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SOME EXTENSIONS TO THE PROBABILISTIC, BIASED MODEL OF TIMED CSP

Gavin Lowe

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Oxford University Computing Laboratory
11 Keble Road, Oxford OX1 3QD

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Abstract

In a previous paper [Low92] we presented two languages based upon timed CSP, with associated semantic models. The first language included biased operators in the syntax, which could be used to model different priorities on actions. The second language extended this by adding a probabilistic choice operator to the syntax; this allowed us to give a semantics which modelled the probabilities of different behaviours occurring.

In this paper we aim to extend these two languages, to bring them up to the full expressive strength of timed CSP. We add environments (variable bindings) to our semantics; this allows us to give a more satisfactory model of recursion and allows us to model mutual recursion.

Our previous language did not allow us to model communication of values over a channel. We extend our language and semantic model to overcome this.

We prove a new property that holds of all behaviours of a process; this basically says that if a process can refuse two actions seperately then it can refuse their union.

We also add infinite probabilistic choice, timeout, timed transfer, event interrupt and communicating parallel operators to our syntax, and give corresponding semantic definitions.

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1 Introduction

In [Low92] we presented two new languages based upon timed CSP. The first language was completely deterministic and made use of prioritized operators. The second language also included a probabilistic choice operator, which allows one to describe probabilistic processes. These two languages were given semantic models which recorded the priorities upon actions; the semantic model for the probabilistic language also gave the probabilities of different behaviours occurring.

In this paper we present a few additions to the probabilistic language and its semantic model. These additions are designed to extend the language to give it the same expressive power as timed CSP.

We begin by reviewing the probabilistic, biased model in section 2. In section 3 we introduce a model with environments (variable bindings); an environment will associate a member of the semantic space with each free variable.

In section 4 we prove a new property that holds of all behaviours of a process. If a process can refuse the offers w and w' while performing v, then it could refuse $w \uplus w'$.

The language described in [Low92] only allowed *binary* probabilistic choice; in section 5 we generalise this by introducing an infinite probabilistic choice operator which can choose between a countably infinite number of processes, according to some probability distribution.

In section 6 we describe a method of modelling channels transmitting data; we take care to ensure that our semantics does not give processes an arbitrary preference between different items of data input on the same channel — we do not want to say that a process prefers to input a 1 to a 2, say; instead we specify that the process has no preference between different inputs.

In [Low92] we gave a semantics to the recursive process $\mu X \bullet F(X)$; this process acts like the process $F(X)$ where the variable X represents recursive calls; there is a delay of length δ associated with all recursive calls. In section 7 we use our treatment of environments to generalise this so that the body of a recursion can be a general PBTCSP term P; the semantics will depend upon the value the environment gives to any free variables within P. We can also define an immediate recursion, written $\mu X \circ P$, where there is no delay involved in recursive calls; for this to be well defined, we have to ensure that the process cannot recurse infinitely often in a finite time. We end this section by giving a semantic definition to mutual recursion, and showing when it is well defined.

In section 8 we present a number of new operators designed to transfer control from one process to another according to certain circumstances: a time-out operator, which transfers control if no action is observer within a certain time; a timed-transfer operator, which transfers control at a certain time regardless of the progress made; and an event interrupt operator, which transfers control when a certain event is observed.

In section 9 we present a probabilistic external choice operator, which is biased in favour of its left- or right-hand argument, probabilistically. We then introduce a communicating parallel operator — a hybrid parallel composition operator which forces synchronisation on events from a particular set, and allows interleaving on all other events.

We give semantic definitions for these new operators in the probabilistic model, confident that it would also be possible to give similar definitions for them in the deterministic model (except, of course, for the new probabilistic operators). We hope that these new operators will prove useful in the specification and implementation of probabilistic, timed processes.

2 Probabilistic, biased, timed CSP

2.1 Syntax

The syntax of probabilistic, biased timed CSP (PBTCSP), as described in [Low92], is as follows:

$$
\begin{array}{lll}
P ::= & STOP \mid SKIP \mid WAIT\ t \mid X \mid & \text{basic processes} \\
& a \xrightarrow{t} P \mid P \fatsemi P \mid WAIT\ t\,;P \mid & \text{sequential composition} \\
& P\ {}_p\sqcap_q\ P \mid & \text{probabilistic choice} \\
& P \mathbin{\Box} P \mid P \mathbin{\boxdot} P \mid & \text{external choice} \\
& P \lfloor\!\lfloor\!\lfloor P \mid P \rfloor\!\rfloor\!\rfloor P \mid P\ {}^{A}\lfloor\!\lfloor\!\lfloor^{B}\ P \mid P\ {}^{A}\rfloor\!\rfloor\!\rfloor^{B}\ P \mid & \text{parallel composition} \\
& P \lfloor\!\lfloor\!\lfloor\!\lfloor P \mid P \rfloor\!\rfloor\!\rfloor\!\rfloor P \mid & \text{interleaving} \\
& P \setminus A \mid f(P) \mid & \text{abstraction and renaming} \\
& \mu X \bullet F(X) & \text{recursion}
\end{array}
$$

where t ranges over the set $TIME$ of times, which we take to be positive real numbers; a ranges over some alphabet Σ of events; and A and B range over $\mathbb{P}\Sigma$. The variables p and q range over $(0,1)$ with the property that $p+q=1$. The renaming function f ranges over $\Sigma \to \Sigma$ and F ranges over functions from variables to processes.

The atomic processes $STOP$, $SKIP$, $WAIT\ t$ and X act like the corresponding timed CSP processes: $STOP$ is the broken process which can perform no visible events; $SKIP$ is the process which can do nothing except terminate by performing the event $\checkmark$; $WAIT\ t$ can terminate after t time units; X is used to represent a recursive call within a recursion of the form $\mu X \bullet F(X)$.

The process $a \xrightarrow{t} P$ is initially willing to perform the visible event a; once it has performed an a, it will act like process P after a delay of length t. If we omit the parameter t we will take its value to be δ — a system constant. The process $P \fatsemi Q$ will initially act like P; if P terminates, the process will then act like Q after a delay of length δ. The process $WAIT\ t\,;P$ will act like P, delayed by t time units.

$P\ {}_p\sqcap_q Q$ performs a probabilistic choice between the processes P and Q; it chooses P with probability p and Q with probability q.

The processes $P \mathbin{\Box} Q$ and $P \mathbin{\boxdot} Q$ offer the environment a choice between the two processes P and Q — they both act like whichever of P and Q is first able to perform an event offered by the environment. The diference between them lies in the way they act when P and Q are first able to perform events offered by the environment simultaneously; in this case the left-biased choice $P \mathbin{\Box} Q$ chooses in favour of P whereas the right-biased choice $P \mathbin{\boxdot} Q$ chooses in favour of Q.

The processes $P \lfloor\!\lfloor\!\lfloor Q$ and $P \rfloor\!\rfloor\!\rfloor Q$ execute P and Q in lockstep parallel, synchronizing on every visible event. The difference between them lies in the way they behave when the processes P and Q have different priorities on the events they offer. The left-biased parallel composition $P \lfloor\!\lfloor\!\lfloor Q$ follows P's priorities whereas the right-biased composition $P \rfloor\!\rfloor\!\rfloor Q$ follows Q's priorities. For example,

$$(a \longrightarrow STOP \mathbin{\Box} b \longrightarrow STOP) \lfloor\!\lfloor\!\lfloor (a \longrightarrow STOP \mathbin{\boxdot} b \longrightarrow STOP)$$

will perform an a if the environment is initially willing to perform either an a or a b — the process on the left is the 'master' of the parallel composition, and so the composition follows its priorities. Similarly, the process

$$(a \longrightarrow STOP \mathbin{\Box} b \longrightarrow STOP) \rfloor\!\rfloor\!\rfloor (a \longrightarrow STOP \mathbin{\boxdot} b \longrightarrow STOP)$$

will perform a b if the environment is initially willing to perform either an a or a b. The processes $P\ {}^{A}\lfloor\!\lfloor\!\lfloor^{B}\ Q$ and $P\ {}^{A}\rfloor\!\rfloor\!\rfloor^{B}\ Q$ execute P and Q in parallel; P can only perform events from the set A, and Q can only perform events from the set B; they must synchronize on events from the set $A \cap B$. The priorities of $P\ {}^{A}\lfloor\!\lfloor\!\lfloor^{B}\ Q$ follow the priorities of P on events from A, and follow Q's

priorities on events from $B \setminus A$. With $P \,{}^{A}\!\!\Vdash^{B}\, Q$ it is the other way round: the priorities are those of Q on events from B, and those of P on $A \setminus B$.

$P \leftthreetimes Q$ and $P \rightthreetimes Q$ interleave P and Q — they are executed in parallel without synchronization. In the left-biased interleaving $P \leftthreetimes Q$ the process P is the master, so

- the process $a \longrightarrow P \leftthreetimes a \longrightarrow Q$ will perform the a on the left if only one a is offered by the environment;
- the process $a \longrightarrow P \leftthreetimes b \longrightarrow Q$ will prefer to perform the a to the b if the environment is only willing to perform one or the other, although the process would prefer to perform both events;
- the process $(a \longrightarrow STOP \leftthreetimes b \longrightarrow STOP) \Vdash (a \longrightarrow STOP \,\Box\, b \longrightarrow STOP)$ will prefer to perform a b to an a since the process on the right — the master — prefers a b.

Abstraction is achieved via the hiding operator. The process $P \setminus A$ acts like P except all the events from the set A occur silently; the environment's cooperation is not necessary for the events from A to occur, so they happen as soon as the process is willing to perform them. The process $f(P)$ acts like P except all the external events are renamed by the function f.

The process $\mu X \bullet F(X)$ is a recursive process. It acts like the process given by $F(X)$ where the variable X represents a recursive call. There is a delay of length δ associated with all recursive calls.

2.2 Semantics

The semantic model for our language represents a process as a pair consisting of:

- a set A containing the behaviours that the process can perform;
- a function f which gives the probabilities of behaviours occurring.

Before discussing these further, we introduce some more terms.

Fundamental to our semantics is the notion of an *offer*. This is simply a pair (t, χ) where t is a member of the set $TIME$ of times, and χ is a bag of events. The pair (t, χ) represents a willingness to perform all the events in χ at time t. We define the space of offers by

$$OFF \mathrel{\widehat=} TIME \times \operatorname{bag} \Sigma$$

Note that a process will often be able to perform any one of several different offers.

We represent behaviours (or observations) as triples.

- As in most process algebras, we include a *trace*, i.e. a record of the visible events performed. We model this as a function s from an initial segment of time to actions

  $$TT \mathrel{\widehat=} \{s : TIME \pfun \operatorname{bag} \Sigma \mid \exists \tau : TIME \bullet \operatorname{dom} s = [0, \tau]\}$$

  $s(t)$ is the action performed at time t. For ease of notation, we will often write traces as sequences of timed events within angle brackets, $\prec$ and $\succ$, for example $\prec(1, a), (2, b), (2, c)\succ$.
- Since we are dealing with a prioritized model, we want to include some record of the different priorities given to different actions. We do this via a relation $\sqsubseteq$ between offers, which we call an *offer relation*, of type $OFFREL$. Informally, $(t, \chi) \sqsubseteq (t, \psi)$ if the process would prefer to perform action ψ to action χ at time t.
- It is also useful to include a record of the time τ that the process is observed up until.

We therefore model behaviours as triples $(\tau, \sqsubseteq, s)$, where τ is a time, $\sqsubseteq$ is an offer relation, and s is a timed trace. Before discussing which behaviours are possible, we introduce some notation.

Notation

We have a number of operations and relations on offers.

$$\begin{array}{rcll} (t,\chi) \uplus (t,\psi) & \widehat{=} & (t, \chi \uplus \psi) & \text{bag union} \\ (t,\chi) \lfloor A & \widehat{=} & (t, \{\!| a : \chi \mid a \in A |\!\}) & \text{projection} \\ (t,\chi) \subseteq (t,\psi) & \Leftrightarrow & \chi \subseteq \psi & \text{sub-offer relation} \end{array}$$

The function I returns the set of times associated with a trace or with an offer relation.

$$Is \mathrel{\widehat{=}} \operatorname{dom} s \qquad I{\sqsubseteq} \mathrel{\widehat{=}} \{ t \mid \exists \chi, \psi \bullet (t,\chi) \sqsubseteq (t,\psi) \}$$

The '*times*' function returns the set of times at which events are performed in a trace.

$$times\ s \mathrel{\widehat{=}} \{ t \mid s(t) \neq \{\!|\}\!\} \}$$

The *during* operator $\upharpoonright$ restricts its first argument to the times specified by its second argument.

$$\begin{array}{c} s \upharpoonright I \mathrel{\widehat{=}} \{ t \mapsto s(t) \mid t \in I \} \\ \sqsubseteq \upharpoonright I \mathrel{\widehat{=}} \sqsubseteq' \quad \text{where } (t,\chi) \sqsubseteq' (t,\psi) \Leftrightarrow (t,\chi) \sqsubseteq (t,\psi) \wedge t \in I \end{array}$$

This can be used to define *before*, *strictly before*, *after*, *strictly after* and *at* operators: $\lceil$, $\lmoustache$, $\rceil$, $\rmoustache$ and $\uparrow$.

$$\begin{array}{rcl@{\qquad\qquad}rcl} s \lceil t & \widehat{=} & s \upharpoonright [0,t] & \sqsubseteq \lceil t & \widehat{=} & \sqsubseteq \upharpoonright [0,t] \\ s \lmoustache t & \widehat{=} & s \upharpoonright [0,t) & \sqsubseteq \lmoustache t & \widehat{=} & \sqsubseteq \upharpoonright [0,t) \\ s \rceil t & \widehat{=} & s \upharpoonright [t,\infty) & \sqsubseteq \rceil t & \widehat{=} & \sqsubseteq \upharpoonright [t,\infty) \\ s \rmoustache t & \widehat{=} & s \upharpoonright (t,\infty) & \sqsubseteq \rmoustache t & \widehat{=} & \sqsubseteq \upharpoonright (t,\infty) \\ s \uparrow t & \widehat{=} & s \upharpoonright \{t\} & \sqsubseteq \uparrow t & \widehat{=} & \sqsubseteq \upharpoonright \{t\} \end{array}$$

The projection operator $\lfloor$ projects a trace onto a set of events:

$$s \lfloor A \mathrel{\widehat{=}} \{ t \mapsto \{\!| a : s(t) \mid a \in A |\!\} \mid t \in Is \}$$

The function Σ returns the set of visible events performed during a trace.

$$\Sigma s \mathrel{\widehat{=}} \{ a \mid \exists t \bullet a \in s(t) \}$$

The functions *begin* and *end* return the times of the first and last events from a trace.

$$begin\ s \mathrel{\widehat{=}} \begin{cases} \inf(times\ t) & \text{if } s \neq \prec\!\succ \\ \infty & \text{if } s = \prec\!\succ \end{cases} \qquad end\ s \mathrel{\widehat{=}} \begin{cases} \sup(times\ t) & \text{if } s \neq \prec\!\succ \\ -\infty & \text{if } s = \prec\!\succ \end{cases}$$

The functions *first* and *last* return the first and last actions from a trace.

$$first\ s \mathrel{\widehat{=}} s(begin\ s) \qquad last\ s \mathrel{\widehat{=}} s(end\ s)$$

The functions *head* and *foot* return the first and last non-empty timed actions from a trace.

$$head\ s \mathrel{\widehat{=}} (begin\ s, first\ s) \qquad foot\ s \mathrel{\widehat{=}} (end\ s, last\ s)$$

Concatenation of traces or offer relations is written using $\frown$.

$$\begin{array}{rcll} s_1 \frown s_2 & \widehat{=} & s_1 \cup s_2 & \text{if } Is_1 \cap Is_2 = \{\} \wedge \exists \tau \bullet Is_1 \cup Is_2 = [0,\tau] \\ \sqsubseteq_1 \frown \sqsubseteq_2 & \widehat{=} & \sqsubseteq_1 \cup \sqsubseteq_2 & \text{if } I\sqsubseteq_1 \cap I\sqsubseteq_2 = \{\} \wedge \exists \tau \bullet I\sqsubseteq_1 \cup I\sqsubseteq_2 = [0,\tau] \end{array}$$

We define two temporal shifting operators:

$$\begin{array}{rcl} s + t & \widehat{=} & \{t + t' \mapsto s(t') \mid t' \in Is\} \\ s - t & \widehat{=} & \{t' - t \mapsto s(t') \mid t' \in Is \wedge t' \geqslant t\} \\ \sqsubseteq + t & \widehat{=} & \sqsubseteq' \quad \text{where } (t + t', \chi) \sqsubseteq' (t + t', \psi) \Leftrightarrow (t, \chi) \sqsubseteq (t, \psi) \\ \sqsubseteq - t & \widehat{=} & \sqsubseteq' \quad \text{where } (t' - t, \chi) \sqsubseteq' (t' - t, \psi) \Leftrightarrow (t, \chi) \sqsubseteq (t, \psi) \wedge t' \geqslant t \end{array}$$

The function items returns the set of offers made by a process:

$$\text{items} \sqsubseteq \widehat{=} \{v \mid \exists\, w \bullet v \sqsubseteq w\}$$

The operator $\otimes$ can be used for enumerating offer relations. $I \otimes \langle \chi_0, \chi_1, \ldots, \chi_{n-1} \rangle$ represents the offer relation $\sqsubseteq$, such that for all times t during I, $(t, \chi_0) \sqsupset (t, \chi_1) \sqsupset (t, \chi_2) \sqsupset \ldots \sqsupset (t, \chi_{n-1})$.

$$\begin{array}{l} \forall\, I : \mathbb{P}(TIME)\ ;\ d : \text{seq}(\text{bag}\, \Sigma) \bullet I \otimes d = \sqsubseteq \\ \quad \text{where } (t, \chi) \sqsubseteq (t', \psi) \Leftrightarrow t = t' \in I \wedge \exists\, i, j \bullet 0 \leqslant i \leqslant j < \# d \wedge d(i) = \psi \wedge d(j) = \chi \end{array}$$

We will write $\sum \{ f(x) \mid p(x) \}$ to represent the sum of the $f(x)$, where the sum is taken over all x such that $p(x)$ holds; we will use this for adding probabilities.

Possible behaviours

Every behaviour $(\tau, \sqsubseteq, s)$ satisfies five healthiness conditions:

1. $I \sqsubseteq = Is = [0, \tau]$
2. $\forall\, t \leqslant \tau \bullet s \upharpoonright t \in \text{items} \sqsubseteq$
3. $t_0 < t_1 \wedge \{(t, \chi) \mid t \in (t_0, t_1)\} \subseteq \text{items} \sqsubseteq \Rightarrow (t_0, \chi) \uplus s \upharpoonright t_0 \sqsupseteq s \upharpoonright t_0$
4. $(t, \chi) \in \text{items} \sqsubseteq \wedge \psi \subseteq \chi \Rightarrow (t, \psi) \in \text{items} \sqsubseteq$
5. $(t < t' \wedge \forall\, t'' \in [t, t') \bullet (t'', \chi) \in \text{items} \sqsubseteq) \Rightarrow (t', \chi) \in \text{items} \sqsubseteq$

We define the space BEH of behaviours to be those triples $(\tau, \sqsubseteq, s)$ of type $TIME \times OFFREL \times TT$ satisfying the conditions. The conditions state:

1. If a process is observed up until time τ, then no events will be seen to be performed or offered after time τ.
2. items$\sqsubseteq$ is the set of timed actions that the process is willing to perform; hence the process can only perform actions from items$\sqsubseteq$.
3. If a process is willing to perform an action χ throughout some open time interval beginning at t_0, then at t_0 it could have performed this action as well as what it did perform; further it would have preferred to perform χ and what it did perform to just what it did perform. This condition is necessary to avoid processes such as the one which offers an a during $(0, 1)$; if the environment offers an a from time 0 onwards, then there is no sensible choice of when the a should be performed.
4. The offers of a process are sub-offer closed.
5. If a process offers χ at all times just before t', then it also offers χ at t'.

Environments

We consider an environment for a process P to be another process Q running in parallel with P and synchronizing on all visible events. We think of the environmental offer as being the set of offers of Q. We model environmental offers as sets of offers: $EOFF \subseteq \mathbb{P}\, OFF$

Not every environment could have brought about a particular behaviour of a process. We shall say that a behaviour $(\tau, \sqsubseteq, s)$ is compatible with an environmental offer Ω, written $(\tau, \sqsubseteq, s)$ *compat* Ω, if $(\tau, \sqsubseteq, s)$ could have resulted from the environment offering Ω. Formally:

1. $I\Omega = [0, \tau]$
2. $\forall\, t \bullet s \uparrow t = \sqcup_{\sqsubseteq} \Omega \uparrow t$
3. $(\forall\, t \in (t_0, t_1) \bullet (t, \chi) \in \Omega) \Rightarrow (t_0, \chi) \uplus s \uparrow t_0 \in \Omega$

where $TINT$ is the set off all time intervals (open, closed or half open), and $\sqcup_{\sqsubseteq} \Omega \uparrow t$ is the element of $\Omega \uparrow t$ which is maximal with respect to the offer relation $\sqsubseteq$; the operator $\sqcup$ satisfies the two axioms

$$\forall\, t \bullet \sqcup_{\sqsubseteq} \Omega \uparrow t \in \Omega \uparrow t \cap \text{items} \sqsubseteq \qquad \forall (t, \chi) \in \Omega \bullet (t, \chi) \sqsubseteq \sqcup_{\sqsubseteq} \Omega \uparrow t$$

The three conditions state that:

1. The duration of the environmental offer is the same as the duration of the behaviour;
2. At all times, the process performs the element of the environmental offer which is maximal under its offer relation — in other words, the process picks the offer which it prefers;
3. If an action χ is offered throughout some open time interval beginning at t_0, then at t_0 the environment must have offered χ along with the events of s. This condition is necessary to avoid an environment such as the one that offers an a during the period $(0, 1]$; if a process that is willing to perform an a from time 0 onwards is placed in this environment, then there is no sensible choice as to to when the event should be performed.

