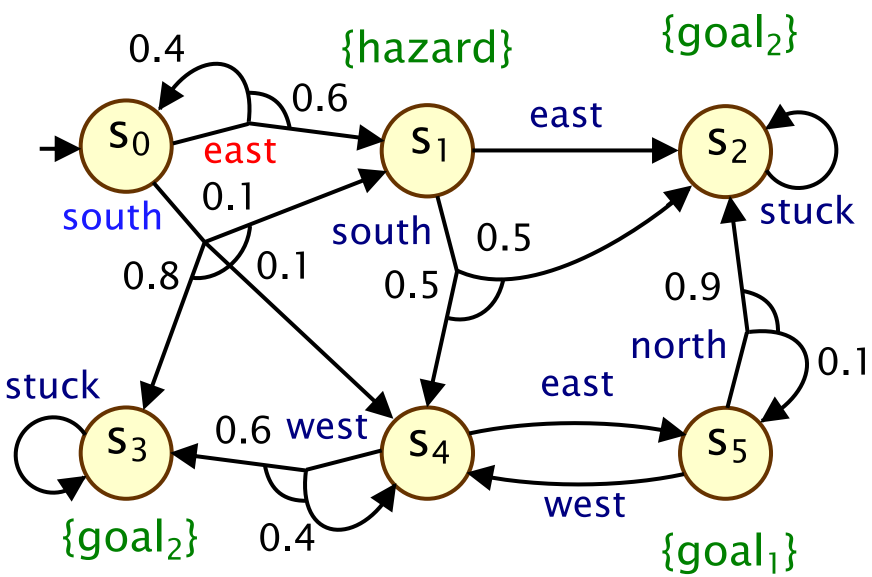
- Multi-objective queries
  - e.g. $\langle\langle * \rangle\rangle$ ($P_{max=?}$ [GF $goal_1$], $P_{\geq 0.7}$ [G ¬hazard])
  - max. objective 1 subject to constrained objective 2
  - also: achievability & Pareto queries

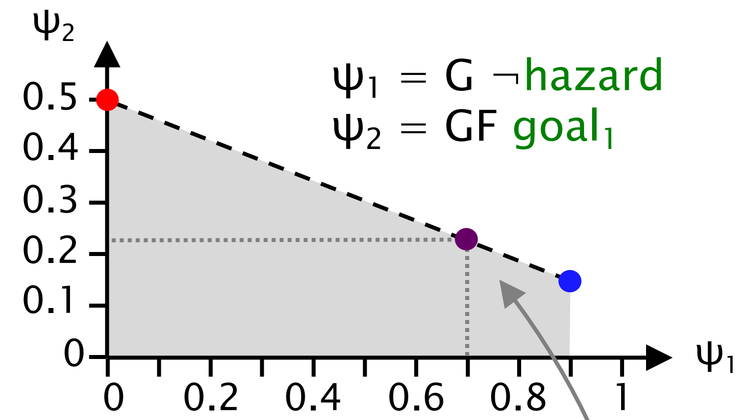


- Nested (branching-time) queries
  - e.g. $R_{min=?}$ [$P_{\geq 0.99}$ [$F^{\leq 10}$ base] U ($zone_1$ ∧ (F $zone_4$))]
  - "minimise expected time to visit zones 1 then 4, whilst (initially) ensuring the base can always be reliably reached

- And more
  - cost-bounded, conditional probabilities, quantiles
  - metric temporal logic, signal temporal logic
  - …

Multi-objective specifications





randomised, finite-memory optimal policy

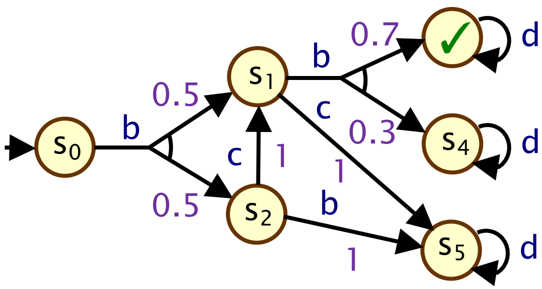
- Achievability query
  - $P_{\geq 0.7}$ [G ¬hazard] ∧ $P_{\geq 0.2}$ [GF goal$_1$] ?
- Numerical query
  - $P_{max=?}$ [GF goal$_1$] such that $P_{\geq 0.7}$ [G ¬hazard] ?
- Pareto query
  - for $P_{max=?}$ [G ¬hazard], $P_{max=?}$ [GF goal$_1$] ?

