Logic and Proof Hilary 2024

Exercise Sheet 3

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1. Write down formulas in first-order logic with equality expressing the following requirements.

(a) Unary function f is an onto function.

(b) Binary relation R is an equivalence relation.

(c) Unary predicate P holds for exactly three different elements.

2. Consider the following structures over a signature with a single binary relation symbol R:

$$U_{\mathcal{A}} = \mathbb{N} \text{ and } R_{\mathcal{A}} = \{(n,m) \in \mathbb{N} \times \mathbb{N} : n < m\}$$
$$U_{\mathcal{B}} = \mathbb{Z} \text{ and } R_{\mathcal{B}} = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} : n < m\}$$
$$U_{\mathcal{C}} = \mathbb{Q} \text{ and } R_{\mathcal{C}} = \{(n,m) \in \mathbb{Q} \times \mathbb{Q} : n < m\}$$

Give a formula that is satisfied by $\mathcal{B}$ but not by $\mathcal{A}$, and a formula that is satisfied by $\mathcal{C}$ but not by $\mathcal{B}$.

3. Translate the following formula to rectified form, then to prenex form, and finally to Skolem form:

$$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg \forall x\, P(x)) \wedge \neg \forall z\, \exists x\, \neg R(f(x,z), z)\,.$$

4. Are the following claims correct? Justify your answers.

(a) For any formula F and term t, if F is valid then $F[t/x]$ is valid.

(b) $\exists x\, (P(x) \rightarrow \forall y\, P(y))$ is valid.

(c) For any formula F and constant symbol c, if $F[c/x]$ is valid and c does not appear in F then $\forall x\, F$ is valid.

5. Let σ be a signature with finitely many relation and constant symbols, but no function symbols.

(a) Given σ-formulas $G_1, \ldots, G_n$ and a propositional formula F that mentions variables $P_1, \ldots, P_n$, let $F[G_1/P_1, \ldots, G_n/P_n]$ denote the σ-formula obtained by substituting G_i for all occurrences of P_i in F. Give a formal definition of $F[G_1/P_1, \ldots, G_n/P_n]$.

(b) Given propositional formulas $F \equiv F'$, both over variables $P_1, \ldots, P_n$, and σ-formulas $G_1 \equiv G'_1, \ldots, G_n \equiv G'_n$, show that $F[G_1/P_1, \ldots, G_n/P_n] \equiv F'[G'_1/P_1, \ldots, G'_n/P_n]$.

(c) Fix $n \in \mathbb{N}$. Show that up to logical equivalence there are only finitely many quantifier-free σ-formulas that use first-order variables $x_1, \ldots, x_n$.

(d) Fix $n, k \in \mathbb{N}$. Show that up to logical equivalence there are only finitely many σ-formulas of quantifier depth at most k that use first-order variables $x_1, \ldots, x_n$.

6. Fix a signature σ. Consider a relation $\sim$ on σ-assignments that satisfies the following two properties:

(P1) If $\mathcal{A} \sim \mathcal{B}$ then for every atomic formula F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.

(P2) If $\mathcal{A} \sim \mathcal{B}$ then for each variable x we have (i) for each $a \in U_{\mathcal{A}}$ there exists $b \in U_{\mathcal{B}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$, and (ii) for all $b \in U_{\mathcal{B}}$ there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$.

Prove that if $\mathcal{A} \sim \mathcal{B}$ then for any formula F, $\mathcal{A} \models F$ if and only if $\mathcal{B} \models F$. You may assume that F is built from atomic formulas using the connectives $\wedge$ and $\neg$ and the quantifier $\exists$.

7. In this question we work with first-order logic without equality.

(a) Consider a signature σ containing only a binary relation symbol R. For each positive integer n show that there is a satisfiable σ-formula F_n such that every model $\mathcal{A}$ of F_n has at least n elements.

(b) Consider a signature σ containing only unary predicate symbols $P_1, \ldots, P_k$. Using Question 6, or otherwise, show that any satisfiable σ-formula has a model with at most 2^k elements.