Of these, condition 2 is perhaps the most important. It describes the way that a process chooses the events it performs. At each instant the environment is willing to perform any one of a number of bags of events; the process takes its pick from these by choosing the bag which is strongest under its offer relation.

The semantic space

We represent a process by a pair, (A, f), where A gives the set of behaviours that a process can perform, and f is a probability function; $f(\tau, \sqsubseteq, s)$ is the probability of $(\tau, \sqsubseteq, s)$ occurring, given a suitable environment, i.e. any environment Ω such that $(\tau, \sqsubseteq, s)$ is compatible with Ω. We define the spaces TS_B of sets of behaviours and PTF_B of probability functions by

$$TS_B \mathrel{\widehat{=}} \mathbb{P}(BEH) \qquad PTF_B \mathrel{\widehat{=}} BEH \to PROB$$

where $PROB$ is the set of probabilities:

$$PROB \mathrel{\widehat{=}} [0, 1]$$

Our processes are members of the space PTP_B defined by

$$PTP_B \mathrel{\widehat{=}} TS_B \times PTF_B$$

We define the space PTM_B, the *Probabilistic, Timed Model using Biases* to be those pairs (A, f) in PTP_B satisfying a number of healthiness conditions:

1. $\forall \tau \geqslant 0 \bullet \exists\, n(\tau) \bullet (\tau, \sqsubseteq, s) \in A \Rightarrow \#s \leqslant n(\tau)$
2. $\forall \tau \geqslant 0 \bullet \exists\, n(\tau) \bullet (\tau, \sqsubseteq, s) \in A \Rightarrow \exists\, k \leqslant n(\tau)\ ;\ I_0, \ldots, I_{k-1} \in TINT \bullet$
   $I_0, \ldots I_{k-1}$ partition $[0, \tau]$
   $\wedge\ \forall\, i : 0 \,..\, k-1\ ;\ t, t' : I_i\ ;\ \chi, \psi : \mathrm{bag}\, \Sigma \bullet (t, \chi) \sqsubseteq (t, \psi) \Leftrightarrow (t', \chi) \sqsubseteq (t', \psi)$
3. $(\tau, \sqsubseteq, s) \in A \wedge (t, \chi) \in \mathrm{items} \sqsubseteq \Rightarrow (t, \sqsubseteq \upharpoonright t, s \setminus t^\frown (t, \chi)) \in A$
4. $f(\tau, \sqsubseteq, s) > 0 \Rightarrow (\tau, \sqsubseteq, s) \in A$
5. $\sum \left\{ f(0, \sqsubseteq, \prec\!\!\succ) \mid \sqsubseteq \in OFFREL \right\} = 1$
6. $\forall\, s : TT\ ; \sqsubseteq : OFFREL\ ; \Omega : EOFF\ ; \tau, \tau' : TIME \bullet Is = [0, \tau] \wedge I\Omega = [\tau, \tau'] \Rightarrow$
   $f(\tau, \sqsubseteq, s) = \sum \left\{ f(\tau', \sqsubseteq', s \setminus \tau^\frown \sqcup_{\sqsubseteq'} \Omega) \mid \sqsubseteq' \upharpoonright \tau = \sqsubseteq \right\}$

These axioms state:

1. The number of events that a process can perform in a finite time is bounded.
2. The number of times at which an offer relation can change in a finite time is bounded.
3. If a process offers action χ at time t, then it could have performed χ at t.
4. If the probability of a process having a certain behaviour is non zero, then that behaviour is possible.
5. If the environment offers no events at time 0, then the empty trace occurs with probability one.
6. The probability of a process displaying some behaviour up to time τ is the same as the sum of the probabilities of the extensions of this behaviour that could have resulted from the environment offering Ω between times τ and τ'.

It is worth stressing the relationship between the probability function f and the environment Ω. $f(\tau, \sqsubseteq, s)$ is the probability of the process performing $(\tau, \sqsubseteq, s)$ *if* $(\tau, \sqsubseteq, s)$ is compatible with the environmental offer Ω; any behaviour not compatible with Ω occurs with probability zero. We can use this to define a probability function f_Ω (for each environment Ω) that gives the probabilities of each behaviour, given that the environment offers Ω.

$$f_\Omega(\tau, \sqsubseteq, s) \mathrel{\widehat{=}} \begin{cases} f(\tau, \sqsubseteq, s) & \text{if } (\tau, \sqsubseteq, s)\ \mathit{compat}\ \Omega \\ 0 & \text{otherwise} \end{cases}$$

Semantic functions

We can define a semantic function $\mathcal{F}_{BT} : PBTCSP \to PTP_B$ such that for all processes P, the corresponding semantic value is given by $\mathcal{F}_{BT}[\![P]\!]$. We can define semantic functions $\mathcal{A}_{BT} : PBTCSP \to TS_B$ and $\mathcal{P}_{BT} : PBTCSP \to PTF_B$ that return respectively the set of possible behaviours and the probability function of a term. So

$$\forall\, P : PBTCSP \bullet \mathcal{F}_{BT}[\![P]\!] = (\mathcal{A}_{BT}[\![P]\!], \mathcal{P}_{BT}[\![P]\!])$$

The semantic definitions for all the operators were derived in [Low92]; in that paper we showed that if P is a process then the corresponding semantic value $\mathcal{F}_{BT}[\![P]\!]$ satisfies the healthiness conditions of PTM_B. In appendix B we give the semantic definitions, including the semantic definitions for the new operators presented in this paper.

3 Environments

We want to extend the syntax of our language by adding a clause X, representing a variable from a set VAR, and by allowing the body of a recursion to be a general term:

$$P ::= \ldots \mid X \mid \mu X \bullet P$$

To give a semantics to a variable we need to include an environment which models the variable bindings. We define a space ENV of environments which contains all functions from VAR, the set of variables, to (behaviour sets, behaviour probability function) pairs:

$$ENV \mathrel{\widehat{=}} VAR \to PTP_B$$

We will write $\rho[\![X]\!]$ for the value assigned to variable X in environment ρ.
We can now define semantic functions that make use of the environments:

$$\begin{aligned} \mathcal{F}_{BT} &\mathrel{\widehat{=}} PBTCSP \to ENV \to PTP_B \\ \mathcal{A}_{BT} &\mathrel{\widehat{=}} PBTCSP \to ENV \to TS_B \\ \mathcal{P}_{BT} &\mathrel{\widehat{=}} PBTCSP \to ENV \to PTF_B \end{aligned}$$

We will write $\mathcal{F}_{BT}[\![P]\!]\rho$ to denote the semantic value of term P in environment ρ; similarly, we will write $\mathcal{A}_{BT}[\![P]\!]\rho$ and $\mathcal{P}_{BT}[\![P]\!]\rho$ to denote the set of possible behaviours and the probability function, respectively. These are related by

$$\mathcal{F}_{BT}[\![P]\!]\rho \mathrel{\widehat{=}} (\mathcal{A}_{BT}[\![P]\!]\rho, \mathcal{P}_{BT}[\![P]\!]\rho)$$

We give semantics to the clause X in the obvious way:

$$\mathcal{F}_{BT}[\![X]\!]\rho \mathrel{\widehat{=}} \rho[\![X]\!]$$

The set of behaviours and probability function are given respectively by

$$\mathcal{A}_{BT}[\![X]\!]\rho \mathrel{\widehat{=}} \pi_1(\rho[\![X]\!]) \qquad \mathcal{P}_{BT}[\![X]\!]\rho \mathrel{\widehat{=}} \pi_2(\rho[\![X]\!])$$

where π_1 and π_2 are the projection functions that return the first and second components of their arguments, respectively:

$$\pi_1(A, f) = A \qquad \pi_2(A, f) = f$$

This allows us to give semantics to syntactic substitution:

$$\mathcal{A}_{BT}[\![P[Q/X]]\!]\rho = \mathcal{A}_{BT}[\![P]\!]\rho[\mathcal{A}_{BT}[\![Q]\!]\rho/X]$$

where $\rho[Y/X]$ is the environment obtained from ρ by setting Y to X:

$$\rho[Y/X][\![Z]\!] \mathrel{\widehat{=}} \begin{cases} Y & \text{if } Z = X \\ \rho[\![Z]\!] & \text{otherwise} \end{cases}$$

Most of the semantic definitions are virtually the same as before, with only the addition of environments; for example:

$$\begin{aligned} \mathcal{A}_{BT}[\![P \nparallel Q]\!]\rho &\mathrel{\widehat{=}} \{(\tau, \sqsubseteq_P \nparallel \sqsubseteq_Q, s) \mid (\tau, \sqsubseteq_P, s) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau, \sqsubseteq_Q, s) \in \mathcal{A}_{BT}[\![Q]\!]\rho\} \\ \mathcal{P}_{BT}[\![P \nparallel Q]\!]\rho(\tau, \sqsubseteq, s) &\mathrel{\widehat{=}} \sum\left\{\mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq_P, s).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau, \sqsubseteq_Q, s) \mid \sqsubseteq_P \nparallel \sqsubseteq_Q = \sqsubseteq\right\} \end{aligned}$$

We will deal with recursion in section 7.

In [Low92], we showed that all of our operators respect the healthiness conditions of the semantic space PTM_B; so if the environment maps all variables to members of PTM_B, then the semantics of a term is a member of PTM_B.

A PBTCSP process is a PBTCSP term with no free variables. Its semantics will be independent of the environment, and so in this case it is sensible to omit reference to the environment.

4 A new property of behaviours

In this section we prove a new healthiness condition that holds of all behaviours of processes. To motivate the condition we consider a behaviour $(\tau, \sqsubseteq, s)$ such that

$$(t, \{\!|a, b, b'|\!\}) \sqsupset (t, \{\!|a|\!\}) \sqsupset (t, \{\!|a, b|\!\}) \sqsupset (t, \{\!|a, b'|\!\})$$

and such that $s(t) = \{\!|a|\!\}$. This process would refuse a b at time t, because $\{\!|a, b|\!\}$ is not offered stronger than $\{\!|a|\!\}$. Similarly it would refuse a b'. However, it would not refuse a b and a b' if they were offered together, since $\{\!|a, b, b'|\!\}$ *is* offered stronger than $\{\!|a|\!\}$. This goes against our intuitions. Our new condition will outlaw such behaviours.

An early formulation of the condition was

$$v \uplus w \uplus w' \sqsupseteq v \Rightarrow (v \uplus w \sqsupseteq v \vee v \uplus w' \sqsupseteq v)$$

If the process prefers the offers w and w' along with v to just v, then it will prefer either w or w' along with v to just v. This condition is equivalent to

$$v \uplus w \not\sqsupseteq v \wedge v \uplus w' \not\sqsupseteq v \Rightarrow v \uplus w \uplus w' \not\sqsupseteq v$$

Suppose the process performs the offer v; the condition says that if it can refuse w and w' seperately (i.e. if it would rather not perform them in addition to the v), then it can refuse them together.

However, it turns out that this condition is not quite strong enough to prove directly by structural induction. Instead we prove it as a consequence of the following stronger condition:

$$v \uplus w \uplus w' \sqsupseteq v' \wedge (v \uplus w \uplus w') \cap v' = v \cap v' \Rightarrow (v \uplus w \sqsupseteq v' \vee v \uplus w' \sqsupseteq v') \qquad (*)$$

Note that this implies our initial condition in the case $v = v'$.

The equation $(v \uplus w \uplus w') \cap v' = v \cap v'$ says that v is a subset of $v \uplus w \uplus w'$ that contains as many members of v' as possible (and possibly events from outside v' as well). It is worth noting that this condition is always satisfied if $v' \subseteq v$.

It is also worth noting that $(*)$ is equivalent to the following condition, where the offer relations have become strict:

$$v \uplus w \uplus w' \sqsupset v' \wedge (v \uplus w \uplus w') \cap v' = v \cap v' \Rightarrow (v \uplus w \sqsupset v' \vee v \uplus w' \sqsupset v') \qquad (\dagger)$$

For suppose $(*)$ holds. Then if $v \uplus w \uplus w' \sqsupset v' \wedge (v \uplus w \uplus w') \cap v' = v \cap v'$ then we have $v \uplus w \sqsupseteq v' \vee v \uplus w' \sqsupseteq v'$. Suppose, without loss of generality, that the former holds, and suppose $v \uplus w = v'$; then $w = (Iw, \{\!|\!|\!\})$ (because of the condition $(v \uplus w \uplus w') \cap v' = v \cap v'$) so $v \uplus w' \sqsupset v'$. Conversely, suppose $(\dagger)$ holds, and suppose $v \uplus w \uplus w' \sqsupseteq v' \wedge (v \uplus w \uplus w') \cap v' = v \cap v'$. If $v \uplus w \uplus w' = v'$ then $w = w' = (Iv, \{\!|\!|\!\})$ so $v \uplus w, v \uplus w' = v'$. Alternatively, if $v \uplus w \uplus w' \sqsupset v'$ then by $(\dagger)$ we have $v \uplus w \sqsupset v' \vee v \uplus w' \sqsupset v'$.

We propose to change our definition of the space BEH by adding $(*)$ to the list of healthiness conditions. Note that it is not implied by the other healthiness conditions — it actually does say something new; we can find behaviours such as the one described in the first paragraph that satisfy all the healthiness conditions except $(*)$.

We prove that $(*)$ holds of all process behaviours by structural induction on the syntax of PBTCSP. The base cases are easy. For probabilistic choice, at each instant the offer relation is equal to the offer relation of either P or Q; the result then follows from the inductive hypothesis. Similar arguments hold for most of the other cases. The only cases worth proving in detail are those of external choice, parallel composition and hiding.

Case external choice:

Let $\sqsubseteq = \sqsubseteq_P \mathbin{\square} \sqsubseteq_Q$ and suppose $v \uplus w \uplus w' \sqsupseteq v' \wedge (v \uplus w \uplus w') \cap v' = v \cap v'$. The cases where w or w' are empty are trivial, so suppose otherwise. Let $Iv = t$. We proceed by a case analysis.

- Case $v \uplus w \uplus w' \sqsupset_P v' \sqsupseteq_P (t, \{\!|\}\!)$. Then by the inductive hypothesis we have $v \uplus w \sqsupset_P v'$ or $v \uplus w' \sqsupset_P v'$, giving $v \uplus w \sqsupset v'$ and $v \uplus w' \sqsupset v'$ respectively, by the definition of $\sqsubseteq_P \mathbin{\square} \sqsubseteq_Q$.
- Case $v \uplus w \uplus w' \sqsupset_P (t, \{\!|\}\!) \wedge v' \not\sqsupseteq_P (t, \{\!|\}\!) \wedge (v' \in \mathrm{items} \sqsubseteq_Q \vee v' \sqsubset_P (t, \{\!|\}\!))$. Using the inductive hypothesis with $(t, \{\!|\}\!)$ in the place of v' we have $v \uplus w \sqsupset_P (t, \{\!|\}\!)$ or $v \uplus w' \sqsupset_P (t, \{\!|\}\!)$, giving $v \uplus w \sqsupset v'$ or $v \uplus w' \sqsupset v'$.
- Case $v \uplus w \uplus w', v' \not\sqsupseteq_P (t, \{\!|\}\!) \wedge v \uplus w \uplus w' \sqsupset_Q v'$. Then by the inductive hypothesis $v \uplus w \sqsupset_Q v'$ or $v \uplus w' \sqsupset_Q v'$, giving $v \uplus w \sqsupset v'$ and $v \uplus w' \sqsupset v'$ respectively, whether or not $v \uplus w$ and $v \uplus w'$ are in $\mathrm{items} \sqsubseteq_P$.
- Case $v \uplus w \uplus w' \not\sqsupseteq_P (t, \{\!|\}\!) \sqsupset v' \wedge v \uplus w \uplus w' \in \mathrm{items} \sqsubseteq_Q \wedge v' \notin \mathrm{items} \sqsubseteq_Q$. Then by condition 4 on behaviours $v \uplus w, v \uplus w' \in \mathrm{items} \sqsubseteq_Q$ so $v \uplus w \sqsupset v'$ and $v \uplus w' \sqsupset v'$, whether or not $v \uplus w$ and $v \uplus w'$ are in $\mathrm{items} \sqsubseteq_P$.
- Case $(t, \{\!|\}\!) \sqsupset_P v \uplus w \uplus w' \sqsupset_P v'$ and $v \uplus w \uplus w', v' \notin \mathrm{items} \sqsubseteq_Q$. Then as before $v \uplus w \sqsupset_P v'$ or $v \uplus w' \sqsupset_P v'$ giving $v \uplus w \sqsupset v'$ or $v \uplus w' \sqsupset v'$.

This completes the case. □

Case parallel composition:

Let $\sqsubseteq = \sqsubseteq_P {}^{A}\!\!\parallel^{B} \sqsubseteq_Q$ and suppose $v \uplus w \uplus w' \sqsupseteq v' \wedge (v \uplus w \uplus w') \cap v' = v \cap v'$. We proceed by a case analysis.

- Case $(v \uplus w \uplus w') \lfloor A \sqsupset_P v' \lfloor A$ and $(v \uplus w \uplus w') \lfloor B, v' \lfloor B \in \mathrm{items} \sqsubseteq_Q$. Then by the form (†) of the inductive hypothesis (noting that $(v \uplus w \uplus w') \lfloor A \cap v' \lfloor A = v \lfloor B \cap v' \lfloor A$) we have $(v \uplus w) \lfloor A \sqsupset_P v' \lfloor A$ or $(v \uplus w') \lfloor A \sqsupset_P v' \lfloor A$. Also $(v \uplus w) \lfloor B, (v \uplus w') \lfloor B \in \mathrm{items} \sqsubseteq_Q$ by condition 4 on behaviours. Hence $v \uplus w \sqsupset v'$ or $v \uplus w' \sqsupset v'$, by the definition of $\sqsubseteq_P {}^{A}\!\!\parallel^{B} \sqsubseteq_Q$.
- Case $(v \uplus w \uplus w') \lfloor A = v' \lfloor A \in \mathrm{items} \sqsubseteq_P$ and $(v \uplus w \uplus w') \lfloor B \sqsupseteq_Q v' \lfloor B$. Then by the inductive hypothesis $(v \uplus w) \lfloor B \sqsupseteq_Q v' \lfloor B$ or $(v \uplus w') \lfloor B \sqsupseteq_Q v' \lfloor B$. Also $((v \uplus w \uplus w') \cap v') \lfloor A = (v \cap v') \lfloor A$ so $(w \uplus w') \lfloor A = (Iv, \{\!|\}\!)$, and so $(v \uplus w) \lfloor A = (v \uplus w') \lfloor A = v' \lfloor A$. Hence we have $v \uplus w \sqsupseteq v'$ or $v \uplus w' \sqsupseteq v'$.

This completes the case. □

Case hiding:

Let $\sqsubseteq = \sqsubseteq' \setminus A$ and suppose $v \uplus w \uplus w' \sqsupseteq v' \wedge (v \uplus w \uplus w') \cap v' = v \cap v'$. Then by the definition of hiding we have $\Uparrow_{\sqsubseteq'}^{\setminus A}(v \uplus w \uplus w') \sqsupseteq' \Uparrow_{\sqsubseteq'}^{\setminus A} v'$. Let $u \mathrel{\widehat{=}} \Uparrow_{\sqsubseteq'}^{\setminus A}(v \uplus w \uplus w') - w - w'$. Then we have $u \setminus A = v$ and $u \uplus w \uplus w' \sqsupseteq' \Uparrow_{\sqsubseteq'}^{\setminus A} v'$. We claim that $(u \uplus w \uplus w') \cap \Uparrow_{\sqsubseteq'}^{\setminus A} v' = u \cap \Uparrow_{\sqsubseteq'}^{\setminus A} v'$. We obviously have the right hand side contained in the left hand side. For the converse, suppose $(t, a) \in (u \uplus w \uplus w') \cap \Uparrow_{\sqsubseteq'}^{\setminus A} v'$; we perform a case analysis:

- If $a \in A$ then we must have $(t, a) \in u$ so $a \in u \cap \Uparrow_{\sqsubseteq'}^{\setminus A} v'$.
- If $a \notin A$ then

$$(t, a) \in (\Uparrow_{\sqsubseteq'}^{\setminus A}(v \uplus w \uplus w') \cap \Uparrow_{\sqsubseteq'}^{\setminus A} v') \setminus A = (v \uplus w \uplus w') \cap v' = v \cap v' \subseteq u \cap \Uparrow_{\sqsubseteq'}^{\setminus A} v'$$

Hence we may apply the inductive hypothesis to deduce that $u \uplus w \sqsupseteq' \Uparrow_{\sqsubseteq'}^{\setminus A} v'$ or $u \uplus w' \sqsupseteq' \Uparrow_{\sqsubseteq'}^{\setminus A} v'$. Without loss of generality, assume that the first case holds. Then we have $(u \uplus w) \setminus A = v \uplus w$ and so $\Uparrow_{\sqsubseteq'}^{\setminus A}(v \uplus w) \sqsupseteq' u \uplus w \sqsupseteq' \Uparrow_{\sqsubseteq'}^{\setminus A} v'$; hence $v \uplus w \sqsupseteq v'$, as required. □

5 Infinite probabilistic choice

In this section we will consider a process that can probabilistically choose between any countable number of processes. If the number of processes is finite, then this is definable in terms of the binary probabilistic choice operator; our new operator will just simplify our notation. In the infinite case we will have to place a restriction upon the sets of processes from which the choice can be made.

