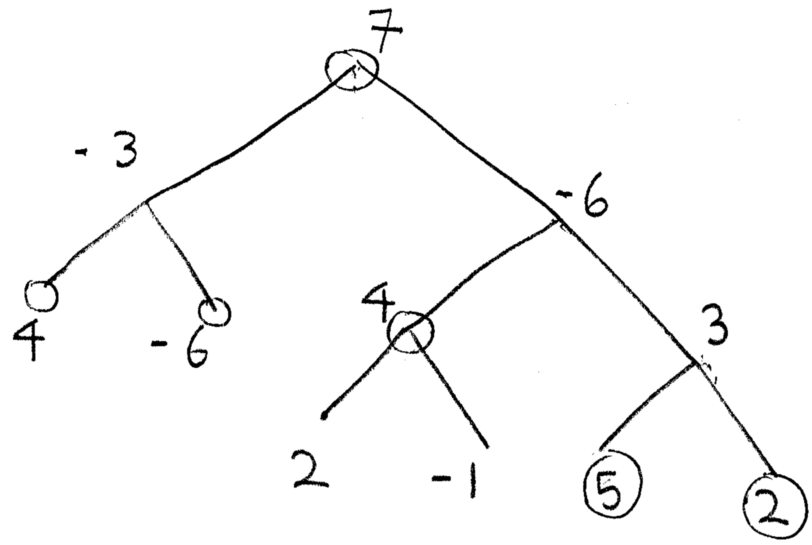
Richard Bird. WG21, Blackheath, April 2000 (1)




```
atig (Leaf (n,b)) = True
atig (Fork (n,b) x y) =
        if b then btig x ∧ btig y
             else atig x ∧ atig y
btig (Leaf (n,b)) = ¬b
btig (Fork (n,b) x y) =
        ¬b ∧ atig x ∧ atig y
```

```
abtig x = (atig x, btig x)

abtig = foldTree base step
base (n,b) = (True, ¬b)
step (n,b) (a1,b1) (a2,b2)
    = ( b ∧ b1 ∧ b2 ∨ ¬b ∧ a1 ∧ a2,
                        ¬b ∧ a1 ∧ a2 )
atig = fst · abtig
```

$mmp : T\,Int \rightarrow Int$

$mmp = max(\leq) \cdot \wedge(value \cdot sat\,p \cdot T\,mark)$

1) $mark : Int \xrightarrow{m} Int \times Bool$

$mark\ n = (n, True) \,\square\, (n, False)$

2) $T : (X \xrightarrow{m} Y) \rightarrow (TX \xrightarrow{m} TY)$

– defined later

3) $sat : (X \rightarrow Bool) \rightarrow (X \xrightarrow{m} X)$

$sat\ p = ok \cdot \langle id, p \rangle$

$ok\,(n, True) = n$

4) $\text{value} : T(\text{Int} \times \text{Bool}) \to \text{Int}$

$\text{value} = \text{sum} \cdot T\,\text{val}$

$\text{val}\,(n, b) = \underline{\text{if}}\; b\; \underline{\text{then}}\; n\; \underline{\text{else}}\; 0$

$\text{sum} : T\,\text{Int} \to \text{Int}$

(defined later)

5) $\Lambda : (X \xrightarrow{m} Y) \to (X \to \text{Set}\,Y)$

$\Lambda f x = \{ y \mid y \leftarrow f x \}$

6) $\text{max} : (X \to X \to \text{Bool}) \to (\text{Set}\,X \xrightarrow{m} X)$

$x \leftarrow \text{max}\,(\trianglelefteq)\, xs$ if

$x \in xs \wedge (\forall y \in xs : y \trianglelefteq x)$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
F(A, TA) \arrow[r, bend right, "in"'] \arrow[d, "F(id,\, \mathrm{fold}\, f)"', "m"] & TA \arrow[l, bend right, "out"'] \arrow[d, "\mathrm{fold}\, f", "m"'] \\
F(A,B) \arrow[r, "m", "f"'] & B
\end{tikzcd}$$

$$id = \mathrm{fold}\ in$$

$$Tf = \mathrm{fold}\,(in \cdot F(f, id))$$

Functor fusion

$$\mathrm{fold}\, f \cdot Tg = \mathrm{fold}\,(f \cdot F(g, id))$$

p mutumorphism

$$p = prop \cdot \mathrm{fold} \langle p_1, \ldots, p_k \rangle$$

$$p_i : F(\mathrm{Int} \times \mathrm{Bool},\ \mathrm{Bool}^k) \to \mathrm{Bool}$$

