Categorical approaches to computations in contextuality and causality

Sander Uijlen

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Promotor:
Prof. dr. B.P.F. Jacobs

Copromotoren:
Dr. A.R. Kissinger & dr. S. Staton (University of Oxford, Verenigd Koninkrijk)

Manuscriptcommissie:

Prof. dr. S. Abramsky
(University of Oxford, Verenigd Koninkrijk)
Prof. dr. J.R.B. Cockett
(University of Calgary, Canada)
Prof. dr. N.P. Landsman
Prof. dr. S.J.L. Smets
(Universiteit van Amsterdam)
Dr. R. Duncan
(University of Strathclyde, Verenigd Koninkrijk)

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Doctoral Thesis

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Sander Marijn Uijlen

born on July 6, 1987
in 's-Hertogenbosch (the Netherlands)

Supervisor:
Prof. dr. B.P.F. Jacobs

Co-supervisors:
Dr. A.R. Kissinger & dr. S. Staton (University of Oxford, UK)

Doctoral Thesis Committee:

Prof. dr. S. Abramsky
(University of Oxford, UK)

Prof. dr. J.R.B. Cockett
(University of Calgary, Canada)

Prof. dr. N.P. Landsman

Prof. dr. S.J.L. Smets
(University of Amsterdam)

Dr. R. Duncan
(University of Strathclyde, UK)

THNX

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To everyone who has taught me

Buy the ticket, take the ride
-Hunter S. Thompson, Fear and Loathing in Las Vegas

Abstract

This thesis presents two new approaches to two topics in quantum foundations: Part I: The study of causal structures and higher order processes, i.e., transformations of processes. Part II: The study of non-locality and contextuality using effect algebras.

In Part I we give a categorical semantics for higher order causal processes. The 'usual' framework of compact closed symmetric monoidal categories is insufficient for describing higher order causality as everything collapses to first order. We give a construction on a class of compact closed categories, which we call *precausal*, such that the resulting category is endowed with a type theory describing causal relations between the inputs and outputs of processes. The axioms of a precausal category then make sure these causal types also correspond to higher order causality. This way the resulting category can be seen as a category of higher order causal processes. Since the categories used for quantum theory and probability theory are precausal, we obtain categories of higher order quantum processes and higher order stochastic processes. These resulting categories have the structure of a *-autonomous category and the connectives in such a category have natural interpretations in the causal type theory. We use this framework to describe no-signalling and one-way signalling processes, their multipartite generalizations such as n-combs and more generally causal orders given by directed acyclic graphs, as well as their duals. Some special attention is given to those processes which exhibit *indefinite causal order* such as the quantum switch, the OCB W-matrix and the classical example by Baumeler and Wolf.

In Part II we show how effect algebras can be used as a way to study and reason about non-locality and contextuality. The standard framework of probability theory cannot explain certain measurement results arising in quantum mechanics as marginals of a joint probability distribution. This is called a *paradox*. We give two generalizations of probability theory, using partial monoids (in particular effect algebras) and using presheaves. The advantage of using effect algebras is that the interval of probabilities, $[0, 1]$, is itself an effect algebra, which allows us to stay inside the category of effect algebras. When going to the probabilistic case (Hardy) we can still use a partial monoid as outcome space. The paradoxes can then be described as a non-factorization through objects corresponding to classical probability theory (Boolean algebras and representable functors). Effect algebras

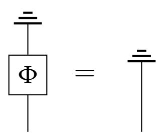
embed in a category of presheaves and this allows us to make a connection between the two approaches using adjunctions. In particular, we are able to relate our effect algebraic framework to the established presheaf approach of Abramsky and Brandenburger. In a similar fashion we relate effect algebras to test spaces. We describe explicitly the examples of Bell, Kochen-Specker, Hardy and GHZ.

Contents

0.1 Introduction

Categorical quantum mechanics (CQM) is the study of quantum theory using category theory. Interest is less focussed on aspects of single (quantum) systems, but more on logical, structural and compositional questions. This thesis does precisely this in two parts. The first part fits the approach initiated by Abramsky and Coecke ([3] [4], [62]). We study causality, in particular higher order causality, using symmetric monoidal categories. The second part of this thesis fits what I hope to be called the Nijmegen approach. Partial monoids and effect algebras in particular are used to describe non-locality and contextuality.

Symmetric monoidal categories (SMCs) describe process theories. Objects and morphisms describe systems and their transformations, seen as processes. The tensor product is a way to combine systems into one. In particular, we start with a compact closed SMC with a preferred discarding effect (⏚) for every object. This allows us to say when a processes is causal; namely when it preserves the discarding process.



This notion of causality then allows us to make precise a notion of causal ordering for processes (Definition 1.1.1) by considering which systems can influence which other systems. This is done via describing operational behaviour of processes using diagrammatic reasoning. In the special case of one-way signalling processes, where one output system is in the causal past of another input system, such a process can be used as a transformation of processes, rather than of systems (see Section 2.2). This strongly implies that a good understanding of to the theory of higher order processes or supermaps is needed ([25] [24]). Moreover, such processes for example play a big role in quantum communication protocols ([47]). The standard framework of compact closed categories allows one to easily obtain higher order processes, however, in general these will not preserve causality (see Section 1.2 and Example 3.1.7). This calls for a better understanding of higher order causality. In Part I of this thesis we solve this problem by giving a construction which turns a certain type of compact closed category, called precausal (Definition 3.1.1), into a category of higher order causal processes. This new category can be seen as a refinement of the old one, by endowing objects with sets of states which we think of as generalized causal states. Since the original category is compact closed, the Choi-Jamiołkowski isomorphism (also known as process-state duality, see Section 2.1.5) allows for sets of processes to be seen as a generalization of the set of causal states. This way we obtain objects representing, among others, causal processes, one-way signalling processes or no-signalling processes. These objects can then be seen as types in a causal type theory. A process has a certain type if and only if it is compatible with a certain causal ordering. Categorically, this category of higher order causal processes is ISO-mix *-autonomous (Theorem 3.3.16). Consequently we have two monoidal products, $\otimes$ and $\parr$. These coincide when we consider first order systems, which are objects where the set of generalized causal states is precisely the set of all causal states (see Section 4.1). However, when we consider joint systems we find interesting results linking the connectives to causal orders. For example:

$$\begin{aligned} (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \quad &\leftarrow \text{ no-signalling processes} \\ (\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}') \quad &\leftarrow \text{ all causal processes} \end{aligned}$$

We then classify the one-way signalling processes, which lie in between no-signalling and all causal processes. From this we turn to the abstract characterisation of a family of higher-order processes called combs, which have previously been studied in quantum information and foundations ([47], [24]). We give two classifications of these combs (Section 4.3). The first is inductive, combs are processes which send combs to other combs. The second operational, discarding the last output of a comb splits the process into a smaller comb and a discard effect. As such, combs correspond to processes with linear causal order. Using a pullback construction we can then give types for all causal orders described by directed acyclic graphs using the types of combs (Section 4.4). Interestingly enough, it has been shown relatively recently that there exist (dual) processes which do not conform to any predefined causal order between the input processes ([73], [23]). Moreover, these processes allow for computational speed up ([21] [81]). While originally formulated as a theory of quantum mechanics with no fixed causal background, such processes even occur in the classical case ([13]). Using simple diagrammatic arguments, we show in Section 4.5 that all these processes are instances of second order causal processes and hence fit our framework. We use this to show that every process with indefinite causal order is an affine combination of processes which have a fixed causal order.

Part I is structured as follows: in Chapter 1 give an introduction to causality, causal orders and higher order processes and discuss their physical meaning. In Chapter 2 we present the needed mathematical background of processes theories (i.e., symmetric monoidal categories), including *-autonomous categories. We also show how the string diagrammatic language makes life a lot easier. Furthermore, we make precise the notion of signalling needed to describe causal orders and show that the usual notion of compact closed categories is not sufficient for our goal of talking about higher order causality. Then in Chapter 3 we define our notion of a precausal category (Definition 3.1.1). We show that the leading examples of categories for quantum theory, **CPM**, and probability theory $\mathbf{Mat}(\mathbb{R}_+)$, are indeed precausal. In the last past of this chapter we give the construction of turning a precausal category into a category of higher order causal processes and it is shown this category is *-autonomous. Chapter 4 shows how the newly constructed types relate to causal orders. We explicitly consider linear causal orders, which correspond to combs and show how to obtain more general causal orders via a pullback construction. We end this chapter with a discussion on indefinite causal orders. Finally we give a conclusion and consider some future work.

The work in this part is mostly based on the paper *A categorical semantics for causal structure* and its extended journal version ([64], [66]), written together with Aleks Kissinger. In turn, this is based on the work by Paulo Perinotti [17] who gave a uniform description of higher-order *quantum* operations in terms of generalised Choi operators. However, rather than relying on the linear structure of spaces of operators, we work purely in terms of the *-autonomous structure and the precausal axioms, which concern the compositional behaviour of discarding processes.

Part II concerns non-locality and causality. Incompatibility of certain measurements can give rise to outcomes or probability distributions over outcomes which cannot arise from a joint distribution. We call this a paradox. This incompatibility naturally gives rise to partial structures. Partial monoids, in particular effect algebras, can be used to give a generalization of probability theory which can account for this partial structure. We consider a simple Bell scenario where two observers each have two measurement settings with two possible outcomes. In this setting we can define an effect algebra E for each observer such that a morphism $E \to [0,1]$ describes a family of probability distributions over jointly measurable settings. Taking the tensor product over these algebras we can precisely describe joint distributions with no-signalling (Proposition 8.2.2). The 'classical' situation where all measurements are compatible is described by a Boolean algebra. The paradox is now formulated as a non-factorization of a generalized probability distribution through Boolean algebras (Proposition 8.3.7):

$$\begin{tikzcd} E_{\mathrm{A}} \otimes E_{\mathrm{B}} \arrow[rr, "t"] \arrow[dr, "i"'] & & {[0,1]} \\ & B_{\mathrm{A}} \otimes B_{\mathrm{B}} \arrow[ur, "/" marking] & \end{tikzcd} \tag{1}$$

Effect algebras embed fully and faithfully in a category of presheaves over natural numbers by taking tests (Section 7.5). This leads to a new generalization of probability theory using presheaves and allows us to reformulate the paradox in terms of a non-factorization of certain presheaves through a representable presheaf. In particular, we make a connection to the presheaf framework of Abramsky and Brandenburger ([2]) via a chain of adjunctions which allow us to transport the paradox between effect algebras and a presheaf category over measurements (Section 8.6). Replacing the interval $[0,1]$ with other (partial) monoids allows us generalize probabilities. Specifically, it allows us to consider possibilities as in the Hardy paradox (Section 9.1). We give an effect algebraic formulation of the Kochen-Specker theorem as the non-existence of a certain morphism. Again using a chain of adjunctions to transport this paradox, this time between effect algebras and a presheaf category over commutative subalgebras of the bounded operators on a Hilbert space, we link this formulation to the presheaf formulation of Hamilton, Isham and Butterfield ([48], Section 9.1.2). Finally, we consider an adjunction between effect algebras and test spaces ([44],[45]). We use this adjunction to transport the Bell paradox and the GHZ paradox.

As the nature of non-locality and contextuality is still very mysterious and quantum mechanics is notoriously counter intuitive, it is important to have good mathematical tools to do reasoning with, especially since non-locality and contextuality give rise to computational speed-ups ([51], [16]). Effect algebras provide such a tool. The adjunctions between effect algebras and other categories ensure that non-factorization results can be transported between these categories (Lemma 8.29). At the same time, the algebraic nature of effect algebras gives a clear conceptual interpretation of what is going on. A further advantage comes from *cohomology*. Simply stated, presheaves deal with 'glueing together' local pieces of information. The paradoxes now arise from local pieces of information which cannot be glued together globally. Cohomology tries to find obstructions to explain why this global gluing is impossible. This is used to study non-locality and contextuality in the presheaf framework ([1], [5]). However, in this approach it is possible to obtain 'false positives'. There is a sufficient condition for contextuality, however, it is not necessary ([5])). While this can be solved ([20]), cohomology for effect algebras, developed by Frank Roumen in his PhD thesis [79], does not have such false positives.

Part II is structured as follows: Chapter 6 gives a quick overview of non-locality and contextuality on the hand of the Bell and Kochen-Specker theorems. In Chapter 7 we give the needed mathematical background on partial monoids, particularly effect algebras, and presheaves. We show that effect algebras embed in a presheaf category and use this to give two generalizations of probability theory. In Chapter 8 we show how probability tables as in Bell scenarios arise as effect algebra morphisms on a tensor product and discuss how tables can be realized. We rephrase the paradox in terms of presheaves and link the effect algebraic formulation to the Abramsky Brandenburger approach using adjunctions to transport the paradox. Chapter 9 shows how other paradoxes have an effect algebraic formulation and link them to existing formulations. We do this for the Hardy paradox, Kochen-Specker theorem and GHZ scenario. The latter is linked to the test spaces approach via an adjunction. Finally we give a conclusion and consider some future work.

The work in this second part is mostly based on the paper *Effect Algebras, Presheaves, Non-locality and Contextuality* and its extended journal version ([87], [86]), written together with Sam Staton.

0.2 Prerequisites

Throughout this thesis we will assume a basic working knowledge of category theory (see for example [10] or [71]). This includes a basic understanding of (co)limits, in particular (co)products and pullbacks. We also assume a basic understanding of adjunctions, but not much more than that. In particular, mathematical background on monoidal categories and their diagrammatic interpretation (Chapter 2), including *-autonomous categories (Section 2.4), partial monoids, including effect algebras (Chapter 7), and some presheaf theory (Section 7.4) is given.

As the two topics addressed in this thesis find their origin in quantum theory, some understanding of this goes a long way in appreciating the results, but is by no means necessary. The standard framework for quantum theory has been formalized by von Neumann ([93]). For a more mathematical background we point to the classics [61], [89] and the more recent [68]. A more physics oriented story can be found, for example, in [43] and [80].

Part I

Higher order processes and causality

Chapter 1

Casual causality: an introduction to causal orders

Correlation does not imply causation. Indeed, over the past, say, 200 years, the average temperature on Earth has gone up while the number of pirates has gone down, yet there is no reason that pirates should slow down global warming.[1] Similarly, it would be dubious to conclude that firemen/women cause burn damage, even though there is a strong correlation between the number of fire fighters and the damage caused by a fire. Even worse, by just considering correlations we might not be able to distinguish between cause and effect: the sun shines *because* I get sunburned, the wind blows harder *because* windmills turn faster.

The fact that correlation does not imply causation of course does not imply that correlation implies no causation. Strong winds do let windmills turn faster and strong sun does inflict sunburn. However, there might also be other causal relations between events which explain the correlations. The correlation between pirates and global warming could just be a coincidence, or maybe there is an increase in wealth which diminished the pirate population as well as warmed up the Earth (both effects which are themselves again not necessarily directly causally related).

A description of such events is given by Bayesian networks and a treatment especially related to causality can be found in the work of Pearl ([75]) or Spirtes et al. ([85]). There the main ingredient to describe a causal structure is a *directed acyclic graph* (DAG), which, as the name suggests, is a

[1] Or is tharrr?

graph where the edges are directed and in which there are no cyclic paths. So for example

$$\begin{tikzcd}
 & D & \\
B \arrow[ur] & & C \arrow[ul] \\
 & A \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

is a valid DAG, but

$$\begin{tikzcd}
 & D \arrow[dl] & \\
B \arrow[dr] & & C \arrow[ul] \\
 & A \arrow[ur] &
\end{tikzcd}$$

is not as it contains path from A to itself. Because of this acyclicity, from now on, we will leave out explicit directions on the edges and read DAGs from bottom-to-top. The first DAG above will thus be drawn as

$$\begin{tikzcd}
 & D & \\
B \arrow[ur, dash] & & C \arrow[ul, dash] \\
 & A \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd} \tag{1.1}$$

Before discussing the interpretation of a DAG, it is good to have some terminology.

Definition 1.0.1. Let A be an event (i.e., vertex) in a DAG $\mathcal{G}$ and denote the set of events of $\mathcal{G}$ by $evt(\mathcal{G})$. The *parents* of A are those events from which there is an edge to A:

$$Par(A) = \{B \in evt(\mathcal{G}) | \text{There exists an edge from } B \text{ to } A\}$$

The *ancestors* of A are those events B from which there is a path (of any length) from B to A:

$$Anc(A) = \{B \in evt(\mathcal{G}) | \text{There exists a path from } B \text{ to } A\}$$

So the ancestors of an event A can be viewed as what happens in the past of A and the parents are those events which can directly influence A.

Note that any event is not a parent of itself, but since paths of lentgh 0 are allowed in the definition of $Anc(A)$, it is an ancestor of itself. This can also be thought of in terms of posets. A DAG is related to a partial order via the above ancestry relation: for vertices A, B we have $A \leq B$ in the partial order if and only if A is an ancestor of B in the DAG. So the ancestors of an event A is the set $Anc(A) = \{B \in evt(\mathcal{G}) | B \leq A)\}$. Because we are dealing with causal relations, we will use the terms 'ancestors' and 'past' synonymously.

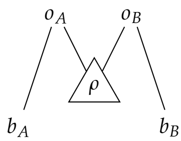
In the Bayesian network framework a DAG can be interpreted as giving the possible causal influences between the vertices. In particular, the absence of an arrow between events means there is no influence between these events. For example, there will be an arrow from the event *the sun is shining* to *I got a sunburn*, even though I do not get a sunburn every time the sun shines. Likewise, me getting a sunburn has no influence on the sun, so there is no arrow in the other direction. If we wish to say that a DAG describes a causal order, then any event should only be causally influenced by its parents. This is made precise as follows: an event, such as *the sun shines*, has values, such as 'true' or 'false' and we can consider probability distributions over these values. For each vertex A_i in a DAG we can consider a value of that event, x_i, and consider the probability that every event is in some specific value: $P(x_1, \ldots, x_n)$.

Definition 1.0.2. A probability function P is compatible with a DAG $\mathcal{G}$ (also called P is *Markov* relative to $\mathcal{G}$ or P *represents* $\mathcal{G}$), if P admits a factorization of the form

$$P(x_1, \ldots, x_n) = \prod_i P(x_i | Par(A_i)) \tag{1.2}$$

So for example, a probability function is compatible with the DAG (1.1) if it factors as $P(x_A, x_B, x_C, x_D) = P(x_D | x_B, x_C) \cdot P(x_C | x_A) \cdot P(x_B | x_A) \cdot P(x_A)$

Now let us apply this causality framework to the following (quantum) setting: A bipartite state ρ is shared between everyone's favourite scientists, Alice and Bob, who also obtain some input bit, b_A, b_B, respectively. Based on the value of this input bit, Alice (Bob) performs one or another measurement on her (his) part of the state ρ and obtains an outcome o_A (o_B). This setup is called the *Bell scenario* and we will consider this in great detail in Part II of this thesis, but for now we just notice that the causal structure would be the following



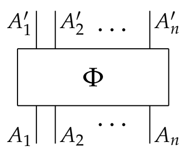
(1.3)

where we need to take special care of ρ as it is unobserved and hence we need to sum over all its values. For the corresponding probability functions we then expect a factorization of the form

$$P(o_A, o_B | b_A, b_B) = \sum_{\lambda} P(\lambda) P(o_A | b_A, \lambda) P(o_B | b_B, \lambda) \tag{1.4}$$

However, any probability distribution that satisfies this factorization must satisfy certain constrains on the correlations of the outcomes of measurements w.r.t. the different settings, known as the *Bell inequalities* ([15] and we consider this in more detail in Part II). Yet quantum mechanics predicts, and experiment verifies, that these Bell inequalities are not always satisfied. This shows that classical probabilistic causal reasoning can be problematic for quantum theory. Therefore, in order to study causal orders, it is important to understand how causation and correlation are related in a quantum mechanical setting. This is done, for example, in [7] where the correlations related to a common cause are explored and in [31] where a framework for quantum causal modelling is developed.

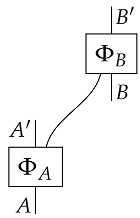
Here, in this part of the thesis, we wish to explore causal structures from a *process theoretic* point of view. We make this precise in Chapter 2. The basic idea is the following. We consider *processes* , which are like (non-deterministic) functions, from input systems to output systems



(1.5)

Such a diagram is to be read as if time goes from bottom to top, wires represent systems and when one or more wires enter a box, a process happens on those systems and some new systems are put out.

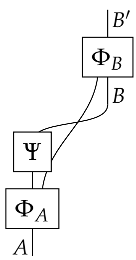
We are then interested in the *causal structure* of such a process. That is to say, what are the causal relations between the inputs and outputs of the process. Which inputs influence which outputs and which outputs could be used as an input to the process. For example, suppose that our process is of the following form:



(1.6)

Here some process Φ_A acts on a system A and outputs a system A' as well as some other system which then serves, together with some input system B, as an input to a second process Φ_B which outputs a system B'. Just by looking at the diagram in (1.6) we can immediately suspect three things:

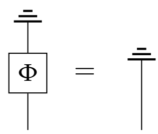
1. The output A' can only depend on the input A.
2. The output B' can depend on the input B as well as the input A and the process Φ_A.
3. It might be possible to take the output system A', perform on it some operation Ψ of which the output is B and use this as input for the process Φ_B, like so:



(1.7)

These statements imply a possibility of signalling from Φ_A to Φ_B, or more precisely, the impossibility of signalling from Φ_B to Φ_A. As such, ignoring some technicalities which we address next, we call a process such as the one in diagram (1.6) a *one-way signalling process* (see Definition 2.2.2 in the next chapter).

So what are these technicalities that we need in order to make the above statements precise? It turns out that the main ingredient is a way to 'forget about' or 'get rid of' a system. This is done by introducing a *discard effect* (see Section 2.1.6). In particular, especially in the quantum setting, performing a (demolition) measurement and forgetting about the outcome, or performing a measurement with only a single outcome discards the system. We draw the discard effect on a system A by using a 'ground symbol': $\bar{\top}_A$. Using the discard effect allows us to distinguish a particular set of processes, namely the *causal* ones: We say a process Φ is *causal* if

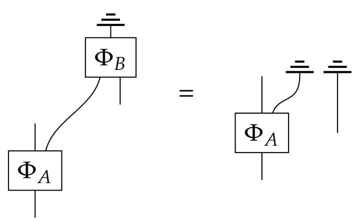


(1.8)

That is

A process is causal if discarding its output is the same as discarding its input, i.e., as if the process never happened.

Consider again the situation in diagram (1.6) and suppose that Φ_B is causal. We focus purely on the output of system A', meaning that we ignore the other outputs, for example because the process Φ_B is in the future of Φ_A or because Φ_B is so far away from Φ_A that it is not possible to send information to Φ_A (see Section 1.4). That is, we are in the situation where we discard the system B', so that



(1.9)

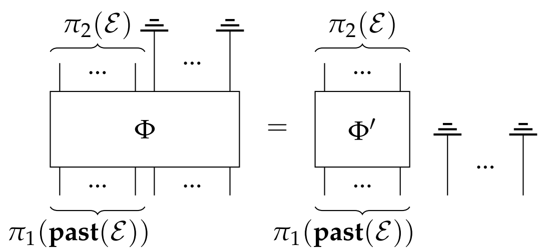
It follows that it is as if the process Φ_B never happened and therefore indeed cannot influence the outcome of Φ_A. This shows, under the condition that Φ_B is causal, part (1) from the list of suspected properties of the process (1.6), but more importantly, it shows a general approach to find which systems can influence the outcomes of a process by discarding the other output systems. Indeed, given a process as a black box as in (1.5), we focus on an output system A'_i by discarding all other output systems and then see which input systems can still influence the resulting process. This approach is explored further in Section 1.1. Finally, statement (2) is immediately clear once one can 'read diagrams' (Chapter 2) while (3) is a bit more subtle, so we address it in Section 1.4.

1.1 Causal inference

Our goal is to say something about the causal structure of a process. But what exactly do we mean by this? When a diagram is given in the form of, for example, (1.6), it is easy to see what the causal order of this process is: process Φ_A (associated with systems A and A') takes place in the past of Φ_B (associated with systems B and B'). However, for a general process we usually do not have such a factorization at hand. What we can do is consider pairs of input/output systems, which we think of the input and output of some local laboratory where some process takes place. We call an input/output pair an *event* and a causal order is then given by a DAG with the events as vertices. Note that events are not uniquely related to a process. For example, we may consider a process $\Phi : A \otimes B \to A' \otimes B'$ as having the 'obvious' events (A, A') and (B, B'), but also as having events (A, I), (B, I), (I, A') and (I, B'), or the rather trivial single event $(A \otimes B, A' \otimes B')$. Since these have different numbers of events, they need to be described by different DAGs. In what follows, however, we will always assume that such a partition in events has already been given in the 'obvious' way.

Recall the Definition of ancestors for a vertex in a DAG (Definition 1.0.1). It is straightforward to extend this from a single vertex to a set of vertices, $\mathcal{E}$, by taking the union of the ancestors of vertices in $\mathcal{E}$. If the vertices of the DAG now correspond to events of input/output pairs related to some process, we write $\mathbf{past}(\mathcal{E})$ for the set of all events which are the ancestors of the events in $\mathcal{E}$. Recall in particular that $\mathcal{E} \subset \mathbf{past}(\mathcal{E})$. Write π_1 and π_2 for the projections of the inputs and outputs of the events, respectively. We can then say what it means for a process to satisfy some causal order:

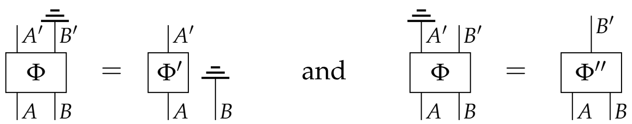
Definition 1.1.1. A process $\Phi : A_1 \otimes \ldots \otimes A_n \to A'_1 \otimes \ldots \otimes A'_n$ is *consistent* with a causal ordering $\mathcal{G}$ (written $\Phi \vDash \mathcal{G}$) if for all subsets of events $\mathcal{E} \subseteq \mathcal{G}$, the outputs of $\mathcal{E}$ only depend on the inputs of the ancestors of $\mathcal{E}$. That is, there exists a process Φ' such that:



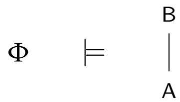
(1.10)

Let us consider two examples. First, an easy example as a sanity check.

Example 1.1.2. Consider our example of the process (1.6). Since there exist Φ' and Φ'' such that



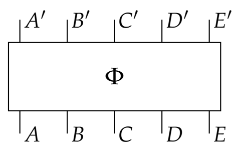
we indeed see that this process is compatible with the causal order of the event $\mathsf{A} := (A, A')$ before $\mathsf{B} := (B, B')$:



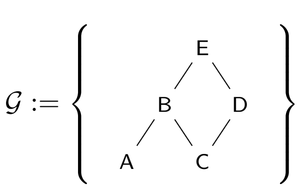
but generally not with the order B before A.

Second, a more general example.

Example 1.1.3. Consider the following process with 5 inputs and outputs:

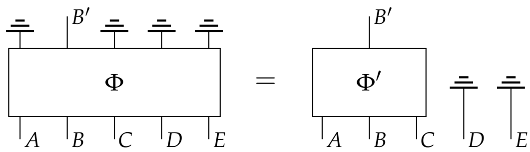


and the following causal ordering on input/output pairs of Φ:

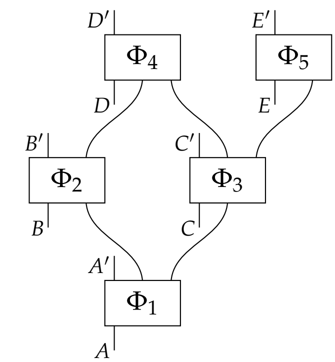


where
$$\begin{cases} \mathsf{A} := (A, A') \\ \mathsf{B} := (B, B') \\ \mathsf{C} := (C, C') \\ \mathsf{D} := (D, D') \\ \mathsf{E} := (E, E') \end{cases}$$

where the ordering is depicted from bottom-to-top, e.g., $\mathsf{A} \preceq \mathsf{B}$. Then, $\Phi \vDash \mathcal{G}$ if for all $\mathcal{E} \subseteq \mathcal{G}$, (1.10) is satisfied. For example, taking $\mathcal{E} := \{\mathsf{B}\}$, we have $\mathbf{past}(\{\mathsf{B}\}) = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$. So, condition (1.10) requires that there exists $\Phi' : A \otimes B \otimes C \to B'$ such that:

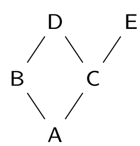


The fact that we could guess the causal order of the process (1.6) is an instance of something more general, which is closely related to causality as in Equation (1.8). It is shown in [63], that in a process theory every output of a process can only be influenced by the inputs in its past if and only if every process in that theory is terminal. It follows from this that any process whose diagram is given in the form of a DAG build up from causal processes satisfies the associated causal order given by that DAG. For instance, the process



(1.11)

satisfies the DAG



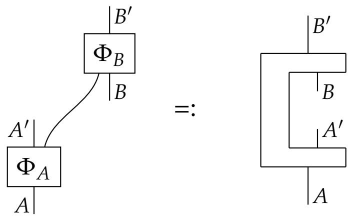
where e.g., $\mathsf{A} = (A, A')$, if and only if the processes $\Phi_1, \ldots, \Phi_5$ are causal.

1.2 Higher order processes

In the third statement about process (1.6) we noted the possibility of using a one-way signalling process as something to ‘plug in’ some other process.

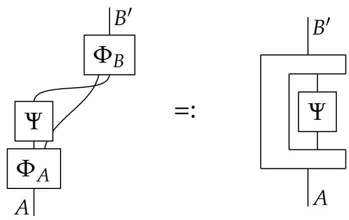
As such, one-way signalling processes play a double role. Both as regular processes from two input systems to two output systems, and as 'transformations' of processes with a single input and output system. This leads to a new kind of process theory where we can have *higher order processes*, sometimes also called *supermaps*, where the inputs and outputs can themselves be processes. We will develop idea formally in Section 2.3.

To accentuate the fact that the one-way signalling process (1.6) can take a process from A' to B as an input, we will draw it as a 'box with a hole':



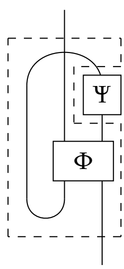
(1.12)

The result of inputting a process Ψ (the exact meaning of this composition is given in diagram (2.18)) is then a process from A to B'



(1.13)

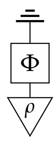
Moreover, for causal processes Φ_A and Φ_B, the resulting process of (1.13) is causal whenever the input Ψ is causal. This follows easily by discarding the output B' and using the properties of the discard effect (Section 2.1.6). In contrast, suppose we have a way to 'bend' a wire to make a feedback loop, that is, suppose we could use an output of a process as an input of itself. Then we could have taken any bipartite process Φ and used it to transform some process Ψ as such:



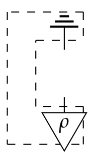
(1.14)

Thinking of the wires as information flow, it is as if we are sending information back in time. This should raise some alarms regarding causality. Indeed, the resulting process in (1.14) is generally not causal even if the processes Ψ and Φ are. We show this explicitly in Example 3.1.7. The reason is that a bent a wire, now seen as an effect, is not causal (see Lemma 2.1.21). Mathematically, bending wires corresponds to a property of categories called *compact closedness*, explained in Section 2.1.4. So we see that we not only have higher order processes, but also a notion of *higher order causality* , i.e., maps which are causal when the inputs are also causal processes (Definition 2.3.1). Our main goal is to develop a semantics for these higher order causal processes and this is done in Chapters 3 and 4. For now, let us consider an example and look at some relations between higher order processes and causal structures.

Example 1.2.1. In a general measurement scenario we start with some state ρ, possible perform some process Φ on it and then perform a measurement.



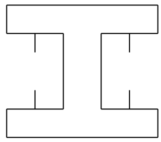
Technically we should keep track of which outcome occurs and to do this we should consider the discard effect as a collection of effects representing these outcomes (see [62]), we will gloss over this here. If we now take the starting state and the measurement fixed, but allow for different processes in between, we are left with a map which sends processes to probability distributions:



(1.15)

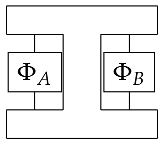
If the process that is used as input now is causal, the resulting probability distribution is normalized. We may therefore say that the process (1.15) is *normalized on causal processes*.

So far, we have seen higher order causal processes which take in a single causal process and whose output is again a causal process or a normalized probability distribution, which is a special case of a causal process with no input or output. Of course, we can also consider higher order processes where the inputs are multiple processes or processes with multiple inputs and outputs. For example, for two input/output pairs a general picture of a higher order causal map with no final input/output system can be drawn as an 'I-shaped beam':



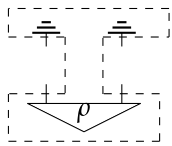
(1.16)

where we can, for example, input processes Φ_A and Φ_B



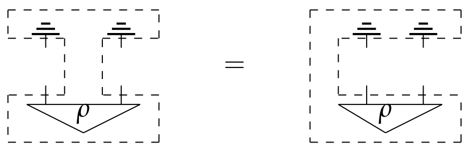
(1.17)

We can still consider multipartite measurement scenarios



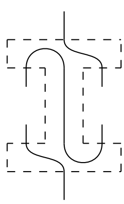
(1.18)

however, the situation here is the same as before. We just consider the two systems as a single joint system.

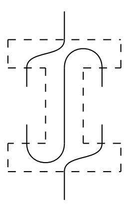


We now consider an example which shows that higher order processes are closely related to causal orders.

Example 1.2.2. Consider the following processes:

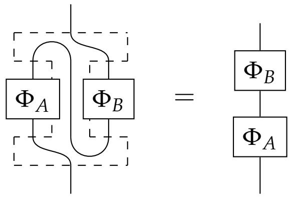


and



(1.19)

For the process on the left side, if we input processes Φ_A and Φ_B the result is their composition:



whereas taking the process on the right hand side would result in the converse composition $\Phi_A \circ \Phi_B$. So depending on whether we take the left or the right process in (1.19), either Φ_A happens first or Φ_B happens first. We come back to a modified version of these examples in Section 1.3.

Instead of inputting two single processes in the diagrams in (1.19), we can also consider what happens when we input a single bipartite process. Plugging the one-way signalling process of diagram (1.6) in the left hand side of (1.19) results in a causal process - and thus in normalized probability distributions after measurement - whereas plugging it in the right hand side does not! So in terms of higher order causality, some higher order processes are compatible with processes satisfying a certain causal order and others are not. In the semantics we will develop in the next chapters, we will attach *types* to processes. Such a type holds information about the causal structure of the process. For instance, a causal process from A to A' will be assigned type $A \multimap A'$, indicating that normalized states of type A are sent to normalized states of type A'. A one-way signalling process will be of type

$$A \multimap (A' \multimap B) \multimap B'$$

which tells us that it can take in a normalized state of system A, a causal map from A' to B and the final output is a normalized state of type B (Theorem 4.3.1). In contrast, just any causal bipartite map will be of type

$$(A \otimes B) \multimap (A' \otimes B')$$

which indicates an input of the joint system $A \otimes B$ and an output of $A' \otimes B'$ (Theorem 4.2.7). Of course, any one-way signalling process is also a causal process, so we expect that one-way signalling processes also satisfy the the type of any causal process. Indeed, it can be shown (Proposition 4.3.2) that the type of one-way signalling processes embeds in the type of causal processes.

The type system also applies to higher order processes and give information about which kind of processes can be used as inputs and what kind of process the output is. A map that sends causal processes to causal processes will be of type

$$(A \multimap A') \multimap (B \multimap B')$$

which can be shown to be isomorphic to the type of one-way signalling processes (via equation (2.35)). A map such as the left hand side of (1.19), which sends one-way signalling processes to a causal process can be assigned type

$$[A \multimap (A' \multimap B) \multimap B'] \multimap (C \multimap C').$$

In general, maps sending processes of type X to processes of type Y are of from $X \multimap Y$. In the special case, such as in diagram (1.18), where there is no final input/output system, or more precisely, the final input/output system is trivial, we obtain a *dual* type: $X \multimap I =: X^*$. Here, I is the *tensor unit* representing the trivial system introduced in Section 2.1.1.