If I is a countable set, and $\sum_{i \in I} p_i = 1$ then we will write $\bigsqcap_{i \in I} [p_i] P_i$ to represent the process that, for all i, acts like process P_i with probability p_i.

We give a semantics to this process in the obvious way:

$$\mathcal{A}_{BT}[\![\bigsqcap_{i \in I} [p_i] P_i]\!] \rho \;\widehat{=}\; \bigcup \{ \mathcal{A}_{BT}[\![P_i]\!] \rho \mid i \in I \}$$

$$\mathcal{P}_{BT}[\![\bigsqcap_{i \in I} [p_i] P_i]\!] \rho(\tau, \sqsubseteq, s) \;\widehat{=}\; \sum \left\{ p_i \times \mathcal{P}_{BT}[\![P_i]\!] \rho(\tau, \sqsubseteq, s) \mid i \in I \right\}$$

As in the timed failures model, we need a restriction upon the sets of processes over which the choice can be made.

Definition 5.1: The set $\{P_i \mid i \in I\}$ is *uniformly bounded* if

$$\forall \tau \geqslant 0 \bullet \exists\, n(\tau) \bullet \forall\, i \in I\, ; \rho \in ENV \bullet (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P_i]\!] \rho \Rightarrow \#s \leqslant n(\tau)$$

and

$$\begin{aligned} &\forall \tau \geqslant 0 \bullet \exists\, n(\tau) \bullet \forall\, i \in I\, ; \rho \in ENV \bullet (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P_i]\!] \rho \Rightarrow \exists\, k \leqslant n(\tau)\, ; J_0, \ldots, J_{k-1} : TINT \bullet \\ &\quad J_0, \ldots, J_{k-1} \text{ partition } [0, \tau] \\ &\quad \wedge\, \forall\, j : 0 \,..\, k-1\, ; t, t' : J_j\, ; \chi, \psi : \mathrm{bag}\, \Sigma \bullet (t, \chi) \sqsubseteq (t, \psi) \Leftrightarrow (t', \chi) \sqsubseteq (t', \psi) \end{aligned}$$

♢

These two conditions correspond to axioms 1 and 2 of the semantic space. The first condition states that there is a uniform bound on the number of event that any of the processes can perform within time τ; the second condition states that there is a uniform bound on the number of times that the offer relation can change shape within time τ.

Subject to these conditions, the semantics of $\bigsqcap_{i \in I} [p_i] P_i$ is well defined:

Theorem 5.2: If $\{P_i \mid i \in I\}$ is uniformly bounded, then the semantic definition of $\bigsqcap_{i \in I} [p_i] P_i$ satisfies the axioms of the semantic space PTM_B. ♡

The reader should be aware that this method does not always effectively model nondeterminism that does not manifest itself in a finite amount of time. For example, consider the process P_n that can perform n as:

$$P_0 \;\widehat{=}\; STOP \qquad P_{n+1} \;\widehat{=}\; a \xrightarrow{1} P_n$$

Let P be the process which chooses probabilistically between the P_n with a geometric distribution:

$$P \;\widehat{=}\; \bigsqcap_{n \in \mathbb{N}} [(1/2)^{n+1}] P_n$$

P can perform any finite number of as. We would expect this to be different from the process P' that can perform an arbitrary number of as:

$$P' \;\widehat{=}\; a \xrightarrow{1} P' \;{}_{1/2}\sqcap_{1/2}\; STOP$$

However, our semantics gives the same value to both of these processes. It is interesting that the behaviours of P' that our model does not adequately represent — namely those where an infinite number of as are performed — occur with zero probability.

6 Communication over channels

In this section we consider how we can model processes that communicate along channels. Some models of communication have achieved this by considering communications of different values to be fundamentally different events — for example if c is a channel inputting integers, then the events $c?1$ and $c?2$ would be treated as completely different. This is not adequate in a model where we want to place priorities upon actions — we do not want to have to make arbitrary decisions such as specifying that the process would prefer to input a 1 than a 2. We want to model the fact that processes have no preference as to which event they input along a channel. We will therefore arrange that the offer relation just records the fact that the process is willing to input *something* on a channel and says nothing about the values passed.

Another problem arises from processes such as

$$(c!1 \xrightarrow{1} P \leftfatsemi c!2 \xrightarrow{1} P') \ {}^{X}\!\leftfatsemi^{Y} \ (c?x \xrightarrow{1} Q(x) \leftfatsemi c?y \xrightarrow{1} Q'(y))$$

Here it is impossible to tell which process on the right inputs the 1 and which inputs the 2. To overcome this we shall insist that no process tries to write two things onto the same channel simultaneously or tries to read two things from the same channel simultaneously. This seems a reasonable assumption when one considers the physical nature of the channels — no wire can pass two messages simultaneously. It will be a requirement of anyone writing a process definition in PBTCSP to check that this condition is satisfied. Fortunately the following lemma simplifies this.

Lemma 6.1: If a process P is such that

- no interleaving within the definition of the process has both sides able to write to the same channel, or has both sides able to read from the same channel; *and*
- all renaming within the process definition is one–one on channel names

then P does not try to write two things to any channel simultaneously or try to read two things from a channel simultaneously. ♡

One further problem arises from hiding of input channels. In normal timed CSP we have the identity

$$(c?x : X \xrightarrow{1} P_x) \setminus c = \bigsqcap_{x \in X} P_x$$

However, we are unable to assign probabilities to the nondeterministic choice on the right. We overcome this problem by not allowing input channels to be hidden; again this seems a reasonable assumption.

We let $CHAN$ be the set of all channel names. If $type(c)$ is the type of data transmitted over channel c then we insist that

$$\forall\, c : CHAN ; x : type(c) \bullet c.x \in \Sigma$$

i.e. all communications are visible events. We will write $c?x$ to represent the input of value x on channel c.

We define an *action* to be a pair consisting of a bag of events and a set of channels.

Definition 6.2: The set ACT of actions is defined by

$$ACT \mathrel{\widehat=} \operatorname{bag} \Sigma \times \mathbb{P}(CHAN)$$

♢

We will write α, β for typical members of ACT, χ, ψ for typical members of $\mathrm{bag}\,\Sigma$ and ζ, η for typical members of $\mathbb{P}(CHAN)$. The pair (χ, ζ) will represent the performance of the events of χ, and the input of events from the channels of ζ. We can now define the space OFF of offers, which are basically timed actions.

Definition 6.3: The set OFF of offers is defined by

$$OFF \mathrel{\widehat=} TIME \times ACT$$

$\diamondsuit$

We will write v, w for typical members of OFF. The pair $(t, (\chi, \zeta))$ will represent that the process is willing to perform the events of χ and input on the channels of ζ at time t. So, for example, we will write $(3, (\{\!|a, a, b|\!\}, \{c?x, d?y\}))$ to represent the willingness of a process to perform two as and a b, and to input on channels c and d at time 3. For ease of notation, we will often write the elements of a particular action within bag brackets, so we will write the above offer as $(3, \{\!|a, a, b, c?x, d?y|\!\})$, for example.

As in the model without communication, we can now define the space $OFFREL$ of offer relations as being those relations $\sqsubseteq$ of type $OFF \times OFF$ satisfying

1. $(t, \alpha) \sqsubseteq (t', \beta) \Rightarrow t = t'$ (comparable events occur at the same time);
2. $w \sqsubseteq w' \wedge w' \sqsubseteq w'' \Rightarrow w \sqsubseteq w''$ (transitivity);
3. $w \sqsubseteq w' \wedge w' \sqsubseteq w \Rightarrow w = w'$ (antisymmetry);
4. $w \in \mathrm{items}\sqsubseteq \Rightarrow w \sqsubseteq w$ (reflexivity on $\mathrm{items}\sqsubseteq$);
5. $(t, \alpha), (t, \beta) \in \mathrm{items}\sqsubseteq \Rightarrow (t, \alpha) \sqsubseteq (t, \beta) \vee (t, \beta) \sqsubseteq (t, \alpha)$ (totality on $\mathrm{items}\sqsubseteq$)

where $\mathrm{items}\sqsubseteq \mathrel{\widehat=} \{ w \mid \exists\, v \bullet v \sqsubseteq w \vee w \sqsubseteq v \}$.

Similarly, we now define the space TT of timed traces by

$$TT \mathrel{\widehat=} \{ s : TIME \nrightarrow ACT \mid \exists\, \tau \bullet \mathrm{dom}\, s = [0, \tau] \}$$

i.e. functions from times to actions.

Our semantic model for our language remains largely unchanged by this extension — the only change being the new definition of the space $OFFREL$.

We can now define a new operator, prefix choice. The process $c?a : A \xrightarrow{0} P_a$ is willing to input any value a on channel c, and then act like the process P_a, where P_a will, in general, depend on the value a input. In order to fit with our intuitions about causality, we will insist that the processes P_a are unable to perform any events at time 0. A behaviour of $c?a : A \xrightarrow{0} P_a$ will be either:

- a behaviour where nothing is performed, and the process is willing to input on channel c at any time; *or*
- a behaviour where an element $\hat{a}$ of A is input at time t, and the process then acts like $P_{\hat{a}}$.

This gives the following definition.

$$\begin{aligned}
&\mathcal{A}_{BT}[\![c?a : A \xrightarrow{0} P_a]\!]\rho \mathrel{\widehat=} \\
&\quad \{(\tau, [0, \tau] \otimes \langle \{\!|c?a|\!\}, \{\!|\!\}\rangle, \prec\!\succ) \mid \tau \in TIME\} \\
&\quad \cup \{(\tau, [0, t] \otimes \langle \{\!|c?a|\!\}, \{\!|\!\}\rangle^\frown \sqsubseteq + t, (t, c?\hat{a})^\frown s + t) \mid \\
&\qquad \hat{a} \in A \wedge t \leqslant \tau \wedge (\tau - t, \{0\} \otimes \langle \{\!|\!\}\rangle^\frown \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P_{\hat{a}}]\!]\rho\}
\end{aligned}$$

For the probability function, behaviours of the first sort occur with probability one (if the environment is unwilling to communicate on c), and the probability of a behaviour of the second sort is the probability of $P_{\hat{a}}$ performing the corresponding behaviour starting at time t (if the first communication the environment is willing to make on channel c is $\hat{a}$ at time t).

$$\begin{aligned}&\mathcal{P}_{BT}[\![c?a : A \xrightarrow{0} P_a]\!]\rho \mathrel{\widehat{=}} \\ &\quad \text{fillout}(\{(\tau, [0,\tau] \otimes \langle\{\!|c?a|\!\}, \{\!|\}\!\rangle, \prec\!\succ) \mapsto 1 \mid \tau \in TIME\} \\ &\qquad \cup \{(\tau, [0,t] \otimes \langle\{\!|c?a|\!\}, \{\!|\}\!\rangle^\frown \sqsubseteq + t, (t, c?\hat{a})^\frown s + t) \mapsto \mathcal{P}_{BT}[\![P_{\hat{a}}]\!]\rho(\tau - t, \{0\} \otimes \langle\{\!|\}\!\rangle^\frown \sqsubseteq, s) \mid \\ &\qquad\quad \hat{a} \in A \wedge t \leqslant \tau\})\end{aligned}$$

These definitions are well defined (i.e. they satisfy the healthiness conditions of the semantic space) if the set of processes is uniformly bounded in the sense of section 5.

Lemma 6.4: The definition of $c?a : A \xrightarrow{0} P_a$ satisfies the axioms of the semantic space if the set $\{P_a \mid a \in A\}$ is uniformly bounded. ♡

We can now define the general prefix choice operator. The process $c?a : A \xrightarrow{t_a} P_a$ inputs a value a on channel c, and then acts like process P_a after a delay of length t_a, where t_a may depend on the value a input. We can define this process by

$$c?a : A \xrightarrow{t_a} P_a \mathrel{\widehat{=}} c?a : A \xrightarrow{0} (WAIT\ t_a\ ; P_a)$$

7 Recursive processes

In this section we describe new ways of defining recursive processes. Our treatment of environments, as described in section 3, will allow us to give a neater semantic definition of the recursive process $\mu X \bullet P$. It will also allow us to define an immediate recursion operator, which we write as $\mu X \circ P$; unlike the delayed recursion, this process has no delay associated with recursive calls to X. We will then go on to define mutual recursion; $\langle X_i = P_i \rangle_j$ represents the jth component of the vector of mutually recursive processes X_i with values P_i.

$$P ::= \ldots \mid \mu X \circ P \mid \langle X_i = P_i \rangle_j$$

Our method follows that of Davies and Schneider [DS90].

7.1 Constructive processes

The semantics of a PBTCSP term P is a function of the free variables appearing in the definition of P. If P is the body of a recursive process, then the recursion is well defined if P corresponds to a contraction mapping in PTM_B. For this to be true, it is sufficient for P to be *constructive* for the bound variable.

Definition 7.1: Term P is *t-constructive* for X iff

$$\forall\, t_0 : T\ ; \rho : ENV \bullet \mathcal{F}_{BT}[\![P]\!]\rho \upharpoonright t_0 + t = \mathcal{F}_{BT}[\![P]\!]\rho[\rho[\![X]\!] \upharpoonright t_0 / X] \upharpoonright t_0 + t$$

♢

P is t-constructive for X if its behaviour up until time $t_0 + t$ is independent of the value of X after t_0.

Definition 7.2: Term P is *constructive* for X iff there is some strictly positive t such that P is t-constructive for X. ♢

From the semantic equations for the PBTCSP operators we can derive a number of results about constructive terms.

Lemma 7.3: For any X and t,

1. $STOP$, $SKIP$ and $WAIT\ t'$ are t-constructive for X;
2. X is 0-constructive for X, and t-constructive for $Y \neq X$;
3. $\mu\, X \bullet P$ is t-constructive for X

♡

Lemma 7.4: If P is t-constructive for X then

1. $a \xrightarrow{t'} P$ and $WAIT\ t' ; P$ are $t + t'$-constructive for X;
2. $\mu\ Y \bullet P$, $P \setminus A$ and $f(P)$ are t-constructive for X;
3. P is t'-constructive for X, for any $t' < t$.

♡

Lemma 7.5: If P is t_1-constructive for X and Q is t_2-constructive for Y then

1. $P \,\Box\, Q$, $P \,\Box\, Q$, $P \sqcap_q Q$, $P \nleftarrow Q$, $P \nrightarrow Q$, $P \overset{A}{\nleftarrow}\overset{B}{} Q$, $P \overset{A}{\nrightarrow}\overset{B}{} Q$, $P \nleftarrow\!\!\!\!\!\!\rightarrow Q$ and $P \rightarrow\!\!\!\!\!\!\leftarrow Q$ are all $t_1 \sqcap t_2$-constructive for X;
2. $P \mathbin{;} Q$ is $t_1 \sqcap t_2 + \delta$-constructive for X.

♡

7.2 Recursive processes

The semantics of a term P with free variable X may be thought of as a function of the semantics of X; it is the function that associates with each member Y of the semantic space PTP_B, the value of P evaluated in an environment where X is bound to Y. We represent this function by $M(X, P)\rho$:

Definition 7.6: If P is a term, X is a variable, and Y is a member of PTP_B, then

$$M(X, P)\rho \mathrel{\widehat{=}} \lambda\ Y \bullet \mathcal{F}_{BT}[\![P]\!]\rho[Y/X]$$

♢

Note that the environment ρ supplies the bindings for any variables other than X. We use this mapping to give a semantics to the immediate recursion operator:

$$\mathcal{F}_{BT}[\![\mu\, X \circ P]\!]\rho \mathrel{\widehat{=}} \text{the unique fixed point of the mapping } M(X, P)\rho$$

To give a semantics to the delayed recursion operator, we consider the composition of this function with the function W_δ which delays its argument by δ.

Definition 7.7: If P is a term, X is a variable, and Y is a member of PTP_B, then

$$M_\delta(X, P)\rho \mathrel{\widehat{=}} M(X, P)\rho \circ W_\delta$$

where

$$W_\delta \mathrel{\widehat{=}} \lambda\ Y \bullet \mathcal{F}_{BT}[\![WAIT\ \delta\ ; X]\!]\rho[Y/X]$$

♢

The W_δ reflects the delay associated with this recursion. We may now give semantics for the delayed recursion operator:

$$\mathcal{F}_{BT}[\![\, \mu X \bullet P]\!]\rho \mathrel{\widehat{=}} \text{the unique fixed point of the mapping } M_\delta(X, P)\rho$$

It can be shown, as in [Low92], that the delayed recursion operator always has a well defined semantics. This is not true for the immediate recursion operator; we show that if P is constructive for X then $\mu X \circ P$ has a well defined semantics.

Lemma 7.8: If term P is constructive for variable X, then $M(X, P)\rho$ is a contraction mapping on the semantic space PTP_B. ♡

Proof: Let $F \mathrel{\widehat{=}} M(X, P)\rho$; F is a contraction mapping iff

$$\exists\, r < 1 \bullet \forall\, S, T : PTP_B \bullet d(F(S), F(T)) \leqslant r.d(S, T)$$

where d is the metric defined by

$$d(S, T) \mathrel{\widehat{=}} \inf\{2^{-t} \mid S \restriction t = T \restriction t\}$$

Pick S and T in PTP_B. If $S = T$ then both sides of the above equation are zero. Otherwise, let $d(S, T) = 2^{-t}$. Now F is constructive, so there is a strictly positive t' such that

$$S \restriction t = T \restriction t \Rightarrow F(S) \restriction t + t' = F(T) \restriction t + t'$$

so

$$d(F(S), F(T)) \leqslant 2^{-(t+t')} = 2^{-t'}.d(S, T)$$

Hence F is a contraction mapping because $2^{-t'} < 1$. □

We have shown that the mapping corresponding to a constructive term is a contraction mapping on PTP_B. To show that it has a unique fixed point, we require the following result from [Sut75]:

Theorem 7.9 (Banach's Fixed Point Theorem) If (M, d) is a complete metric space and the mapping $F : M \to M$ is a contraction mapping, then F has a unique fixed point $\mathit{fix}(F)$. Furthermore, for all $S \in M$ we have $\mathit{fix}(F) = \lim_{n\to\infty} F^n(S)$. ♡

Corollary 7.10: If $F : PTP_B \to PTP_B$ is a contraction mapping that maps PTM_B into itself, then F has a unique fixed point, which lies in PTM_B. ♡

Proof: This follows immediately from the previous result, because PTM_B is a complete metric space, and a contraction mapping on PTP_B is a contraction mapping on PTM_B. □

Lemmas 7.8 and 7.10 can be combined to give the following result:

Theorem 7.11: If term P is constructive for variable X, then the semantics for $\mathcal{F}_{BT}[\![\, \mu X \circ P]\!]\rho$ is well defined in all environments ρ. ♡

The semantic definition gives rise to the following equivalence:

Theorem 7.12: $\mu X \circ F(X) = F(\mu X \circ F(X))$ ♡

This result justifies the use of recursive equations, such as $X \mathrel{\widehat{=}} a \xrightarrow{1} X$, as process definitions.

7.3 Mutual recursion

We can define a PBTCSP term in terms of a vector of mutually recursive equations; $\langle X_i = P_i \mid i \in I\rangle_j$ represents the jth component of the vector of terms defined by the set of equations $\{X_i = P_i\}$. We shall write $\underline{P}$ for the vector $\langle P_i \mid i \in I\rangle$, etc.

In order to give semantics to $\langle X_i = P_i \mid i \in I\rangle_j$ we consider the semantic domain PTP_B^I, i.e. the product space with one copy of PTP_B for each element of I. We define a metric on this space by

$$\underline{d}(\underline{U}, \underline{V}) \mathrel{\widehat{=}} \sup\{ d(U_i, V_i) \mid i \in I\}$$

We define the semantic mapping relating to a vector of terms:

Definition 7.13: If $\underline{P}$ is a vector of PBTCSP terms, X is a vector of variables, and Y a vector of members of PTP_B, all indexed by set I, then the mapping on PTP_B^I corresponding to $\underline{X}$ and $\underline{P}$ is given by

$$M(\underline{X}, \underline{P})\rho \mathrel{\widehat{=}} \lambda\, \underline{Y} \bullet \mathcal{F}_{BT}[\![\underline{P}]\!]\rho[Y_i/X_i \mid i \in I]$$

$\diamondsuit$

We can now give a semantics to mutual recursion.

Definition 7.14: If $\underline{P}$ is a vector of PBTCSP terms, then

$$\mathcal{F}_{BT}[\![\langle X_i = P_i \mid i \in I\rangle_j]\!]\rho \mathrel{\widehat{=}} S_j \text{ where } \underline{S} \text{ is a fixed point of } M(\underline{X}, \underline{P})\rho$$

$\diamondsuit$

This is well-defined when all fixed points of $M(\underline{X}, \underline{P})\rho$ agree on the jth component. In the rest of this section, we study under what circumstances $M(\underline{X}, \underline{P})\rho$ has a *unique* fixed point. We define what it means for a vector of terms $\underline{P}$ to be *constructive* for a vector of variables $\underline{X}$, and then show that this is a sufficient condition for $M(\underline{X}, \underline{P})\rho$ to have a unique fixed point.

