Recursive Coalgebras from Comonads

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Motivation

- We want to program using (general) recursion, but when is this justified, i.e, in which situations can we be sure that the equation we want to employ has a unique solution?
- Approaches: inductive, coinductive types, structured recursion, corecursion schemes, guarded-by-destructors recursion, guarded-by-constructors corecursion; general totality/termination/productivity analysis methodologies.
- Not so well recognized: For guarded-by-destructors recursion, there does not have to be an inductive type around.
- This talk: recursive coalgebras (as opposed to initial algebras) as a framework to deal with guarded-by-destructors generically.

Recursive coalgebras: motivation

- Consider quicksort: Let Z be a set linearly ordered by $\leq$. One usually defines quicksort recursively.

$$\begin{array}{lcl} \mathsf{qsort} \colon \mathsf{List}\, Z \to \mathsf{List}\, Z & & \\ \mathsf{qsort}\ [\,] & = & [\,] \\ \mathsf{qsort}\ (x : l) & = & \mathsf{qsort}(l_{\leq x}) +\!\!+ (x : \mathsf{qsort}(l_{>x})) \end{array}$$

- Why does this recursive (a priori dubious) definition actually make sense as a definition, i.e., how do we know the underlying equation has a unique solution?

Recursive coalgebras: motivation

- The equation has the form

$$qsort = \mathsf{qmerge} \circ \mathsf{BT}\, qsort \circ \mathsf{qsplit}$$

where $\mathsf{BT}_Z\ X = 1 + Z \times X \times X$, and

$$\begin{array}{lcl} \multicolumn{3}{l}{\mathsf{qsplit} \colon \mathsf{List}\, Z \to 1 + Z \times \mathsf{List}\, Z \times \mathsf{List}\, Z} \\ \mathsf{qsplit}\ [\,] & = & \mathsf{inl}(*) \\ \mathsf{qsplit}\ (x : l) & = & \mathsf{inr}(\langle x, l_{\leq x}, l_{>x} \rangle) \end{array}$$

$$\begin{array}{lcl} \multicolumn{3}{l}{\mathsf{qmerge} \colon 1 + Z \times \mathsf{List}\, Z \times \mathsf{List}\, Z \to \mathsf{List}\, Z} \\ \mathsf{qmerge}\ \mathsf{inl}(*) & = & [\,] \\ \mathsf{qmerge}\ \mathsf{inr}(\langle x, l_1, l_2 \rangle) & = & l_1 +\!\!+ (x : l_2) \end{array}$$

Recursive coalgebras: motivation

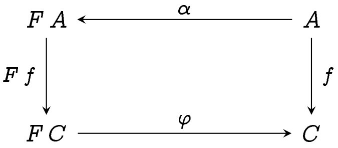
- So why does the equation make sense as a definition?

$$\begin{array}{ccc}
1 + Z \times \mathsf{List}\, Z \times \mathsf{List}\, Z & \xleftarrow{\mathsf{qsplit}} & \mathsf{List}\, Z \\
{\scriptstyle \mathsf{id} + \mathsf{id} \times qsort \times qsort}\downarrow & & \downarrow{\scriptstyle qsort} \\
1 + Z \times \mathsf{List}\, Z \times \mathsf{List}\, Z & \xrightarrow{\mathsf{qmerge}} & \mathsf{List}\, Z
\end{array}$$

- Because qsplit sends a list to a container of strictly shorter lists.
- Note, the fact that the result type was $\mathsf{List}\, Z$ and that the assembling function was qmerge did not play any role, we can replace them with something else and the equation is still a definition.