Dual types also allow us to obtain information about the causal structure of processes. Indeed, if Ψ is a process and for every (dual) process w of type X^* we have that the composition of Ψ with w is normalized, then Ψ is itself of type X. Reversing this argument, suppose we wish to show that some process Φ is *not* of type X. Then it suffices to find a process w of type X^* such that the composition of Ψ and w is not normalized. In fact, this is what we (informally) did when we considered one-way signalling processes and the processes in (1.19).

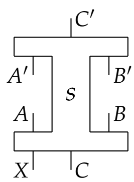
So the type system gives a lot of information about the causal structure of processes, however, it does not give all information. In the next section we will discuss processes which give rise to probability distributions which cannot arise in (quantum) theories with a fixed causal background.

1.3 Indefinite causal structures

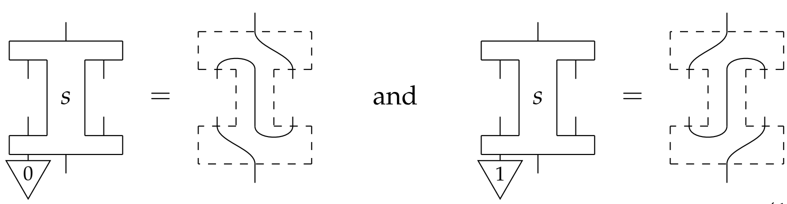
Following a slight simplification of [73] we consider a game where two players, Alice and Bob, each have a lab and are given a random bit. They then get a system coming into their lab, possibly perform some process and send the system out again. Their task is then to guess the value of the bit of the other person. In a fixed causal background where, say, Alice is before Bob, that is, Alice first gets the system and then Bob gets the transformed system, she can send the value of her bit to Bob encoded in the system. Bob will then always guess the right answer for Alice, whereas Alice has a 50/50 chance of getting Bob's bit right. It is then shown that this success rate is optimal, even if the causal order between Alice and Bob is allowed to change every time we do a run of this game. However, when we drop the assumption of a global fixed causal order, there are processes which can break this bound.

So what is meant by 'a global fixed causal order'? In Section 4.3 we will define a class of processes called *combs*. These processes are essentially circuits of systems and channels with possibly holes in them. As we will see, such a comb is compatible with a total causal order where for any two events we can say which one comes before the other. The process used in the above game in order to obtain probabilities from processes is more general than what we can obtain from even probabilistic combinations of these combs. Therefore we say this process is not compatible with a fixed causal order, or say it has *indefinite* causal order. Let us consider another example of a process which also cannot be realised using circuits, where the link with causal orders is nicely visible.

Example 1.3.1 (quantum switch)**.** Consider a process where the input is a state, ρ, of a qubit and two causal processes, f and g, with the same input and output system. Then, if the qubit is in state given by the density matrix $|0\rangle\langle 0|$, the output is the composition $g \circ f$, whereas when the qubit is in state $|1\rangle\langle 1|$ the output is the composition $f \circ g$. We can draw the switch as follows

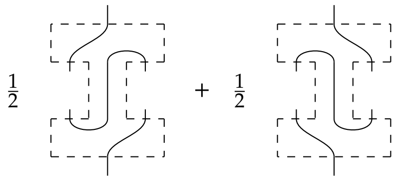


Recalling Example 1.2.2 we than have



(1.20)

This process is called the *quantum switch*, introduced in [23]. The reason that it is not compatible with a particular causal order can easily be visualized using the diagrams in equation (1.20). Indeed, consider the input qubit to be in the mixed state $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$. Then the output is the convex combination of causal orders



(1.21)

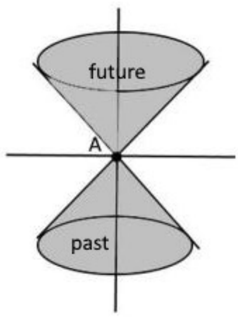
which is clearly not compatible with a single causal structure. However, in this scenario, every time we perform a run of this switch experiment, one of the causal orders occurs with probability one half, so it is compatible with a convex combination of causal structures.

It is at this point where we should note that the conditions defining the switch do not fully specify it. Indeed, the density matrices corresponding to 0 and 1 are not a basis for the four dimensional state-space of the qubit. In this sense, whenever we mention the switch map, we rather mean a class of switch maps. Still, we find some intuition for the indefinite causal order of the switch. Suppose we input a qubit in the pure state $|+\rangle\langle+|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Then, the final process again contains the terms in diagram (1.21), so we can think of this as a coherent superposition of causal orders. Heuristically, the idea is that if we perform no 'measurement' on the global causal structure, it can be in such a superposition in the same way that the outcome of an observable is not well defined before the measurement. We will come back to the concept of indefinite causal structures in Section 4.5 where we will also consider the switch in different categories.

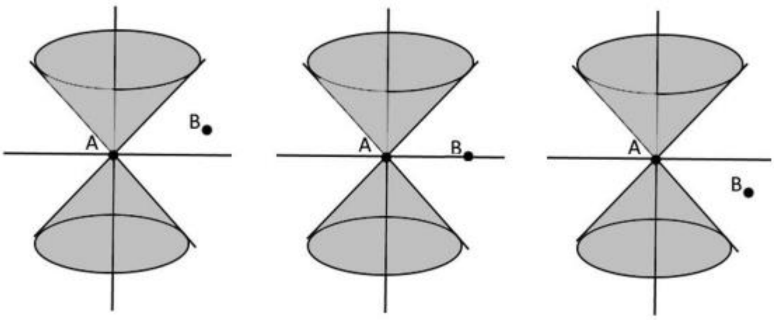
As the 'bit guessing game' at the beginning of this section shows, indefinite causal structures can lead to computational advantages. The switch also leads to computational advantage. In [21] it is shown that the switch can perfectly discriminate between no-signalling channels whereas this cannot be done by any circuit with fixed causal order and in [81] it is shown that even when the inputs to the switch are completely depolarising, it is still possible to send quantum information with a non-zero probability. We might expect that these advantages are due to the quantum nature of these examples, however, in [13] it is shown that a game similar to the bit guessing game can even be won in a setting which is locally probabilistic if there are at least three parties involved, showing that 'quantum weirdness' is not necessary and the gain really comes from the indefinite causal order. These occurrences in both quantum theory and probability theory ask for a general overarching approach to indefinite causal structures and more generally higher order processes. In this part of the theses we provide this setting and develop semantics regarding these higher order processes, but before we do that, we first make some relations to physics and relativity theory in particular.

1.4 Relations with relativity

We have been considering all processes now as purely mathematical entities. When we wish to implement such processes this happens in a space-time arena and is therefore subject to the rules of the game: physics and relativity theory in particular. The theory of relativity, introduced by Einstein in 1905, describes how different observers observe space-time. One of the starting points of this theory is that the speed of light is the same for every observer. A consequence is then that it is not possible to send information with a speed exceeding the speed of light. As such, for every point A in space-time there is a *future light-cone* of coordinates which can be influenced from this point and a *past light-cone* of points which can influence this point.

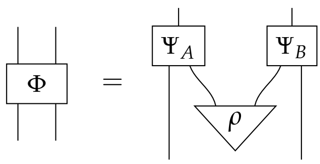


Whenever two space-time points are not in each others light-cones, they are called space-like separated and there can be no information transfer between them. If influence between two space-like separated points was possible, then it would be as if we could send information to the past. To see this, consider two events at space-time point A and B such that for some observer B is in the future of A, but not in the light-cone of A. Then we can always find other observers, related with Lorentz boosts, i.e., the transformations which relate the different space-time coordinates for different observers, such that, depending on the observer, A happens before B, A and B happen simultaneously, or B happens before A:



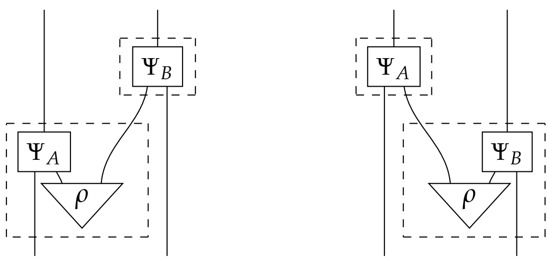
Now consider the following process:

Example 1.4.1. The two systems of some bipartite state ρ are sent to two far apart (space-like separated) laboratories managed by Alice and Bob. After the systems arrive at their respective labs, some processes Ψ_A and Ψ_B are performed on these systems and some local inputs and the resulting system is sent out again. The overall process then looks like this:



(1.22)

Now, since these labs are space-like separated, there can be no signalling between the events related to Ψ_A and Ψ_B. We can show this by explicitly making the process one-way signalling as in diagram (1.6).



(1.23)

Processes of this form have been called *strongly no-signalling* in [65] and *localizable* in [14]. They play an important part in the Bell theorem which we address in Part II of this thesis.

We now wish to use this process as a higher order process by taking one of the outputs, applying a process and using the output of this process as local input again, similar to diagram (1.7). While this is completely fine from a mathematical point of view (we just calculate the composition of the processes), physically we cannot implement this because it implies signalling between space-like separated points. A way to implement this restriction is to not allow non-trivial processes between space-like separated points, an idea introduced in [30]. Throughout this theses, however, we will not care about this restriction and only consider the processes on their own, not caring about any particular implementation or space-time positioning. That said, we do note that one can also inverse the above argument. Instead of letting light-cones define which events can influence which other events, we can also construct a 'generalized light-cone structure' by considering which events can influence each other.

Finally, we consider the switch map of Example 1.3.1 again. Since superpositions are a quantum phenomenon and causal orders are more related to relativity, having a super position of causal orders is right on the intersection of quantum theory and relativity. While there currently is no theory that describes both quantum and relativity, we do know from the theory of general relativity that clocks tick slower in the presence of gravitation. That is, time slows down when large masses are around. We can then image we have a superposition of the position of such a large mass, like a planet or a black hole. Consequently, we have a superposition of slowing down time in different space-time areas. In [96] this idea is used to build a switch map which can achieve computational advantages by breaking certain bounds that cannot otherwise be broken.

Chapter 2

Categorical quantum mechanics

2.1 Process theories

It is a remarkable feature of science that in order to understand something, it is often useful to take a step back and look at the subject from a more abstract point of view. Not disturbed by specifics, one can better study the underlying, more general, principles of the theory. For quantum mechanics, it has proven to be extremely fruitful to consider it as a *process theory* as has been done first in [4]. In process theories, we consider systems and transformations, or processes, between these systems. This simple setup describes a very wide variety of theories, among which finite dimensional quantum mechanics, finite probability theory and even cooking (a potato is a system and peeling it is a process [62]).

2.1.1 Monoidal categories

The mathematical framework of process theories is that of monoidal categories ([71, 60]). Our goal in this section is to give the basics of this theory and in particular develop a diagrammatic notation (Section 2.1.2) to work with such categories [62]. We start with basic monoidal categories and gradually add more structure, such as symmetry 2.1.3, compact closure 2.1.4 and discarding 2.1.6.

A process theory is a monoidal category together with an interpretation of objects as systems and morphisms as processes.

Monoidal categories are essentially categories where, in addition to the usual sequential composition, there is an additional parallel composition of objects and morphisms via the tensor, $\otimes$. This is subject to some canonical *coherence conditions*, related to the associative structure ($(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$) and the unit object ($A \otimes I \cong A \cong I \otimes A$). As the name suggests, the coherence conditions between these isomorphisms ensure that basically everything works as expected. Since writing these isomorphisms everywhere is a tiresome job, we will not do so and take them to be the identity. Such categories are called *strict*. By a theorem of MacLane [71], every monoidal category is equivalent to a strict one, so we lose no generality in doing this.

Definition 2.1.1. A *strict monoidal category* consists of a triple $(\mathcal{C}, \otimes, I)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is an associative bifunctor called *tensor* and I is an object of $\mathcal{C}$, called the *tensor unit*, satisfying

$$I \otimes A = A = A \otimes I, \tag{2.1}$$

for any object A of $\mathcal{C}$.

Instead of writing the triple $(\mathcal{C}, \otimes, I)$, we usually only write $\mathcal{C}$.

Throughout this part of the thesis, we consider two main examples related to quantum theory and probability theory.

Example 2.1.2 (Quantum theory)**.** Finite dimensional quantum theory is an example of a process theory. Systems represent operator algebras associated to the physical systems in consideration. Processes are those maps which send states to states and are therefore given by completely positive maps. From this we obtain the following category [83]:

Definition 2.1.3. The monoidal category **CPM** has objects finite dimensional Hilbert spaces $H, K, \ldots$. A morphism between objects H, K is a completely positive maps $\phi : B(H) \to B(K)$ between the corresponding operator algebras of bounded operators on the Hilbert spaces. That is, both $\phi : B(H) \to B(K)$ and $\phi \otimes id_{B(L)} : B(H) \otimes B(L) \to B(K) \otimes B(L)$ are positive, where $id_{B(L)}$ is the identity on $B(L)$.

The monoidal product is the Hilbert space tensor product. In terms of the algebras we have $B(H) \otimes B(K) \cong B(H \otimes K)$. The tensor unit is the complex numbers $I := \mathbb{C} \cong B(\mathbb{C})$.

Equivalently we could have defined the objects of **CPM** to be the operator algebras themselves, or, since Hilbert spaces of the same dimension are isomorphic, by natural numbers. Each of these choices would lead to an equivalent category. Strictly speaking, the category is only strict when the objects are natural numbers, where $n \otimes m = n \cdot m$, but we gloss over this and keep to the more original definition from [83].

Example 2.1.4 (Probability theory)**.** Our second main example is that of finite probability theory. The relevant monoidal category is the following:

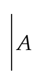
Definition 2.1.5. The category $\mathbf{Mat}(\mathbb{R}_+)$ has as objects natural numbers $n, m, \ldots$. A morphisms $n \to m$ is a $m \times n$ matrix with non-negative entries and composition is matrix multiplication. The monoidal product is multiplication, $n \otimes m = n \cdot m$, and the tensor unit is the number 1.

The interpretation of an object n in $\mathbf{Mat}(\mathbb{R}_+)$ is of course a finite set with n points. Maps from such a finite set to $\mathbb{R}_+$ can be seen as 'vectors' in $(\mathbb{R}_+)^n$, which we interpret as (possibly non-normalized) probability distributions. Using the discard effect from Section 2.1.6, we can then restrict to normalized probability distributions. Morphisms then act as matrices on these vectors, sending distributions to distributions.

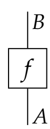
2.1.2 Diagrams

Monoidal categories have a particularly nice presentation in the form of diagrams [62] and throughout this work we will use a mixture of diagrammatic and analytical notions at our convenience. This is allowed because diagrammatic reasoning is sound and complete, see [83]. We start with a diagrammatic notation for monoidal categories, which corresponds to planar graphs. Processes are vertices and systems are non-intersecting edges. In the following, whenever we add new structure to our monoidal categories, we will see that we also obtain new diagrammatic notation. A survey of this can be found in [84].

We draw a system, or rather, the identity on this system, as a wire:

systems: $A := id_A =$ 

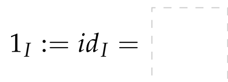
Processes between systems are drawn as boxes between their wires:

processes: $f : A \to B :=$ 

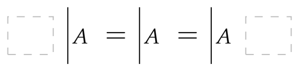
Whenever it is clear which systems are involved, we will not label the wires.

The tensor product of systems is given by parallel wires and the tensor product of processes is given by parallel boxes. We call this *parallel composition*.

The tensor unit is given by no wire:



Equation (2.1) now becomes a diagrammatic tautology



There are special maps out of and into the tensor unit:

A morphism $I \to I$ is both a state and an effect and is called a *scalar*. These are diagrams with no inputs and no outputs and will often be written as a number: $\lambda, \mu, \ldots$

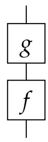
We now consider what states and effects are in our main examples **CPM** and $\mathbf{Mat}(\mathbb{R}_+)$:

Example 2.1.6. In **CPM**, a state on $B(H)$ is a completely positive map $\rho : \mathbb{C} \to B(H)$, which we identify with the (unnormalized) density matrix $\rho(1)$. An effect in **CPM** is a completely positive map $\sigma : B(H) \to \mathbb{C}$. Such a map is always of the form $\sigma(a) = tr(\rho_\sigma a)$ where tr is the trace on $B(H)$ and ρ_σ is some operator corresponding to σ. Scalars are positive maps from $\mathbb{C}$ to $\mathbb{C}$ and thus correspond to the non-negative reals.

Example 2.1.7. In $\mathbf{Mat}(\mathbb{R}_+)$, a state on n is a $1 \times n$ matrix with positive entries. Hence a state is either the zero-vector or it is an (unnormalized) probability distribution. Effects in $\mathbf{Mat}(\mathbb{R}_+)$ are $n \times 1$ matrices with positive entries, which we regard as row vectors. If ρ is a state and σ is an effect in $\mathbf{Mat}(\mathbb{R}_+)$, then the scalar $\sigma \circ \rho$ is the inner product of the corresponding vectors, $\sigma \circ \rho = \sum_i \sigma^i \rho_i$. Scalars are therefore the positive reals $\mathbb{R}_+$.

If $f : A \to B$ and $g : B \to C$ are processes, then their composition $g \circ f : A \to C$ is the process:

$$g \circ f :=$$

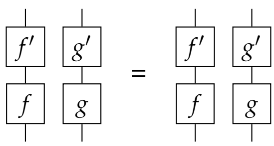


We call this *sequential composition*.

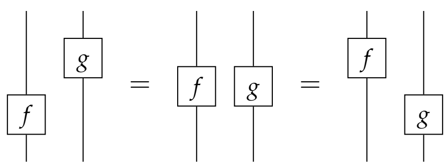
As a further example of the advantage of the diagrammatic method, consider four processes: $f : A \to A'$, $f' : A' \to A''$ and $g : B \to B'$, $g' : B' \to B''$. We can either do parallel composition followed by sequential composition, or we first compose sequential and then parallel. It is then easy, yet non-trivial, to show these are the same.

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$$

However, in diagrammatic language this equation is a tautology:



Whenever we introduce a new concept, we will also consider the diagrammatic aspects. At first, we will be rather pedantic in working with diagrams, but as we go on, we will see all that matters is the connectivity of diagrams. If two diagrams can be deformed into each other without changing which inputs and outputs are connected, the processes represented by these diagrams are the same. For example, we may may 'slide' morphisms over wires.



corresponding to the identities

$$(f \otimes 1) \circ (1 \otimes g) = (f \otimes g) = (1 \otimes g) \circ (f \otimes f)$$

2.1.3 Symmetry

If we think of parallel composition as taking joint systems, we expect to find a relation between $A \otimes B$ and $B \otimes A$, as both are a joint system of systems A and B. This relation is expressed as an isomorphism between the two joint systems.

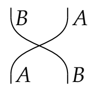
Definition 2.1.8. We call a monoidal category $\mathcal{C}$ a *symmetric monoidal category* (SMC) if there is a natural 'swap' isomorphism $\sigma_{A,B} : A \otimes B \cong B \otimes A$ for all objects A, B in $\mathcal{C}$, satisfying

$$\sigma_{B,A} \circ \sigma_{A,B} = id_{A \otimes B}. \tag{2.2}$$

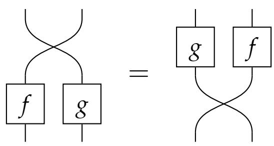
and

$$\sigma_{A,B\otimes C} = id_B \otimes \sigma_{A,C} \circ \sigma_{A,B} \otimes id_C \tag{2.3}$$

We draw the swap map as

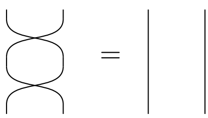
swap: $\sigma_{A,B} :=$ 

Naturality means we can slide morphisms over swap:



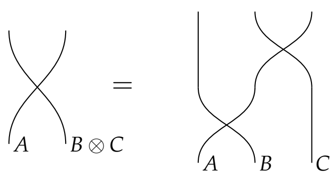
(2.4)

and equation (2.2) says that swapping twice is the identity:



(2.2′)

Finally, equation (2.3) gives the interaction of swap with the tensor:



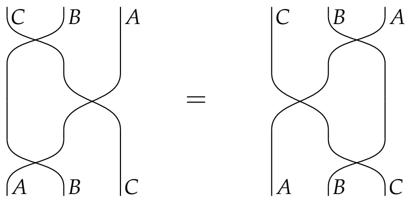
(2.3′)

Equation (2.3), or equivalently (2.3′), is called the *hexagon identity*, since taking the coherence conditions explicitly into account would lead to a commutative diagram involving six morphisms.

Swapping around three systems can essentially be done in two ways. It follows from naturality, (2.8), and the hexagon identity (2.3′) that these are the same:

$$(\sigma \otimes 1) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) = (1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma)$$

or in diagrammatic terms:



(2.5)

This equality is also known as the *Yang-Baxter* equation and we will need it in the next section.

Example 2.1.9. In both **CPM** and **Mat**$(\mathbb{R}_+)$, swap is given by linear extension of $a \otimes b \mapsto b \otimes a$.

Diagrams in a SMC correspond to acyclic graphs. Clearly the swap allows us to cross wires. In the next section we introduce a way to bend wires, allowing us to write diagrams corresponding to any kind of graph.

2.1.4 Compact closure

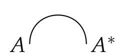
Compact closure introduces *duals* in a category. These can be thought of as generalizations of duals of finite dimensional vector spaces, in the sense that states of such a dual space corresponds to effects of the original space. Here we focus on the basic property of compact closed categories and see how compact closure works diagrammatically.

Definition 2.1.10. A symmetric monoidal category $\mathcal{C}$ is called *compact closed* if for every object A there exists a *dual object* A^*. That is, for every A there exists morphisms $\eta_A : I \to A^* \otimes A$ and $\epsilon_A : A \otimes A^* \to I$, satisfying:

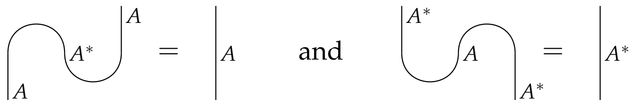
$$(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A \qquad (1_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes 1_{A^*}) = 1_{A^*} \tag{2.6}$$

Diagrammatically, the compact closed structure is given by a *cup* and *cap*, respectively:

$\eta_A :=$ 

$\epsilon_A :=$ 

and the equations (2.6) become



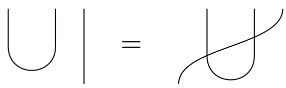
(2.7)

We call these the *yanking equations* as they represent the wires being yanked to a straight line. The advantage of equation (2.7) over (2.6) is that we can visualize wires representing a dual space as representing the original space, but going from top to bottom.

A priori, cup is a map $I \to A^* \otimes A$ and we need to be mindful of the order of A and its dual A^*. However, swap allows us to define new maps $\eta'_A := \sigma_{A^*,A} \circ \eta_A : I \to A \otimes A^*$ and $\epsilon'_A := \epsilon_A \circ \sigma_{A,A^*} : A^* \otimes A \to I$.

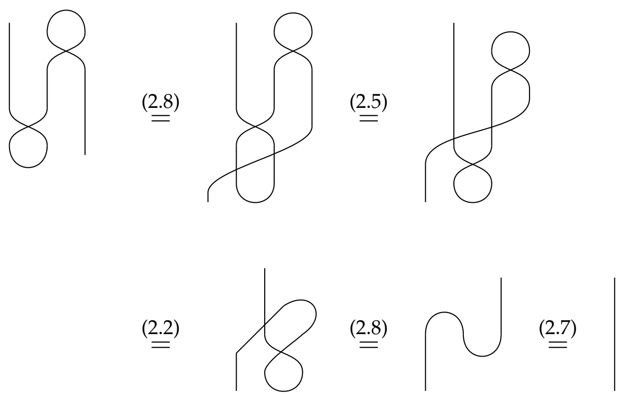
Lemma 2.1.11. If η and ϵ form a cup/cap pair, i.e., satisfy the equations of (2.7), then $\eta' = \sigma \circ \eta$ and $\epsilon' = \epsilon \circ \sigma$ also form a cup/cap pair.

Proof. By naturality of swap with respect to η and the identity, we have



(2.8)

Similar horizontally and vertically reflected equations are obtained by interchanging the identity and η and considering naturality w.r.t. ϵ. Then:

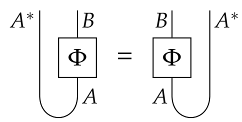


□

A similar calculation shows that we always have canonical isomorphisms $A \cong A^{**}$ and we we can always choose cup/cap pairs such that this relation is strict:

$$A = A^{**} \tag{2.9}$$

In the following we wish to 'bend' wires by applying cups and caps. The above results show that we do not need to be careful about which cups and caps we use and with a little bit of abuse of notation, we for example write



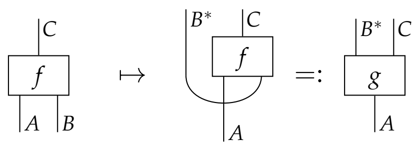
Here we not only relax our cups and caps, we even neglect the order of the systems, thus really conforming to the mantra that

Only connectivity matters.

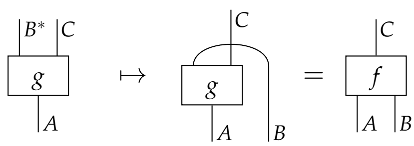
As the name suggests, compact closed categories are closed. That is, the functor $(-) \otimes B$ has a right adjoint given $B^* \otimes (-)$. Perhaps more familiar, defining for now $A \Rightarrow B := A^* \otimes B$, as the 'internal hom', there is a natural isomorphism

$$Hom(A \otimes B, C) \cong Hom(A, B \Rightarrow C) \tag{2.10}$$

given by



whose inverse is:



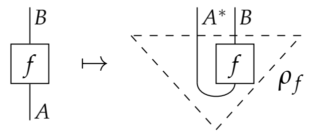
In particular we have

$$Hom(A, B) \cong Hom(I, A^* \otimes B) \tag{2.11}$$

so maps from A to B correspond to states on $A^* \otimes B$, which justifies the term internal hom. We take a closer look at this remarkable property now.

2.1.5 Process-state duality

As we just saw, compact closure gives a way to view maps $f : A \to B$ as states ρ_f on the joint system $A^* \otimes B$ by 'bending the wire':



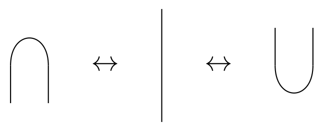
(2.12)

This correspondence is known in the literature as the Choi-Jamiołkowski isomorphism [70] or as *process-state duality*.

Considering the adjunction (2.11) where $B = A$, we find a chain of isomorphisms

$$Hom(A \otimes A^*, I) \cong Hom(A, A) \cong Hom(I, A^* \otimes A) \tag{2.13}$$

Applying this to the the identity, $id_A : A \to A$, we find cup and cap as the unit and counit of the adjunction (2.11), respectively.



(2.14)

In general, process-state duality allows us to consider any diagram as a state ρ on some space X by bending all inputs to outputs. We will often use the abbreviation $\rho : X$ to mean $\rho : I \to X$ is a state on X.

Before we consider cup/cap pairs in **CPM** and **Mat**$(\mathbb{R}_+)$, we consider the relation between dual and tensor.

Lemma 2.1.12. In a compact closed category we can always choose cups and caps in such a way that the dual distributes strictly over the tensor:

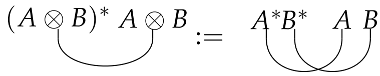
$$(A \otimes B)^* = A^* \otimes B^* \tag{2.15}$$

Proof. First we use the isomorphism (2.10) twice to show that $(A \otimes B)^*$ and $A^* \otimes B^*$ are always isomorphic:

$$\begin{aligned} Hom(C, (A \otimes B)^*) &\cong Hom(C \otimes A \otimes B, I) \\ &\cong Hom(C \otimes A, B^*) \\ &\cong Hom(C, A^* \otimes B^*) \end{aligned}$$

Hence by the Yoneda lemma $(A \otimes B)^* \cong A^* \otimes B^*$.

Now for given cups on A and B, define a cup on $A \otimes B$ as



Defining cap in a similar way, the new cup/cap pair satisfies yanking because the cup/cap pairs on A and B do. Hence, under this choice we have $(A \otimes B)^* = A^* \otimes B^*$ $\square$

Example 2.1.13. We consider the compact closed structure of **Mat**$(\mathbb{R}_+)$. Let n be an object in **Mat**$(\mathbb{R}_+)$, which we identify with $\mathbb{R}_+^n$. Write $|i\rangle$ for the i'th standard basis vector. We identify n^* with $(\mathbb{R}_+^n)^*$, the space of row vectors, and write $\langle i|$ to be the i'th dual basis vector, i.e., $\langle i| \, |j\rangle = \delta_{i,j}$. Then

$$\begin{aligned} id &= \sum_i |i\rangle \langle i| \\ cup &= \sum_j |j\rangle \, |j\rangle \\ cap &= \sum_k \langle k| \, \langle k| \end{aligned}$$

The yanking equations (2.7) now become

$$\begin{aligned} \left(\sum_k \langle k| \, \langle k| \otimes \sum_l |l\rangle \langle l| \right) \circ \left(\sum_i |i\rangle \langle i| \otimes |j\rangle \, |j\rangle \right) &= \sum_{ijkl} \delta_{k,i}\delta_{k,j}\delta_{l,j} \, |l\rangle \langle i| \\ &= \sum_i |i\rangle \langle i| \\ &= id \end{aligned}$$

and similarly for the other yanking equation.

Example 2.1.14. For **CPM** we note that any operator $a \in B(H)$ can be written as $a = \sum_{i,j} \langle i| \, a \, |j\rangle \, |i\rangle \langle j|$, where $\{|i\rangle\}$ is some orthonormal basis for H. From this we find $B(H) \cong H^* \otimes H$ via the almost tautology $|i\rangle \langle j| \leftrightarrow |i\rangle \langle j|$ [70]. Furthermore, any Hilbert space is self dual [94], $H \cong H^*$, via $|i\rangle \leftrightarrow \langle i|$.

Therefore we have $B(H) \cong H \otimes H \cong B(H)^*$. We now define

$$\begin{aligned} id &= \sum_{i,j} |ij\rangle \langle ij| \\ cup &= \sum_{k,l} |kl\rangle |kl\rangle \\ cap &= \sum_{m,n} \langle mn| \langle mn| \end{aligned}$$

The yanking equations are then similar to those of $\mathbf{Mat}(\mathbb{R}_+)$.

There is another way to consider the compact closure of **CPM** which is more related to the Choi-Jamiołkowski isomorphism as often encountered in the literature. We define a cup as a map η as $\eta = |\Phi^+\rangle \langle \Phi^+| : \mathbb{C} \to B(H \otimes H)$, where $|\Phi^+\rangle = \sum_i |ii\rangle \in H \otimes H$. The image $\eta(1)$ is an operator $|\Phi^+\rangle \langle \Phi^+| = \sum_{i,j} |ii\rangle \langle jj|$ in $B(H \otimes H)$. We then define cap as $\epsilon(\rho) = tr(\rho |\Phi^+\rangle \langle \Phi^+|)$, for $\rho \in B(H \otimes H)$.

Then for an operator $\rho = \sum_{i,j} \langle i| \rho |j\rangle |i\rangle \langle j| \in B(H)$, we have

$$\begin{aligned} (\epsilon \otimes id) \circ (id \otimes \eta)(\rho) &= (\epsilon \otimes id) \circ (\rho \otimes \sum_{k,l} |kk\rangle \langle ll|) \\ &= \sum_{k,l} tr(\rho \otimes |k\rangle \langle l| |\Phi^+\rangle \langle \Phi^+|) |k\rangle \langle l| \\ &= \sum_{k,l,i,j} tr(|ik\rangle \langle jl| |\Phi^+\rangle \langle \Phi^+|) \langle i| \rho |j\rangle |k\rangle \langle l| \\ &= \sum_{k,l,i,j} \delta_{i,k}\delta_{j,l} \langle i| \rho |j\rangle |k\rangle \langle l| \\ &= \sum_{i,j} \langle i| \rho |j\rangle |i\rangle \langle j| \\ &= \rho \end{aligned}$$

We now see that equation (2.12) corresponds precisely to the map

$$f \mapsto (id \otimes f) \circ \left(|\Phi^+\rangle \langle \Phi^+|\right)$$

which is, up to a transpose of f, the Choi-Jamiołkowski isomorphism in for example [25]. This transpose is then compensated for in the inverse isomorphism, which in diagrammatic notation is just bending the wire back down.

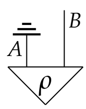
2.1.6 Discarding

In Chapter 1 we noted that we needed a way to disregard a system. Therefore our main interest will be SMCs with *discarding*. That is, for every object A, there exists a special effect $\stackrel{=}{\top}_A : A \to I$ which are together compatible with the monoidal structure, i.e., they satisfy

$$\stackrel{=}{\top}_{A\otimes B} = \stackrel{=}{\top}_A \stackrel{=}{\top}_B \qquad \text{and} \qquad \stackrel{=}{\top}_I = id_I := 1$$

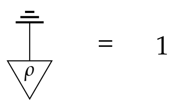
We think of discarding a system as forgetting about it, 'throwing the system away', marginalising or having no access to the system. When we only forget one part of a composite system, we obtain the notion of marginal.

Definition 2.1.15. Let $\rho : A \otimes B$ be a bipartite state on A and B. The *marginal state* of ρ at B, or just *marginal* when the context is clear, is the state on B that arises from discarding A:



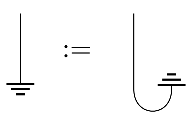
Given a state $\rho : A$ on $A = A \otimes I$, we can calculate the marginal of ρ at the monoidal unit I.

Definition 2.1.16. A state $\rho : A$ is *normalized* or *normal* if its marginal at I equals $1 = id_I$:



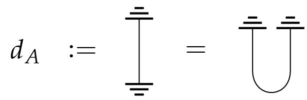
That is, discarding the state gives 1.

Making use of process-state duality, we find an (unnormalized) state corresponding to discarding:

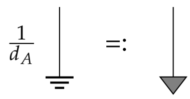


This state is called the *maximally mixed state*. Discarding the maximally mixed state gives the following:

Definition 2.1.17. The *dimension* d_A of a system A is the scalar defined by



Suppose that the dimension d_A of a system A is invertible, then we obtain a normalized version of the maximally mixed state:



In Section 2.2 we will show the importance of discarding in the study of causal structures.

Example 2.1.18. The discard effect in **CPM** on a system $B(H)$ is given by taking the trace: $\bar{\top} = tr(-)$. Hence a state ρ is normalized if and only if $tr(\rho) = 1$.

In **Mat**$(\mathbb{R}_+)$, discarding is given by the effect $(1, 1, \ldots, 1) = \sum_i \langle i|$. A state $\rho = \sum_i \rho_i |i\rangle$ is therefore normalized if $\sum_i \rho_i = 1$.

The causality condition (1.8) can now be made explicit in **Mat**$(\mathbb{R}_+)$ and **CPM**:

Proposition 2.1.19. In **Mat**$(\mathbb{R}_+)$, processes satisfying equation (1.8) are stochastic matrices.

Proof. Indeed, the causality condition for a process $\Phi : n \to m$ reads

$$\overbrace{(1\,1\ldots1)}^{m} \circ \Phi = \overbrace{(1\,1\ldots1)}^{n}$$

This tells us precisely that every column sums up to 1 and since by definition the entries of Φ are positive, the results follows. $\square$

Proposition 2.1.20. In **CPM**, processes satisfying equation (1.8) are trace preserving maps.

Proof. For an operator a in $B(H)$, the causality condition for a map $\Phi : B(H) \to B(K)$ reads

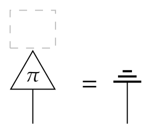
$$tr(\Phi(a)) = tr(a)$$

So Φ preserves the trace. □

Since a state ρ on a system A is a process $\rho : I \to A$, and discard on I is id_I, we find that causal states are normal states. Considering causal effects we find the following:

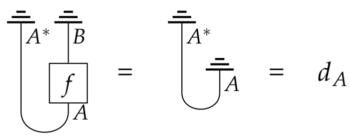
Lemma 2.1.21. An effect π is causal if and only if it is the discard effect.

Proof. Suppose π is causal, then



Hence discard is the *unique* causal effect. □

Finally, some care is needed when considering causal processes when process-state duality is involved. Indeed, if a process $f : A \to B$ is causal, i.e., it preserves discard, then its corresponding state is in general not causal:

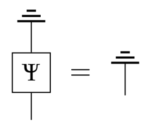


which in general does not equal 1.