Techniques, tools & languages

Verification techniques

- Probabilistic model checking techniques
  - automata + graph analysis + numerical solution
  - often more focus on exhaustive/"exact"/optimal methods
  - e.g., for MDPs: value iteration (VI), linear programming

- Example (MDPs):
  - max. probability of reaching ✓
  - values $p(s) = \sup_\sigma Pr_s^\sigma(F ✓)$ are the least fixed point of:

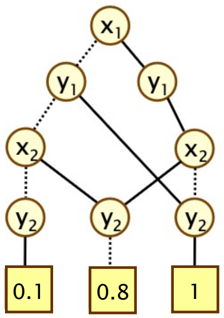


$$p(s) = \begin{cases} 1 & \text{if } s \models ✓ \\ \max_a \sum_{s'} \delta(s,a)(s') \cdot p(s') & \text{otherwise} \end{cases}$$

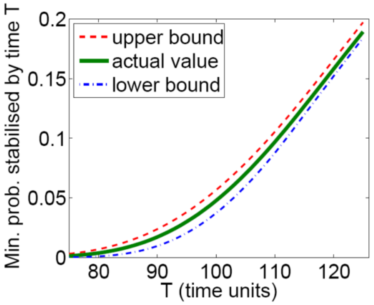
- But: VI has known accuracy and convergence issues
  - interval iteration, sound VI, optimistic VI
  - separate convergence from above and below

Scalability & efficiency

- Scalability & efficiency are always key challenges
  - many approaches investigated…
- Symbolic probabilistic model checking
  - i.e., (multi-terminal) binary decision diagrams



- Model reductions
  - bisimulation minimisation
  - abstraction + sound bounds (property driven)



- Sampling (simulation) based methods
  - statistical model checking, PAC guarantees, heuristics, …
- Trade-off: scalability/efficiency vs. accuracy/guarantees
  - spectrum of “correctness” : exact, floating-point correct, ε-correct, probably ε-correct, often ε-correct

Probabilistic verification tools

- Probabilistic verification software
  - PRISM (and PRISM-games), Storm, Modest toolset, ePMC
  - general purpose probabilistic model checking tools
  - wide range of models (Markov chains, (PO)MDPs, games), many temporal logics & solution techniques

- Also many other specialised tools…
  - PET (partial exploration)
  - FAUST^2, StocHy (continuous space/hybrid systems)
  - MultiGain (multi-objective + mean payoff)
  - Tempest (permissive + shielding)
  - PAYNT (POMDPs + probabilistic programs)
  - Prophesy (parametric techniques)

Modelling languages

- Example formal modelling languages
  - PRISM: textual language, based on guarded commands
  - Modest: expressive language for stochastic hybrid automata
- Some key modelling language features
  - nondeterministic + probabilistic behaviour
  - compositional model specifications
    - components, parallel composition, communication
  - parameterised models
    - probabilities, sizes, components

PRISM modelling language

- PRISM modelling language
  - de-facto standard for probabilistic model checkers
  - key ingredients: modules, variables, guarded commands
  - language features: nondeterminism + probability, parallel composition, costs/rewards, parameters

PRISM modelling language

Example (PRISM-games)

- PRISM modelling language
  - de
  - ke ... ands
  - lar pa

```
csg // Model type: concurrent stochastic game
player p1 user1 endplayer    player p2 user2 endplayer
// Parameters
const int emax;  const double q1;  const double q2 = 0.9 * q1;
// Modules: users (senders) + channel
module user1
        s1 : [0..1] init 0; // has player 1 sent?
        e1 : [0..emax] init emax; // energy level of player 1
        [w1] true -> (s1'=0); // wait
        [t1]  e1>0 -> (s1'=c' ? 0 : 1) & (e1'=e1-1); // transmit
endmodule
module user2 = user1 [ s1=s2, e1=e2, w1=w2, t1=t2 ] endmodule
module channel
        c : bool init false; // is there a collision?
        [t1,w2] true -> q1 : (c'=false) + (1-q1) : (c'=true); // only user 1 transmits
        [w1,t2] true -> q1 : (c'=false) + (1-q1) : (c'=true); // only user 2 transmits
        [t1,t2]  true -> q2 : (c'=false) + (1-q2) : (c'=true); // both users transmit
endmodule
// Reward structures: energy usage
rewards "energy" [t1] true: 1.5; [t2] true: 1.2; endrewards
```

PRISM modelling language

- PRISM modelling language
  - de-facto standard for probabilistic model checkers
  - key ingredients: modules, variables, guarded commands
  - language features: nondeterminism + probability, parallel composition, costs/rewards, parameters
- Quite simplistic, low-level
  - e.g., no control flow, functions, mostly finite variables, …
- But:
  - uniform language for many types of probabilistic model
  - many translations exist from more expressive languages
  - forces users to confront state space explosion?
  - well suited to symbolic methods (NB: but not to simulation)