Recursive coalgebras: definition

- Let (A, α) be a F-coalgebra and (C, φ) an F-algebra
- A morphism $f : A \to C$ is a coalgebra-to-algebra morphism, if



- A F-coalgebra (A, α) is recursive, if there is a unique coalgebra-to-algebra morphism from it into any F-algebra
  - Denote: $f = \mathsf{fix}_{F,\alpha}(\varphi)$
- (An F-algebra (C, φ) is corecursive, if there is a unique coalgebra-to-algebra morphism into it from any F-coalgebra)

Recursive coalgebras: examples

- Let $F : \mathcal{C} \to \mathcal{C}$ be a functor with an initial algebra, $(\mu F, \mathsf{in}_F)$.
- Iteration: $(\mu F, \mathsf{in}_F^{-1})$ is a recursive F-coalgebra

$$
\begin{array}{ccc}
F\mu F & \xleftarrow{\mathsf{in}_F^{-1}} & \mu F \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\
FC & \xrightarrow{\varphi} & C
\end{array}
$$

- Primitive recursion: $(\mu F, F\langle \mathsf{id}_{\mu F}, \mathsf{id}_{\mu F} \rangle \circ \mathsf{in}_F^{-1})$ is a recursive $F(\mathsf{Id} \times K_{\mu F})$-coalgebra

$$
\begin{array}{ccccc}
F(\mu F \times \mu F) & \xleftarrow{F\langle \mathsf{id}_{\mu F}, \mathsf{id}_{\mu F} \rangle} & F\mu F & \xleftarrow{\mathsf{in}_F^{-1}} & \mu F \\
{\scriptstyle F(f \times \mathsf{id}_{\mu F})}\downarrow & & & & \downarrow{\scriptstyle f} \\
F(C \times \mu F) & & \xrightarrow{\varphi} & & C
\end{array}
$$

Recursive coalgebras: examples

- Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the covariant powerset function.
- A $\mathcal{P}$-coalgebra (A, α) is a binary relation $(A, \prec)$:

$$\begin{aligned} \alpha(a) &= \{x \in A \mid x \prec a\} \\ x \prec a &\text{ iff } x \in \alpha(a) \end{aligned}$$

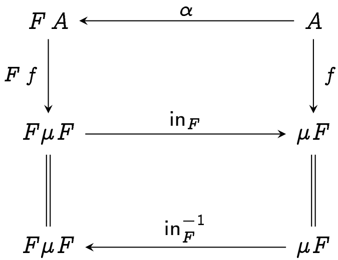
- A $\mathcal{P}$-coalgebra-to-algebra morphism from (A, α) to (C, φ) is a function $f : A \to C$ such that $f = \varphi \circ \mathcal{P}f \circ \alpha$ i.e., such that, for any $a \in A$,

$$f(a) = \varphi(\{f(x) \mid x \prec a\})$$

  Such a morphism exists uniquely for any (C, φ) iff $\prec$ is wellfounded.
  So: (A, α) is recursive iff $(A, \prec)$ is wellfounded.

Recursive coalgebras: basic properties

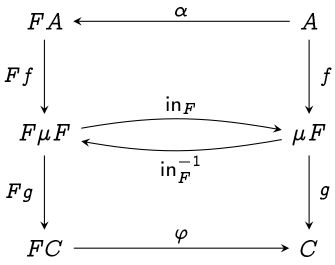
- Let $F : \mathcal{C} \to \mathcal{C}$ be a functor with an initial algebra, $(\mu F, \mathsf{in}_F)$.
- **Prop.** $(\mu F, \mathsf{in}_F^{-1})$ is a final recursive F-coalgebra.



Recursive coalgebras: basic properties

- Let $F : \mathcal{C} \to \mathcal{C}$ be a functor with an initial algebra, $(\mu F, \mathsf{in}_F)$.
- Let (A, α) be a recursive F-coalgebra
- **Cor.** Then, for any F-algebra (C, φ), the unique coalgebra-to-algebra morphism factorizes through the initial algebra

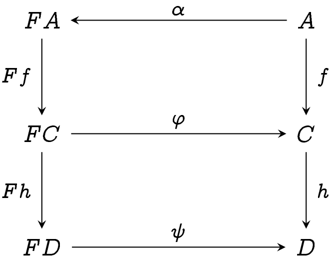
$$\mathsf{fix}_{F,\alpha}(\varphi) = \mathsf{fix}_{F,\mathsf{in}_F^{-1}}(\varphi) \circ \mathsf{fix}_{F,\alpha}(\mathsf{in}_F)$$



