# Is Your RBox Safe?
## Technical Report

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**Abstract.** The Description Logics underpinning OWL impose a well-known syntactic restriction in order to preserve decidability: they do not allow to use non-simple roles—that is, transitive roles or their super-roles—in number restrictions. When modeling composite objects, for example in bio-medical ontologies, this restriction can pose problems.
Therefore, we take a closer look at the problem of counting over non-simple roles. On the one hand, we sharpen the known undecidability results and demonstrate that: (i) for DLs with inverse roles, counting over non-simple roles leads to undecidability even when there is only one role in the language; (ii) for DLs without inverses, two transitive and an arbitrary role are sufficient for undecidability. On the other hand, we demonstrate that counting over non-simple roles does not compromise decidability in the absence of inverse roles provided that certain restrictions on role inclusion axioms are satisfied.

## 1 Introduction

Recently, Description Logics (DLs) [1] have attracted increasing attention, partially due to their usage as logical underpinning of ontology languages such as OIL, DAML+OIL, and OWL[1] [6]. All these languages are based on DLs of the $\mathcal{SHQ}$ family, which are decidable fragments of first order logic and close relatives of modal logics. In DLs, unary predicates/propositional variables are usually called concepts, binary predicates/modal parameters are called roles and, in a nutshell, $\mathcal{SHQ}$ extends $\mathcal{ALC}$ (a notational variant of multi-modal $\mathsf{K}$) with transitivity and role inclusion axioms and with *number restrictions*: these are concepts of the form $(\leqslant n\, R.C)$ for $n$ a non-negative integer, $R$ a role, and $C$ a possibly complex concept. Number restrictions are heavily used to define concepts, e.g., the following expression makes use of standard DL notation to define the concept $\mathtt{Human}$ as featherless bipeds:

$$\mathtt{Human} = \mathtt{Mammal} \sqcap \forall \mathtt{hasPart}.\neg\mathtt{Feather} \sqcap (\geqslant 2\, \mathtt{hasPart}.\mathtt{Leg}) \sqcap (\leqslant 2\, \mathtt{hasPart}.\mathtt{Leg})$$

We find numerous more convincing yet less readable such applications of number restrictions in bio-informatics and medical applications, e.g., they are used to restrict the number of certain components of proteins [10].

[1] OWL comes in three flavours, OWL Lite, OWL DL, and OWL Full. Here, we are only concerned with the first two.

Other heavily used features are the above mentioned *transitivity* and *role inclusion* axioms. They allow to express, e.g., that hasPart must be interpreted as a transitive relation (which is closely related to the modal logic $\mathsf{K4}$) and that hasComponent implies hasPart.

Now ontology design and maintenance is a non-trivial task, especially since ontologies can be quite large: e.g., SNOMED and the National Cancer Institute ontology have over 300,000 resp. 17,000 defined concepts. In order to check for consistency and compute the (implicit) concept hierarchy w.r.t. the subsumption relationship, ontology editors make use of DL reasoners[2] which implement decision procedures for concept satisfiability and subsumption w.r.t. DL axioms. For this to be possible, i.e., for these reasoning problems to be decidable for $\mathcal{SHQ}$, their designer had to impose a syntactic restriction: in number restrictions, one can neither use transitive roles nor super-roles of transitive roles, i.e., number restrictions can only be used on *simple roles*. For example, if we want to make use of our definition of Human, we have to either refrain from making hasPart a transitive role or use, e.g., a (non-transitive) subrole such as hasComp of hasPart in its number restrictions. Both options are sub-optimal since they result in the loss of other, useful consequences. For the first option, e.g., we could add the following definition of HumanBird without causing a (useful) inconsistency:

$$\mathtt{HumanBird} = \mathtt{Human} \sqcap \exists \mathtt{hasPart}.(\mathtt{Wing} \sqcap \exists \mathtt{hasPart}.\mathtt{Feather}).$$

For the second option, e.g., we could add the following definition of 3LHuman without causing an inconsistency (please note that here we use twice the sub-role hasComp of hasPart and only once hasPart):

$$\begin{aligned}\mathtt{3LHuman} = \mathtt{Human} &\sqcap \exists \mathtt{hasComp}.(\mathtt{Leg} \sqcap \mathtt{Left}) \sqcap \exists \mathtt{hasComp}.(\mathtt{Leg} \sqcap \mathtt{Right} \sqcap \neg \mathtt{Left}) \\ &\sqcap \exists \mathtt{hasPart}.(\mathtt{Leg} \sqcap \neg \mathtt{Right} \sqcap \neg \mathtt{Left}).\end{aligned}$$

In [7], it is shown that satisfiability of concepts in $\mathcal{SHQ}$ (even in its sublogic $\mathcal{SHN}$) is undecidable if non-simple roles (i.e., transitive roles or their super-roles) are used in number restrictions. In this paper, we explore this area more thoroughly with the goal of finding a more expressive but still decidable DL where we can use non-simple roles in number restrictions. Our contributions are two-fold: on the one hand, we sharpen the above undecidability result and demonstrate that: (i) for DLs such as $\mathcal{SHIN}$ (which extends $\mathcal{SHN}$ with inverse roles), counting over non-simple roles leads to undecidability even with only one role in the language; (ii) for DLs without inverses such as $\mathcal{SHN}$, two transitive and a third role are sufficient for undecidability. On the other hand, we demonstrate that, in the absence of inverse roles, counting over non-simple roles does not compromise decidability provided that they satisfy certain other restrictions regarding role inclusion axioms. Roughly speaking, as long as any two transitive roles are either completely unrelated w.r.t. inclusion or one of them implies the other, we can use them in number restrictions without losing decidability. We believe that the latter result will turn out to be useful in practice since it allows, for example, to capture a transitive role hasPart alongside other, possibly transitive roles such as hasComp or hasSegment *and* to use them all in number restrictions—as long as any two of these transitive roles are related by a (bi)-implication.

[2] See http://www.cs.man.ac.uk/~sattler/reasoners.html for a list.

## 2 Preliminaries and Known Results

The vocabulary of a DL consists of disjoint infinite sets of *concept names* $\mathsf{CN}$, *role names* $\mathsf{RN}$, and *individual names* $\mathsf{IN}$. A *role* is an expression of the form $r$ or $r^-$, where $r$ is a role name. For convenience, we introduce a syntactic operator defined on roles: $\mathsf{Inv}(R) := r^-$, if $R$ is a role name $r$; and $\mathsf{Inv}(R) := r$, if $R = r^-$ for some role name $r$. Finally, we use $\mathsf{Card}(M)$ for the cardinality of a set $M$.

**Definition 1 (RBox).** An *RBox* $\mathcal{R}$ is a finite collection of *transitivity axioms* of the form $\mathsf{Tr}(R)$ and *role inclusion axioms* of the form $R \sqsubseteq S$, where $R, S$ are roles.

An *interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty set $\Delta^\mathcal{I}$, its *domain*, and an interpretation function $\cdot^\mathcal{I}$ that maps each role name $r \in \mathsf{RN}$ to a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$; $\mathcal{I}$ is *finite* if the domain of $\mathcal{I}$ is finite. We define $(r^-)^\mathcal{I} := \{\langle x, y\rangle \mid \langle y, x\rangle \in r^\mathcal{I}\}$. We define whether $\mathcal{I}$ *satisfies* an axiom $\alpha$, written $\mathcal{I} \models \alpha$ as follows: $\mathcal{I} \models \mathsf{Tr}(R)$ iff $R^\mathcal{I}$ is transitive, and $\mathcal{I} \models R \sqsubseteq S$ iff $R^\mathcal{I} \subseteq S^\mathcal{I}$. An interpretation satisfying all axioms in $\mathcal{R}$ is called a *model* of $\mathcal{R}$. An RBox $\mathcal{R}$ *entails* an axiom $\alpha$, written $\mathcal{R} \models \alpha$, if all models of $\mathcal{R}$ satisfy $\alpha$.

The deductive *closure* $[\mathcal{R}]$ of $\mathcal{R}$ is the minimal set that contains $\mathcal{R}$ and axioms $R \sqsubseteq R$, for all roles $R$ in $\mathcal{R}$, and that is closed under the following rules:

$$\frac{R \sqsubseteq S \quad S \sqsubseteq T}{R \sqsubseteq T} \qquad \frac{R \sqsubseteq S}{\mathsf{Inv}(R) \sqsubseteq \mathsf{Inv}(S)} \qquad \frac{T \sqsubseteq S \sqsubseteq T \quad \mathsf{Tr}(T)}{\mathsf{Tr}(S)} \qquad \frac{\mathsf{Tr}(T)}{\mathsf{Tr}(\mathsf{Inv}(T))}$$

We write $\mathcal{R} \vdash \alpha$ as an alternative notation for $\alpha \in [\mathcal{R}]$, where $\alpha$ is an RBox axiom.

**Lemma 1.** *Let $\mathcal{R}$ be an RBox. Then*

1. *the number of axioms in $[\mathcal{R}]$ is polynomial in the number of axioms in $\mathcal{R}$, and*
2. *$\mathcal{R} \vdash \alpha$ iff $\mathcal{R} \models \alpha$.*

**Definition 2.** The set of concepts in DL $\mathcal{ALCIQ}$ is defined by the grammar:

$$\mathbf{C} ::= \ \bot \mid A \mid \neg C \mid C \sqcap D \mid \exists R.C \mid \leqslant n\, S.C,$$

where $A \in \mathsf{CN}$, $C, D \in \mathbf{C}$, $R$ and $S$ are roles, and $n$ is a non-negative integer.

The interpretation function $\cdot^\mathcal{I}$ maps, additionally, each concept name $C \in \mathsf{CN}$ to a subset $C^\mathcal{I} \subseteq \Delta^\mathcal{I}$, and $\cdot^\mathcal{I}$ is extended to complex concepts inductively as follows:

$$\begin{gathered}\bot^\mathcal{I} = \varnothing,\ \ (\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I},\ \ (C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}, \\ (\exists R.C)^\mathcal{I} = \{\, e \in \Delta^\mathcal{I} \mid \text{ there exists } d \in C^\mathcal{I} \text{ such that } \langle e, d\rangle \in R^\mathcal{I}\}, \\ (\leqslant n\, S.C)^\mathcal{I} = \{\, e \in \Delta^\mathcal{I} \mid \mathsf{Card}(\{d \in C^\mathcal{I} \mid \langle e, d\rangle \in S^\mathcal{I}\}) \leqslant n\}.\end{gathered}$$

For $C$ and $D$ $\mathcal{ALCIQ}$ concepts, $C \sqsubseteq D$ is a *general concept inclusion* (GCI), and a finite set of GCIs is called a *TBox*. An interpretation $\mathcal{I}$ *satisfies* a GCI $C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$. An interpretation is a *model* of a TBox if it satisfies all its axioms. If a (finite) interpretation $\mathcal{I}$ is a model of an RBox $\mathcal{R}$ and a TBox $\mathcal{T}$, then we say that $\mathcal{I}$ is a *(finite) model* of $\langle \mathcal{R}, \mathcal{T}\rangle$, or $\langle \mathcal{R}, \mathcal{T}\rangle$ is *(finitely) satisfiable*. A concept $C$ is *(finitely) satisfiable w.r.t.* $\langle \mathcal{R}, \mathcal{T}\rangle$ if there exists a (finite) model $\mathcal{I}$ of $\langle \mathcal{R}, \mathcal{T}\rangle$ such that $C^\mathcal{I} \neq \varnothing$.

As usual, the concept expressions $\top$, $C_1 \sqcup C_2$, $\forall R.C$ and $\geqslant n\, S.C$ are assumed to be abbreviations for $\neg\bot$, $\neg(\neg C_1 \sqcap \neg C_2)$, $\neg(\exists R.\neg C)$ and $\neg(\leqslant (n-1)\, S.\neg C)$ respectively. Concepts of $\mathcal{ALCIQ}$ that do not use number restrictions $(\leqslant n\, R.C)$, or inverse roles, or both, will be called $\mathcal{ALCI}$-, $\mathcal{ALCQ}$-, and $\mathcal{ALC}$ concepts, resp. The letter $\mathcal{N}$ in the name of a DL indicates that this DL supports only number restrictions of the form $(\leqslant n\, R.\top)$.

Please note that, so far, we have introduced RBoxes and $\mathcal{ALCIQ}$ TBoxes separately, i.e., we did not put them into a single logic, which is slightly unusual. Recall that in [7] a role $S$ is called *simple w.r.t.* $\mathcal{R}$ if there is no transitive subrole of $S$ in $\mathcal{R}$. Traditionally, the DL that allows for

- an RBox without inverse roles and an $\mathcal{ALCQ}$ TBox where all roles in number restrictions are simple is called $\mathcal{SHQ}$, and
- an RBox and an $\mathcal{ALCIQ}$ TBox where all roles in number restrictions are simple is called $\mathcal{SHIQ}$.

For $\mathcal{SHIQ}$ and related DLs, roles in number restrictions are restricted to simple ones to ensure decidability of concept satisfiability w.r.t. a TBox and an RBox: in $\mathcal{SHN}$ (and hence $\mathcal{SHIQ}$), non-simple roles in number restrictions lead to the undecidability of the satisfiability problem [7]. Our aim is to find conditions under which we can relax or even get rid of this restriction to simple roles in number restrictions while preserving decidability. This aim can be achieved by extending the notion of a simple role in such a way that it covers, besides roles that are usually called simple, also some transitive roles or their super-roles. In this paper, we focus on a sub-problem, namely, we are looking for conditions on an RBox under which one can use all its roles in number restrictions and still have a decidable logic. Therefore, we introduce the following notion.

**Definition 3.** Let $\mathcal{L}$ be a logic between $\mathcal{ALC}$ and $\mathcal{ALCIQ}$ and $\mathcal{R}$ an RBox. The problem of $\mathcal{L}(\mathcal{R})$-satisfiability is to determine, given an $\mathcal{L}$-concept $C$ and an $\mathcal{L}$-TBox $\mathcal{T}$, whether $C$ is satisfiable w.r.t. $\langle \mathcal{R}, \mathcal{T} \rangle$. We say that an RBox $\mathcal{R}$ is $\mathcal{L}$*-safe* (or *safe for* $\mathcal{L}$) if $\mathcal{L}(\mathcal{R})$-satisfiability is decidable, and $\mathcal{L}$*-unsafe* otherwise.

*Remark 1.* Note that the problem of $\mathcal{L}$-safety for an RBox $\mathcal{R}$, can be equivalently reformulated using the problem of satisfiability for pairs $\langle \mathcal{R}, \mathcal{T} \rangle$ instead of satisfiability of an $\mathcal{L}$-concept $C$ w.r.t. $\langle \mathcal{R}, \mathcal{T} \rangle$, since a concept $C$ is satisfiable w.r.t. $\mathcal{T}$ and $\mathcal{R}$ iff the pair $\langle \mathcal{R}, \mathcal{T} \cup \{\top \sqsubseteq \exists R.C\} \rangle$ is satisfiable, where $R$ is a fresh role.

Any RBox is $\mathcal{ALCI}$-safe because (i) neither $\mathcal{ALCI}$ nor $\mathcal{SHI}$ support number restrictions, and (ii) since a concept $C$ and a TBox $\mathcal{T}$ are $\mathcal{ALCI}(\mathcal{R})$-satisfiable iff $C$ is satisfiable w.r.t. $\langle \mathcal{R}, \mathcal{T} \rangle$, we have that $\mathcal{ALCI}(\mathcal{R})$ satisfiability can be viewed as the standard $\mathcal{SHI}$ satisfiability problem which is known to be decidable [7]. With a similar argument, any RBox $\mathcal{R}$ without transitivity axioms is $\mathcal{ALCIQ}$-safe because (i) all roles are simple in this case, and (ii) $\mathcal{ALCIQ}(\mathcal{R})$-satisfiability can be viewed as the standard $\mathcal{SHIQ}$ satisfiability problem which is known to be decidable [7]. There are numerous other restrictions on the syntax that could possibly lead to decidability, for example to use only number restrictions of the form $(\leqslant 1\, R)$.

At the same time, we know from [7] that the following RBox $\mathsf{Star}_4$ (with eight roles, of which four are transitive) is $\mathcal{ALCN}$-unsafe:

$$\mathsf{Star}_4 = \{\, s_i \sqsubseteq t_{ij},\, r_j \sqsubseteq t_{ij},\, \mathsf{Tr}(t_{ij}) \mid 0 \leqslant i, j \leqslant 1 \,\}.$$

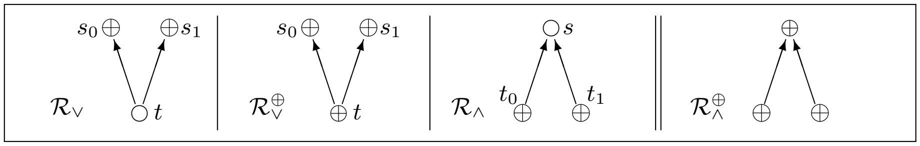


**Fig. 1.** The first three RBoxes are unsafe (Theorem 1) and for the last one the problem is open.

In what follows, we show that

- there is a large class of RBoxes involving role inclusions and transitivity axioms that are $\mathcal{ALCQ}$-safe (Theorem 4),
- there exists an $\mathcal{ALCIN}$-unsafe RBox with only one transitive role (Theorem 2),
- there exist $\mathcal{ALCN}$-unsafe RBoxes involving only three roles (Theorem 1).

## 3 Undecidability Results

Here we show that three roles are sufficient for building an unsafe RBox for $\mathcal{ALCQ}$, whereas for $\mathcal{ALCIQ}$, even one role is sufficient for that. We give fine-grained formulations of results by indicating, as a subscript to the name of a logic, the maximal number $n$ occurring in number restrictions $(\leqslant n\, R.C)$ in the proof. The undecidability results are proved by reduction from the undecidable *domino problem* (see, e.g., [3]).

**Definition 4 (Domino).** A *domino system* is a triple $\mathcal{D} = \langle D, H, V\rangle$, where $D = \{d_1, \ldots, d_n\}$ is a finite set of *tile types* and $H, V \subseteq D \times D$ are horizontal and vertical *matching* relations. We say that $\mathcal{D}$ *tiles* $\mathbb{N}\times\mathbb{N}$ if there exists a $\mathcal{D}$*-tiling*, i.e., a mapping $\tau\colon \mathbb{N}\times\mathbb{N} \to D$ such that, for all $i, j \in \mathbb{N}$, the following *compatibility* conditions hold: $\langle \tau(i,j), \tau(i{+}1, j)\rangle \in H$ and $\langle \tau(i,j), \tau(i, j{+}1)\rangle \in V$. The *domino problem* is to check, given a domino system $\mathcal{D}$, whether $\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$.

Our proofs follow the usual pattern: in order to show $\mathcal{L}$-unsafety of a given RBox $\mathcal{R}$, we first build an $\mathcal{L}$-TBox $\mathcal{T}_{\mathsf{grid}}$ that together with $\mathcal{R}$ "encodes" a $\mathbb{N}\times\mathbb{N}$ grid. Then, given a domino system $\mathcal{D}$, we build (efficiently) an $\mathcal{ALC}$-TBox $\mathcal{T}_{\mathcal{D}}$ that "tiles" the grid and "ensures" the compatibility conditions. Finally, we prove that $\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$ iff some concept (usually a concept name) $C$ is satisfiable w.r.t. $\mathcal{R} + \mathcal{T}_{\mathsf{grid}} + \mathcal{T}_{\mathcal{D}}$.

To save space and make presentation easier to understand, we depicture RBoxes (without inverse roles) as directed graphs whose nodes are non-transitive ($\bigcirc$) and transitive ($\oplus$) roles and arrows represent implications between roles.