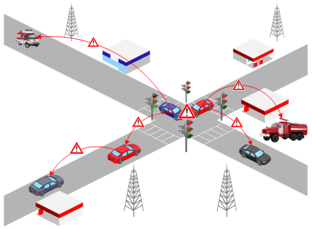
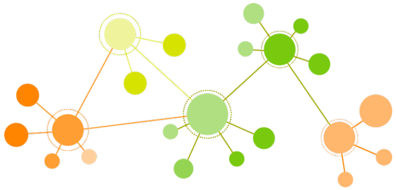
Modelling languages

- Example formal modelling languages
  - PRISM: textual language, based on guarded commands
  - Modest: expressive language for stochastic hybrid automata
- Some key modelling language features
  - nondeterministic + probabilistic behaviour
  - compositional model specifications
    - components, parallel composition, communication
  - parameterised models
    - probabilities, sizes, components
- Challenges
  - language/tool interoperability
    - e.g., JANI (models), PPDDL (planning), HOAF (automata), tool APIs
  - modelling stochasticity/uncertainty
    - probabilistic programming languages?

Multi-agent verification

Verification with stochastic games

- How do we plan rigorously with…
  - multiple autonomous agents acting concurrently
  - competitive or collaborative behaviour between agents, possibly with differing/opposing goals
  - e.g. security protocols, algorithms for distributed consensus, energy management, autonomous robotics, auctions

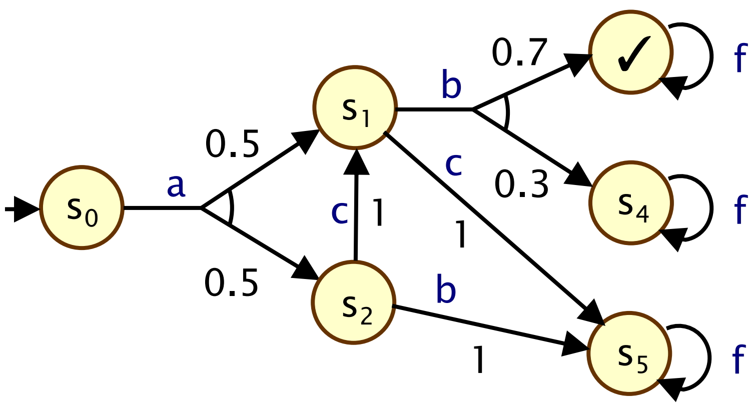




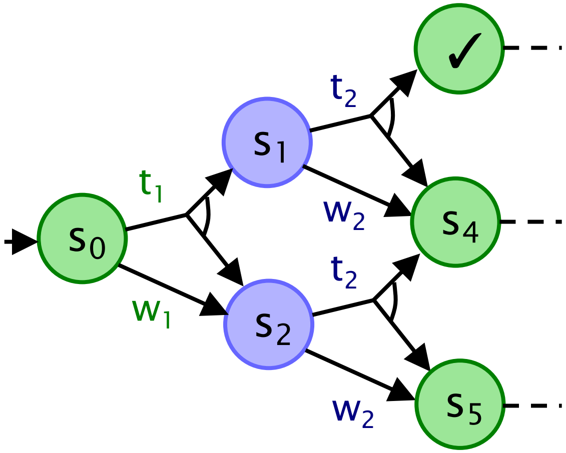
- Verification with stochastic multi-player games
  - verification (and synthesis) of strategies that are robust in adversarial settings and stochastic environments

Stochastic multi-player games

- Stochastic multi-player games
  - strategies + probability + multiple players
  - for now: turn-based (player i controls states S_i)

Markov decision processes (MDPs) → Turn-based stochastic games (TSGs)



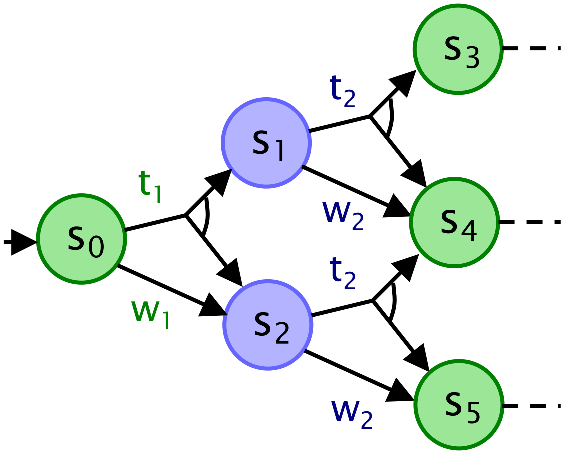


Property specification: rPATL

- rPATL (reward probabilistic alternating temporal logic)
  - branching-time temporal logic for stochastic games
- CTL, extended with:
  - coalition operator $\langle\langle C \rangle\rangle$ of ATL
  - probabilistic operator P of PCTL
  - generalised (expected) reward operator R from PRISM
- In short:
  - zero-sum, probabilistic reachability + expected total reward
- Example:
  - $\langle\langle \{robot_1, robot_3\} \rangle\rangle \, P_{>0.99} \, [\, F^{\leq 10} \, (goal_1 \vee goal_3) \,]$
  - "robots 1 and 3 have a strategy to ensure that the probability of reaching the goal location within 10 steps is >0.99, regardless of the strategies of other players"