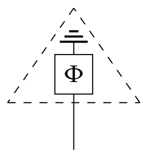
2.2 Signalling

The causal discovery scheme in Chapter 1 relied on the fact that when we discard certain outputs of a process, the process splits into a new process and some discard effects on the other input systems (Definition 1.1.1). These discarded inputs systems are then disconnected from the remaining process and hence cannot influence it. It is this influencing, or signalling, or rather, no-signalling, that we wish to make precise here. We start with the causality equation (1.8)

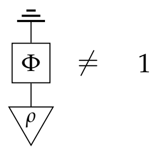
Definition 2.2.1. A process $\Psi : A \to B$ is *causal* if $\bar{\top}_B \circ \Psi = \bar{\top}_A$:



To understand what the implication is of a process being causal, it is most instructive to understand what happens when a process is not causal. So suppose some process Φ is not causal. Then, when we discard its output, we are left with some effect



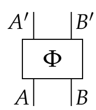
Now in the type of category we will consider in the next chapter, for such an effect there always exists some state ρ such that



This means that if the process Φ is applied to ρ, it induces an overall non-unit factor, which then influences all other measurements. Because of this and the reasoning around equation (1.9), causality is also called *no signalling from the future*.

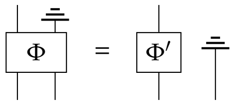
We now consider signalling properties of bipartite processes. In light of Definition 1.1.1 the following should not come as a surprise

Definition 2.2.2. Let $\Phi : A \otimes B \to A' \otimes B'$ be a causal process:



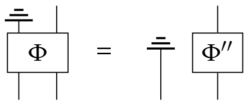
Then Φ is *one-way signalling* with (A, A') before (B, B') (written with a small abbreviation as $\mathsf{A} \preceq \mathsf{B}$) if there exists a causal map $\Phi' : A \to A'$

such that $(id_A \otimes \bar{\top}_{B'}) \circ \Phi = \Phi' \otimes \bar{\top}_B$. That is, discarding B' splits Φ into a causal process on A and discard on B:



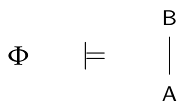
(2.16)

Similarly we say Φ is one-way signalling with $\mathsf{B} \preceq \mathsf{A}$ if there exists a causal $\Phi'' : B \to B'$ such that

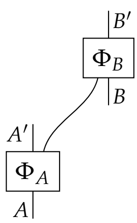


(2.17)

One-way signalling processes are thus those processes satisfying a causal order given by



and in particular, by Example 1.1.2, processes of the form



which we already called one-way signalling in Chapter 1 are indeed one-way signalling. Moreover, we can view these kind of processes as a canonical version of one-way signalling processes. Indeed, in the type of category we will consider in the next chapter (Definition 3.1.1) every one-way signalling process will be of this form (Proposition 3.1.4).

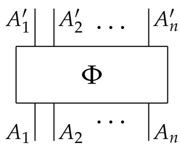
In Example 1.4.1 we considered a process which we called strongly non-signalling. As we showed, such a process has both factorizations related to one-way signalling processes, so in particular is one-way signalling in both ways.

Definition 2.2.3. If Φ is one-way signalling with both $\mathsf{A} \preceq \mathsf{B}$ and $\mathsf{B} \preceq \mathsf{A}$, we call Φ *no-signalling*.

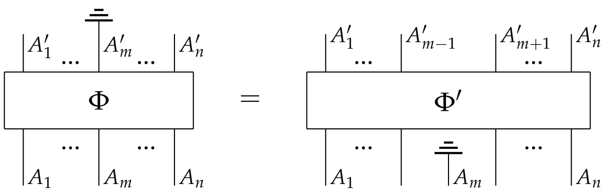
An interesting question is now whether every no-signalling process is strongly no-signalling. This turns out not to be the case and an example can be found in [14].

Up to now we have only considered bipartite processes. The generalization to multiple inputs and outputs is straightforward, but for sake of completeness we still present it now.

Let $\Phi : A_1 \otimes \ldots \otimes A_n \to A_1' \otimes \ldots \otimes A_n'$ be a causal process.



Then Φ is no-signalling if discarding some A_m' for some m results in discard on A_m and some process Φ' on the left over systems.

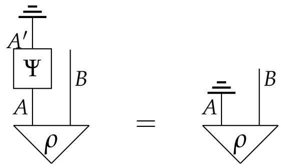


It is one-way signalling with $\mathsf{A}_1 \preceq \ldots \preceq \mathsf{A}_{n-1} \preceq \mathsf{A}_n$ if discarding A_n' results in a one-way signalling process Φ' with $\mathsf{A}_1 \preceq \ldots \preceq \mathsf{A}_{n-1}$ and discard on A_n.

When we say Φ is one-way signalling without specifying a specific order, we mean that the inputs and outputs of Φ can be permuted with swaps in order to obtain the order $\mathsf{A}_1 \preceq \ldots \preceq \mathsf{A}_n$.

We end this section with a slightly different view on causal processes, related to marginals as in Definition 2.1.15

Definition 2.2.4. A process $\Psi : A \to A'$ is called *marginal preserving* if for all $\rho : A \otimes B$ we have

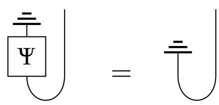


Preserving the marginal can also be interpreted as a type of *no signalling from the future* condition. Indeed, suppose Alice and Bob each get their share of the bipartite state ρ and Bob wants to perform some experiment on his system. Then, if Alice could perform some process which did not preserve the marginal, she would not only be able to influence the outcome of Bob's experiment from a distance, which is in contrast with special relativity, but she could even do so after Bob has done his experiment, in contrast with causality.

So both preservation of the marginal (equation (2.2.4)) and the causality condition (equation (1.8)) can be interpreted as no signalling from the future. Obviously, any causal map is marginal preserving. For the converse:

Lemma 2.2.5. If a SMC is compact closed then marginal preserving implies causal.

Proof. Let Ψ be a marginal preserving process. Then letting $\rho : A^* \otimes A$ be the cup we have

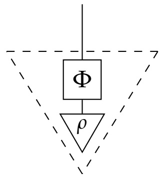


The result now follows from bending the wire down. □

2.3 Higher order processes

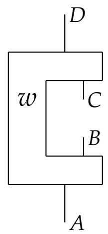
By now we have introduced SMCs with discarding and considered signalling properties of processes. In this section we will consider higher order processes.

Heuristically, higher order processes are maps between processes. Just as processes $\Phi : B \to C$ take states on B to states on C:

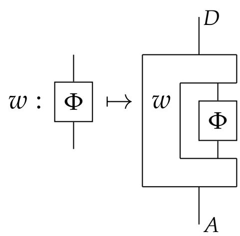


higher order processes, or supermaps, will take processes to processes.

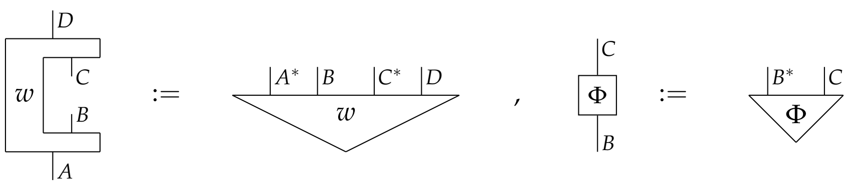
If $\Phi : B \to C$ is a process, we can construct a new process by applying a higher order process w to Φ. We draw w suggestively as



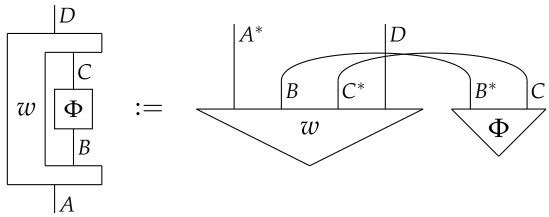
Then this map acts on Φ as



The result is a process from A to D. We make this precise now. Using process-state duality we can represent all processes as states. The process $\Phi : B \to C$ becomes a state on $B^* \otimes C$ and the supermap w is a state on $A^* \otimes B \otimes C^* \otimes D$.



The action of w on Φ is just applying the appropriate caps:

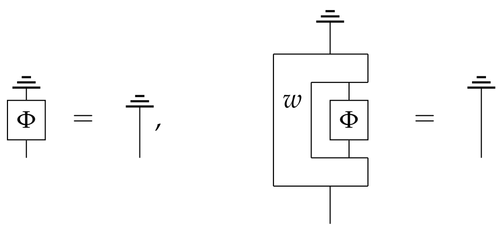


(2.18)

Not only do we want these second order processes to be causal maps (i.e., discarding B and D results in discard on A and C), we also want these maps to preserve the causality of the input. Hence we come to the the concept of *second order causality*:

Definition 2.3.1. A process $w : A \otimes C \to B \otimes D$ is called *second order causal* if for all causal processes $\Phi : B \to C$ the resulting process of 'plugging Φ into w' is a causal map. That is:

$\forall\Phi$ s.t.



2.3.1 Higher order systems in a compact closed category

As easy as compact closure is to work with, it turns out that categories which are compact closed do not have the right structure to study higher order processes. As an intermediate notation, let us denote by

$$A \Rightarrow B := A^* \otimes B,$$

the object representing maps from system A to B. [1] That is, the internal hom as the right adjoint to the tensor, as in (2.10). We can then consider the space of maps from this system $A \Rightarrow B$ to some system C. These are in some sense the simplest higher order maps one can consider. However, for

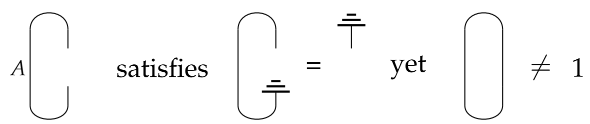
[1]We note that our $\Rightarrow$ is not equal to the one that occurs in linear logic literature. There one often defines $A \Rightarrow B := (!A) \multimap B$, which has the interpretation of 'B is caused by some iteration of A' [39].

these maps we have the following:

$$\begin{aligned}(A \Rightarrow B) \Rightarrow C &\cong (A^* \otimes B)^* \otimes C \\ &\cong A \otimes B^* \otimes C \\ &\cong B^* \otimes A \otimes C \\ &\cong B \Rightarrow (A \otimes C) \qquad (2.19)\end{aligned}$$

That is, this space of higher order processes is a space of regular first order processes. Hence we lose the ability to distinguish higher order processes from ordinary ones and the next example shows these processes are genuinely different.

Example 2.3.2. Taking $C = I$ in equation (2.19) implies that $(A \Rightarrow B)^* \cong B \Rightarrow A$. Now consider the process which is the identity from the future to the past: It is causal, but not second order causal as can be seen by plugging in the identity.



We come back to this example in Theorem 3.1.6

To remedy this problem of all processes collapsing to first order, we will need to drop the requirement that the dual distributes over the tensor. In Theorem 3.3.16 we will see that the type of category we then obtain is what is called $*$-*autonomous*. A short introduction to these categories will be given in the next section.

2.4 *-Autonomous categories

$*$-Autonomous categories were originally introduced by Michael Barr ([11]) in 1979 when studying topological vector spaces. Their name stems from the somewhat outdated term of autonomous category to mean a symmetric closed monoidal category. A *-autonomous category is then an autonomous category with a distinguished star functor, which gives duals in an appropriate sense.

Definition 2.4.1. A **-autonomous category* is a symmetric monoidal closed category, $(\mathcal{C}, \otimes, I, \multimap)$, with a full and faithful functor $(-)^* : \mathcal{C} \to \mathcal{C}^{op}$ such that

$$\mathcal{C}(A \otimes B, C^*) \cong \mathcal{C}(A, (B \otimes C)^*) \tag{2.20}$$

We recall the difference between 'internal' and 'external' homsets, as is standard with closed categories. We use $\mathcal{C}(A, B)$ (a.k.a. $Hom_{\mathcal{C}}(A, B)$) to mean the set of morphisms from A to B in $\mathcal{C}$. This is the homset as seen from outside the category, i.e., external. In contrast, the internal hom, $A \multimap B$, given by the monoidal closure

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, B \multimap C) \tag{2.21}$$

is an object in the category $\mathcal{C}$. Their relation is given by

$$\mathcal{C}(A, B) \cong \mathcal{C}(I, A \multimap B) \tag{2.22}$$

cf. (2.10) and (2.11).

Comparing the monoidal closure $\multimap$ to the *-autonomous equation (2.20), we immediately find

$$B \multimap C^* \cong (B \otimes C)^* \tag{2.23}$$

and we shall take this to be the definition of the closure. In particular we find

$$A \multimap I^* \cong A^* \tag{2.24}$$

Furthermore, since the star is full and faithful, we have

$$\begin{aligned} \mathcal{C}(A, B) &\cong \mathcal{C}^{op}(A^*, B^*) \\ &\cong \mathcal{C}(B^*, A^*) \end{aligned} \tag{2.25}$$

It also follows from equation (2.20), and using the swap, that

$$\mathcal{C}(B, A^*) \cong \mathcal{C}(A \otimes B, I^*) \cong \mathcal{C}(A, B^*) \tag{2.26}$$

Combining these we obtain

$$\mathcal{C}(A, B) \cong \mathcal{C}(B^*, A^*) \cong \mathcal{C}(A, B^{**}) \tag{2.27}$$

and thus, by a corollary of the Yoneda lemma (see Corollary 7.4.3 in Part II) we have

$$B \cong B^{**} \tag{2.28}$$

and we shall also take this to be strict. We can now rewrite the *-autonomous equation (2.20), using equation (2.28), in a way that might be more familair to some:

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, (B \otimes C^*)^*) \tag{2.29}$$

and this relates to the closure $\multimap$ again via (2.23).

Example 2.4.2. Any compact closed category is *-autonomous. Indeed, with the monoidal closure given by $A \multimap B = A^* \otimes B$ we have by Lemma 2.1.12

$$\mathcal{C}(A \otimes B, C^*) \cong \mathcal{C}(A, B^* \otimes C^*) \cong \mathcal{C}(A, (B \otimes C)^*) \tag{2.30}$$

Crucial in this example is the distribution of the star over the tensor. And this was precisely what we used in the derivation of (2.19) to show that higher order processes collapse to first order in a compact closed category. In a *-autonomous category, this distribution does not hold in general. This allows us to define a new monoidal product, called *par*, as a de Morgan dual of the tensor:

$$A \parr B := (A^* \otimes B^*)^* \tag{2.31}$$

Associativity and commutativity of par follow directly from those of the tensor:

$$A \parr B \cong B \parr A \qquad (A \parr B) \parr C = A \parr (B \parr C) \tag{2.32}$$

Moreover, since I is the unit for $\otimes$, it follows that I^* is the unit for $\parr$:

$$\begin{aligned} A \parr I^* &= (A^* \otimes I^{**})^* \\ &\cong (A^* \otimes I)^* \\ &\cong A^{**} \\ &\cong A \end{aligned} \tag{2.33}$$

Hence we find a second monoidal structure $(\mathcal{C}, \parr, I^*)$.

Example 2.4.3. A compact closed category is a degenerate *-autonomous category in the sense that the two monoidal products coincide, $\otimes = \parr$, and that $I \cong I^*$. Indeed, $A \parr B = (A^* \otimes B^*)^* = A \otimes B$ and $I = I \parr I^* = I \otimes I^* = I^*$.

The internal hom, or *linear implication*, sometimes called 'lolly', $\multimap$, is obviously not symmetric or associative, however, we do have the following for any object X:

$$\begin{aligned} \mathcal{C}(X, A \multimap (B \multimap C)) &\cong \mathcal{C}(X \otimes A, B \multimap C) \\ &\cong \mathcal{C}(X \otimes A \otimes B, C) \\ &\cong \mathcal{C}(X, (A \otimes B) \multimap C) \end{aligned}$$

Hence

$$A \multimap (B \multimap C) \cong (A \otimes B) \multimap C \tag{2.34}$$

Symmetry of tensor then implies

$$A \multimap (B \multimap C) \cong B \multimap (A \multimap C) \tag{2.35}$$

It will be good to have the relations between $\otimes$, $\parr$ and $\multimap$ spelled out. We list them in this table:

	$\otimes$	$\parr$	$\multimap$
$\otimes$		$A \otimes B \cong (A^* \parr B^*)^*$	$A \otimes B \cong (A \multimap B^*)^*$
$\parr$	$A \parr B := (A^* \otimes B^*)^*$		$A \parr B \cong A^* \multimap B$
$\multimap$	$A \multimap B = (A \otimes B^*)^*$	$A \multimap B \cong A^* \parr B$	

Next we consider the relation between the monoidal structures $(\mathcal{C}, \otimes, I)$ and $(\mathcal{C}, \parr, I^*)$ a bit more.

Proposition 2.4.4. Let $\mathcal{C}$ be a *-autonomous category. Then there is a canonical 'distribution' morphism

$$\delta : A \otimes (B \parr C) \to (A \otimes B) \parr C \tag{2.36}$$

Proof. We have the following isomorphisms:

$$\begin{aligned} \mathcal{C}(A \multimap B^*, A \multimap B^*) &\cong \mathcal{C}((A \multimap B^*) \otimes A, B^*) \\ &\cong \mathcal{C}(A \otimes (A \multimap B^*), B^*) \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \mathcal{C}(B^* \multimap C, B^* \multimap C) &\cong \mathcal{C}(B^* \otimes (B^* \multimap c), C) \\ &\cong \mathcal{C}(B^*, (B^* \multimap C) \multimap C) \end{aligned} \tag{2.38}$$

Starting from the identity in both cases we end up with maps

$$A \otimes (A \multimap B^*) \to B^* \to (B^* \multimap C) \multimap C$$

Then using that

$$\mathcal{C}(A \otimes X, Y \multimap C) \cong \mathcal{C}(A \otimes Y, X \multimap C)$$

we obtain the desired map

$$A \otimes (B^* \multimap C) \to (A \otimes B)^* \multimap C \tag{2.39}$$

$\square$

Categorically, this means that *-autonomous categories are a special case of *linear distributive category*[2] (See [28], particularly Theorem 4.5). We interpret the left hand side of equation (2.36) as the system A together with maps from B^* to C, whereas we interpret the right hand side as maps from $(A \otimes B)^*$ to C. In this sense, in the r.h.s., the system A can 'influence' system C, whereas in the l.h.s. it can not.

In Theorem 3.3.16 we will see that the type of category we construct has the property that $I \cong I^*$, just as with compact closed categories (Example 2.4.3). However, in general, there is no clear relation between the monoidal units I and I^*.

Definition 2.4.5 ([27])**.** A *-autonomous category $\mathcal{C}$ is called *MIX* if there is a *mix map* $m : I^* \to I$ making the following diagram, made up from coherence maps and m, commute:

$$\begin{tikzcd}
A \otimes B \arrow[r, "\cong"] \arrow[d, "\cong"'] & A \otimes (I^* \parr B) \arrow[r, "m''"] & A \otimes (I \parr B) \arrow[d, "\delta"] \\
(A \parr I^*) \otimes B \arrow[d, "\delta'"'] & & (A \otimes I) \parr B \arrow[d, "\cong"] \\
A \parr (I^* \otimes B) \arrow[r, "m'"] & A \parr (I \otimes B) \arrow[r, "\cong"] & A \parr B
\end{tikzcd} \tag{2.40}$$

Here δ' is the map coming from δ and the appropriate swaps and m' and m'' are the obvious maps induced by the mix map m. The resulting map

$$mx_{A,B} : A \otimes B \to A \parr B \tag{2.41}$$

is called a *mixor* ([29]).

The category is called *isoMIX* if $m : I^* \to I$ is an isomorphism.

It was shown in 1989 by Robert Seely ([82]) that $*$-autonomous categories are a semantics for classical multiplicative linear logic (MLL). This is a fragment of linear logic, introduced by Jean-Yves Girard ([40]), and deals with logic for systems which are resource sensitive. This is often explained in terms of cooking. One can use a beer in a stew or drink it, but not both. This is reminiscent of the no cloning theorem [95] in quantum theory and hence it should not be a big surprise that these categories pop up in categorical quantum research. They are for example also related to entanglement in Hilbert spaces [32]. The intuition is that $\otimes$ behaves like

[2] A slightly outdated term for linearly distributive is *weakly distributive*.

'and' and $(-)^*$ is a negation. It than follows that $\parr$ behaves like 'or' (not (not a and not b) = a or b) and $\multimap$ behaves like implication (not a or b = a implies b). The main difference is that instead of distribution (a and (b or c) = (a and b) or (a and c)) we have the linearized version equation (2.36).

From now on we will adopt the convention that $\otimes$ has precedence over $\multimap$ and that $\multimap$ associates to the right:

$$A \otimes B \multimap C \quad := \quad (A \otimes B) \multimap C \tag{2.42}$$

$$A \multimap B \multimap C \quad := \quad A \multimap (B \multimap C) \tag{2.43}$$

We end with a small preview of what is to come. In the next chapter we start with a compact closed category $\mathcal{C}$. By process state duality (Section 2.1.5) we can therefore consider processes and higher order processes as states on some system. From this we construct a category whoso objects are related to sets of causal states in this generalized sense. Theorem 3.3.16 shows that this category is isoMIX *-autonomous and in Section 3.3 we see how the two monoidal products are different ways to combine two of these sets. The system $A \otimes B$ is a joint state-space of A and B, whereas $A \parr B$ should be seen as a map from A^* to B. These are the same for a class of systems called *first order* (Section 4.1), but differ in general and allow us to capture different signalling properties of processes in Chapter 4.

Chapter 3

Precausal categories and Caus$[\mathcal{C}]$

In this chapter we will use the framework built up in the previous chapter to define a class of categories which we call *precausal* (Definition 3.1.1). Such precausal categories will serve as a kind of 'base' category to which we will assign a new category (Section 3.3) whose objects represent *higher order causal types* (Chapter 4).

3.1 Precausal categories

In order to construct a theory for higher order processes, we need to start with the right kind of 'base' category. These base categories will be certain compact closed SMCs together with some natural additional requirements that ensure a well behaved construction of a type system describing higher order systems. We will first introduce these base categories, called *precausal categories*, here. After that we will introduce the construction to define a category with a refined type system.

Recall that second order causal processes are those morphisms which send causal processes to causal processes (Definition 2.3.1). We now define the class of categories in which all process will take place.

Definition 3.1.1. A *precausal category* is a symmetric monoidal category $\mathcal{C}$, which is compact closed and satisfies the following four properties:

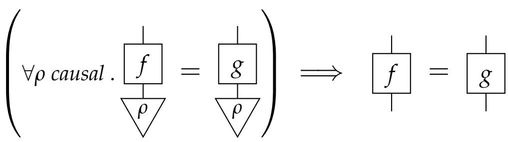
(C1) $\mathcal{C}$ has discarding.

(C2) For every non-zero system A, the dimension of A (cf. Definition 2.1.17)

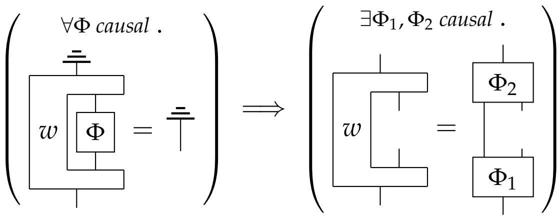
$d_A =$ 

is an invertible scalar.

(C3) $\mathcal{C}$ has *enough causal states*:

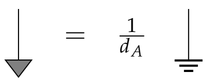


(C4) *Second-order causal* processes factorise:



For the rest of this section we will investigate properties of precausal categories.

Since causality is defined in terms of discarding (1.8), condition (C1) will be no surprise. Condition (C2) gives us information about the scalars, $\mathcal{I} := Hom(I, I)$, in a precausal category $\mathcal{C}$. It says that for every object A, which is not the categorical zero object, the scalars d_A and d_A^{-1} are elements of $\mathcal{I}$ and hence their products are as well. This implies that we can always normalize the maximally mixed state for every non-zero system A (cf. equation (2.1.6))

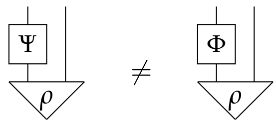


(3.1)

This condition shows us that there always exists at least one normalized state on any (non-zero) object. We will need this to make sense of Definition 4.1.1 when we consider causal types.

At first glance, the enough causal states axiom (C3) might seem like a triviality: if two functions give the same output on all inputs, they must be the same. However, in general there are theories where two processes

Ψ and Φ can agree on all states, but still are not equal as there exists some additional system and a bipartite state ρ such that

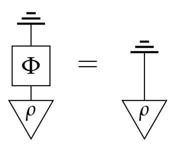


An example of such a theory is *real quantum theory* (see for example [22]). In a nutshell, real quantum theory is quantum theory, but with real Hilbert spaces and algebras. In particular, when we consider a qubit, the Pauli matrix σ_y, related to the spin in the y-direction, has complex entries and is therefore not part of real quantum theory. However, this operator tensored with itself, $\sigma_y \otimes \sigma_y$, is real. Recall that that Pauli matrices, together with the identity, form a basis for the self adjoint elements for a qubit. Consider now two processes Ψ and Φ on a single qubit which act different on this σ_y component, but the same on all other components. Now consider a bipartite state ρ with a $\sigma_y \otimes \sigma_y$ component, then the non-equality (3.1) holds, but Ψ and Φ do act the same on all single qubit states.

In Lemma 2.2.5 we saw that compact closedness is a sufficient condition to relate preservation of the marginal to causality. We now see that having enough causal states is also sufficient.

Proposition 3.1.2. Let $\mathcal{C}$ be a SMC with discarding and enough causal states, then preserving the marginal (equation (2.2.4)) is equivalent to the causality condition (equation (1.8)).

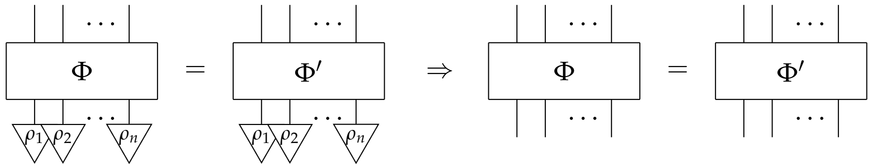
Proof. We have already noted that causality trivially implies preservation of the marginal. The other way around, since preservation of the marginal must hold for all states $\rho : A \otimes B$, we may take B to be trivial. Then



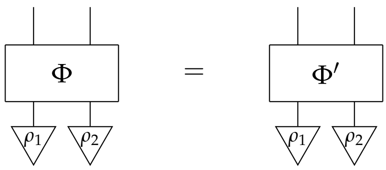
and by enough causal states the result now follows. □

In the formulation of Axiom (C3) we consider all causal states on some system. Of course that system could be some composite system. It turns out that in this case we need not consider all causal states, but it suffices to consider only the product states.

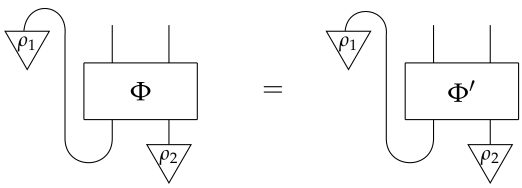
Lemma 3.1.3. Let $\mathcal{C}$ be a compact closed SMC with enough causal states. Then



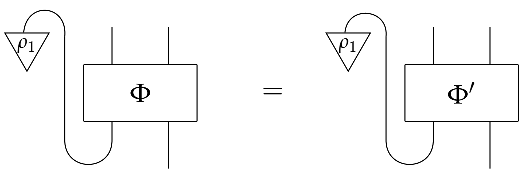
Proof. In the bipartite case we have that if



then by compact closure



Using the fact that there are enough states once allows us to get rid of ρ_2.

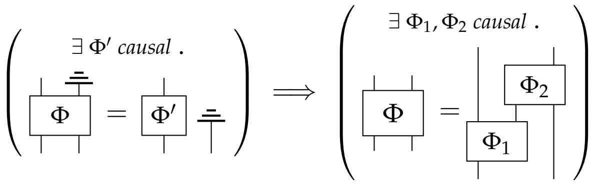


Repeating this process by bending the other wires shows $\Phi = \Phi'$. The general case then can then be found via induction. $\square$

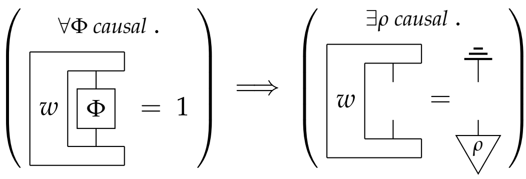
The last axiom (C4) states that all second order causal processes are channels with memory (also see [25]). This condition can be broken down into two easier conditions.

Proposition 3.1.4. For a compact closed SMC $\mathcal{C}$ satisfying (C1), (C2), and (C3), condition (C4) is equivalent to the following two conditions:

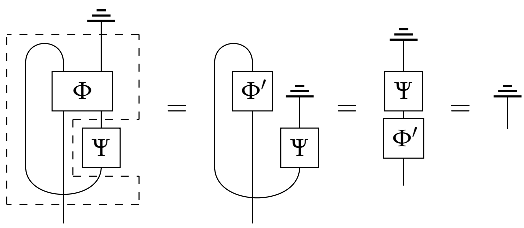
(C4′) Causal *one-way signalling* processes factorise:



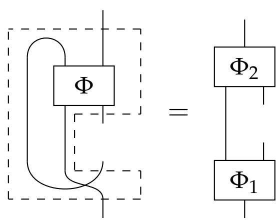
(C5′) *Second order causal effects* factorise. That is, for all $w : A \otimes B^*$ there exists some causal state ρ such that:



Proof. First suppose (C4) holds and let Φ be a causal one-way signalling map as in the premise of (C4′). Then, for any causal Ψ

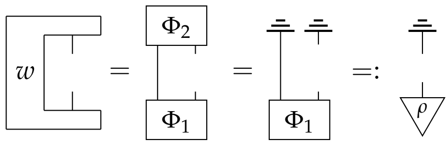


Hence, by (C4) there exist causal Φ_1', Φ_2 such that:



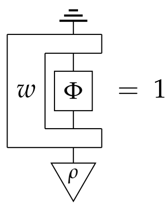
Deforming then gives the factorisation in (C4′).

To get to (C5′) from (C4), we take the input and output systems of w trivial. Then because discarding is the unique causal effect, Φ_2 has to be the discarding effect and we get:

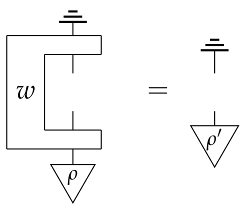


Causality of ρ follows directly from the fact that Φ_1 is causal.

The other way around, assume that (C4′) and (C5′) hold and suppose that w is second order causal. Then for any causal state ρ we have that for any causal map Φ the following holds:

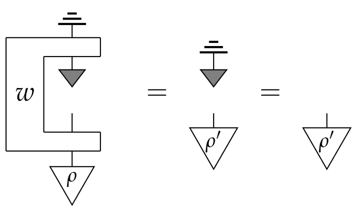


Hence by (C5′) we have that

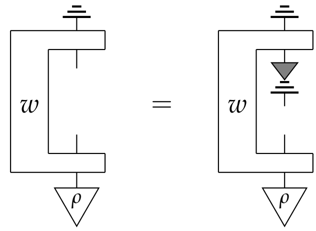


(3.2)

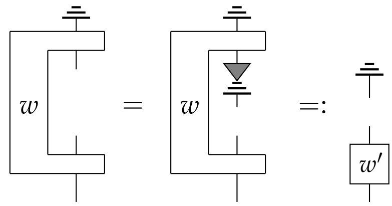
Recall the normalized version of the maximally mixed state (3.1). Then since non-zero dimensions are invertible, we obtain an expression for ρ':



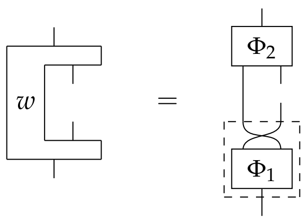
Substituting this result for ρ' back into the equality (3.2), we find



As this holds for all causal states ρ, we have by enough causal states, (C3), that



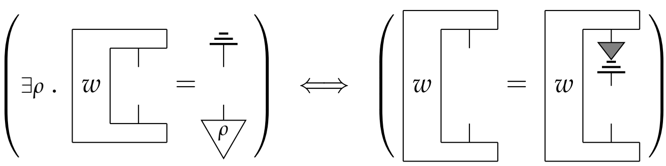
Hence w is one-way signalling as in (C4′), with $\Phi := w$ and $\Phi' := w'$. From this the factorization as in (C4) follows from the factorization of (C4′) by diagram deformation:



□

The proof of Proposition 3.1.4 reveals an interesting fact which is worth mentioning explicitly:

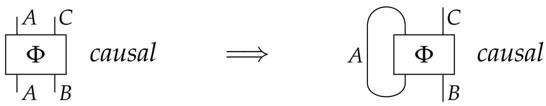
Lemma 3.1.5. For any $w : A \otimes B^*$:



In Example 1.1.2 we showed that the factorization in condition (C4′) was sufficient to be one-way signalling. By definition, in a precausal category all one-way signalling processes are of this form. This of course spawns the question whether there are categories which have one-way signalling processes which do not factor in this way. Unfortunately, to the knowledge of the author, this is not known. See Example 3.2.4 for a possible candidate.

Before looking at some examples of precausal categories in the next section, we end this section by showing that precausal categories do not admit feedback loops, a property which me might think of as no time-travelling.

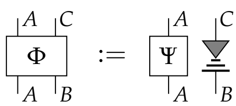
Theorem 3.1.6 (No time travel)**.** No non-trivial system A in a precausal category $\mathcal{C}$ admits *time travel*. That is, if there exist systems B and C such that for all processes $\Phi : A \otimes B \to A \otimes C$ we have:



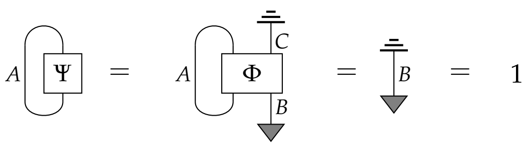
(3.3)

then $A \cong I$.

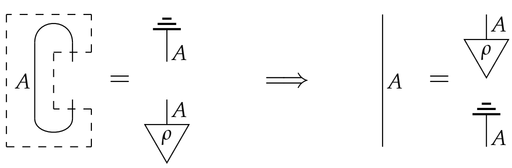
Proof. For any causal process $\Psi : A \to A$, we can define:



which is also a causal process. Then implication (3.3) gives:

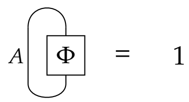


Applying (C5′), we have:



for some causal state $\rho : I \to A$. That is, $\rho \circ \bar{\top} = 1_A$, and by definition of causality for ρ, $\bar{\top} \circ \rho = 1_I$, so $A \cong I$. □

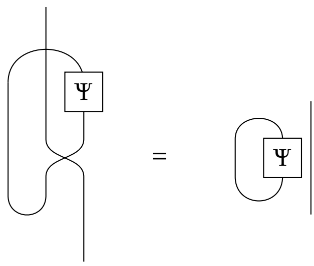
Note that a special case of Theorem 3.1.6 implies that if for all causal processes $\Phi : A \to A$ we have



then $A \cong I$.

We can now also explicitly show that the diagram (1.14) cannot be causal for any process Φ.

Example 3.1.7. Take Φ to be the swap map in (1.14), then



which by the no time-travel theorem is only causal for all Ψ if its input and output are trivial.

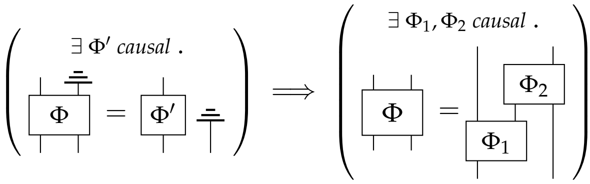
3.2 Examples of precausal categories

It should come as no surprise that the categories $\mathbf{Mat}(\mathbb{R}_+)$ and **CPM** are precausal. We show this explicitly.

Theorem 3.2.1. $\mathbf{Mat}(\mathbb{R}_+)$ is a precausal category.

Proof. (C1) was given in Example 2.1.18. (C2) is immediate, and (C3) follows from the fact that one can always construct a basis for a vector space out of probability distributions, e.g., by taking the point distributions. To show (C4), we will decompose it into (C4′) and (C5′) via Proposition 3.1.4.