Recursive coalgebras: basic properties

- Let (A, α) be a recursive F-coalgebra
- Then

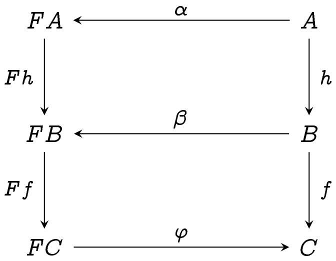
$$h \circ \varphi = \psi \circ Fh \quad \Rightarrow \quad h \circ \mathsf{fix}_{F,\alpha}(\varphi) = \mathsf{fix}_{F,\alpha}(\psi)$$



Recursive coalgebras: basic properties

- Let (A, α) and (B, β) be recursive F-coalgebras
- Then

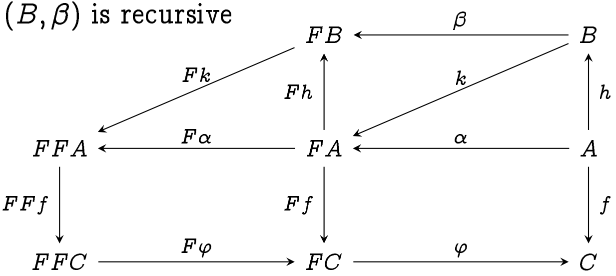
$$\beta \circ h = Fh \circ \alpha \quad \Rightarrow \quad \mathsf{fix}_{F,\beta}(\varphi) \circ h = \mathsf{fix}_{F,\alpha}(\varphi)$$



- Let $F = \mathcal{P} : \mathbf{Set} \to \mathbf{Set}$, then, (B, β) is recursive and from coalgebra (A, α) there is a homomorphism into it, then (A, α) is recursive [Osius, Taylor]
- Does not hold in general :-(

Recursive coalgebras: basic properties

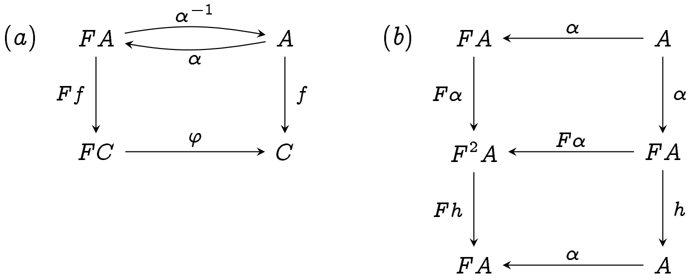
- Let (A, α) be a recursive F-coalgebra and (B, β) a F-coalgebra
- Let $h : (A, \alpha) \to (B, \beta)$ and $k : (B, \beta) \to (FA, F\alpha)$ be homomorphisms s.t., $\beta = Fh \circ k$
- Then (B, β) is recursive



- **Prop.** If (A, α) is recursive, then $(FA, F\alpha)$ is recursive

Recursive coalgebras: basic properties

- Let (A, α) be a recursive F-coalgebra.
- **(a)** If α is iso, then (A, α^{-1}) is an initial F-algebra.
- **(b)** If (A, α) is a final recursive F-coalgebra, then α is iso (both as a morphism and as a coalgebra morphism) (and hence (A, α^{-1}) is an initial F-algebra).



Recursive coalgebras: basic properties

- Let (A, α) be a recursive F-coalgebra
- Then $(A, F\alpha \circ \alpha)$ is a recursive F^2-coalgebra