Constructive vectors

We will need some definitions relating to partial orders.

Definition 7.15: A partial order $\prec$ on a set S is a well-ordering if there is no infinite descending chain $\langle s_i \mid i \in \mathbb{N}\rangle$ such that $s_{i+1} \prec s_i$, for all $i \in \mathbb{N}$. The initial segment of s in $(S, \prec)$ is the set of elements less than s; i.e. $\text{seg}(s) \mathrel{\widehat{=}} \{s' : S \mid s' \prec s\}$. $\diamondsuit$

We can now define a constructive vector of terms.

Definition 7.16: A vector of terms $\langle P_i \mid i \in I\rangle$ is *t-constructive* for vector of variables $\langle X_i \mid i \in I\rangle$ if there is a well-ordering $\prec$ on I such that

$$\forall\, i, j : I \bullet j \notin seg(i) \Rightarrow P_i \ t\text{-constructive for } X_j$$

$\diamondsuit$

Definition 7.17: A vector of terms $\underline{P}$ is *constructive* for vector of variables $\underline{X}$ if there is a strictly positive t such that $\underline{P}$ is t-constructive for $\underline{X}$. $\diamondsuit$

If $\underline{P}$ is constructive for $\underline{X}$ then all unguarded recursive calls from term P_i are to a variable X_j such that $j \prec i$. Any sequence of unguarded recursive calls must correspond to a decreasing sequence of I, and so must be finite.

It will normally be possible to show that *all* terms P_i are constructive for *all* variables X_j.

Definition 7.18: A vector of terms $\langle P_i \mid i \in I \rangle$ is *uniformly t-constructive* for the vector of variables $\langle X_i \mid i \in I \rangle$ if P_i is t-constructive for X_j, for all i and j in I. ♢

Definition 7.19: A vector of terms $\underline{P}$ is *uniformly constructive* for vector of variables $\underline{X}$ if there is some strictly positive t such that $\underline{P}$ is uniformly t-constructive for $\underline{X}$. ♢

Lemma 7.20: If $\underline{P}$ is uniformly constructive for $\underline{X}$, then $\underline{P}$ is constructive for $\underline{X}$. ♡

Well-definedness of mutual recursion

In [DS90], Davies and Schneider show that in the timed-failures model if a vector of terms $\underline{P}$ is constructive for vector of variables $\underline{X}$, then the mapping $M(\underline{X}, \underline{P})\rho$ has a unique fixed point in TS_F^I. We prove a similar result for the probabilistic model; unfortunately our proof is complicated by the fact that our semantic model consists of pairs (of sets of behaviours and probability functions) rather that just sets of behaviours. We sketch those parts of the proof that closely follow Davies and Schneider, and concentrate on those parts that differ significantly.

The presence of a behaviour $(\tau, \sqsubseteq, s)$ in the semantic set of a term P depends only upon the values the environment assigns to a finite set of variables. The probability of behaviour $(\tau, \sqsubseteq, s)$ depends, to any given degree of accuracy, only on the values the environment assigns to a finite set of variables.

Theorem 7.21 (Finite Dependency Theorem) If P is a PBTCSP term, possibly containing free occurrences of process variables from the set $\{X_i \mid i \in I\}$, and ρ is an environment, then

$$(\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho \Rightarrow \exists\, N : \mathbb{F}\, I \bullet \forall\, \rho' : ENV \bullet \\ (\forall\, i : N \bullet \rho[\![X_i]\!] = \rho'[\![X_i]\!]) \Rightarrow (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho'$$

and

$$\forall (\tau, \sqsubseteq, s) : BEH \bullet \forall\, \varepsilon > 0 \bullet \exists\, N : \mathbb{F}\, I \bullet \forall\, \rho' : ENV \bullet \\ (\forall\, i \in N \bullet \rho[\![X_i]\!] = \rho'[\![X_i]\!]) \Rightarrow \left| \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq, s) - \mathcal{P}_{BT}[\![P]\!]\rho'(\tau, \sqsubseteq, s) \right| < \varepsilon$$

♡

The proof of this theorem is given in appendix A. In [DS90] a result corresponding to the first equation is proved. We would have liked the corresponding result for the probability functions to have been of the form

$$\forall (\tau, \sqsubseteq, s) : BEH \bullet \exists\, N : \mathbb{F}\, I \bullet \forall\, \rho' : ENV \bullet \\ (\forall\, i \in N \bullet \rho[\![X_i]\!] = \rho'[\![X_i]\!]) \Rightarrow \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq, s) = \mathcal{P}_{BT}[\![P]\!]\rho'(\tau, \sqsubseteq, s)$$

To see why this is not true, consider the process $P \mathrel{\widehat{=}} \prod_{i \in \mathbb{N}} [p_i] X_i$ (with p_i non-zero, for all i). The probability of any given behaviour $(\tau, \sqsubseteq, s)$ occurring obviously depends on the values the environment assigns to *all* the X_i, so the above equation does not hold. However, if two environments agree on sufficiently many of the X_i, then the probabilities in the different environments will agree to within ε; it is enough for the environments to agree on all X_i for i in some set N such that $\sum_{i \in N} p_i > 1 - \varepsilon$.

We may restate this theorem in a more applicable form:

Corollary 7.22: If P is a PBTCSP term, possibly containing free occurrences of process variables from the set $\{X_i \mid i \in I\}$, and ρ is an environment, then

$$(\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho \Rightarrow \exists\, N : \mathbb{F}\, I \bullet \forall\, \rho' : ENV \bullet (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho'[\rho[\![X_i]\!]/X_i \mid i \in N]$$

and

$$\begin{aligned}\forall (\tau, \sqsubseteq, s) : BEH \bullet \forall\, \varepsilon > 0 \bullet \exists\, N : \mathbb{F}\, I \bullet \forall\, \rho' : ENV \bullet \\ \left|\mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq, s) - \mathcal{P}_{BT}[\![P]\!]\rho'[\rho[\![X_i]\!]/X_i \mid i \in N](\tau, \sqsubseteq, s)\right| < \varepsilon\end{aligned}$$

♡

We say that a term P is t-constructive for a vector of variables $\underline{X}$ if the semantics up until time $t_0 + t$ is independent of every component of $\underline{X}$ after time t_0.

Definition 7.23: If P is a PBTCSP term then P is *t-constructive* for $\langle X_i \mid i \in I\rangle$ if

$$\forall\, t_0 : TIME\ ; \rho : ENV \bullet \mathcal{F}_{BT}[\![P]\!]\rho \upharpoonright t_0 + t = \mathcal{F}_{BT}[\![P]\!]\rho\,[\rho[\![X_i]\!] \upharpoonright t_0/X_i \mid i \in I] \upharpoonright t_0 + t$$

♢

Using this definition, we will show that if a term P is t-constructive for each variable X_i for $i \in I$, then it is constructive for the vector $\langle X_i \mid i \in I\rangle$. The case where the indexing set is finite is easily established by induction on the length of the vector.

Lemma 7.24: If N is a finite set, and term P is t-constructive for each variable X_i for $i \in N$, then P is t-constructive for vector $\langle X_i \mid i \in N\rangle$. ♡

We prove the general case using the Finite Dependency Theorem.

Theorem 7.25: If term P is t-constructive for each variable X_i for $i \in I$, then P is t-constructive for vector $\langle X_i \mid i \in I\rangle$. ♡

Proof: To show that P is t-constructive for $\langle X_i \mid i \in I\rangle$, we show

- $\mathcal{A}_{BT}[\![P]\!]\rho \upharpoonright t_0 + t = \mathcal{A}_{BT}[\![P]\!]\rho[\rho[\![X_i]\!] \upharpoonright t_0/X_i \mid i \in I] \upharpoonright t_0 + t$
- $\mathcal{P}_{BT}[\![P]\!]\rho \upharpoonright t_0 + t = \mathcal{P}_{BT}[\![P]\!]\rho[\rho[\![X_i]\!] \upharpoonright t_0/X_i \mid i \in I] \upharpoonright t_0 + t$

For the first equation, let $(\tau, \sqsubseteq, s)$ be a member of $\mathcal{A}_{BT}[\![P]\!]\rho \upharpoonright t_0 + t$. Then we have from the corollary to the Finite Dependency Theorem that there exists some finite set of indices N such that

$$(\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho'[\rho[\![X_i]\!]/X_i \mid i \in N] \upharpoonright t_0 + t$$

for any environment ρ'. If we define ρ' by $\rho' \mathrel{\widehat{=}} \rho[\rho[\![X_i]\!] \upharpoonright t_0/X_i \mid i \in I \setminus N]$ then we have

$$\begin{aligned} & \mathcal{A}_{BT}[\![P]\!]\rho'[\rho[\![X_i]\!]/X_i \mid i \in N] \upharpoonright t_0 + t \\ = & \left\langle \text{previous lemma} \right\rangle \\ & \mathcal{A}_{BT}[\![P]\!]\rho'[\rho[\![X_i]\!] \upharpoonright t_0/X_i \mid i \in N] \upharpoonright t_0 + t \\ = & \left\langle \text{definition of } \rho' \right\rangle \\ & \mathcal{A}_{BT}[\![P]\!]\rho[\rho[\![X_i]\!] \upharpoonright t_0/X_i \mid i \in I] \upharpoonright t_0 + t \end{aligned}$$

so $(\tau, \sqsubseteq, s)$ is a member of $\mathcal{A}_{BT}[\![P]\!]\rho[\rho[\![X_i]\!] \upharpoonright t_0/X_i \mid i \in I] \upharpoonright t_0 + t$, as required.

For the probability function, pick $(\tau, \sqsubseteq, s) \in BEH$ and let $\tau \leqslant t_0 + t$. We must show

$$\mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq, s) = \mathcal{P}_{BT}[\![P]\!]\rho[\rho[\![X_i]\!] \upharpoonright t_0/X_i \mid i \in I](\tau, \sqsubseteq, s)$$

Pick $\varepsilon > 0$. Then by the corollary to the Finite Dependency Theorem, there exists some finite set of indices N such that

$$\left|\mathcal{P}_{BT}[\![P]\!]\rho(\tau,\sqsubseteq,s) - \mathcal{P}_{BT}[\![P]\!]\rho'[\rho[\![X_i]\!]/X_i \mid i \in N](\tau,\sqsubseteq,s)\right| < \varepsilon$$

for any environment ρ'. If we define ρ', as before, by $\rho' \mathrel{\widehat{=}} \rho[\rho[\![X_i]\!] \restriction t_0/X_i \mid i \in I \setminus N]$ then we have

$$\begin{array}{ll} & \mathcal{P}_{BT}[\![P]\!]\rho'[\rho[\![X_i]\!]/X_i \mid i \in N](\tau,\sqsubseteq,s) \\ = & \left\langle \text{previous lemma} \right\rangle \\ & \mathcal{P}_{BT}[\![P]\!]\rho'[\rho[\![X_i]\!] \restriction t_0/X_i \mid i \in N](\tau,\sqsubseteq,s) \\ = & \left\langle \text{definition of } \rho' \right\rangle \\ & \mathcal{P}_{BT}[\![P]\!]\rho[\rho[\![X_i]\!] \restriction t_0/X_i \mid i \in I](\tau,\sqsubseteq,s) \end{array}$$

So we have shown that for all $\varepsilon > 0$,

$$\left|\mathcal{P}_{BT}[\![P]\!]\rho(\tau,\sqsubseteq,s) - \mathcal{P}_{BT}[\![P]\!]\rho[\rho[\![X_i]\!] \restriction t_0/X_i \mid i \in I](\tau,\sqsubseteq,s)\right| < \varepsilon$$

Hence we have that

$$\mathcal{P}_{BT}[\![P]\!]\rho(\tau,\sqsubseteq,s) = \mathcal{P}_{BT}[\![P]\!]\rho[\rho[\![X_i]\!] \restriction t_0/X_i \mid i \in I](\tau,\sqsubseteq,s)$$

as required. □

We are now ready to prove that the mapping corresponding to a mutual recursion has a unique fixed point if the vector of terms is constructive for the vector of variables.

Theorem 7.26: If vector of terms $\underline{P}$ is constructive for vector of variables $\underline{X}$, then the mapping $M(\underline{X},\underline{P})\rho$ has a unique fixed point in PTP_B^I. ♡

Proof: The proof proceeds as follows: we define a second vector of terms $\underline{Q}$; we show that this vector is well defined; we show that the corresponding mapping $M(\underline{X},\underline{Q})\rho$ is a contraction mapping and so has a unique fixed point; we show that this fixed point is also a fixed point of $M(\underline{X},\underline{P})\rho$; we show that this fixed point is unique.

The vector of terms $\underline{Q}$ is defined by

$$Q_i \mathrel{\widehat{=}} P_i[Q_j/X_j \mid j \in seg(i)]$$

We proceed via a series of lemmas.

Lemma A: The vector $\underline{Q}$ is well defined. ♡

Proof of lemma: This can be proved by transfinite recursion, precisely as in [DS90]. □

Lemma B: The mapping $M(\underline{X},\underline{Q})\rho$ is a contraction mapping on PTP_B^I and so has a unique fixed point. ♡

Proof of lemma: By analogy with lemma 7.8, to show that $M(\underline{X},\underline{Q})\rho$ is a contraction mapping we assume that $S \restriction t_0 = T \restriction T_0$ and prove that

$$M(\underline{X},\underline{Q})\rho S \restriction t_0 + t = M(\underline{X},\underline{Q})\rho T \restriction t_0 + t$$

which is equivalent to

$$\mathcal{F}_{BT}[\![Q]\!]\rho[\underline{S}/\underline{X}] \restriction t_0 + t = \mathcal{F}_{BT}[\![Q]\!]\rho[\underline{T}/\underline{X}] \restriction t_0 + t$$

We employ the following result from [End77]:

Transfinite Induction Principle: If $\prec$ is a well-ordering on set I, and J is a subset of I with the property

$$\forall\, i : I \bullet seg(i) \subseteq J \Rightarrow i \in J$$

then J coincides with I. $\Diamond$

We define J by

$$J \mathrel{\widehat{=}} \{i : I \mid \mathcal{F}_{BT}[\![Q_i]\!]\rho[\underline{S}/\underline{X}] \restriction t_0 + t = \mathcal{F}_{BT}[\![Q_i]\!]\rho[\underline{T}/\underline{X}] \restriction t_0 + t\}$$

and assume for induction that $seg(i) \subseteq J$; we must show $i \in J$, i.e.

$$\mathcal{F}_{BT}[\![Q_i]\!]\rho_S \restriction t_0 + t = \mathcal{F}_{BT}[\![Q_i]\!]\rho_T \restriction t_0 + t$$

where

$$\rho_S \mathrel{\widehat{=}} \rho[\underline{S}/\underline{X}] \qquad \rho_T \mathrel{\widehat{=}} \rho[\underline{T}/\underline{X}]$$

From the definition of $\underline{Q}$, we obtain

$$\mathcal{F}_{BT}[\![Q_i]\!]\rho_S = \mathcal{F}_{BT}[\![P_i]\!]\rho_S[\mathcal{F}_{BT}[\![Q_j]\!]\rho_S/X_j \mid j \in seg(i)]$$

and similarly for ρ_T. Letting

$$\rho'_S \mathrel{\widehat{=}} \rho_S[\mathcal{F}_{BT}[\![Q_j]\!]\rho_S/X_j \mid j \in seg(i)] \qquad \rho'_T \mathrel{\widehat{=}} \rho_T[\mathcal{F}_{BT}[\![Q_j]\!]\rho_T/X_j \mid j \in seg(i)]$$

we reduce the proof obligation to

$$\mathcal{F}_{BT}[\![P_i]\!]\rho'_S \restriction t_0 + t = \mathcal{F}_{BT}[\![P_i]\!]\rho'_T \restriction t_0 + t$$

Now, we have

$$\begin{array}{ll} & \mathcal{F}_{BT}[\![P_i]\!]\rho'_S \restriction t_0 + t \\ = & \left\langle \text{expanding} \right\rangle \\ & \mathcal{F}_{BT}[\![P_i]\!]\,(\rho'_T\,[\rho'_S[\![X_k]\!]/X_k \mid k \in seg(i)]\,[\rho'_S[\![X_k]\!]/X_k \mid k \in I \setminus seg(i)]) \restriction t_0 + t \\ = & \left\langle \begin{array}{l} P_i \text{ is 0-constructive for } X_k \text{ for } k \in seg(i), \text{ and } t\text{-constructive} \\ \text{for } X_k \text{ for } k \in I \setminus seg(i); \text{ theorem 7.25} \end{array} \right\rangle \\ & \mathcal{F}_{BT}[\![P_i]\!]\,(\rho'_T\,[\rho'_S[\![X_k]\!] \restriction t_0 + t/X_k \mid k \in seg(i)]\,[\rho'_S[\![X_k]\!] \restriction t_0/X_k \mid k \in I \setminus seg(i)]) \restriction t_0 + t \qquad (*) \end{array}$$

For $k \in seg(i)$, we have

$$\begin{array}{ll} & \rho'_S[\![X_k]\!] \restriction t_0 + t \\ = & \left\langle \text{definition} \right\rangle \\ & \mathcal{F}_{BT}[\![Q_k]\!]\rho_S \restriction t_0 + t \\ = & \left\langle \text{inductive hypothesis} \right\rangle \\ & \mathcal{F}_{BT}[\![Q_k]\!]\rho_T \restriction t_0 + t \\ = & \left\langle \text{definition} \right\rangle \\ & \rho'_T[\![X_k]\!] \restriction t_0 + t \end{array}$$

and for $k \in I \setminus seg(i)$, we have $\rho'_S[\![X_k]\!] \restriction t_0 = \rho'_T[\![X_k]\!] \restriction t_0$, since $S \restriction t_0 = T \restriction t_0$, by hypothesis. Hence, $(*)$ equals

$$\begin{array}{ll} & \mathcal{F}_{BT}[\![P_i]\!](\rho'_T[\rho'_T[\![X_k]\!] \restriction t_0 + t/X_k \mid k \in seg(i)][\rho'_T[\![X_k]\!] \restriction t_0/X_k \mid k \in I \setminus seg(i)]) \restriction t_0 + t \\ = & \left\langle \begin{array}{l} P_i \text{ is 0-constructive for } X_k \text{ for } k \in seg(i), \text{ and } t\text{-constructive} \\ \text{for } X_k \text{ for } k \in I \setminus seg(i); \text{ theorem 7.25} \end{array} \right\rangle \\ & \mathcal{F}_{BT}[\![P_i]\!]\rho'_T \restriction t_0 + t \end{array}$$

as required. Hence $M(\underline{X}, \underline{Q})$ is a contraction mapping and so has a unique fixed point. $\square$

Lemma C: The fixed point of $M(\underline{X}, \underline{Q})\rho$ is a fixed point of $M(\underline{X}, \underline{P})\rho$. ♡

Proof of lemma: Let $\underline{S}$ be the unique fixed point of $M(\underline{X}, \underline{Q})\rho$. Then

$$
\begin{array}{ll}
 & (M(\underline{X}, \underline{P})\rho(\underline{S}))_i \\
= & \left\langle \text{definition of } M(\underline{X}, \underline{P})\rho \right\rangle \\
 & (\mathcal{F}_{BT}[\![\underline{P}]\!]\rho[\underline{S}/\underline{X}])_i \\
= & \left\langle \underline{S} \text{ is a fixed point of } M(\underline{X}, \underline{Q})\rho \right\rangle \\
 & \mathcal{F}_{BT}[\![P_i]\!]\rho[(M(\underline{X}, \underline{Q})\rho(\underline{S}))_j/X_j \mid j \in seg(i)][S_j/X_j \mid j \notin seg(i)] \\
= & \left\langle \text{definition of } M(\underline{X}, \underline{Q})\rho \right\rangle \\
 & \mathcal{F}_{BT}[\![P_i]\!]\rho[\mathcal{F}_{BT}[\![Q_j]\!]\rho[\underline{S}/\underline{X}]/X_j \mid j \in seg(i)][S_j/X_j \mid j \notin seg(i)] \\
= & \left\langle \text{definition of substitution} \right\rangle \\
 & \mathcal{F}_{BT}[\![P_i[Q_j/X_j \mid j \in seg(i)]]\!]\rho[\underline{S}/\underline{X}] \\
= & \left\langle \text{definition of } Q_i \right\rangle \\
 & \mathcal{F}_{BT}[\![Q_i]\!]\rho[\underline{S}/\underline{X}] \\
= & \left\langle \text{definition of } M(\underline{X}, \underline{Q})\rho \right\rangle \\
 & (M(\underline{X}, \underline{Q})\rho(\underline{S}))_i \\
= & \left\langle \underline{S} \text{ is the fixed point of } M(\underline{X}, \underline{Q})\rho \right\rangle \\
 & S_i
\end{array}
$$

Hence $\underline{S}$ is a fixed point of $M(\underline{X}, \underline{P})\rho$. □

Lemma D: The fixed point of $M(\underline{X}, \underline{Q})\rho$ is the *unique* fixed point of $M(\underline{X}, \underline{P})\rho$. ♡

Proof of lemma: The proof of this is identical to the proof in the timed failures model, given in [DS90]. □

We conclude that $M(X, P)\rho$ has a unique fixed point in PTP_B^I. □

From this theorem, we can deduce the following corollary:

Corollary 7.27: If vector of processes P is constructive for vector of variables X, then the semantics of $\langle X_i = P_i \rangle_j$ is well defined. ♡

8 Transfer operators

In this section we introduce three new operators, which transfer control from one process to another according to different circumstances.