**Theorem 1.** *The RBoxes $\mathcal{R}_\wedge$, $\mathcal{R}_\vee$, and $\mathcal{R}_\vee^\oplus$ shown in Fig. 1 are unsafe for $\mathcal{ALCN}$; more precisely, they are unsafe for $\mathcal{ALCN}_9$ and $\mathcal{ALCQ}_1$.*

*Proof.* We use 16 concept names $A_{ij}$, $0 \leqslant i, j \leqslant 3$, place them on an $\mathbb{N}\times\mathbb{N}$ grid (by repeating a $[0,3]\times[0,3]$ pattern periodically) and link them with $R$- and $S$-edges as shown in Fig. 2a. We will refer to edges in this grid as $\langle A, r, B\rangle$, where $A, B \in \{A_{ij} \mid 0 \leqslant i, j \leqslant 3\}$ and $r \in \{R, S\}$. Having this picture in mind, we add the following axioms to an $\mathcal{ALCQ}$-TBox $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$ and an $\mathcal{ALCN}$-TBox $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$. First, we assert that all concepts

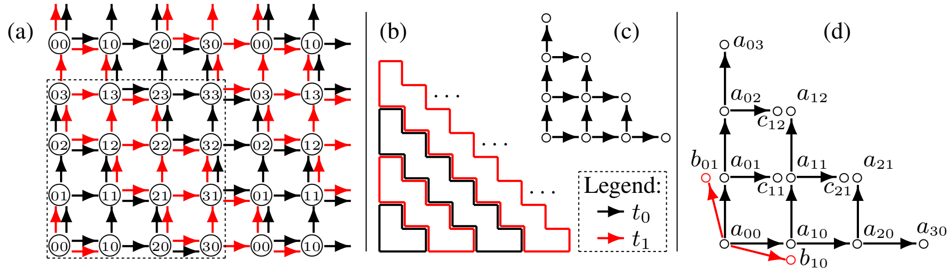


**Fig. 2.** A grid for Theorem 1: (a) A detailed view of the grid. (b) A grid at a glance. (c) Accessibility relation for $R$ (and similarly for $S$). (d) A pre-grid for $\mathcal{ALCN}$.

$A_{ij}$ are pairwise disjoint: $A_{ij} \sqcap A_{k\ell} \sqsubseteq \bot$, for all $\langle i,j\rangle \neq \langle k,\ell\rangle$. From now on, let us carry out the proof for the RBox $\mathcal{R}_\wedge$ first. We add the following axioms (a), (b), (c) to $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$ and axioms (a), (d) $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$, where we denote $i \oplus j := (i+j) \bmod 4$:

(a) $A \sqsubseteq \exists r.B$ for each edge $\langle A, r, B\rangle$, where $r \in \{R, S\}$;
(b) $A \sqsubseteq \leqslant 1\, Q.B$ for each double edge $\langle A, R, B\rangle$ and $\langle A, S, B\rangle$;
(c) $A_{ij} \sqsubseteq \leqslant 1\, Q.A_{i\oplus 1, j\oplus 1}$ for all $0 \leqslant i, j \leqslant 3$;
(d) $A_{ij} \sqsubseteq \leqslant 9\, Q$ for all $0 \leqslant i, j \leqslant 3$ such that $i + j$ is even.

For instance, we have axioms $A_{10} \sqsubseteq \exists R.A_{11}$ from (a), $A_{11} \sqsubseteq \leqslant 1\, Q.A_{12}$ from (b), $A_{32} \sqsubseteq \leqslant 1\, Q.A_{03}$ from (c) in $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$, and $A_{13} \sqsubseteq \leqslant 9\, Q$ from (d) in $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$.

Next, given a domino system $\mathcal{D} = \langle D, H, V\rangle$ with $D = \{d_1, \ldots, d_n\}$, we introduce fresh concept names $D_1, \ldots, D_n$ and add the following $\mathcal{ALC}$-axioms to a TBox $\mathcal{T}_{\mathcal{D}}$:

(e) $\top \sqsubseteq D_1 \sqcup \ldots \sqcup D_n$;
(f) $D_k \sqcap D_\ell \sqsubseteq \bot$, for all $1 \leqslant k < \ell \leqslant n$;
(g) $A \sqcap D_k \sqsubseteq \forall r.\big(B \to \bigsqcup_{\ell:\, \langle d_k, d_\ell\rangle \in H} D_\ell\big)$ for each horizontal edge $\langle A, r, B\rangle$;
(h) $A \sqcap D_k \sqsubseteq \forall r.\big(B \to \bigsqcup_{\ell:\, \langle d_k, d_\ell\rangle \in V} D_\ell\big)$ for each vertical edge $\langle A, r, B\rangle$.

Now, for $\mathcal{X} \in \{\mathcal{Q}, \mathcal{N}\}$, we set $\mathcal{K}^{\mathcal{X}}_{\mathcal{D}} := \mathcal{R}_\wedge \cup \mathcal{T}^{\mathcal{X}}_{\mathsf{grid}} \cup \mathcal{T}_{\mathcal{D}}$ and prove the following lemma.

**Lemma 1.1** **(For $\mathcal{R}_\wedge$).** *The concept $A_{00}$ is satisfiable w.r.t. $\mathcal{K}^{\mathcal{X}}_{\mathcal{D}}$ iff $\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$.*

($\Leftarrow$) Given a tiling $\tau\colon \mathbb{N}\times\mathbb{N} \to D$, we build a model $\mathcal{I}$ as follows: set $\Delta^{\mathcal{I}} := \mathbb{N}\times\mathbb{N}$, interpret concepts $A_{ij}$ exactly as in Fig. 2a, i.e., $A^{\mathcal{I}}_{ij} = \{\langle i+4k, j+4\ell\rangle \mid k,\ell \in \mathbb{N}\}$; roles $R, S$ as the transitive closures of the relations depicted by arrows in Fig. 2a; set $Q^{\mathcal{I}} := R^{\mathcal{I}} \cup S^{\mathcal{I}}$; and set $\langle i,j\rangle \in D^{\mathcal{I}}_k$ iff $\tau(i,j) = d_k$. Then $A_{00} \neq \varnothing$, as $\langle 0,0\rangle \in A^{\mathcal{I}}_{00}$.

It remains to check that $\mathcal{I}$ is a model of $\mathcal{K}^{\mathcal{X}}_{\mathcal{D}}$. Clearly, $\mathcal{I} \models \mathcal{R}_\wedge$. Notice that the relation $Q^{\mathcal{I}}$ is not transitive, and $\mathcal{R}_\wedge$ has no transitivity axiom for $Q$. Axioms (a)–(d) are true in $\mathcal{I}$, since they were "read-off" directly from Fig. 2a. In particular, to verify axioms (c) and (d), observe the following main property of our model: any two elements are linked via $Q^{\mathcal{I}}$ iff they are linked via either a chain of $R$-edges, or a chain of $S$-edges. Therefore, any element $\langle i,j\rangle$ with even $i+j$ has exactly 9 $Q$-successors, namely $\langle i+k, j+\ell\rangle$ with $0 < k+\ell \leqslant 3$ (hence axiom (d) is true), of which only one, namely $\langle i+k, j+\ell\rangle$, belongs to $A^{\mathcal{I}}_{i\oplus k, j\oplus \ell}$ (hence axiom (c) is true).

Axioms (e) and (f) are true in $\mathcal{I}$, since $\tau$ is a total function. It remains to show that axioms (g) and (h) are true in $\mathcal{I}$. We check it only for an instance of axiom (g) taken for a horisontal $R$-edge $\langle A, R, B\rangle$, where $A = A_{ij}$ and $B = A_{i\oplus 1,j}$, for some $0 \leqslant i, j \leqslant 3$. Take any element $a \in (A \sqcap D_k)^{\mathcal{I}}$; it is of the form $a = \langle s, t\rangle$, for some $s, t \in \mathbb{N}$. Then $i = s \bmod 4$ and $j = t \bmod 4$, by construction of $A^{\mathcal{I}}_{ij}$. To show that the element $a$ belongs to the r.h.s. of axiom (g), take any $b \in \Delta^{\mathcal{I}}$ such that $\langle a, b\rangle \in R^{\mathcal{I}}$ and suppose that $b \in B^{\mathcal{I}}$. As mentioned above, there is only one $Q$-successor (in our case, $R$-successor) of $a$ that belongs to $B = A_{i\oplus 1,j}$, namely, $b = \langle s{+}1, t\rangle$. Since $\langle s, t\rangle \in D^{\mathcal{I}}_k$, we have $\tau(s, t) = d_k$. Now let $d_\ell := \tau(s{+}1, t)$, then $b \in D^{\mathcal{I}}_\ell$, by construction of $D^{\mathcal{I}}_\ell$. By compatibility conditions for $\tau$, we have $\langle d_k, d_\ell\rangle \in H$. Thus, $b \in (\bigsqcup_{\ell\colon \langle d_k,d_\ell\rangle\in H} D_\ell)^{\mathcal{I}}$, so we are done.

($\Rightarrow$) Suppose that $\mathcal{I} \models \mathcal{K}^{\mathcal{X}}_{\mathcal{D}}$ and $A^{\mathcal{I}}_{00}$ is nonempty, say $a_{00} \in A^{\mathcal{I}}_{00}$. Then we extract a grid from $\mathcal{I}$, i.e., a subset of (not necessarily distinct) elements $\mathsf{G} := \{a_{ij} \mid i, j \in \mathbb{N}\} \subseteq \Delta^{\mathcal{I}}$, such that $\mathcal{I}$ restricted to $\mathsf{G}$ looks as in Fig. 2a, and build a $\mathcal{D}$-tiling of $\mathbb{N}\times\mathbb{N}$. The proof is different for $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$ and $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$.

For $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$: by axioms (a), there exist $a_{10}, a'_{10} \in A^{\mathcal{I}}_{10}$ such that $\langle a_{00}, a_{10}\rangle \in R^{\mathcal{I}}$ and $\langle a_{00}, a'_{10}\rangle \in S^{\mathcal{I}}$. By role inclusions in $\mathcal{R}_\wedge$, these pairs are in $Q^{\mathcal{I}}$, and axiom (b) implies $a_{10} = a'_{10}$. By a similar argument, there exists an element $a_{01} \in A^{\mathcal{I}}_{01}$ such that $\langle a_{00}, a_{01}\rangle \in R^{\mathcal{I}} \cap S^{\mathcal{I}}$. Furthermore, by axioms (a), there are $a_{11}, a'_{11} \in A^{\mathcal{I}}_{11}$ such that $\langle a_{10}, a_{11}\rangle, \langle a_{01}, a'_{11}\rangle \in R^{\mathcal{I}}$. By transitivity of $R$, both $a_{11}$ and $a'_{11}$ are $R$-successors (and hence $Q$-successors) of $a_{00}$. Now we apply axiom (c) $A_{00} \sqsubseteq\, \leqslant 1\, Q.A_{11}$ to conclude that $a_{11} = a'_{11}$. Thus, we have constructed the first cell of the grid. By repeating this argument, we can build all elements $\{a_{ij} \mid i, j \in \mathbb{N}\}$.

For $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$: starting with $a_{00}$ and applying axiom (a), we first build a "pre-grid" depicted in Fig. 2d, which consists of 9 elements $a_{ij}$ with $0 < i + j \leqslant 3$, 3 elements $c_{11}, c_{21}, c_{12}$, and 2 elements $b_{10}, b_{01}$. By role inclusions $R \sqsubseteq Q$ and $S \sqsubseteq Q$, all these elements are $Q$-successors of $a_{00}$. Now we apply axiom (d) $A_{00} \sqsubseteq\, \leqslant 9\, Q$ and recall that $A^{\mathcal{I}}_{ij}$ are pairwise disjoint. Therefore, in each of the 9 concepts $A^{\mathcal{I}}_{ij}$, with $0 < i + j \leqslant 3$, elements merge into a single one: $c_{ij} = a_{ij}$ and $b_{ij} = a_{ij}$ (for suitable $i, j$), which yields a structure shown in Fig. 2c. Next we apply a similar argument (but with $R$ and $S$ swapped) to each element $a_{ij}$ with $i + j = 2$ (in any order), then to each $a_{ij}$ with $i + j = 4$, and so on, until we build the whole grid.

Once we have built the set $\{a_{ij} \mid i, j \in \mathbb{N}\} \subseteq \Delta^{\mathcal{I}}$, we define $\tau\colon \mathbb{N}\times\mathbb{N} \to D$ by putting $\tau(i, j) := d_k$ iff $a_{ij} \in D^{\mathcal{I}}_k$. By (e) and (f), $\tau$ is a total function; and compatibility conditions easily follow from (g) and (h). Thus $\tau$ is indeed a $\mathcal{D}$-tiling of $\mathbb{N}\times\mathbb{N}$. This completes the proof of Lemma 1.1. $\dashv$

For the remaining two RBoxes $\mathcal{R}_\vee$ and $\mathcal{R}^{\oplus}_\vee$, we add the following axioms (a'), (b'), (c') in $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$ and axioms (a'), (b'), (d') in $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$, where $i, j$ range over $\{0, 1, 2, 3\}$:

(a') $A \sqsubseteq \exists r.B$ for each *single* edge $\langle A, r, B\rangle$, where $r \in \{R, S\}$;

(b') $A \sqsubseteq \exists Q.B$ for each *double* edge $\langle A, R, B\rangle$ and $\langle A, S, B\rangle$;

(c') $A_{ij} \sqsubseteq\, \leqslant 1\, r.A_{i\oplus 1, j\oplus 1}$, where $r = \begin{cases} R, & \text{if } i \oplus j \in \{0, 1\},\\ S, & \text{if } i \oplus j \in \{2, 3\};\end{cases}$

(d') $A_{ij} \sqsubseteq\, \leqslant 9\, r$, where $r = R$ if $i \oplus j = 0$, and $r = S$ if $i \oplus j = 2$.

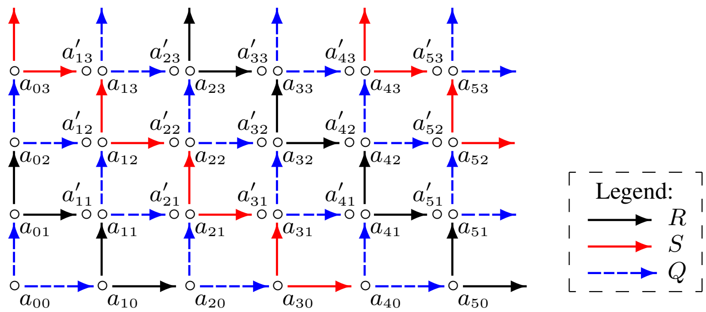


**Fig. 3.** A pre-grid for the proof of $\mathcal{ALCQ}$-unsafety of the RBoxes $\mathcal{R}_\vee$ and $\mathcal{R}_\vee^\oplus$.

For $\mathcal{X} \in \{\mathcal{Q}, \mathcal{N}\}$, we set $\mathcal{K}_\mathcal{D}^\mathcal{X} := \mathcal{R} + \mathcal{T}_{\mathsf{grid}}^\mathcal{X} + \mathcal{T}_\mathcal{D}$, where the TBox $\mathcal{T}_\mathcal{D}$ is defined as above, and prove the following lemma for both $\mathcal{R}_\vee$ and $\mathcal{R}_\vee^\oplus$.

**Lemma 1.2 (For $\mathcal{R}_\vee^{(\oplus)}$).** *The concept $A_{00}$ is satisfiable w.r.t. $\mathcal{K}_\mathcal{D}^\mathcal{X}$ iff $\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$.*

**($\Leftarrow$)** As in the proof of Lemma 1.1, but now we set $Q^\mathcal{I} := R^\mathcal{I} \cap S^\mathcal{I}$. Then $Q^\mathcal{I}$ is transitive as the intersection of transitive relations. Hence $\mathcal{I}$ satisfies $\mathcal{R}_\vee$ and even $\mathcal{R}_\vee^\oplus$.
**($\Rightarrow$)** Again, the proof is similar to that of Lemma 1.1, but here it is more convenient to present it in a different style. Suppose that $\mathcal{I} \models \mathcal{K}_\mathcal{D}^\mathcal{X}$ and $A_{00}^\mathcal{I}$ is nonempty, say $a_{00} \in A_{00}^\mathcal{I}$. Then we extract a grid $\mathsf{G} := \{a_{ij} \mid i, j \in \mathbb{N}\} \subseteq \Delta^\mathcal{I}$, using axioms from the TBox $\mathcal{T}_{\mathsf{grid}}^\mathcal{X}$, and then build a $\mathcal{D}$-tiling of $\mathbb{N}\times\mathbb{N}$, using axioms from $\mathcal{T}_\mathcal{D}$. In the first part of the proof, which is common for both $\mathcal{X} \in \{\mathcal{Q}, \mathcal{N}\}$, we build a "pre-grid" depicted in Fig. 3, using $\mathcal{ALC}$-axioms (a') and (b') only, in 3 steps.

**Step 1.** Starting from $a_{00}$, build a horisontal axis $\{a_{i0} \mid i > 0\}$, such that, for all $i \geqslant 0$,

$$\langle a_{i,0}, a_{i+1,0} \rangle \in \begin{cases} Q^\mathcal{I}, & \text{if } i \text{ is even,} \\ R^\mathcal{I}, & \text{if } i \bmod 4 = 1, \\ S^\mathcal{I}, & \text{if } i \bmod 4 = 3. \end{cases}$$

**Step 2.** Starting from each $a_{i0}$, build a vertical axis $\{a_{ij} \mid j > 0\}$, s.t. for all $i, j \geqslant 0$,

$$\langle a_{i,j}, a_{i,j+1} \rangle \in \begin{cases} Q^\mathcal{I}, & \text{if } i \oplus j \text{ is even,} \\ R^\mathcal{I}, & \text{if } i \oplus j = 1, \\ S^\mathcal{I}, & \text{if } i \oplus j = 3. \end{cases}$$

Note that we have no edges $\langle a_{i,j}, a_{i+1,j} \rangle$ yet, for $j > 0$.
**Step 3.** Build elements $\{a'_{ij} \mid i, j > 0\}$ such that, for all $i \geqslant 0$, $j > 0$,

$$\langle a_{i,j}, a'_{i+1,j} \rangle \in \begin{cases} Q^\mathcal{I}, & \text{if } i \oplus j \text{ is even,} \\ R^\mathcal{I}, & \text{if } i \oplus j = 1, \\ S^\mathcal{I}, & \text{if } i \oplus j = 3. \end{cases}$$

Now we turn this pre-grid into a grid using the remaining axioms from $\mathcal{T}_{\mathsf{grid}}^\mathcal{X}$, and hence the rest of the proof is different for $\mathcal{X} = \mathcal{Q}$ and $\mathcal{X} = \mathcal{N}$.

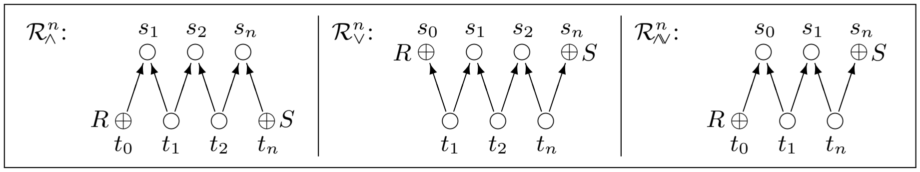


**Fig. 4.** The chain RBoxes $\mathcal{R}^n_\wedge$, $\mathcal{R}^n_\vee$, $\mathcal{R}^n_{\wedge\!\!\!\vee}$, for $n \geqslant 1$, are $\mathcal{ALCN}$-unsafe.

**For $\mathcal{X} = \mathcal{Q}$:** by the axioms $Q \sqsubseteq R$ and $\mathsf{Tr}(R)$, the element $a_{ij}$ with $i \oplus j \in \{0, 1\}$, has $R$-successors $a_{i+1,j+1}$ and $a'_{i+1,j+1}$ in $A^{\mathcal{I}}_{i\oplus1,j\oplus1}$. By axiom (c'), they are equal: $a_{i+1,j+1} = a'_{i+1,j+1}$. Similarly for $i \oplus j \in \{2, 3\}$ and the role $S$. Thus, we obtain a grid $\{a_{ij} \mid i, j \in \mathbb{N}\}$ as in Fig 2a.

**For $\mathcal{X} = \mathcal{N}$:** by the axioms $Q \sqsubseteq R$ and $\mathsf{Tr}(R)$, the element $a_{00}$ has the following $R$-successors in the pre-grid: 9 elements $a_{ij}$ with $0 < i + j \leqslant 3$, and 3 elements $a'_{11}, a'_{12}, a'_{21}$. Now we apply axiom (d') $A_{00} \sqsubseteq \leqslant 9\, R$ and recall that all $A^{\mathcal{I}}_{ij}$ are pairwise disjoint. Then $a_{11} = a'_{11}$, $a_{12} = a'_{12}$, and $a_{21} = a'_{21}$. Notice that the axiom $\mathsf{Tr}(Q)$ has no effect on our argument, since the relation $Q^{\mathcal{I}}$ restricted to the pre-grid has no chains of the length greater than 1 at all. Next we carry out the same argument (but with $R$ and $S$ swapped), starting from each element $a_{ij}$ with $i + j = 2$ (in any order), then with $i + j = 4$, and so on.