Model checking rPATL

- Main task: checking individual P and R operators
  - reduces to solving a (zero-sum) stochastic 2-player game
  - e.g. max/min reachability probability: $\sup_{\sigma_1} \inf_{\sigma_2} \Pr_s^{\sigma_1,\sigma_2}(F\checkmark)$
  - complexity: NP ∩ coNP (if we omit some reward operators)



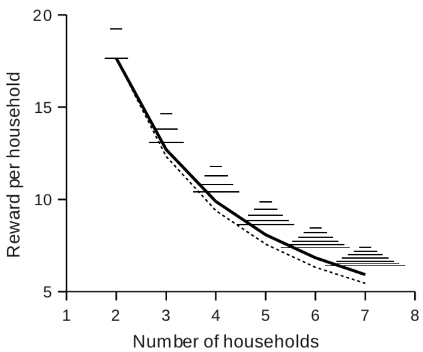
- We again use value iteration
  - values p(s) are the least fixed point of:

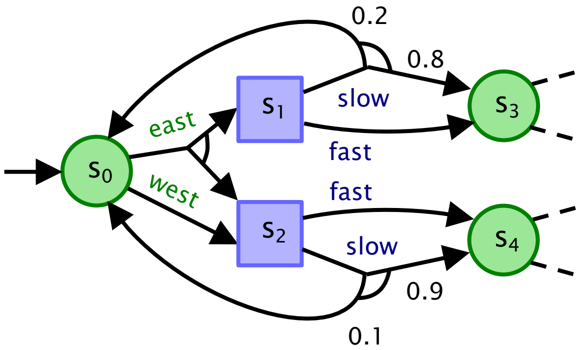
$$p(s) = \begin{cases} 1 & \text{if } s \models \checkmark \\ \max_a \sum_{s'} \delta(s,a)(s') \cdot p(s') & \text{if } s \not\models \checkmark \text{ and } s \in S_1 \\ \min_a \sum_{s'} \delta(s,a)(s') \cdot p(s') & \text{if } s \not\models \checkmark \text{ and } s \in S_2 \end{cases}$$

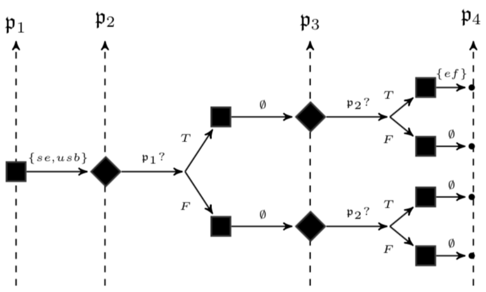
  - and more: graph-algorithms, sequences of fixed points, …

Applications

- Example application domains (PRISM-games)
  - collective decision making and team formation protocols
  - security: attack-defence trees; network protocols
  - human-in-the-loop UAV mission planning
  - autonomous urban driving
  - self-adaptive software architectures