8.1 A timeout operator

In [Sch90], Schneider defines a timeout operator by

$$P \overset{t}{\triangleright} Q \mathrel{\widehat{=}} (P \,\Box\, WAIT\ t\ ;\ trig \longrightarrow Q) \setminus trig$$

where *trig* is an event not in the alphabets of P or Q. This begins by acting like P; if no visible event has occurred by time t then the process times out by performing the event *trig* silently, and after a delay of length δ acts like Q. If P is able to perform its first visible event precisely at time t, then it is nondeterministic whether or not the timeout occurs.

We will define our timeout operator by refining the external choice in the process definition to either a left- or right-biased choice. We consider the effects of these two different choices.

- If we choose a left-biased choice, then if the process P is willing to do its first event precisely at time t, then that event is offered stronger than the silent *trig*, and so will occur if the environment is willing to perform it.
- If we choose a right-biased choice, then if the process P is willing to do its first event precisely at time t, then that event is offered weaker than the silent *trig*, and so will occur only if the environment is *not* willing to idle.

The first choice produces the simpler operator, and seems to be more useful in practice: the process $P \stackrel{t}{\rhd} Q$ will be willing to act like P at all times up to *and including* time t. We therefore have the following definition:

Definition 8.1 (Timeout) The process $P \stackrel{t}{\rhd} Q$ is defined by

$$P \stackrel{t}{\rhd} Q \mathrel{\widehat{=}} (P \,\Box\, WAIT\ t\,;\, trig \longrightarrow Q) \setminus trig$$

where *trig* is an event not in the alphabets of P and Q. ♢

We can use this definition to give semantic equations for the timeout operator.

$$\begin{aligned}\mathcal{A}_{BT}[\![P \stackrel{t}{\rhd} Q]\!]\rho = {} & \{(\tau, \sqsubseteq, s) \mid (\tau \leqslant t \vee \mathit{begin}\, s \leqslant t) \wedge (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho\} \\ & \cup \{(\tau, \sqsubseteq^\frown (t, \tau] \otimes \langle \{|\!|\} \rangle, \prec\!\!\succ) \mid t < \tau < t + \delta \wedge (t, \sqsubseteq, \prec\!\!\succ) \in \mathcal{A}_{BT}[\![P]\!]\rho\} \\ & \cup \{(\tau, \sqsubseteq_P{}^\frown (t, t+\delta) \otimes \langle \{|\!|\} \rangle^\frown \sqsubseteq_Q + t + \delta, \prec\!\!\succ^\frown s_Q + t + \delta) \mid \\ & \qquad \tau \geqslant t + \delta \wedge (t, \sqsubseteq_P, \prec\!\!\succ) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau - t - \delta, \sqsubseteq_Q, s_Q) \in \mathcal{A}_{BT}[\![Q]\!]\rho\}\end{aligned}$$

A behaviour of $P \stackrel{t}{\rhd} Q$ can either be:

- a behaviour of P that either ends before time t or where a visible event has occurred by time t;
- a behaviour of P in which no visible events are observed up until time t, followed by a short period during which control is being transferred to Q;
- a behaviour of P in which no visible events are observed up until time t, followed by a behaviour of Q.

For the probability function, writing f_P for $\mathcal{P}_{BT}[\![P]\!]\rho$ and f_Q for $\mathcal{P}_{BT}[\![Q]\!]\rho$, we have

$$\mathcal{P}_{BT}[\![P \stackrel{t}{\rhd} Q]\!]\rho(\tau, \sqsubseteq, s) = \begin{cases} f_P(\tau, \sqsubseteq, s) & \text{if } \tau \leqslant t \vee \mathit{begin}\, s \leqslant t \\ f_P(t, \sqsubseteq \restriction t, \prec\!\!\succ) & \text{if } t < \tau < t + \delta \wedge s = \prec\!\!\succ \wedge \sqsubseteq \setminus t = (t, \tau] \otimes \langle \{|\!|\} \rangle \\ f_P(t, \sqsubseteq \restriction t, \prec\!\!\succ).f_Q((\tau, \sqsubseteq, s) - t - \delta) & \text{if } \tau \geqslant t + \delta \wedge s \restriction t + \delta = \{|\!|\} \\ & \quad \wedge \sqsubseteq \uparrow (t, t+\delta) = (t, t+\delta) \otimes \langle \{|\!|\} \rangle \\ 0 & \text{otherwise} \end{cases}$$

These definitions satisfy the healthiness conditions on our semantic space because the timeout operator is defined in terms of other operators that themselves satisfy the healthiness conditions.

8.2 Timed transfer

The process $P \underset{t}{\lightning} Q$ acts like P up until time t, at which time control is passed to Q (after a short delay) regardless of the progress made by P. This differs slightly from the definition given in [Sch90], where control was not transferred to Q if P terminated normally before time t. The set of possible behaviours of this process is as follows.

$$\begin{aligned}\mathcal{A}_{BT}[\![P \underset{t}{\lightning} Q]\!]\rho \mathrel{\widehat{=}} & \{(\tau, \sqsubseteq, s) \mid \tau \leqslant t \wedge (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho\} \\ & \cup \{(\tau, \sqsubseteq^\frown (t, \tau] \otimes \langle \{\!|\}\!\rangle, s^\frown \prec\!\!\succ) \mid t < \tau < t + \delta \wedge (t, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho\} \\ & \cup \{(\tau, \sqsubseteq_P^\frown (t, t+\delta) \otimes \langle \{\!|\}\!\rangle^\frown \sqsubseteq_Q + t + \delta, s_P^\frown \prec\!\!\succ^\frown s_Q + t + \delta) \mid \\ & \qquad \tau \geqslant t + \delta \wedge (t, \sqsubseteq_P, s_P) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau - t - \delta, \sqsubseteq_Q, s_Q) \in \mathcal{A}_{BT}[\![Q]\!]\rho\}\end{aligned}$$

A behaviour of $P \underset{t}{\lightning} Q$ can be either:

- a behaviour of P which ends no later than t; *or*
- a behaviour of P up to time t, followed by a short delay during which control is being transfered to Q; *or*
- a behaviour of P up to time t, followed by a short delay, followed by a behaviour of Q starting at time $t + \delta$.

For the probability function, writing f_P for $\mathcal{P}_{BT}[\![P]\!]\rho$ and f_Q for $\mathcal{P}_{BT}[\![Q]\!]\rho$, we have

$$\mathcal{P}_{BT}[\![P \underset{t}{\lightning} Q]\!]\rho(\tau, \sqsubseteq, s) \mathrel{\widehat{=}} \begin{cases} f_P(\tau, \sqsubseteq, s) & \text{if } \tau \leqslant t \\ f_P((\tau, \sqsubseteq, s) \restriction t) & \text{if } t < \tau < t + \delta \wedge \sqsubseteq \natural t = (t, \tau] \otimes \langle \{\!|\}\!\rangle \wedge s \natural t = \prec\!\!\succ \\ f_P((\tau, \sqsubseteq, s) \restriction t).f_Q((\tau, \sqsubseteq, s) - t - \delta) & \text{if } \tau \geqslant t \wedge s \uparrow (t, t+\delta) = \prec\!\!\succ \\ & \quad \wedge \sqsubseteq \uparrow (t, t+\delta) = (t, t+\delta) \otimes \langle \{\!|\}\!\rangle \\ 0 & \text{otherwise} \end{cases}$$

It is easy to show that these definitions satisfy the healthiness conditions on our semantic space.

8.3 Interrupts

The process $P \underset{i}{\triangledown} Q$ initially acts like P except it is always willing to perform the interrupt event i. If an i occurs, control is passed to the interrupt handler Q, after a delay of length δ. We assume that P cannot perform the event i — it cannot interrupt itself.

Before the event i occurs, the process should always offer the event i; it should be willing to perform an i *in addition* to whatever actions P offers. For example, if P has an offer relation $\sqsubseteq$ with $(t, \{\!|b|\!\}) \sqsupset (t, \{\!|a|\!\}) \sqsupset (t, \{\!|\}\!\})$, then $P \underset{i}{\triangledown} Q$ should have an offer relation $\sqsubseteq'$ with

$$(t, \{\!|i, b|\!\}) \sqsupset' (t, \{\!|i, a|\!\}) \sqsupset' (t, \{\!|i|\!\}) \sqsupset' (t, \{\!|b|\!\}) \sqsupset' (t, \{\!|a|\!\}) \sqsupset' (t, \{\!|\}\!\})$$

In general, if P has offer relation $\sqsubseteq$, then before the interrupt occurs $P \underset{i}{\triangledown} Q$ should have offer relation $\sqsubseteq \oplus i$ given by

$$\sqsubseteq \oplus i \mathrel{\widehat{=}} \sqsubseteq \mathrel{-\!\!\!\Vdash} I \sqsubseteq \otimes \langle \{\!|i|\!\}, \{\!|\}\!\} \rangle$$

The set of possible behaviours of $P \underset{i}{\triangledown} Q$ is given by

$$\begin{aligned}
&\mathcal{A}_{BT}[\![P \underset{i}{\triangledown} Q]\!]\rho \mathrel{\widehat{=}} \\
&\quad \{(\tau, \sqsubseteq \oplus i, s) \mid (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge i \notin \Sigma s\} \\
&\quad \cup \{(\tau, \sqsubseteq \oplus i^\frown(t, \tau] \otimes \langle\{\!|\}\!\rangle, s^\frown\prec(t, i)\succ) \mid t \leqslant \tau < t + \delta \wedge i \notin \Sigma s \wedge (t, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho\} \\
&\quad \cup \{(\tau, \sqsubseteq_P \oplus i^\frown(t, t+\delta) \otimes \langle\{\!|\}\!\rangle^\frown\sqsubseteq_Q + t + \delta, s_P{}^\frown\prec(t, i)\succ{}^\frown s_Q + t + \delta) \mid \\
&\qquad \tau \geqslant t + \delta \wedge i \notin \Sigma s_P \wedge (t, \sqsubseteq_P, s_P) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau - t - \delta, \sqsubseteq_Q, s_Q) \in \mathcal{A}_{BT}[\![Q]\!]\rho\}
\end{aligned}$$

A behaviour of $P \underset{i}{\triangledown} Q$ can be either:

- a behaviour of P where an additional i is offered at all times, and no i occurs; *or*
- a behaviour of P where an additional i is offered at all times and an i first occurs at time t, followed by a short pause during which control is being transferred to Q; *or*
- a behaviour of P where an additional i is offered at all times and an i first occurs at time t, followed by a behaviour of Q after a delay of length δ.

For the probability function, writing f_P for $\mathcal{P}_{BT}[\![P]\!]\rho$, and f_Q for $\mathcal{P}_{BT}[\![Q]\!]\rho$, we have

$$\mathcal{P}_{BT}[\![P \underset{i}{\triangledown} Q]\!]\rho(\tau, \sqsubseteq, s) \mathrel{\widehat{=}} \begin{cases}
f_P(\tau, \sqsubseteq_P, s) & \text{if } \sqsubseteq = \sqsubseteq_P \oplus i \wedge i \notin \Sigma s \\
f_P(t, \sqsubseteq_P, s_P) & \text{if } t \leqslant \tau < t \wedge s = s_P{}^\frown\prec(t, i)\succ \wedge i \notin \Sigma s_P \\
 & \quad \wedge \sqsubseteq = \sqsubseteq_P \oplus i^\frown(t, \tau) \otimes \langle\{\!|\}\!\rangle \\
f_P(t, \sqsubseteq_P, s_P).f_Q((\tau, \sqsubseteq, s) - t - \delta) & \text{if } \tau \geqslant t + \delta \wedge s \upharpoonright t + \delta = s_P{}^\frown\prec(t, i)\succ \wedge i \notin \Sigma s_P \\
 & \quad \wedge \sqsubseteq \upharpoonright t + \delta = \sqsubseteq_P \oplus i^\frown(t, t+\delta) \otimes \langle\{\!|\}\!\rangle \\
0 & \text{otherwise}
\end{cases}$$

This is well defined since $\_ \oplus i$ has a unique (partial) inverse.

It is easy to show that the definitions of the interrupt operator satisfy the healthiness conditions on our semantic space.

9 Some more operators

In this section we introduce two new operators: a probabilistic external choice operator, and a hybrid parallel operator that allows processes to act independently on all but a given set of events.

9.1 A probabilistic external choice operator

In this section we introduce a probabilistic choice operator ${}_p\Box_q$ such that $P \;{}_p\Box_q\; Q$ offers an external choice between P and Q that is biased in favour of P with probability p, and biased in favour of Q with probability q. For all processes P and Q, the probabilistic choice $P \;{}_p\Box_q\; Q$ is defined by

$$P \;{}_p\Box_q\; Q \mathrel{\widehat{=}} (P \,\Box\, Q) \;{}_p\sqcap_q\; (P \,\Box\, Q)$$

$P \;{}_p\Box_q\; Q$ acts like $P \,\Box\, Q$ with probability p, and like $P \,\Box\, Q$ with probability q. This process satisfies the axioms of the semantic space because it is defined in terms of simpler operators that preserve the axioms.

This operator is very similar to the probabilistic choice operator defined in most probabilistic models of CCS, for example in [vGSST90]. An external choice between processes P and Q is written $[p]P + [q]Q$

— if the environment can perform events of both P and Q then the choice is made in favour of P with probability p and in favour of Q with probability q. This can then be used to define a probabilistic internal choice between two processes by $[p]\tau.P + [q]\tau.Q$, where τ represents an internal action. Our approach has been the other way round; we have defined biased external choice operators and a probabilistic internal choice operator, and used these to define a probabilistic external choice operator. We believe that it is more natural to define separate internal and external choice operators since these are very different operations. A language with more operators, while being harder to reason *about*, is easier to reason *with*.

9.2 Communicating parallel

The process $P \underset{C}{\lhd\!\parallel\!\rhd} Q$ executes the processes P and Q in parallel, synchronising on events from C, and interleaving on all other events. It can be defined by

$$P \underset{C}{\lhd\!\parallel\!\rhd} Q \mathrel{\widehat{=}} c(l(P) \overset{A}{\lhd\!\parallel\!\rhd} r(Q))$$

where

$$l(a) \mathrel{\widehat{=}} \begin{cases} a & \text{if } a \in C \\ l.a & \text{otherwise} \end{cases} \qquad r(a) \mathrel{\widehat{=}} \begin{cases} a & \text{if } a \in C \\ r.a & \text{otherwise} \end{cases} \qquad \begin{array}{r} c(a) \mathrel{\widehat{=}} a \ \text{ if } a \in C \\ c(l.a) \mathrel{\widehat{=}} a \ \text{ if } a \notin C \\ c(r.a) \mathrel{\widehat{=}} a \ \text{ if } a \notin C \end{array}$$

and

$$A \mathrel{\widehat{=}} l(\Sigma - C) \cup C \qquad B \mathrel{\widehat{=}} r(\Sigma - C) \cup C$$

and we assume $l(\Sigma) \cap C = r(\Sigma) \cap C = \{\}$.

In order to give a semantic definition to this operator, we first consider what offers P and Q perform when $P \underset{C}{\lhd\!\parallel\!\rhd} Q$ performs some offer v. By analogy with the definition of interleaving, we claim that P and Q perform $\underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} v$ and $v - \underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} v$, respectively, where the operator $\underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q}$ is defined by

$$\underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} v \mathrel{\widehat{=}} \sqcup_{\sqsubseteq_P} \{ v_P \subseteq v \mid v_P \lfloor C = v \lfloor C \wedge v - (v_P \setminus C) \in \mathrm{items} \sqsubseteq_Q \}$$
$$\text{if } \exists\, v_P \in \mathrm{items} \sqsubseteq_P ; v_Q \in \mathrm{items} \sqsubseteq_Q \bullet v_P \lfloor C = v_Q \lfloor C = v \lfloor C \wedge v_P \uplus (v_Q \setminus C) = v$$

P performs a subset of v that contains all of $v \lfloor C$, and such that Q can perform the rest of v along with $v \lfloor C$; subject to these conditions, it performs the subset of v that is maximal with respect to its offer relation.

If P and Q have offer relations $\sqsubseteq_P$ and $\sqsubseteq_Q$ then $P \underset{C}{\lhd\!\parallel\!\rhd} Q$ will have offer relation $\sqsubseteq_P \underset{C}{\lhd\!\parallel\!\rhd} \sqsubseteq_Q$, defined by

$$\begin{array}{l} v(\sqsubseteq_P \underset{C}{\lhd\!\parallel\!\rhd} \sqsubseteq_Q) w \Leftrightarrow \exists\, v'_P, w'_P \in \mathrm{items} \sqsubseteq_P ; v'_Q, w'_Q \in \mathrm{items} \sqsubseteq_Q \bullet \\ \qquad v'_P \lfloor C = v'_Q \lfloor C = v \lfloor C \wedge w'_P \lfloor C = w'_Q \lfloor C = w \lfloor C \\ \qquad \wedge\ v'_P \uplus (v'_Q \setminus C) = v \wedge w'_P \uplus (w'_Q \setminus C) = w \\ \qquad \wedge\ (v_P \sqsubset_P w_P \vee v_P = w_P \wedge v_Q \sqsubseteq_Q w_Q) \\ \qquad \text{where } v_P = \underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} v \qquad w_P = \underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} w \qquad v_Q = v - v_P \qquad w_Q = w - w_P \end{array}$$

$\sqsubseteq_P \underset{C}{\lhd\!\parallel\!\rhd} \sqsubseteq_Q$ is the lexicographical ordering on the corresponding projections of its arguments.

We can now give the semantics for the $\underset{C}{\lhd\!\parallel\!\rhd}$ operator.

$$\begin{array}{l} \mathcal{A}_{BT}[\![P \underset{C}{\lhd\!\parallel\!\rhd} Q]\!]\rho \mathrel{\widehat{=}} \\ \quad \{(\tau, \sqsubseteq_P \underset{C}{\lhd\!\parallel\!\rhd} \sqsubseteq_Q, s) \mid (\tau, \sqsubseteq_P, \underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} s) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau, \sqsubseteq_Q, s - \underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} s) \in \mathcal{A}_{BT}[\![Q]\!]\rho\} \end{array}$$

where the $\underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q}$ operator on traces is defined by

$$\underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} s \mathrel{\widehat{=}} \{t \mapsto \underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q}(s \uparrow t) \mid t \in Is\}$$

The probability function is given by

$$\mathcal{P}_{BT}[\![P \underset{C}{\Leftarrow\!\!\!\!\!\!\parallel} Q]\!]\rho(\tau, \sqsubseteq, s) \mathrel{\widehat{=}}$$
$$\sum\left\{\left| \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq_P, \underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} s).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau, \sqsubseteq_Q, s - \underset{C}{\uparrow}_{\sqsubseteq_P,\sqsubseteq_Q} s) \mid \sqsubseteq = \sqsubseteq_P \underset{C}{\Leftarrow\!\!\!\!\!\!\parallel} \sqsubseteq_Q \right.\right\}$$

We can define a right-biased communicating parallel operator by

$$P \underset{C}{\Rrightarrow\!\!\!\!\!\!\parallel} Q \mathrel{\widehat{=}} c(l(P) \;{}^{A}\!\Rrightarrow\!\!\!\!\!\!\parallel^{B}\; r(Q))$$

where l, r, c, A and B are defined as above. Note that if $\Sigma P \subseteq A$ and $\Sigma Q \subseteq B$ for some sets A and B such that $A \cap B = C$ then $P \underset{C}{\Leftarrow\!\!\!\!\!\!\parallel} Q = P \;{}^{A}\!\Leftarrow\!\!\!\!\!\!\parallel^{B}\; Q$ and $P \underset{C}{\Rrightarrow\!\!\!\!\!\!\parallel} Q = P \;{}^{A}\!\Rrightarrow\!\!\!\!\!\!\parallel^{B}\; Q$.

10 Conclusions

In this paper we have introduced a number of new operators for probabilistic, biased timed CSP. We have introduced a notion of an environment which is used to give a semantics to free variables. We have used this to give a more elegant semantics to recursion, and to define immediate and mutual recursion. We have proved a property that holds of all behaviours of a process. We have defined an infinite probabilistic internal choice operator and a probabilistic external choice operator. We have shown how to adapt the semantic space so as to be able to handle communication of values along channels. We have defined three new operators that transfer control from one process to another: a timeout operator, a timed transfer operator and an interrupt operator. We have introduced a hybrid parallel composition operator that forces its arguments to synchronise on events taken from a certain set, and allows interleaving on events from outside that set. We hope that these changes and additions will increase the expressive power of our language.

10.1 Future work

We believe that the language first presented in [Low92] and extended in this paper is now powerful enough to describe most aspects of prioritized, probabilistic, timed, communicating systems. We now want to investigate ways of using the language and the semantic model for specifying processes, implementing them, and then proving the implementation correct. We could, in theory, do this by applying the semantic equations directly. However, the semantic definitions are rather complicated and difficult to reason with. We therefore need a mechanism for simplifying our proofs. We intend to develop a proof system for arguing about processes, in the style of Davies and Schneider [DS89].