We start with (C4′):



Write $P(B_1B_2|A_1A_2)$ for the probability of obtaining outcomes B_1 and B_2, given inputs A_1 and A_2. In terms of a conditional probability distribution, the premise above amounts to the usual no-signalling condition:

$$P(B_1|A_1, A_2) := \sum_{B_2} P(A_1, A_2|B_1, B_2) = P(B_1|A_1)$$

Hence the conclusion follows from the product rule:

$$\begin{aligned} P(B_1, B_2|A_1, A_2) &= P(B_1|A_1, A_2)P(B_2|B_1, A_1, A_2) \\ &= P(B_1|A_1)P(B_2|B_1, A_1, A_2) \end{aligned}$$

More precisely, suppose Φ_{ij}^{kl} is a stochastic matrix such that there exists another stochastic matrix $(\Phi')_i^k$ where:

$$\sum_l \Phi_{ij}^{kl} = (\Phi')_i^k$$

Then, let:

$$\begin{aligned} (\Phi_1)_i^{ki'k'} &= (\Phi')_i^k \delta_{ii'}\delta_{kk'} \\ (\Phi_2)_{i'k'j}^l &= \begin{cases} \delta_{0l} & \text{if } (\Phi')_i^{k'} = 0 \\ \Phi_{ij}^{k'l}/(\Phi')_i^{k'} & \text{otherwise} \end{cases} \end{aligned}$$

where δ_{ij} is the Kronecker delta. One can straightforwardly verify that these are both stochastic matrices. Let Ψ_{ij}^{kl} be the result of plugging outputs i', k' of Φ_1 into those inputs for Φ_2, i.e.

$$\Psi_{ij}^{kl} := \sum_{i'k'} (\Phi_1)_i^{ki'k'} (\Phi_2)_{i'k'j}^l = (\Phi')_i^k (\Phi_2)_{ikj}^l$$

If $(\Phi')_i^k = 0$, then both Φ_{ij}^{kl} and Ψ_{ij}^{kl} are 0 for all j, l. So, suppose $(\Phi')_i^k \neq 0$. Then:

$$\Psi_{ij}^{kl} = (\Phi')_i^k (\Phi_{ij}^{kl}/(\Phi')_i^k) = \Phi_{ij}^{kl}$$

For (C5′), let w_j^i be the matrix of a second-order causal effect $w : A \otimes B^*$. Then for all stochastic matrices Φ_i^j, we have:

$$\sum_{ij} w_j^i \Phi_i^j = 1$$

For some fixed column m, and fixed rows $n \neq n'$, the following matrix:

$$\Phi_i^j = \begin{cases} p & i = m, j = n \\ 1 - p & i = m, j = n' \\ 0 & i = m, j \neq n, j \neq n' \\ \delta_i^j & i \neq m \end{cases}$$

defines a stochastic map for any $p \in [0, 1]$. Then:

$$\sum_{ij} w_j^i \Phi_i^j = p w_n^m + (1 - p) w_{n'}^m + K = 1$$

where K doesn't depend on p. Since we can freely vary p between 0 and 1, the only way to preserve normalisation is if $w_n^m = w_{n'}^m$. Hence, for all j, we have $w_j^i = w_0^i$. Defining $\rho^i := w_0^i$ gives the factorisation (C5′). □

Theorem 3.2.2. The category **CPM** is a precausal category.

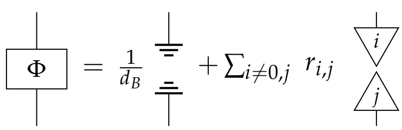
Proof. Discard is given by the trace (Example 2.1.18). For (C2), $d_{\mathcal{B}(H)} = \dim(H)$, which is invertible whenever $\dim(H) \neq 0$. (C3) follows from the fact that density operators span $\mathcal{B}(H)$.

For (C4), we shall show (C4′) and (C5′). Condition (C4′) states that

$$tr_{B'}(\Phi) = \Phi' \implies \Phi = (1_{A'} \otimes \Phi_2) \circ (\Phi_1 \otimes 1_B)$$

This is precisely the result of [34] and is based on the fact that minimal Stinespring dilations are related by a unitary.

For (C5′), a causal map $\Phi : A \to B$ in **CPM** is a completely positive trace preserving map which we may consider as a state in $A^* \otimes B$. Recall that there exists a basis of A^* and of B which contain the identity and are orthogonal w.r.t. the trace. We can then write



Now if any second order causal effect w does not split as in (C5′), we can always change the value of some of the $r_{i,j}$ such that Φ is still positive, but $w(\Phi) \neq 1$. □

As we will now see, the category **REL** of sets and relations does not satisfy axiom (C5′) and hence not (C4), but it does satisfy all other axioms.

Example 3.2.3. Let **REL** be the category whose objects are sets and whose morphisms $Hom(A, B)$ are relations $R \subset A \times B$. It is convenient to think of a relation R as a matrix over the Booleans, where $R_{b,a} = 1$ if $a \in A$ and $b \in B$ are in relation to each other and $R_{b,a} = 0$ otherwise. This category is compact closed as we see below. The monoidal product on objects is the direct product and on morphisms it is the Kronecker product. The tensor unit I is a singleton set $\{*\}$. Composition then just becomes matrix multiplication over the Booleans.

Hence a state $\rho : I \to A$ can be thought of as a column-vector of length $|A|$ and an effect $\sigma : B \to I$ as a row-vector of width $|B|$. The entries of these vectors are either 1 or 0 depending on whether the corresponding element is in relation to $*$ or not. Discard is given by the row-vector with all 1's as entries and any state which has at least one 1 as an entry is causal. Indeed,

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{|A|} \end{pmatrix} = \rho_1 \vee \rho_2 \vee \ldots \vee \rho_{|A|}$$

which equals 1 if at least one of the ρ_i equals 1. Note that we by no means mean that A is countable. This is just notation. By a similar argument, a morphism in **REL** is causal if and only if every column of its matrix representation contains at least one 1.

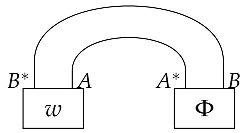
The compact closed structure is as follows: Let $A^* = A$ as set. Cup is the relation $R \subset \{*\} \times (A \times A)$ which relates $*$ to all elements of the form (a, a). Cap is the relation which relates all elements of the form (a, a) to $*$.

In order to investigate what a second order causal effect is in **REL**, we first write any process as a sum of tensor products of states and effects. Let e_a be the state on A which relates $*$ to $a \in A$ and nothing else. Similarly, let e^a be the effect on A which relates $a \in A$ to $*$ and nothing else. Now any matrix, and hence any relation, R can be decomposed in rows or columns. That is, we can find states ρ_b on B or effects σ_a on A such that

$$R = \sum_{b \in B} \rho_b \otimes e^b = \sum_{a \in A} e_a \otimes \sigma_a$$

In order for R to correspond to a causal map now, we need that all ρ_b are causal states, or similarly, that all σ_a contain at least one 1.

Now suppose w is a second order causal effect, so for all maps f



$$= 1 \tag{3.4}$$

Write $w = \sum_{a \in A} \sigma_a \otimes e_a$ and write $f = \sum_{a' \in A} e^{a'} \otimes \rho_{a'}$. Calculating the the caps in diagram (3.4) is now very easy as it can be done component wise. First doing the cap over system A will only contribute if $a = a'$. Hence the resulting cap over system B can be calculated as $\sum_{a \in A} \langle \sigma_a, \rho_a \rangle$, where $\langle \cdot, \cdot \rangle >$ is the inner product between states and effects. As this has to be equal to 1 for every choice of causal ρ_a, we conclude that there exists $\tilde{a} \in A$ such that $\sigma_{\tilde{a}} = (1\,1 \ldots 1) = \bar{\top}_B$. In contrast, any element of the form $\rho \otimes \bar{\top}$ as in (C5$'$) must have all columns equal.

We know $\mathbf{Mat}(\mathbb{R}_+)$ is a precausal category. An interesting question is what happens when we drop positivity.

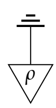
Example 3.2.4. Just like $\mathbf{Mat}(\mathbb{R}_+)$, let $\mathbf{Mat}(\mathbb{R})$ be the category whose objects are natural numbers, but whose morphisms are matrices over the reals. Then by the same arguments as for $\mathbf{Mat}(\mathbb{R}_+)$, $\mathbf{Mat}(\mathbb{R})$ satisfies (C1), (C2), (C3), and (C5$'$). However, (C4$'$) does not carry over since we explicitly used positivity in deducing that both Φ^{kl}_{ij} and Ψ^{kl}_{ij} are zero when $(\Phi')^k_i$ is.

As such, $\mathbf{Mat}(\mathbb{R})$ could be an example of a category where one-way signalling processes do not factor. Whether this is the case or not is unknown.

3.3 The construction Caus[$\mathcal{C}$]

In the previous section we introduced a type of 'well behaved' SMCs. In this section we will take such a category and built from it a new category of which we consider the objects to be *higher order causal types*. This construction will go via sets of generalized causal states, so we will begin by considering properties of causal or normalized states.

Given a precausal category $\mathcal{C}$ we can consider for every object A the set of normalized states of A:

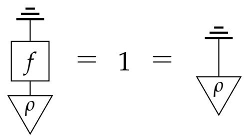
$$C = \left\{ \rho : A \;\middle|\; \right.$$


$$\left. = \quad 1 \right\} \tag{3.5}$$

Causal processes, or more specifically first order causal processes, are precisely those maps which preserve this set:

Lemma 3.3.1. A map $f : A \to B$ in a precausal category is causal if and only if for all causal states ρ in A, $f \circ \rho$ is causal in B.

Proof. If f is causal, then obviously $f \circ \rho$ is causal whenever ρ is. Now suppose $\rho \circ f$ is causal for every causal ρ:



Then by enough causal states (C3) f is causal. □

This characterization leads us to consider sets of generalized causal states for the objects of $\mathcal{C}$ and then consider processes which preserve these sets.

Definition 3.3.2. Let X be any set of states on an object A in a precausal category. The *dual set* of X, written X^*, is the set of effects which normalize X:

$$X^* = \{\pi : A^* \mid \forall x \in X : \pi \circ x = 1\}$$

Since we identify A with its double dual A^{**}, the double dual of a set X is again a set of states on A. We now have the following results:

Lemma 3.3.3. For any set of states X on a system A the following hold:

(i) $X \subset X^{**}$,

(ii) If $X \subset Y$, then $Y^* \subset X^*$,

(iii) $X^* = X^{***}$,

(iv) Taking the double dual is a closure operation, i.e., $X^{**} = X^{****}$.

Proof. For (i), we have that if $x \in X$, then for all $\pi \in X^*$ we have $\pi \circ x = 1$, hence $x \in X^{**}$. Now if $\pi \in Y^*$ normalizes Y it certainly normalizes X whenever $X \subset Y$, which shows (ii). To show (iii), we note that by (i), $X^* \subset X^{***}$. Also by (i), $X \subset X^{**}$ so by (ii), $X^{***} \subset X^*$. Finally, (iv) follows directly from (iii). □

Consider again the set C of causal states on a system A (3.5). Using duals, we can rewrite this set to

$$C = \{\rho : A \mid \rho \in \{\bar{\top}_A\}^*\} \tag{3.6}$$

Indeed, causal states are exactly those states which are normalized by the discard effect.

Now consider the dual of C, C^*. Obviously $\bar{\top}_A \in C^*$. Now suppose there was some other effect σ such that σ normalizes all elements in C, then by enough causal states we have $\sigma = \bar{\top}_A$. We have thus proven:

Lemma 3.3.4. Let A be an object in a precausal category. The dual of the set C in (3.6) is the set

$$C^* = \{\bar{\top}_A\}$$

It then follows:

Corollary 3.3.5. The set (3.6) satisfies $C = C^{**}$.

Furthermore, since

$$\frac{1}{d_A} \quad \overset{\bar{\top}}{\underset{\bar{\perp}}{|}}\, A \;= 1,$$

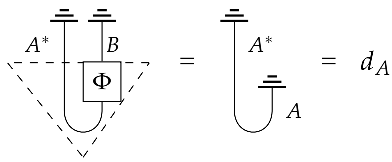
we have that the normalized maximally mixed state is an element of C and the discard effect is an element of C^*.

This leads us to the following:

Definition 3.3.6. A set of states c on an object A in a precausal category $\mathcal{C}$ is called

- *Closed* if $c^{**} = c$,
- *Flat* if there exist scalars λ, μ such that $\lambda \bar{\perp} \in c$ and $\mu \bar{\top} \in c^*$.

The reason that we allow the maximally mixed state and the discard effect up to a scalar becomes clear when we consider the space of causal maps from A to B seen as states on $A^* \otimes B$. Discarding such a state coming from a map Φ, by applying $\bar{\top}_{A^*} \bar{\top}_B$ gives d_{A^*} which equals d_A.



So $\frac{1}{d_A} \bar{\top}_{A^*} \bar{\top}_B$ is in C^* and $\frac{1}{d_B} \underline{\perp}_{A^*} \underline{\perp}_B$ is in C.

Lemma 3.3.7. The scalars λ, μ from Definition 3.3.6 are invertible.

Proof. Since $\mu \bar{\top}_A$ is in the dual of $\lambda \underline{\perp}$, it follows that

$$1 = \mu \bar{\top}_A \circ \lambda \underline{\perp} = \lambda\mu \; \overset{\bar{\top}}{\underset{\underline{\perp}}{|}}{}_A = \lambda\mu d_A$$

Hence λ and μ are invertible. $\square$

We are now ready to present the construction of the category of higher order causal types from a precausal category $\mathcal{C}$. We denote this higher order causal category by Caus[$\mathcal{C}$].

Definition 3.3.8. Given a precausal category $\mathcal{C}$, the category Caus[$\mathcal{C}$] is the category where

- Objects are pairs $\boldsymbol{A} := (A, c_{\boldsymbol{A}})$ where A is an object of $\mathcal{C}$ and $c_{\boldsymbol{A}}$ is a set of states which is flat and closed,
- Morphisms $f : (A, c_{\boldsymbol{A}}) \to (B, c_{\boldsymbol{B}})$ in Caus[$\mathcal{C}$] corresponds to morphisms $f : A \to B$ in $\mathcal{C}$ such that

$$\rho \in c_{\boldsymbol{A}} \Rightarrow f \circ \rho \in c_{\boldsymbol{B}}.$$

We will use bold letters to denote objects in Caus[$\mathcal{C}$]. That is, $\boldsymbol{A} := (A, c_{\boldsymbol{A}})$. It should be noted here that $c_{\boldsymbol{A}}$ can be any set of states, as long as it is closed and flat. In particular, there could be multiple objects which have the same underlying space, but different sets of 'generalized causal states': $\boldsymbol{A} = (A, c_{\boldsymbol{A}})$ and $\boldsymbol{A'} = (A, c_{\boldsymbol{A'}})$. We will give the objects unique names and attach this name in subscript to the set c to distinguish these different sets.

The condition on morphisms in the definition of Caus[$\mathcal{C}$] is given in terms of states, but from the definition of duals we can express it in terms of effects or scalars as well.

Lemma 3.3.9. Let $\boldsymbol{A} = (A, c_{\boldsymbol{A}})$ and $\boldsymbol{B} = (B, c_{\boldsymbol{B}})$ be objects in Caus[$\mathcal{C}$] and let $f : A \to B$ be a map in $\mathcal{C}$. Then the following are equivalent:

(i) $\rho \in c_{\boldsymbol{A}} \implies f \circ \rho \in c_{\boldsymbol{B}}$,

(ii) $\pi \in c_{\boldsymbol{B}}^* \implies \pi \circ f \in c_{\boldsymbol{A}}^*$,

(iii) $\rho \in c_{\boldsymbol{A}}, \pi \in c_{\boldsymbol{B}}^* \implies \pi \circ f \circ \rho = 1$.

Proof. Assume (i) and let $\pi \in c_{\boldsymbol{B}}^*$. Then $\pi \circ f \in c_{\boldsymbol{A}}^*$ if for every $\rho \in c_{\boldsymbol{A}}$ we have $(\pi \circ f) \circ \rho = 1$, but this is the case because $f \circ \rho \in c_{\boldsymbol{B}}$.
Now assume (ii) and let $\rho \in c_{\boldsymbol{A}}, \pi \in c_{\boldsymbol{B}}^*$. Then $\pi \circ f \circ \rho = 1$ by definition of $(-)^*$.
Finally assume (iii) and let $\rho \in c_{\boldsymbol{A}}$. For $\pi \in c_{\boldsymbol{B}}^*$ we have $\pi \circ f \circ \rho = 1$ and hence $f \circ \rho \in c_{\boldsymbol{B}}^{**} = c_{\boldsymbol{B}}$. □

We are now going to show that the resulting category Caus[$\mathcal{C}$] is $*$-autonomous. To this end we first show it is an SMC. For this we will first need to define the tensor. Given objects $\boldsymbol{A} = (A, c_{\boldsymbol{A}})$ and $\boldsymbol{B} = (B, c_{\boldsymbol{B}})$, let $c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}}$ be the set of product states of $c_{\boldsymbol{A}}$ and $c_{\boldsymbol{B}}$. That is:

$$c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}} := \{\rho_A \otimes \rho_B \mid \rho_A \in c_{\boldsymbol{A}}, \rho_B \in c_{\boldsymbol{B}}\}$$

Definition 3.3.10. For objects $\boldsymbol{A} = (A, c_{\boldsymbol{A}})$ and $\boldsymbol{B} = (B, c_{\boldsymbol{B}})$, their tensor product $\boldsymbol{A} \otimes \boldsymbol{B}$ is the object

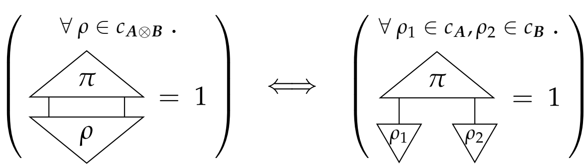
$$\boldsymbol{A} \otimes \boldsymbol{B} = (A \otimes B, c_{\boldsymbol{A}\otimes\boldsymbol{B}})$$

where the set of states $c_{\boldsymbol{A}\otimes\boldsymbol{B}}$ is the closure of the set of product states:

$$c_{\boldsymbol{A}\otimes\boldsymbol{B}} = (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**}$$

Note that we do not actually know if this tensor is a valid object of Caus[$\mathcal{C}$]. We show this in Lemma 3.3.12, but first we need the following result which shows that effects that normalize all product states automatically normalize all states.

Lemma 3.3.11. For any effect $\pi : A^* \otimes B^*$ in $\mathcal{C}$:



(3.7)

Proof. The LHS of (3.7) states that

$$\pi \in c_{\boldsymbol{A}\otimes\boldsymbol{B}}^* := ((c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**})^* = (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{***}$$

whereas the RHS states that $\pi \in (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^*$. Hence, (3.7) follows from Lemma 3.3.3. □

With this we can now show that indeed the tensor is an object in Caus[$\mathcal{C}$].

Lemma 3.3.12. Let $(A, c_{\boldsymbol{A}})$ and $(B, c_{\boldsymbol{B}})$ be objects in Caus[$\mathcal{C}$]. Then the set $c_{\boldsymbol{A}\otimes\boldsymbol{B}}$ is closed and flat.

Proof. By Lemma 3.3.3, $c_{\boldsymbol{A}\otimes\boldsymbol{B}}$ is obviously closed. Since $c_{\boldsymbol{A}}$ and $c_{\boldsymbol{B}}$ are flat, we have that for some λ, λ':

$$\lambda \perp \in c_{\boldsymbol{A}},\ \lambda' \perp \in c_{\boldsymbol{B}}$$

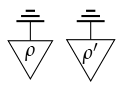
hence:

$$\lambda\lambda' \perp \perp \in c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}} \subseteq c_{\boldsymbol{A}\otimes\boldsymbol{B}}$$

Similarly, for some μ, μ':

$$\mu \top \in c^*_{\boldsymbol{A}},\ \mu' \top \in c^*_{\boldsymbol{B}}$$

So, for all $\rho \in c_{\boldsymbol{A}}$, $\rho' \in c_{\boldsymbol{B}}$, we have:

$\mu\mu'$  $= 1$

which implies, by Lemma 3.3.11:

$$\mu\mu' \top \top \in (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^* = (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{***} = c^*_{\boldsymbol{A}\otimes\boldsymbol{B}}$$

□

In order to show that Caus[$\mathcal{C}$] is indeed an SMC, we need to check a lot of details. We do this in the following theorem.

Theorem 3.3.13. Let $\mathcal{C}$ be a precausal category. Then Caus[$\mathcal{C}$] is a symmetric monoidal category where

$$\boldsymbol{A} \otimes \boldsymbol{B} = (A \otimes B, c_{\boldsymbol{A}\otimes\boldsymbol{B}})$$

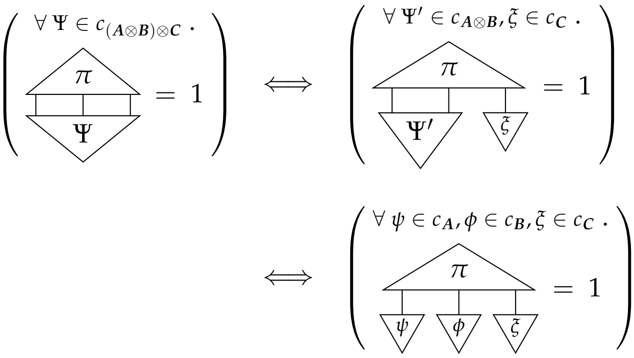
and

$$\boldsymbol{I} = (I, \{1\})$$

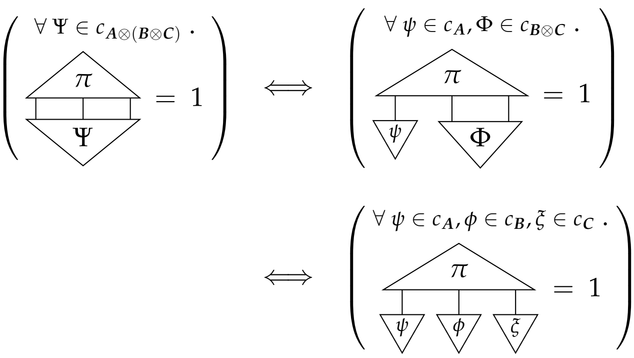
Proof. First we show that $\boldsymbol{A} \otimes \boldsymbol{B}$ and $\boldsymbol{I}$ are indeed objects in Caus[$\mathcal{C}$], namely the $c_{\boldsymbol{A}\otimes\boldsymbol{B}}$ and $c_{\boldsymbol{I}}$ are flat and closed. For $c_{\boldsymbol{A}\otimes\boldsymbol{B}}$ this is Lemma 3.3.12 and for $c_{\boldsymbol{I}}$ we just have to note that $Hom(I, I)$ are the states on I as well as the effects on I and that $\top_I = 1 = \perp$.

Next, we show associativity and unit laws for $\otimes$. For any object $\boldsymbol{A}$, the unit laws $\boldsymbol{A} \otimes \boldsymbol{I} = \boldsymbol{A} = \boldsymbol{I} \otimes \boldsymbol{A}$ follow from the closure of $c_{\boldsymbol{A}}$.

For associativity, we will show that $c^*_{(\boldsymbol{A}\otimes\boldsymbol{B})\otimes\boldsymbol{C}} = c^*_{\boldsymbol{A}\otimes(\boldsymbol{B}\otimes\boldsymbol{C})}$. Applying Lemma 3.3.11 twice to an effect $\pi \in c^*_{(\boldsymbol{A}\otimes\boldsymbol{B})\otimes\boldsymbol{C}}$ gives:

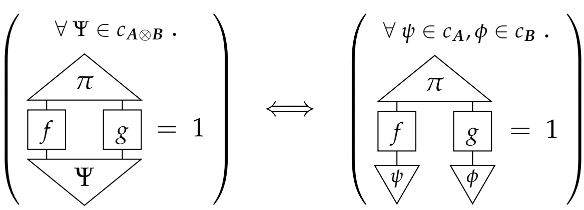


Similarly, for $\pi \in c^*_{\boldsymbol{A}\otimes(\boldsymbol{B}\otimes\boldsymbol{C})}$



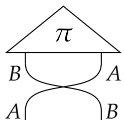
Hence $c^*_{(\boldsymbol{A}\otimes\boldsymbol{B})\otimes\boldsymbol{C}} = c^*_{\boldsymbol{A}\otimes(\boldsymbol{B}\otimes\boldsymbol{C})}$ and so $c_{(\boldsymbol{A}\otimes\boldsymbol{B})\otimes\boldsymbol{C}} = c_{\boldsymbol{A}\otimes(\boldsymbol{B}\otimes\boldsymbol{C})}$.

Next we show that $\otimes$ is well-defined on morphisms. For morphisms $f : \boldsymbol{A} \to \boldsymbol{A}'$, $g : \boldsymbol{B} \to \boldsymbol{B}'$, and an effect $\pi \in c^*_{\boldsymbol{A}'\otimes\boldsymbol{B}'}$, we have by Lemma 3.3.11:



The RHS holds since $f \circ \psi \in c_{\boldsymbol{A}'}$ and $g \circ \phi \in c_{\boldsymbol{B}'}$. From the LHS above, we can conclude that $f \otimes g : \boldsymbol{A} \otimes \boldsymbol{B} \to \boldsymbol{A}' \otimes \boldsymbol{B}'$ is a morphism in Caus[$\mathcal{C}$].

Finally, it remains to show that swap is a morphism in Caus[$\mathcal{C}$]. By Lemma 3.3.9, this is the case when, for all $\pi \in c^*_{\boldsymbol{B}\otimes\boldsymbol{A}}$, we have:



$\in c^*_{\boldsymbol{A}\otimes\boldsymbol{B}}$

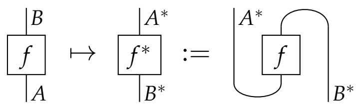
This again follows by relying on Lemma 3.3.11. □

We now focus on the involution, $(-)^*$. Since $\mathcal{C}$ is compact closed, we have duals in $\mathcal{C}$. Furthermore, we have duals on sets of states. This leads us to consider the involution on Caus[$\mathcal{C}$] as

$$\boldsymbol{A}^* = (A, c_{\boldsymbol{A}})^* := (A^*, c^*_{\boldsymbol{A}})$$

Lemma 3.3.14. The transposition functor $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$:

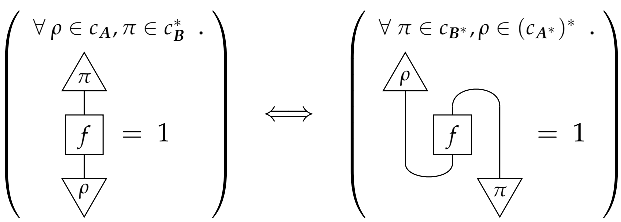
$A \mapsto A^*$



(3.8)

lifts to a full and faithful functor $(-)^* : \mathrm{Caus}[\mathcal{C}]^{\mathrm{op}} \to \mathrm{Caus}[\mathcal{C}]$, where $\boldsymbol{A}^* := (A^*, c_{\boldsymbol{A}^*} := c^*_{\boldsymbol{A}})$.

Proof. Since $c^*_{\boldsymbol{B}} = c_{\boldsymbol{B}^*}$, by definition, and $c_{\boldsymbol{A}} = c^{**}_{\boldsymbol{A}} = (c_{\boldsymbol{A}^*})^*$, we have, for $f : \boldsymbol{A} \to \boldsymbol{B}$ that:



so $f^* : \boldsymbol{B}^* \to \boldsymbol{A}^*$ is a morphism in Caus[$\mathcal{C}$]. Just as with the functor $(-)^*$ in $\mathcal{C}$, $((-)^*)^* = \mathrm{Id}_{\mathrm{Caus}[\mathcal{C}]}$, so fullness and faithfulness is immediate. □

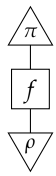
Consider a map $f : \boldsymbol{A} \to \boldsymbol{I}$ in Caus[$\mathcal{C}$]. That is, $f : A \to I$ in $\mathcal{C}$ and for all $\rho \in c_{\boldsymbol{A}}$ we have $f \circ \rho = 1$. From this we see that $\boldsymbol{A}^*$ can be identified with the space of maps from $\boldsymbol{A}$ to $\boldsymbol{I}$. We make this precise. Recall that in a $*$-autonomous category the object $A \multimap B$ is defined by $(A \otimes B^*)^*$ (equation

(2.23)). While we do not yet know Caus[$\mathcal{C}$] is $*$-autonomous, we do have enough structure to define

$$\boldsymbol{A} \multimap \boldsymbol{B} := (\boldsymbol{A} \otimes \boldsymbol{B}^*)^* \tag{3.9}$$

Explicitly we have:

Lemma 3.3.15. For objects $\boldsymbol{A}, \boldsymbol{B} \in$ Caus[$\mathcal{C}$]:

$$c_{\boldsymbol{A} \multimap \boldsymbol{B}} = \left\{ f : A^* \otimes B \;\middle|\; \forall \rho \in c_{\boldsymbol{A}}, \pi \in c_{\boldsymbol{B}}^* \,.\right.$$

$$\left. = 1 \right\}$$

Proof. This follows by simplifying:

$$c_{(\boldsymbol{A}\otimes\boldsymbol{B}^*)^*} = c^*_{(\boldsymbol{A}\otimes\boldsymbol{B}^*)} = (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}^*})^{***} = (c_{\boldsymbol{A}} \otimes c^*_{\boldsymbol{B}})^*$$

and noting that $f \in (c_{\boldsymbol{A}} \otimes c^*_{\boldsymbol{B}})^*$ is precisely the statement given in the lemma. □

We now show that Caus[$\mathcal{C}$] always has the structure of an isoMIX $*$-autonomous category, i.e., it satisfies $\boldsymbol{I} = \boldsymbol{I}^*$ ([26]).

Theorem 3.3.16. For any precausal category $\mathcal{C}$, the category Caus[$\mathcal{C}$] is $*$-autonomous category with the additional property that $\boldsymbol{I} = \boldsymbol{I}^*$.

Proof. We have already shown that Caus[$\mathcal{C}$] is an SMC (Theorem 3.3.13) with a full and faithful functor $(-)^* : \text{Caus}[\mathcal{C}]^{\text{op}} \to \text{Caus}[\mathcal{C}]$ (Lemma 3.3.14). Consider objects $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ in Caus[$\mathcal{C}$]. The underlying object of $\boldsymbol{B} \multimap \boldsymbol{C}$ is:

$$(B \otimes C^*)^* = B^* \otimes C^{**} = B^* \otimes C$$

Recall that since $\mathcal{C}$ is compact closed, there is a natural isomorphism:

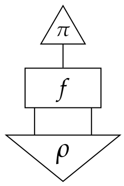
$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, B^* \otimes C)$$

obtained from bending a wire, which we saw in (2.10). Thus, it suffices to show that:

$$f \in \text{Caus}[\mathcal{C}](\boldsymbol{A} \otimes \boldsymbol{B}, \boldsymbol{C}) \iff g \in \text{Caus}[\mathcal{C}](\boldsymbol{A}, \boldsymbol{B} \multimap \boldsymbol{C})$$

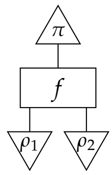
This follows from Lemma (3.3.11):

$$f \in \mathrm{Caus}[\mathcal{C}](\boldsymbol{A} \otimes \boldsymbol{B}, \boldsymbol{C}) \iff \Big(\forall \rho \in c_{\boldsymbol{A}\otimes\boldsymbol{B}}, \pi \in c^*_{\boldsymbol{C}} \,.$$



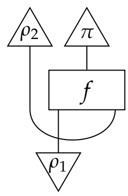
$$= 1 \Big)$$

$$\iff \Big(\forall \rho_1 \in c_{\boldsymbol{A}}, \rho_2 \in c_{\boldsymbol{B}}, \pi \in c^*_{\boldsymbol{C}} \,.$$



$$= 1 \Big)$$

$$\iff \Big(\forall \rho_1 \in c_{\boldsymbol{A}}, \rho_2 \in c^*_{\boldsymbol{B}^*}, \pi \in c^*_{\boldsymbol{C}} \,.$$



$$= 1 \Big)$$

$$\iff \; g \in \mathrm{Caus}[\mathcal{C}](\boldsymbol{A}, \boldsymbol{B} \multimap \boldsymbol{C})$$

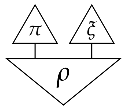
Finally, $\boldsymbol{I} = \boldsymbol{I}^*$ follows from the fact that $I = I^*$ and

$$c^*_{\boldsymbol{I}} = \{\lambda \mid 1\lambda = 1\} = \{1\} = c_{\boldsymbol{I}}$$

$\square$

Now that we know Caus[$\mathcal{C}$] is $*$-autonomous, we explicitly consider the second monoidal structure, $\parr$. Recall that $\boldsymbol{A} \parr \boldsymbol{B} := (\boldsymbol{A}^* \otimes \boldsymbol{B}^*)^*$. That is, $\boldsymbol{A} \parr \boldsymbol{B} = (A \otimes B, c_{\boldsymbol{A}\parr\boldsymbol{B}})$, where

$$c_{\boldsymbol{A}\parr\boldsymbol{B}} = \left\{ \rho : A \otimes B \;\middle|\; \forall \pi \in c^*_{\boldsymbol{A}}, \xi \in c^*_{\boldsymbol{B}} \,.\right. \tag{3.10}$$



$$\left. = 1 \right\}$$

Hence $\boldsymbol{A} \parr \boldsymbol{B}$ is the object whose 'causal states' are precisely those states which are normalized by all product effects. Since the causal states of the tensor product are those states normalized by all effects, we have the following embedding:

$$\boldsymbol{A} \otimes \boldsymbol{B} \hookrightarrow \boldsymbol{A} \parr \boldsymbol{B} \tag{3.11}$$

The way Caus[$\mathcal{C}$] is defined, it comes with a natural type theory which is finer than that of $\mathcal{C}$ itself. We say a state ρ has type $\boldsymbol{X}$, written $\rho : \boldsymbol{X}$, if $\rho \in c_{\boldsymbol{X}}$. Now the notation $\rho : \boldsymbol{X}$ is used in two ways. Once as the abbreviation that ρ is a state on $\boldsymbol{X}$ and once to note that ρ has type $\boldsymbol{X}$. However, it turns out this actually means the same!

Lemma 3.3.17. For an object $\boldsymbol{X}$ in Caus[$\mathcal{C}$] and a state ρ on $\boldsymbol{X}$ we have

$$\rho : \boldsymbol{X} \Leftrightarrow \rho \in c_{\boldsymbol{X}}$$

That is, a state on a system $\boldsymbol{X}$ has type $\boldsymbol{X}$.

Proof. Per definition, a state ρ is a morphism $\rho : \boldsymbol{I} \to \boldsymbol{X}$. Since $\boldsymbol{I} = (I, \{1\})$ we must, by definition of morphism in Caus[$\mathcal{C}$], have that $\rho = \rho \circ 1 \in c_{\boldsymbol{X}}$. $\square$

In the next chapter we will relate certain types to (higher order) causal orders. If we then have some diagram Φ in $\mathcal{C}$, we can always consider it as a state ρ_Φ by bending the input wires. We can then consider the objects $\boldsymbol{A}$ such that $\rho_\Phi \in c_{\boldsymbol{A}}$. If any of these types correspond to ones which we related to some causal order, we know that Φ satisfies this order.

In the next section we consider our leading examples **Mat**($\mathbb{R}_+$) and **CPM** and show that they are indeed precausal, so that the constructions Caus[**Mat**($\mathbb{R}_+$)] and Caus[**CPM**] give valid categories of 'higher order probability theory' and 'higher order quantum theory', respectively.

For now, we end this section with some remarks.

The type of a system is in general not unique. If $c_{\boldsymbol{X}} \subset c_{\boldsymbol{X}'}$ are sets of states which are both flat and closed, and $\rho : \boldsymbol{X}$ then $\rho : \boldsymbol{X}'$. In fact, for every invertible scalar λ, there is a poset of flat and closed sets of states with corresponding normalization, under inclusion. The set $\{\lambda \,\overline{\overline{\top}}\}$ is the bottom element and $\{\mu \,\overline{\overline{\top}}\}^*$ is the top element.