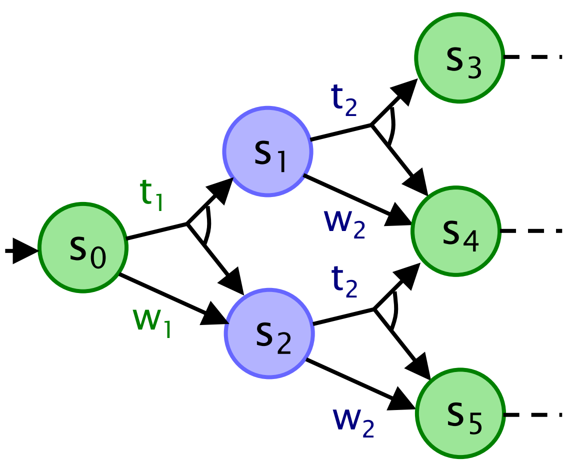


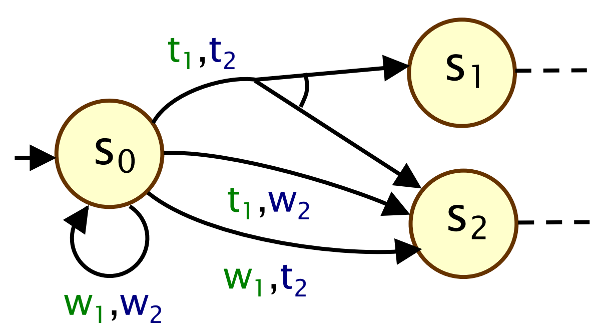


Concurrent stochastic games

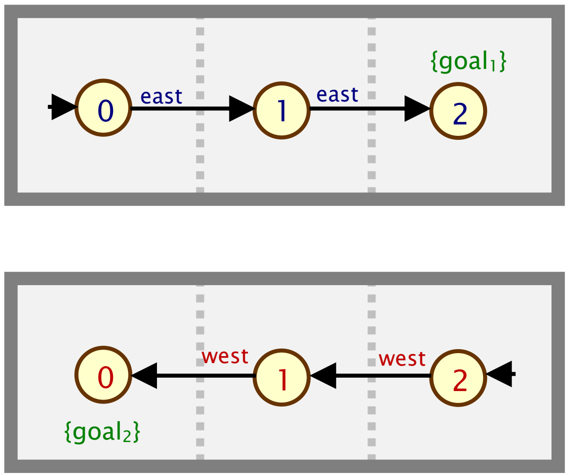
- Motivation:
  - more realistic model of components operating concurrently, making action choices <u>without</u> knowledge of others

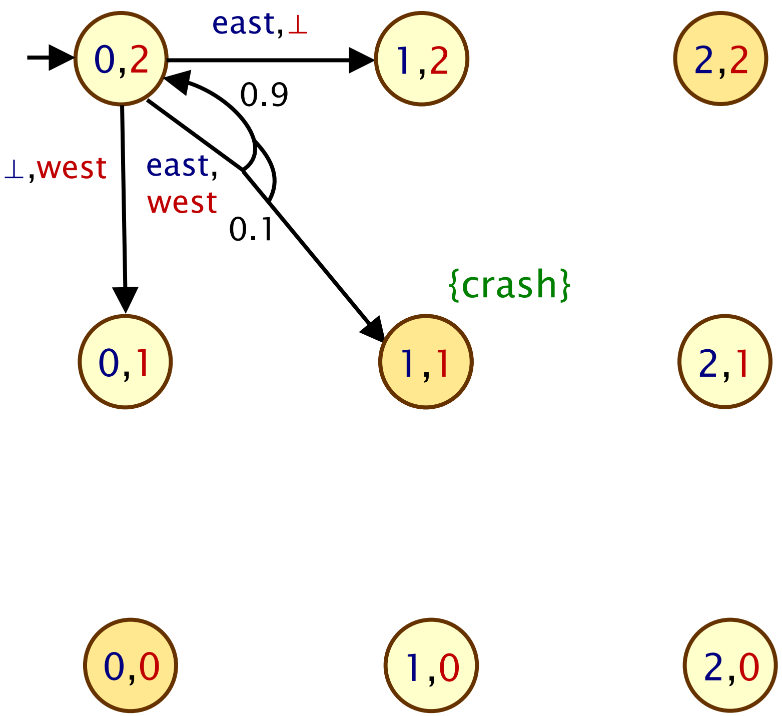
Turn-based stochastic games (TSGs) → **Concurrent stochastic games (CSGs)**