We will begin by considering non-probabilistic specifications, i.e. specifications that state that *all* behaviours of a process satisfy some specification. We will write 'P $\mathbf{sat}_\rho$ $S(\tau, \sqsubseteq, s)$' to specify that, in variable binding ρ, all behaviours of process P satisfy the predicate $S(\tau, \sqsubseteq, s)$. We will then develop a number of inference rules for reducing a proof obligation on a composite process to proof obligations on the subcomponents. For example, the proof rule

$$\frac{\begin{array}{l} P \;\mathbf{sat}_\rho\; S_P(\tau, \sqsubseteq, s) \\ Q \;\mathbf{sat}_\rho\; S_Q(\tau, \sqsubseteq, s) \\ S_P(\tau, \sqsubseteq_P, s) \wedge S_Q(\tau, \sqsubseteq_Q, s) \Rightarrow S(\tau, \sqsubseteq_P \underset{C}{\Leftarrow\!\!\!\!\!\!\parallel} \sqsubseteq_Q, s) \end{array}}{P \underset{C}{\Leftarrow\!\!\!\!\!\!\parallel} Q \;\mathbf{sat}_\rho\; S(\tau, \sqsubseteq, s)}$$

allows the proof obligation $P \between Q \ \mathbf{sat}_\rho \ S(\tau, \sqsubseteq, s)$ to be reduced to two proof obligations on simpler processes, $P \ \mathbf{sat}_\rho \ S_P(\tau, \sqsubseteq, s)$ and $Q \ \mathbf{sat}_\rho \ S_Q(\tau, \sqsubseteq, s)$, whenever the predicates S_P and S_Q are such that $S_P(\tau, \sqsubseteq_P, s) \wedge S_Q(\tau, \sqsubseteq_Q, s) \Rightarrow S(\tau, \sqsubseteq_P \between \sqsubseteq_Q, s)$.

We will then extend these ideas to develop a proof system for reasoning about probabilistic processes. We will write $P \ \mathbf{sat}_\rho^{\geqslant p} \ S(\tau, \sqsubseteq, s)$ to indicate that, in variable binding ρ, and given any environmental offer, P performs a behaviour $(\tau, \sqsubseteq, s)$ that satisfies predicate $S(\tau, \sqsubseteq, s)$ with probability at least p. The probabilistic version of the above rule is then

$$\frac{\begin{array}{l} P \ \mathbf{sat}_\rho^{\geqslant p} \ S_P(\tau, \sqsubseteq, s) \\ Q \ \mathbf{sat}_\rho^{\geqslant q} \ S_Q(\tau, \sqsubseteq, s) \\ S_P(\tau, \sqsubseteq_P, s) \wedge S_Q(\tau, \sqsubseteq_Q, s) \Rightarrow S(\tau, \sqsubseteq_P \between \sqsubseteq_Q, s) \end{array}}{P \between Q \ \mathbf{sat}_\rho^{\geqslant pq} \ S(\tau, \sqsubseteq, s)}$$

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A The Finite Dependency Theorem

In this appendix we prove the following theorem, which was used in section 7.

Theorem A.1 (Finite Dependency Theorem) If P is a PBTCSP term, possibly containing free occurrences of process variables from the set $\{X_i \mid i \in I\}$, and ρ is an environment, then

$$(\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho \Rightarrow \exists N : \mathbb{F} I \bullet \forall \rho' : ENV \bullet \\ (\forall i : N \bullet \rho[\![X_i]\!] = \rho'[\![X_i]\!]) \Rightarrow (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho'$$

and

$$\forall(\tau, \sqsubseteq, s) : BEH \bullet \forall\, \varepsilon > 0 \bullet \exists\, N : \mathbb{F}\, I \bullet \forall\, \rho' : ENV \bullet$$
$$(\forall\, i \in N \bullet \rho[\![X_i]\!] = \rho'[\![X_i]\!]) \Rightarrow \left|\mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq, s) - \mathcal{P}_{BT}[\![P]\!]\rho'(\tau, \sqsubseteq, s)\right| < \varepsilon$$

♡

This result is a consequence of the following two lemmas.

Lemma A.2: If b_0 is a member of $\mathcal{A}_{BT}[\![P]\!]\rho$ then

$$\exists\, M \in \mathbb{F}(VAR \times BEH) \bullet PROP_A(M, b_0, P, \rho)$$

where

$$PROP_A(M, b_0, P, \rho) \mathrel{\widehat{=}}$$
$$(\forall(X, b) \in M \bullet b \in \pi_1(\rho[\![X]\!]))$$
$$\wedge\, \forall\, \rho' \in ENV \bullet (\forall(X, b) \in M \bullet b \in \pi_1(\rho'[\![X]\!])) \Rightarrow b_0 \in \mathcal{A}_{BT}[\![P]\!]\rho'$$

♡

We may find a finite set M of (variable, behaviour) pairs such that the inclusion of behaviour b_0 in the semantic set depends only upon the elements of M. This result can be proved by structural induction as in [DS90].

Lemma A.3: For all behaviours b_0, for all $\varepsilon_0 > 0$, and for all environments ρ, if we define p_0 by $p_0 = \mathcal{P}_{BT}[\![P]\!]\rho b_0$, then

$$\exists\, M \in \mathbb{F}(VAR \times BEH \times PROB \times \mathbb{R}^+) \bullet PROP_P(M, b_0, P, \rho, p_0, \varepsilon_0)$$

where

$$PROP_P(M, b_0, P, \rho, p_0, \varepsilon_0) \mathrel{\widehat{=}}$$
$$(\forall(X, b, p, \varepsilon) \in M \bullet \pi_2(\rho[\![X]\!])\, b = p)$$
$$\wedge\, \forall\, \rho' \in ENV \bullet (\forall(X, b, p, \varepsilon) \in M \bullet |\pi_2(\rho'[\![X]\!])\, b - p| < \varepsilon) \Rightarrow \left|\mathcal{P}_{BT}[\![P]\!]\rho' b_0 - p_0\right| < \varepsilon_0$$

♡

For all ε_0, we may find a finite set M of (variable, behaviour, probability, tolerance) quadruples, such that if environment ρ' is such that for all members (X, b, p, ε) of M, $\rho'[\![X]\!]$ gives probability p to behaviour b to within tolerance ε, then the probability of b_0 in ρ' is p_0 to within ε_0.

Proof: The following real analysis result will be useful.

Lemma A.3.1: If I is a countable set, f is a function from I to $PROB$ and $\varepsilon > 0$, then there exists a finite set $N \subseteq I$ such that

$$0 \leqslant \sum\left\{\!\!\left| f(x) \mid x \in I \right|\!\!\right\} - \sum\left\{\!\!\left| f(x) \mid x \in N \right|\!\!\right\} < \varepsilon$$

♡

Proof of lemma: $f(x) \geqslant 0$ for all x, so the first inequality holds for any $N \subseteq I$. For the second inequality, note that I is countable, so we may enumerate it: $I = \{x_0, x_1, \ldots\}$. Then by definition,

$$\sum\left\{\!\!\left| f(x) \mid x \in I \right|\!\!\right\} = \lim_{n \to \infty} \sum\left\{\!\!\left| f(x_i) \mid i \in 0\,..\,n-1 \right|\!\!\right\}$$

So there exists some k such that

$$\sum\left\{\!\!\left| f(x) \mid x \in I \right|\!\!\right\} - \sum\left\{\!\!\left| f(x_i) \mid i \in 0\,..\,k-1 \right|\!\!\right\} < \varepsilon$$

Taking $N = \{x_0, \ldots, x_{k-1}\}$ we have the result. □

We can now prove the main result by structural induction. Most of the base cases are trivial because they are closed terms, where the semantic values do not depend upon the environment. The remaining case is the variable clause:

Case X:

For any $b_0 \in BEH$, $\varepsilon_0 > 0$, $\rho \in ENV$, define M to be the singleton set given by

$$M \mathrel{\widehat{=}} \{(X, b_0, \pi_2(\rho[\![X]\!])\, b_0, \varepsilon_0)\}$$

The result follows immediately. □

For the inductive step, we give the proofs in three example cases, namely indexed probabilistic choice, parallel composition, and mutual recursion.

Case $\bigsqcap_{i\in I}[p_i]P_i$:

Pick $b_0 \in BEH$, $\varepsilon_0 > 0$, and $\rho \in ENV$, and let $p_0 = \mathcal{P}_{BT}[\![\bigsqcap_{i\in I}[p_i]P_i]\!]\rho b_0$. Recall that

$$\mathcal{P}_{BT}[\![\bigsqcap_{i\in I}[p_i]P_i]\!]\rho b_0 = \sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho b_0 \mid i \in I \,|\!\}$$

By lemma A.3.1 there exists a finite set $N \subseteq I$ such that

$$\begin{aligned} 0 &\leqslant \sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho b_0 \mid i \in I \,|\!\} - \sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho b_0 \mid i \in N \,|\!\} < \varepsilon_0/3 \\ 0 &\leqslant \sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho' b_0 \mid i \in I \,|\!\} - \sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho' b_0 \mid i \in N \,|\!\} < \varepsilon_0/3 \end{aligned} \qquad (*)$$

By the inductive hypothesis, for all $i \in N$

$$\exists\, M_i \in \mathbb{F}(VAR \times BEH \times PROB \times \mathbb{R}^+) \bullet PROP_P(M_i, b_0, P_i, \rho, \mathcal{P}_{BT}[\![P_i]\!]\rho b_0, \varepsilon_0/3)$$

in other words,

$$\begin{aligned} &\forall (X, b, p, \varepsilon) \in M_i \bullet \pi_2(\rho[\![X]\!])\, b = p \\ &\wedge\, \forall\, \rho' \in ENV \bullet (\forall (X, b, p, \varepsilon) \in M_i \bullet |\pi_2(\rho'[\![X]\!])\, b - p| < \varepsilon) \Rightarrow \\ &\qquad\qquad \left|\mathcal{P}_{BT}[\![P_i]\!]\rho' b_0 - \mathcal{P}_{BT}[\![P_i]\!]\rho b_0\right| < \varepsilon_0/3 \end{aligned}$$

Let $M \mathrel{\widehat{=}} \bigcup\{M_i \mid i \in N\}$. Note that M is finite because N and each of the M_i are. We claim that

$$PROP_P(M, b_0, \bigsqcap_{i\in I}[p_i]P_i, \rho, p_0, \varepsilon_0)$$

Note first of all that

$$\forall (X, b, p, \varepsilon) \in M \bullet \pi_2(\rho[\![X]\!])\, b = p$$

since each (X, b, p, ε) is in one of the M_i. Suppose then that environment ρ' is such that

$$\forall (X, b, p, \varepsilon) \in M \bullet |\pi_2(\rho'[\![X]\!])\, b - p| < \varepsilon$$

Then for each $i \in N$,

$$\left|\mathcal{P}_{BT}[\![P_i]\!]\rho' b_0 - \mathcal{P}_{BT}[\![P_i]\!]\rho b_0\right| < \varepsilon_0/3$$

so

$$\begin{aligned} &\left|\sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho' b_0 \mid i \in N \,|\!\} - \sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho b_0 \mid i \in N \,|\!\}\right| \\ &< \sum\{\!|\, p_i.\varepsilon_0/3 \mid i \in N \,|\!\} \\ &\leqslant \varepsilon_0/3 \end{aligned}$$

Combining this with $(*)$ we see that

$$\left|\sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho' b_0 \mid i \in I \,|\!\} - \sum\{\!|\, p_i.\mathcal{P}_{BT}[\![P_i]\!]\rho b_0 \mid i \in I \,|\!\}\right| < \varepsilon_0$$

which means that

$$\left|\mathcal{P}_{BT}[\![p_i.\bigsqcap_{i\in I}[p_i]P_i]\!]\rho' b_0 - \mathcal{P}_{BT}[\![p_i.\bigsqcap_{i\in I}[p_i]P_i]\!]\rho b_0\right| < \varepsilon_0$$

as required. □

Case $P \nparallel Q$:

Pick $b_0 = (\tau_0, \sqsubseteq_0, s_0) \in BEH$, $\varepsilon_0 > 0$, and $\rho \in ENV$, and let $p_0 = \mathcal{P}_{BT}[\![P \nparallel Q]\!]\rho b_0$. Recall that

$$\mathcal{P}_{BT}[\![P \nparallel Q]\!]\rho(\tau_0, \sqsubseteq_0, s_0) = \sum\left\{\mathcal{P}_{BT}[\![P]\!]\rho(\tau_0, \sqsubseteq_P, s_0).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau_0, \sqsubseteq_Q, s_0) \mid \sqsubseteq_P \nparallel \sqsubseteq_Q = \sqsubseteq_0\right\}$$

Note that the set of pairs

$$\{(\sqsubseteq_P, \sqsubseteq_Q) \mid \sqsubseteq_P \nparallel \sqsubseteq_Q = \sqsubseteq_0 \wedge \mathcal{P}_{BT}[\![P]\!]\rho(\tau_0, \sqsubseteq_P, s_0).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau_0, \sqsubseteq_Q, s_0) > 0\}$$

is countable, so we can use lemma A.3.1 to show that for any environment ρ' there exists a finite set $N : OFFREL \times OFFREL$ such that

$$\begin{aligned}
&0 \leqslant \mathcal{P}_{BT}[\![P \nparallel Q]\!]\rho b_0 - \sum\left\{\mathcal{P}_{BT}[\![P]\!]\rho(\tau_0, \sqsubseteq_P, s_0).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau_0, \sqsubseteq_Q, s_0) \mid (\sqsubseteq_P, \sqsubseteq_Q) \in N\right\} \\
&\quad < \varepsilon_0/4 \\
&0 \leqslant \mathcal{P}_{BT}[\![P \nparallel Q]\!]\rho' b_0 - \sum\left\{\mathcal{P}_{BT}[\![P]\!]\rho'(\tau_0, \sqsubseteq_P, s_0).\mathcal{P}_{BT}[\![Q]\!]\rho'(\tau_0, \sqsubseteq_Q, s_0) \mid (\sqsubseteq_P, \sqsubseteq_Q) \in N\right\} \\
&\quad < \varepsilon_0/4 \qquad (*)
\end{aligned}$$

For simplicity assume that $N = \{(\sqsubseteq_{P,i}, \sqsubseteq_{Q,i}) \mid i \in 0\,..\,k-1\}$. By the inductive hypothesis, for all $i \in 0\,..\,k-1$,

$$\begin{aligned}
&\exists\, M_{P,i} \in \mathbb{F}(VAR \times BEH \times PROB \times \mathbb{R}^+) \bullet \\
&\quad PROP_P(M_{P,i}, (\tau_0, \sqsubseteq_{P,i}, s_0), P, \rho, \mathcal{P}_{BT}[\![P]\!]\rho(\tau_0, \sqsubseteq_{P,i}, s_0), \varepsilon_0/2^{i+3}) \\
&\exists\, M_{Q,i} \in \mathbb{F}(VAR \times BEH \times PROB \times \mathbb{R}^+) \bullet \\
&\quad PROP_P(M_{Q,i}, (\tau_0, \sqsubseteq_{Q,i}, s_0), Q, \rho, \mathcal{P}_{BT}[\![Q]\!]\rho(\tau_0, \sqsubseteq_{Q,i}, s_0), \varepsilon_0/2^{i+3})
\end{aligned}$$

Let $M \mathrel{\widehat{=}} \bigcup\{M_{P,i} \mid i \in 0\,..\,k-1\} \cup \bigcup\{M_{Q,i} \mid i \in 0\,..\,k-1\}$. Note that M is finite because the indexing set and each of the $M_{P,i}$ and $M_{Q,i}$ are. We claim that

$$PROP_P(M, (\tau_0, \sqsubseteq_0, s_0), P \nparallel Q, \rho, p_0, \varepsilon_0)$$

Note first off all that

$$\forall (X, b, p, \varepsilon) \in M \bullet \pi_2(\rho[\![X]\!])\, b = p$$

because $PROP_P$ holds for each of the $M_{P,i}$ and $M_{Q,i}$. Suppose then that environment ρ' is such that

$$\forall (X, b, p, \varepsilon) \in M \bullet |\pi_2(\rho'[\![X]\!])\, b - p| < \varepsilon$$

Then for each $i \in 0\,..\,k-1$ we have

$$\begin{aligned}
\left|\mathcal{P}_{BT}[\![P]\!]\rho'(\tau_0, \sqsubseteq_{P,i}, s_0) - \mathcal{P}_{BT}[\![P]\!]\rho(\tau_0, \sqsubseteq_{P,i}, s_0)\right| &< \varepsilon_0/2^{i+3} \\
\left|\mathcal{P}_{BT}[\![Q]\!]\rho'(\tau_0, \sqsubseteq_{Q,i}, s_0) - \mathcal{P}_{BT}[\![Q]\!]\rho(\tau_0, \sqsubseteq_{Q,i}, s_0)\right| &< \varepsilon_0/2^{i+3}
\end{aligned}$$

because $PROP_P$ holds for each of the $M_{P,i}$ and $M_{Q,i}$. So

$$\begin{aligned}
&\left|\mathcal{P}_{BT}[\![P]\!]\rho'(\tau_0, \sqsubseteq_{P,i}, s_0).\mathcal{P}_{BT}[\![Q]\!]\rho'(\tau_0, \sqsubseteq_{Q,i}, s_0) - \mathcal{P}_{BT}[\![P]\!]\rho(\tau_0, \sqsubseteq_{P,i}, s_0).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau_0, \sqsubseteq_{Q,i}, s_0)\right| \\
&\quad < \varepsilon_0/2^{i+2}
\end{aligned}$$

using the fact that we are dealing with quantities in the range $[0, 1]$. Hence

$$\begin{array}{l}\Big|\sum\Big\{\mathcal{P}_{BT}[\![P]\!]\rho'(\tau_0, \sqsubseteq_P, s_0).\mathcal{P}_{BT}[\![Q]\!]\rho'(\tau_0, \sqsubseteq_Q, s_0) \mid (\sqsubseteq_P, \sqsubseteq_Q) \in N\Big\} \\ \quad -\sum\Big\{\mathcal{P}_{BT}[\![P]\!]\rho(\tau_0, \sqsubseteq_P, s_0).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau_0, \sqsubseteq_Q, s_0) \mid (\sqsubseteq_P, \sqsubseteq_Q) \in N\Big\}\Big| \\ < \sum\Big\{\varepsilon_0/2^{i+2} \mid i \in 0\,..\,k-1\Big\} \\ < \varepsilon_0/2\end{array}$$

Combining this result with $(*)$, we see that

$$\left|\mathcal{P}_{BT}[\![P \nVdash Q]\!]\rho' b_0 - \mathcal{P}_{BT}[\![P \nVdash Q]\!]\rho b_0\right| < \varepsilon_0$$

as required. □

Case $\langle X_i = P_i\rangle_j$**:**

We assume that the vector $\underline{P}$ of terms is t-constructive for the vector X of variables. Pick $b_0 = (\tau_0, \sqsubseteq_0, s_0) \in BEH$, $\varepsilon_0 > 0$, $\rho \in ENV$ and let $p_0 = \mathcal{P}_{BT}[\![\langle X_i = P_i\rangle_j]\!]\rho b_0$.

Firstly, for each term P_l, behaviour $b_1 = (\tau_1, \sqsubseteq_1, s_1)$, $\varepsilon_1 > 0$, and environment ρ, if we define p_1 by $p_1 = \mathcal{P}_{BT}[\![P_l]\!]\rho b_1$ then from the inductive hypothesis we have

$$\exists\, M(b_1, l, p_1, \varepsilon_1) \in \mathbb{F}(VAR \times BEH \times PROB \times \mathbb{R}^+) \bullet PROP_P(M(b_1, l, p_1, \varepsilon_1), b_1, P_l, \rho, p_1, \varepsilon_1)$$

We make use of the following lemma:

Lemma A.3.2: If $PROP_P(M, b_0, P, \rho, p_0, \varepsilon_0)$ and $b_0 = (\tau_0, \sqsubseteq_0, s_0)$ then

$$\begin{array}{l}\exists\, M' \subseteq M \bullet PROP_P(M', b_0, P, \rho, p_0, \varepsilon_0) \\ \qquad\qquad \wedge\, \forall (X, (\tau, \sqsubseteq, s), p, \varepsilon) \in M' \bullet P\ t_X\text{-constructive for } X \Rightarrow \tau + t_X \leqslant \tau_0\end{array}$$

♡

We can find a subset M' of M such that $PROP_P$ holds of M', and all behaviours corresponding to variables for which P is t-constructive end at least t before b_0.