Related to the previous fact is this: the category Caus[$\mathcal{C}$] is constructed from $\mathcal{C}$ and it is often good to think about the objects of Caus[$\mathcal{C}$] as refinements of the objects of $\mathcal{C}$. Given a state ρ in $\mathcal{C}$, there is not just a single system in Caus[$\mathcal{C}$] to which it belongs. There could be many objects $\boldsymbol{X}_i$ in

Caus[$\mathcal{C}$] such that $\rho : \boldsymbol{X}_i$. Some of these types are related to certain causal structures as we will see in the next section.

Whenever we draw a diagram, we consider this diagram to be in the precausal category $\mathcal{C}$. We then say whether or not this diagram, seen as a state, satisfies some type in Caus[$\mathcal{C}$]. In particular, whenever we draw parallel wires, we always mean that this is the tensor of the systems involved, as this is the only monoidal structure in $\mathcal{C}$.

For a SMC, in particular a precausal category, $\mathcal{C}$, the homset of morphisms from the tensor unit to itself form a monoid $\mathcal{I} = \mathcal{C}(I, I)$. Let $\mathcal{M} \subset \mathcal{I}$ be a submonoid of this monoid. We can then define a dual with respect to this submonoid. For a set of states X on an system A,

$$X^* = \{\pi : A \to I \mid \forall x \in X, \pi \circ x \in \mathcal{M}\}$$

We then recover our original definition of dual by picking $\mathcal{M} = \{1\}$, but other choices are possible. For example, if $\mathcal{C}$ has a zero-object we might consider $\mathcal{M} = \{0, 1\}$ or in the examples of **CPM** or **Mat**$(\mathbb{R}_+)$ one could take $\mathcal{M} = [0, 1]$.

The construction of Caus[$\mathcal{C}$] from a precausal category $\mathcal{C}$ is, upto the flatness condition (Definition 3.3.6), an example of what is called double glueing [52]. In their language, we start with the double glued category glued along the Hom-functors $\mathcal{C}(I, -) : \mathcal{C} \to \mathbf{Set}$ and $\mathcal{C}(-, I)$ seen as a functor $\mathcal{C} \to \mathbf{Set}^{op}$. Objects of this category are triples (A, X, Y) with A an object in $\mathcal{C}$, $X \subset \mathcal{C}(I, A)$ and $Y \subset \mathcal{C}(A, I) \cong \mathcal{C}(I, A^*)$. We then define an *orthogonality*, $\perp$, on maps $I \to A$ and maps $A \to I$, where we say $\rho \perp \sigma$ precisely when $\sigma \circ \rho = 1$. Note that this is opposed to the 'usual' notion of orthogonality where we would say $\rho \perp \sigma$ if $\sigma \circ \rho = 0$. For a set $X \subset \mathcal{C}(I, A)$ we then define its *orthogonal* X^o as the set

$$X^o = \{\sigma \in \mathcal{C}(I, A) | \forall \rho \in X : \sigma \perp \rho\} \tag{3.12}$$

And similarly we obtain an orthogonal for effects. The *tight orthogonal subcategory* of this double glued category is then the full subcategory for which the objects satisfy $X^o = Y$ and $Y^o = X$. This is precisely our closedness condition (Definition 3.3.6) where $X^o := X^*$. Relating to the previous comment, we obtain an orthogonality by taking a set $F \subset \mathcal{C}(I, I)$ by setting $\rho \perp \sigma$ when $\sigma \circ \rho \in F$. This orthogonality is called *focussed with focus F*. In our case we therefore obtain a focussed tight orthogonality subcategory with focus $\{1\}$.

This double glueing construction gives rise to a nice possible application. Since our construction starts with a precausal category, which is compact closed, and gives a *-autonomous category which is not compact closed, we technically cannot iterate this construction. However, the double gluing construction can be iterated. Heuristically, this would give some doubly fine grained causal category whose objects are now triples (A, c, c') where (A, c) is an object in Caus$[\mathcal{C}]$ and $c' \subset c$ is another set of states. Morphisms $(A, c, c') \to (B, d, d')$ would then be morphisms $f : A \to B$ such that not only $f(c) \subset d$, but also $f(c') \subset d'$. The details and exact applications of this are left as future research.

Chapter 4

Higher order processes and causal orders

In the previous chapters we introduced the concept of a precausal category (Definition 3.1.1): a category which behaves well with regard to defining causal structures. We have seen how to turn a precausal category into a category of higher order processes in Section 3.3. Here we are going to study some actual causal structure and see how the type theory gives us information about this causal structure. We will identify several interesting types and relate them to causal orders.

4.1 First order systems

Let A be an object in a precausal category $\mathcal{C}$. Then the dual of the discard effect, $\{\,\bar{\top}_A\}^*$, is closed by Lemma 3.3.3 and flat because $\frac{1}{d_A}\,\underline{\perp} \in \{\,\bar{\top}_A\}^*$. Hence $(A, \{\,\bar{\top}_A\}^*)$ is an object of $\mathrm{Caus}[\mathcal{C}]$. This leads us to the following:

Definition 4.1.1. An object $\boldsymbol{A}$ is *first order* if it is of the form $(A, \{\,\bar{\top}_A\}^*)$.

First order systems are thus precisely those systems whose set of generalized causal states are the actual causal states. We think of first order systems as the canonical systems which come from the original precausal category. We make this precise now. Let $\mathcal{C}_c$ be the category whose objects are the (non-zero) objects of a precausal category $\mathcal{C}$, but whose morphisms are those morphisms of $\mathcal{C}$ that preserve discarding, i.e., the causal morphisms, so $\mathcal{C}_c$ is the causal subcategory of $\mathcal{C}$. Note that if $\mathcal{C}$ has a zero object 0, then id_0 is not causal, so 0 is not an object of $\mathcal{C}_c$. Then there is an identity-on-morphisms embedding $\mathcal{C}_c \hookrightarrow \mathrm{Caus}[\mathcal{C}]$ given by

$$A \mapsto (A, \{\bar{\top}_A\}^*)$$

Indeed:

Proposition 4.1.2. Let $\boldsymbol{A} = (A, \{\bar{\top}_A\}^*)$ and $\boldsymbol{B} = \{\bar{\top}_B\}^*)$ be first order systems in $\mathrm{Caus}[\mathcal{C}]$. Then a morphism $f : \boldsymbol{A} \to \boldsymbol{B}$ is a morphism in $\mathrm{Caus}[\mathcal{C}]$ if and only if $f : A \to B$ in $\mathcal{C}$ is causal.

Proof. Suppose f is a morphism in $\mathrm{Caus}[\mathcal{C}]$. By definition this means that $\rho \in \{\bar{\top}_A\}^* \Rightarrow f \circ \rho \in \{\bar{\top}_B\}^*$ and hence f is causal by Lemma 3.3.1. The other way around, suppose f is causal, then it preserves the causal states and hence is a morphism $\boldsymbol{A} \to \boldsymbol{B}$ in $\mathrm{Caus}[\mathcal{C}]$. □

So the 'causal part' of a precausal $\mathcal{C}$ category embeds in the category of higher order causal processes $\mathrm{Caus}[\mathcal{C}]$. The following shows that this embedding is monoidal.

Proposition 4.1.3. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be first order systems. Then $\boldsymbol{A} \otimes \boldsymbol{B}$ is again first order. That is

$$\boldsymbol{A} \otimes \boldsymbol{B} = (A \otimes B, \{\bar{\top}_A \bar{\top}_B\}^*)$$

Proof. By definition, $c_{\boldsymbol{A}\otimes\boldsymbol{B}}$ is the set of states $\rho : A \otimes B$ which are normalized for all effects π that normalize all product states on $A \otimes B$:

By Lemma 3.1.3 this implies that $\pi = \bar{\top}_{A\otimes B} = \bar{\top}_A \bar{\top}_B$. □

The above results can be summarized in the following:

Proposition 4.1.4. There exists a full, faithful and monoidal embedding of the category $\mathcal{C}_c$ of causal processes into $\mathrm{Caus}[\mathcal{C}]$ given by

$$A \mapsto (A, \{\bar{\top}_A\}^*) \qquad\qquad f \mapsto f$$

Corollary 4.1.5. For first order systems $\boldsymbol{A}$ and $\boldsymbol{B}$ we have

$$\boldsymbol{A} \otimes \boldsymbol{B} = \boldsymbol{A} \parr \boldsymbol{B}$$

Proof. This follows from a simple calculation

$$c_{\boldsymbol{A} \parr \boldsymbol{B}} = (c^*_{\boldsymbol{A}} \otimes c^*_{\boldsymbol{B}})^* = \{\bar{\top}_A \bar{\top}_B\}^* = c_{\boldsymbol{A} \otimes \boldsymbol{B}}$$

□

Up to now we have not seen any difference between the two monoidal products $\otimes$ and $\parr$. Only when we will consider higher order systems in the next section this difference will become apparent. First order systems are just state spaces and behave the expected way in Caus$[\mathcal{C}]$. The fact that first order systems are closed under tensor product should therefore not come as a surprise. The joint space of two state spaces is just a state space again. The fact that tensor and par are equal on first order spaces is somewhat more of a surprise, but it does explain why the par has not shown up that much in research regarding causality or state spaces before.

4.2 Higher order systems

In this section we will use the connectives $\otimes$, $\parr$ and $\multimap$, as well as the dual operation $(-)^*$, to construct some types in Caus$[\mathcal{C}]$ which represent causal orders. Notably, we will therefore also show that Caus$[\mathcal{C}]$ really contains more than just the first order segment.

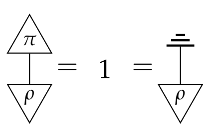
We begin with some simple types.

Lemma 4.2.1. Let $\boldsymbol{A}$ be a first order system. Then $\boldsymbol{A}^*$ is the system

$$\boldsymbol{A}^* = (A^*, \{\bar{\top}_A\})$$

where we see $\bar{\top}_A : A \to I$ as a state $\bar{\top}_A : I \to A^*$.

Proof. Let $\pi \in c_{\boldsymbol{A}^*} = c^*_{\boldsymbol{A}}$. Then for every $\rho \in \{\bar{\top}_A\}^*$ we have



So by (C3) we have $\pi = \bar{\top}_A$. □

Using Corollary 4.1.5, it follows that combining $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ for first order $\boldsymbol{A}, \boldsymbol{B}$ using tensor or par gives the same result.

Corollary 4.2.2. For first order $\boldsymbol{A}$ and $\boldsymbol{B}$ we have

$$\boldsymbol{A}^* \otimes \boldsymbol{B}^* = \boldsymbol{A}^* \parr \boldsymbol{B}^* = (\boldsymbol{A} \otimes \boldsymbol{B})^* = (\boldsymbol{A} \parr \boldsymbol{B})^*$$

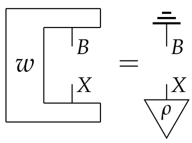
So starting from two first order systems $\boldsymbol{A}$ and $\boldsymbol{B}$ we have seen that $\boldsymbol{A} \otimes \boldsymbol{B} = \boldsymbol{A} \parr \boldsymbol{B}$ and this is again a first order system. Invoking the dual there are two ways of combining $\boldsymbol{A}^*$ with $\boldsymbol{B}$. We have already seen that for first order systems $\boldsymbol{A}$ and $\boldsymbol{B}$ states on the system $\boldsymbol{A} \multimap \boldsymbol{B}$ corresponds bijectively to morphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$ in $\mathrm{Caus}[\mathcal{C}]$. Furthermore, we know $\boldsymbol{A} \multimap \boldsymbol{B} = \boldsymbol{A}^* \parr \boldsymbol{B}$. Finally we can look at $\boldsymbol{A}^* \otimes \boldsymbol{B}$. We then have

$$\boldsymbol{A}^* \otimes \boldsymbol{B} = (\boldsymbol{A} \parr \boldsymbol{B}^*)^* = (\boldsymbol{B} \multimap \boldsymbol{A})^*$$

We therefore wish to know what the dual of a space of maps is. By (C4) we know this splits into a state and discard.

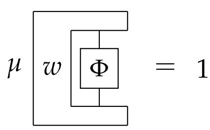
We can generalize this result.

Lemma 4.2.3. Let $\boldsymbol{X}$ be any system and let $\boldsymbol{B}$ be first order. Then for any process $w : (\boldsymbol{X} \multimap \boldsymbol{B})^*$ there exists a state $\rho : \boldsymbol{X}$ such that

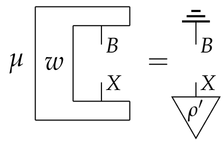


(4.1)

Proof. As $c_{\boldsymbol{X}}$ is flat, there is some invertible scalar μ such that $\mu \bar{\top}_X \in c^*_{\boldsymbol{X}}$. Then for any causal map $\Phi : X \to B$ in $\mathcal{C}$ we have $\bar{\top}_B \circ \mu\Phi = \mu \bar{\top}_X \in c^*_{\boldsymbol{X}}$. Hence $\mu\Phi : \boldsymbol{X} \multimap \boldsymbol{B}$. This implies that for $w : (\boldsymbol{X} \multimap \boldsymbol{B})^*$ we have

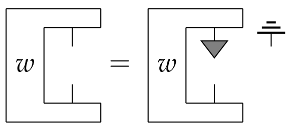


In other words, μw sends all causal maps to 1 and hence splits as some state ρ' and discard:



Now take $\rho = \mu^{-1}\rho'$ to conclude the proof. □

Recall that discarding the (normalized) maximal mixed state gives 1. Then we can give an explicit characterization of the state ρ in Lemma 4.2.3 just as in Lemma 3.1.5. The difference is that we are now working in Caus$[\mathcal{C}]$ instead of $\mathcal{C}$. Any $w : (\boldsymbol{X} \multimap \boldsymbol{A})^*$ splits as



(4.2)

We summarize the results so far in the following table:

System	States	Interpretation
$\boldsymbol{A}$	$\{\bar{\top}_A\}^*$	State space
$\boldsymbol{A}^*$	$\{\bar{\top}_A\}$	Discard effect
$\boldsymbol{A} \otimes \boldsymbol{B} = \boldsymbol{A} \parr \boldsymbol{B}$	$\{\bar{\top}_{A\otimes B}\}^* = \{\bar{\top}_A \bar{\top}_B\}^*$	Joint state space
$\boldsymbol{A}^* \otimes \boldsymbol{B}^* = \boldsymbol{A}^* \parr \boldsymbol{B}^*$	$\{\bar{\top}_{A\otimes B}\} = \{\bar{\top}_A \bar{\top}_B\}$	Joint discard effect
$\boldsymbol{A}^* \parr \boldsymbol{B} = \boldsymbol{A} \multimap \boldsymbol{B}$	$\{f : A \to B \mid \bar{\top}_B \circ f = \bar{\top}_A\}$	Causal maps
$\boldsymbol{A}^* \otimes \boldsymbol{B} = (\boldsymbol{B} \multimap \boldsymbol{A})^*$	$\{\rho_B \bar{\top}_A\}$	State and discard

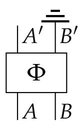
Admittedly, these types are not very interesting. They show nothing about internal structure of processes or causal ordering. The reason for this is of course that there are not enough systems involved to even consider causal orders between the inputs and outputs. Throughout the rest of this section we will consider some interesting causal orders for multipartite systems, but first we give an example of an object where there is only one system involved which is not first order.

Example 4.2.4. Consider the object $(\mathbb{C}^2, \{|0\rangle\langle 0|, \frac{1}{2}\underline{\perp}\}^{**})$ in Caus[**CPM**]. The generalized set of causal states of this object is, by construction, the smallest closed and flat set containing the state $|0\rangle\langle 0|$. Explicit calculation shows that $\{|0\rangle\langle 0|, \underline{\perp}\}^{**})$ is the set of convex combinations $p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$. Indeed, the dual of $\{|0\rangle\langle 0|, \underline{\perp}\}$ is the set containing the discard effect, the effects related to the vector $|0\rangle + e^{i\theta}|1\rangle$, and convex combinations of these. As such, this dual may be identified with the plane through the equator on the Bloch ball and contains more than just the discard effect, so our object is not first order.

As promised, we now consider how to combine two spaces of functions. For $\boldsymbol{A}, \boldsymbol{A}', \boldsymbol{B}$ and $\boldsymbol{B}'$ first order, we can combine $\boldsymbol{A} \multimap \boldsymbol{A}'$ and $\boldsymbol{B} \multimap \boldsymbol{B}'$ in (at least) two different ways; using tensor and using par. We will show that these ways of combining the function spaces lead to causal non-signalling processes and causal processes, respectively.

Theorem 4.2.5. For first-order systems $\boldsymbol{A}, \boldsymbol{A}', \boldsymbol{B}, \boldsymbol{B}'$, a process Φ is of type $(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$ if and only if it is causal and no-signalling.

Proof. First assume that $\Phi : (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$. Then, since discarding B' is causal, we can regard it as a morphism $\bar{\top}_{B'} : \boldsymbol{B}' \to \boldsymbol{I}$. Hence by functoriality of $\otimes$ and $\multimap$, we have:

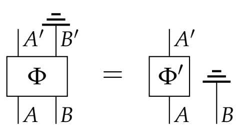


$: \ (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{I})$

Then we can transform to an equivalent type as follows:

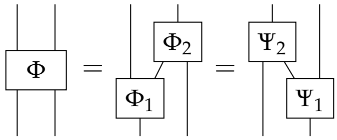
$$\begin{aligned}
(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{I}) &\cong (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes \boldsymbol{B}^* \\
&\cong ((\boldsymbol{A} \multimap \boldsymbol{A}')^* \parr \boldsymbol{B})^* \\
&\cong ((\boldsymbol{A} \multimap \boldsymbol{A}') \multimap \boldsymbol{B})^*
\end{aligned}$$

Hence, by Lemma 3.1.5, Φ splits as $\bar{\top}_B : \boldsymbol{B}$ and $\Phi' : \boldsymbol{A} \multimap \boldsymbol{A}'$. This gives exactly the first no-signalling equation:



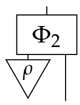
The second equation is shown similarly, by plugging in $\bar{\top}_{A'}$.

Conversely, suppose that Φ is causal and no-signalling. Then it satisfies the two no-signalling equations in Definition 2.2.2. Hence by (C4′), it can be factored in two ways:



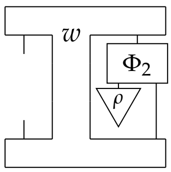
for causal processes Φ_i, Ψ_i.

Now, take any effect $w : ((\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}'))^* \cong (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap (\boldsymbol{A} \multimap \boldsymbol{A}')^*$. For any causal state ρ,



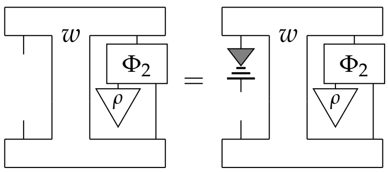
$: \boldsymbol{B} \multimap \boldsymbol{B}'$

Plugging this into one side of w gives:

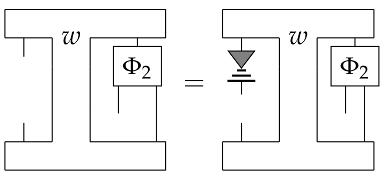


$: (\boldsymbol{A} \multimap \boldsymbol{A}')^*$

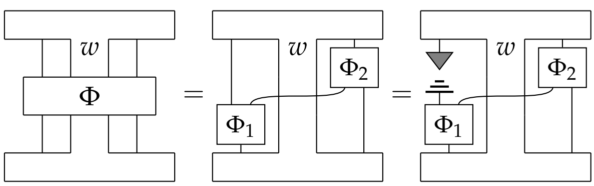
Applying equation (4.2) gives:

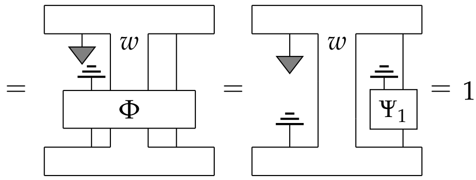


Hence by enough causal states we have



It then follows that





Therefore $\Phi : ((A \multimap A') \otimes (B \multimap B'))^{**} = (A \multimap A') \otimes (B \multimap B')$. $\square$

Recalling when a multipartite system is no-signalling from Section 2.2, we can extend this theorem to multipartite systems:

Corollary 4.2.6. For first order systems A_i, A'_i, $i = 1, \ldots, n$, a process $\Phi : A_1 \otimes \ldots \otimes A_n \to A'_1 \otimes \ldots \otimes A'_n$ is causal and no-signalling if and only if it is of type $(A_1 \multimap A'_1) \otimes \ldots \otimes (A_n \multimap A'_n)$.

Theorem 4.2.7. For first-order systems A, A', B, B', a process Φ is of type $(A \multimap A') \parr (B \multimap B')$ if and only if it is causal. That is:

$$(A \multimap A') \parr (B \multimap B') \cong A \otimes B \multimap A \otimes B'$$

Proof. We rely on the relationship between $\multimap$ and $\parr$:

$$\begin{aligned}
(A \multimap A') \parr (B \multimap B') &\cong A^* \parr A' \parr B^* \parr B' \\
&\cong A^* \parr B^* \parr A' \parr B' \\
&\cong (A^* \parr B^*)^* \multimap A' \parr B' \\
&\cong A \otimes B \multimap A' \parr B'
\end{aligned}$$

Then, since A' and B' are first-order, $A' \parr B' \cong A' \otimes B'$, which completes the proof. $\square$

Extending to multipartite maps we find:

Corollary 4.2.8. For first order systems A_i, A'_i, $i = 1, \ldots, n$, a process $\Phi : A_1 \otimes \ldots \otimes A_n \to A'_1 \otimes \ldots \otimes A'_n$ is causal if and only if it is of type $(A_1 \multimap A'_1) \parr \ldots \parr (A_n \multimap A'_n)$.

So the tensor of two function spaces correspond to all no-signalling causal maps whereas the par of these spaces contain *all* causal maps. We thus see tensor and par as the extremes to combine two systems. We either have no signalling between the subsystems or we impose no no-signalling conditions whatsoever.

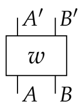
In the next section we will consider a type in between signalling and no-signalling: the one-way signalling processes. We will find the type corresponding to one-way signalling processes and their extension to multipartite systems, thereby developing the type theory for quantum combs.

4.3 One-way signalling and combs

From a physical point of view, one-way signalling is a notion which is in between no-signalling and any kind of signalling. Any process that is no-signalling definitely cannot signal in a particular direction and is thus one-way signalling, and any one-way signalling process is certainly a process. Diagrammatically we easily obtain the same conclusion. Indeed, the no-signalling condition is defined via one-way signalling conditions. One of the main goals of this section is to show that the type theoretic framework developed here gives us these inclusions for free.

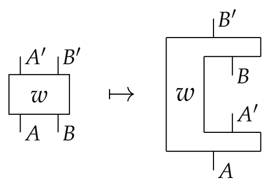
The first thing we will do now is find the type of one-way signalling processes.

Theorem 4.3.1. For first order systems $\boldsymbol{A}, \boldsymbol{A}', \boldsymbol{B}, \boldsymbol{B}'$, a process w is one-way signalling ($A \preceq B$) if and only if:

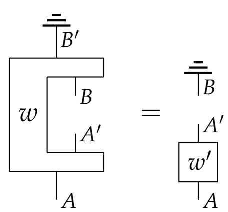


$$: \ \boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}'$$

Proof. Suppose w is one-way signalling with $A \preceq B$. First, for our convenience, we deform w in order to put the two A-labelled systems below the two B-labelled systems:

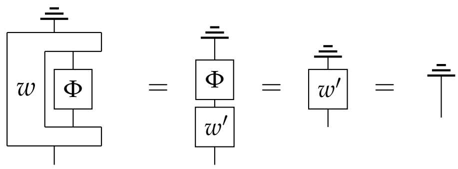


The one-way signalling equation (2.16) then becomes:



(4.3)

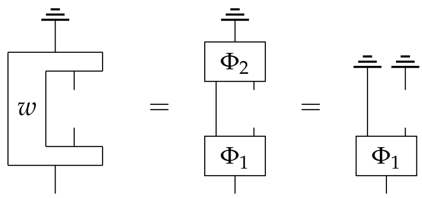
Now let $\Phi : \boldsymbol{A'} \multimap \boldsymbol{B}$ be any causal map. It then follows directly from equation (4.3) that w is second order causal:



So maps of type $\boldsymbol{A'} \multimap \boldsymbol{B}$ are sent to maps of type $\boldsymbol{A} \multimap \boldsymbol{B'}$. That is, $w : (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{A} \multimap \boldsymbol{B'}$. Finally, by (2.35) we have:

$$(\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{A} \multimap \boldsymbol{B'} \cong \boldsymbol{A} \multimap (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{B'}$$

Conversely, if w is of type $\boldsymbol{A} \multimap (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{B'}$, it sends causal processes to causal processes. Therefore it factorises as in (C4) and we thus find the one-way signalling equation (4.3) via:



where $w' = (\bar{\top} \otimes id) \circ \Phi_1$. □

Before we go on to multipartite one-way signalling processes, we summarize the results for bipartite processes in the following table:

Signalling conditions	Type
Causal no-signalling	$(\boldsymbol{A} \multimap \boldsymbol{A'}) \otimes (\boldsymbol{B} \multimap \boldsymbol{B'})$
Causal one-way signalling	$\boldsymbol{A} \multimap (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{B'}$
Causal	$(\boldsymbol{A} \multimap \boldsymbol{A'}) \parr (\boldsymbol{B} \multimap \boldsymbol{B'})$

Of course as expected, these types embed into each other and we can show this on the type theoretic level. To this end we make use of the linear distributivity property of $*$-autonomous categories (Proposition 2.4.4 and [28]). That is, in any $*$-autonomous category, there exists a canonical mapping:

$$(\boldsymbol{A} \parr \boldsymbol{B}) \otimes \boldsymbol{C} \to \boldsymbol{A} \parr (\boldsymbol{B} \otimes \boldsymbol{C}) \tag{4.4}$$

Proposition 4.3.2. There exists embeddings of no-signalling process into one-way signalling processes into causal processes:

$$\begin{array}{c}(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \\ \downarrow \\ \boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}' \\ \downarrow \\ (\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')\end{array}$$

Proof. For the embedding of no-signalling processes into one-way signalling processes we make use of the linear distributivity (4.4):

$$\begin{aligned}(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') &\cong (\boldsymbol{A}^* \parr \boldsymbol{A}') \otimes (\boldsymbol{B}^* \parr \boldsymbol{B}') \\ &\to \boldsymbol{A}^* \parr (\boldsymbol{A}' \otimes (\boldsymbol{B}^* \parr \boldsymbol{B}')) \\ &\to \boldsymbol{A}^* \parr (\boldsymbol{A}' \otimes \boldsymbol{B}^*) \parr \boldsymbol{B}' \\ &\cong \boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}'\end{aligned}$$

For the embedding of one-way signalling processes into causal processes we use the embedding of tensor into par:

$$\begin{aligned}\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}' &\cong \boldsymbol{A}^* \parr (S\boldsymbol{A}'^* \parr \boldsymbol{B})^* \parr \boldsymbol{B}' \\ &\cong \boldsymbol{A}^* \parr (\boldsymbol{A}' \otimes \boldsymbol{B}^*) \parr \boldsymbol{B}' \\ &\hookrightarrow \boldsymbol{A}^* \parr \boldsymbol{A}' \parr \boldsymbol{B}^* \parr \boldsymbol{B}' \\ &\cong (\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')\end{aligned}$$

□

Just as we transformed one-way signalling processes to processes with holes in them, we do the same in the multipartite case, which leads to a class of processes called *n-combs*, introduced in [24]. One-way signalling processes are such that a regular causal process can be plugged in. We now extend this idea to processes which take in $(n-1)$-combs and send them to causal processes. Hence we obtain the following inductive definition.

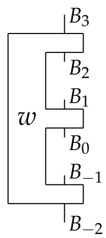
Definition 4.3.3. The *n-combs* C_n are defined by

- $C_0 = \boldsymbol{I}$,
- $C_{i+1} = \boldsymbol{B}_{-i} \multimap C_i \multimap \boldsymbol{B}_{i+1}$.

where all $\boldsymbol{B}_i$ are first order systems.

Note that our indexing of systems goes negative for 2-combs and higher order combs. There is no special meaning to this. It is just convenient to use the integers over the naturals in order to maintain the left-to-right ordering of indices. Because of the inductive nature of the definition of n-combs, we have also changed the names of the systems. This is again highly convenient since at every stage of n-combs, the inputs become outputs and *vice versa*. If need be, we can always switch back to the $\boldsymbol{A}_i$ and $\boldsymbol{A}'_i$ systems by renaming $\boldsymbol{A}_i := \boldsymbol{B}_{2i-n-1}$ and $\boldsymbol{A}'_i := \boldsymbol{B}_{2i-n}$.

A 1-comb has type $\boldsymbol{B}_0 \multimap \boldsymbol{I} \multimap \boldsymbol{B}_1 \cong \boldsymbol{B}_0 \multimap \boldsymbol{B}_1$, so it is just a causal process. A 2-comb has type $\boldsymbol{B}_{-1} \multimap (\boldsymbol{B}_0 \multimap \boldsymbol{B}_1) \multimap \boldsymbol{B}_2$, which we recognize as a one way signalling process, so viewing combs as generalization of one-way signalling processes makes sense. As an example of a new type, a 3-comb has type:



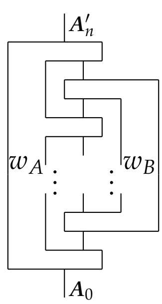
$$: \boldsymbol{B}_{-2} \multimap (\boldsymbol{B}_{-1} \multimap (\boldsymbol{B}_0 \multimap \boldsymbol{B}_1) \multimap \boldsymbol{B}_2) \multimap \boldsymbol{B}_3$$

Or rewritten in terms of the original system names:

$$\boldsymbol{A}_1 \multimap (\boldsymbol{A}'_1 \multimap (\boldsymbol{A}_2 \multimap \boldsymbol{A}'_2) \multimap \boldsymbol{A}_3) \multimap \boldsymbol{A}'_3$$

We see that something of this type sends one way signalling processes to causal maps, which was indeed the reason for the definition.

Combs are important in the context of communication protocols [47]. We can think of an $(n+1)$-comb as a protocol of a party with $n+1$ input/output steps. An n-comb is then an n step communication protocol for another party. Let Alice and Bob be two parties having access to an $(n+1)$-comb w_A and and n-comb w_B, respectively. Plugging them together

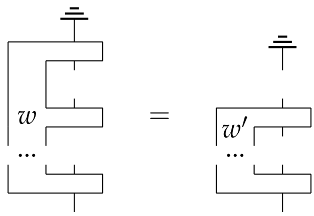


$$: \quad \boldsymbol{A}_0 \multimap \boldsymbol{A}'_n$$

gives a causal map from A_0 to A'_n. This resulting map, acting on a state on A_0 first sends it to a new state on A'_0 and this new state serves as the input for the comb of Bob. He transforms it again to state which serves as the input of the second stage Alice's comb. This goes on until eventually the end is reached and Alice outputs a state of type A'_n. The whole process was then an n-step communication protocol between Alice and Bob.

We introduced combs as a multipartite generalization of one-way signalling processes. However, the only justification for this, up to now, is that 2-combs are one-way signalling processes. The following results will show that combs are indeed one-way signalling processes.

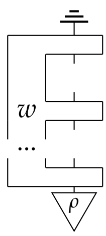
Lemma 4.3.4. Let $w : C_n$ be an n-comb. Discarding the output A'_n separates w as follows:



(4.5)

for some w':

Proof. Plugging any causal state into the first input of w and discarding the last output yields:

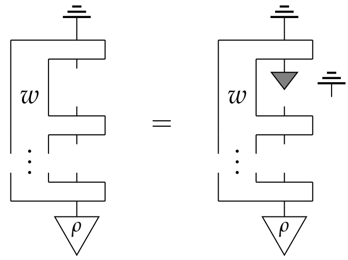


$: C_{n-1} \multimap \boldsymbol{I}$

We then calculate:

$$\begin{aligned} C_{n-1} \multimap \boldsymbol{I} \cong C^*_{n-1} &\cong (\boldsymbol{B}_{-(n-2)} \multimap C_{n-2} \multimap \boldsymbol{B}_{n-1})^* \\ &\cong (\boldsymbol{B}_{-(n-2)} \otimes C_{n-2} \multimap \boldsymbol{B}_{n-1})^* \end{aligned}$$

Hence by Lemma 4.2.3, in particular equation (4.2), we obtain:

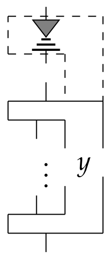


The result then follows from enough causal states. □

Note that we haven't actually said that w' is itself an $(n-1)$-comb. The following theorem will take care of this.

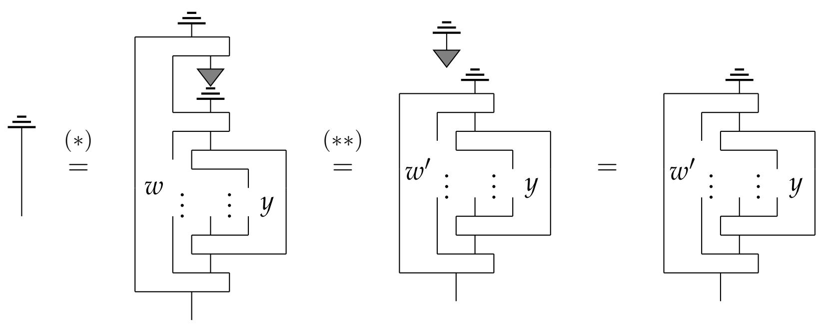
Theorem 4.3.5. A process w is an n-comb, i.e. $w : C_n$, if and only if it separates as in equation (4.5) for some $(n-1)$-comb $w' : C_{n-1}$.

Proof. By induction. For $n = 1$ the theorem is true because a 0-comb is always $\boldsymbol{I}$ by construction. Suppose the theorem is true for n. Let w be an $(n+1)$-comb. We need to show that w' is an n-comb. So let y be any $(n-1)$-comb. Then, if we form the process:



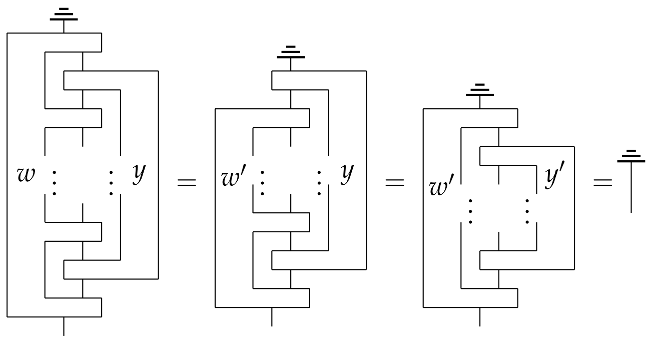
(4.6)

then clearly discarding the top output results in an $(n-1)$ comb (namely y) and a discard on the top input. So by the induction hypotheses, (4.6) is an n-comb. Therefore we have



where $(*)$ follows from the definition of $(n+1)$-comb and $(**)$ is Lemma 4.3.4. Hence w' sends any $(n-1)$-comb to a causal map, so w' is itself an n-comb.

Conversely, let w' in equation (4.5) be an n-comb, and take any n-comb y. Then by the induction hypothesis, discarding the top output of y separates as discarding and an $(n-1)$-comb y'. Hence:



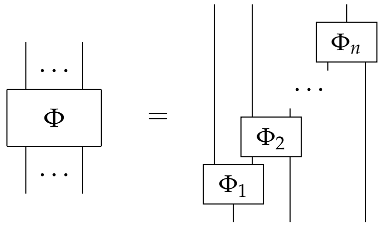
so w is an $(n+1)$-comb. □

Hence, n-combs can be characterised inductively in exactly the same way as n-party one-way signalling processes. Since 1-combs are just causal processes, the following is immediate.