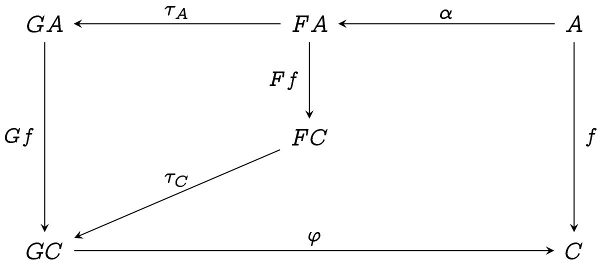
$$
\begin{array}{ccccc}
FFA & \xleftarrow{\quad F\alpha \quad} & FA & \xleftarrow{\quad \alpha \quad} & A \\
\Big\downarrow{\scriptstyle FFg} & & \Big\downarrow{\scriptstyle Fg} & & \Big\downarrow{\scriptstyle g} \\
FF(C \times FC) & \xrightarrow{F\langle\, \varphi \circ F\mathsf{snd}, F\mathsf{fst}\,\rangle} & F(C \times FC) & \xrightarrow{\langle\, \varphi \circ F\mathsf{snd}, F\mathsf{fst}\,\rangle} & C \times FC \\
\Big\downarrow{\scriptstyle FF\mathsf{fst}} & \swarrow{\scriptstyle F\mathsf{snd}} & & & \Big\downarrow{\scriptstyle \mathsf{fst}} \\
FFC & & \xrightarrow{\qquad\qquad \varphi \qquad\qquad} & & C
\end{array}
$$

- Holds also more generally: for any $n \geq 0$, the following is recursive

$$
F^{n+1}A \xleftarrow{F^n\alpha} F^nA \longleftarrow \cdots \longleftarrow F^2A \xleftarrow{F\alpha} FA \xleftarrow{\alpha} A
$$

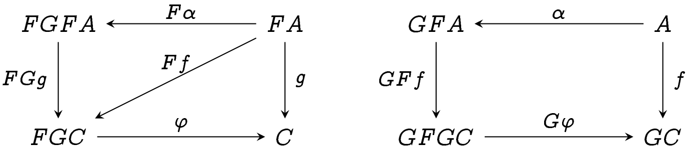
Transposition properties

- Let $F, G : \mathcal{C} \to \mathcal{C}$ be functors.
- Let $\tau : F \dot{\to} G$ be a natural transformation.
- Let (A, α) be a recursive F-coalgebra.
- Then $(A, \tau_A \circ \alpha)$ is a recursive G-coalgebra.



Transposition properties

- Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors.
- Let (A, α) be a recursive GF-coalgebra.
- Then $(FA, F\alpha)$ is a recursive FG-coalgebra.



Transposition properties

- Let $F : \mathcal{C} \to C$, $G : \mathcal{D} \to \mathcal{D}$ be functors.
- Let $L : \mathcal{C} \to \mathcal{D}$ be a functor with a right adjoint.
- Let $\tau : LF \stackrel{.}{\to} GL$ be a natural transformation.
- Let (A, α) be a recursive F-coalgebra.
- Then $(LA, \tau_A \circ L\alpha)$ is a recursive G-coalgebra.

Variations of recursiveness

- Let $\mathcal{C}$ be cartesian and $F : \mathcal{C} \to \mathcal{C}$ a functor with a strength σ.
- An F-coalgebra (A, α) is strongly recursive iff, for any object Γ of $\mathcal{C}$ and F-algebra (C, φ), there is a unique morphism $f : \Gamma \times A \to C$ satisfying

$$
\begin{array}{ccccc}
F(\Gamma \times A) & \xleftarrow{\sigma_{\Gamma,A}} & \Gamma \times FA & \xleftarrow{\mathrm{id}_\Gamma \times \alpha} & \Gamma \times A \\
\Big\downarrow{\scriptstyle Ff} & & & & \Big\downarrow{\scriptstyle f} \\
C & & \xleftarrow{\qquad\qquad\varphi\qquad\qquad} & & FC
\end{array}
$$

  i.e., iff, for any object Γ, the F-coalgebra $(\Gamma \times A, \sigma_{\Gamma,A} \circ (\mathrm{id}_\Gamma \times \alpha))$ is recursive.
- A strongly recursive F-coalgebra (A, α) is also a recursive F-coalgebra.
- For the converse, it is sufficient that $\mathcal{C}$ is cartesian closed.