Proof of lemma: We define M' by

$$M' \mathrel{\widehat{=}} \{(X, (\tau, \sqsubseteq, s), p, \varepsilon) \in M \mid P\ t_X\text{-constructive for } X \Rightarrow \tau + t_X \leqslant \tau_0\}$$

We show that $PROP_P(M', b_0, P, \rho, p_0, \varepsilon_0)$ holds. The first conjunct is immediate as $M' \subseteq M$. For the second conjunct, suppose that ρ' is such that

$$\forall (X, b, p, \varepsilon) \in M' \bullet |\pi_2(\rho'[\![X]\!])b - p| < \varepsilon$$

Define ρ'' by

$$\rho''[\![X]\!] \mathrel{\widehat{=}} (\pi_1(\rho'[\![X]\!]), \pi_2(\rho'[\![X]\!]) \oplus \{b \mapsto p \mid (X, b, p, \varepsilon) \in M \setminus M'\})$$

Note that $\forall (X, b, p, \varepsilon) \in M \bullet |\pi_2(\rho''[\![X]\!])b - p| < \varepsilon$, by the definition of ρ'' and by the assumption about ρ', so from $PROP_P(M, b_0, P, \rho, p_0, \varepsilon_0)$ we have $\left|\mathcal{P}_{BT}[\![P]\!]\rho'' b_0 - p_0\right| < \varepsilon_0$. Note also that

$$\begin{array}{l}\pi_2(\rho''\,[\rho''[\![X]\!] \upharpoonright \tau_0 - t_X/X \mid (X, b, p, \varepsilon) \in M]\,[\![X_0]\!]) = \\ \quad \begin{cases}\pi_2(\rho'[\![X_0]\!]) & \text{if } \nexists b, p, \varepsilon \bullet (X_0, b, p, \varepsilon) \in M \\ \pi_2(\rho'[\![X_0]\!]) \oplus \{b \mapsto p \mid (X_0, b, p, \varepsilon) \in M \setminus M'\}) \upharpoonright \tau_0 - t_X & \text{otherwise}\end{cases}\end{array}$$

Now, if $(X_0, b, p, \varepsilon) \in M \setminus M'$ then $end\, b > \tau_0 - t_X$ by the definition of M', so

$$\rho''[\rho''[\![X]\!] \restriction \tau_0 - t_X / X \mid (X, b, p, \varepsilon) \in M] = \rho'[\rho'[\![X]\!] \restriction \tau_0 - t_X / X \mid (X, b, p, \varepsilon) \in M] \qquad (\dagger)$$

Hence we have

$$\begin{array}{ll} & \mathcal{P}_{BT}[\![P]\!]\rho'' b_0 \\ = & \left\langle b_0 = (\tau_0, \sqsubseteq_0, s_0) \right\rangle \\ & (\mathcal{P}_{BT}[\![P]\!]\rho'' \restriction \tau_0)\, b_0 \\ = & \left\langle \text{definition of constructivity} \right\rangle \\ & (\mathcal{P}_{BT}[\![P]\!]\rho''[\rho''[\![X]\!] \restriction \tau_0 - t_X / X \mid (X, b, p, \varepsilon) \in M] \restriction \tau_0)\, b_0 \\ = & \left\langle \text{by } (\dagger) \right\rangle \\ & (\mathcal{P}_{BT}[\![P]\!]\rho'[\rho'[\![X]\!] \restriction \tau_0 - t_X / X \mid (X, b, p, \varepsilon) \in M] \restriction \tau_0)\, b_0 \\ = & \left\langle \text{constructivity} \right\rangle \\ & \mathcal{P}_{BT}[\![P]\!]\rho' b_0 \end{array}$$

so $\left|\mathcal{P}_{BT}[\![P]\!]\rho' b_0 - p_0\right| < \varepsilon_0$ as required. □

Applying this lemma, we have

$$\begin{array}{l} \exists\, M'(b_1, l, p_1, \varepsilon_1) \subseteq M(b_1, l, p_1, \varepsilon_1) \bullet \\ \quad PROP_P(M', b_1, P_l, \rho, p_1, \varepsilon_1) \\ \quad \wedge\, \forall (X, (\tau, \sqsubseteq, s), p, \varepsilon) \in M' \bullet P_l \ t\text{-constructive for } X \Rightarrow \tau + t_X \leqslant \tau_1 \end{array}$$

We define a function $m : VAR \times BEH \times PROB \times \mathbb{R}^+ \to \mathbb{P}(VAR \times BEH \times PROB \times \mathbb{R}^+)$ by

$$m(X, b, p, \varepsilon) \mathrel{\widehat{=}} \begin{cases} \{\} & \text{if } \nexists i \in I \bullet X = X_i \\ M'(b, l, p, \varepsilon) & \text{if } X = X_l \wedge l \in I \end{cases}$$

We define a relation R of type $(VAR \times BEH \times PROB \times \mathbb{R}^+) \times (VAR \times BEH \times PROB \times \mathbb{R}^+)$ by

$$Q_1\ R\ Q_2 \Leftrightarrow Q_1 \in m(Q_2)$$

for all quadruples Q_1 and Q_2 of type $VAR \times BEH \times PROB \times \mathbb{R}^+$. We state a lemma about this relation; the proof is identical to the corresponding proof in [DS90].

Lemma A.3.3: The relation R is a well founded, finite-to-one relation; that is:

- there are no infinite chains $\{Q_n\}$ such that $\forall\, n \bullet Q_{n+1}\ R\ Q_n$;
- for any Q, the set $\{Q' \mid Q'\ R\ Q\}$ is finite.

♡

We now use the following result from [End77].

Königs Lemma: If R is a well founded, finite-to-one relation then $\forall\, y \bullet \{x \mid x\ R^t\ y\}$ is finite. ♢

Applying this, we see that the set

$$M \mathrel{\widehat{=}} \{Q \mid Q\ R^t\ (X_j, b_0, p_0, \varepsilon_0)\}$$

is finite. We claim that $PROP_P(M, b_0, \langle X_i = P_i \rangle_j, \rho, p_0, \varepsilon_0)$.

For the first conjunct of $PROP_P$, pick $(X, b, p, \varepsilon) \in M$. Then for some X', b', p', ε' we have $(X, b, p, \varepsilon) \in m(X', b', p', \varepsilon')$, so for some l, we have $(X, b, p, \varepsilon) \in M'(b', l, p', \varepsilon')$. Now, $PROP_P(M'(b', l, p', \varepsilon'), b', P_l, \rho, p', \varepsilon')$ so $\pi_2(\rho[\![X]\!])\, b = p$, as required.

For the second conjunct, pick $\rho' \in ENV$ such that

$$\forall (X, b, p, \varepsilon) \in M \bullet |\pi_2(\rho'[\![X]\!])\, b - p| < \varepsilon \qquad (*)$$

we want to show that $\left|\mathcal{P}_{BT}[\![\langle X_i = P_i \rangle_j]\!]\rho'\, b_0 - p_0\right| < \varepsilon_0$. We make use of the following lemma:

Lemma A.3.4: If environment ρ' satisfies $(*)$, then

$$\forall (X, b, p, \varepsilon) \in M \bullet HYP_P(X, b, p, \varepsilon)$$

where

$$HYP_P(X, b, p, \varepsilon) \mathrel{\widehat{=}} \left|\pi_2(\rho'[\mathcal{F}_{BT}[\![\langle X_i = P_i \rangle_k]\!]\rho'/X_k][\![X]\!])\, b - p\right| < \varepsilon$$

♡

Proof of lemma: It is enough to show that for any $(X, b, p, \varepsilon) \in M$,

$$(\forall (X', b', p', \varepsilon') \in m(X, b, p, \varepsilon) \bullet HYP(X', b', p', \varepsilon')) \Rightarrow HYP(X, b, p, \varepsilon)$$

Assume, then, the left hand side of this implication, and consider the variable X.

- If $\nexists i \in I \bullet X = X_i$ then $\rho'[\mathcal{F}_{BT}[\![\langle X_i = P_i \rangle_k]\!]\rho'/X_k][\![X]\!] = \rho'[\![X]\!]$, so

  $$\left|\pi_2(\rho'[\mathcal{F}_{BT}[\![\langle X_i = P_i \rangle_k]\!]\rho'/X_k][\![X]\!])\, b - p\right| = |\pi_2(\rho'[\![X]\!])\, b - p| < \varepsilon$$

  by $(*)$, as required.
- If $X = X_l$ for $l \in I$ then $m(X, b, p, \varepsilon) = M'(b, l, p, \varepsilon)$. Hence, by the hypothesis,

  $$\forall (X', b', p', \varepsilon') \in M'(X, b, p, \varepsilon) \bullet \left|\pi_2(\rho'[\mathcal{F}_{BT}[\![\langle X_i = P_i \rangle_k]\!]\rho'/X_k][\![X']\!])\, b' - p'\right| < \varepsilon'$$

  So by $PROP_P(M'(b, l, p, \varepsilon), b, P_l, \rho, p, \varepsilon)$,

  $$\left|\mathcal{P}_{BT}[\![P_l]\!]\rho'[\mathcal{F}_{BT}[\![\langle X_i = P_i \rangle_k]\!]\rho'/X_k]\, b - p\right| < \varepsilon$$

  Hence by the definition of recursion,

  $$\left|\pi_2(\rho'[\mathcal{F}_{BT}[\![\langle X_i = P_i \rangle_k]\!]\rho'/X_k][\![X_l]\!])\, b - p\right| < \varepsilon$$

  as required.

□

Applying this lemma, we have, in particular, $HYP_P(X_j, b_0, p_0, \varepsilon_0)$, i.e.

$$\left|\pi_2(\rho'[\mathcal{F}_{BT}[\![\langle X_i = P_i \rangle_j]\!]\rho'/X_k][\![X_j]\!])\, b_0 - p_0\right| < \varepsilon_0$$

giving

$$\left|\mathcal{P}_{BT}[\![\langle X_i = P_i \rangle_k]\!]\rho'\, b_0 - p_0\right| < \varepsilon_0$$

as required. This establishes the result for the case of mutual recursion. □

This completes the proof. □

B Definitions of the operators

B.1 Subsidiary functions

$$\text{fillout} f\ (\tau, \sqsubseteq, s) \ \widehat{=}\ \begin{cases} f(\tau, \sqsubseteq, s) & \text{if } (\tau, \sqsubseteq, s) \in \text{dom} f \\ 0 & \text{if } (\tau, \sqsubseteq, s) \notin \text{dom} f \end{cases}$$

$$\Uparrow_{\sqsubseteq_P, \sqsubseteq_Q} w \ \widehat{=}\ \sqcup_{\sqsubseteq_P} \{ w'_P \in \text{items} \sqsubseteq_P \mid w'_P \subseteq w \wedge w - w'_P \in \text{items} \sqsubseteq_Q \}$$
$$\text{if } \exists\, w_P \in \text{items} \sqsubseteq_P, w_Q \in \text{items} \sqsubseteq_Q \bullet w = w_P \uplus w_Q$$

$$\Uparrow^g_{\sqsubseteq} w \ \widehat{=}\ \sqcup_{\sqsubseteq} \{ w' \in \text{items} \sqsubseteq \mid gw' = w \}, \quad \text{if } \exists\, w' \in \text{items} \sqsubseteq \bullet gw' = w$$

$$M(X, P)\rho \ \widehat{=}\ \lambda\, Y \bullet \mathcal{F}_{BT}[\![P]\!]\rho[Y/X]$$

$$W_\delta \ \widehat{=}\ \lambda\, Y \bullet \mathcal{F}_{BT}[\![WAIT\ \delta\ ; X]\!]\rho[Y/X]$$

$$M_\delta(X, P)\rho \ \widehat{=}\ M(X, P)\rho \circ W_\delta$$

$$M(\underline{X}, \underline{P})\rho \ \widehat{=}\ \lambda\, \underline{Y} \bullet \mathcal{F}_{BT}[\![\underline{P}]\!]\rho[Y_i/X_i \mid i \in I]$$

B.2 Operations on offer relations

$$\begin{aligned} v(\sqsubseteq_P \,\square\, \sqsubseteq_Q)\, w \ \Leftrightarrow\ & v \sqsubseteq_P w \\ & \vee\ w \in \text{items} \sqsubseteq_P \wedge \Sigma w \neq \{\!|\}\!\} \wedge v \in \text{items} \sqsubseteq_Q \setminus \text{items} \sqsubseteq_P \\ & \vee\ v, w \notin \text{items} \sqsubseteq_P \wedge v \sqsubseteq_Q w \end{aligned}$$

$$\begin{aligned} v(\sqsubseteq_P {}^{X}\!\!\not\Vdash^{Y} \sqsubseteq_Q)\, w \ \Leftrightarrow\ & (v \lfloor X \sqsubset_P w \lfloor X \vee v \lfloor X = w \lfloor X \wedge v \lfloor Y \sqsubseteq_Q w \lfloor Y) \\ & \wedge\ v \lfloor X, w \lfloor X \in \text{items} \sqsubseteq_P \wedge v \lfloor Y, w \lfloor Y \in \text{items} \sqsubseteq_Q \wedge \Sigma v, \Sigma w \subseteq X \cup Y \end{aligned}$$

$$\begin{aligned} v(\sqsubseteq_P \not\Vdash \sqsubseteq_Q)\, w \ \Leftrightarrow\ & \exists\, v'_P \in \text{items} \sqsubseteq_P, v'_Q \in \text{items} \sqsubseteq_Q \bullet v = v'_P \uplus v'_Q \\ & \wedge\ \exists\, w'_P \in \text{items} \sqsubseteq_P, w'_Q \in \text{items} \sqsubseteq_Q \bullet w = w'_P \uplus w'_Q \\ & \wedge\ v_P \sqsubset_P w_P \vee v_P = w_P \wedge v_Q \sqsubseteq_Q w_Q \\ & \text{where } v_P = \Uparrow_{\sqsubseteq_P, \sqsubseteq_Q} v \quad v_Q = v - v_P \quad w_P = \Uparrow_{\sqsubseteq_P, \sqsubseteq_Q} w \quad w_Q = w - w_P \end{aligned}$$

$$v(\sqsubseteq \setminus X)\, w \ \Leftrightarrow\ \exists\, v', w' \in \text{items} \sqsubseteq \bullet v' \setminus X = v \wedge w' \setminus X = w \wedge \Uparrow^{\setminus X}_{\overline{\sqsubseteq}} v \sqsubseteq \Uparrow^{\setminus X}_{\overline{\sqsubseteq}} w$$

$$v(g \odot \sqsubseteq)\, w \ \Leftrightarrow\ \exists\, v' \in \text{items} \sqsubseteq \bullet gv' = v \wedge \exists\, w' \in \text{items} \sqsubseteq \bullet gw' = w \wedge \Uparrow^g_{\sqsubseteq} v \sqsubseteq \Uparrow^g_{\sqsubseteq} w$$

$$\sqsubseteq \oplus a \ \widehat{=}\ \sqsubseteq \twoheadrightarrow I \sqsubseteq \otimes \langle \{\!| a |\!\}, \{\!|\}\!\} \rangle$$

B.3 Definitions of the operators

$$\mathcal{A}_{BT}[\![STOP]\!]\rho \,\widehat{=}\, \{(\tau, [0, \tau] \otimes \langle \{\!|\}\!\}\rangle, \{\!|\}\!\})\}$$

$$\mathcal{P}_{BT}[\![STOP]\!]\rho \,\widehat{=}\, \text{fillout}\{(\tau, [0, \tau] \otimes \langle \{\!|\}\!\}\rangle, \{\!|\}\!\}) \mapsto 1\}$$

$$\begin{aligned} & \mathcal{A}_{BT}[\![WAIT\ t]\!]\rho \,\widehat{=} \\ & \quad \{(\tau, [0, \tau] \otimes \langle \{\!|\}\!\}\rangle, \{\!|\}\!\}) \mid \tau < t\} \\ & \quad \cup \{(\tau, [0, t) \otimes \langle \{\!|\}\!\}\rangle^\frown [t, \tau] \otimes \langle \{\!|\surd|\!\}, \{\!|\}\!\}\rangle, \prec\!\succ) \mid \tau \geqslant t\} \\ & \quad \cup \{(\tau, [0, t) \otimes \langle \{\!|\}\!\}\rangle^\frown [t, t'] \otimes \langle \{\!|\surd|\!\}, \{\!|\}\!\}\rangle^\frown (t', \tau] \otimes \langle \{\!|\}\!\}\rangle, \prec(t', \surd)\succ) \mid t \leqslant t' \leqslant \tau\} \end{aligned}$$

$$\mathcal{P}_{BT}[\![WAIT\ t]\!]\rho \mathrel{\widehat{=}}$$
$$\quad \text{fillout}\,(\{(\tau,[0,\tau]\otimes\langle\{\!|\}\!\}\rangle,\prec\!\!\succ) \mapsto 1 \mid \tau < t\}$$
$$\qquad \cup\{(\tau,[0,t)\otimes\langle\{\!|\}\!\}\rangle^\frown[t,\tau]\otimes\langle\{\!|\surd\}\!\},\{\!|\}\!\}\rangle,\prec\!\!\succ) \mapsto 1 \mid \tau \geqslant t\}$$
$$\qquad \cup\{(\tau,[0,t)\otimes\langle\{\!|\}\!\}\rangle^\frown[t,t']\otimes\langle\{\!|\surd\}\!\},\{\!|\}\!\}\rangle^\frown(t',\tau]\otimes\langle\{\!|\}\!\}\rangle,\prec(t',\surd)\succ) \mapsto 1 \mid t \leqslant t' \leqslant \tau\})$$

$$\mathcal{F}_{BT}[\![X]\!]\rho \mathrel{\widehat{=}} \rho[\![X]\!]$$

$$\mathcal{A}_{BT}[\![a \xrightarrow{0} P]\!]\rho \mathrel{\widehat{=}}$$
$$\quad \{(\tau,[0,\tau]\otimes\langle\{\!|a|\}\!\},\{\!|\}\!\}\rangle,\prec\!\!\succ)\}$$
$$\quad \cup\{(\tau,[0,t]\otimes\langle\{\!|a|\}\!\},\{\!|\}\!\}\rangle^\frown\sqsubseteq_P + t,(t,a)^\frown s_P + t) \mid (\tau - t, 0\otimes\langle\{\!|\}\!\}\rangle^\frown\sqsubseteq_P, s_P) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge \tau \geqslant t\}$$

$$\mathcal{P}_{BT}[\![a \xrightarrow{0} P]\!]\rho \mathrel{\widehat{=}}$$
$$\quad \text{fillout}\,(\{(\tau,[0,\tau]\otimes\langle\{\!|a|\}\!\},\{\!|\}\!\}\rangle,\prec\!\!\succ) \mapsto 1\}$$
$$\qquad \cup\{(\tau,[0,t]\otimes\langle\{\!|a|\}\!\},\{\!|\}\!\}\rangle^\frown\sqsubseteq_P + t,(t,a)^\frown s + t) \mapsto \mathcal{P}_{BT}[\![P]\!]\rho(\tau - t, 0\otimes\langle\{\!|\}\!\}\rangle^\frown\sqsubseteq_P, s) \mid \tau \geqslant t\})$$

$$\mathcal{A}_{BT}[\![P \fatsemi Q]\!]\rho \mathrel{\widehat{=}}$$
$$\quad \{(\tau,\sqsubseteq_P,s_P) \mid \forall\, t \bullet s_P \uparrow t \in \text{items}\sqsubseteq_P$$
$$\qquad\qquad \wedge\ \exists\sqsubseteq'_P \bullet \sqsubseteq'_P \setminus \surd = \sqsubseteq_P \wedge (\tau,\sqsubseteq'_P,\Uparrow^{\setminus\surd}_{\sqsubseteq'_P} s_P) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge \surd \notin \Sigma(\Uparrow^{\setminus\surd}_{\sqsubseteq'_P} s_P)\}$$
$$\quad \cup\{(\tau,\sqsubseteq_P{}^\frown(t,\tau]\otimes\langle\{\!|\}\!\}\rangle,s_P) \mid$$
$$\qquad t \leqslant \tau < t+\delta \wedge \forall\, t' \bullet s_P \uparrow t' \in \text{items}\sqsubseteq_P$$
$$\qquad \wedge\ \exists\sqsubseteq'_P \bullet \sqsubseteq'_P \setminus \surd = \sqsubseteq_P \wedge (t,\sqsubseteq'_P,\Uparrow^{\setminus\surd}_{\sqsubseteq'_P} s_P) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge \mathit{begin}((\Uparrow^{\setminus\surd}_{\sqsubseteq'_P} s_P)\lfloor\surd) = t\}$$
$$\quad \cup\{(\tau,\sqsubseteq_P{}^\frown(t,t+\delta)\otimes\langle\{\!|\}\!\}\rangle^\frown\sqsubseteq_Q + t+\delta, s_P{}^\frown s_Q + t+\delta) \mid$$
$$\qquad t \leqslant \tau - \delta \wedge \forall\, t' \bullet s_P \uparrow t' \in \text{items}\sqsubseteq_P$$
$$\qquad \wedge\ \exists\sqsubseteq'_P \bullet \sqsubseteq'_P \setminus \surd = \sqsubseteq_P \wedge (t,\sqsubseteq'_P,\Uparrow^{\setminus\surd}_{\sqsubseteq'_P} s_P) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge \mathit{begin}((\Uparrow^{\setminus\surd}_{\sqsubseteq'_P} s_P)\lfloor\surd) = t$$
$$\qquad \wedge\ (\tau - (t+\delta),\sqsubseteq_Q, s_Q) \in \mathcal{A}_{BT}[\![Q]\!]\rho\}$$

$$\mathcal{P}_{BT}[\![P \fatsemi Q]\!]\rho \mathrel{\widehat{=}}$$
$$\quad \text{fillout}\,\{(\tau,\sqsubseteq,s) \mapsto$$
$$\qquad \sum\left\{\mathcal{P}_{BT}[\![P]\!]\rho(\tau,\sqsubseteq_P,\Uparrow^{\setminus\surd}_{\sqsubseteq_P} s) \mid \sqsubseteq_P \setminus \surd = \sqsubseteq \wedge \surd \notin \Sigma(\Uparrow^{\setminus\surd}_{\sqsubseteq_P} s)\right\}$$
$$\qquad + \sum\left\{\begin{array}{l}\mathcal{P}_{BT}[\![P]\!]\rho(t,\sqsubseteq_P,\Uparrow^{\setminus\surd}_{\sqsubseteq_P} s) \mid \\ \quad \sqsubseteq = \sqsubseteq_P \setminus \surd{}^\frown(t,\tau]\otimes\langle\{\!|\}\!\}\rangle \wedge \mathit{begin}((\Uparrow^{\setminus\surd}_{\sqsubseteq_P} s)\lfloor\surd) = t \wedge t \leqslant \tau < t+\delta\end{array}\right\}$$
$$\qquad + \sum\left\{\begin{array}{l}\mathcal{P}_{BT}[\![P]\!]\rho(t,\sqsubseteq_P,\Uparrow^{\setminus\surd}_{\sqsubseteq_P}(s\restriction t)).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau-(t+\delta),\sqsubseteq_Q, s-t-\delta) \mid \\ \quad \sqsubseteq = \sqsubseteq_P \setminus \surd{}^\frown(t,t+\delta)\otimes\langle\{\!|\}\!\}\rangle^\frown\sqsubseteq_Q + t+\delta \\ \quad \wedge\ \mathit{begin}((\Uparrow^{\setminus\surd}_{\sqsubseteq_P} s)\lfloor\surd) = t \wedge t \leqslant \tau - \delta\end{array}\right\}[]$$
$$\qquad \forall\, t \bullet s \uparrow t \in \text{items}\sqsubseteq\}$$

