Some Applications of Modal Semirings

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1 Survey

- aim: give algebraic semantics to some modal logics
- such as multiagent common knowledge logic
- and preference logic
- apply the algebra to some examples

2 The Wise Men Puzzle

wise men puzzle:

- a king wants to test the wisdom of his three wise men
- they have to sit on three chairs behind each other, all facing the same direction
- the king puts a hat on each head, either red or black
- he announces that at least one hat is red
- he asks the wise man in the back if he knows his hat colour
- that one denies
- he asks the middle one who denies, too
- now he says to the front one: “If you are really wise, you should now know the colour of your hat.”

formalisation:

- rules of the puzzle represented as individual knowledge $K_j\varphi$ of man i or common knowledge $C\varphi$ where ψ are certain formulas
- let r_i mean that i's hat is red (numbering in order of questioning, i.e. from back to front)
- every man can only see the hats before him

$$C(r_i \to K_j r_i) \qquad C(\neg r_i \to K_j \neg r_i) \qquad (j < i)$$

- at least one hat is red

$$C(r_1 \vee r_2 \vee r_3)$$

- after the king's questions

$$C(\neg K_i r_i \wedge \neg K_i \neg r_i) \qquad (i = 1, 2)$$

- can we infer anything about $K_3 r_3$ from that?

3 Modelling Knowledge (Epistemic Modal Logic)

Kripke semantics for modal logic:

- set of possible worlds
- predicates characterise subsets of possible worlds
- access relation between worlds
- the worlds accessible from a current world w are called the *epistemic neighbours* of w
- box/diamond act as universal/existential quantifiers over the neighbour worlds
- knowing p means that p holds in all neighbour worlds

- setting: systems with several agents
- each has its own access relation with associated box operator K_i
- now $K_i p$ is interpreted as "agent i knows p"
- corresponding special properties :

$K_i p \leq p$	if i knows p, it's actually true
$K_i p \leq K_i K_i p$	if i knows p, she knows that she knows p, positive introspection
$\neg K_i p \leq K_i \neg K_i p$	negative introspection

4 Algebraic Semantics

abstraction:

- use a test semiring (see appendix for precise definitions)
- tests (= monotoypes = coreflexives) play the role of predicates or sets of worlds
- $0 \leftrightarrow \mathit{false} \leftrightarrow \emptyset$ $\qquad 1 \leftrightarrow \mathit{true} \leftrightarrow$ set of all worlds
- $\leq$ is implication (or subsethood)
- general elements play the role of access relations
- compositions pa and ap of an access element a with a test p mean restriction of a on the input/output side
- hence paq is the part of a that takes p-elements to q-elements

- informal definition of the box operator:
- a world w satisfies $[a]q$ iff all worlds accessible from w via a satisfy (or guarantee) p
- for the algebraic characterisation we lift this to sets of worlds
- all p-worlds satisfy $[a]q$ iff there is no a-connection from p-worlds to $\neg q$-worlds:

$$p \leq [a]q \overset{\text{def}}{\Leftrightarrow} p\,a\,\neg q \leq 0$$

- the diamond is the de Morgan dual of box:

$$\langle a \rangle q \overset{\text{def}}{\Leftrightarrow} \neg[a]\neg q$$

consequences of the definition:

- box is *normal*, i.e.

$$[a]1 = 1 \qquad [a](p \to q) \leq [a](p) \to [a](q)$$

- consequently, box is conjunctive, hence isotone,
- diamond is disjunctive, hence isotone
- box is anti-disjunctive

$$[a + b]p \; = \; [a]p \; \wedge \; [b]p$$

additional axiom for composition:

$$[ab] = [a][b]$$

in a Kleene algebra this entails box star induction:

$$q \leq p \wedge q \leq [a]q \Rightarrow q \leq [a^*]p$$

modelling common knowledge:

- assume agents $i \in I = \{1, \ldots, n\}$
- *agent group* $G \subseteq I$
- two operators for expressing common knowledge:
- $E_G p$: everyone in group G knows p
- $C_G p$: everyone knows that everyone knows that ...