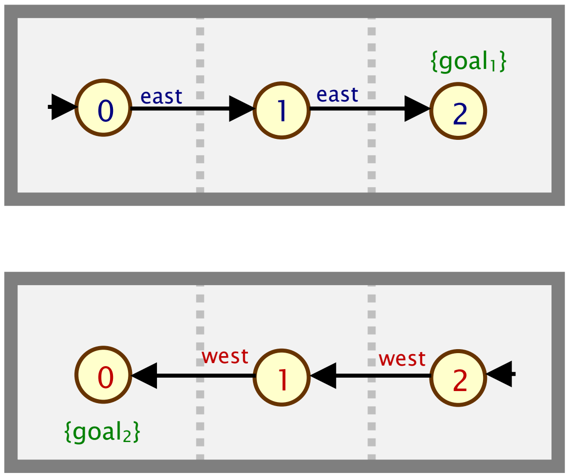


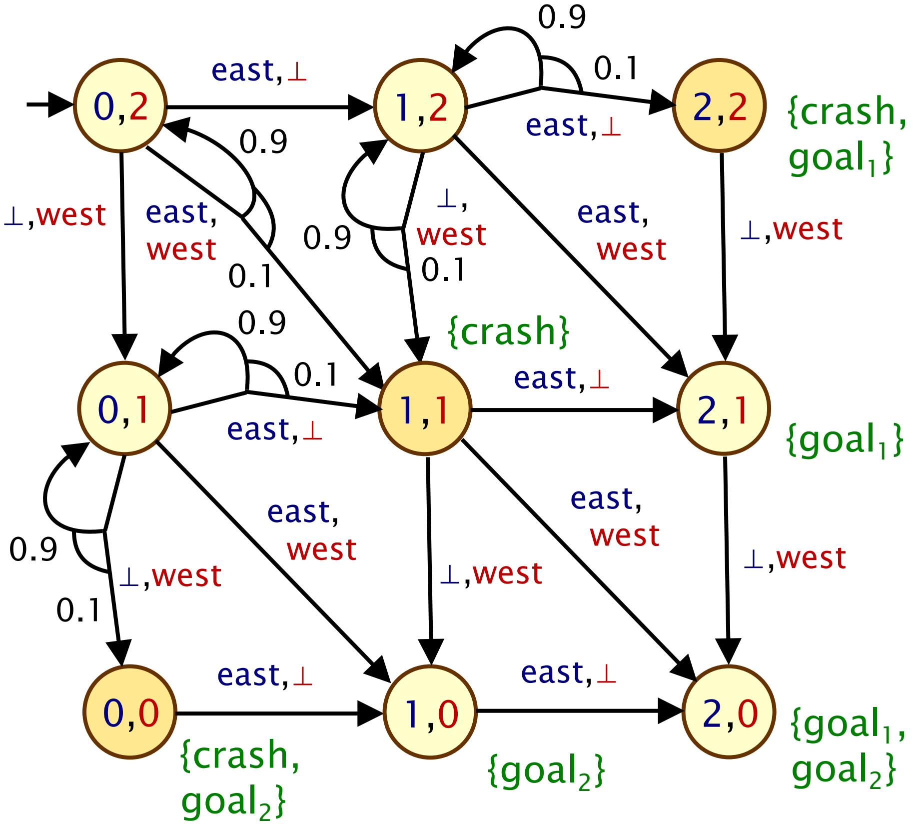
CSG for 2 robots on a 3x1 grid





CSG for 2 robots on a 3x1 grid





Concurrent stochastic games

- Concurrent stochastic games (CSGs)
  - players choose actions concurrently & independently
  - jointly determines (probabilistic) successor state
  - $\delta : S \times (A_1 \cup \{\bot\}) \times \ldots \times (A_n \cup \{\bot\}) \rightarrow Dist(S)$
  - generalises turn-based stochastic games

- We again use the logic rPATL for properties

- Same overall rPATL model checking algorithm [QEST'18]
  - key ingredient is now solving (zero-sum) 2-player CSGs
  - this problem is in PSPACE
  - note that optimal strategies are now randomised

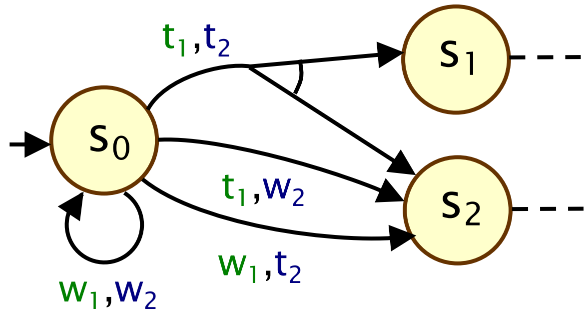
rPATL model checking for CSGs

- We again use a value iteration based approach
  - e.g. max/min reachability probabilities
  - $\sup_{\sigma_1} \inf_{\sigma_2} \Pr_s^{\sigma_1,\sigma_2}(\mathrm{F}\ \checkmark)$ for all states s
  - values p(s) are the least fixed point of:

$$p(s) = \begin{cases} 1 & \text{if } s \models \checkmark \\ \mathrm{val}(Z) & \text{if } s \not\models \checkmark \end{cases}$$

  - where Z is the matrix game with $z_{ij} = \Sigma_{s'}\ \delta(s,(a_i,b_j))(s') \cdot p(s')$

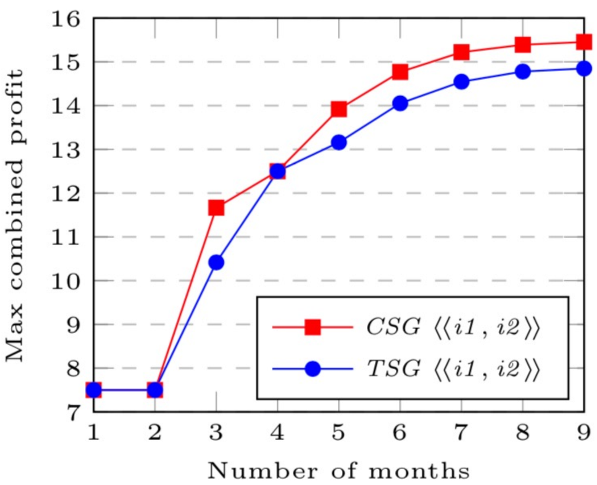
- So each iteration solves a matrix game for each state
  - LP problem of size |A|, where A = action set