The rest of the proof is the same as in Lemma 1.1, so Lemma 1.2 is proved. $\dashv$

This completes the proof of Theorem 1. $\dashv$

We conjecture that the fourth RBox $\mathcal{R}^{\oplus}_{\wedge}$ shown in Fig. 1 is $\mathcal{ALCQ}$-safe. The same construction as in Theorem 1 allows us to obtain the following generalisation.

**Theorem 2.** *The RBoxes* $\{\oplus{\to}\bigcirc{\leftarrow}\bigcirc{\to}\oplus\}$, $\{\oplus{\to}\bigcirc{\leftarrow}\bigcirc{\to}\bigcirc{\leftarrow}\oplus\}$ *are $\mathcal{ALCN}$-unsafe, as well as longer chains with any number of non-transitive roles between two transitive roles and with interleaving direction of role inclusions.*

*Proof.* These chains can be of 3 kinds: where the ending transitive roles are both minimal, or both maximal, or one is minimal and another maximal. More precisely, we consider 3 series of RBoxes, for $n \geqslant 1$, depicted in Fig. 4:

$$\begin{aligned}
\mathcal{R}^n_\wedge &:= \{\mathsf{Tr}(Q_0), \mathsf{Tr}(Q_n)\} \cup \{Q_{k-1} \sqsubseteq T_k,\, T_k \sqsupseteq Q_k \mid 1 \leqslant k \leqslant n\},\\
\mathcal{R}^n_\vee &:= \{\mathsf{Tr}(T_0),\, \mathsf{Tr}(T_n)\} \cup \{T_{k-1} \sqsupseteq Q_k, Q_k \sqsubseteq T_k \mid 1 \leqslant k \leqslant n\},\\
\mathcal{R}^n_{\wedge\!\!\!\vee} &:= \{\mathsf{Tr}(Q_0), \mathsf{Tr}(T_n)\} \cup \{T_{k-1} \sqsupseteq Q_k,\, Q_\ell \sqsubseteq T_\ell \mid 1 \leqslant k \leqslant n,\, 0 \leqslant \ell \leqslant n\}.
\end{aligned}$$

Note that $\mathcal{R}^1_\wedge = \mathcal{R}_\wedge$ and $\mathcal{R}^1_\vee = \mathcal{R}_\vee$, so Theorem 1 is a special case of this theorem. For proving unsafety of these RBoxes, we use the same construction of a grid as shown in Fig. 2a; to comply with that picture, we denote the ending transitive roles in our RBoxes by $R$ and $S$. Additionally, denote by $T_z$ the first of the roles $T_k$, i.e., $T_0$ in $\mathcal{R}^n_\vee$ and $T_1$ in $\mathcal{R}^n_\wedge$ and $\mathcal{R}^n_{\wedge\!\!\!\vee}$; ans similarly $Q_z$. To encode a grid, we add the following axioms (a), (b), (c), (d) to an $\mathcal{ALCQ}$-TBox $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$ and axioms (a), (b), (c'), (d') to an $\mathcal{ALCN}$-TBox $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$ (as above, we denote $i \oplus j := (i + j) \bmod 4$):

(a) All 16 concept names $A_{ij}$, $0 \leqslant i, j \leqslant 3$, are pairwise disjoint;

(b) $A \sqsubseteq \exists r.B$ for each *single* edge $\langle A, r, B\rangle$, where $r \in \{R, S\}$;
$A \sqsubseteq \exists Q_k.B$ for each *double* edge $\langle A, {}^{R}_{S}, B\rangle$ and each $Q_k$;

(c) $A \sqsubseteq \leqslant 1\, T_k.B$ for each *double* edge $\langle A, {}^{R}_{S}, B\rangle$ and each $T_k$ that has two subroles;

(c') $A_{ij} \sqsubseteq \leqslant 2\, T_k$ for each $T_k$ that has two subroles except for $T_z$ if $i \oplus j = 0$, or except for $T_n$ if $i \oplus j = 2$;

(d) $A_{ij} \sqsubseteq \leqslant 1\, r.A_{i\oplus 1, j\oplus 1}$, where $r = \begin{cases} R, & \text{if } i \oplus j \in \{0, 1\}, \\ S, & \text{if } i \oplus j \in \{2, 3\}; \end{cases}$

(d') $A_{ij} \sqsubseteq \leqslant 9\, T_z$ for $i \oplus j = 0$, and $A_{ij} \sqsubseteq \leqslant 9\, T_n$ for $i \oplus j = 2$.

For $\mathcal{X} \in \{\mathcal{Q}, \mathcal{N}\}$, we set $\mathcal{K}^{\mathcal{X}}_{\mathcal{D}} := \mathcal{R} + \mathcal{T}^{\mathcal{X}}_{\mathsf{grid}} + \mathcal{T}_{\mathcal{D}}$, where the TBox $\mathcal{T}_{\mathcal{D}}$ is defined as in Theorem 1, and prove the following lemma simultaneously for $\mathcal{R}^n_{\wedge}$, $\mathcal{R}^n_{\vee}$, and $\mathcal{R}^n_{\wedge\!\!\!\vee}$.

**Lemma 2.1.** *The concept $A_{00}$ is satisfiable w.r.t. $\mathcal{K}^{\mathcal{X}}_{\mathcal{D}}$ iff $\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$.*

($\Leftarrow$) Similar to Lemma 1.1, but here we interpret roles on $\Delta^{\mathcal{I}}$ as follows: roles $R$ and $S$ as in Lemma 1.1; all $Q_k$ that differ from $R$ and $S$ are interpreted as $R^{\mathcal{I}} \cap S^{\mathcal{I}}$; finally, all $T_k$ that differ from $R$ and $S$ are interpreted as as the union of the interpretations of their two subroles, i.e., $Q^{\mathcal{I}}_{k-1} \cup Q^{\mathcal{I}}_k$ in $\mathcal{R}^n_{\wedge}$ and $\mathcal{R}^n_{\wedge\!\!\!\vee}$, and $Q^{\mathcal{I}}_k \cup Q^{\mathcal{I}}_{k+1}$ in $\mathcal{R}^n_{\vee}$. Observe that all $Q_k$ that differ from $R$ and $S$ are interpreted as the double edges in Fig. 2a. It is straightforward to show that $\mathcal{I} \models \mathcal{K}^{\mathcal{X}}_{\mathcal{D}}$ and $A^{\mathcal{I}}_{00} \neq \varnothing$.

($\Rightarrow$) The proof follows the same pattern as in Lemma 1.2: assuming that $\mathcal{I} \models \mathcal{K}^{\mathcal{X}}_{\mathcal{D}}$ and $a_{00} \in A^{\mathcal{I}}_{00}$, we build a grid $\{a_{ij} \mid i, j \in \mathbb{N}\} \subseteq \Delta^{\mathcal{I}}$ depicted in Fig. 2a, and then build a $\mathcal{D}$-tiling of $\mathbb{N}\times\mathbb{N}$. To this end, first we build, using $\mathcal{ALC}$ axioms (b) only, a pre-grid similar to that shown in Fig. 3, but with the following difference: for each $a_{ij}$ with even $i + j$, instead of one horisontal and one vertical $Q$-edges outgoing from $a_{ij}$, now we will have $N$ horisontal and $N$ vertical edges outgoing from $a_{ij}$ that correspond to all the roles $Q_k$, where $N$ is the number of roles $Q_k$ in the RBox. The $Q$-edges outgoing from $a_{ij}$ in Fig. 3 should be identified with $Q_z$ resp. $Q_n$, for $i \oplus j = 0$ resp. 2.

Next, by applying axiom (c) or (c') (depending on our logic), we merge all horisontal (and similarly for vertical) $Q_k$-edges outgoing from $a_{ij}$ with even $i + j$ into a single "$Q$-edge". Indeed, e.g. for $\mathcal{R}^n_{\vee}$ and $a_{00}$, we have $\langle a_{00}, a^k_{10}\rangle \in Q^{\mathcal{I}}_k$ for $1 \leqslant k \leqslant n$. Then $\langle a_{00}, a^k_{10}\rangle, \langle a_{00}, a^{k+1}_{10}\rangle \in T^{\mathcal{I}}_k$, for all $0 < k < n$, i.e., exactly for all $k$ such that $T_k$ has two subroles. Now applying axiom (c), we conclude that $a^k_{10} = a^{k+1}_{10}$, for all $0 < k < n$, i.e., all horisontal $Q_k$-edges merged in a single edge. Similarly we merge all vertical $Q_k$-edges outgoing from $a_{ij}$ with even $i + j$.

To do the same job again for $\mathcal{R}^n_{\vee}$ and $a_{00}$, but with the help of axiom (c'), consider in addition $n$ vertical edges $\langle a_{00}, a^k_{01}\rangle \in Q^{\mathcal{I}}_k$ for $1 \leqslant k \leqslant n$. Then, for each $0 < k < n$, $a_{00}$ has 2 $T_k$-successors in $A^{\mathcal{I}}_{10}$ and 2 $T_k$-successors in $A^{\mathcal{I}}_{01}$. Since these two concepts are disjoint, applying axiom (c') yields that $a^k_{10} = a^{k+1}_{10}$ and $a^k_{01} = a^{k+1}_{01}$, for all $0 < k < n$.

Now we have exactly the pre-grid shown in Fig. 3, and we need to turn it into a grid using the remaining axioms in $\mathcal{T}^{\mathcal{X}}_{\mathsf{grid}}$. The remainder of the proof repeats that of Lemma 1.2. This completes the proof of Lemma 2.1 and hence of Theorem 2. $\dashv$

**Theorem 3.** *The RBox $\mathcal{R} := \{\mathsf{Tr}(R)\}$ is unsafe for $\mathcal{ALCIN}$ (more precisely, for $\mathcal{ALCIN}_8$ and $\mathcal{ALCIQ}_1$), even for TBoxes with a single role name $R$.*

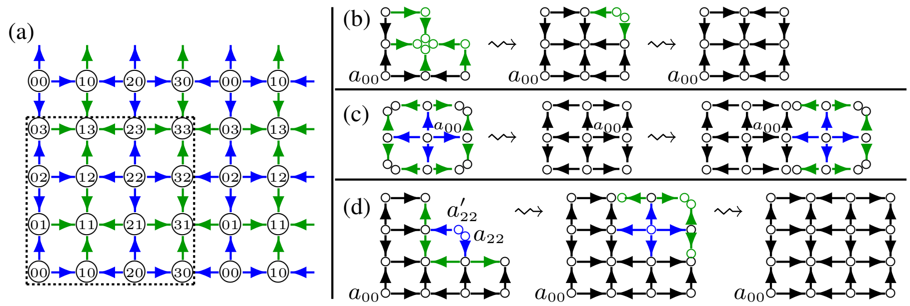


**Fig. 5.** (a) A grid for Theorem 3 (coloured for decoration only, as we have only 1 role). (b) A pre-grid for $\mathcal{ALCIQ}$. (c) Building a horisontal axis in $\mathcal{ALCIN}$. (d) Building new cells in $\mathcal{ALCIN}$.

*Proof.* Take 16 concept names $A_{ij}$, $0 \leqslant i, j \leqslant 3$. Place them on the $\mathbb{N}\times\mathbb{N}$ grid (by repeating a $[0,3]\times[0,3]$ pattern periodically) and link them with $R$-edges in accordance with Fig. 5a. Now, having this picture in mind (we refer to its edges as $\langle A, R, B\rangle$, where $A, B$ are concept names), we add the following axioms (a)–(c) to an $\mathcal{ALCIQ}$-TBox $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$ and axioms (a) and (d) to an $\mathcal{ALCIN}$-TBox $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$:

(a) All 16 concept names $A_{ij}$, $0 \leqslant i, j \leqslant 3$, are pairwise disjoint;
(b) $A \sqsubseteq \exists R.B$ and $B \sqsubseteq \exists R^-.A$, for each edge $\langle A, R, B\rangle$;
(c) $A_{ij} \sqsubseteq \leqslant 1\, R.A_{k\ell}$ and $A_{k\ell} \sqsubseteq \leqslant 1\, R^-.A_{ij}$, for all even $i, j$ and odd $k, \ell$;
(d) $A_{ij} \sqsubseteq \leqslant 8\, R$ and $A_{k\ell} \sqsubseteq \leqslant 8\, R^-$, for all even $i, j$ and odd $k, \ell$.

Given a domino system $\mathcal{D}$, we build a $\mathcal{ALC}$-TBox $\mathcal{T}_{\mathcal{D}}$: axioms (e) and (f) are the same as in the proof of Theorem 1, whereas (g) is the following (and (h) is analogous):

(g) $A \sqcap D_k \sqsubseteq \forall R.\big(B \to \bigsqcup_{\ell:\, \langle d_k, d_\ell\rangle \in H} D_\ell\big)$ for each right-going edge $\langle A, R, B\rangle$;
$A \sqcap D_\ell \sqsubseteq \forall R.\big(B \to \bigsqcup_{k:\, \langle d_k, d_\ell\rangle \in H} D_k\big)$ for each left-going edge $\langle A, R, B\rangle$.

Finally, for each $\mathcal{X} \in \{\mathcal{Q}, \mathcal{N}\}$, we set $\mathcal{K}^{\mathcal{X}}_{\mathcal{D}} := \mathcal{R} + \mathcal{T}^{\mathcal{X}}_{\mathsf{grid}} + \mathcal{T}_{\mathcal{D}}$. It remains to prove

**Lemma 3.1.** *The concept $A_0$ is satisfiable w.r.t. $\mathcal{K}^{\mathcal{X}}_{\mathcal{D}}$ iff $\mathcal{D}$ tiles $\mathbb{N}\times\mathbb{N}$.*

The implication '$\Leftarrow$' is proved as in Theorem 1. To prove '$\Rightarrow$', suppose that $\mathcal{I} \models \mathcal{K}^{\mathcal{X}}_{\mathcal{D}}$ and $a_{00} \in A^{\mathcal{I}}_{00}$. Then we extract a grid from $\mathcal{I}$, i.e., a subset of (not necessarily distinct) elements $\mathsf{G} := \{a_{ij} \mid i, j \in \mathbb{N}\} \subseteq \Delta^{\mathcal{I}}$, such that the restriction of $\mathcal{I}$ to $\mathsf{G}$ looks like Fig. 5a, and then build a $\mathcal{D}$-tiling of $\mathbb{N}\times\mathbb{N}$.

For $\mathcal{T}^{\mathcal{Q}}_{\mathsf{grid}}$: We start with $a_{00}$ and apply axiom (b) to build a horizontal and a vertical axes that consist of elements $a_{i,0} \in A^{\mathcal{I}}_{i \bmod 4, 0}$ and $a_{0,j} \in A^{\mathcal{I}}_{0, j \bmod 4}$ ($i, j \geqslant 1$), which are linked via $R$-edges of interleaving orientation (as in Fig. 5a). Then, whenever we have 2 horizontal and 2 vertical edges, say, $a_{00} R a_{10} R^- a_{20}$ and $a_{00} R a_{01} R^- a_{02}$, we complete them into a $3\times 3$ grid as follows. First, using (b), we build 10 new elements that form a "pre-grid" depicted in Fig. 5b, namely, $a_{20}, a_{02}, a_{21}, a_{12}, a_{22}, a'_{22}, a_{11}, a'_{11}, a''_{11}, a'''_{11}$. Next, by axiom (c), only 1 element in $A^{\mathcal{I}}_{11}$ is $R$-reachable from each of $a_{00}$, $a_{20}$, and $a_{02}$, thus: $a_{11} = a'_{11} = a''_{11} = a'''_{11}$. Finally, we apply axiom (c) to $a_{11}$ and conclude that $a_{22} = a'_{22}$. Continuing this process infinitely, we build all elements $a_{ij}$, for $i, j \in \mathbb{N}$.

For $\mathcal{T}^{\mathcal{N}}_{\mathsf{grid}}$: For organising the elements $a_{ij}$ ($i, j \geqslant 0$) into a grid, we need two extra rows $a_{i,-1}$ and $a_{-1,j}$ with $i, j \geqslant -1$, see Fig. 5c. First, applying axiom (b) for suitable $B$'s to the element $a_{00}$, we create 4 elements: $a_{0,\pm 1}$ and $a_{\pm 1, 0}$. Again by (b), each of them in turn can $R$-reach 2 elements, so we create 8 new elements: $a_{\pm 1,\pm 1}$ and $a'_{\pm 1,\pm 1}$. Note that, in the above passage, whenever we create an element $a_{ij}$, we mean that it belongs to $A^{\mathcal{I}}_{i \bmod 4, j \bmod 4}$. Now apply axiom (d) and recall that $A^{\mathcal{I}}_{ij}$ are pairwise disjoint, hence $a_{\pm 1,\pm 1} = a'_{\pm 1,\pm 1}$. Thus we obtain a 3×3 grid of elements $a_{ij}$ with $-1 \leqslant i, j \leqslant 1$.

Next, we create an element $a_{20}$ by axiom (b) $A_{10} \sqsubseteq \exists R^-.A_{20}$ and surround it by a 3×3 grid as before. Repeating this process, we build a horizontal axis $a_{i,0}$ surrounded by two axes $a_{i,\pm 1}$ (for $i \geqslant -1$) and, similarly, a vertical axis $a_{0,j}$ surrounded by two axes $a_{\pm 1,j}$ (for $j \geqslant -1$). Note that number restrictions on $R^-$ were not used so far.

Now, whenever we have three adjacent 3×3 grids, say, with the centres $a_{00}$, $a_{20}$, and $a_{02}$, we build the fourth one, with the centre $a_{22}$, as follows. First, apply axiom (b) $A \sqsubseteq \exists R^-.A_{22}$ to $a_{21}$ and $a_{12}$, thus creating two new elements $a_{22}, a'_{22} \in A^{\mathcal{I}}_{22}$ such that $\langle a_{22}, a_{21}\rangle, \langle a'_{22}, a_{12}\rangle \in R^{\mathcal{I}}$ (see Fig. 5d). Now apply axiom (d) $A_{11} \sqsubseteq \leqslant 8\, R^-$ to the element $a_{11}$ and observe that $a_{11}$ can $R^-$-reach 9 neighbours which belong to 8 disjoint concepts; therefore $a_{22} = a'_{22}$. Finally, we surround $a_{22}$ by a 3×3 grids similarly to the above (with the only difference that 5 of 8 elements surrounding $a_{22}$ are already built).

Once the set $\{a_{ij} \mid i, j \in \mathbb{N}\}$ is ready, we complete the proof as in Theorem 1. $\dashv$

In the proof of Theorem 3, we essentially used number restrictions on the role $R$ *and* on its inverse. An open question is whether allowing number restrictions in one direction only (w.l.o.g., on $R$) would make the logic decidable. For some applications, this expressivity would be sufficient, e.g., we will be able to express a class of proteins that contain at least / at most $n$ amino-acids of a certain type [10], but usually we do not need to talk about amino-acids that are contained in a given number of proteins.

## 4 Internalization of RBoxes in TBoxes Using Extended Roles

In order to study safety of RBoxes for different DLs, it is somewhat inconvenient to work separately with RBoxes and TBoxes. Therefore, in this section, we demonstrate how RBoxes can be internalized into TBoxes, provided additional role constructors—role unions and transitive closure operator—can be used. We also demonstrate that it is sufficient to focus only on TBoxes of some simple form. The results of this section can be applied to any logic $\mathcal{L}$ between $\mathcal{ALC}$ and $\mathcal{ALCIQ}$.

