Calculating Circular Programs

Alberto Pardo
Instituto de Computación, Universidad de la República, Uruguay

joint work with

João Fernandes and João Saraiva
Departamento de Informática, Universidade do Minho, Portugal

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Bird's *repmin*

$$\textbf{data}\ \mathit{Tree} = \mathit{Leaf}\ \mathit{Int} \mid \mathit{Join}\ \mathit{Tree}\ \mathit{Tree}$$

$$\begin{array}{l}
\mathit{transform} :: \mathit{Tree} \rightarrow \mathit{Tree} \\
\mathit{transform}\ t = \mathit{replace}\ t\ (\mathit{tmin}\ t)
\end{array}$$

$$\begin{array}{l}
\mathit{replace} :: \mathit{Tree} \rightarrow \mathit{Int} \rightarrow \mathit{Tree} \\
\mathit{replace}\ (\mathit{Leaf}\ n)\ m = \mathit{Leaf}\ m \\
\mathit{replace}\ (\mathit{Join}\ l\ r)\ m = \mathit{Join}\ (\mathit{replace}\ l\ m)\ (\mathit{replace}\ r\ m)
\end{array}$$

$$\begin{array}{l}
\mathit{tmint} :: \mathit{Tree} \rightarrow \mathit{Int} \\
\mathit{tmint}\ (\mathit{Leaf}\ n) = n \\
\mathit{tmint}\ (\mathit{Join}\ l\ r) = \mathit{min}\ (\mathit{tmin}\ l)\ (\mathit{tmin}\ r)
\end{array}$$

Derivation of single-pass definition

$$repmin\ t\ m = (replace\ t\ m, tmin\ t)$$

$$\Downarrow$$

$$\begin{aligned}
&repmin\ (Leaf\ n)\ m = (Leaf\ m, n)\\
&repmin\ (Join\ l\ r)\ m = (Join\ l'\ r', min\ ml\ mr)\\
&\qquad\qquad\qquad\quad \textbf{where}\ (l', ml) = repmin\ l\ m\\
&\qquad\qquad\qquad\qquad\qquad (r', mr) = repmin\ r\ m
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
&transform\ t = t'\\
&\quad \textbf{where}\ (t', m) = repmin\ t\ m
\end{aligned}$$

Problem reformulation

$$\textbf{data}\ Tree = Leaf\ Int \mid Join\ Tree\ Tree$$

$$\textbf{data}\ STree = SLeaf \mid SJoin\ STree\ STree$$

$$\begin{aligned}
&transform :: Tree \rightarrow Tree\\
&transform\ t = replace\ (shapeMin\ t)
\end{aligned}$$

$$\begin{aligned}
&replace :: STree\ \times\ Int \rightarrow Tree\\
&replace\ SLeaf\ m = Leaf\ m\\
&replace\ (SJoin\ l\ r)\ m = Join\ (replace\ l\ m)\ (replace\ r\ m)
\end{aligned}$$

$$\begin{aligned}
&shapeMin :: Tree \rightarrow STree\ \times\ Int\\
&shapeMin\ (Leaf\ n) = (SLeaf, n)\\
&shapeMin\ (Join\ l\ r) = (SJoin\ l'\ r', min\ ml\ mr)\\
&\qquad\qquad\textbf{where}\ (l', ml) = shapeMin\ l\\
&\qquad\qquad\qquad\quad\ (r', mr) = shapeMin\ r
\end{aligned}$$

Our transformation

$$transform\ t = replace\ (shapeMin\ t)$$

$$\Downarrow$$

$$shapeMin = g\ (SLeaf, SJoin)$$

$$
\begin{aligned}
& g :: (a, a \to a \to a) \to Tree \to a\ \times\ Int \\
& g\ (sleaf, sjoin)\ (Leaf\ \ n) = (sleaf, n) \\
& g\ (sleaf, sjoin)\ (Join\ l\ r) = (sjoin\ l'\ r', min\ ml\ mr) \\
& \qquad\qquad\qquad \mathbf{where}\ (l', ml) = g\ (sleaf, sjoin)\ l \\
& \qquad\qquad\qquad\qquad\quad\ (r', mr) = g\ (sleaf, sjoin)\ r
\end{aligned}
$$