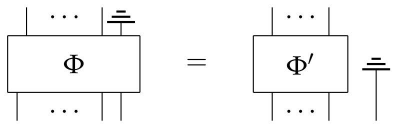
Corollary 4.3.6. For first order systems $\boldsymbol{A}_1, \boldsymbol{A}'_1, \ldots, \boldsymbol{A}_n, \boldsymbol{A}'_n$, a map $w : A_1 \otimes \ldots \otimes A_n \to A'_1 \otimes \ldots \otimes A'_n$ is one-way signalling ($A_1 \preceq \ldots \preceq A_n$) if and only if it is of type $\boldsymbol{A}_1 \multimap (\boldsymbol{A}'_1 \multimap (\ldots) \multimap \boldsymbol{A}_n) \multimap \boldsymbol{A}'_n$. That is, it is an n-comb.

We know that in precausal categories, bipartite one-way signalling processes factor as causal processes with memory. Indeed, this is axiom (C4$'$). Hence the same holds for 2-combs. We now show that any multipartite one-way signalling process, and hence any n-comb, factors in a similar way, which can be seen as yet another characterization of combs or one-way signalling processes.

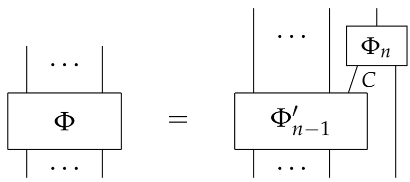
Proposition 4.3.7. Let Φ be one-way signalling with $A_1 \preceq \ldots \preceq A_n$, then there exists $\Phi_1, \ldots, \Phi_n$ such that



Proof. For $n = 2$, this is just (C4$'$). Suppose the proposition is true for $n-1$. Then because



for some one way signalling process Φ' with $A_1 \preceq \ldots \preceq A_{n-1}$, we have by (C4$'$) that there exists Φ'_{n-1} and Φ_n such that

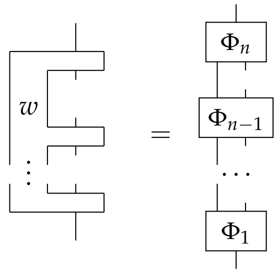


It follows that Φ' equals Φ'_{n-1} with the C system discarded, so that $\Phi'_{n-1} : A_1 \otimes \ldots \otimes A_{n-1} \to A'_1 \otimes \ldots \otimes (A'_{n-1} \otimes C)$ is again one-way signalling. By assumption Φ'_{n-1} now factors and hence so does Φ. □

So we have three characterizations of one-way signalling processes.

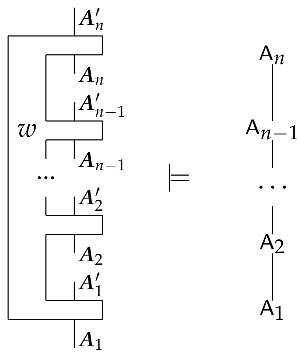
1. *Operational*: discarding the last output splits the process in a smaller process and a discard.
2. *Combs*: inductively defined by sending smaller combs to causal processes.
3. *Processes with memory*: a sequence of processes where the first process has both an output and a channel to the next process.

In this section we have shown that in a precausal category these three characterizations coincide. In particular, combs are given by channels with memory:



4.4 Causal orders given by DAGs

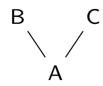
In Chapter 1 we introduced the notion of processes being compatible with a causal order given by a directed acyclic graph (DAG, Definition 1.1.1). In the previous section, we considered combs, which can be represented as channels with memory. As such, they represent a linear causal order.



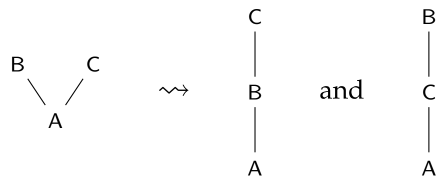
In this section we wish to extend the type theory from combs to DAGs. We will do this by considering totalizations of the DAG and then *intersecting* the types corresponding to these linear orders (Theorem 4.4.8). A crucial point to extending the type theory to DAGs is being able to 'refine' a partial order to a total order.

Definition 4.4.1. Let $\mathcal{G}$ be a DAG. A DAG $\mathcal{G}'$ is called a *totalization* of $\mathcal{G}$ if $\mathcal{G}$ and $\mathcal{G}'$ have the same vertices, $\mathcal{G}'$ is a total order (meaning that for any vertices e, e' in $\mathcal{G}'$, either $e \leq e'$ or $e' \leq e$) and $e \leq e'$ in $\mathcal{G}$ implies $e \leq e'$ in $\mathcal{G}'$.

Example 4.4.2. As a simple example, consider the DAG where A is before B and C, but B and C have no causal relation.



A totalization for this DAG is given by either putting B before C or *vice versa*. In either case, A comes before both B and C just as in the original DAG.

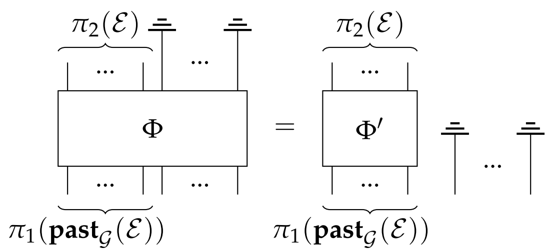


This example also shows that a totalization is not unique.

Now consider some process Φ and suppose it is consistent with a causal order $\mathcal{G}$. Then it is not hard to see that Φ is also consistent with any totalization of $\mathcal{G}$.

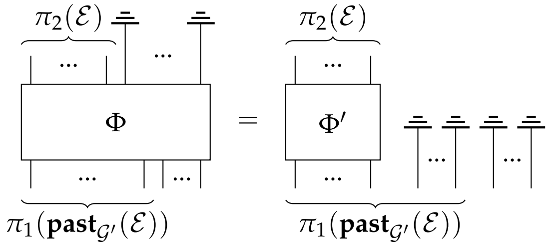
Lemma 4.4.3. For a process Φ, a causal ordering $\mathcal{G}$, and a totalization $\mathcal{G}'$ of $\mathcal{G}$, we have $\Phi \vDash \mathcal{G} \implies \Phi \vDash \mathcal{G}'$.

Proof. Assume $\Phi \vDash \mathcal{G}$. Then, for any subset $\mathcal{E} \subseteq \mathcal{G}$, there exists a process Φ' such that:



(4.7)

where π_1, π_2 are the projections on the input/output systems, respectively and $\mathbf{past}_{\mathcal{G}}(\mathcal{E})$ are the ancestors of $\mathcal{E}$ with respect to the DAG $\mathcal{G}$. But now, since $e \preceq_{\mathcal{G}} e' \implies e \preceq_{\mathcal{G}'} e'$, we have $\mathbf{past}_{\mathcal{G}}(\mathcal{E}) \subseteq \mathbf{past}_{\mathcal{G}'}(\mathcal{E})$. Hence (4.7) implies that Φ factors as required by Definition 1.1.1:

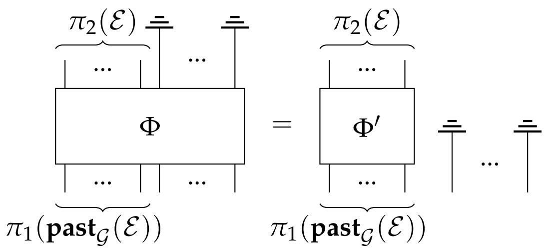


Therefore $\Phi \vDash \mathcal{G}'$. $\square$

The following theorem shows that the converse also holds and we can therefore describe causal orders given by DAGs by a collection of linear orders, namely the totalizations of the DAG.

Theorem 4.4.4. For a process Φ and a causal ordering $\mathcal{G}$, $\Phi \vDash \mathcal{G}$ if and only if, for every totalization $\mathcal{G}'$ of $\mathcal{G}$, $\Phi \vDash \mathcal{G}'$.

Proof. $(\Rightarrow)$ follows immediately from Lemma 4.4.3. For $(\Leftarrow)$, let $\mathcal{E} \subseteq \mathcal{G}$ be any subset. Split $\mathcal{G}$ into two parts, $\mathcal{G}_1 := \mathbf{past}_{\mathcal{G}}(\mathcal{E})$ and $\mathcal{G}_2 := \mathcal{G} \backslash \mathbf{past}_{\mathcal{G}}(\mathcal{E})$. Define a total ordering $\mathcal{G}'$ on $\mathcal{G}_1 \cup \mathcal{G}_2$ by requiring that every element in $\mathcal{G}_1$ is below every element in $\mathcal{G}_2$ and taking any totalization on $\mathcal{G}_1$ and $\mathcal{G}_2$. This order refines $\mathcal{G}$ because $\mathbf{past}_{\mathcal{G}}(\mathcal{E})$ is downward-closed, and by construction $\mathbf{past}_{\mathcal{G}}(\mathcal{E}) = \mathbf{past}_{\mathcal{G}'}(\mathcal{E})$. Hence $\Phi \vDash \mathcal{G}'$ implies:



Since we can find a suitable totalization to give the equation above for any subset $\mathcal{E} \subseteq \mathcal{G}$, we have $\Phi \vDash \mathcal{G}$. $\square$

So causal orders given by DAGs can equivalently be described as given by a collection of linear orders and we know from Section 4.3 what the type a of comb is. For example combs that satisfy the linear orders in Example 4.4.2 have the types

$$\boldsymbol{X} := \boldsymbol{A} \multimap (\boldsymbol{A}' \multimap (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap \boldsymbol{C}) \multimap \boldsymbol{C}' \tag{4.8}$$

and

$$\boldsymbol{X}' := \boldsymbol{A} \multimap (\boldsymbol{A}' \multimap (\boldsymbol{C} \multimap \boldsymbol{C}') \multimap \boldsymbol{B}) \multimap \boldsymbol{B}', \tag{4.9}$$

respectively.

Now since types correspond to sets of states, having multiple types should correspond to the intersection of these sets of states. This leads us to consider *intersection types* (also see [17]) such as $\boldsymbol{X} \cap \boldsymbol{X}' := (X, c_{\boldsymbol{X}} \cap c_{\boldsymbol{X}'})$ We make this precise by describing intersection types as pullbacks in Theorem 4.4.8.

We first need to take care of the fact that technically the different types of combs have different carriers, related by rearranging the systems with swaps. Since we can express the lolly, $\multimap$, in terms of $\parr$ and dual, it suffices to show the following lemma.

Lemma 4.4.5. Any object $\boldsymbol{X}$ which is built inductively from first order systems $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_n$ and duals of first order systems $\boldsymbol{A}_1^*, \ldots, \boldsymbol{A}_m^*$, using $\otimes$ and $\parr$ has a canonical embedding of the form:

$$e : \boldsymbol{X} \to (\boldsymbol{A}_1 \otimes \ldots \otimes \boldsymbol{A}_m \multimap \boldsymbol{B}_1 \otimes \ldots \otimes \boldsymbol{B}_n)$$

whose underlying $\mathcal{C}$-morphism is just a permutation of systems.

Proof. The proof is a straightforward application of the embedding $\boldsymbol{A} \otimes \boldsymbol{B} \hookrightarrow \boldsymbol{A} \parr \boldsymbol{B}$ (equation (3.11)) and the special property of first order systems that $\boldsymbol{A} \otimes \boldsymbol{B} \cong \boldsymbol{A} \parr \boldsymbol{B}$ (Corollary 4.1.5).

Given $\boldsymbol{X}$ built inductively from first-order types via the $*$-autonomous structure, $\otimes$, $\parr$, and dual, we can push the $(-)^*$ inside as far as possible via application of the following isomorphisms from left-to-right:

$$(\boldsymbol{A} \otimes \boldsymbol{B})^* \cong \boldsymbol{A}^* \parr \boldsymbol{B}^* \qquad (\boldsymbol{A} \parr \boldsymbol{B})^* \cong \boldsymbol{A}^* \otimes \boldsymbol{B}^*$$

We can then apply $\boldsymbol{A}^{**} \cong \boldsymbol{A}$ to reduce $\boldsymbol{X}$ to an expression consisting of either first-order types or their duals, combined with $\otimes$ and $\parr$.

We can then use the embedding $\boldsymbol{A} \otimes \boldsymbol{B} \hookrightarrow \boldsymbol{A} \parr \boldsymbol{B}$ to change all $\otimes$'s into $\parr$'s and use commutativity to permute all dual systems to the left. This gives the embedding

$$\begin{aligned}
\boldsymbol{X} &\hookrightarrow \boldsymbol{A}_1^* \parr \ldots \parr \boldsymbol{A}_m^* \parr \boldsymbol{B}_1 \parr \ldots \parr \boldsymbol{B}_n \\
&\cong (\boldsymbol{A}_1 \otimes \ldots \otimes \boldsymbol{A}_m)^* \parr (\boldsymbol{B}_1 \parr \ldots \parr \boldsymbol{B}_n) \\
&\cong (\boldsymbol{A}_1 \otimes \ldots \otimes \boldsymbol{A}_m) \multimap (\boldsymbol{B}_1 \parr \ldots \parr \boldsymbol{B}_n) \\
&\cong (\boldsymbol{A}_1 \otimes \ldots \otimes \boldsymbol{A}_m) \multimap (\boldsymbol{B}_1 \otimes \ldots \otimes \boldsymbol{B}_n)
\end{aligned} \tag{4.10}$$

We end by noting that all isomorphisms and embeddings arise from the identity morphism in $\mathcal{C}$. The only step that did not come from the identity were the permutations. □

In practice, one will often encounter a type $\boldsymbol{X}$ built up from first order systems A_i, where it is not a priori clear which systems are duals and which are not. Therefore it is helpful to consider an example.

Example 4.4.6. Consider the type $\boldsymbol{X} := \boldsymbol{A}_1 \multimap (\boldsymbol{A}_1' \multimap \boldsymbol{A}_2) \multimap \boldsymbol{A}_2'$. Getting rid of the lollies, $\multimap$, pushing the dual inwards and using $\boldsymbol{A}^{**} \cong \boldsymbol{A}$ we find

$$\begin{aligned}
\boldsymbol{X} &:= \boldsymbol{A}_1 \multimap (\boldsymbol{A}_1' \multimap \boldsymbol{A}_2) \multimap \boldsymbol{A}_2' \\
&\cong \boldsymbol{A}_1^* \parr ((\boldsymbol{A}_1')^* \parr \boldsymbol{A}_2)^* \parr \boldsymbol{A}_2' \\
&\cong \boldsymbol{A}_1^* \parr ((\boldsymbol{A}_1')^{**} \otimes \boldsymbol{A}_2^*) \parr \boldsymbol{A}_2' \\
&\cong \boldsymbol{A}_1^* \parr (\boldsymbol{A}_1' \otimes \boldsymbol{A}_2^*) \parr \boldsymbol{A}_2'
\end{aligned}$$

We can then use the embedding $\boldsymbol{A} \otimes \boldsymbol{B} \hookrightarrow \boldsymbol{A} \parr \boldsymbol{B}$ and permute the systems.

$$\begin{aligned}
\boldsymbol{X} &\cong \boldsymbol{A}_1^* \parr (\boldsymbol{A}_1' \otimes \boldsymbol{A}_2^*) \parr \boldsymbol{A}_2' \\
&\hookrightarrow \boldsymbol{A}_1^* \parr \boldsymbol{A}_1' \parr \boldsymbol{A}_2^* \parr \boldsymbol{A}_2' \\
&\cong \boldsymbol{A}_1^* \parr \boldsymbol{A}_2^* \parr \boldsymbol{A}_1' \parr \boldsymbol{A}_2'
\end{aligned}$$

Finally we bring the expression to its canonical form in the same way in (4.10).

We will now construct $\boldsymbol{X} \cap \boldsymbol{X}'$ essentially in terms of a set-theoretic intersection of their associated states $c_{\boldsymbol{X}}$ and $c_{\boldsymbol{X}'}$. The next lemma shows that this intersection of states is itself again flat and closed.

Lemma 4.4.7. Let c and d be sets of states for the same object A which are flat, closed, and furthermore satisfy the property that $\lambda \,\overline{\top} \in c$ and $\lambda \,\overline{\top} \in d$ for a fixed scalar λ. Then $c \cap d$ is also flat and closed.

Proof. For both properties, we rely on the fact that $(-)^*$ is order-reversing. That is, $a \subseteq b \Rightarrow b^* \subseteq a^*$. For flatness, we have by assumption that

$\lambda \underset{=}{\perp} \in c \cap d$. Since $c \cap d \subseteq c$, we have $c^* \subseteq (c \cap d)^*$. So by flatness of c, $\mu \overset{=}{\top} \in c^*$ for some μ. Hence $\mu \overset{=}{\top} \in (c \cap d)^*$ and $c \cap d$ is flat.

For closure, first note that any set of states is contained in its double dual, so we have $c \cap d \subseteq (c \cap d)^{**}$. For the converse, $c \cap d \subseteq c$ implies $c^* \subseteq (c \cap d)^*$ and similarly $d^* \subseteq (c \cap d)^*$. Hence, $c^* \cup d^* \subseteq (c \cap d)^*$, so $(c \cap d)^{**} \subseteq (c^* \cup d^*)^*$. It therefore suffices to show that $(c^* \cup d^*)^* \subseteq c \cap d$. This follows from the fact that $c^* \subseteq c^* \cup d^*$, so $(c^* \cup d^*)^* \subseteq c^{**} = c$ and similarly $(c^* \cup d^*)^* \subseteq d^{**} = d$. $\square$

The next theorem shows that we can obtain the intersection as a pullback.

Theorem 4.4.8. Let $\boldsymbol{X}, \boldsymbol{X}'$ be objects with canonical embeddings e, f into a fixed system $\boldsymbol{Y} := \boldsymbol{A}_1 \otimes \ldots \otimes \boldsymbol{A}_m \multimap \boldsymbol{A}'_1 \otimes \ldots \otimes \boldsymbol{A}'_n$, as in Lemma 4.4.5. Then there exists an object $\boldsymbol{X} \cap \boldsymbol{X}'$ and morphisms p_1, p_2 in $\mathrm{Caus}[\mathcal{C}]$ making the following pullback:

$$\begin{tikzcd}
\boldsymbol{X} \cap \boldsymbol{X}' \arrow[r, "p_1"] \arrow[d, "p_2"] & \boldsymbol{X} \arrow[d, "e"] \\
\boldsymbol{X}' \arrow[r, "f"] & \boldsymbol{Y}
\end{tikzcd} \tag{4.11}$$

Proof. Let $\boldsymbol{Y} := (Y, c_{\boldsymbol{Y}})$ and define the following two sets of states for $\boldsymbol{Y}$:

$$\bar{c}_{\boldsymbol{X}} := \{e \circ \rho \,|\, \rho \in c_{\boldsymbol{X}}\}$$
$$\bar{c}_{\boldsymbol{X}'} := \{f \circ \rho \,|\, \rho \in c_{\boldsymbol{X}'}\}$$

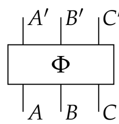
Since e and f are just permutations of systems, it is straightforward to show that both of these sets are flat, closed, and both contain $\lambda \underset{=}{\perp}$ for some fixed λ. Hence, applying Lemma 4.4.7, we have that $\bar{c}_{\boldsymbol{X}} \cap \bar{c}_{\boldsymbol{X}'}$ is flat and closed. Then, let $\boldsymbol{X} \cap \boldsymbol{X}' := (Y, \bar{c}_{\boldsymbol{X}} \cap \bar{c}_{\boldsymbol{X}'})$, $p_1 := e^{-1}$ and $p_2 := f^{-1}$. It is straightforward to check that p_1, p_2 are indeed $\mathrm{Caus}[\mathcal{C}]$-morphisms and diagram (4.11) clearly commutes.

It only remains to show that, for any $g : \boldsymbol{Z} \to \boldsymbol{X}$ and $h : \boldsymbol{Z} \to \boldsymbol{X}'$ such that $e \circ g = f \circ h$, there is a unique mediating morphism $z : \boldsymbol{Z} \to \boldsymbol{X} \cap \boldsymbol{X}'$:

$$\begin{tikzcd}
\boldsymbol{Z} \arrow[dr, dotted, "z"] \arrow[drr, "g"] \arrow[ddr, "h"'] & & \\
& \boldsymbol{X} \cap \boldsymbol{X}' \arrow[r, "e^{-1}"'] \arrow[d, "f^{-1}"] & \boldsymbol{X} \arrow[d, "e"] \\
& \boldsymbol{X}' \arrow[r, "f"] & \boldsymbol{Y}
\end{tikzcd}$$

Since e and f are isomorphisms, the only possibility is $z := e \circ g = f \circ h$. So, it suffices to show that z is a morphism in $\mathrm{Caus}[\mathcal{C}]$. For any $\rho \in c_{\mathbf{Z}}$, $g \circ \rho \in c_{\mathbf{X}}$, so $z \circ \rho = e \circ g \circ \rho \in \bar{c}_{\mathbf{X}}$. Similarly, $z \circ \rho = f \circ h \circ \rho \in \bar{c}_{\mathbf{X}'}$. Hence $z \circ \rho \in \bar{c}_{\mathbf{X}} \cap \bar{c}_{\mathbf{X}'}$, which completes the proof. $\square$

Since combs embed into all causal processes, it immediately follows that we can take intersections of combs via pullback. A process satisfying the types (4.8) and (4.9) related to Example 4.4.2 can thus equivalently be given the type



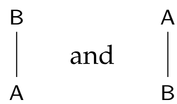
$$:$$

$$(\boldsymbol{A} \multimap (\boldsymbol{A'} \multimap (\boldsymbol{B} \multimap \boldsymbol{B'}) \multimap \boldsymbol{C}) \multimap \boldsymbol{C'}) \cap (\boldsymbol{A} \multimap (\boldsymbol{A'} \multimap (\boldsymbol{C} \multimap \boldsymbol{C'}) \multimap \boldsymbol{B}) \multimap \boldsymbol{B'})$$

and this generalises in the obvious way to any causal ordering given by a DAG. In particular, consider the following DAG related to no-signalling processes:

A   B

where there are no arrows between events A and B. The totalizations of this DAG:



are precisely those of one way signalling processes. Hence we have

Proposition 4.4.9. The intersection of the types of one-way signalling processes $(A \preceq B)$ and $(B \preceq A)$ is the type of no-signalling processes.

$$\boldsymbol{A} \multimap (\boldsymbol{A'} \multimap \boldsymbol{B}) \multimap \boldsymbol{B'} \cap \boldsymbol{B} \multimap (\boldsymbol{B'} \multimap \boldsymbol{A}) \multimap \boldsymbol{A'} \cong (\boldsymbol{A} \multimap \boldsymbol{A'}) \otimes (\boldsymbol{B} \multimap \boldsymbol{B'}) \quad (4.12)$$

Finally, we note a categorical consequence. The fact that $\cap$ arises as a pullback also gives us some properties of intersections 'for free'. For instance, any functor with a left adjoint necessarily preserves limits. By the definition of $*$-autonomous categories, $(\boldsymbol{B} \multimap -)$ has a left adjoint given by $(- \otimes \boldsymbol{B})$, so the following is immediate:

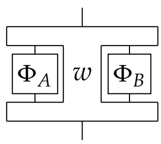
Corollary 4.4.10. For objects $\boldsymbol{A}, \boldsymbol{A'}$ and $\boldsymbol{B}$ in $\mathrm{Caus}[\mathcal{C}]$ we have

$$\boldsymbol{B} \multimap (\boldsymbol{A} \cap \boldsymbol{A'}) \cong (\boldsymbol{B} \multimap \boldsymbol{A}) \cap (\boldsymbol{B} \multimap \boldsymbol{A'})$$

4.5 Indefinite causal orders

In the previous section we considered combs, which represent linear causal orders. In particular, a process built up from causal maps in the form of a DAG, i.e., a circuit with holes, such as in (1.11), can be presented as a comb. In contrast, the quantum switch of Example 1.3.1 could not be given as a comb, or a linear combination thereof, and was said to have an indefinite causal structure. This leads us to consider what are called *process matrices*, introduced in [73], which are introduced as processes which take in two causal processes as input and output a new causal process.

Definition 4.5.1. A process $w : (A^* \otimes A') \otimes (B^* \otimes B') \to C^* \otimes C$ is called *bipartite second-order causal* (SOC_2) if for all causal Φ_A, Φ_B the following map is causal:



Such processes were called bipartite second-order causal in [65]. So SOC_2 maps send products of causal processes to a causal process. The following shows that SOC_2 processes are actually normalized on *all* non-signalling maps, not just product maps.

Theorem 4.5.2. For first order systems $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}', \mathbf{C}, \mathbf{C}'$, a process w is SOC_2 if and only if it is of type $(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') \multimap (\mathbf{C} \multimap \mathbf{C}')$.

Proof. Since products of causal processes are no-signalling, they have type $(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')$, so any process sending no-signalling processes to causal processes is certainly SOC_2.

For the converse, let π be an effect of type $(\mathbf{C} \multimap \mathbf{C}')^*$ (which by Lemma 4.2.3 is a state on $\mathbf{C}$ and discard on $\mathbf{C}'$). Then $\pi \circ w$ is an effect on products of causal processes. Now Lemma 3.3.11 states that $\pi \circ w$ yields 1 for product states if and only if it yields 1 for any state in the tensor product. Hence it is an effect for $(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')$, i.e., the no-signalling maps. By Lemma 3.3.9 this means $w : (\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') \multimap (\mathbf{C} \multimap \mathbf{C}')$. $\square$

This represents a significant strengthening of the result in [65], where it was shown that SOC_2 extends to strongly no-signalling processes, i.e., processes of the form (1.22).

Theorem 4.5.2 also extends naturally to a characterisation of n-partite second-order causal processes (SOC_n) via:

$$(\boldsymbol{A}_1 \multimap \boldsymbol{A}_1') \otimes \ldots \otimes (\boldsymbol{A}_n \multimap \boldsymbol{A}_n') \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$$

It is easy to see, either by using Theorem 4.5.2 or by an embedding as in Proposition 4.3.2, that the 3-combs, which arise from fixing a causal ordering between $\boldsymbol{A}$ and $\boldsymbol{B}$ are SOC_2:

$$\boldsymbol{C} \multimap (\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}') \multimap \boldsymbol{C}'$$
$$\boldsymbol{C} \multimap (\boldsymbol{B} \multimap (\boldsymbol{B}' \multimap \boldsymbol{A}) \multimap \boldsymbol{A}') \multimap \boldsymbol{C}'$$

However, the most interesting SOC_2 processes are those which do not arise from combs.

Definition 4.5.3. A process is said to have *definite causal order*, or to be *causally separable*, if it can be obtained as a convex combination of combs. Otherwise, we say it has *indefinite causal order*.

Note that in order to consider convex combinations, we need to assume that the interval $[0,1]$ is contained in $Hom(I,I)$ and that we can take sums, as is the case in our examples $\mathbf{Mat}(\mathbb{R}_+)$ and **CPM**. As such, throughout this section we will assume this is the case.

Let us now consider what the type theory developed here can tell us about these orders. We should start with a confession: while causal types are very useful to understand the signalling properties of processes, they do not distinguish between definite and indefinite causal orders. This might not come as a surprise, similar as to how we cannot distinguish strongly no-signalling processes from other no-signalling processes. That said, there are still some interesting observations to be made.

As a bit of shorthand notation, we introduce

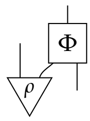
$$\begin{aligned} \boldsymbol{A} \preceq \boldsymbol{B} \quad &:= \quad \boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}' \\ \boldsymbol{B} \preceq \boldsymbol{A} \quad &:= \quad \boldsymbol{B} \multimap (\boldsymbol{B}' \multimap \boldsymbol{A}) \multimap \boldsymbol{A}' \end{aligned}$$

to denote the types of one-way signalling processes. We also consider their duals. For example, for the dual of $\boldsymbol{A} \preceq \boldsymbol{B}$, we have by equation (2.34)

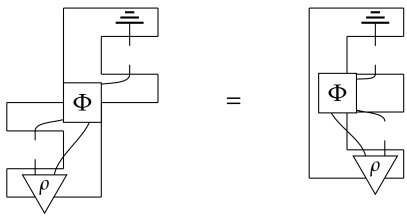
$$[\boldsymbol{A} \preceq \boldsymbol{B}]^* \quad = \quad [\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}']^* \tag{4.13}$$
$$\cong \quad [\boldsymbol{A} \otimes (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}']^* \tag{4.14}$$

Now by Lemma 4.2.3, every state related to such this object splits into a state of type $\boldsymbol{A} \otimes (\boldsymbol{A}' \multimap \boldsymbol{B})$ and discard on $\boldsymbol{B}'$.

We can then characterize states of type $\boldsymbol{A} \otimes (\boldsymbol{A}' \multimap \boldsymbol{B})$, by noting that by Proposition 4.2.5, these correspond precisely to no-signalling processes $(\boldsymbol{X} \multimap \boldsymbol{A}) \otimes (\boldsymbol{A}' \multimap \boldsymbol{B})$ with trivial input system $\boldsymbol{X} = \boldsymbol{I}$. Hence by (C4) these processes are of the form



for some causal state ρ and causal process Φ. We thus find that the process matrices compatible with the fixed causal order where Alice is before Bob are of the form



where we explicitly drew the higher order process to indicate that one party is before the other. This should come as no surprise as these are precisely 3-combs with trivial input and output. We find a similar expression for the dual of one-way signalling processes $\boldsymbol{B} \leq \boldsymbol{A}$ where Bob is before Alice. The union of these sets of states are then all process matrices which are compatible with either causal order, however, this set is not closed. We can close it by taking the double dual and obtain the object $(A^* \otimes A' \otimes B^* \otimes B', [c_{(\boldsymbol{B} \leq \boldsymbol{A})^*} \cup c_{(\boldsymbol{B} \leq \boldsymbol{A})^*}]^{**})$.

Recall that by Proposition 4.4.9 the type of no-signalling processes is the intersection $(\boldsymbol{A} \preceq \boldsymbol{B}) \cap (\boldsymbol{B} \preceq \boldsymbol{A})$. As a shorthand notation we will write $\mathbf{NS}$ for the type of no-signalling processes.

Theorem 4.5.4. The closure of the union of the duals of one-way signalling processes with either causal ordering equals the dual of the no-signalling processes. That is (with some abuse of notation):

$$[(\boldsymbol{A} \preceq \boldsymbol{B})^* \cup (\boldsymbol{B} \preceq \boldsymbol{A})^*]^{**} = \mathbf{NS}^*$$

Proof. For any closed sets of states c, d we have $c^* \subset (c^* \cup d^*)$. Since duals are order reversing, i.e., $a \subset b \Rightarrow b^* \subset a^*$, we have $(c^* \cup d^*)^* \subset c$ and $(c^* \cup d^*)^* \subset d$. Hence $(c^* \cup d^*)^* \subset c \cap d$.

Applying this to $c = \boldsymbol{A} \preceq \boldsymbol{B}$, $d = \boldsymbol{B} \preceq \boldsymbol{A}$ and taking the dual on both sides gives

$$\mathbf{NS}^* \subset [(\boldsymbol{A} \preceq \boldsymbol{B})^* \cup (\boldsymbol{B} \preceq \boldsymbol{A})^*]^{**}$$

For the converse inclusion, it suffices to show that

$$\mathbf{NS} \subset [(\boldsymbol{A} \preceq \boldsymbol{B})^* \cup (\boldsymbol{B} \preceq \boldsymbol{A})^*]^*$$

Again for any closed sets c, d we have $c \cap d \subset c$, so $c^* \subset (c \cap d)^*$. Similarly $d^* \subset (c \cap d)^*$, so $c^* \cup d^* \subset (c \cap d)^*$. Taking duals again we find $c \cap d \subset (c^* \cup d^*)^*$. Again taking $c = \boldsymbol{A} \preceq \boldsymbol{B}$ and $d = \boldsymbol{B} \preceq \boldsymbol{A}$, we find the desired result. $\square$

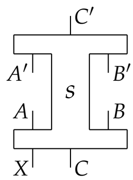
In other words, the smallest closed set which contains the process matrices which are compatible with either causal order is precisely the set of all process matrices. In the case of (finite dimensional) quantum theory, where the closure corresponds to affine combinations ([17]), this shows that any process matrix is an affine combination of process matrices which are compatible with a fixed causal structure. In particular, this contains the process matrices which are a *convex* combination of causal orders, i.e., the causally separable process matrices, but also contains more, such as the OCB W-matrix from [73] (Example 4.5.8), related to the guess your neighbours input game of Section 1.3. This situation is reminiscent of the difference between general combinations of product states opposed to convex combinations of product states, leading to the difference between entangled and disentangled states. This leads us to informally say:

$$\frac{\text{definite}}{\text{indefinite}} \simeq \frac{\text{separable}}{\text{entangled}}$$

This correspondence is taken a step further in [13] where a tripartite higher order process inspired by the W-state is given which also breaks a causal bound. We consider this process in Example 4.5.9. Furthermore, just as there are witnesses for entanglement, there are also witnesses for indefinite causal order ([8]). These witnesses can be thought of to 'pick out' the negative component in the decomposition in combs It should be noted that these witnesses are general hermitian operators, so one needs to go outside of the category **CPM**.

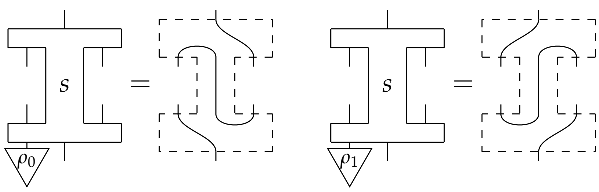
Let us consider some examples of processes with indefinite causal order. To start, we come back to the switch of Example 1.3.1. There we considered it as a higher order quantum process, here we consider it more general and give the switch in both $\mathbf{Mat}(\mathbb{R}_+)$ as a higher order stochastic map and in **CPM** as higher order quantum channel.

Definition 4.5.5. For first-order systems $\boldsymbol{X}$ and $\boldsymbol{A} = \boldsymbol{A}' = \boldsymbol{B} = \boldsymbol{B}' = \boldsymbol{C} = \boldsymbol{C}'$, a *switch* is a process of type:



$$: \boldsymbol{X} \otimes \boldsymbol{C} \multimap (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap \boldsymbol{C}' \tag{4.15}$$

in Caus$[\mathcal{C}]$, such that for distinct states $\rho_0, \rho_1 : \boldsymbol{X}$, we have:

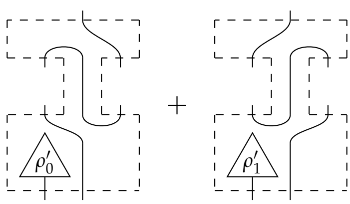


(4.16)

Example 4.5.6. For $\mathcal{C} = \mathbf{Mat}(\mathbb{R}_+)$, the *classical switch* process is uniquely fixed by (4.16) if we let $\boldsymbol{X} = 2$ and:

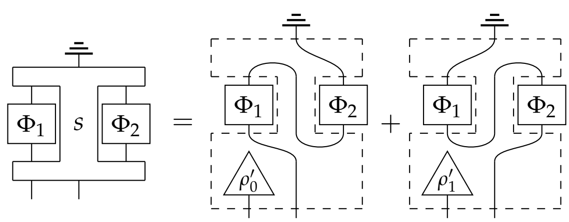
$$\rho_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \rho_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

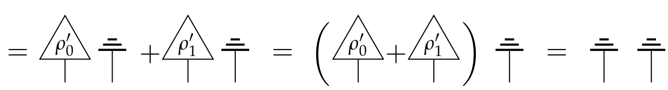
Indeed, s is given by:



(4.17)

where $\rho_i' := \rho_i^T$. Then, since $\rho_0' + \rho_1' = \top$, we have:





Hence s has the correct type shown in (4.15).

Example 4.5.7. For $\mathcal{C} = \mathbf{CPM}$, a switch can be defined similar to (4.17), by letting $\boldsymbol{X} = \mathcal{L}(\mathbb{C}^2)$, the (bounded) linear transformations on $\mathbb{C}^2$, and replacing ρ_i and ρ_i^T with the appropriate qubit projections and their associated quantum effects:

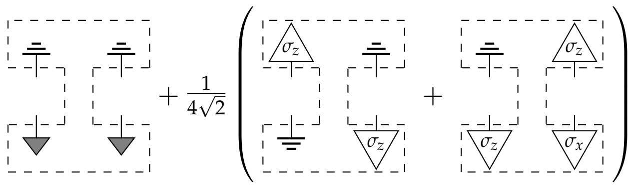
$$\rho_i := |i\rangle\langle i| \qquad\qquad \rho_i'(-) := \mathrm{Tr}(|i\rangle\langle i| -)$$

This is precisely the $\mathcal{Z}$ superoperator defined in [23], which defines a (decoherent) switch for quantum channels. As noted in Section 1.3, the quantum switch is not fully specified by (4.16), since ρ_0, ρ_1 do not form a basis for $\mathcal{L}(\mathbb{C}^2)$.