**Definition 5.** We say that an $\mathcal{L}$-TBox $\mathcal{T}$ is in a *simple form* if all axioms in $\mathcal{T}$ have the following forms, where $A_{(i)}$, $B_{(j)}$ are concept names, $m$, $n$ integers, and $S$ a role:

$$\textstyle\bigsqcap A_i \sqcap \bigsqcap \neg B_j \sqsubseteq \bot \tag{1}$$

$$A \sqsubseteq \geqslant n\, S.B \tag{2}$$

$$A \sqsubseteq \leqslant m\, S.B \tag{3}$$

**Lemma 2 (Simplification of $\mathcal{L}$-TBoxes).** *Given an $\mathcal{L}$-TBox $\mathcal{T}$, one can construct in polynomial time an $\mathcal{L}$-TBox $\mathcal{T}_{sf}$ in simple form such that, for every RBox $\mathcal{R}$, $\langle \mathcal{T}, \mathcal{R}\rangle$ is (finitely) satisfiable iff $\langle \mathcal{R}, \mathcal{T}_{sf}\rangle$ is (finitely) satisfiable.*

*Proof.* The transformation to the simple form can be done by applying the usual structural transformation for DLs (see e.g. [8]). $\dashv$

**Definition 6.** The set of *extended roles* $\mathbf{R}^{\sqcup,+}$ is defined by the following grammar:

$$\mathbf{R}^{\sqcup,+} ::= R \mid \rho_1 \sqcup \rho_2 \mid \rho^+, \quad \text{where } R \text{ is a role and } \rho_{(i)} \in \mathbf{R}^{\sqcup,+}.$$

The additional role constructors are interpreted as follows: $(\rho_1 \sqcup \rho_2)^{\mathcal{I}} = \rho_1^{\mathcal{I}} \cup \rho_2^{\mathcal{I}}$, $(\rho^+)^{\mathcal{I}} = (\rho^{\mathcal{I}})^+$, where $(\cdot) \cup (\cdot)$ and $(\cdot)^+$ are usual operators of union and transitive closure on binary relations. Concepts of $\mathcal{L}(\sqcup,+)$ are defined as for $\mathcal{L}$ except that extended roles can be used in place of roles. The semantics of $\mathcal{L}(\sqcup,+)$ is defined as for $\mathcal{L}$, where the interpretation of extended roles is used.

Our goal is to demonstrate that every RBox can be internalized in a simple $\mathcal{L}$-TBox producing an $\mathcal{L}(\sqcup,+)$-TBox of a certain simple form:

**Definition 7 (Simple $\mathcal{L}(\sqcup,+)$-TBox).** We say that an $\mathcal{L}(\sqcup,+)$-TBox $\mathcal{T}$ is *simple* if every axiom from $\mathcal{T}$ is either of the form (1), (2), or:

$$A \sqsubseteq \leqslant m\,(\textstyle\bigsqcup u_i^+ \sqcup v).B \tag{4}$$

where $A_{(i)}$, $B_{(j)}$ are concept names, $m$, $n$ integers, and $u_i$ and $v$ are disjunctions of roles: $u_i, v = \bigsqcup R_i$. For a simple TBox $\mathcal{T}$, we denote by $K(\mathcal{T})$ the number of axioms of type (4) in $\mathcal{T}$, by $N(\mathcal{T})$ and $M(\mathcal{T})$ the sum of all numbers $n$, resp. $m$, over all axioms of type (2), resp. (4), by $C(\mathcal{T})$ the number of concept names in $\mathcal{T}$.

In order to speak about the relationship between roles induced by RBoxes, we introduce additional terminology and notation:

**Definition 8.** Given an RBox $\mathcal{R}$ and two roles $S$ and $S'$, we say that $S$ *is a subrole of* $S'$ *in* $\mathcal{R}$ if $\mathcal{R} \vdash S \sqsubseteq S'$; $S$ *is equivalent to* $S'$ *in* $\mathcal{R}$ (notation: $\mathcal{R} \vdash S \equiv S'$) if $\mathcal{R} \vdash S \sqsubseteq S'$ and $\mathcal{R} \vdash S' \sqsubseteq S$; $S$ *is directly related to* $S'$ *in* $\mathcal{R}$, or $S$ *and* $S'$ *are comparable in* $\mathcal{R}$ (notation: $\mathcal{R} \vdash S \sim S'$) if $\mathcal{R} \vdash S \sqsubseteq S'$ or $\mathcal{R} \vdash S' \sqsubseteq S$; finally, $S$ *is related to* $S'$ *in* $\mathcal{R}$, or $S$ *and* $S'$ *are connected in* $\mathcal{R}$ (notation $\mathcal{R} \vdash S \overset{*}{\sim} S'$) if there exists a sequence of roles $S = S_1, \ldots, S_n = S'$ with $n \geqslant 1$ such that $\mathcal{R} \vdash S_i \sim S_{i+1}$ for all $1 \leqslant i < n$.

We say that $T$ is *transitive* in $\mathcal{R}$ if $\mathcal{R} \vdash \mathsf{Tr}(T)$; if additionally $T$ is a subrole of $S$ and for every transitive role $T'$ in $\mathcal{R}$, we have $\mathcal{R} \vdash T \sqsubseteq T' \sqsubseteq S$ implies $\mathcal{R} \vdash T \equiv T'$, then $T$ *is a maximal transitive subrole of* $S$ *in* $\mathcal{R}$.

**Definition 9 ($\mathcal{R}$-extension).** Given an RBox $\mathcal{R}$, an *extension* of a role $S$ in $\mathcal{R}$ (or the $\mathcal{R}$*-extension* of $S$, for short) is an extended role $\mathcal{R}(S) \in \mathbf{R}^{\sqcup,+}$ defined as follows:

- If $S$ is transitive in $\mathcal{R}$ then $\mathcal{R}(S) := (\bigsqcup S_i)^+$, where $\{S_i\}$ is the set of all subroles of $S$ in $\mathcal{R}$ (including $S$ itself);
- If $S$ is not transitive, then $\mathcal{R}(S) := \bigsqcup \mathcal{R}(T_i) \sqcup \bigsqcup S_j$, where $\{T_i\}$ is exactly the set of all maximal transitive subroles of $S$, and $\{S_j\}$ is the set of all subroles of $S$.

The $\mathcal{R}$*-extension* of an interpretation $\mathcal{I}$ is an interpretation $\mathcal{J} = \mathcal{R}(\mathcal{I})$ which is defined as $A^{\mathcal{J}} := A^{\mathcal{I}}$, for each concept name $A$, and $r^{\mathcal{J}} := (\mathcal{R}(r))^{\mathcal{I}}$, for each atomic role $r$.

*Remark 2.* Note that $\mathcal{R}(S)$ can be computed in polynomial time in the size of $\mathcal{R}$.

**Lemma 3 (Semantic Properties of $\mathcal{R}$-extensions).**
*Let $\mathcal{R}$ be an RBox and $\mathcal{I}$, $\mathcal{J}$ interpretations. Then: (1) $\mathcal{R}(\mathcal{I}) \models \mathcal{R}$; (2) $\mathcal{I} \models \mathcal{R}$ implies $\mathcal{R}(\mathcal{I}) = \mathcal{I}$; (3) $S^{\mathcal{I}} \subseteq S^{\mathcal{R}(\mathcal{I})}$ for every $S$; and (4) if $S^{\mathcal{I}} \subseteq S^{\mathcal{J}}$ for every $S$, then $S^{\mathcal{R}(\mathcal{I})} \subseteq S^{\mathcal{R}(\mathcal{J})}$ for every $S$ (monotonicity property).*
*Proof.* $\dashv$

**Definition 10 (Internalization of an RBox in an $\mathcal{L}$-TBox).**
Let $\mathcal{R}$ be an RBox and $\mathcal{T}$ be a simple $\mathcal{L}$-TBox. The internalization of $\mathcal{R}$ in $\mathcal{T}$ is a simple $\mathcal{L}(\sqcup, +)$-TBox $\mathcal{R}(\mathcal{T}) := \{\mathcal{R}(\alpha) \mid \alpha \in \mathcal{T}\}$, where:

- $\mathcal{R}(\alpha) := \alpha$ if $\alpha$ is of the form (1) or (2), and
- $\mathcal{R}(\alpha) := A \sqsubseteq \leqslant m\, (\mathcal{R}(S)).B$ if $\alpha = A \sqsubseteq \leqslant m\, S.B$ is of the form (3).

**Lemma 4.** *Let $\mathcal{R}$ be an RBox and $\mathcal{T}$ a simple $\mathcal{L}$-TBox. Then $\langle \mathcal{R}, \mathcal{T} \rangle$ is (finitely) satisfiable iff $\mathcal{R}(\mathcal{T})$ is (finitely) satisfiable.*
*Proof.* $\dashv$

## 5 Decidability Results

As we have demonstrated in Theorem 3, an RBox consisting of just one transitivity axiom is already unsafe for $\mathcal{ALCIN}$. In fact, we demonstrate in Section 6 that this is true for every RBox containing one transitive non-symmetric role. Hence, there is a little room left for non-trivial safe RBoxes for $\mathcal{ALCIN}$. In contrast, the undecidability results in Section 3 for $\mathcal{ALCN}$ require a certain interaction between several transitive roles. This poses a question about safety of those RBoxes that do not fit such a pattern. In this section, we investigate this question and define a relatively large class of so-called admissible RBoxes that, as we will prove, are safe for $\mathcal{ALCQ}$. Since we focus on $\mathcal{ALCQ}$, within this section we assume that there are no inverse roles in RBoxes.

**Definition 11.** For a TBox $\mathcal{T}$, RBox $\mathcal{R}$, or an axiom $\alpha$, let $\mathsf{RN}(\mathcal{T})$, $\mathsf{RN}(\mathcal{T})$, $\mathsf{RN}(\alpha)$ denote the set of role names that occur in $\mathcal{T}$, $\mathcal{R}$, $\alpha$, respectively.

An RBox $\mathcal{R}$ is *strongly admissible* if, for every two transitive roles $T_1, T_2 \in \mathsf{RN}(\mathcal{R})$, we have $\mathcal{R} \vdash T_1 \sim T_2$. An RBox $\mathcal{R}$ is *admissible* if $\mathcal{R} = \bigcup \mathcal{R}_i$ where (1) each $\mathcal{R}_i$ is strongly admissible and (2) $\mathsf{RN}(\mathcal{R}_i) \cap \mathsf{RN}(\mathcal{R}_j) = \varnothing$ for all $i \neq j$.

Under the terminology of Definition 8, we can view an RBox as a directed graph, where nodes correspond to roles and edges correspond to role inclusion axioms in RBox. Under this correspondence, we can describe the class admissible RBoxes as those in which transitive roles are linearly ordered in every connected component.

In the remainder of this section, we prove the following Theorem:

**Theorem 4.** *Every admissible RBox is $\mathcal{ALCQ}$-safe.*

*Note 1.* For $\mathcal{R} = \{\mathsf{Tr}(r)\}$, this result corresponds to the decidability of the graded variant of the modal logic $\mathbf{K4}$ (called $\mathbf{GrK4}$), which has already been addressed in [4]via the finite model property (FMP). It seems, however, that the proof in this paper is incorrect already for $\mathbf{K4}$ (although works fine for Graded $\mathbf{K}$): the paper claims that any

$\mathbf{K4}$-model of any formula $\varphi$ has a finite $\mathbf{K4}$-submodel, i.e., can be restricted to a finite model by removing all but finitely many elements from its domain. However, this is not the case, say, for $\varphi = \Diamond p \wedge \Box(p \to \Diamond p)$, whose $\mathbf{K4}$-model $(\mathbb{N}, <)$ has no finite $\mathbf{K4}$-submodels of $\varphi$, although it has a finite $\mathbf{K}$-submodel of $\varphi$, namely $\langle W, R\rangle$, where $W = \{0, 1, 2\}$ and $R = \{\langle 0, 1\rangle, \langle 1, 2\rangle\}$. Note that it is well-known that $\mathbf{K4}$ enjoys FMP: in our example, $\varphi$ has a finite $\mathbf{K4}$-model $\langle W, R\rangle$, where $W = \{0\}$ and $R = \{\langle 0, 0\rangle\}$; however, it cannot be obtained as a submodel of $(\mathbb{N}, <)$.

In this paper we re-establish decidability of $\mathbf{GrK4}$ as a special case of Theorem 4. In fact, our result will imply that $\mathbf{GrK4}$ indeed enjoys finite model property.

First of all, we demonstrate that, for the purpose of proving safety, it is sufficient to focus only on strongly admissible RBoxes.

**Lemma 5 (Modularity).** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be RBoxes with $\mathsf{RN}(\mathcal{R}_1) \cap \mathsf{RN}(\mathcal{R}_2) = \varnothing$ and $\mathcal{L}$ is between $\mathcal{ALC}$ and $\mathcal{ALCIQ}$. Then $\mathcal{R}_1 \cup \mathcal{R}_2$ is $\mathcal{L}$-safe iff $\mathcal{R}_1$ and $\mathcal{R}_2$ are $\mathcal{L}$-safe.*

*Proof.* The '$\Rightarrow$' part of the lemma is obvious. In order to prove the '$\Leftarrow$' part, we use the results about fusions of DLs from [2]. For any fixed RBox $\mathcal{R}$, the logic $\mathcal{L}(\mathcal{R})$ can be expounded as an *abstract description system* introduced therein. Indeed, the $\mathcal{L}$-component determines the syntax and semantics for concepts, whereas $\mathcal{R}$ restricts the class of $\mathcal{L}$-models (to those where all axioms from $\mathcal{R}$ hold). Moreover, this class of $\mathcal{L}$-models is closed under disjoint unions, hence $\mathcal{L}(\mathcal{R})$ is *local* according to the definition from [2]. Finally, $\mathcal{L}(\mathcal{R}_1 \cup \mathcal{R}_2)$ is precisely the *fusion* $\mathcal{L}(\mathcal{R}_1) \otimes \mathcal{L}(\mathcal{R}_2)$ as defined in that paper, since the restrictions imposed by the logic $\mathcal{L}(\mathcal{R}_1 \cup \mathcal{R}_2)$ on the class of $\mathcal{L}$-models are independent for roles from $\mathcal{R}_1$ and $\mathcal{R}_2$ (here we use the fact that $\mathcal{R}_1$ and $\mathcal{R}_2$ do not share role names). Since we deal with the problem of concept satisfiability w.r.t. general TBoxes, our lemma follows from Corollary 23 in [2]. $\dashv$

**Corollary 1.** *Let $\mathcal{L}$ be a logic between $\mathcal{ALC}$ and $\mathcal{ALCIQ}$. Then every admissible RBox is $\mathcal{L}$-safe provided every strongly admissible RBox is $\mathcal{L}$-safe.*

In order to prove that every strongly admissible RBox $\mathcal{R}$ is safe, according to Remark 1, it is sufficient to show that the problem of satisfiability of a pair $\langle \mathcal{R}, \mathcal{T}\rangle$, with $\mathcal{T}$ an $\mathcal{L}$-TBox, is decidable. To this end, we first simplify the TBox $\mathcal{T}$ using Proposition 2 and then internalize RBox $\mathcal{R}$ using Definition 10, which will result in some $\mathcal{L}(\sqcup, +)$-TBox of a restricted form, which we call admissible. We then demonstrate that satisfiability of admissible $\mathcal{L}(\sqcup, +)$-TBoxes is decidable.

In what follows, for convenience, we often identify an extended role $u = \bigsqcup R_i$ with the set $\bigcup\{R_i\}$. Using this convention, we can write $r \in u$ or $u \subseteq u'$ for disjunction of roles $u$ and $u'$, as well as $u^{\mathcal{I}}$ for sets of roles $u$.

**Definition 12.** A simple $\mathcal{L}(\sqcup, +)$ TBox $\mathcal{T}$ is *admissible* if $(i)$ all axioms of form (4) are of the forms (5) and (6) below, and $(ii)$ for every two axioms $A_1 \sqsubseteq\, \leqslant m_1\, (u_1^+ \sqcup v_1).B_1$ and $A_2 \sqsubseteq\, \leqslant m_2\, (u_2^+ \sqcup v_2).B_2$ of form (6), we have that either $u_1 \subseteq u_2$, or $u_2 \subseteq u_1$.

$$A \sqsubseteq\, \leqslant m\, (v).B \tag{5}$$

$$A \sqsubseteq\, \leqslant m\, (u^+ \sqcup v).B \tag{6}$$

In other words, a simple $\mathcal{L}(\sqcup,+)$-TBox is admissible if in every axiom of form (4) there is at most one occurrence of a transitively closed disjunction of roles.

**Lemma 6.** *Let $\mathcal{T}$ be a simple $\mathcal{L}$-TBox and $\mathcal{R}$ a strongly admissible RBox. Then $\mathcal{R}(\mathcal{T})$ is a simple admissible $\mathcal{L}(\sqcup,+)$-TBox.*
*Proof.* $\dashv$

The condition $(ii)$ from Definition 12 can be alternatively formulated as follows:

**Proposition 1.** *Let $\mathcal{T}$ be a simple admissible $\mathcal{L}(\sqcup,+)$-TBox. Then all roles in $\mathcal{T}$ can be ordered as $r_1,\ldots,r_n$ in such a way that for every axiom $A \sqsubseteq\, \leqslant m\,(u^+ \sqcup v).B$ of type* (6) *and every $1 \leqslant i \leqslant j \leqslant n$, we have that $r_j \in u$ implies $r_i \in u$.*
*Proof.* $\dashv$

We prove that satisfiability of simple admissible $\mathcal{L}(\sqcup,+)$-TBoxes is decidable by demonstrating the finite model property (FMP) for such TBoxes. The key property that will guarantee FMP is that, in every model of a simple admissible TBox, it is possible to "loop back" every sufficiently long chain of elements connected via roles. Therefore, given a possibly infinite model of a TBox, we can consider its finite part of a bounded branching and up to a certain depth in order to build a finite model. This idea is reminiscent to blocking conditions in tableau decision procedures for modal and description logics [7].

The next lemma states that every model of a simple $\mathcal{L}(\sqcup,+)$ TBox can be reduced to a model with bounded branching degree by removing edges that are not "required" by axioms of type (2).

**Definition 13 (Branching Degree of an Interpretation).**
Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an $\mathcal{L}$-interpretation. A *branching degree* of an element $x \in \Delta^{\mathcal{I}}$ in $\mathcal{I}$ is $\mathsf{deg}(\mathcal{I},x) = \mathsf{Card}\{y \mid \langle x,y\rangle \in r^{\mathcal{I}} \text{ for some } r\}$. A branching degree of $\mathcal{I}$ is $\mathsf{deg}(\mathcal{I}) = \max\{\mathsf{deg}(\mathcal{I},x) \mid x \in \Delta^{\mathcal{I}}\}$.

**Lemma 7.** *Any satisfiable simple $\mathcal{L}(\sqcup,+)$-TBox $\mathcal{T}$ has a model $\mathcal{I}$ with $\mathsf{deg}(\mathcal{I}) \leqslant N(\mathcal{T})$.*
*Proof.* $\dashv$

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. For each axiom $\alpha$ of type (6) in $\mathcal{T}$, we introduce a function $\delta_\alpha^{\mathcal{I}}(x,y)$ defined on elements of $\Delta^{\mathcal{I}}$ as follows:

$$\delta_\alpha^{\mathcal{I}}(x,y) = \begin{cases} \mathsf{Card}\{x' \mid x' \in B^{\mathcal{I}},\ \langle x,x'\rangle \in (u^+)^{\mathcal{I}},\ \langle y,x'\rangle \notin (u^+)^{\mathcal{I}}\} & \\ \qquad\qquad\qquad\qquad\qquad \text{if there exists } y' \in A^{\mathcal{I}} \text{ with } \langle y',y\rangle \in (u^+)^{\mathcal{I}} & \\ 0 \qquad\qquad\qquad\qquad\quad\ \text{otherwise} & \end{cases}$$

In other words, if $y$ has a $u^+$ predecessor in which $A$ holds, $\delta_\alpha^{\mathcal{I}}(x,y)$ equals to the number of elements in which $B$ holds and that are reachable via $u^+$ from $x$ but not from $y$ (see Fig. 6a). The value of $\delta_\alpha^{\mathcal{I}}(x,y)$ intuitively indicates the number of new $u^+$ successors of $y$ that might appear and potentially violate the axiom $\alpha$ (at the points, where $A$ holds), if $x$ becomes reachable from $y$ via $u^+$.

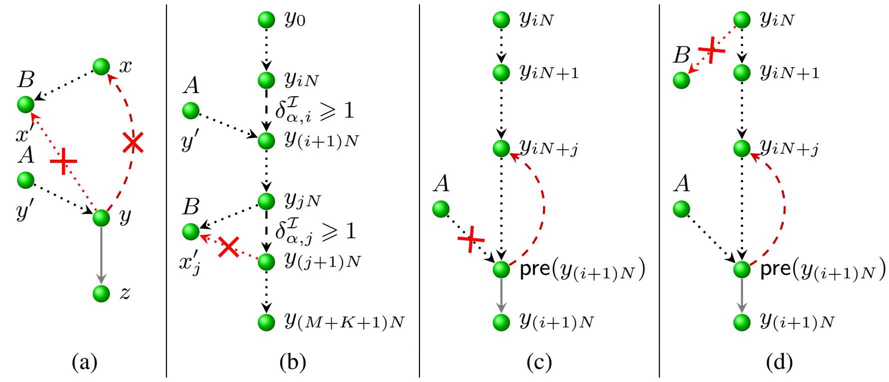


**Fig. 6.** Looping long chains in a model back

**Definition 14.** Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. For an element $x \in \Delta^{\mathcal{I}}$, let $\mathsf{CN}^{\mathcal{I}}(x) := \{A \in \mathsf{CN} \mid x \in A^{\mathcal{I}}\}$ denote the set of concept names that hold at $x$ in $\mathcal{I}$.