$$\Downarrow$$

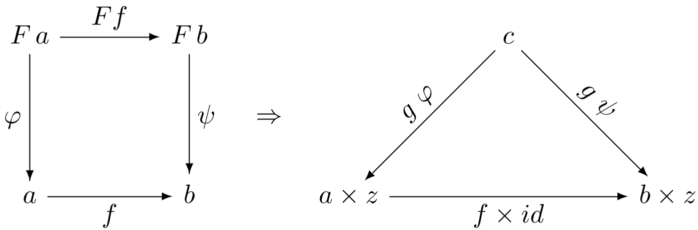
$$
\begin{array}{l}
\mathit{transform}\ t = t' \\
\qquad \textbf{where}\ (t', m) = g\ (\mathit{fleaf}, \mathit{fjoin})\ t \\
\qquad\qquad \mathit{fleaf} = \mathit{Leaf}\ m \\
\qquad\qquad \mathit{fjoin}\ l\ r = \mathit{Join}\ l\ r
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{l}
\mathit{transform}\ t = t' \\
\qquad \textbf{where}\ (t', m) = \mathit{repmin}\ t \\
\qquad\qquad \mathit{repmin}\ (\mathit{Leaf}\ n) = (\mathit{Leaf}\ m, n) \\
\qquad\qquad \mathit{repmin}\ (\mathit{Join}\ l\ r) = (\mathit{Join}\ l'\ r', \mathit{min}\ \mathit{ml}\ \mathit{mr}) \\
\qquad\qquad\quad \textbf{where}\ (l', \mathit{ml}) = \mathit{repmin}\ l \\
\qquad\qquad\qquad\qquad (r', \mathit{mr}) = \mathit{repmin}\ r
\end{array}
$$

Free Theorem

$$g :: \forall\ a\ .\ (F\ a \to a) \to c \to a\ \times\ z$$



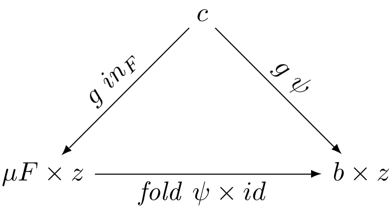
Free Theorem

Taking

$$\varphi = in_F : F\mu F \to \mu F$$
$$f = fold\ \psi : \mu F \to b$$

we obtain

$$(fold\ \psi\ \times\ id) \circ g\ in_F = g\ \psi$$

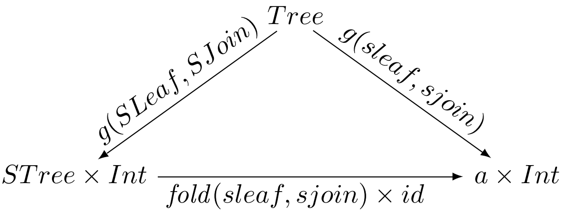


Free Theorem for our g

$$g :: \forall\, a\ .\ (a, a \to a \to a) \to \mathit{Tree} \to a\ \times\ \mathit{Int}$$

$\Rightarrow$

$$(\mathit{fold}\ (\mathit{sleaf}, \mathit{sjoin})\ \times\ \mathit{id}) \circ g\ (\mathit{SLeaf}, \mathit{SJoin}) = g\ (\mathit{sleaf}, \mathit{sjoin})$$



Fold and pfold for STree

$$
\begin{aligned}
&\mathit{fold} :: (a, a \rightarrow a \rightarrow a) \rightarrow \mathit{STree} \rightarrow a \\
&\mathit{fold}\ (\mathit{sleaf}, \mathit{sjoin}) = f \\
&\qquad\qquad \mathbf{where}\ f\ \mathit{SLeaf} = \mathit{sleaf} \\
&\qquad\qquad\qquad\quad f\ (\mathit{SJoin}\ l\ r) = \mathit{sjoin}\ (f\ l)\ (f\ r)
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{pfold} :: (z \rightarrow a, a \rightarrow a \rightarrow z \rightarrow a) \rightarrow \mathit{STree}\ \times\ z \rightarrow a \\
&\mathit{pfold}\ (\mathit{pleaf}, \mathit{pjoin}) = f \\
&\qquad\quad \mathbf{where}\ f\ (\mathit{SLeaf}, z) = \mathit{pleaf}\ z \\
&\qquad\qquad\qquad f\ (\mathit{SJoin}\ l\ r, z) = \mathit{pjoin}\ (f\ (l, z))\ (f\ (r, z))\ z
\end{aligned}
$$