One can also define a *coherent* quantum switch which also satisfies (4.16), but where inputting the state $|+\rangle\langle +|$ into $\boldsymbol{X}$ yields a quantum superposition of causal orderings. See [23] for details.

Next we will take a look at the process leading to violation of the classical bound in the guess your neighbours input game.

Example 4.5.8. The *OCB process* is defined as follows:

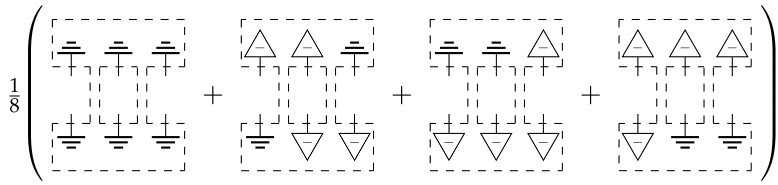


where σ_x, σ_z are Pauli matrices and associated effects. While the individual summands are not positive, the result is, thus yielding a process in **CPM**. The fact that it is an SOC_2 process in Caus[**CPM**] follows straightforwardly from the fact that the Pauli matrices are trace-free.

It is shown in [73] that this process indeed breaks a causal bound. In a nutshell, the idea is that one of the parties can, as it were, can choose a causal ordering between the parties *a posteriori* by a choice of quantum measurement.

Finally, we consider a probabilistic higher order process without definite causal structure.

Example 4.5.9. Not all processes exhibiting indefinite causal order are quantum. Indeed the following process:



is an SOC_3 process in $\mathrm{Caus}[\mathbf{Mat}(\mathbb{R}_+)]$, where the '$-$' labelled states and effects are column vectors and row vectors with values $(1, -1)$, respectively. It was shown in [13] that this process, as well as a generalisation to an SOC_n process for odd n, is incompatible with any pre-defined causal order.

We end with two remarks linking our framework to recent research regarding causality. First, we consider transformations of process matrices. General relativity tells us that time slows down near massive bodies (e.g., a black hole). In a theory of quantum gravity ([50]), such a massive body could be in a superposition of positions. A clock following a path near these positions would therefore undergo a superposition of time dilation. We then have something similar to the switch, depending on the state of a system, there is a different causal background. If the mass distribution now also changes over time, the process induced by this mass distribution also changes. This leads one to consider transformations of process matrices ([36]). In the case where there is no input or output, such a transformation is of type

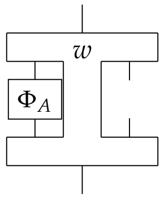
$$[(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')]^* \multimap [(\boldsymbol{C} \multimap \boldsymbol{C}') \otimes (\boldsymbol{D} \multimap \boldsymbol{D}')]^* \tag{4.18}$$

But then it is clear, using $(X^* \multimap Y^*) \cong (Y \multimap X)$, that such transformations are equally described by processes of type

$$(\boldsymbol{C} \multimap \boldsymbol{C}') \otimes (\boldsymbol{D} \multimap \boldsymbol{D}') \multimap (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$$

Hence this simple application of the type theory tells us that transformations of process matrices are just transformations of the no-signalling maps on which they act.

Second, consider a process matrix and think of the inputs as labs for Alice and Bob. Once they choose which process they use as input, the result is a causal process from the overall input to the overall output. We can then wonder what Alice and Bob themselves see. If Alice chooses some process Φ_A, then Bob sees the following process:



(4.19)

Now w is of type $[(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')] \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$ and $\Phi_A : \boldsymbol{A} \multimap \boldsymbol{A}'$. Hence the result for Bob is of type $(\boldsymbol{B} \multimap \boldsymbol{B}') \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$, i.e., a one way signalling processes, which factors as in Axiom (C4) of a precausal category. See [72] for a more detailed description.

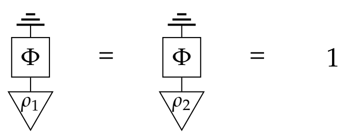
Chapter 5

Conclusion and future work

In order to solve the problem that compact closed categories are insufficient to study higher order causality, we have given a categorical construction which takes a precausal category, $\mathcal{C}$, to a category of higher order causal processes, Caus$[\mathcal{C}]$. This new category has enough fine-grained structure that its objects form a type theory describing causal types. We have shown, using the *-autonomous structure of Caus$[\mathcal{C}]$, how types for causal structures given by directed acyclic graphs can be obtained in this type theory. This includes causal processes, one-way signalling and no-signalling processes. As a special case we have been able to give two characterizations of combs, inductive and operational. Finally, we considered processes exhibiting indefinite causal order.

All types related to causal orders have been built up from first order systems, the *-autonomous structure and taking pullbacks. However, there are objects with closed and flat sets of generalized causal states, which are not of this form (Example 4.2.4). It is an interesting question to what extent these objects have a physical or causal meaning. For example, a map from an object X into the object of Example 4.2.4 sends every state in c_X to a state that is a convex combination of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, which can be seen as a probability distribution on two points. Also interesting is the relation with multiplicative linear logic. The category Caus$[\mathcal{C}]$ is *-autonomous and *-autonomous categories are a model of multiplicative linear logic (MLL). Hence the logic of higher order causal categories is given by MLL. This opens up a toolbox, including automation with linear logic provers such as `llprover` ([90]). Such a prover allows for easy automatic checks of for example whether two types are equal or whether one type is contained in an other. Furthermore, we have shown in Theorem 4.5.4 that if a type contains duals of one way signalling processes, it automatically contains all duals of no-signalling processes. This means that there are no types which explicitly capture indefinite causal order. Nevertheless, using the framework of causal types, we can say when a process has a particular definite causal type, or is the dual thereof. Hence we can show that a process is *not* of a certain causal order if it does not have the corresponding type. This way we can at least rule out certain processes having indefinite causal order. We may say that types can impose definite causal order, but not indefinite causal order. Hopefully the causal framework in this thesis can shed some more light on the nature of indefinite causal structures.

As a further topic, we mention the study of the category of precausal categories and higher order causal categories as well as the relation between them. An open question in this area is the precise relation to the double glueing construction we mentioned at the end of Section 3.3. As mentioned there, closedness comes very natural, but flatness needs to be additionally imposed. Considering the other axioms for a precausal category, it is also interesting to mention enough causal states ((C3)) when it comes to iterating the Caus construction. Indeed, a precausal category has enough causal states, but in the category of higher order processes this is not the case any more. We can see this by considering second order effects as in (C5′). For every causal maps Φ, we have that for causal states ρ_1 and ρ_2, that



Hence the states on $\boldsymbol{A} \multimap \boldsymbol{B}$ cannot distinguish between different maps of type $\boldsymbol{A} \multimap \boldsymbol{B} \to \boldsymbol{I}$. Even more elementary, since the only causal process from a first order object $\boldsymbol{A}$ to $\boldsymbol{I}$ is the discard effect, there are already not enough causal states in this simple scenario.

Further questions are, for example, how functors between precausal categories relate to functors between the higher order categories. Given precausal categories $\mathbb{X}$ and $\mathbb{Y}$ there are embeddings of the causal subcategories of $\mathbb{X}$ and $\mathbb{Y}$ into Caus$[\mathbb{X}]$ and Caus$[\mathbb{Y}]$, respectively (Section 4.1). A functor $F : \mathbb{X} \to \mathbb{Y}$ now restricts to a functor between these causal subcategories. We can then wonder whether this restricted functor lifts to a functor between Caus$[\mathbb{X}]$ and Caus$[\mathbb{Y}]$. Similarly, the question arises whether every functor between Caus$[\mathbb{X}]$ and Caus$[\mathbb{Y}]$ comes from a functor between the underlying precausal categories $\mathbb{X}$ and $\mathbb{Y}$. A third question, also in a similar vain, is which isoMIX *-autonomous categories can be obtained via the Caus construction, starting with a precausal category. A possible first step here, as well as interest in its own right, lies in finding more examples of precausal categories and their higher order equivalents. For example, what about $\mathbf{Mat}(\mathbb{R})$ (Example 3.2.4)?

Finally we mention the question whether a category of higher order causal processes can have a non-trivial *core* ([18]). That is, the core of a *-autonomous category are those objects A for which $A \otimes - \cong A \parr -$. It is clear that in the case of a category Caus$[\mathcal{C}]$, the unit $\boldsymbol{I}$ is in the core. The question is then whether there can be other objects in the core. We know that for first order objects $\boldsymbol{A}, \boldsymbol{B}$ we have $\boldsymbol{A} \otimes \boldsymbol{B} \cong \boldsymbol{A} \parr \boldsymbol{B}$ (Corollary 4.1.5), however, this is not enough for $\boldsymbol{A}$ to be in the core as this should hold for all objects $\boldsymbol{B}$, not just first order objects. As a necessary condition for such an object to be in the core we note that that we must then have $(\boldsymbol{A} \multimap \boldsymbol{A})^* \cong \boldsymbol{A} \otimes \boldsymbol{A}^* \cong \boldsymbol{A} \parr \boldsymbol{A}^* \cong (\boldsymbol{A} \multimap \boldsymbol{A})$.

Part II

Non locality and contextuality

Chapter 6

Introduction: non-locality and contextuality

When measuring a quantum observable on a state which is not in an eigenstate of that observable, the outcome of this measurement cannot be predicted with certainty by quantum theory. All we can do is find the probabilities with which the possible outcomes occur. A natural question which then arises is whether this is an artifact of the theory or a fact of life. Indeed, it is easy to imagine that quantum mechanics is at its core an incomplete theory and that there is some underlying theory which is able to predict all outcomes of all measurements. Perhaps a theory where we replace the quantum mechanical states (wave-functions or density matrices) by some other object which carries *all* the information of a system, including outcomes of measurements. We call such an object a *hidden variable* and the idea that the outcomes of all measurements can be predicted *realism*. An immediate question is then whether, or to what extent, a realist quantum theory is possible. Over the past decades there have been several *no-go* results which put severe boundaries on such realist theories. Our goal in this chapter is to consider the two most influential no-go theorems - by Bell and Kochen-Specker - and see how they are related. In a sentence, we may say that the heart of these no-go theorems is some 'paradox' regarding the non-existence of a joint distribution of outcomes of measurements whereas the partial distributions do exist. We then use this general point of view to develop an algebraic framework to describe these paradoxes and relate this method to other methods using adjunctions.

6.1 Bell's theorem and locality

Consider two parties, Alice and Bob, each in their own lab, sharing a bipartite state as in Chapter 1. They are each given a measurement device which has two settings; a_0, a_1 for Alice and b_0, b_1 for Bob. When they get their part of the bipartite state, say an entangled qubit pair, they can each perform one *and only one* of these measurements on their system and each then obtains some outcome, say 0 or 1. This situation is precisely the situation depicted in diagram (1.22), where the local inputs are used to decide the measurement settings and the outputs are the classical outcomes. Repeating this experiment many times and recording the outcomes given the settings gives rise to some probability distributions over the measurement settings and outcomes of Alice and Bob. These statistics can be put into a table which might look like this:

	$a_0{:}0$	$a_0{:}1$	$a_1{:}0$	$a_1{:}1$
$b_0{:}0$	$1/2$	0	$3/8$	$1/8$
$b_0{:}1$	0	$1/2$	$1/8$	$3/8$
$b_1{:}0$	$3/8$	$1/8$	$1/8$	$3/8$
$b_1{:}1$	$1/8$	$3/8$	$3/8$	$1/8$

(6.1)

where for example $a_0{:}1$ stands for 'Alice chose setting a_0 and got outcome 1. Equivalently, we might set the outcomes out against the measurement settings:

	$0,0$	$0,1$	$1,0$	$1,1$
a_0b_0	$1/2$	0	0	$1/2$
a_0b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_0	$3/8$	$1/8$	$1/8$	$3/8$
a_1b_1	$1/8$	$3/8$	$3/8$	$1/8$

(6.2)

Note that the rows of this table add to one, as they should since they form probability distributions. In fact, this particular table can actually be obtained by doing measurements on qubits (see Definition 8.3.1). This result is well known, but for completeness, we will show how to obtain it.

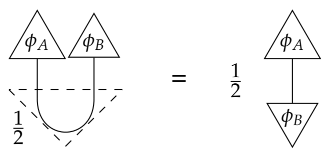
Proposition 6.1.1. The probabilities of table (6.2) can be obtained by performing measurements on a quantum system. That is, there exist a state and POVMs such that the corresponding probabilities make up table (6.2).

Proof. Consider the bipartite maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$. The corresponding density matrix is, up to normalization and order of subsystems, the cup from Example 2.1.14. The probability that Alice and Bob get some particular joint outcome can then be found by applying the POVMs corresponding to:

$$
\begin{array}{rcl}
\mathrm{a}_0 & = & \{|0\rangle\langle 0|, |\pi\rangle\langle\pi|\}, \\
\mathrm{a}_1 & = & \{|\pi/3\rangle\langle\pi/3|, |4\pi/3\rangle\langle 4\pi/3|\}, \\
\mathrm{b}_0 & = & \{|0\rangle\langle 0|, |\pi\rangle\langle\pi|\}, \\
\mathrm{b}_1 & = & \{|-\pi/3\rangle\langle -\pi/3|, |2\pi/3\rangle\langle 2\pi/3|\}
\end{array}
$$

where $|\theta\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ (so that by some mixture of notation $|0\rangle = |0\rangle$ and $|\pi\rangle = |1\rangle$) and where the first effect corresponds to outcome 0 and the second to outcome 1.

Using the diagrammatic notation, we can now calculate the probability of a particular joint outcome as



where the factor $\frac{1}{2}$ is the normalisation of the cup. This expressions comes down to

$$
\begin{array}{rcl}
\frac{1}{2}(\langle\theta_A|\,|\theta_B\rangle)^2 & = & \frac{1}{2}(\cos(\phi_A/2)\cos(\phi_B/2) + \sin(\phi_A/2)\sin(\phi_B/2))^2 \\
 & = & \frac{1}{2}\cos^2(\frac{\phi_A - \phi_B}{2})
\end{array}
$$

Plugging in the angles then gives the above table. □

Now suppose that, in contrast to the situation above, we would be able to know the outcome of all measurements at the same time. That is, suppose some realist point of view. Then we would consider a state space S_A for Alice consisting of maps $\{\mathrm{a}_0, \mathrm{a}_1\} \to \{0, 1\}$ mapping settings to outcomes and a similar state space S_B for Bob. The product space $S_A \times S_B$ is then the space of all outcomes for these measurements. Let us now fix some notation: if Alice chooses setting a_i and obtains outcome j, we denote this by $\mathrm{a}_i{:}j$. Similarly, we have $\mathrm{b}_k{:}l$ if Bob chooses setting k and obtains outcome l. We denote the occurrence of both events as $\mathrm{a}_i{:}j \wedge \mathrm{b}_k{:}l$. A hidden variable in this case can then be seen as the occurrence of the four events at once and we denote these by $\mathrm{a}_0{:}i \wedge \mathrm{a}_1{:}j \wedge \mathrm{b}_0{:}k \wedge \mathrm{b}_1{:}l$.

Now suppose that in a particular run of the experiment Alice chose setting a_0 and obtained outcome 0, whereas Bob chose b_1 and got 1. That is, they find $a_0{:}0 \wedge b_1{:}1$. Then, if we assume that these outcomes come from an underlying hidden variable, we know that it must have been one of the following ones: $a_0{:}1 \wedge a_1{:}j \wedge b_0{:}k \wedge b_1{:}l$ for some $j, k \in \{0,1\}$. However, we do not exactly know which one. Likewise, suppose that just as in the Bell table the situation $a_0{:}0 \wedge b_0{:}1$ never occurs. Then all the hidden variables $a_0{:}0 \wedge a_1{:}j \wedge b_0{:}1 \wedge b_1{:}k$ for any $j, k \in \{0,1\}$, must have probably 0.

We can then wonder if there is some probability distribution over all hidden variables, such that we recover the measurement statistics related to some given table. Moreover, we want this probability distribution not to depend on the actual settings Alice and Bob choose to measure. This is because we assume that Alice and Bob can choose their settings at the very last moment. If the hidden variable is to depend on the settings, it should either know these settings beforehand, which implies some sort of super determinism, or information about the settings should be able to travel instantaneously, in contrast with special relativity and causality as in Section 1.4. Not only must the hidden variables not depend on the measurement settings, the outcomes of Alice and Bob must also be independent of the other's settings for the same reason.

Let us spell out what this independence of choice of measurements means here. Suppose that Alice chooses her setting a_0. Then the probability that she obtains some outcome should be the same regardless of Bob's choice of measurement. That is, marginalizing over Bob's outcomes must be independent of Bob's settings:

$$P(a_0{:}0 \mid b_0{:}0) + P(a_0{:}0 \mid b_0{:}1) \quad = \quad P(a_0{:}0 \mid b_1{:}0) + P(a_0{:}0 \mid b_1{:}1) \tag{6.3}$$
$$P(a_0{:}1 \mid b_0{:}0) + P(a_0{:}1 \mid b_0{:}1) \quad = \quad P(a_0{:}1 \mid b_1{:}0) + P(a_0{:}1 \mid b_1{:}1) \tag{6.4}$$

And similar equations hold for the other settings of Alice and Bob. This no-signalling principle is also often called *locality*. Let us consider what happens when these equations are not satisfied.

Example 6.1.2. Suppose that the following table describes some measurement statistics:

	0,0	0,1	1,0	1,1
a_0b_0	1/2	0	1/2	0
a_0b_1	1/2	0	1/2	0
a_1b_0	0	1/2	0	1/2
a_1b_1	0	1/2	0	1/2

(6.5)

If Alice now chooses setting a_0, Bob will always get the outcome 0, whereas if Alice chooses a_1, Bob will get outcome 1 with certainty. This means that Alice can *instantaneously* send a bit of information to Bob by choosing a measurement setting.

This brings us back to the situation of the causal diagram (1.3); a probability distribution over some local hidden variables in the common past the measurements of Alice and Bob. In Proposition 8.3.7 we will see that the above Bell table indeed cannot be obtained by a probability distribution over the hidden variables, thus showing that:

A local hidden variable theory cannot explain the measurement statistics of quantum mechanics!

This result was first discovered by John Bell, and is often called *Bell's theorem* [15]. For completeness we note that Bell's theorem is *not* a statement about hidden variables. It is a statement about *local* hidden variables. Indeed, if we allow non-local hidden variables, we may just take the quantum state to be the hidden variable, in which case we obviously obtain the statistics of any quantum measurements. Another option is Bohmian mechanics, where the initial state of a system may be seen as a non-local hidden variable (See for example [74]).

6.2 The Kochen-Specker theorem and contextuality

Bell's theorem is a *no-go theorem*, showing that realism comes at a price. However, it is probabilistic, in the sense that it deals with the measurement statistics and not with the measurement outcomes in a particular run of an experiment. The *Kochen-Specker theorem* [67] is another no-go theorem against realism, involving only one party and no statistics.

Recall that a massive spin 1 particle has 3 degrees of freedom and is therefore described by a quantum system (Hilbert space) of dimension 3. This is also called a *qutrit*. If we now choose a direction in space, call it x, and measure the spin of this particle in this direction, S_x, we obtain as outcome either -1, 0, or 1. Pick two other directions y and z such that the three directions form an orthogonal basis. Then quantum mechanics tells us that S_x, S_y and S_z cannot be simultaneously measured (they do not commute), however, their squares can be simultaneously measured and satisfy

$$S_x^2 + S_y^2 + S_z^2 = 2 \tag{6.6}$$

Replacing S_i^2 with $1 - S_i^2$, these new operators add to 1. Moreover, since S_i has outcomes in $\{-1, 0, 1\}$, their squares can only take the value 0 or 1. We conclude that if we measure $1 - S_i^2$ in three pairwise orthogonal directions, we always obtain $0, 0, 1$ as the outcomes in some order.

Now imagine we have a *non-contextual* realist theory of quantum mechanics. This means that the outcome in a direction x does not depend on its *context*. That is, it does not matter whether we consider the measurements in the basis $\{x, y, z\}$ or some other basis $\{x, y', z'\}$. This theory would then, for every direction, assign either 0 or 1, such that every orthonormal basis is assigned $0, 0, 1$ in some order. The Kochen-Specker theorem shows this is impossible by finding a finite number of directions such that, no matter which choices we make for the values, we obtain a contradiction.

A few words on the proof of the Kochen-Specker theorem based on work done together with Bas Westerbaan in [92]. The original proof constructed a set of 117 directions to come to a contradiction [67]. There is a very nice proof with high symmetry, found independently by Penrose and Peres, using 33 directions [76]. The current proof with the smallest number of directions comes from Conway and uses 31 directions [77]. Instead of finding smaller proofs of the Kochen-Specker theorem, one can also go the other way around and show that some number of directions can *not* lead to a contradiction. This is originally done in [9] where it was shown that a proof of the Kochen-Specker theorem in 3 dimensions needs at least 18 vectors. This lower bound was then raised in [92], where it was shown that one needs at least 22 directions. What happens between 23 and 31 is still unknown. When we consider Hilbert spaces of dimension at least 4, there are smaller proofs and the smallest uses 18 [19].

Hence the Kochen-Specker theorem tells us that either certain quantum mechanical measurements have no predefined outcomes before the actual measurement happens, or the outcome of the measurements is dependant on the context in which it happens. We will come back to the Kochen-Specker theorem in Section 9.1.1.

6.3 A general perspective

Bell's theorem showed us that we cannot reproduce the statistics of quantum mechanics in a realist way such that the outcomes of one party are independent of the settings of the other party. The Kochen-Specker theorem showed us that we cannot assign values to all measurements in such a way that these values are independent of the other settings.

In both scenarios we consider specific measurements, (a_i, b_j) in the Bell scenario and $1 - S_i^2$ in the Kochen-Specker scenario, which are jointly measurable, or compatible, with some other measurements, but incompatible with others. For Bell, a_i is compatible with b_j, but a_0 and a_1, for example, are incompatible. For Kochen-Specker, any two such operators are compatible if their directions are orthogonal and incompatible otherwise. We call a maximal set of compatible operators a *context*. The no-go theorems, often called paradoxes, now have to do with the fact that for every context we can find well defined outcomes, or probability distributions thereof, but these are not consistent over the different contexts. In this sense, the non-locality of Bell's theorem is a special case of the contextuality of the Kochen-Specker theorem and this opens up the possibilities of describing these phenomenons in a unified manner.

In [2], Abramsky and Brandenburger have done this by using a sheaf-theoretic setting, which we explain in Section 8.6. In the next chapter we will develop the theory of *effect algebras* for this purpose, which also capture the concept of incompatibility of contexts. Then, in Chapters 8 we will consider these paradoxes again in an effect algebraic setting. To this end we will generalize the concept of probability distribution in two ways; using effect algebras and using functors. These generalizations are related by an adjunction and we use this to link the sheaf theoretic approach to our effect algebraic approach.

Chapter 7

Effect algebras

In this chapter we introduce the basics of the theory of partial monoids and in particular *effect algebras* as a special class. Effect algebras were introduced by D. Foulis and M. Bennett ([38]) as an abstraction of effects on a Hilbert space. They also occur naturally in categorical treatments of quantum logic as the space of *predicates*; maps from some object to $1 + 1$, seen as the truth values true and false (see for example [55]). For a comprehensive treatment of effect algebras, we refer to [33].

7.1 Boolean algebras

In classical propositional logic, we have a set of propositions where we can take the conjunction or disjunction of every pair of propositions, we have special true and a false propositions, and for every proposition there is a negated proposition. This structure is captured by that of a Boolean algebra.

Definition 7.1.1. A *Boolean algebra* is a distributive lattice with top and bottom and where every element has a complement. In detail: a Boolean algebra $(B, \wedge, \vee, \neg, 0, 1)$ consists of a set B together with two associative, commutative binary operations $\vee, \wedge$ (*meet, join*), a unary operation $\neg$ (*not*), and two special elements $0, 1$ (*bottom, top*) which are the units for $\vee$ and $\wedge$, respectively, satisfying for all $a, b, c \in B$:

- $a \wedge (a \vee b) = a$,

- $a \vee (a \wedge b) = a$,
- $a \vee \neg a = 1$,
- $a \wedge \neg a = 0$,
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

A morphism of Boolean algebras is a map of the underlying sets which preserves $\vee$, $\wedge$, 0 and 1. The resulting category of Boolean algebras is denoted **BA** and the subcategory of finite Boolean algebras is denoted **FinBA**.

It is a well known result by M.H. Stone ([88]) that every Boolean algebra is isomorphic to some subset of the powerset $\mathcal{P}(X)$ of some set X, where $1 = X$, $0 = \emptyset$, $\wedge = \cap$, $\vee = \cup$, and $\neg$ is the set-theoretic complement. See for example [41] for a proof. Moreover, any finite Boolean algebra is isomorphic to the powerset of some finite set (see Lemma 7.3.1). Now consider again the Bell scenario. The propositions $a_1{:}0$ and $a_0{:}0$ are valid propositions, but their conjunction $a_1{:}0 \wedge a_0{:}0$ doesn't seem to make sense as it would imply some hidden variable for Alice. This leads us to consider partial structures where certain operations are not defined for all elements.

7.2 Partial monoids and effect algebras

We start with a very general partial structure of which effect algebras will be a special instance. For most quantum paradoxes it suffices to consider just effect algebras, however, when we will consider the Hardy paradox (Section 9.1, [49]), we will see that we need a bit more generality and have to resort to this more general structure.

Definition 7.2.1. A *pointed partial commutative monoid* (or PPCM) is a quadruple $(E, \varovee, 0, 1)$, where E is a set, $0, 1 \in E$ and $\varovee : E \times E \to E$ is a partial function satisfying for all $x, y, z \in E$:

- If $x \varovee y$ is defined, then $y \varovee x$ is defined and $x \varovee y = y \varovee x$,
- If $x \varovee (y \varovee z)$ is defined, then $(x \varovee y) \varovee z$ is defined and $x \varovee (y \varovee z) = (x \varovee y) \varovee z$,

- $x \ovee 0$ is always defined and $x \ovee 0 = x$.

We write $x \perp y$ if $x \ovee y$ is defined and we call $x \ovee y$ the sum of x and y. This is sometimes called 'x is perpendicular to y' in the literature.

For A and E PPCMs, a function $f : A \to E$ is a morphism of PPCMs if

- $f(0) = 0$,
- If $a_1 \perp a_2$ in A, then $f(a_1) \perp f(a_2)$ in E and $f(a_1 \ovee a_2) = f(a_1) \ovee f(a_2)$,
- $f(1) = 1$.

The resulting category of pointed partial commutative monoids and their morphisms will be denoted **PPCM**.

Note that the point 1 in a PPCM has no special properties. It is just their to make a pointed version of partial commutative monoids. In particular, it need not be a top element for some order on the PPCM, as is the case with 1 as effect on a Hilbert space or *true* in a Boolean algebra. Usually we will omit writing $x \perp y$ and tacitly assume this is the case whenever we write $x \ovee y$. Instead of writing out $(E, \ovee, 0, 1)$ we will also usually just write E. We now consider a special class of PPCMs in which 1 does play a special role. This is motivated by the fact that when we consider effects related to Hilbert spaces (Example 7.2.5), the effect 1 corresponds to the single outcome measurement, which is given by the trace, which is precisely the unique causal effect of Lemma 2.1.21.

Definition 7.2.2 (effect algebra). An *effect algebra* $(E, \ovee, 0, 1)$ is a PPCM such that

- For every $x \in E$ there exists a unique $x^\perp \in E$ such that $x \perp x^\perp$ and $x \ovee x^\perp = 1$,
- $x \perp 1$ implies $x = 0$.

A morphism of effect algebras is a morphism of the underlying PPCMs.

The category of effect algebras will be denoted **EA**.

The element $x^\perp$ is called the orthocomplement of x. While it is not an explicit requirement for orthocomplements to be preserved by morphisms, this is still the case.

Lemma 7.2.3. Let A and E be effect algebras. A morphism $f : A \to E$ satisfies

$$f(a^{\perp}) = f(a)^{\perp}$$

for all $a \in A$.

Proof. We have

$$\begin{aligned} 1 &= f(1) \\ &= f(a \otimes a^{\perp}) \\ &= f(a) \otimes f(a^{\perp}) \end{aligned}$$

Hence by uniqueness of the orthocomplement we have $f(a^{\perp}) = f(a)^{\perp}$. $\square$

We consider some simple examples:

Example 7.2.4.
- Obviously every effect algebra is a PPCM.
- The two element set $\{0, 1\}$, where $0 \otimes 0 = 0, 0 \otimes 1 = 1 \otimes 0 = 1$ and $1 \otimes 1$ is undefined, is an effect algebra. It is the initial effect algebra and initial PPCM.
- The two element set, but now with $1 \otimes 1 = 1$ is a PPCM, but not an effect algebra.
- The one element set $\{0\}$ with $0 \otimes 0 = 0$ is the final effect algebra and final PPCM. Note that in this case $0 = 1$.
- The interval $[0, 1]$ is an effect algebra if we set $x \perp y$ if and only if $x + y \leq 1$ in which case $x \otimes y = x + y$. The orthocomplement is given by $x^{\perp} = 1 - x$.

In effect algebras, the partial sum gives rise to a partial order given by

$$x \leq y \Leftrightarrow \exists\, z \text{ such that } x \otimes z = y.$$

Since for any element x in an effect algebra E we have $0 \leq x \leq 1$, the bottom of this order is 0 and 1 is the top. This order cannot be defined for general PPCMs. Indeed, consider the PPCM $\{0, 1\}$ with $1 \otimes 1 = 0$ (i.e., the 2-element group). Then $0 \leq 1$ and $1 \leq 0$, yet $0 \neq 1$.

Example 7.2.5. The following examples will show how effect algebras are indeed a generalization of well known concepts.

- Every Boolean algebra $(B, \vee, \wedge, \neg, 0, 1)$ can be understood as an effect algebra if we set $x \perp y$ if and only of $x \wedge y = 0$. In that case, $x \oslash y = x \vee y$ and $x^{\perp} = \neg x$.
- Let H be a Hilbert space and $B(H)$ the bounded operators on H. There is a canonical ordering on the positive elements of $B(H)$ given by $a \leq b$ if $a - b$ is positive. Let $[0,1]_{B(H)} \subset B(H)$ be the subset of effects, i.e., the positive elements below 1. Then $[0,1]_{B(H)}$ is an effect algebra where $a \perp b$ if $a + b \leq 1$ and $a^{\perp} = 1 - a$.
- Likewise, let $Proj(H)$ be the projections on H, then $Proj(H)$ is an effect algebra in a similar way as the effects, where for projections p, q we have $p + q \leq 1$ if the ranges of p and q are orthogonal.

Finally, we consider some categorical properties of effect algebras.

Proposition 7.2.6. Let E and A be effect algebras. Then the product and coproduct of E and A exist.

Proof. We give the relevant constructions. A detailed proof can be found in [56].

- For the product, let
$$E \times A = \{(e, a) \mid e \in E, a \in A\}.$$
Let $0 = (0, 0)$ and $1 = (1, 1)$. Set $(e_1, a_1) \perp (e_2, a_2)$ if $e_1 \perp e_2$ and $a_1 \perp a_2$, in this case $(e_1, a_1) \oslash (e_2, a_2) = (e_1 \oslash e_2, a_1 \oslash a_2)$. The projections are the obvious ones: $(e, a) \mapsto e$ and $(e, a) \mapsto a$.
- For the coproduct, also called *horizontal sum* in [33], assume E and A are not the final effect algebra. Let
$$E \oplus A = (E \backslash \{0, 1\}) \cup (A \backslash \{0, 1\}) \cup \{0, 1\}.$$
Orthogonality is given by $x \perp y$ if x or y equals 0, $x \perp y$ in E or $x \perp y$ in A. If E or A is final, then $E \oplus A$ is also final. The injections are the obvious ones.

$\square$

In fact, we have the following stronger result:

Proposition 7.2.7. The category of effect algebras is complete and cocomplete.

Proof. See [56]. □

7.3 Functors and tests

Effect algebras are a generalization of Boolean algebras and every finite Boolean algebra is isomorphic to the powerset of some finite set. In this section we make two more useful generalizations. First we consider how effect algebras give rise to probability distributions and then we consider how effect algebras themselves further embed in a functor category. Considering probability distributions in an effect algebraic setting allow us to treat non-locality and contextuality using effect algebras as we will see in the next chapter. The embedding in a functor category allows us to then compare the effect algebraic approach to the existing presheaf approach as in [2] in Section 8.6. Both these generalizations rely on the concept of tests on an effect algebra (Definition 7.3.3), but before we do this, we first consider the relation between finite sets and Boolean algebras.

Let **FinSet** be the category of finite sets and functions between them. In order not to deal with isomorphic finite sets, let $\mathbb{F}$ be the skeleton of **FinSet**. That is, the objects of $\mathbb{F}$ are the sets $\underline{n} := \{0, 1, \ldots, n-1\}$ and morphisms are functions between these sets. The relation between finite sets and Boolean algebras can now be states as follows:

Lemma 7.3.1. The contravariant powerset functor, $\mathcal{P}$, induces an equivalence between the opposite of $\mathbb{F}$ and finite Boolean algebras:

$$\mathcal{P} : \mathbb{F}^{op} \simeq \mathbf{FinBA} \tag{7.1}$$

Proof. This follows from [41]. We show what happens with morphisms. Let $f : \underline{n} \to \underline{m}$ be a function. Then we obtain a morphism of Boolean algebras $\mathcal{P}(f) : \mathcal{P}(\underline{m}) \to \mathcal{P}(\underline{n})$, given by

$$\mathcal{P}(f)(X) = f^{-1}(X) = \{i \in \underline{n} \mid f(i) \in X\}$$

Conversely, if $g : \mathcal{P}(\underline{m}) \to \mathcal{P}(\underline{n})$ is a morphism of Boolean algebras, such that $g(\{i\}) = X$, then we obtain a map $\underline{n} \to \underline{m}$ where $j \mapsto i$ if $j \in X$. □

Now finite Boolean algebras embed in Boolean algebras which in turn embed in effect algebras and PPCMs.

$$\mathbf{FinBA} \hookrightarrow \mathbf{BA} \hookrightarrow \mathbf{EA} \hookrightarrow \mathbf{PPCM}$$

Hence we can view the powerset functor either as $\mathcal{P} : \mathbb{F}^{op} \to \mathbf{FinBA}$, $\mathcal{P} : \mathbb{F}^{op} \to \mathbf{BA}$, $\mathcal{P} : \mathbb{F}^{op} \to \mathbf{EA}$ or as $\mathcal{P} : \mathbb{F}^{op} \to \mathbf{PPCM}$. Whenever we write $\mathcal{P}$, it will be clear from context which (possibly multiple) of these is meant. Since $\mathcal{P} : \mathbb{F}^{op} \to \mathbf{FinBA}$ is an equivalence, it is in particular full and faithful. Furthermore, since any PPCM morphism between $\mathcal{P}(\underline{n})$ and $\mathcal{P}(\underline{m})$ is fixed once the images of the singletons are known, any PPCM morphism between $\mathcal{P}(\underline{n})$ and $\mathcal{P}(\underline{m})$ is also a Boolean algebra morphism. Hence we find

Corollary 7.3.2. The powerset functor, $\mathcal{P}$, is full and faithful.

We now introduce the concept of a test, which can be seen as a generalization of a PVM or POVM.

Definition 7.3.3 (n-test)**.** Let E be an effect algebra (or more generally any PPCM). An ordered list of elements $(e_1, \ldots, e_n), e_i \in E$ is called an *n-test* if $e_1 \varovee \ldots \varovee e_n = 1$. The set of all n-tests on E is denoted $T(E)(n)$. A *test* is an n-test for some n.

Example 7.3.4.