Given a simple admissible $\mathcal{L}(\sqcup, +)$-TBox $\mathcal{T}$, an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, and $x, y, z \in \Delta^{\mathcal{I}}$, we say that $x$ *can foster* $z$ *for* $y$ *in* $\mathcal{I}$ *(w.r.t.* $\mathcal{T}$*)* if $(i)$ $\mathsf{CN}^{\mathcal{I}}(z) = \mathsf{CN}^{\mathcal{I}}(x)$, $(ii)$ $\langle y, x\rangle \in r^{\mathcal{I}}$ for no atomic role $r$, and $(iii)$ for every axiom $\alpha$ of type (6) in $\mathcal{T}$, if $\langle y, z\rangle \in r^{\mathcal{I}}$ for some role $r \in u_\alpha = u$, then $\delta^{\mathcal{I}}_\alpha(x, y) = 0$.

**Lemma 8 (Model Transformation).** *Let* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *be a model of a simple admissible* $\mathcal{L}(\sqcup, +)$*-TBox* $\mathcal{T}$ *and* $x$*,* $y$*,* $z$ *elements of* $\Delta^{\mathcal{I}}$ *such that* $x$ *can foster* $z$ *for* $y$ *in* $\mathcal{I}$ *w.r.t.* $\mathcal{T}$*. Let* $\mathcal{J} = \mathsf{swap}(\mathcal{I}, x, y, z)$ *be obtained from* $\mathcal{I}$ *by setting* $A^{\mathcal{J}} := A^{\mathcal{I}}$ *and,* $r^{\mathcal{J}} := r^{\mathcal{I}} \setminus \{\langle y, z\rangle\} \cup \{\langle y, x\rangle\}$ *if* $\langle y, z\rangle \in r^{\mathcal{I}}$*, and* $r^{\mathcal{J}} := r^{\mathcal{I}}$ *otherwise, for every concept name* $A$ *and role name* $r$*. Then* $\mathcal{J}$ *is a model of* $\mathcal{T}$*.*

*Proof.* Since for every $x \in \Delta^{\mathcal{I}}$, we have $\mathsf{CN}^{\mathcal{J}}(x) = \mathsf{CN}^{\mathcal{I}}(x)$, all axioms of type (1) are satisfied in $\mathcal{J}$. Since $\mathsf{CN}^{\mathcal{I}}(x) = \mathsf{CN}^{\mathcal{I}}(z)$ and, for every role name $r$, we have $\langle y, z\rangle \in r^{\mathcal{I}}$ iff $\langle y, x\rangle \in r^{\mathcal{J}}$ and $\langle y, x\rangle \in r^{\mathcal{I}}$ iff $\langle y, z\rangle \in r^{\mathcal{J}}$, all axioms of type (2) and (5) are satisfied in $\mathcal{J}$. Finally, the only possibility for an axiom $\alpha$ of type (6) to be violated in $\mathcal{J}$ is when the new roles between $y$ and $x$ have resulted in new $u^+$ successors for some elements where $A$ holds. In this case, $\langle y, z\rangle \in r^{\mathcal{I}}$ for some role $r \in u$ and $\delta^{\mathcal{I}}_\alpha(x, y) \geqslant 1$, which is impossible by the conditions of the lemma. ⊣

Our main lemma states that, in every model of simple admissible $\mathcal{L}(\sqcup, +)$-TBox $\mathcal{T}$, every sufficiently long chain $x_0, \dots, x_p$ of elements connected with roles contains two elements $x_i$ and $x_j$ with $i < j$ such that $x_i$ can foster $x_j$ for the predecessor $x_{j-1}$ of $x_j$ w.r.t. $\mathcal{T}$. Thus, every sufficiently long chain can be "looped back" using the transformation described in Lemma 8.

**Lemma 9 (Main Lemma).** *Let* $\mathcal{T}$ *be a simple admissible* $\mathcal{L}(\sqcup, +)$*-TBox and* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *a model for* $\mathcal{T}$ *with* $\mathsf{deg}(\mathcal{I}) \leqslant N$*. Let* $r_1, \dots, r_n$ *be all the role names in* $\mathcal{T}$ *enumerated according to Proposition 1,* $k$ *an integer with* $1 \leqslant k \leqslant n$*, and* $x_0, \dots, x_p$ *a sequence of distinct elements in* $\Delta^{\mathcal{I}}$ *such that, for every* $i \geqslant 1$*, there exists* $\ell \leqslant k$ *such that* $\langle x_{i-1}, x_i\rangle \in r^{\mathcal{I}}_\ell$*. Then there exist* $i$ *and* $j$ *with* $1 \leqslant i < j \leqslant p$ *such that* $x_i$ *can foster*

$x_j$ *for* $x_{j-1}$*, provided that* $p \geqslant p_k := ((M+K+1)N \cdot 2^C + 1)^k$*, where* $M = M(\mathcal{T})$*,* $K = K(\mathcal{T})$*, and* $C = C(\mathcal{T})$ *as defined in Definition 7.*

Before proving Lemma 9, we demonstrate the following auxiliary property. For convenience, if $x$ is an element of the sequence $x_0, \dots, x_p$, i.e., $x = x_i$ for some $i$, then its predecessor *in this sequence* will be denoted by $\mathsf{pre}(x) := x_{i-1}$.

**Lemma 10 (Auxiliary Lemma).** *Let a TBox* $\mathcal{T}$*, a model* $\mathcal{I}$*, and a sequence* $x_0, \dots, x_p$ *be as in Lemma 9. Let* $y_0, \dots, y_q$ *be a sub-sequence in* $x_1, \dots, x_p$ *such that* $(i)$ $q \geqslant (M+K+1)N$*,* $(ii)$ $\mathsf{CN}^{\mathcal{I}}(y_0) = \dots = \mathsf{CN}^{\mathcal{I}}(y_q)$*,* $(iii)$ $\langle \mathsf{pre}(y_i), y_i \rangle \notin r_\ell^{\mathcal{I}}$ *for all* $0 \leqslant i \leqslant q$ *and* $\ell < k$*. Then for some* $0 \leqslant i < j \leqslant q$*,* $y_i$ *can foster* $y_j$ *for* $\mathsf{pre}(y_j)$*.*

*Proof.* Let $U_k$ be the set of axioms $\alpha = (A \sqsubseteq \leqslant m\,(u^+ \sqcup v).B) \in \mathcal{T}$ of type (6) such that $r_k \in u$. Take any axiom $\alpha \in U_k$ and consider a sequence of values $\delta^{\mathcal{I}}_{\alpha,i} := \delta^{\mathcal{I}}_{\alpha}(y_{iN}, y_{(i+1)N})$ for $0 \leqslant i \leqslant M+K$ (see Fig. 6b). We claim that at most $m+1$ of values $\delta^{\mathcal{I}}_{\alpha,i}$ are positive.

Indeed, for the first $i$ with $\delta^{\mathcal{I}}_{\alpha,i} \geqslant 1$, by definition of $\delta^{\mathcal{I}}_{\alpha}(x,y)$, there exists $y' \in A^{\mathcal{I}}$ with $\langle y', y_{(i+1)N} \rangle \in (u^+)^{\mathcal{I}}$. For all subsequent $j > i$ with $d^j_\alpha \geqslant 1$, there exists an element $x'_j$ such that $\langle y_{jN}, x'_j \rangle \in (u^+)^{\mathcal{I}}$, but $\langle y_{(j+1)N}, x'_j \rangle \notin (u^+)^{\mathcal{I}}$. In particular, all such $x_j$ are distinct for different $j$. Note that since $r_k \in u$, by Proposition 1, $r_\ell \in u$ for all $\ell \leqslant k$. Hence $\langle y', x'_j \rangle \in (u^+)^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\alpha$, the number of such different $j$ can be at most $m$.

Hence, the number of different $i$ such that, for some $\alpha \in U_k$, $\delta^{\mathcal{I}}_{\alpha,i} \geqslant 1$, is at most $\sum_{\alpha \in U_k} (m_\alpha + 1) \leqslant M + K$. Since $q \geqslant (M+K+1)N$, there exists at least one $i$ such that $\delta^{\mathcal{I}}_{\alpha,i} = 0$ for all $\alpha \in U_k$. For every $\alpha \in U_k$, there are two cases possible: either (1) there exists no $y' \in A^{\mathcal{I}}$ such that $\langle y', y_{(i+1)N} \rangle \in (u^+)^{\mathcal{I}}$ (see Fig. 6c), or (2) such a $y'$ exists, but there exists no $x' \in B^{\mathcal{I}}$ with $\langle y_{iN}, x' \rangle \in (u^+)^{\mathcal{I}}$ (see Fig. 6d). Hence, in particular, $\delta^{\mathcal{I}}_{\alpha}(y_{iN+j}, \mathsf{pre}(y_{(i+1)N})) = 0$ for all $j < N$ and all $\alpha \in U_k$.

Since $\mathsf{deg}(\mathcal{I}) \leqslant N$ and $\langle \mathsf{pre}(y_{(i+1)N}), y_{(i+1)N} \rangle \in r_\ell^{\mathcal{I}}$ for $\ell \leqslant k$, there exists $j < N$ such that $\langle \mathsf{pre}(y_{(i+1)N}), y_{iN+j} \rangle \notin r^{\mathcal{I}}$, for every $r$. Since, by condition $(iii)$, we have $\langle \mathsf{pre}(y_{(i+1)N}), y_{(i+1)N} \rangle \notin r_l^{\mathcal{I}}$ for each $\ell < k$, we have $\delta^{\mathcal{I}}_{\alpha}(y_{iN+j}, \mathsf{pre}(y_{(i+1)N})) = 0$ for each axiom $\alpha$ of type (6) such that $\langle \mathsf{pre}(y_{(i+1)N}), y_{(i+1)N} \rangle \in r_\ell^{\mathcal{I}}$ and $r_\ell \in u$. Indeed, those are exactly $\alpha \in U_k$, because $r_\ell \in u$ implies $r_k \in u$ for every $\ell \geqslant k$ by Proposition 1. Hence, by Definition 14, $y_{iN+j}$ can foster $y_{(i+1)N}$ for $\mathsf{pre}(y_{(i+1)N})$. $\dashv$

*Proof (of Lemma 9).* We prove the lemma by induction on $k$, using Lemma 10 both in induction base and induction step. Denote $L := (M+K+1)N$ for short.

*Induction base:* For $k = 1$, we have a sequence of elements $x_0, \dots, x_p \in \Delta^{\mathcal{I}}$ with $p \geqslant p_1 := L \cdot 2^C + 1$ such that $\langle x_{i-1}, x_i \rangle \in r_1^{\mathcal{I}}$, for all $1 \leqslant i \leqslant p$. We claim that there exists a subsequence $y_0, \dots, y_q$ in $x_1, \dots, x_p$ with $q \geqslant L$ such that $\mathsf{CN}^{\mathcal{I}}(y_0) = \dots = \mathsf{CN}^{\mathcal{I}}(y_q)$. Indeed, otherwise, since the number of different values of $\mathsf{CN}^{\mathcal{I}}(x)$ is bounded by $2^C$, and the number of elements $x$ in $x_1, \dots, x_p$ with the same value of $\mathsf{CN}^{\mathcal{I}}(x)$ is at most $L$, the total number of elements in $x_1, \dots, x_p$ cannot exceed $L \cdot 2^C$, which contradicts to $p \geqslant p_1$. Now, Lemma 10 can be applied to the sequence $y_0, \dots, y_q$, since there are no roles $r_\ell$ with $\ell < k = 1$. By Lemma 10 there exist elements $y_i$ and $y_j$ in this sequence with $0 \leqslant i < j \leqslant q$, such that $y_i$ can foster $y_j$ for $\mathsf{pre}(y_j)$.

*Induction Step:* Assume that the lemma holds for $k-1$. Two cases are possible:

(A) There exists a sub-sequence of consecutive elements $x_i, x_{i+1}, \dots, x_{i+p_{k-1}}$ with $p_{k-1} = (L \cdot 2^C + 1)^{k-1}$ and for each $j$ with $1 \leqslant j \leqslant p_{k-1}$, there exists $\ell \leqslant k-1$ such that $\langle x_{i+j-1}, x_{i+j} \rangle \in r_\ell^{\mathcal{I}}$. In this case the lemma holds by the induction hypothesis.

(B) Otherwise, in every sequence $x_{ip_{k-1}}, x_{ip_{k-1}+1} \dots, x_{(i+1)p_{k-1}}$ of consecutive elements with $0 \leqslant i \leqslant p_1 - 1 = L \cdot 2^C$, there exists an element $x'_i = x_{ip_{k-1}+j}$, with $1 \leqslant j \leqslant p_{k-1}$, such that $\langle \mathsf{pre}(x'_i), x'_i \rangle \notin r_\ell^{\mathcal{I}}$ for all $\ell \leqslant k-1$. By applying a combinatorial argument as in the induction base, from the sequence $x'_0, \dots, x'_{p_1-1}$ of $p_1 = L \cdot 2^C$ distinct elements one can select a subsequence $y_0, \dots, y_q$ with $q \geqslant L$ such that $\mathsf{CN}^{\mathcal{I}}(y_0) = \dots = \mathsf{CN}^{\mathcal{I}}(y_q)$. Hence the claim of the lemma follows from Lemma 10 applied to the sequence $y_0, \dots, y_q$. $\dashv$

**Theorem 5 (Finite Model Property for Admissible $\mathcal{ALCQ}(\sqcup, +)$-TBoxes).** *An admissible $\mathcal{ALCQ}(\sqcup, +)$-TBox $\mathcal{T}$ is satisfiable iff $\mathcal{T}$ has a finite model.*

*Proof.* The "if" direction of the theorem is trivial, so we focus on the "only if" part. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of $\mathcal{T}$. By Lemma 7, without loss of generality, we may assume that $\mathsf{deg}(\mathcal{I}) \leqslant N = N(\mathcal{T})$. Given an element $x \in \Delta^{\mathcal{I}}$ and an integer $d$, we define a *$d$-neighbourhood of $x$ in $\mathcal{I}$* as the set $\omega_d^{\mathcal{I}}(x)$ of all elements from the domain of $\mathcal{I}$ reachable from $x$ in at most $d$ steps. Formally,

$$\omega_0^{\mathcal{I}}(x) = \{x\}, \quad \omega_{d+1}^{\mathcal{I}}(x) := \omega_d^{\mathcal{I}}(x) \cup \{y \mid \exists x' \in \omega_d^{\mathcal{I}}(x) : \langle x', y \rangle \in r^{\mathcal{I}} \text{ for some } r\}.$$

Since $\mathsf{deg}(\mathcal{I}) \leqslant N$, it is easy to show by induction that $|\omega_d^{\mathcal{I}}(x)| \leqslant (N+1)^d$. In particular, $\omega_d^{\mathcal{I}}(x)$ is finite. We write $\mathsf{dist}^{\mathcal{I}}(x, y) = d$ iff $y \in \omega_d^{\mathcal{I}}(x)$, but $y \notin \omega_{d-1}^{\mathcal{I}}(x)$.

Let $x_0 \in \Delta^{\mathcal{I}}$, $r_1, \dots, r_n$ be all role names from $\mathcal{T}$ enumerated according to Proposition 1, and $p_n$ be defined as $p_k$ in Lemma 9 for $k = n$. We will construct a model $\mathcal{J}$ of $\mathcal{T}$ whose domain will consist of (finitely many) elements from $\omega_{p_n-1}^{\mathcal{I}}(x_0)$, i.e., those that are reachable from $x_0$ in less than $p_n$ steps. Intuitively, the model $\mathcal{J}$ is constructed first by applying a sequence of transformations $\mathcal{I} = \mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_m$ of model $\mathcal{I}$ by "looping back" all paths from $x_0$ to elements at the distance $p_n$ from $x_0$ using Lemma 8 and Lemma 9 until none of them left, and then by removing the elements in $\mathcal{I}_m$ that became disconnected from $x_0$ after such transformations.

Formally, assume that $\mathcal{I}_\ell$ is a model of $\mathcal{T}$ and there exists $x_{p_n}$ with $\mathsf{dist}^{\mathcal{I}_\ell}(x_0, x_{p_n}) = p_n$. Then there exists a sequence of elements $x_0, x_1, \dots, x_{p_n}$ in $\mathcal{I}_\ell$ such that for every $i$ with $1 \leqslant i \leqslant p$, we have $\langle x_{i-1}, x_i \rangle \in r^{\mathcal{I}}$ for some $r$. By Lemma 9, there exist $i$ and $j$ with $1 \leqslant i < j \leqslant p_n$ such that $x_i$ can foster $x_j$ for $x_{j-1}$. Let $\mathcal{I}_{\ell+1} := \mathsf{swap}(\mathcal{I}_\ell, x_i, x_{j-1}, x_j)$ be obtained from $\mathcal{I}$ like in Lemma 8. Since $\mathcal{I}_\ell$ is a model of $\mathcal{T}$, by Lemma 8, $\mathcal{I}_{\ell+1}$ is a model for $\mathcal{T}$. Note also that $\mathsf{deg}(\mathcal{I}_{\ell+1}) = \mathsf{deg}(\mathcal{I}_\ell)$.

Our goal is now to show that by repeatedly applying the transformations above starting with $\mathcal{I}_0 = \mathcal{I}$, we eventually obtain a model $\mathcal{I}_m$ in which there is no element $x$ with $\mathsf{dist}^{\mathcal{I}_\mathsf{m}}(x_0, x_{p_n}) = p_n$. For this we need an additional property:

*Claim.* Let $\mathcal{I}_{l+1} = \mathsf{swap}(\mathcal{I}_\ell, x_i, x_{j-1}, x_j)$ be obtained from $\mathcal{I}_\ell$ as described above. Then for every $d \geqslant 0$ we have $\omega_d^{\mathcal{I}_{\ell+1}}(x_0) \subseteq \omega_d^{\mathcal{I}_\ell}(x_0)$.

Indeed, assume to the contrary, that there exists $y$ such that $y \in \omega_d^{\mathcal{I}_{\ell+1}}(x_0)$, but $y \notin \omega_d^{\mathcal{I}_\ell}(x_0)$. Then there exists a sequence of distinct elements $x_0 = y_0, y_1, \dots, y_d = y$ such that for every $i'$ with $1 \leqslant i' \leqslant d$, we have $\langle y_{i'-1}, y_{i'} \rangle \in r^{\mathcal{I}_{\ell+1}}$ for some $r$, but for some $j'$ with $1 \leqslant j' \leqslant d$, we have $\langle y_{j'-1}, y_{j'} \rangle \notin r^{\mathcal{I}_\ell}$ for every $r$. Since $\mathcal{I}_{\ell+1} = \mathsf{swap}(\mathcal{I}_\ell, x_i, x_{j-1}, x_j)$, this is possible only if $y_{j'-1} = x_{j-1}$ and $y_{j'} = x_i$ (all other relations in $\mathcal{I}_{\ell+1}$ occur in $\mathcal{I}_\ell$ as well). Moreover, $j' \geqslant j$, since otherwise $\mathsf{dist}^{\mathcal{I}}(x_0, p_n) < p_n$ as $x_0 = y_0, y_1, \dots, y_{j'-1} = x_{j-1}, x_j, \dots, x_{p_n}$ is a sequence of elements connected with roles in $\mathcal{I}_\ell$ which has a smaller length than the sequence $x_0, x_1, \dots, x_{p_n}$ used in the transformation. But then $x_0, \dots, x_i = y_{j'}, \dots, y_d$ is a sequence of elements connected with roles in $\mathcal{I}_\ell$ consisting of $d$ elements since $i < j \leqslant j'$. Hence $y = y_d \in \omega_d^{\mathcal{I}_\ell}(x_0)$ contrary to what has been assumed.