Our case

$$transform\ t = replace\ (shapeMin\ t)$$

$$\Downarrow$$

$$shapeMin = g\ (SLeaf, SJoin)$$

$$g :: (a, a \to a \to a) \to Tree \to a \times Int$$

$$\begin{aligned} &replace :: STree \times Int \to Tree \\ &replace = pfold\ (pleaf, pjoin) \\ &\qquad\qquad \textbf{where}\ pleaf\ m = Leaf\ m \\ &\qquad\qquad\qquad\ \ pjoin\ l\ r\ m = Join\ l\ r \end{aligned}$$

The rule

$$transform = pfold\ (pleaf, pjoin) \circ g\ (SLeaf, SJoin)$$

$$\Downarrow$$

$$\begin{aligned} &transform\ t = t' \\ &\quad \textbf{where}\ (t', m) = g\ (fleaf, fjoin) \\ &\qquad\qquad fleaf = pleaf\ m \\ &\qquad\qquad fjoin\ l\ r = pjoin\ l\ r\ m \end{aligned}$$

Proof

$$transform\ t = pfold\ (pleaf, pjoin)\ (g\ (SLeaf, SJoin)\ t)$$

$$\Downarrow$$

$$transform\ t = pfold\ (pleaf, pjoin) \circ \langle \pi_1 \circ g\ (SLeaf, SJoin), \pi_2 \circ g\ (SLeaf, SJoin) \rangle\ \$\ t$$

$$\Downarrow$$

$$\begin{aligned} transform\ t &= fold\ (fleaf, fjoin) \circ \pi_1 \circ g\ (SLeaf, SJoin)\ \$\ t \\ \textbf{where}\ m &= \pi_2 \circ g\ (SLeaf, SJoin)\ \$\ t \\ fleaf &= pleaf\ m \\ fjoin\ l\ r &= pjoin\ l\ r\ m \end{aligned}$$

$$\Downarrow \qquad \pi_1 \textit{ natural transformation}$$

$$\begin{aligned} &\mathit{transform}\ t = \pi_1 \circ (\mathit{fold}\ (\mathit{fleaf}, \mathit{fjoin})\ \times\ \mathit{id}) \circ g\ (\mathit{SLeaf}, \mathit{SJoin})\ \$\ t \\ &\qquad \textbf{where}\ m = \pi_2 \circ g\ (\mathit{SLeaf}, \mathit{SJoin})\ \$\ t \\ &\qquad\qquad \mathit{fleaf} = \mathit{pleaf}\ m \\ &\qquad\qquad \mathit{fjoin}\ l\ r = \mathit{pjoin}\ l\ r\ m \end{aligned}$$

$$\Downarrow \qquad \textit{free theorem}$$

$$\begin{aligned} &\mathit{transform}\ t = \pi_1 \circ g\ (\mathit{fleaf}, \mathit{fjoin})\ \$\ t \\ &\qquad \textbf{where}\ m = \pi_2 \circ g\ (\mathit{SLeaf}, \mathit{SJoin})\ \$\ t \\ &\qquad\qquad \mathit{fleaf} = \mathit{pleaf}\ m \\ &\qquad\qquad \mathit{fjoin}\ l\ r = \mathit{pjoin}\ l\ r\ m \end{aligned}$$

$$\Downarrow \qquad \pi_2 \circ g\ (\mathit{SLeaf}, \mathit{SJoin}) = \pi_2 \circ g\ (\mathit{fleaf}, \mathit{fjoin})$$

$$
\begin{aligned}
&\mathit{transform}\ t = \pi_1 \circ g\ (\mathit{fleaf}, \mathit{fjoin})\ \$\ t \\
&\qquad \textbf{where}\ m = \pi_2 \circ g\ (\mathit{fleaf}, \mathit{fjoin})\ \$\ t \\
&\qquad\qquad \mathit{fleaf} = \mathit{pleaf}\ m \\
&\qquad\qquad \mathit{fjoin}\ l\ r = \mathit{pjoin}\ l\ r\ m
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\mathit{transform}\ t = t' \\
&\qquad \textbf{where}\ (t', m) = g\ (\mathit{fleaf}, \mathit{fjoin})\ t \\
&\qquad\qquad \mathit{fleaf} = \mathit{pleaf}\ m \\
&\qquad\qquad \mathit{fjoin}\ l\ r = \mathit{pjoin}\ l\ r\ m
\end{aligned}
$$