$$\mathcal{A}_{BT}[\![WAIT\ t\,;P]\!]\rho \mathrel{\widehat{=}}$$
$$\quad \{(\tau,[0,\tau]\otimes\langle\{\!|\}\!\}\rangle,\{\!|\}\!\}) \mid t > \tau\} \cup \{(\tau,\sqsubseteq+t,s+t) \mid t \leqslant \tau \wedge (\tau - t,\sqsubseteq,s) \in \mathcal{A}_{BT}[\![P]\!]\rho\}$$

$$\mathcal{P}_{BT}[\![WAIT\ t\,;P]\!]\rho \mathrel{\widehat{=}}$$
$$\quad \text{fillout}\,(\{(\tau,[0,\tau]\otimes\langle\{\!|\}\!\}\rangle,\{\!|\}\!\}) \mapsto 1 \mid t > \tau\}$$
$$\qquad \cup\{(\tau,\sqsubseteq,s) \mapsto \mathcal{P}_{BT}[\![P]\!]\rho(\tau - t,\sqsubseteq - t, s - t) \mid t \leqslant \tau \wedge s \natural t = \prec\!\!\succ \wedge \sqsubseteq \natural t = [0,t)\otimes\langle\{\!|\}\!\}\rangle\})$$

$$\mathcal{A}_{BT}[\![P\ {}_p\sqcap_q\ Q]\!]\rho \mathrel{\widehat{=}} \mathcal{A}_{BT}[\![P]\!]\rho \cup \mathcal{A}_{BT}[\![Q]\!]\rho$$

$$\mathcal{P}_{BT}[\![P\ {}_p\sqcap_q\ Q]\!]\rho(\tau,\sqsubseteq,s) \mathrel{\widehat{=}} p.\mathcal{P}_{BT}[\![P]\!]\rho(\tau,\sqsubseteq,s) + q.\mathcal{P}_{BT}[\![Q]\!]\rho(\tau,\sqsubseteq,s)$$

$$\mathcal{A}_{BT}[\![\textstyle\bigsqcap_{i\in I}[p_i]P_i]\!]\rho \mathrel{\widehat{=}} \bigcup\{\mathcal{A}_{BT}[\![P_i]\!]\rho \mid i \in I\}$$

$$\mathcal{P}_{BT}[\![\textstyle\bigsqcap_{i\in I}[p_i]P_i]\!]\rho(\tau,\sqsubseteq,s) \mathrel{\widehat{=}} \sum\left\{p_i \times \mathcal{P}_{BT}[\![P_i]\!]\rho(\tau,\sqsubseteq,s) \mid i \in I\right\}$$

$$\begin{aligned}
&\mathcal{A}_{BT}[\![P \,\square\, Q]\!]\rho \mathrel{\widehat{=}} \\
&\quad \{(\tau, \sqsubseteq_P \,\square\, \sqsubseteq_Q, \prec\!\succ) \mid (\tau, \sqsubseteq_P, \prec\!\succ) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau, \sqsubseteq_Q, \prec\!\succ) \in \mathcal{A}_{BT}[\![Q]\!]\rho\} \\
&\quad \cup \{(\tau, \sqsubseteq_P \,\square\, \sqsubseteq_Q, s) \mid \\
&\qquad s \neq \prec\!\succ \wedge \mathit{begin}\ s = t \wedge (t, \sqsubseteq_P, \prec\!\succ) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau, \sqsubseteq_Q, s) \in \mathcal{A}_{BT}[\![Q]\!]\rho \wedge s \uparrow t \not\sqsupseteq_P (t, \{\!|\}\!)\} \\
&\quad \cup \{(\tau, \sqsubseteq_P \,\square\, \sqsubseteq_Q, s) \mid \\
&\qquad s \neq \prec\!\succ \wedge \mathit{begin}\ s = t \wedge (\tau, \sqsubseteq_P, s) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (t, \sqsubseteq_Q, \prec\!\succ) \in \mathcal{A}_{BT}[\![Q]\!]\rho \\
&\qquad \wedge (s \uparrow t \sqsupset_P (t, \{\!|\}\!) \vee s \uparrow t \notin \mathrm{items} \sqsubseteq_Q)\}
\end{aligned}$$

$$\mathcal{P}_{BT}[\![P \,\square\, Q]\!]\rho(\tau, \sqsubseteq, \prec\!\succ) \mathrel{\widehat{=}} \sum \left\{ \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq_P, \prec\!\succ).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau, \sqsubseteq_Q, \prec\!\succ) \mid \sqsubseteq = \sqsubseteq_P \,\square\, \sqsubseteq_Q \right\}$$

$$\begin{aligned}
&\mathcal{P}_{BT}[\![P \,\square\, Q]\!]\rho(\tau, \sqsubseteq, s) \mathrel{\widehat{=}} \\
&\quad \sum \left\{ \mathcal{P}_{BT}[\![P]\!]\rho(t, \sqsubseteq_P, \prec\!\succ).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau, \sqsubseteq_Q, s) \mid \sqsubseteq = \sqsubseteq_P \,\square\, \sqsubseteq_Q \wedge s \uparrow t \not\sqsupseteq_P (t, \{\!|\}\!) \right\} \\
&\quad + \sum \left\{ \begin{array}{l} \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq_P, s).\mathcal{P}_{BT}[\![Q]\!]\rho(t, \sqsubseteq_Q, \prec\!\succ) \mid \\ \quad \sqsubseteq = \sqsubseteq_P \,\square\, \sqsubseteq_Q \wedge (s \uparrow t \sqsupset_P (t, \{\!|\}\!) \vee s \uparrow t \notin \mathrm{items} \sqsubseteq_Q) \end{array} \right\} \\
&\qquad \text{if } s \neq \prec\!\succ \wedge \mathit{begin}\ s = t
\end{aligned}$$

$$\begin{aligned}
&\mathcal{A}_{BT}[\![c?a : A \xrightarrow{0} P_a]\!]\rho \mathrel{\widehat{=}} \\
&\quad \{(\tau, [0, \tau] \otimes \langle \{\!| c?a |\!\}, \{\!|\}\!\rangle, \prec\!\succ) \mid \tau \in \mathit{TIME}\} \\
&\quad \cup \{(\tau, [0, t] \otimes \langle \{\!| c?a |\!\}, \{\!|\}\!\rangle^\frown \sqsubseteq + t, (t, c?\hat{a})^\frown s + t) \mid \\
&\qquad \hat{a} \in A \wedge t \leqslant \tau \wedge (\tau - t, \{0\} \otimes \langle \{\!|\}\!\rangle^\frown \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P_{\hat{a}}]\!]\rho\}
\end{aligned}$$

$$\begin{aligned}
&\mathcal{P}_{BT}[\![c?a : A \xrightarrow{0} P_a]\!]\rho \mathrel{\widehat{=}} \\
&\quad \mathrm{fillout}(\{(\tau, [0, \tau] \otimes \langle \{\!| c?a |\!\}, \{\!|\}\!\rangle, \prec\!\succ) \mapsto 1 \mid \tau \in \mathit{TIME}\} \\
&\qquad \cup \{(\tau, [0, t] \otimes \langle \{\!| c?a |\!\}, \{\!|\}\!\rangle^\frown \sqsubseteq + t, (t, c?\hat{a})^\frown s + t) \mapsto \mathcal{P}_{BT}[\![P_{\hat{a}}]\!]\rho(\tau - t, \{0\} \otimes \langle \{\!|\}\!\rangle^\frown \sqsubseteq, s) \mid \\
&\qquad\quad \hat{a} \in A \wedge t \leqslant \tau\})
\end{aligned}$$

$$\begin{aligned}
&\mathcal{A}_{BT}[\![P \,{}^{X}\!\!\not\Vdash^{Y}\, Q]\!]\rho \mathrel{\widehat{=}} \\
&\quad \{(\tau, \sqsubseteq_P \,{}^{X}\!\!\not\Vdash^{Y}\, \sqsubseteq_Q, s) \mid (\tau, \sqsubseteq_P, s \downharpoonright X) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau, \sqsubseteq_Q, s \downharpoonright Y) \in \mathcal{A}_{BT}[\![Q]\!]\rho \wedge \Sigma s \subseteq X \cup Y\}
\end{aligned}$$

$$\begin{aligned}
&\mathcal{P}_{BT}[\![P \,{}^{X}\!\!\not\Vdash^{Y}\, Q]\!]\rho \mathrel{\widehat{=}} \\
&\quad \mathrm{fillout}\left\{(\tau, \sqsubseteq, s) \mapsto \sum \left\{ \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq_P, s \downharpoonright X).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau, \sqsubseteq_Q, s \downharpoonright Y) \mid \sqsubseteq = \sqsubseteq_P \,{}^{X}\!\!\not\Vdash^{Y}\, \sqsubseteq_Q \right\} \,\middle|\, \right. \\
&\qquad \left. \Sigma s \subseteq X \cup Y \right\}
\end{aligned}$$

$$\begin{aligned}
&\mathcal{A}_{BT}[\![P \leftVdash Q]\!]\rho \mathrel{\widehat{=}} \{(\tau, \sqsubseteq_P \leftVdash \sqsubseteq_Q, s) \mid (\tau, \sqsubseteq_P, \Uparrow_{\sqsubseteq_P, \sqsubseteq_Q} s) \in \mathcal{A}_{BT}[\![P]\!]\rho \\
&\qquad\qquad \wedge (\tau, \sqsubseteq_Q, s - \Uparrow_{\sqsubseteq_P, \sqsubseteq_Q} s) \in \mathcal{A}_{BT}[\![Q]\!]\rho\}
\end{aligned}$$

$$\begin{aligned}
&\mathcal{P}_{BT}[\![P \leftVdash Q]\!]\rho(\tau, \sqsubseteq, s) \mathrel{\widehat{=}} \\
&\quad \sum \left\{ \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq_P, \Uparrow_{\sqsubseteq_P, \sqsubseteq_Q} s).\mathcal{P}_{BT}[\![Q]\!]\rho(\tau, \sqsubseteq_Q, s - \Uparrow_{\sqsubseteq_P, \sqsubseteq_Q} s) \mid \sqsubseteq_P \leftVdash \sqsubseteq_Q = \sqsubseteq \right\}
\end{aligned}$$

$$\mathcal{A}_{BT}[\![P \setminus X]\!]\rho \mathrel{\widehat{=}} \{(\tau, \sqsubseteq, s) \mid \forall\, t \bullet s \uparrow t \in \mathrm{items} \sqsubseteq \wedge \exists \sqsubseteq' \bullet \sqsubseteq' \setminus X = \sqsubseteq \wedge (\tau, \sqsubseteq', \Uparrow_{\sqsubseteq'}^{\setminus X} s) \in \mathcal{A}_{BT}[\![P]\!]\rho\}$$

$$\mathcal{P}_{BT}[\![P \setminus X]\!]\rho \mathrel{\widehat{=}} \mathrm{fillout}\{(\tau, \sqsubseteq, s) \mapsto \sum \left\{ \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq', \Uparrow_{\sqsubseteq'}^{\setminus X} s) \mid \sqsubseteq' \setminus X = \sqsubseteq \right\} \mid \forall\, t \bullet s \uparrow t \in \mathrm{items} \sqsubseteq\}$$

$$\mathcal{A}_{BT}[\![gP]\!]\rho \mathrel{\widehat{=}} \{(\tau, \sqsubseteq, s) \mid \forall\, t \bullet s \uparrow t \in \mathrm{items} \sqsubseteq \wedge \exists \sqsubseteq' \bullet g \odot \sqsubseteq' = \sqsubseteq \wedge (\tau, \sqsubseteq', \Uparrow_{\sqsubseteq'}^{g} s) \in \mathcal{A}_{BT}[\![P]\!]\rho\}$$

$$\mathcal{P}_{BT}[\![gP]\!]\rho \mathrel{\widehat{=}} \mathrm{fillout}\{(\tau, \sqsubseteq, s) \mapsto \sum \left\{ \mathcal{P}_{BT}[\![P]\!]\rho(\tau, \sqsubseteq', \Uparrow_{\sqsubseteq'}^{g} s) \mid g \odot \sqsubseteq' = \sqsubseteq \right\} \mid \forall\, t \bullet s \uparrow t \in \mathrm{items} \sqsubseteq\}$$

$$\begin{aligned}
&\mathcal{A}_{BT}[\![P \,{}^{t}_{t}\, Q]\!]\rho \mathrel{\widehat{=}} \{(\tau, \sqsubseteq, s) \mid \tau \leqslant t \wedge (\tau, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho\} \\
&\qquad \cup \{(\tau, \sqsubseteq^\frown (t, \tau] \otimes \langle \{\!|\}\!\rangle, s^\frown \prec\!\succ) \mid t < \tau < t + \delta \wedge (t, \sqsubseteq, s) \in \mathcal{A}_{BT}[\![P]\!]\rho\} \\
&\qquad \cup \{(\tau, \sqsubseteq_P{}^\frown (t, t + \delta) \otimes \langle \{\!|\}\!\rangle^\frown \sqsubseteq_Q + t + \delta, s_P{}^\frown \prec\!\succ{}^\frown s_Q + t + \delta) \mid \\
&\qquad\quad \tau \geqslant t + \delta \wedge (t, \sqsubseteq_P, s_P) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau - t - \delta, \sqsubseteq_Q, s_Q) \in \mathcal{A}_{BT}[\![Q]\!]\rho\}
\end{aligned}$$

$$\mathcal{P}_{BT}[\![P \underset{t}{\nwarrow} Q]\!]\rho(\tau,\sqsubseteq,s) \mathrel{\widehat{=}}$$

$$\begin{cases} \mathcal{P}_{BT}[\![P]\!]\rho(\tau,\sqsubseteq,s) & \text{if } \tau \leqslant t \\ \mathcal{P}_{BT}[\![P]\!]\rho((\tau,\sqsubseteq,s) \upharpoonright t) & \text{if } t < \tau < t+\delta \wedge s \nwarrow t = \prec\!\succ \\ & \quad \wedge \sqsubseteq \nwarrow t = (t,\tau] \otimes \langle\{|\}\rangle \\ \mathcal{P}_{BT}[\![P]\!]\rho((\tau,\sqsubseteq,s) \upharpoonright t).\mathcal{P}_{BT}[\![Q]\!]\rho((\tau,\sqsubseteq,s) - t - \delta) & \text{if } \tau \geqslant t \wedge s \uparrow (t,t+\delta) = \prec\!\succ \\ & \quad \wedge \sqsubseteq \uparrow (t,t+\delta) = (t,t+\delta) \otimes \langle\{|\}\rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{A}_{BT}[\![P \underset{i}{\triangledown} Q]\!]\rho \mathrel{\widehat{=}}$$

$$\begin{array}{l} \{(\tau,\sqsubseteq \oplus i,s) \mid (\tau,\sqsubseteq,s) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge i \notin \Sigma s\} \\ \cup\{(\tau,\sqsubseteq \oplus i^\frown(t,\tau] \otimes \langle\{|\}\rangle, s^\frown\!\prec\!(t,i)\!\succ) \mid t \leqslant \tau < t+\delta \wedge i \notin \Sigma s \wedge (t,\sqsubseteq,s) \in \mathcal{A}_{BT}[\![P]\!]\rho\} \\ \cup\{(\tau,\sqsubseteq_P \oplus i^\frown(t,t+\delta) \otimes \langle\{|\}\rangle^\frown\sqsubseteq_Q + t + \delta, s_P{}^\frown\!\prec\!(t,i)\!\succ^\frown s_Q + t + \delta) \mid \\ \qquad \tau \geqslant t+\delta \wedge i \notin \Sigma s_P \wedge (t,\sqsubseteq_P,s_P) \in \mathcal{A}_{BT}[\![P]\!]\rho \wedge (\tau - t - \delta,\sqsubseteq_Q,s_Q) \in \mathcal{A}_{BT}[\![Q]\!]\rho\} \end{array}$$

$$\mathcal{P}_{BT}[\![P \underset{i}{\triangledown} Q]\!]\rho(\tau,\sqsubseteq,s) \mathrel{\widehat{=}}$$

$$\begin{cases} \mathcal{P}_{BT}[\![P]\!]\rho(\tau,\sqsubseteq_P,s) & \text{if } \sqsubseteq = \sqsubseteq_P \oplus i \wedge i \notin \Sigma s \\ \mathcal{P}_{BT}[\![P]\!]\rho(t,\sqsubseteq_P,s_P) & \text{if } t \leqslant \tau < t \wedge s = s_P{}^\frown\!\prec\!(t,i)\!\succ \wedge i \notin \Sigma s_P \\ & \quad \wedge \sqsubseteq = \sqsubseteq_P \oplus i^\frown(t,\tau) \otimes \langle\{|\}\rangle \\ \mathcal{P}_{BT}[\![P]\!]\rho(t,\sqsubseteq_P,s_P).\mathcal{P}_{BT}[\![Q]\!]\rho((\tau,\sqsubseteq,s) - t - \delta) & \text{if } \tau \geqslant t+\delta \wedge i \notin \Sigma s_P \\ & \quad \wedge s \nwarrow t+\delta = s_P{}^\frown\!\prec\!(t,i)\!\succ \\ & \quad \wedge \sqsubseteq \nwarrow t+\delta = \sqsubseteq_P \oplus i^\frown(t,t+\delta) \otimes \langle\{|\}\rangle \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{F}_{BT}[\![\,\mu X \bullet P]\!]\rho \mathrel{\widehat{=}}$ the unique fixed point of the mapping $M_\delta(X,P)\rho$

$\mathcal{F}_{BT}[\![\,\mu X \circ P]\!]\rho \mathrel{\widehat{=}}$ the unique fixed point of the mapping $M(X,P)\rho$

$\mathcal{F}_{BT}[\![\langle X_i = P_i \mid i \in I\rangle_j]\!]\rho \mathrel{\widehat{=}} S_j$ where $\underline{S}$ is a fixed point of $M(\underline{X},\underline{P})\rho$

B.4 Derived operators

$$\begin{array}{rcl} SKIP & \mathrel{\widehat{=}} & WAIT\ 0 \\ a \xrightarrow{t} P & \mathrel{\widehat{=}} & a \xrightarrow{0} WAIT\ t\,;P \\ P \,\square\, Q & \mathrel{\widehat{=}} & Q \,\square\, P \\ P\ {}_p\Box_q\ Q & \mathrel{\widehat{=}} & (P \,\square\, Q)\ {}_p\sqcap_q\ (P \,\square\, Q) \\ P \nparallel Q & \mathrel{\widehat{=}} & P\ {}^{\Sigma}\!\nparallel^{\Sigma Q} \\ P \nparallel\!\!\nparallel Q & \mathrel{\widehat{=}} & Q\ {}^{\Sigma}\!\nparallel^{\Sigma P} \\ P\ {}^{X}\!\nparallel\!\!\nparallel^{Y}\ Q & \mathrel{\widehat{=}} & Q\ {}^{Y}\!\nparallel^{X}\ P \\ P \nparallel\!\!\nparallel\!\!\rightarrow Q & \mathrel{\widehat{=}} & Q \leftarrow\!\!\nparallel\!\!\nparallel P \\ P \overset{t}{\triangleright} Q & \mathrel{\widehat{=}} & (P \,\square\, WAIT\ t\,;trig \longrightarrow Q) \setminus trig \end{array}$$

and

$$P \underset{C}{\nparallel} Q \mathrel{\widehat{=}} c(l(P)\ {}^{A}\!\nparallel^{B}\ r(Q)) \qquad P \underset{C}{\nparallel\!\!\nparallel} Q \mathrel{\widehat{=}} c(l(P)\ {}^{A}\!\nparallel\!\!\nparallel^{B}\ r(Q))$$

$$\text{where } l(a) \mathrel{\widehat{=}} \begin{cases} a & \text{if } a \in C \\ l.a & \text{otherwise} \end{cases} \qquad r(a) \mathrel{\widehat{=}} \begin{cases} a & \text{if } a \in C \\ r.a & \text{otherwise} \end{cases}$$

$$\begin{array}{rl} c(a) \mathrel{\widehat{=}} a & \text{if } a \in C \\ c(l.a) \mathrel{\widehat{=}} a & \text{if } a \notin C \\ c(r.a) \mathrel{\widehat{=}} a & \text{if } a \notin C \end{array}$$

and $A \mathrel{\widehat{=}} l(\Sigma - C) \cup C \qquad B \mathrel{\widehat{=}} r(\Sigma - C) \cup C \qquad$ and $l(\Sigma) \cap C = r(\Sigma) \cap C = \{\}$