Now let us define $e_d^{\mathcal{I}}(x_0) = \{\langle x, y \rangle \mid x, y \in \omega_d(x_0), \langle x, y \rangle \in r^{\mathcal{I}} \text{ for some } r\}$. Note that $|e_d^{\mathcal{I}}(x_0)| \leqslant |\omega_d^{\mathcal{I}}(x_0)|^2$. Let $\mathcal{I} = \mathcal{I}_0, \dots, \mathcal{I}_\ell, \mathcal{I}_{\ell+1}, \dots$ be a sequence of interpretations obtained by transformations described above. We claim that during every transformation step $\mathcal{I}_{\ell+1} = \mathsf{swap}(\mathcal{I}_\ell, x_i, x_{j-1}, x_j)$ either:

(1) $\omega_d^{\mathcal{I}_{\ell+1}}(x_0) \subseteq \omega_d^{\mathcal{I}_\ell}(x_0)$ for every $d \leqslant p_n$, but for some $d \leqslant p_n$ we have $\omega_d^{\mathcal{I}_{\ell+1}}(x_0) \subsetneq \omega_d^{\mathcal{I}_\ell}(x_0)$, or
(2) $\omega_d^{\mathcal{I}_{\ell+1}}(x_0) = \omega_d^{\mathcal{I}_\ell}(x_0)$ and $|e_d^{\mathcal{I}_{\ell+1}}(x_0)| \geqslant |e_d^{\mathcal{I}_\ell}(x_0)|$ for every $d \leqslant p_n$, but for some $d \leqslant p_n$ we have $|e_d^{\mathcal{I}_{\ell+1}}(x_0)| > |e_d^{\mathcal{I}_\ell}(x_0)|$.

Indeed, the properties above hold since when $\omega_d^{\mathcal{I}_{\ell+1}}(x_0) = \omega_d^{\mathcal{I}_\ell}(x_0)$, we have:

$$e_d^{\mathcal{I}_{\ell+1}}(x_0) = \begin{cases} e_d^{\mathcal{I}_\ell}(x_0) & \text{if } d < j-1, \\ e_d^{\mathcal{I}_\ell}(x_0) \cup \{\langle x_{j-1}, x_i \rangle\} & \text{if } d = j-1, \\ e_d^{\mathcal{I}_\ell}(x_0) \cup \{\langle x_{j-1}, x_i \rangle\} \setminus \{\langle x_{j-1}, x_j \rangle\} & \text{if } d > j-1. \end{cases}$$

Since $\mathsf{deg}(\mathcal{I}_{\ell+1}) = \mathsf{deg}(\mathcal{I}_\ell) < N$, we have $|\omega_d^{\mathcal{I}}(x_0)| \leqslant (N+1)^d$ and $|e_d^{\mathcal{I}}(x_0)| \leqslant |\omega_d^{\mathcal{I}}(x_0)|^2 \leqslant (N+1)^{2d}$, so the transformation process $\mathcal{I}_0, \dots, \mathcal{I}_\ell, \mathcal{I}_{\ell+1}, \dots$ necessarily terminates with some $\mathcal{I}_m$. Now define $\mathcal{J} := \mathcal{I}_m|_{\omega_{p_n-1}^{\mathcal{I}_m}(x_0)}$ to be the restriction of $\mathcal{I}_m$ to the elements that are reachable from $x_0$ in at most $p_n - 1$ steps (and hence, reachable from $x_0$ at all). It is easy to see that $\mathcal{J}$ remains a model of $\mathcal{T}$ since the axioms of all types (1), (2) and (6) remain satisfied. Since $\omega_{p_n-1}^{\mathcal{I}_m}(x_0)$ consists of finitely many elements, $\mathcal{J}$ is a finite model for $\mathcal{T}$. $\dashv$

*Remark 3.* Note that in the proof of Theorem 5, we have demonstrated not only the finite model property for admissible $\mathcal{ALCQ}(\sqcup, +)$-TBoxes, but also a so-called *small model property*. Namely, that the bound on the size of the finite model can be computed a-priory for a given TBox $\mathcal{T}$. It is easy to see from the proof of Theorem 5 that every satisfiable TBox $\mathcal{T}$ has a model with at most $|\omega_{p_n-1}^{\mathcal{I}_m}(x_0)| \leqslant (N+1)^{p_n-1}$ elements, where $N = N(\mathcal{T})$ and $p_n$ can be computed from $\mathcal{T}$.

Now it is time to harvest our decidability results (see [9] for all proofs).

**Theorem 6 (Finite Model Property for Strongly Admissible RBoxes in $\mathcal{ALCQ}$).** *Let $\mathcal{T}$ be an $\mathcal{ALCQ}$-TBox and $\mathcal{R}$ a strongly admissible RBox. Then $\langle \mathcal{R}, \mathcal{T} \rangle$ is satisfiable iff it has a finite model.*

*Proof.* By Lemma 2, one can effectively convert TBox $\mathcal{T}$ into simple form $\mathcal{T}_{\text{sf}}$ such that $\langle\mathcal{R},\mathcal{T}\rangle$ is (finitely) satisfiable iff $\langle\mathcal{R},\mathcal{T}_{\text{sf}}\rangle$ is (finitely) satisfiable. By Lemma 4, $\langle\mathcal{R},\mathcal{T}_{\text{sf}}\rangle$ is (finitely) satisfiable iff $\mathcal{R}(\mathcal{T}_{\text{sf}})$ is (finitely) satisfiable. Since $\mathcal{R}$ is an admissible RBox and $\mathcal{T}_{\text{sf}}$ a simple $\mathcal{ALCQ}$-TBox, by Lemma 6, $\mathcal{R}(\mathcal{T}_{\text{sf}})$ is a simple admissible $\mathcal{ALCQ}(\sqcup,+)$-TBox. Now, if $\langle\mathcal{R},\mathcal{T}\rangle$ is satisfiable, then $\mathcal{R}(\mathcal{T}_{\text{sf}})$ is satisfiable and hence by Theorem 5, has a finite model. Hence $\langle\mathcal{R},\mathcal{T}\rangle$ has a finite model. $\dashv$

**Corollary 2 (Finite Model Property for $\mathbf{GrK4}$).**
*Every satisfiable* $\mathbf{GrK4}$*-formula has a finite model*

*Proof.* Every $\mathbf{GrK4}$-formula $\varphi$ can be translated into an $\mathcal{ALC}$-TBox $\mathcal{T}$ containing only one role $r$, such that for an RBox $\mathcal{R}=\{\mathsf{Tr}(r)\}$, $\varphi$ is (finitely) satisfiable iff $\langle\mathcal{R},\mathcal{T}\rangle$ is (finitely) satisfiable. Since $\mathcal{R}$ is a strongly admissible RBox, by Theorem 6 $\langle\mathcal{R},\mathcal{T}\rangle$ is satisfiable iff $\langle\mathcal{R},\mathcal{T}\rangle$ has a finite model. Hence $\varphi$ is satisfiable iff $\varphi$ has a finite model. Using Remark 3, it is also easy to show that $\mathbf{GrK4}$ has a small model property. $\dashv$

**Corollary 3.** *Every strongly admissible RBox is safe for $\mathcal{ALCQ}$.*

*Proof.* Given a strongly admissible RBox $\mathcal{R}$ and an $\mathcal{ALCQ}$ TBox $\mathcal{T}$, we can decide satisfiability of $\langle\mathcal{R},\mathcal{T}\rangle$ by first converting $\mathcal{T}$ into a simple form $\mathcal{T}_{\text{sf}}$ and then checking if there exists a finite model for $\mathcal{R}(\mathcal{T}_{\text{sf}})$ of the size at most $(N+1)^{p_n-1}$ by enumerating all such (finitely many) models, where $N$ and $p_n$ can be effectively computed from $\mathcal{R}(\mathcal{T}_{\text{sf}})$ as mentioned in Remark 3. $\dashv$

**Corollary 4 (Theorem 4).** *Every admissible RBox is safe for $\mathcal{ALCQ}$.*

*Proof.* If $\mathcal{R}$ is an admissible RBox, then, by to Definition 11, $\mathcal{R}=\bigcup\mathcal{R}_i$, where $\mathcal{R}_i$ are strongly admissible RBoxes. By Corollary 3, every $\mathcal{R}_i$ is safe for $\mathcal{ALCQ}$. Hence, by Lemma 5, $\mathcal{R}$ is safe for $\mathcal{ALCQ}$. $\dashv$

## 6 Extending RBoxes

In Section 5 we have described a rather large class of $\mathcal{ALCQ}$-safe RBoxes. However, so far, only few RBoxes were shown to be unsafe for $\mathcal{ALCN}$ and $\mathcal{ALCIN}$ in Section 3. In this section we are concerned with a question whether every RBox "containing" any of the patterns described in Section 3 is necessarily unsafe? Or, in general, what happens to the (un)safety of an RBox when the RBox are extended?

It is clear that adding axioms may turn a safe RBox into unsafe and vice versa: an $\mathcal{ALCN}$-safe RBox $\{\mathsf{Tr}(r)\}$ can be extended to an $\mathcal{ALCN}$-unsafe RBox $\mathcal{R}_\wedge$ from Theorem 1; adding to $\mathcal{R}_\wedge$ an inclusion between its incomparable transitive roles yields an $\mathcal{ALCN}$-safe RBox by Theorem 4. So it is not sufficient for an RBox $\mathcal{R}'$ to be unsafe if it contains some unsafe RBox $\mathcal{R}$. The question now is: what additional property an extension $\mathcal{R}'$ of $\mathcal{R}$ should fulfill so that unsafety of $\mathcal{R}$ can be transferred to $\mathcal{R}'$. In this section we demonstrate that it is sufficient to require that $\mathcal{R}'$ is *semantically conservative* over $\mathcal{R}$.

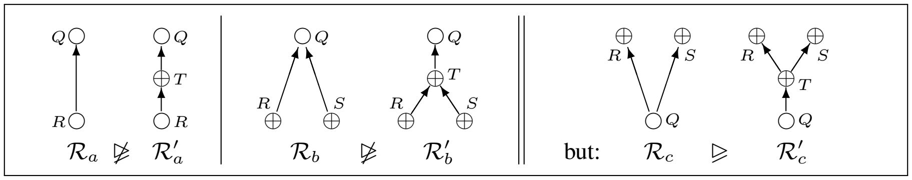


**Fig. 7.** For $i \in \{a, b\}$, we have $\mathcal{R}_i \sqsubseteq \mathcal{R}'_i$ and $\mathcal{R}_i \not\trianglerighteq \mathcal{R}'_i$; whereas $\mathcal{R}_c \trianglerighteq \mathcal{R}'_c$.

### 6.1 Conservativity of RBoxes

In this section we define several notions of conservativity for RBoxes. The notion of conservativity (or specifically of conservative extensions) is well-known from Logic [], and been recently applied in the context of Description Logic [].

**Definition 15.** Let $\mathcal{R}$ and $\mathcal{R}'$ be two RBoxes, and $\Sigma$ a set of role names. For given an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ let $\mathcal{I}|_\Sigma = (\Delta^{\mathcal{I}|_\Sigma}, \cdot^{\mathcal{I}|_\Sigma})$ be an interpretation defined by: $\Delta^{\mathcal{I}|_\Sigma} := \Delta^{\mathcal{I}}$; for every concept name $A$, we have $A^{\mathcal{I}|_\Sigma} := A^{\mathcal{I}}$; and for every role name $r$, we have $r^{\mathcal{I}|_\Sigma} := r^{\mathcal{I}}$ if $r \in \Sigma$, and $r^{\mathcal{I}|_\Sigma} := \emptyset$ otherwise.

We say that $\mathcal{R}'$ *is deductively conservative over* $\mathcal{R}$ *w.r.t.* $\Sigma$ if for every role axiom $\alpha$ with $\mathsf{RN}(\alpha) \subseteq \Sigma$, we have $\mathcal{R}' \models \alpha$ implies $\mathcal{R} \models \alpha$.

We say that $\mathcal{R}'$ *is semantically conservative over* $\mathcal{R}$ *w.r.t.* $\Sigma$ (notation: $\mathcal{R} \trianglerighteq_\Sigma \mathcal{R}'$), if for every model $\mathcal{I}$ of $\mathcal{R}$ there exists a model $\mathcal{I}'$ of $\mathcal{R}'$ such that $\mathcal{I}'|_\Sigma = \mathcal{I}|_\Sigma$.

We say that $\mathcal{R}'$ is a deductive (semantic) conservative *extension* of $\mathcal{R}$ w.r.t. $\Sigma$ if, additionally, $[\mathcal{R}] \subseteq [\mathcal{R}']$. We omit $\Sigma$ when $\Sigma = \mathsf{RN}(\mathcal{R})$.

*Remark 4.* It is easy to see that the relation of $\mathcal{R} \trianglerighteq_\Sigma \mathcal{R}'$ is transitive reflexive w.r.t. $\mathcal{R}'$ and $\mathcal{R}$, and monotonic w.r.t. $\Sigma$.

Semantic conservativity always implies deductive conservativity (but not vice versa, as the example below shows). Indeed, assume that $\mathcal{R} \trianglerighteq_\Sigma \mathcal{R}'$. In order to show that $\mathcal{R}'$ is a deductively conservative over $\mathcal{R}$ w.r.t. $\Sigma$, take any axiom $\alpha$ with $\mathsf{RN}(\alpha) \subseteq \Sigma$ such that $\mathcal{R}' \models \alpha$. According to Definition 15, we need to demonstrate that $\mathcal{R} \models \alpha$, or, equivalently, that for every model $\mathcal{I}$ of $\mathcal{R}$ we have $\mathcal{I} \models \alpha$. Since $\mathcal{R} \trianglerighteq_\Sigma \mathcal{R}'$, and $\mathcal{I}$ is a model of $\mathcal{R}$, by Definition 15, there exists a model $\mathcal{I}'$ of $\mathcal{R}'$ such that $\mathcal{I}'|_\Sigma = \mathcal{I}|_\Sigma$. Since $\mathcal{R}' \models \alpha$, we have $\mathcal{I}' \models \alpha$. Since $\mathsf{RN}(\alpha) \subseteq \Sigma$ and $\mathcal{I}'|_\Sigma = \mathcal{I}|_\Sigma$, we have $\mathcal{I} \models \alpha$.

**Example 1.** Consider RBoxes depicted in Fig. 7. We have $\mathcal{R}_c \trianglerighteq \mathcal{R}'_c$. Indeed, given any model $\mathcal{I} \models \mathcal{R}_c$, we define $\mathcal{I}'$ by setting $R^{\mathcal{I}'} := R^{\mathcal{I}}$, $S^{\mathcal{I}'} := S^{\mathcal{I}}$, and $T^{\mathcal{I}'} := R^{\mathcal{I}} \cap S^{\mathcal{I}}$. Then $\mathcal{I}' \models \mathcal{R}'_c$; in particular, $T^{\mathcal{I}'}$ is transitive as the intersection of transitive relations.

At the same time, for $i \in \{a, b\}$, we have that $\mathcal{R}'_i$ is deductively conservative over $\mathcal{R}_i$, but $\mathcal{R}_i \not\trianglerighteq \mathcal{R}'_i$. Indeed,$\mathcal{R}_a \not\trianglerighteq \mathcal{R}'_a$, since one can easily construct two non-transitive relations on some set that have no transitive relations between them. To show that $\mathcal{R}_b \not\trianglerighteq \mathcal{R}'_b$, take $\Delta^{\mathcal{I}} = \{0, 1, 2\}$ and set $R^{\mathcal{I}} := \{\langle 0, 1\rangle\}$, $S^{\mathcal{I}} := \{\langle 1, 2\rangle\}$, and $Q^{\mathcal{I}} := R^{\mathcal{I}} \cup S^{\mathcal{I}}$. Then we cannot interpret the transitive role $T$ to satisfy $\mathcal{R}'_b$.

Comparing $\mathcal{R}_c \trianglerighteq \mathcal{R}'_c$ with $\mathcal{R}_b \not\trianglerighteq \mathcal{R}'_b$, we can observe that inverting role inclusions can invalidate semantic conservativity. On the other hand, deductive conservativity is always preserved under this operation. This easily follows from the observation that if $\mathcal{R} \vdash R \sqsubseteq S$ and an RBox $\mathcal{R}_1$ is obtained from $\mathcal{R}$ by inverting all role inclusion axioms, then $\mathcal{R}_1 \vdash S \sqsubseteq R$.

**Theorem 7 (Preservation of Unsafety under Conservative Extensions of RBoxes).** *If $\mathcal{R}'$ is a conservative extension of $\mathcal{R}$ and $\mathcal{R}$ is $\mathcal{L}$-unsafe, then $\mathcal{R}'$ is $\mathcal{L}$-unsafe.*

*Proof.* The proof is by contra-position. Suppose that $[\mathcal{R}] \subseteq [\mathcal{R}']$ and $\mathcal{R} \trianglerighteq \mathcal{R}'$, and assume that $\mathcal{R}'$ is $\mathcal{L}$-safe, then let us prove that $\mathcal{R}$ is also $\mathcal{L}$-safe.

To this end, according to Definition 3 and Remark 1, we need to demonstrate how to decide satisfiability of pairs of the form $\langle \mathcal{R}, \mathcal{T} \rangle$, for $\mathcal{L}$-TBoxes $\mathcal{T}$. Let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by renaming all role names from $\mathsf{RN}(\mathcal{R}') \setminus \mathsf{RN}(\mathcal{R})$ with fresh ones. We claim that $\langle \mathcal{R}, \mathcal{T} \rangle$ is satisfiable iff $\langle \mathcal{R}', \mathcal{T}' \rangle$ is satisfiable. Indeed:

(1) $\langle \mathcal{R}, \mathcal{T} \rangle$ is satisfiable iff $\langle \mathcal{R}, \mathcal{T}' \rangle$ is satisfiable.
This is because $\langle \mathcal{R}, \mathcal{T}' \rangle$ is obtained from $\langle \mathcal{R}, \mathcal{T} \rangle$ by renaming of role names;

(2) $\langle \mathcal{R}, \mathcal{T}' \rangle$ is satisfiable iff $\langle \mathcal{R}', \mathcal{T}' \rangle$ is satisfiable.
The implication '$\Leftarrow$' is trivial due to $[\mathcal{R}] \subseteq [\mathcal{R}']$.
The implication '$\Rightarrow$' can be proved as follows. Suppose that $\mathcal{I} \models \langle \mathcal{R}, \mathcal{T}' \rangle$. Since $\mathcal{R} \trianglerighteq \mathcal{R}'$, by Definition 15 there exists $\mathcal{I}' \models \mathcal{R}'$ such that $\mathcal{I}'|_{\mathsf{RN}(\mathcal{R})} = \mathcal{I}|_{\mathsf{RN}(\mathcal{R})}$. Let $\mathcal{J}$ be an interpretation such that $\mathcal{J}|_{\mathsf{RN}(\mathcal{T}')} = \mathcal{I}|_{\mathsf{RN}(\mathcal{T}')}$ and $\mathcal{J}|_{\mathsf{RN}(\mathcal{R}')} = \mathcal{I}'|_{\mathsf{RN}(\mathcal{R}')}$. Since $\mathsf{RN}(\mathcal{T}') \cap \mathsf{RN}(\mathcal{R}') \subseteq \mathsf{RN}(\mathcal{R})$ and $\mathcal{I}'|_{\mathsf{RN}(\mathcal{R})} = \mathcal{I}|_{\mathsf{RN}(\mathcal{R})}$, such interpretation $\mathcal{J}$ always exists. Moreover, $\mathcal{J} \models \mathcal{R}'$ since $\mathcal{I}' \models \mathcal{R}'$ and $\mathcal{J}|_{\mathsf{RN}(\mathcal{R}')} = \mathcal{I}'|_{\mathsf{RN}(\mathcal{R}')}$, and $\mathcal{J} \models \mathcal{T}'$ since $\mathcal{I} \models \mathcal{T}'$ and $\mathcal{J}|_{\mathsf{RN}(\mathcal{T}')} = \mathcal{I}|_{\mathsf{RN}(\mathcal{T}')}$. Hence, $\mathcal{J}$ is a model of $\langle \mathcal{R}', \mathcal{T}' \rangle$, and therefore $\langle \mathcal{R}', \mathcal{T}' \rangle$ is satisfiable.