$$prop : \mathrm{Bool}^k \to \mathrm{Bool}$$

$$sum = \mathrm{fold}\ plus$$

$$plus : F(\mathrm{Int}, \mathrm{Int}) \to \mathrm{Int}$$

$$
\begin{aligned}
& \mathit{value} \cdot \mathit{sat}\ p \\
= \quad & \{\text{above}\} \\
& \mathit{value} \cdot \mathit{ok} \cdot \langle id, p \rangle \\
= \quad & \{\text{claim}\} \\
& \mathit{ok} \cdot \langle \mathit{value}, p \rangle \\
= \quad & \{\text{defn}\} \\
& \mathit{ok} \cdot \langle \mathit{fold}\ \mathit{plus} \cdot T\mathit{val}, p \rangle \\
= \quad & \{\text{functor fusion}\} \\
& \mathit{ok} \cdot \langle \mathit{fold}\ (\mathit{plus} \cdot F(\mathit{val}, id)), \mathit{prop} \cdot \mathit{fold} \langle p_1 \cdots p_k \rangle \rangle \\
= \quad & \{ op = ok \cdot (id \times prop) \} \\
& op \cdot \langle \mathit{fold}\ (\mathit{plus} \cdot F(\mathit{val}, id)), \mathit{fold} \langle p_1 \cdots p_k \rangle \rangle \\
= \quad & \{\text{banana-split}\} \\
& op \cdot \mathit{fold}\ f
\end{aligned}
$$

where $f = \langle \mathit{plus} \cdot F(\mathit{val}, \mathit{fst}), \langle p_1 \cdots p_k \rangle \cdot F(id, \mathit{snd}) \rangle$

$$\begin{aligned}
& value \cdot sat\ p \cdot T\ mark \\
= \quad & \{\text{above}\} \\
& op \cdot fold\ f \cdot T\ mark \\
= \quad & \{\text{functor fusion}\} \\
& op \cdot fold\ g
\end{aligned}$$

where $g = f \cdot F(mark, id)$

Hence

$$\begin{aligned}
& max\,(\leq) \cdot \Lambda\,(value \cdot sat\ p \cdot T\ mark) \\
= \quad & \{\text{above}\} \\
& max\,(\leq) \cdot \Lambda\,(op \cdot fold\ g) \\
= \quad & \{\text{claim}\} \\
& op \cdot \underline{\underline{max\,(\leq') \cdot \Lambda(fold\ g)}}
\end{aligned}$$

where $(m, bs) \leq' (n, cs)$

$\hat{=} \quad prop\ cs \wedge (\neg\, prop\ bs \vee m \leq n)$

correction!

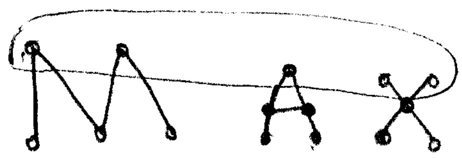
$(m, bs) \leq' (n, cs)$

$\equiv \quad (prop\ bs, m) \leq_L (prop\ cs, n)$

Thinning – a method for solving $\max(\leq) \cdot \Lambda\, \mathrm{fold}\, f$

$$\mathrm{thin} : (X \to X \to \mathrm{Bool}) \to (\mathrm{Set}\, X \xrightarrow{m} \mathrm{Set}\, X)$$

$$ys \leftarrow \mathrm{thin}\, (\trianglelefteq)\, xs \quad \text{if}$$
$$ys \subseteq xs \;\wedge$$
$$(\forall x \in xs : \exists y \in ys : x \trianglelefteq y)$$



Theorem

$$\max(\leq) \cdot \Lambda\, \mathrm{fold}\, f \supseteq \max(\leq) \cdot \mathrm{fold}\, g$$

where $g : F(X, \mathrm{Set}\, Y) \xrightarrow{m} \mathrm{Set}\, Y$

$$g = \mathrm{thin}\, (\trianglelefteq) \cdot \Lambda(f \cdot F(id, \mathrm{choose}))$$

provided

(i) $x \trianglelefteq y \Rightarrow x \leq y$

(ii) $f : F(X, Y) \xrightarrow{m} Y$ is monotonic under $\trianglelefteq$

$f : F(X, Y) \xrightarrow{m} Y$ monotonic

under $(\trianglelefteq) : Y \to Y \to Bool$

if $x \trianglelefteq_F y \wedge u \leftarrow f x$

then $(\exists v : v \leftarrow f y : u \trianglelefteq v)$

$(\trianglelefteq_F) : F(X, Y) \to F(X, Y) \to Bool$

For our problem:

$f = \langle plus \cdot F(val \cdot mark, fst), \langle P_1 \cdots P_k \rangle \cdot F(mark, snd) \rangle$

and it is monotonic under $\trianglelefteq$ where

$(m, bs) \trianglelefteq (n, cs)$

$\hat{=} \; (m \leq n \wedge bs = cs)$

$(m, bs) \trianglelefteq (n, cs)$

$\Rightarrow (prop\ bs, m) \leq_L (prop\ cs, n)$

In thinning wrt $\trianglelefteq$, we need keep no more than 2^k partial solutions.

Where is the program?

$$mmp = maxlist\ r \cdot fold\ (thinlist\ q \cdot extend)$$

$$extend = concat \cdot map\ lambdaf \cdot cplist$$

$$cplist : F(A, List\ B) \longrightarrow List\ (F(A, B))$$

$$cplist \subseteq listify \cdot \Lambda(F(id, choose))$$

$$lambdaf : F(Int, Int \times Bool^k) \rightarrow List\ (Int \times Bool^k)$$

$$lambdaf \subseteq listify \cdot \Lambda f$$

plug in F, $plus$, $P_1 \cdots P_k$,

and play!

$\boxed{\text{Isao Sasano et al}}$