Variations of recursiveness

- Let $\mathcal{C}$ be cartesian and $F : \mathcal{C} \to \mathcal{C}$ a functor.
- An F-coalgebra (A, α) is parametrically recursive iff, for any $(K_A \times F)$-algebra (C, φ), there is a unique morphism $f : A \to C$ satisfying

$$\begin{array}{ccc} A \times FA & \xleftarrow{\langle \mathsf{id}_A, \alpha \rangle} & A \\ {\scriptstyle \mathsf{id}_A \times Ff}\downarrow & & \downarrow{\scriptstyle f} \\ A \times FC & \xrightarrow{\varphi} & C \end{array}$$

  i.e., iff the $(K_A \times F)$-coalgebra $(A, \langle \mathsf{id}_A, \alpha \rangle)$ is recursive.
- A parametrically recursive F-coalgebra (A, α) is necessarily recursive, but the converse does not hold in general.

Comonads and coalgebras

- A comonad is a triple $\mathbf{N} = (N, \varepsilon, \delta)$, where N is a endofunctor, $\varepsilon : N \stackrel{.}{\to} \mathsf{Id}$ and $\delta : N \stackrel{.}{\to} NN$ are natural transformations, s.t.:

$$
\begin{array}{ccc}
NA & \xrightarrow{\delta_A} & NNA \\
{\scriptstyle \delta_A}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \varepsilon_{NA}} \\
NNA & \xrightarrow{N\varepsilon_A} & NA
\end{array}
\qquad
\begin{array}{ccc}
NA & \xrightarrow{\delta_A} & NNA \\
{\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle \delta_{NA}} \\
NNA & \xrightarrow{N\delta_A} & NNNA
\end{array}
$$

- A comonadic coalgebra is a N-coalgebra (A, i), s.t.:

$$
\begin{array}{ccc}
A & & \\
{\scriptstyle \mathsf{i}}\downarrow & \diagdown\!\!\diagdown & \\
NA & \xrightarrow{\varepsilon_A} & A
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\mathsf{i}} & NA \\
{\scriptstyle \mathsf{i}}\downarrow & & \downarrow{\scriptstyle \delta_A} \\
NA & \xrightarrow{N\mathsf{i}} & NNA
\end{array}
$$

Distributive comonads

- Let F be an endofunctor and $\boldsymbol{N} = (N, \varepsilon, \delta)$ a comonad
- Distributivity is a natural transformation $\kappa : FN \stackrel{\cdot}{\to} NF$, st.:

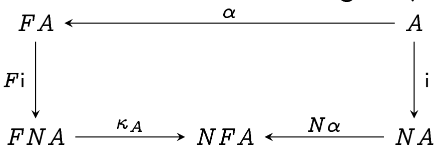
$$
\begin{array}{ccc}
FNA & \xrightarrow{\kappa_A} & NFA \\
{\scriptstyle F\varepsilon_A}\downarrow & & \downarrow{\scriptstyle \varepsilon_{FA}} \\
FA & = & FA
\end{array}
\qquad
\begin{array}{ccccc}
FNA & & \xrightarrow{\kappa_A} & & NFA \\
{\scriptstyle F\delta_A}\downarrow & & & & \downarrow{\scriptstyle \delta_{FA}} \\
FNNA & \xrightarrow{\kappa_{NA}} & NFNA & \xrightarrow{N\kappa_A} & NNFA
\end{array}
$$

- Let $f : FNA \to B$ be a morphism, then it's extension $f^{\ddagger} : FNA \to NB$ is defined as:

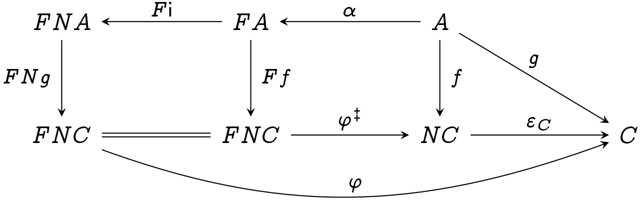
$$
\begin{array}{c}
FNA \\
{\scriptstyle f}\downarrow \\
B
\end{array}
\qquad
\begin{array}{ccc}
FNA & \xrightarrow{F\delta_A} & FNNA \\
{\scriptstyle f^{\ddagger}}\downarrow & & \downarrow{\scriptstyle \kappa_{NA}} \\
NB & \xleftarrow{Nf} & NFNA
\end{array}
$$