Now, if $\mathcal{R}'$ if $\mathcal{L}$-safe then satisfiability of $\langle \mathcal{R}', \mathcal{T}' \rangle$ is decidable, and therefore one can decide satisfiability of $\langle \mathcal{R}', \mathcal{T}' \rangle$, which proves that $\mathcal{R}$ is $\mathcal{L}$-safe. $\dashv$

As a consequence, if $\mathcal{R}'$ is a semantic conservative extension of any RBox "of the form" $\mathcal{R}_{\vee}^{\oplus}, \mathcal{R}_{\wedge}^{n}, \mathcal{R}_{\vee}^{n}, \mathcal{R}_{\curlywedge}^{n}$ depicted in Fig. 1 and 4, then $\mathcal{R}$ is $\mathcal{ALCQ}$-unsafe. To formulate this statement rigorously, denote by $\mathcal{F} := \{\mathcal{R}_{\vee}^{\oplus}\} \cup \{\mathcal{R}_{\wedge}^{n}, \mathcal{R}_{\vee}^{n}, \mathcal{R}_{\curlywedge}^{n} \mid n \geqslant 1\}$ the family of RBoxes (see. Fig. 4) that were shown to be $\mathcal{ALCQ}$-unsafe in Sect. 3.

**Definition 16.** An RBox $\mathcal{R}'$ is *stronger than* $\mathcal{R}$ (written $\mathcal{R} \hookrightarrow \mathcal{R}'$) if there is a one-to-one renaming of role names $\sigma\colon \mathsf{RN} \to \mathsf{RN}$ such that $\mathcal{R}'$ is a conservative extension of $\sigma(\mathcal{R})$, where $\sigma(\mathcal{R})$ is an RBox obtained from $\mathcal{R}$ by replacing every role $S$ with $\sigma(S)$.

Theorem 7 implies that if $\mathcal{R}$ is $\mathcal{L}$-unsafe and $\mathcal{R} \hookrightarrow \mathcal{R}'$ then $\mathcal{R}'$ is $\mathcal{L}$-unsafe.

**Corollary 5.** *(1) Any $\mathcal{R}'$ that is stronger than $\mathcal{R} = \{\mathsf{Tr}(r)\}$, is $\mathcal{ALCIQ}$-unsafe.*
*(2) Any $\mathcal{R}'$ that is stronger than some $\mathcal{R} \in \mathcal{F}$ is $\mathcal{ALCQ}$-unsafe.*

*Conjecture 1.* Corollary 5 describes *all* RBoxes that are unsafe for $\mathcal{ALCIQ}$ and $\mathcal{ALCQ}$.

Corollary 5 gives a structural description of (some) unsafe RBoxes: the relation '$\hookrightarrow$' is a partial order on RBoxes which preserves unsafety, $\mathcal{F}$ is the set of minimal pairwise incomparable (w.r.t. $\hookrightarrow$) RBoxes, and an RBox $\mathcal{R}'$ is $\mathcal{ALCQ}$-unsafe if (and only if, as we conjecture) we have $\mathcal{R} \hookrightarrow \mathcal{R}'$, for some $\mathcal{R} \in \mathcal{F}$. [3] However, it is not obvious how to verify the above condition for unsafety. Therefore, in what follows we provide an efficient procedure for checking semantic conservativity and, as a consequence, for checking the above sufficient conditions for unsafety of an RBox.

### 6.2 Checking Semantic Conservativity of RBoxes

Theorem 7 provides a missing condition, namely semantic conservativity, which guarantees that an unsafe RBox remains unsafe after extension with new axioms. However, it is not yet clear how to this condition for given two RBoxes. In this section we present a general procedure for checking semantic conservativity using extensions of roles introduced in Definition 9.

**Definition 17.** Given two extended roles $\rho'$ and $\rho$, we write $\mathcal{I} \models \rho' \sqsubseteq_\Sigma \rho$ iff $\mathcal{I}|_\Sigma \models \rho' \sqsubseteq \rho$. We write $\models \rho' \sqsubseteq_\Sigma \rho$ if for every interpretation $\mathcal{I}$, we have $\mathcal{I} \models \rho' \sqsubseteq_\Sigma \rho$.

*Remark 5.* Note that, for every extended role $\rho$, we have the inclusion $\rho^{\mathcal{I}|_\Sigma} \subseteq \rho^{\mathcal{I}}$.

**Lemma 11 (Criterion for Semantic Conservativity of RBoxes).**
*Let $\mathcal{R}$ and $\mathcal{R}'$ be two RBoxes and $\Sigma$ a set of role names. Then $\mathcal{R} \geqslant_\Sigma \mathcal{R}'$ iff, for every role name $s \in \Sigma$, we have $\models \mathcal{R}'(s) \sqsubseteq_\Sigma \mathcal{R}(s)$.*

*Proof.* **($\Leftarrow$)** Suppose that $\models \mathcal{R}'(s) \sqsubseteq_\Sigma \mathcal{R}(s)$ for every $s \in \Sigma$. In order to prove that $\mathcal{R} \geqslant_\Sigma \mathcal{R}'$ we need to construct, for every model $\mathcal{I}$ of $\mathcal{R}$, a model $\mathcal{I}'$ of $\mathcal{R}'$ such that $s^{\mathcal{I}} = s^{\mathcal{I}'}$ for every $s \in \Sigma$.

Let $\mathcal{I}$ be a model of $\mathcal{R}$. Define $\mathcal{I}' := \mathcal{R}'(\mathcal{I}|_\Sigma)$. By Lemma 3 (1), $\mathcal{I}'$ is a model of $\mathcal{R}'$. We need to show that $s^{\mathcal{I}} = s^{\mathcal{I}'}$, for every $s \in \Sigma$. Take any $s \in \Sigma$.

By Definition 17 and Lemma 3 (3) we have:

$$s^{\mathcal{I}} = s^{\mathcal{I}|_\Sigma} \subseteq s^{\mathcal{R}'(\mathcal{I}|_\Sigma)} = s^{\mathcal{I}'}. \tag{7}$$

Since $\models \mathcal{R}'(s) \sqsubseteq_\Sigma \mathcal{R}(s)$, by Definition 17, assumption $\models \mathcal{R}'(s) \sqsubseteq_\Sigma \mathcal{R}(s)$, Remark 5 and Lemma 3 (2), we have:

$$s^{\mathcal{I}'} = s^{\mathcal{R}'(\mathcal{I}|_\Sigma)} = (\mathcal{R}'(s))^{\mathcal{I}|_\Sigma} \subseteq (\mathcal{R}(s))^{\mathcal{I}|_\Sigma} \subseteq (\mathcal{R}(s))^{\mathcal{I}} = s^{\mathcal{R}(\mathcal{I})} = s^{\mathcal{I}} \tag{8}$$

From (7) and (8), we have $s^{\mathcal{I}} = s^{\mathcal{I}'}$.

**($\Rightarrow$)** Suppose that $\mathcal{R} \geqslant_\Sigma \mathcal{R}'$. In order to show that $\models \mathcal{R}'(s) \sqsubseteq_\Sigma \mathcal{R}(s)$ for every $s \in \Sigma$, let $\mathcal{I}$ be an arbitrary interpretation. According to Definition 17, we need to prove that $\mathcal{R}'(s)^{\mathcal{I}|_\Sigma} \subseteq \mathcal{R}(s)^{\mathcal{I}|_\Sigma}$, or, equivalently, by Definition 9, that $s^{\mathcal{R}'(\mathcal{I}|_\Sigma)} \subseteq s^{\mathcal{R}(\mathcal{I}|_\Sigma)}$.

Since, by Lemma 3 (1), $\mathcal{R}(\mathcal{I}|_\Sigma)$ is a model of $\mathcal{R}$, and $\mathcal{R} \geqslant_\Sigma \mathcal{R}'$, there exists a model $\mathcal{I}'$ of $\mathcal{R}'$ such that $s^{\mathcal{I}'} = s^{\mathcal{R}(\mathcal{I}|_\Sigma)}$ for every $s \in \Sigma$. Also, since $r^{\mathcal{I}|_\Sigma} \subseteq r^{\mathcal{R}(\mathcal{I}|_\Sigma)} = r^{\mathcal{I}'}$ for every role name $r$, by Lemma 3 (4) we have:

$$s^{\mathcal{R}'(\mathcal{I}|_\Sigma)} \subseteq s^{\mathcal{R}'(\mathcal{I}')} = s^{\mathcal{I}'} = s^{\mathcal{R}(\mathcal{I}|_\Sigma)} \tag{9}$$

That is, $s^{\mathcal{R}'(\mathcal{I}|_\Sigma)} \subseteq s^{\mathcal{R}(\mathcal{I}|_\Sigma)}$, which was required to show. $\dashv$

[3] This resembles Kuratiwski's Theorem in graph theory: a graph $G$ is not planar iff $K_{3,3} \hookrightarrow G$ or $K_5 \hookrightarrow G$, where '$H \hookrightarrow G$' means that $H$ is embeddable into $G$ in a certain sense (cf. [5]).

Lemma 11 effectively reduces the problem of checking semantic conservativity for RBoxes to the problem of checking the entailment $\models \rho' \sqsubseteq_\Sigma \rho$ for extended role expressions $\rho'$ and $\rho$. The next lemma demonstrates how to syntactically check this entailment. Recall that we identify disjunctions of roles with sets of roles, that is we can write $u \subseteq v$ for disjunctions of roles $u$ and $v$, as well as $u^\mathcal{I}$ for sets of roles $u$.

**Lemma 12.** *Let $\rho' = \bigsqcup (u'_i)^+ \sqcup v'$ and $\rho = \bigsqcup (u_j)^+ \sqcup v$ be extended roles, where $u'_i$, $u_j$, $v'$ and $v$ are disjunctions of roles, and $\Sigma$ a set of role names. Let $\overline{\Sigma} := \{r,\, r^- \mid r \in \Sigma\}$. Then $\models \rho' \sqsubseteq_\Sigma \rho$ iff:*

*(1) $v' \cap \overline{\Sigma} \subseteq \bigcup u_j \cup v$, and*
*(2) For every $u'_i$, there exists $u_j$ such that $u'_i \cap \overline{\Sigma} \subseteq u_j$.*

*Proof.* **($\Leftarrow$)** Suppose the conditions (1) and (2) above hold for the given $\rho'$, $\rho$ and $\Sigma$. In order to show that $\models \rho' \sqsubseteq_\Sigma \rho$, take an arbitrary interpretation $\mathcal{I}$. We need to demonstrate that $(\rho')^{\mathcal{I}|_\Sigma} \subseteq \rho^{\mathcal{I}|_\Sigma}$. Indeed, by condition (1), for every $u'_i$ there exists $u_j$ such that $u'_i \cap \overline{\Sigma} \subseteq u_j$, and so, since $(u'_i)^{\mathcal{I}|_\Sigma} = (u'_i \cap \overline{\Sigma})^{\mathcal{I}|_\Sigma} \subseteq (u_j)^{\mathcal{I}|_\Sigma}$, we have $((u'_i)^+)^{\mathcal{I}|_\Sigma} \subseteq ((u_j)^+)^{\mathcal{I}|_\Sigma} \subseteq \rho^{\mathcal{I}|_\Sigma}$. By condition (1), $(v')^{\mathcal{I}|_\Sigma} = (v' \cap \overline{\Sigma})^{\mathcal{I}|_\Sigma} \subseteq (\bigcup u_j \cup v)^{\mathcal{I}|_\Sigma} \subseteq \rho^{\mathcal{I}|_\Sigma}$. Hence $(\rho')^{\mathcal{I}|_\Sigma} \subseteq \rho^{\mathcal{I}|_\Sigma}$, which was required to show.

**($\Rightarrow$)** Suppose that $\models \rho' \sqsubseteq_\Sigma \rho$, but either the condition (1) or the condition (2) do not hold for $\rho$ and $\rho'$. Consider each of these cases:

If condition (1) does not hold, then there exits $s \in v' \cap \overline{\Sigma}$ such that $s \notin \bigcup u_j \cup v$. Consider an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ such that $\Delta^\mathcal{I} = \{0\}$, $r^\mathcal{I} = \emptyset$ for $r \neq s$, and $s^\mathcal{I} = \langle 0, 0\rangle$. Note that $\mathcal{I}|_\Sigma = \mathcal{I}$ since $s \in \Sigma$. It is easy to see that $(\rho')^\mathcal{I} = \{\langle 0, 0\rangle\}$ but $\rho^\mathcal{I} = \emptyset$. Hence $\mathcal{I}|_\Sigma = \mathcal{I} \not\models \rho' \sqsubseteq \rho$, and so, $\not\models \rho' \sqsubseteq_\Sigma \rho$.

If condition (2) does not hold, then there exists $u'_i$ such that $u'_i \cap \overline{\Sigma} \in u_j$ for no $u_j$. Assume that $u'_i \cap \overline{\Sigma} = \{s_1, \ldots, s_p, s^-_{p+1}, \ldots, s^-_q\}$. Consider an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ with $\Delta^\mathcal{I} = \{0, \ldots, q+1\}$ such that $s^\mathcal{I} = \emptyset$ for $s \notin \Sigma$, and for each $s \in \Sigma$:

$$s^\mathcal{I} := \{\langle 0, 1\rangle\} \cup \{\langle i, i+1\rangle \mid s = s_i, 1 \leqslant i \leqslant p\} \cup \{\langle i+1, i\rangle \mid s = s_i, p+1 \leqslant i \leqslant q\}.$$

Note that $\mathcal{I} = \mathcal{I}|_\Sigma$. It is easy to see that $\langle 0, q+1\rangle \in ((u'_i)^+)^\mathcal{I}$ but $\langle 0, q+1\rangle \notin ((u_j)^+)^\mathcal{I}$ for every $u_j$. Since $u'_i$ is non-empty, we have $q \geqslant 1$ and hence $\langle 0, q+1\rangle \notin v^\mathcal{I}$. Thus $\langle 0, q+1\rangle \in (\rho')^\mathcal{I}$ but $\langle 0, q+1\rangle \notin \rho^\mathcal{I}$. Hence $\mathcal{I}|_\Sigma = \mathcal{I} \not\models \rho' \sqsubseteq \rho$, and so, $\not\models \rho' \sqsubseteq_\Sigma \rho$. $\dashv$

**Theorem 8 (Decidability for Semantic Conservativity of RBoxes).**
*There is a polynomial-time procedure that, given two RBoxes $\mathcal{R}$, $\mathcal{R}'$, and a signature $\Sigma$, decides the relation $\mathcal{R} \trianglerighteq_\Sigma \mathcal{R}'$.*

*Proof.* In order to check whether $\mathcal{R} \trianglerighteq_\Sigma \mathcal{R}'$, by Lemma 11, it is sufficient to check whether $\mathcal{R}'(s) \sqsubseteq_\Sigma \mathcal{R}(s)$ for every $s \in \Sigma$. By Remark 2, the extended roles $\rho = \mathcal{R}(s)$ and $\rho' = \mathcal{R}'(s)$ can be computed in polynomial time in the size of $\mathcal{R}$ and $\mathcal{R}'$. By Lemma 12, for checking the entailment $\rho' \sqsubseteq_\Sigma \rho$, it is sufficient to check the properties (1) and (2) for $\rho'$ and $\rho$, which obviously can be done in polynomial time. $\dashv$

### 6.3 Geometric Interpretation of the Criterion for Conservativity

As has been demonstrated in Theorem 8, a combination of Lemma 11 and Lemma 12 provides an effective way of checking semantic conservativity for RBoxes. It is desirable to have a more intuitive "geometric" conditions that are required for an RBox $\mathcal{R}'$ to be conservative over $\mathcal{R}$. Here we give such a simplified formulation for the criterion of conservativity of RBoxes. We apply this criterion to obtain an efficient characterisation of the classes of unsafe RBoxes described in Corollary 5.

**Definition 18.** Let $\mathcal{R}$ be an RBox, $S$ a role and $u$ a set of roles. We write $\mathcal{R} \models u \sqsubseteq S$ if $\mathcal{R} \models R \sqsubseteq S$, for all $R \in u$. We write $\mathcal{R} \models u^+ \sqsubseteq S$ and say that $u$ *is transitively separated from* $S$ *in* $\mathcal{R}$ if, for some role $T$, we have $\mathcal{R} \models \mathsf{Tr}(T)$ and $\mathcal{R} \models u \sqsubseteq T \sqsubseteq S$.

*Remark 6.* Note that in the case when role $S$ is transitive in $\mathcal{R}$, $u$ is transitively separated from $S$ in $\mathcal{R}$ if and only if all roles in $u$ are subroles of $S$. Also note that once the $\mathcal{R}$-extension of a role $S$ is computed: $\mathcal{R}(S) = \bigsqcup u_i^+ \sqcup v$, it allows to easily check whether a set $u$ of roles is transitively separated from $S$ in $\mathcal{R}$. Namely: $\mathcal{R} \models u^+ \sqsubseteq S$ iff $u \subseteq u_i$, for some $i$.

**Theorem 9 (Geometrical Criterion for Conservativity).**
*Let $\mathcal{R}$ and $\mathcal{R}'$ be RBoxes and $\Sigma$ a set of role names. Then $\mathcal{R} \trianglerighteq_\Sigma \mathcal{R}'$ iff for every $s \in \Sigma$:*

*(1) for every role $R \in \overline{\Sigma}$, we have $\mathcal{R}' \models R \sqsubseteq s$ implies $\mathcal{R} \models R \sqsubseteq s$, and*
*(2) for every set $u \subseteq \overline{\Sigma}$, we have $\mathcal{R}' \models u^+ \sqsubseteq s$ implies $\mathcal{R} \models u^+ \sqsubseteq s$.*

*Proof.* By Lemma 11, $\mathcal{R} \trianglerighteq_\Sigma \mathcal{R}'$ iff, for every $s \in \Sigma$, we have $\models \mathcal{R}'(s) \sqsubseteq_\Sigma \mathcal{R}(s)$. Let $\mathcal{R}(s) = \bigsqcup u_i^+ \sqcup v$ and $\mathcal{R}'(s) = \bigsqcup u_j'^+ \sqcup v'$. Recall that, by Definition 9, $v$ and $v'$ consist of exactly all the subroles of $s$ and $s'$, resp. By Lemma 12, $\models \mathcal{R}'(s) \sqsubseteq_\Sigma \mathcal{R}(s)$ iff two conditions hold:

$(a)$ $v' \cap \overline{\Sigma} \subseteq \bigcup u_j \cup v = v$ and
$(b)$ for every $u_i'$, there exists $u_j$ with $u_i' \cap \overline{\Sigma} \subseteq u_j$.

We claim that $(a)$ is equivalent to $(1)$, and $(b)$ is equivalent to (2). Indeed:

$(a) \Leftrightarrow (1)$**:** $v' \cap \overline{\Sigma} \subseteq v$ iff $R \in v'$ implies $R \in v$, for every $R \in \overline{\Sigma}$,
iff (by Definition 9) $\mathcal{R}' \models R \sqsubseteq s$ implies $\mathcal{R} \models R \sqsubseteq s$, for every $R \in \overline{\Sigma}$.

$(b) \Leftrightarrow (2)$**:** for every $u_i'$, there exists $u_j$ with $u_i' \cap \overline{\Sigma} \subseteq u_j$
iff $u \subseteq \overline{\Sigma}$ and $u \subseteq u_i'$ implies $u \subseteq u_j$ for some $u_j$
iff (by Remark 6) $u \subseteq \overline{\Sigma}$, $\mathcal{R}' \models u^+ \sqsubseteq s$ implies $\mathcal{R} \models u^+ \sqsubseteq s$. $\dashv$

A *geometric interpretation* of Theorem 9 is that an RBox $\mathcal{R}'$ is conservative over $\mathcal{R}$ w.r.t. $\Sigma$, if (1) $\mathcal{R}'$ does not entail new inclusions between roles over $\Sigma$ (i.e., $\mathcal{R}'$ does not add to the graph of the RBox any edges $\mathcal{R}$ between roles over $\Sigma$), and (2) $\mathcal{R}'$ does not make any set of roles $u \subseteq \overline{\Sigma}$ transitively separated from some role $S$, unless this was already the case in $\mathcal{R}$. Note that (2) in particular means that no role name $s$ from $\Sigma$ (and hence no role from $\overline{\Sigma}$) cannot become transitive in $\mathcal{R}'$ if it was not in $\mathcal{R}$, since otherwise the condition (2) would not hold for $u = \{s\}$.