formal definition: exploit the algebra of modal operators

- for $G = \{k_1, \ldots, km\}$,

$$\begin{aligned} \mathsf{E}_G p &= \mathsf{K}_{k_1} p \wedge \cdots \wedge \mathsf{K}_{k_m} p \\ &= [a_{k_1}]p \wedge \cdots \wedge [a_{k_m}]p \\ &= [a_{k_1} + \cdots + a_{k_m}]p \\ &= [a_G]p \end{aligned}$$

where $a_G \stackrel{\text{def}}{=} a_{k_1} + \cdots + a_{k_m}$

for $\mathsf{C_G}$ we obtain

$$\begin{aligned} \mathsf{C_G p} &= \mathsf{E_G p} \wedge \mathsf{E_G E_G p} \wedge \mathsf{E_G E_G E_G p} \cdots \\ &= [a_G]p \wedge [a_G][a_G]p \wedge [a_G][a_G][a_G]p \cdots \\ &= [a_G + a_G^2 + a_G^3 \cdots] \\ &= [a_G^+]p \end{aligned}$$

if the underlying semiring is even a Kleene algebra

in sum we have an algebraic version of the multiagent logic $KT45^n$
(see e.g. [HR04])

using common knowledge:

- implication order

$$a \leq b \overset{\text{def}}{\Leftrightarrow} b = a + b$$

- expresses that b offers at least as much transition possibilities as a
- the addition law entails

$$a \leq b \Rightarrow [b]p \leq [a]p$$

(if more choices are offered, one can guarantee less)

- now, since $a_{k_j} \leq a_G \leq a_G^+$ we get

$$\mathsf{C}_G p \leq \mathsf{E}_G p \leq \mathsf{K}_{k_j} p$$

and

$$\mathsf{C}_G p \leq \mathsf{C}_G \mathsf{K}_{k_j} p$$

5 Solving the Wise Men Puzzle

main reasoning principle: isotony of modal operators M

$$p \leq q \;\Rightarrow\; \mathsf{M}p \leq \mathsf{M}q$$

(remember that $\leq$ means implication)

basic equivalence (shunting)

$$p \leq q \;\Leftrightarrow\; 1 \leq p \rightarrow q$$

repetition of the knowledge assertions

$$\mathsf{C}(\mathsf{r_i} \to \mathsf{K_j r_i}) \qquad \mathsf{C}(\neg \mathsf{r_i} \to \mathsf{K_j} \neg \mathsf{r_i}) \qquad (j < i)$$

$$\mathsf{C}(\mathsf{r_1} \vee \mathsf{r_2} \vee \mathsf{r_3})$$

$$\mathsf{C}(\neg \mathsf{K_i r_i} \wedge \neg \mathsf{K_i} \neg \mathsf{r_i}) \qquad (i = 1, 2)$$

before using isotony we take the contrapositives of the first two clauses to have simple literals right of $\to$ and rewrite the third into an implication (fourth unchanged):

$$\mathsf{C}(\neg \mathsf{K_j r_i} \to \neg \mathsf{r_i}) \qquad (1)$$

$$\mathsf{C}(\neg \mathsf{K_j} \neg \mathsf{r_i} \to \mathsf{r_i}) \qquad (2)$$

$$\mathsf{C}(\neg \mathsf{r_2} \wedge \neg \mathsf{r_3} \to \mathsf{r_1}) \qquad (3)$$

$$C(\neg K_j r_i \rightarrow \neg r_i) \qquad (1)$$

$$C(\neg K_j \neg r_i \rightarrow r_i) \qquad (2)$$

$$C(\neg r_2 \wedge \neg r_3) \rightarrow r_1) \qquad (3)$$

now we reason as follows:

$$
\begin{array}{lll}
 & K_1((\neg r_2 \wedge \neg r_3) \rightarrow r_1) & \\
\leq & K_1(\neg r_2 \wedge \neg r_3) \rightarrow K_1 r_1 & \text{normality} \\
= & \neg K_1 r_1 \rightarrow \neg K_1(\neg r_2 \wedge \neg r_3) & \text{contraposition} \\
= & \neg K_1 r_1 \rightarrow (\neg K_1 \neg r_2 \vee \neg K_1 \neg r_3) & \text{conjunctivity, de Morgan} \\
\leq & \neg K_1 r_1 \rightarrow (r_2 \vee r_3) & \text{by (2)}
\end{array}
$$