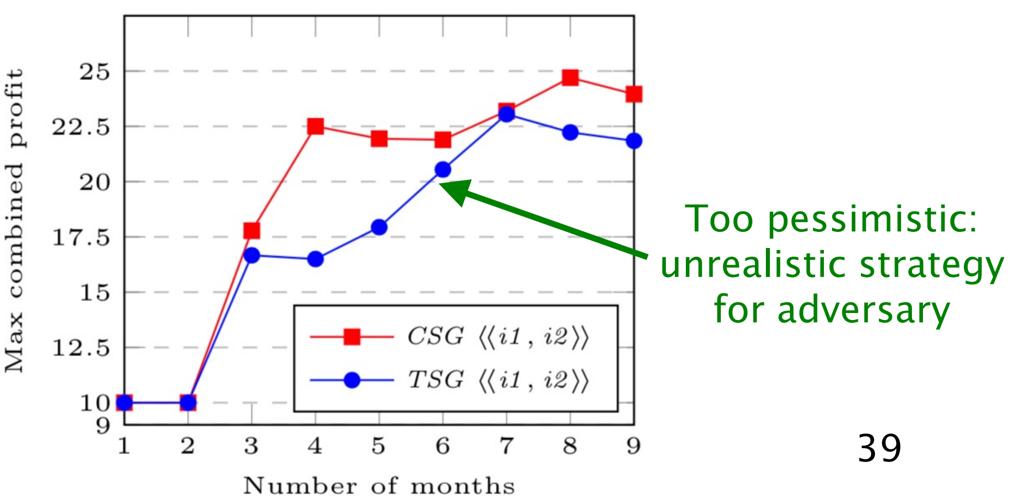


Example: Future markets investor

- Example rPATL query:
  - $\langle\langle investor_1, investor_2 \rangle\rangle\ R^{profit_{1,2}}_{max=?}\ [\ F\ finished_{1,2}\ ]$
  - i.e. maximising joint profit

- Results: with (left) and without (right) fluctuations
  - optimal (randomised) investment strategies synthesised
  - CSG yields more realistic results (market has less power due to limited observation of investor strategies)





Equilibria-based properties

- Motivation:
  - players/components may have distinct objectives but which are not directly opposing (non zero-sum)

Zero-sum properties → Equilibria-based properties

$\langle\langle robot_1 \rangle\rangle_{max=?}$ P [$F^{\leq k}$ $goal_1$]

$\langle\langle robot_1{:}robot_2 \rangle\rangle_{max=?}$ (P [$F^{\leq k}$ $goal_1$]+P [$F^{\leq k}$ $goal_2$])

- We use Nash equilibria (NE)
  - no incentive for any player to unilaterally change strategy
  - actually, we use ϵ-NE, which always exist for CSGs
  - a strategy profile $\sigma=(\sigma_{1,\dots,}\sigma_n)$ for a CSG is an ϵ-NE for state s and objectives $X_1,\dots,X_n$ iff:
  - $Pr_s^{\sigma}(X_i) \geq \sup \{ Pr_s^{\sigma'}(X_i) \mid \sigma'=\sigma_{-i}[\sigma_i'] \text{ and } \sigma_i' \in \Sigma_i \} - \epsilon$ for all i

Social-welfare Nash equilibria

- Key idea: formulate model checking (strategy synthesis) in terms of social-welfare Nash equilibria (SWNE)
  - these are NE which maximise the sum $E_s^\sigma(X_1) + \ldots E_s^\sigma(X_n)$
  - i.e., optimise the players combined goal

- We extend rPATL accordingly

Zero-sum properties

$\langle\langle robot_1 \rangle\rangle_{max=?}\ P\ [\ F^{\leq k}\ goal_1\ ]$

find a robot 1 strategy which maximises the probability of it reaching its goal, regardless of robot 2