### 6.4 Consequences for Unsafety of RBoxes

Now we are ready to show that the conditions given in Corollary 5 that are sufficient for the unsafety of an RBox can be verified efficiently (in polynomial time).

**Lemma 13.** *Let $\mathcal{R} = \{\mathsf{Tr}(s)\}$. Then $\mathcal{R} \trianglerighteq \mathcal{R}'$ iff $\mathcal{R}' \not\models s^- \sqsubseteq s$.*

*Proof.* By Theorem 9, it is sufficient to show that $\mathcal{R}' \not\models s^- \sqsubseteq s$ iff the conditions (1) and (2) from Theorem 9 hold for these $\mathcal{R}$, $\mathcal{R}'$, and $\Sigma = \{s\}$. It is easy to see that (2) holds always since $\mathcal{R} \models \{s\}^+ \sqsubseteq s$. The only non-trivial part of (1) is when $R = s^-$. So $\mathcal{R} \trianglerighteq \mathcal{R}'$ iff ($\mathcal{R}' \models s^- \sqsubseteq s$ implies $\mathcal{R} \models s^- \sqsubseteq s$) iff $\mathcal{R}' \not\models s^- \sqsubseteq s$. ⊣

Lemma 13 and part (1) of Corollary 5 imply that $\{\mathsf{Tr}(s)\} \hookrightarrow \mathcal{R}'$ iff $\mathcal{R}'$ contains a role that is transitive and not symmetric in $\mathcal{R}'$. In order to formulate a similar condition for part (2) in Corollary 5, we need some auxiliary notation and terminology.

**Definition 19.** Given roles $s$, $s'$, $r$ and an RBox $\mathcal{R}$, we write $\mathcal{R} \vdash s \overset{*}{\underset{r}{\sim}} s'$ if there exits a sequence of roles $s = s_1, \ldots, s_n = s'$ with $n \geqslant 1$ such that $\mathcal{R} \vdash s_i \sim s_{i+1}$ for all $1 \leqslant i < n$, and $\mathcal{R} \nvdash r \sqsubseteq s_i$ for every $i$ with $1 \leqslant i \leqslant n$.

In other words, relation $\mathcal{R} \vdash s \overset{*}{\underset{r}{\sim}} s'$ expresses that $s$ and $s'$ are connected in $\mathcal{R}$ with a sequence of roles that are not above $r$.

**Proposition 2.** *Given roles $s$, $s'$, $r$ and an RBox $\mathcal{R}$, it is possible to check in polynomial time whether $\mathcal{R} \vdash s \underset{r}{\sim} s'$.*

*Proof.* Given $r$ and $\mathcal{R}$ we first compute a relation $s_i \underset{r}{\sim} s_j$ for all roles $s_i$ and $s_j$ such that $\mathcal{R} \vdash s_i \sim s_j$, $\mathcal{R} \nvdash s_i \sqsubseteq r$ and $\mathcal{R} \nvdash s_j \sqsubseteq r$. Then the relation $s \overset{*}{\underset{r}{\sim}} s'$ is a transitive closure of the relation $s_i \underset{r}{\sim} s_j$. ⊣

**Lemma 14.** *Let $\mathcal{R}'$ be an RBox. Then $\mathcal{R} \hookrightarrow \mathcal{R}'$, for some RBox $\mathcal{R} \in \mathcal{F}$, iff there exists two transitive role names $t_1$ and $t_2$ in $\mathcal{R}'$ such that $(i)$ $\mathcal{R}' \vdash t_1 \overset{*}{\sim} t_2$ and $(ii)$ for every transitive role name $t$ in $\mathcal{R}'$ with $\mathcal{R}' \vdash t_1 \sqsubseteq t$ and $\mathcal{R}' \vdash t_2 \sqsubseteq t$, we have $\mathcal{R}' \vdash t_1 \overset{*}{\underset{t}{\sim}} t_2$.*

*Proof.* **(⇒):** Suppose that $\mathcal{R} \hookrightarrow \mathcal{R}'$, for some RBox $\mathcal{R} \in \mathcal{F}$. Without loss of generality (by renaming roles), we can assume that $\mathcal{R}'$ is a conservative extension of $\mathcal{R}$ (e.g. $[\mathcal{R}] \subseteq [\mathcal{R}']$ and $\mathcal{R} \trianglerighteq \mathcal{R}'$). Then take $t_1$ and $t_2$ be, respectively, the role names $R, S$ if $\mathcal{R} = \mathcal{R}_\vee^\oplus$, $Q_0, Q_n$ if $\mathcal{R} = \mathcal{R}_\wedge^n$, or $T_0, T_n$ if $\mathcal{R} = \mathcal{R}_\vee^n$, or $Q_0, T_n$ if $\mathcal{R} = \mathcal{R}_{\wedge\!\!\!\vee}^n$ (see Fig. 4). It is easy to see $t_1$ and $t_2$ are transitive roles that are connected in $\mathcal{R}$, and hence in $\mathcal{R}'$ (since $[\mathcal{R}] \subseteq [\mathcal{R}']$). Hence the condition $(i)$ of lemma holds. In order to prove the condition $(ii)$, suppose $\mathcal{R}' \vdash t_1 \sqsubseteq t$ and $\mathcal{R}' \vdash t_2 \sqsubseteq t$ for some transitive role name $t$ in $\mathcal{R}'$, but $\mathcal{R}' \nvdash t_1 \overset{*}{\underset{t}{\sim}} t_2$. Since every connection between $t_1$ and $t_2$ in $\mathcal{R}$ is also a connection in $\mathcal{R}'$ (since $[\mathcal{R}] \subseteq [\mathcal{R}']$), we have $\mathcal{R}' \vdash t \sqsubseteq s$ for some $s \in \mathsf{RN}(\mathcal{R})$. Hence $\mathcal{R}' \models \{t_1, t_2\}^+ \sqsubseteq s$. Since $\mathcal{R} \trianglerighteq \mathcal{R}'$ and $\{t_1, t_2, s\} \subseteq \mathsf{RN}(\mathcal{R})$, by condition (2) of Theorem 9, we have $\mathcal{R} \models \{t_1, t_2\}^+ \sqsubseteq s$ which is possible for neither $\mathcal{R} \in \mathcal{F}$.

**($\Leftarrow$):** First, note that this direction of the lemma holds if there exists tree role names $t'_1$, $t'_2$ and $t'$ that are transitive in $\mathcal{R}'$ and such that $\mathcal{R}' \vdash t' \sqsubseteq t'_1$, $\mathcal{R}' \vdash t' \sqsubseteq t'_2$, and $\mathcal{R}' \nvdash t'_1 \sim t'_2$. In this case $\mathcal{R} \hookrightarrow \mathcal{R}'$ for $\mathcal{R} = \mathcal{R}^{\oplus}_{\vee} \in \mathcal{F}$ using the renaming $\sigma(R) := t'_1$, $\sigma(S) := t'_2$ and $\sigma(Q) := t'$. Indeed, it is easy to see that conditions condition (1) and (2) of Theorem 9 hold for $\sigma(\mathcal{R})$, $\mathcal{R}'$ and $\Sigma = \mathsf{RN}(\sigma(\mathcal{R})) = \{t'_1, t'_2, t'\}$. Hence in the reminder of the proof we assume that for every transitive roles $t'_1$, $t'_2$ and $t'$ in $\mathcal{R}'$:

$$\mathcal{R}' \vdash t' \sqsubseteq t'_1 \quad \text{and} \quad \mathcal{R}' \vdash t' \sqsubseteq t'_2 \quad \text{implies} \quad \mathcal{R}' \vdash t'_1 \sim t'_2 \tag{10}$$

Property (10) essentially means that the inclusions between transitive roles in $\mathcal{R}'$ form a tree structure. In particular, for every two transitive roles $t_1$ and $t_2$ in $\mathcal{R}'$, there exists at most one, up to equivalence, minimal transitive role $t$ such that $\mathcal{R}' \vdash t_1 \sqsubseteq t$ and $\mathcal{R}' \vdash t_2 \sqsubseteq t$; that is, whenever $\mathcal{R}' \vdash t_1 \sqsubseteq t'$ and $\mathcal{R}' \vdash t_2 \sqsubseteq t'$ for some transitive role $t'$, then $\mathcal{R}' \vdash t \sqsubseteq t'$.

Consider the set of $\mathcal{P}$ all sequences $t_1 = s_1, \dots, s_n = t_n$, $n \geqslant 1$, such that $t_1$ and $t_2$ are transitive in $\mathcal{R}'$, $\mathcal{R}' \vdash s_i \sim s_{i+1}$ for every $i$ with $1 \leqslant i < n$, and $\mathcal{R}' \nvdash t \sqsubseteq s_i$ where $t$ is a minimal transitive role such that $\mathcal{R}' \vdash t_1 \sqsubseteq t$ and $\mathcal{R}' \vdash t_2 \sqsubseteq t$. By condition $(ii)$ of this lemma, at least one such sequence exists. Consider a shortest sequence $t_1 = s_1, \dots, s_n = t_n$ from $\mathcal{P}$. We claim that for this sequence the following properties hold: $(a)$ $\mathcal{R}' \nvdash s_i \sim s_j$, for all $1 \leqslant i, j \leqslant n$ with $i + 1 < j$, $(b)$ $\mathcal{R}' \nvdash \{s_{i-1}\}^+ \sqsubseteq s_i$ for $i < n$, and $(c)$ $\mathcal{R}' \nvdash \{s_{j+1}\}^+ \sqsubseteq s_j$ for $i > 1$.

Indeed, if property $(a)$ does not hold, then $t_1 = s_1, \dots, s_i, s_j, \dots, s_n = t_2$ is a shorter sequence in $\mathcal{P}$. If property $(b)$ does not hold then there exists a transitive role $t$ in $\mathcal{R}'$ such that $\mathcal{R}' \vdash s_{i-1} \sqsubseteq t$ and $\mathcal{R}' \vdash t \sqsubseteq s_i$, $i < n$. In this case the $t_1 = s_1, \dots, s_{i-1}, t$ is a shorter sequence in $\mathcal{P}$. Similarly, if condition $(c)$ does not hold then there exists a shorter sequence $t, s_{j+1}, \dots, s_n = t_2$ in $\mathcal{P}$. Note also that $(d)$ $\mathcal{R}' \nvdash t_1 \sim t_2$, since, otherwise $\mathcal{R}' \vdash t_1 \sqsubseteq t$ and $\mathcal{R}' \vdash t_2 \sqsubseteq t$ for $t = t_1$ or $t = t_2$, and so, $\mathcal{R}' \nvdash t \sqsubseteq t_1$ or $\mathcal{R}' \nvdash t \sqsubseteq t_2$, which is not possible by definition of $\mathcal{P}$.

From properties $(a)$–$(d)$, using Theorem 9, it is easy to show that $\mathcal{R}'$ is a conservative extensions of some $\mathcal{R} \in \mathcal{F}$. $\dashv$

In words, Lemma 14 says that an RBox $\mathcal{R}'$ is stronger than some RBox from the family $\mathcal{F}$ iff there are two connected transitive roles $t$ and $t'$ in $\mathcal{R}'$ that are connected below every transitive super-role of $t$ and $t'$.

**Proposition 3.** *Given an RBox $\mathcal{R}'$ it is possible to verify in polynomial time whether $\mathcal{R} \hookrightarrow \mathcal{R}'$ for some $\mathcal{R} \in \mathcal{F}$.*
*Proof.* $\dashv$

Now we put together the results from this section and Section 3 and formulate a sufficient (and, as we conjecture, necessary) condition for unsafety of RBoxes:

**Corollary 6 (Sufficient Condition for Unsafety).** *Let $\mathcal{R}$ be an RBox. Then:*

1. *If there is a role $T$ with $\mathcal{R} \models \mathsf{Tr}(T)$ and $\mathcal{R} \not\models T^- \sqsubseteq T$, then $\mathcal{R}$ is $\mathcal{ALCIN}$-unsafe.*

2. *If there are two transitive role names $t_1$ and $t_2$ in $\mathcal{R}$ such that: $(i)$ $\mathcal{R} \vdash t_1 \overset{*}{\sim} t_2$ and $(ii)$ for every transitive role name $t$ in $\mathcal{R}$ with $\mathcal{R} \vdash t_1 \sqsubseteq t$ and $\mathcal{R} \vdash t_2 \sqsubseteq t$, we have $\mathcal{R} \vdash t_1 \underset{t}{\overset{*}{\sim}} t_2$, then $\mathcal{R}$ is unsafe for $\mathcal{ALCN}$.*

*Proof.* This is a consequence of Theorems 2, 3, 9, and Lemmas 13 and 14. $\dashv$

## 7 Conclusions and Future Work

Driven by applications, we have looked more closely at the effect of non-simple roles in number restrictions on the decidability of standard DL reasoning problems. We have shown that, in the absence of inverse roles, the restriction imposed by $\mathcal{SHQ}$ to non-simple roles in number restrictions can be relaxed substantially and that, in the presence of inverse roles, this restriction turns out to be crucial for decidability.

These results raise numerous further questions. Firstly, given a DL $\mathcal{L}$, can we formulate necessary and sufficient conditions for an RBox to be $\mathcal{L}$-safe? Secondly, for an interesting class of $\mathcal{L}$-safe RBoxes $\mathcal{R}$, what is the computational complexity of deciding $\mathcal{L}(\mathcal{R})$-satisfiability? And can these decision procedures be implemented and used in practice? Thirdly, in the approach taken here, we allow all roles to occur in number restrictions. Given an $\mathcal{L}$-unsafe RBox $\mathcal{R}$, can we extend the notion of simple roles to regain decidability of $\mathcal{L}(\mathcal{R})$? And how applicable would this be in practice? Finally, in the presence of inverse roles, can we restrict the usage of inverse roles in TBoxes so as to re-gain decidability? For example, would disallowing number restrictions on inverse roles whilst allowing number restrictions on transitive role names help? For the list of other interesting open problems, see the accompanying technical report [9].

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## Appendix: Open problems

*Problem 1.* Is the RBox $\mathcal{R}^{\oplus}_{\wedge} = \{\oplus{\to}\oplus{\leftarrow}\oplus\}$ $\mathcal{ALCQ}$-safe? Conjecture: $\mathsf{Yes}$.

*Problem 2 (Criterion for $\mathcal{ALCQ}$-safety).* An RBox $\mathcal{R}$ is $\mathcal{ALCQ}$-unsafe iff $\mathcal{R} \hookrightarrow \mathcal{R}'$ for some $\mathcal{R} \in \mathcal{F}$. Here $\mathcal{F} := \{\mathcal{R}^n_{\wedge}, \mathcal{R}^n_{\vee}, \mathcal{R}^n_{\wedge\!\!\!\vee} \mid n \geqslant 1\}$ (cf. Fig. 4).

Note: the '$\Leftarrow$' is already proved. To prove '$\Rightarrow$', first try to solve Problem 1.

*Problem 3 (Simple roles revised).* In an $\mathcal{ALCQ}$-unsafe RBox $\mathcal{R}$, which roles in $\mathcal{R}$ could be called *simple* and used in number restrictions, while keeping the logic decidable?

Note: the answer may be non-deterministic: you can use in num. restr. either these roles, or these roles... Hence we can talk about a *simple* subset $u \subseteq \mathsf{RN}(\mathcal{R})$ of roles.

Conjecture: a subset $u \subseteq \mathsf{RN}(\mathcal{R})$ is simple if its downward closure in $\mathcal{R}$ is $\mathcal{ALCQ}$-safe. Here by the *downward closure* of a set of roles $u$ in $\mathcal{R}$ we call the minimal subset of roles containing roles from $u$ and all their subroles in $\mathcal{R}$.

For example, in $\mathcal{R} := \{\oplus{\to}\bigcirc{\leftarrow}\oplus\}$, if we allow to use in number restriction the two transitive roles, but not the non-transitive role, will it be decidable? Conjecture: $\mathsf{Yes}$.

*Problem 4.* Is the RBox $\mathcal{R} = \{\mathsf{Tr}(R), R \sqsubseteq R^-\}$ $\mathcal{ALCIQ}$-safe?[4] Conjecture: $\mathsf{Yes}$.

*Problem 5 (Criterion for $\mathcal{ALCIQ}$-safety).* If Problem 4 is answered '$\mathsf{No}$', then the Criterion is: *An RBox is $\mathcal{ALCIQ}$-unsafe iff it contains a transitive role.*

If Problem 4 is answered '$\mathsf{Yes}$', then the Criterion for $\mathcal{ALCIQ}$-safety is:
*An RBox is $\mathcal{ALCIQ}$-unsafe iff it contains a transitive non-symmetric role.*

*Problem 6 (Semi-safety).* An RBox is *$\mathcal{ALCIQ}$-semi-safe* if all its role *names* but not their inverses can be used in number restrictions, while keeping the logic decidable. (We confined the definition to names w.l.o.g., as we can always rename roles.)

Which RBoxes are semi-safe? In particular, is $\{\mathsf{Tr}(R)\}$ $\mathcal{ALCIQ}$-semi-safe?

Conjecture: an RBox (without inverse roles) is semi-safe iff it is $\mathcal{ALCQ}$-safe.

Note: even $\mathcal{ALCIF}$ with general TBoxes does not have the Finite Model Property.

*Problem 7 (Semi-simple roles).* An obvious combination of the previous two problems: how can we divide roles into *simple* (both $r$ and $r^-$ can be used in num. rest.), *semi-simple* (only $r$ but not $r^-$ can be used in num. restr., or vice-versa), and *non-simple*?

Note: since we are in $\mathcal{ALCIQ}$, transitive roles cannot be simple, only semi-simple.

*Problem 8.* Are our decidability results hold for ABox consistency problem as well?

*Problem 9 (Nominals).* What about $\mathcal{ALCOQ}$-safety?

Note that in presence of nominals, the logic does not have the Disjoint Union Model Property, which was used in the proof of Modularity (Lemma 5). Therefore it might be the case that two disjoint safe RBoxes put together yield an unsafe RBox.

*Problem 10.* What about $\mathcal{ALCF}$-safety and $\mathcal{ALCIF}$-safety? Open problem since [7]. Here '$\mathcal{F}$' stands for $(\leqslant 1\, R)$.

Conjecture: all RBoxes are $\mathcal{ALCF}$- and $\mathcal{ALCIF}$-safe (by eliminating transitivity?).

[4] By the way, is it meaningful to consider $\mathcal{ALCQ}$-safety of RBoxes that involve inverses?

*Problem 11 (Empty TBox and new Modal Logics).* Our main notion of a safe RBox (Definition 3) can be written more explicitly as follows: an RBox $\mathcal{R}$ is *safe* for a *logic* $\mathcal{L}$ *with general TBoxes*. In most cases (except for one transitive role in $\mathcal{ALCQ}$), TBoxes are not internalizable. And in our proofs of undecidability, we essentially used GCIs (general concept inclusion axioms). Now, what about (traditional in DLs) acyclic and empty TBoxes? Note that if we are interested in decidability not in complexity, then acyclic TBoxes have the same impact as empty TBoxes. Empty TBoxes yield the following interesting modal logics:

**Graded Transitive Logics:** are logics **GrL**, where $\mathbf{L} \in \{\mathbf{K45}, \mathbf{K4B}, \mathbf{S4}, \mathbf{S5}\}$, decidable? What is their complexity?
Recall: **4** denotes transitivity, **5** euclideanness, **B** symmetricity, **S** reflexivity.
Decidability of **GrK4B** is the retriction of Problem 4 to empty TBoxes. Since the proofs in [4] for transitive modalities are incorrect, these problems are again open.

**Graded Transitive Logics with inverse modalities:** are Graded modal logics with inverse **GrLI** decidable, where $\mathbf{L} \in \{\mathbf{K4}, \mathbf{K45}, \mathbf{K4B}, \mathbf{S4}, \mathbf{S5}\}$? What is their complexity? In addition, consider these logics with counting allowed on modality but not on its inverse. (Check if these logics were already considered in literature.)
Note: Theorem 3 says that $\{\mathsf{Tr}(R)\}$ is $\mathcal{ALCIQ}$-unsafe; but since TBoxes are not internalizable in this logic, undecidability of **GrK4I** does not follow from it.

**Finite Model Property:** which of the logics mentioned above possess FMP?
Note that absense of FMP for $\mathcal{ALCIF}$ with general TBoxes does not imply absense of FMP for **GrK4I**.

**Graded Multimodal Logics:** the same questions for $\mathbf{GrL}_m$ and $\mathbf{GrLI}_m$, **L** as above.