hence

$$
\begin{array}{lll}
 & C(r_1 \vee r_2 \vee r_3) \wedge C(\neg K_1 r_1) & \\
\leq & CK_1(r_1 \vee r_2 \vee r_3) \wedge C(\neg K_1 r_1) & \text{use of common knowledge} \\
\leq & C(\neg K_1 r_1 \rightarrow (r_2 \vee r_3)) \wedge C(\neg K_1 r_1) & \text{previous derivation} \\
= & C(r_2 \vee r_3) & \text{normality, modus ponens}
\end{array}
$$

analogously,

$$C(r_2 \vee r_3) \wedge C(\neg K_2 r_2) \leq C(r_3) \leq K_3(r_3)$$

and we are done

generalised form of the argument: for agent groups G and $H \subseteq G$,

$$C(\bigvee_{j \in G} r_j) \wedge C(\bigwedge_{i \in H} \neg K_i r_i) \wedge C(\bigwedge_{i \in H} \bigwedge_{j \in G - H} r_j \rightarrow K_i r_j) \;\leq\; C(\bigvee_{j \in G - H} r_j)$$

puzzles with a similar structure that should allow re-use of the general result:

- muddy children
- unexpected hangman's paradox
- Mr. S and Mr. P

6 Preferences and Their Upgrade

some agent logics allow expressing preferences between possible worlds, e.g. [BL04]

- since we are completely free in choosing our accessibility elements, we can also include these
- each agent i has her own preference relation $\preceq_i$
- then $[\preceq_i]p$ holds in a world w iff p holds in all worlds that agent i prefers to w under $\preceq_i$
- requirements on $\preceq_i$: preorder, modally expressed by

$$[\preceq_i]p \leq p \qquad \text{reflexivity}$$

$$[\preceq_i]p \leq [\preceq_i][\preceq_i]p \qquad \text{transitivity}$$

- antisymmetry is not required: agent i is *indifferent* about w_1 and w_2 if $w_1 \preceq_i w_2 \wedge w_2 \preceq_i w_1$

some things that can be modelled that way:

- regret: $K_i \neg p \wedge \langle \preceq_i \rangle p$

  although agent i knows her wish p cannot be satisfied, she'd still prefer a world where it could
- the agent system can be updated in various ways
- in *belief revision* agents may discard or add links to epistemic neighbour worlds
- e.g., *public announcement* of property p, denoted $!p$:

  make sure that all agents now know p
- to this end, remove all links between p and $\neg p$ worlds:

$$a_i !p = p\, a_i\, p + \neg p\, a_i\, \neg p$$

- preference upgrade by *suggesting* that p be observed:

$$p \# \preceq_i \stackrel{\text{def}}{=} p \preceq_i p \cup \neg p \preceq_i$$

  now agent a_i no longer prefers $\neg p$ worlds over p ones and so on — the field is vast...

References

[BL04] J. van Benthem, F. Liu: Dynamic logic of preference upgrade. Manuscript 2004. To appear in J. Applied Non-Classical Logics 2006

[HR04] M. Huth, M. Ryan: Logic in Computer Science — Modelling and Reasoning About Systems, 2nd Edition. Cambridge University Press 2004

Appendix: Algebraic Background

Definition 6.1 *semiring*: structure $(S, +, \cdot, 0, 1)$ such that

- $(S, +, 0)$ is a commutative monoid
- $(S, \cdot, 1)$ is a monoid
- the distributive laws hold
- 0 is an annihilator: $0 \cdot a = 0 = a \cdot 0$

if S is idempotent, i.e., $x + x = x$, the relation $a \leq b \overset{\text{def}}{\Leftrightarrow} a + b = b$ is a partial order, the *natural* order

test: element $p \leq 1$ that has a complement $\neg p$ relative to 1

interpretation:

$+$	$\leftrightarrow$	choice,
$\cdot$	$\leftrightarrow$	sequential composition
0	$\leftrightarrow$	empty set of choices
1	$\leftrightarrow$	identity
$\leq$	$\leftrightarrow$	increase in information or in choices
test:	$\leftrightarrow$	assertion/predicate

Kleene star and plus can be added with the usual axioms