Generalized comonadic recursion

- Let (A, α) be a recursive F-coalgebra
- Let $\boldsymbol{N} = (N, \varepsilon, \delta, \kappa)$ be a distributive comonad
- Let $\mathsf{i} : A \to NA$ be a comonadic N-coalgebra, s.t:



- Then $(A, F\mathsf{i} \circ \alpha)$ is a recursive FN-coalgebra



Comonadic recursion

- Let the recursive F-coalgebra (A, α) be $(\mu F, \mathsf{in}_F^{-1})$.
- Let $\boldsymbol{N} = (N, \varepsilon, \delta, \kappa)$ be a distributive comonad.
- Then $\mathsf{i} = (\!|\, N\mathsf{in}_F \circ \kappa_{\mu F} \,|\!) : \mu F \to N\mu F$ is a comonadic N-coalgebra

$$
\begin{array}{ccccc}
F\mu F & & \xrightarrow{\quad \mathsf{in}_F \quad} & & \mu F \\
\Big\downarrow{\scriptstyle F\mathsf{i}} & & & & \Big\downarrow{\scriptstyle \mathsf{i}} \\
FN\mu F & \xrightarrow{\kappa_{\mu F}} & NF\mu F & \xrightarrow{N\mathsf{in}_F} & N\mu F
\end{array}
$$

- For any FN-algebra (C, φ), there is a unique morphism $f : \mu F \to C$ s.t.,

$$f \circ \mathsf{in}_F = \varphi \circ F(Nf \circ \mathsf{i}) \quad \equiv \quad f = \varepsilon_C \circ (\!|\, \varphi^{\ddagger} \,|\!)$$

$$
\begin{array}{ccc}
F\mu F & \xrightarrow{\mathsf{in}_F} & \mu F \\
\Big\downarrow{\scriptstyle F(Nf \circ \mathsf{i})} & & \Big\downarrow{\scriptstyle f} \\
FNC & \xrightarrow{\varphi} & C
\end{array}
\qquad\qquad
\begin{array}{ccccc}
F\mu F & \xrightarrow{\mathsf{in}_F} & \mu F & & \\
\Big\downarrow{\scriptstyle Fg} & & \Big\downarrow{\scriptstyle g} & & \\
FNC & \xrightarrow{\varphi^{\ddagger}} & NC & \xrightarrow{\varepsilon_C} & C
\end{array}
$$

Comonadic recursion

- Primitive recursion as an instance:

$$
\begin{array}{rcl}
N\,A & = & A \times \mu F \\
N\,f & = & f \times \mathsf{id}_{\mu F} \\
\varepsilon_A & = & \mathsf{fst} \\
\delta_A & = & \langle\, \mathsf{id}_{A \times \mu F}, \mathsf{snd} \,\rangle \\
\kappa_A & = & \langle\, F\mathsf{fst}, \mathsf{in}_F \circ F\mathsf{snd} \,\rangle
\end{array}
$$

- Course-of-values iteration as an instance:

$$
\begin{array}{rcl}
N\,A & = & \mathsf{Str}^F A \\
N\,f & = & \mathsf{gen}^F(f \circ \mathsf{hd}^F_A, \mathsf{tl}^F_A) \\
\varepsilon_A & = & \mathsf{hd}^F_A \\
\delta_A & = & \mathsf{gen}^F(\mathsf{id}_{\mathsf{Str}^F A}, \mathsf{tl}^F_A) \\
\kappa_A & = & \mathsf{gen}^F(F\mathsf{hd}^F_A, F\mathsf{tl}^F_A)
\end{array}
$$

Conclusions and future work

- Done: An elegant framework, a generalization of results known for initial algebras and modularization of proofs.
- To do: Develop further methods for checking a coalgebra for recursiveness.
- Relation between recursiveness and wellfoundedness (Paul Taylor's work).