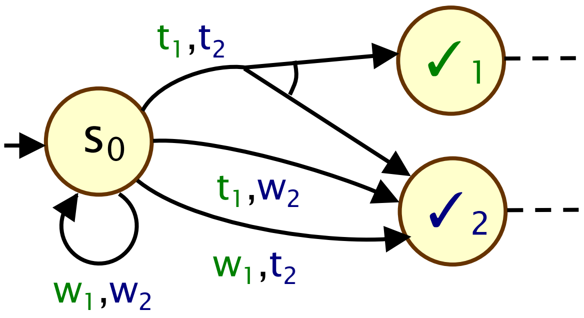
Equilibria-based properties

$\langle\langle robot_1{:}robot_2 \rangle\rangle_{max=?}\ (P\ [\ F^{\leq k}\ goal_1\ ]{+}P\ [F^{\leq k}\ goal_2])$

find (SWNE) strategies for robots 1 and 2 where there is no incentive to change actions and which maximise joint goal probability

Model checking for extended rPATL

- Model checking for CSGs with equilibria
  - first: 2-coalition case [FM'19]
  - needs solution of bimatrix games
  - (basic problem is EXPTIME)
  - we adapt a known approach using labelled polytopes, and implement with an SMT encoding



- We further extend the value iteration approach:

$$p(s) = \begin{cases} (1,1) & \text{if } s \models ✓_1 \wedge ✓_2 \\ (p_{max}(s,✓_2),1) & \text{if } s \models ✓_1 \wedge \neg ✓_2 \\ (1,p_{max}(s,✓_1)) & \text{if } s \models \neg ✓_1 \wedge ✓_2 \\ val(Z_1,Z_2) & \text{if } s \models \neg ✓_1 \wedge \neg ✓_2 \end{cases}$$

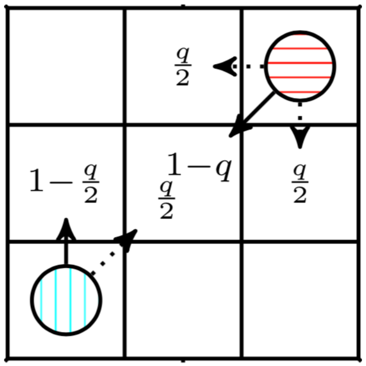
standard MDP analysis (second and third cases)

bimatrix game (fourth case)

  - where Z_1 and Z_2 encode matrix games similar to before

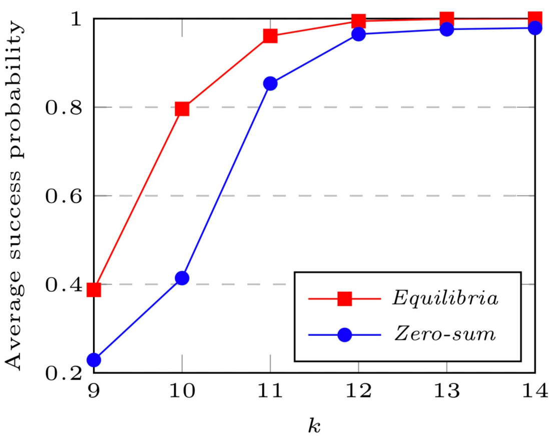
Example: multi-robot coordination

- 2 robots navigating an l x l grid
  - start at opposite corners, goals are to navigate to opposite corners
  - obstacles modelled stochastically: navigation in chosen direction fails with probability q



- We synthesise SWNEs to maximise the average probability of robots reaching their goals within time k
  - $\langle\langle robot_1 : robot_2 \rangle\rangle_{max=?}$ (P [$F^{\leq k}$ $goal_1$]+P [$F^{\leq k}$ $goal_2$])

- Results (10 x 10 grid)
  - better performance obtained than using zero-sum methods, i.e., optimising for robot 1, then robot 2



Conclusions

Conclusions

- Probabilistic model checking
  - temporal logics & automata
  - tools, techniques, modelling languages
  - multi-agent systems

- Challenges
  - partial information/observability
  - managing model uncertainty
  - integration with machine learning
  - continuous variables/state spaces
  - scalability & efficiency vs. accuracy


Annual Review of Control, Robotics, and Autonomous Systems

Probabilistic Model Checking and Autonomy

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probabilistic modeling, temporal logic, model checking, strategy synthesis, stochastic games, equilibria

Abstract

The design and control of autonomous systems that operate in uncertain or adversarial environments can be facilitated by formal modeling and analysis. Probabilistic model checking is a technique to automatically verify, for a given temporal logic specification, that a system model satisfies the specification, as well as to synthesize an optimal strategy for its control. This method has recently been extended to multiagent systems that exhibit competitive or cooperative behavior modeled via stochastic games and synthesis of equilibria strategies. In this article, we provide an overview of probabilistic model checking, focusing on models supported by the PRISM and PRISM-games model checkers. This overview includes fully observable and partially observable Markov decision processes, as well as turn-based and concurrent stochastic games, together with associated probabilistic temporal logics. We demonstrate the applicability of the framework through illustrative examples from autonomous systems. Finally, we highlight research challenges and suggest directions for future work in this area.

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More details and references here