- An n-test of $\mathcal{P}(X)$ is precisely an ordered partition of X, where the empty set can occur multiple times.
- For the initial effect algebra we have $T(\{0,1\})(n) \cong \underline{n}$; a test consist out of exactly one 1, which can be in any of the n entries in a list.
- An n-test in $Proj(\mathcal{H})$ corresponds to a von Neumann measurement (PVM) whereas an n-test in the interval $[0,1]_B(\mathcal{H})$ corresponds to a general measurement (POVM).
- For the interval $[0,1]$, an n-test is a collection of numbers $(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_i \in [0,1]$ and $\sum_i \lambda_i = 1$. That is, an n-test in $[0,1]$ is a probability distribution on an n element set and can therefore also be identified with an n-simplex.

This last example shows that finite probability distributions are obtained as tests on an effect algebra. Normally, to consider (finite) probability distributions, one introduces (finite) measure spaces.

Definition 7.3.5. Let X be a finite set with powerset $\mathcal{P}(X)$ and let Ω be a sub-Boolean algebra of $\mathcal{P}(X)$, i.e., $\Omega \subset \mathcal{P}(X)$ is a Boolean algebra and

$\emptyset, X \in \Omega$. A probability distribution on (X, Ω) is a function $p : \Omega \to [0,1]$ such that $p(X) = 1$ and $p(X_1 \cup \ldots \cup X_n) = p(X_1) + \ldots + p(X_n)$ for disjoint subsets $X_1, \ldots, X_n \subset X$.

When we consider this definition, we find that the surrounding space X actually does not play any significant role. The relevant information is whether or not two subsets in Ω are disjoint or not. We can capture this by noting that $(\Omega, \uplus, \emptyset, X)$ is an effect algebra. A morphism from $(\Omega, \uplus, \emptyset, X)$ to $([0,1], +_{\leq 1}, 0, 1)$ is now precisely a probability distribution. This leads us to consider the following (see also [35]):

A 'generalized probability space' is given by an effect algebra E and a 'generalized probability distribution' is a morphism $E \to [0,1]$.

We now consider the second generalization of embedding effect algebras in a functor category, or category of *presheaves*. We recall some theory.

7.4 Intermezzo: presheaves

Let $\mathcal{C}$ be a small category, meaning that the collection of objects and morphisms both are sets. Let $[\mathcal{C}^{op}, \mathbf{Set}]$ (or $\mathbf{Set}^{\mathcal{C}^{op}}$) be the *presheaf category over* $\mathcal{C}$, i.e., the category of functors from the opposite of $\mathcal{C}$ to **Set** and whose morphisms are natural transformations between these functors. The *Yoneda embedding* $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ sends an object A to the *representable presheaf Representable presheaf* $Hom(-, A)$ and a morphism f to the natural transformation $f \circ -$. The following is a classical result in this setting:

Lemma 7.4.1 (Yoneda lemma)**.** Let $F : \mathcal{C}^{op} \to \mathbf{Set}$ be a functor, then there is a natural isomorphism between natural transformations from a representable presheaf $y(A)$ to F and the set $F(A)$

$$[\mathcal{C}^{op}, \mathbf{Set}](y(A), F) \cong F(A) \tag{7.2}$$

Proof. The idea is that a natural transformation $\eta : y(A) \to F$ is completely determined by the value of $\eta_A(id_A) \in F(A)$ and *vice versa*. Indeed, by naturality we have for any $f : B \to A$ that $\eta_B(f) = F(f)(\eta_A(id_A))$. Details can be found in any book on category theory, for example [10] or [71]. $\square$

Some consequences:

Corollary 7.4.2. The Yoneda embedding is full and faithful.

$$[\mathcal{C}^{op}, \mathbf{Set}](y(A), y(B)) \cong y(B)(A) = Hom(A, B) \tag{7.3}$$

Corollary 7.4.3. If for every $C \in \mathcal{C}$ we have $Hom(C, A) \cong Hom(C, B)$ then $A \cong B$:

$$y(A) \cong y(B) \Leftrightarrow A \cong B \tag{7.4}$$

The next statement is often called the *co-Yoneda lemma*.

Corollary 7.4.4. Any presheaf $F \in [\mathcal{C}^{op}, \mathbf{Set}]$ is a canonical colimit of representable presheaves.

Proof. The proof follows from considering El_F, the *category of elements* of F. That is, objects of El_F are pairs $(C, x \in F(C))$ and morphisms $(C, x) \to (C', y)$ are morphisms $f : C \to C'$ such that $F(f)(y) = x$. Composing the forgetful functor, $(C, x) \mapsto C$, with the Yoneda embedding then gives a diagram of the category of elements in $[\mathcal{C}^{op}, \mathbf{Set}]$. The colimit over this diagram is then isomorphic to F. See [71] or [69] for details. $\square$

So $\mathcal{C}$ embeds fully faithful into its category of presheaves and every object there is a colimit of these representable functors. If fact, $[\mathcal{C}^{op}, \mathbf{Set}]$ is the *free cocompletion* of $\mathcal{C}$. This means that any functor $F : \mathcal{C} \to \mathcal{D}$ into a cocomplete category $\mathcal{D}$ factors though the Yoneda embedding as $F \cong L_F \circ y$. By the above, it is clear what this functor L_F should do on objects. On representable functors $y(c)$ since we must have $L_F(y(c)) = F(c)$. Then since any functor is a colimit of representable functors, we get $L_F(colimy(c)) = colimF(c)$.

On morphisms we must have $L_F(y(f) : y(c) \to y(d)) = F(f)$. Then if $\mu : P \to Q$ is a morphism (i.e., natural transformation) between presheaves, we know that both P and Q are colimits of a diagram of representables, so we know how these diagrams are mapped into $\mathcal{D}$. For a colimit injection $i_c : y(c) \to P$ we define $L_F(i_c)$ to be the colimit injection of $F(c)$ in $L_F P = colimF(c)$.

Let (c, x) be an object in the category of elements of P, El_P, then $x \in P(c) \cong Hom(y(c), P)$. We may therefore write $\mu \circ x$ to be a map from $y(c)$ to Q and hence $(c, \mu \circ x)$ is an object in El_Q. This extends to a functor from El_P to El_Q which is the identity on the underlying objects $y(c)$. So for $(c, x) \in El_P$ we obtain a colimit injection $y(c) \to Q$. This then means that in $\mathcal{D}$ $L_F Q$ is a cone for the diagram over $L_F P$, so by the universal property of the colimit we obtain a morphism $L_F P \to L_F Q$ which we take as the image of μ under L_F.

We now consider this setting from another point of view. Let $\mathcal{D}$ be a locally small category, meaning that for any objects A, B the collection of morphisms $Hom(A, B)$ is a set. A functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $\mathcal{D} \to [\mathcal{C}^{op}, \mathbf{Set}]$ called the *nerve functor*:

Definition 7.4.5. The nerve functor $N : \mathcal{D} \to [\mathcal{C}^{op}, \mathbf{Set}]$ is given by

$$N(d) := \mathcal{D}(F(-), d) \tag{7.5}$$

and the obvious action on morphisms.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "y"] \arrow[d, "F"'] & {[\mathcal{C}^{op}, \mathbf{Set}]} \\ \mathcal{D} \arrow[ur, "N"'] & \end{tikzcd} \tag{7.6}$$

Starting from an object $c \in \mathcal{C}$ we have $y(c) = Hom(-, c)$, whereas going via the bottom gives $c \mapsto F(c) \mapsto Hom(F(-), F(c))$. Hence we see:

Lemma 7.4.6. Diagram 7.6 commutes if and only if F is fully faithful.

We now have two functors $N : \mathcal{D} \rightleftarrows [\mathcal{C}^{op}, \mathbf{Set}] : L_F$. We now have the following:

Proposition 7.4.7. The functors $L_F \dashv N$ form an adjoint pair.

Proof. We give a small calculation, leaving the details.

$$\begin{aligned} Hom(L_F G, d) &\cong Hom(L_F(colim(y(C))), d) \\ &\cong limHom(L_F y(c), d) \\ &\cong limHom(F(c), d) \\ &\cong limHom(y(c), Hom(F(-), d)) \\ &\cong limHom(G, N(d)) \end{aligned}$$

□

The full picture is now as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "y"] \arrow[d, "F"'] & {[\mathcal{C}^{op}, \mathbf{Set}]} \arrow[dl, bend left, "L_F"] \\ \mathcal{D} \arrow[ur, bend left, "N"] \arrow[ur, phantom, "\perp"] & \end{tikzcd} \tag{7.7}$$

The left adjoint is also known as the Yoneda extension of F, which is the left Kan extension of F along y. We refer to [71], [69] and [59] for details and more.

7.5 The effect algebraic situation

In our situation, we have $F : \mathcal{C} \to \mathcal{D} = \mathcal{P} : \mathbb{F}^{op} \to \mathbf{EA}$. The Yoneda embedding then becomes $y : \mathbb{F}^{op} \to [\mathbb{F}, \mathbf{Set}]$, mapping $\underline{n} \mapsto Hom_{\mathbb{F}^{op}}(-, \underline{n}) \cong Hom_{\mathbb{F}}(\underline{n}, -)$. The nerve functor takes an effect algebra E to the presheaf $Hom(\mathcal{P}(-), E)$, which takes $\underline{n}$ to $Hom(\mathcal{P}(\underline{n}), E)$. A function in this Hom-set sends singletons $\{i\} \in \mathcal{P}(\underline{n})$ to elements $e_i \in E$ such that $\bigovee_i e_i = 1$. That is, such a function is precisely an n-test and every n-test gives rise to a function in this way. Because of this we call the nerve functor the *test functor*, lifting Definition 7.3.3 to a functor:

Definition 7.5.1 (Test functor)**.** The *test functor* $T : \mathbf{EA} \to [\mathbb{F}, \mathbf{Set}]$ is the functor $E \mapsto T(E)$ where

$$\begin{aligned} T(E)(\underline{n}) &= \{n\text{-tests on E}\} && (7.8) \\ &\cong Hom(\mathcal{P}(\underline{n}), E) && (7.9) \end{aligned}$$

and for $f : \underline{n} \to \underline{m}$

$$T(E)(f) : (e_1, \ldots, e_n) \mapsto (\bigovee_{j \in f^{-1}(i)} e_j)_{i=1,\ldots m} \tag{7.10}$$

For an effect algebra morphism $\phi : E \to A$, we obtain a natural transformation $T(\phi) : T(E) \to T(A)$ which is defined with components

$$T(\phi)_{\underline{n}} : (e_1, \ldots, e_n) \mapsto (f(e_1), \ldots, f(e_n)) \tag{7.11}$$

By Corollary 7.3.2 the powerset functor is fully faithful, Hence by Lemma 7.4.6 the diagram

$$\begin{tikzcd} \mathbb{F}^{op} \arrow[r, "y"] \arrow[d, "\mathcal{P}"'] & {[\mathbb{F}, \mathbf{Set}]} \\ \mathbf{EA} \arrow[ur, "T"'] & \end{tikzcd} \tag{7.12}$$

is commutative up to isomorphism.

Since **EA** is cocomplete ([56]), we obtain a left adjoint to the test functor as the Yoneda extension of the powerset functor.

Theorem 7.5.2. The test functor $T : \mathbf{EA} \to [\mathbb{F}, \mathbf{Set}]$ has a left adjoint.

$$\begin{tikzcd}
\mathbb{F}^{op} \arrow[r, "y"] \arrow[d, "\mathcal{P}"'] & {[\mathbb{F}, \mathbf{Set}]} \arrow[dl, bend left, "L"] \arrow[dl, phantom, "\perp"] \\
\mathbf{EA} \arrow[ur, bend left, "T"] &
\end{tikzcd} \tag{7.13}$$

We next show that the test functor is fully faithful. Hence effect algebras form a *reflective subcategory* of the presheaf category. In fact, we may even be a bit more general. While the test functor is defined on effect algebras, Definition 7.3.3 shows that tests can also be defined for PPCMs. This also extends to a functor and the next theorem should be seen in that setting.

Theorem 7.5.3. Let A and B be PPCMs. If A is an effect algebra, the induced functor $T_{A,B} : \mathbf{PPCM}(A, B) \to [\mathbb{F}, \mathbf{Set}](TA, TB)$ is a bijection.

Proof. We start by introducing some notation. Denote functions from $\underline{n}$ to $\underline{m}$ as lines from n nodes above to m nodes below. For example, $(\mid\vee) : \{1, 2, 3\} \to \{1, 2\}$ is the map $1 \mapsto 1$ and $2, 3 \mapsto 2$. We will also use this notation for the image of this function under a functor.

Now if A is an effect algebra, every element $a \in A$ is part of a 2-test $(a, a^{\perp})$. It is then clear that $T_{A,B}$ is injective.

To show surjectivity, suppose that $\mu : T(A) \to T(B)$ is some natural transformation. We need to find a PPCM morphism $\psi_\mu : A \to B$ such that $T(\psi_\mu) = \mu$. To this end, for $a \in A$, consider the 2-test $(a, a^{\perp})$ and define $\psi_\mu(a) = x$, where $(x, x') = \mu_2(a, a^{\perp})$. Note that x' is a complement of x, but not necessarily unique. We show that ψ_μ is indeed a PPCM morphism. Let $a \perp b$ in A and let $c = (a \ovee b)^{\perp}$, so that (a, b, c) is a 3-test. Let $\mu_3(a, b, c) = (x, y, z)$, then we immediately see $x \perp y$ in B. Furthermore, by naturality of μ we have

$$\begin{aligned}
\mu_2(a, a^{\perp}) &= \mu_2\,(\mid\vee)\,(a, b, c) \\
&= (\mid\vee)\,\mu_3(a, b, c) \\
&= (x, y \ovee z)
\end{aligned}$$

Therefore $\psi_\mu(a) = x$ and similarly $\psi_\mu(b) = y$. We then calculate

$$\begin{aligned}
\mu_2(a \ovee b, c) &= \mu_2\,(\vee/)\,(a, b, c) \\
&= (\vee/)\,\mu_3(a, b, c) \\
&= (x \ovee y, z)
\end{aligned}$$

which shows that $\psi_\mu(a \otimes b) = x \otimes y = .\psi_\mu(a) \otimes \psi_\mu(b)$.

To show ψ_μ preserves 1 we calculate

$$\begin{aligned}\mu_2(1_A, 0_A) &= \mu_2\,(\prime\cdot)\,(1_A)\\ &= (\prime\cdot)\,\mu_1(1_A)\\ &= (\prime\cdot)\,1_B\\ &= (1_B, 0_B)\end{aligned}$$

Similarly ψ_μ preserves 0. By construction we now indeed have $T_{A,B}(\psi_\mu) = \mu$, so $T_{A,B}$ is indeed a bijection. $\square$

Corollary 7.5.4. The restriction to effect algebras, $T : \mathbf{EA} \to [\mathbb{F}, \mathbf{Set}]$, is full and faithful.

In particular:

Proposition 7.5.5. For an effect algebra E we have $LT(E) \cong E$.

Proof. Let A be any effect algebra. Then since L is a left adjoint and T is full and faithful, we have $Hom(LT(E), A) \cong Hom(T(E), T(A)) \cong Hom(E, A)$. Hence by the Lemma 7.4.3 we have $LT(E) \cong E$. $\square$

We can rephrase this as *'the counit of the adjunction $L \dashv T$ is an isomorphism at effect algebras'*.

Before giving our second generalization of probability theory, we note the following. Because of the equivalence of finite Boolean algebras and $\mathbb{F}^{op}$ (Lemma 7.3.1), we may see $[\mathbb{F}, \mathbf{Set}]$ as the free cocompletion of finite Boolean algebras. The test functor is now an embedding of effect algebras in this free cocompletion. Using the general theory above, we then find that every effect algebra is a canonical colimit of Boolean algebras (this inspired part of the work in [58]). This can be stated as *'finite Boolean algebras are dense in effect algebras'*. Moreover, we remark that our category $\mathbb{F}$ is equivalent to finite Hausdorff spaces, which yields a connection to a related result: *compact Hausdorff spaces are dense in piecewise C^*-algebras* (see [37, Thm 4.5]).

7.6 A second generalization of probability theory

We now present a further generalization of probability spaces and distributions based on functors. Consider again the distributions on a finite set

$\underline{n}$. This is the set

$$\begin{aligned} D(\underline{n}) &= \{(\lambda_0,\ldots,\lambda_{n-1}) | \lambda_i \in [0,1], \sum_i \lambda_i = 1\} \\ &\cong \{d : \underline{n} \to [0,1] | \sum_i d(i) = 1\} \end{aligned} \tag{7.14}$$

It is well known that $D : \mathbb{F} \to \mathbf{Set}$ extends to the *distribution functor*: if $f : \underline{n} \to \underline{m}$ is a function, then

$$D(f)(\lambda_0,\ldots,\lambda_{n-1}) = (\sum_{j\in f^{-1}(i)} \lambda_j)_{i=0,\ldots,m-1}$$

Hence we see that the distribution functor equals the test functor on $[0,1]$ as functors in $[\mathbb{F}, \mathbf{Set}]$:

$$D = T([0,1]) \tag{7.15}$$

Considering a probability distribution on n points a map $\underline{n} \to [0,1]$ we can extend it to an effect algebra morphism $\mathcal{P}(n) \to [0,1]$ via its action on singletons. Taking tests then gives a morphism from $T\mathcal{P}(\underline{n}) \cong y(\underline{n}) \to D = T([0,1])$ (cf. (7.12)). By the Yoneda lemma we now have

$$D(\underline{n}) \cong [\mathbb{F}, \mathbf{Set}](y(\underline{n}), D) \tag{7.16}$$

That is, probability distributions on $\underline{n}$ correspond to natural transformations between the Yoneda embedding of $\underline{n}$ and the distribution functor. This leads us to consider the following generalization:

A 'generalized probability space' is a functor $Q : \mathbb{F} \to \mathbf{Set}$ and a 'generalized probability distribution' is a natural transformation $Q \to D = T([0,1])$, to the distribution functor.

We now have two generalizations of probability theory. Via effect algebras and via presheaves. We know effect algebras embed in presheaves. A natural question is then whether we can characterize the image of this embedding. It turns out this is possible as done in [54] and [53]. This might not come a surprise. Indeed, since every element e in an effect algebra E has a unique orthocomplement, $e^\perp$, the correspondence $e \leftrightarrow (e, e^\perp)$ gives a bijection $T(E)(\underline{2}) \cong E$ as sets. Furthermore, if $a \perp b$ in E, then there exists an element (a, b, c) of $T(E)(\underline{3})$, i.e., a 3-test in E. Hence $T(E)$ also contains all information about the partial structure, $\oslash$, of E. We leave the details to [54] and [53].

Chapter 8

Back to Bell

Here we use the effect algebraic framework developed in the previous chapter to consider the Bell paradox again. We will construct an effect algebra which takes into account the impossibility of measuring the different settings simultaneously and consider probability distributions on this algebra. We start by considering the simple Bell scenario with two observers, who each have two measurement settings which each have two outcomes, but we will generalize this to more general cases in Section 8.4. The main goal here is not to consider non-locality and contextuality in different systems, but rather to give a general framework to study them. We will then relate this effect algebraic framework to other approaches in Section 8.6.

8.1 Tables

Recall that in the Bell scenario Alice and Bob each perform one of two measurements, $\mathrm{a}_0, \mathrm{a}_1$ for Alice and $\mathrm{b}_0, \mathrm{b}_1$ for Bob. Each of them then obtains some outcome in an outcome set, which we for simplicity take to be the set $\{0, 1\}$. Furthermore, recall the notations $\mathrm{a}_i{:}j$ to mean that Alice chooses some setting and gets some outcome and $\mathrm{a}_i{:}j \wedge \mathrm{b}_k{:}l$ to mean that both Alice and Bob choose some setting and both obtain some outcome.

As an example of a possible probability distribution over the joint settings and outcomes we gave the standard Bell table (6.2). In general, we are interested in probability distributions which are normalized and non-signalling. This leads us to consider the following:

Definition 8.1.1. A function

$$\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to [0,1]$$

is called a *probability table*, or just a *table*, if:

- *each experiment certainly has an outcome:* for $i, j \in \{0,1\}$,
$$\sum_{o,o' \in \{0,1\}} \tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') \;=\; 1,$$
- *it has marginalization, that is, it is non signalling:* for all $i, j, o \in \{0,1\}$,
$$\begin{aligned} \tau(\mathrm{a}_i{:}o, \mathrm{b}_0{:}0) + \tau(\mathrm{a}_i{:}o, \mathrm{b}_0{:}1) &= \tau(\mathrm{a}_i{:}o, \mathrm{b}_1{:}0) + \tau(\mathrm{a}_i{:}o, \mathrm{b}_1{:}1), \\ \tau(\mathrm{a}_0{:}0, \mathrm{b}_j{:}o) + \tau(\mathrm{a}_0{:}1, \mathrm{b}_j{:}o) &= \tau(\mathrm{a}_1{:}0, \mathrm{b}_j{:}o) + \tau(\mathrm{a}_1{:}1, \mathrm{b}_j{:}o). \end{aligned}$$

More generally, we can replace the effect algebra $[0,1]$ with any PPCM X. This will allow us to consider for example *possibilities* instead of *probabilities* by replacing $[0,1]$ by the PPCM $\{0,1\}$ where $1 \ovee 1 = 1$ (see Section 9.1). We call such a general table an X-*table*. A table as in Definition 8.1.1 is then just an X-table for $X = [0,1]$.

Tables thus give us information about the likelihood that the observed events occur. One might then be interested to understand how these likelihoods arise. More specifically, we are interested in understanding if a given table can be obtained in a quantum mechanical setting and whether there is a realist explanation for the table. To this end, let us define some algebras for Alice and Bob.

$$E_{\mathrm{A},0} \stackrel{\text{def}}{=} \mathcal{P}(\{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1\}) \qquad E_{\mathrm{A},1} \stackrel{\text{def}}{=} \mathcal{P}(\{\mathrm{a}_1{:}0, \mathrm{a}_1{:}1\}) \tag{8.1}$$

$$E_{\mathrm{B},0} \stackrel{\text{def}}{=} \mathcal{P}(\{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1\}) \qquad E_{\mathrm{B},1} \stackrel{\text{def}}{=} \mathcal{P}(\{\mathrm{b}_1{:}0, \mathrm{b}_1{:}1\}) \tag{8.2}$$

For clarity, we consider the algebra $E_{\mathrm{A},0}$ in detail. This is the Boolean algebra $\{0, \mathrm{a}_0{:}0, \mathrm{a}_0{:}1, 1\}$ where $\mathrm{a}_0{:}0 \wedge \mathrm{a}_0{:}1 = 0$ and $\mathrm{a}_0{:}0 \vee \mathrm{a}_0{:}1 = 1$. This accounts for the fact that 0 and 1 are the only outcomes for the setting a_0 and that they cannot occur at the same time. As such, the algebra $E_{\mathrm{A},0}$ is also an effect algebra. A probability distribution $p : E_{\mathrm{A},0} \to [0,1]$ then gives the probability of each of the outcomes $0, 1$ occurring.

We now consider these algebras from a different point of view. Any Boolean algebra or effect algebra has an underlying set of elements. This induces the forgetful functors $U_B : \mathbf{BA} \to \mathbf{Set}$ and $U_E : \mathbf{EA} \to \mathbf{Set}$ which send an algebra to this underlying set and a morphism to its underlying function. In many categories such a forgetful functor has a left adjoint and the resulting objects of this functor are usually called *free*. Let us consider this for effect algebras and Boolean algebras.

Definition 8.1.2. Let X be a set.

- The *free Boolean algebra*, $F_B X$, on the set X is the Boolean algebra such that for all Boolean algebras B we have

$$Hom(X, U_B B) \cong Hom_{\mathbf{BA}}(F_B X, B)$$

- The *free effect algebra*, $F_E X$, on the set X is the effect algebra such that for all effect algebras A we have

$$Hom(X, U_E A) \cong Hom_{\mathbf{EA}}(F_E X, A)$$

So for every function $f : X \to B$, there is a unique algebra morphism $\hat{f} : FX \to B$ such that f factors via the injection of X in FX.

$$\begin{tikzcd} FX \arrow[r, dashed, "\hat{f}"] & B \\ X \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd} \tag{8.3}$$

We now consider these algebras in the special case where X is finite as this is sufficient for our purposes.

Proposition 8.1.3. Let X be a finite set.

- The free Boolean algebra $F_B X$ is isomorphic to the double powerset $\mathcal{P}\mathcal{P}X \cong 2^{2^X}$.
- The free effect algebra $F_E X$ is isomorphic to the effect algebra $X \cup X^{\perp} \cup \{0, 1\}$, where $X^{\perp}$ is the set $\{x^{\perp} | x \in X\}$ and the only non-trivial sums are $x \ovee x^{\perp} = 1$.

Proof.
- In the Boolean case, note that for $Y \in \mathcal{P}X$ we can define an element $\mathsf{a}_Y = \bigwedge_{x \in Y} x \wedge \bigwedge_{x \in X \setminus Y} \neg x$. These elements are the atoms of $\mathcal{P}\mathcal{P}X$. Then for $Z = \{Y_1, \ldots, Y_n\} \in \mathcal{P}\mathcal{P}X$ we obtain an element $\bigvee_{Y_i \in Z} \mathsf{a}_{Y_i}$. The bottom line is that every element is a join over meets

of elements of X and X can be embedded $X \hookrightarrow \mathcal{P}\mathcal{P}X$ by taking the join of all elements corresponding to sets $\{Y \in \mathcal{P}X | x \in Y\}$.

Now for any map $f : X \to U_B B$ we obtain a morphism $\tilde{f} : \mathcal{P}\mathcal{P}X \to B$ which acts on atoms as $\tilde{f}(\bigwedge_{x \in Y} x \wedge \bigwedge_{x \in X \setminus Y} \neg x) = \bigwedge_{x \in Y} f(x) \wedge \bigwedge_{x \in X \setminus Y} \neg f(x)$. Conversely any $g : FX \to B$ restricts to a map $\tilde{g} : X \to U_B B$.

- In the effect algebra case, there is an canonical embedding $X \hookrightarrow F_E X$, from which the rest follows.

□

Remark: the above proposition does not hold for Boolean algebras when X is infinite, as in this case we cannot take infinite meets and joins. For example, if $X = \mathbb{N}$, the free algebra $F_B\mathbb{N}$ consists of all finite joins of all finite meets of the elements of $\mathbb{N}$ and their complements. Therefore it has cardinality equal to that of the natural numbers themselves.

From the characterizations of the free Boolean and effect algebras, we see that $E_{\mathrm{A},0}$ is the free Boolean algebra generated by the element $\mathrm{a}_0{:}0$, where we identify $\neg\mathrm{a}_0{:}0 = \mathrm{a}_0{:}1$. It is also the free effect algebra generated by $\{\mathrm{a}_0{:}0\}$ under the identification $(\mathrm{a}_0{:}0)^\perp = \mathrm{a}_0{:}1$.

To continue, we need to take into account that both Alice and Bob each have two incompatible settings. For this we take the sum of their effect algebras, i.e., the coproduct from Proposition 7.2.6 . Hence we define:

$$E_{\mathrm{A}} = E_{\mathrm{A},0} \oplus E_{\mathrm{A},1} \tag{8.4}$$

$$E_{\mathrm{B}} = E_{\mathrm{B},0} \oplus E_{\mathrm{B},1} \tag{8.5}$$

Again in detail, the algebra E_{A} has $\{0, \mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1, 1\}$ as its underlying set and the only non-trivial sums are $\mathrm{a}_0{:}0 \ovee \mathrm{a}_0{:}1 = 1$ and $\mathrm{a}_1{:}0 \ovee \mathrm{a}_1{:}1 = 1$. In contrast to the single setting algebras $E_{\mathrm{A},i}$, these algebras are not Boolean. As such, probability distributions $p : E_{\mathrm{A}} \to [0,1]$ do not occur in classical probability theory. That being said, by the universal property of the sum, such a distribution corresponds to a pair of distributions $p_0 : E_{\mathrm{A},0} \to [0,1]$ and $p_1 : E_{\mathrm{A},1} \to [0,1]$, each of which is just a classical distribution.

While E_{A} and E_{B} are not Boolean, they can be embedded in Boolean algebras. Let

$$B_{\mathrm{A}} = \mathcal{P}(\{\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0, \mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1, \mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}0, \mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}1\}) \tag{8.6}$$
$$B_{\mathrm{B}} = \mathcal{P}(\{\mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0, \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1, \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}0, \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}1\}) \tag{8.7}$$

as Boolean algebras. Then the embedding $i_{\mathrm{A}} : E_{\mathrm{A}} \hookrightarrow B_{\mathrm{A}}$ is given by

$$\mathrm{a}_i{:}j \mapsto (\mathrm{a}_i{:}j \wedge \mathrm{a}_{\neg i}{:}0) \vee (\mathrm{a}_i{:}j \wedge \mathrm{a}_{\neg i}{:}1)$$

where $\neg i = 1$ if $i = 0$ and $\neg i = 0$ if $i = 1$.

We can again consider these algebras as free algebras.

Lemma 8.1.4. The algebras B_{A} and E_{A} are the free Boolean algebra and free effect algebra on $\{\mathrm{a}_0{:}0, \mathrm{a}_1{:}0\}$, respectively.

Proof. Since taking the free algebra is a left adjoint to the forgetful functor, it preserves colimits. In particular, it preserves the coproduct, which in **Set** is disjoint union. It follows that for the effect algebras and Boolean algebras for Alice and Bob we have

$$B_{\mathrm{A}} = B_{\mathrm{A},0} \oplus B_{\mathrm{A},1} = F_B\{\mathrm{a}_0{:}0\} \oplus F_B\{\mathrm{a}_1{:}0\} \cong F_B\{\mathrm{a}_0{:}0, \mathrm{a}_1{:}0\} \tag{8.8}$$
$$E_{\mathrm{A}} = E_{\mathrm{A},0} \oplus E_{\mathrm{A},1} = F_E\{\mathrm{a}_0{:}0\} \oplus F_E\{\mathrm{a}_1{:}0\} \cong F_E\{\mathrm{a}_0{:}0, \mathrm{a}_1{:}0\} \tag{8.9}$$

□

The relation between B_{A} and E_{A} is that B_{A} is the free completion of E_{A} to a Boolean algebra. This means that $E_{\mathrm{A}} \subset B_{\mathrm{A}}$ and that every map $E_{\mathrm{A}} \to B$, for a Boolean algebra B, factors through B_{A}. This is a general occurrence:

Lemma 8.1.5. Let $X = \{x_1, \ldots, x_n\}$ be a finite set and let $F_E X$ and $F_B X$ be the free effect algebra and free Boolean algebra on this set, respectively. Let B be a Boolean algebra and suppose $f : F_E X \to B$ is an effect algebra morphism. Then f factors uniquely via the embedding i of $F_E X$ in $F_B X$. That is, there exists a unique $\hat{f} : F_B \to B$ such that $f = \hat{f} \circ i$.

$$\begin{tikzcd} F_E X \arrow[r, "f"] \arrow[d, "i"'] & B \\ F_B X \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd}$$

Moreover, the correspondence between f and $\hat{f}$ is a bijection.

Proof. An atom of $F_B X$ is of the form $x_{\phi(1)} \wedge \ldots \wedge x_{\phi(i)} \wedge \neg x_{\phi(i+1)} \wedge \ldots \wedge \neg x_{\phi(n)}$, where ϕ is some permutation of $\underline{n}$. Define $\hat{f} : F_B X \to B$ as $\hat{f}(x_{\phi(1)} \wedge \ldots \wedge x_{\phi(i)} \wedge \neg x_{\phi(i+1)} \wedge \ldots \wedge \neg x_{\phi(n)}) = f(x_{\phi(1)}) \wedge \ldots \wedge f(x_{\phi(i)}) \wedge f(x^{\perp}_{\phi(i+1)}) \wedge \ldots \wedge f(x^{\perp}_{\phi(n)})$. It is then clear that $f = \hat{f} \circ i$ and $\hat{f}$ is unique precisely because its values are fixed on the elements x_i and $F_B X$ is freely generated by these elements.

Finally, since $U_E B = U_B B =: UB$, we have

$$Hom_{\mathbf{EA}}(F_E X, A) \cong Hom_{\mathbf{Set}}(X, UA) \cong Hom_{\mathbf{BA}}(F_B X, A)$$

so the correspondence is indeed a bijection. □

8.2 Bimorphisms

In Section 6.1 we gave the statistics of a certain measurement as a table. In the previous section we made clear what exactly we meant by a table (Definition 8.1.1). We also introduced algebras for Alice and Bob which describe their measurements. Now of course to describe a table, we need to combine the measurements (and hence the algebras) of Alice and Bob. For this we introduce bimorphisms, which in turn will be related to tensor products.

Definition 8.2.1. Let A, B and C be PPCMs. A *bimorphism* $A, B \to C$ is a function $f : A \times B \to C$ such that for all $\mathrm{a}, \mathrm{a}_1, \mathrm{a}_2 \in A$ and $\mathrm{b}, \mathrm{b}_1, \mathrm{b}_2 \in B$ with $\mathrm{a}_1 \perp \mathrm{a}_2$ and $\mathrm{b}_1 \perp \mathrm{b}_2$ we have

$$f(\mathrm{a}, \mathrm{b}_1 \oslash \mathrm{b}_2) = f(\mathrm{a}, \mathrm{b}_1) \oslash f(\mathrm{a}, \mathrm{b}_2) \qquad f(\mathrm{a}_1 \oslash \mathrm{a}_2, \mathrm{b}) = f(\mathrm{a}_1, \mathrm{b}) \oslash f(\mathrm{a}_2, \mathrm{b})$$
$$f(\mathrm{a}, 0) = f(0, \mathrm{b}) = 0 \qquad f(1, 1) = 1$$

So in particular both $f(-, 1)$ and $f(1, -)$ are morphisms of PPCMs (or effect algebras if A, B and C are effect algebras). A bimorphism then contains information of these two morphisms at the same time, but also assigns values to combinations of elements from the different algebras, while still taking into account the partial structure of these algebras. This relates them to tables. Indeed, a bimorphism $t : E_{\mathrm{A}}, E_{\mathrm{B}} \to [0, 1]$ restricts to a function

$$\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to [0, 1]$$

We now show that this function is indeed a table and that this correspondence is even a bijection. In fact, we can show this for general X-tables (see below Definition 8.1.1):

Proposition 8.2.2. Let X be a PPCM (e.g. $X = [0,1]$). A function

$$\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to X$$

arises as the restriction of a bimorphism $E_\mathrm{A}, E_\mathrm{B} \to X$ if and only if it is an X-table.

Proof. First we show that the restriction of a bimorphism is an X-table. So let $t : E_\mathrm{A}, E_\mathrm{B} \to X$ be a bimorphism and τ its restriction. In particular, for $i, j, k, l \in \{0,1\}$ we have $t(\mathrm{a}_i{:}k, \mathrm{b}_j{:}l) = \tau(\mathrm{a}_i{:}k, \mathrm{b}_j{:}l)$. Then by the properties of bimorphisms it follows that

$$\begin{aligned}
\tau(\mathrm{a}_i{:}0, \mathrm{b}_j{:}0) \ovee \tau(\mathrm{a}_i{:}0, \mathrm{b}_j{:}1) \ovee \tau(\mathrm{a}_i{:}1, \mathrm{b}_j{:}0) \ovee \tau(\mathrm{a}_i{:}1, \mathrm{b}_j{:}1) &= \\
t(\mathrm{a}_i{:}0, \mathrm{b}_j{:}0) \ovee t(\mathrm{a}_i{:}0, \mathrm{b}_j{:}1) \ovee t(\mathrm{a}_i{:}1, \mathrm{b}_j{:}0) \ovee t(\mathrm{a}_i{:}1, \mathrm{b}_j{:}1) &= \\
t(\mathrm{a}_i{:}0 \ovee \mathrm{a}_i{:}1, \mathrm{b}_j{:}0) \ovee t(\mathrm{a}_i{:}0 \ovee \mathrm{a}_i{:}1, \mathrm{b}_j{:}1) &= \\
t(1, \mathrm{b}_j{:}0 \ovee \mathrm{b}_j{:}1) &= \\
t(1,1) &= 1
\end{aligned}$$

The marginalization requirement follows similarly. Indeed, notice that for any bimorphism $t : E_\mathrm{A}, E_\mathrm{B} \to X$ we have, for instance,

$$t(\mathrm{a}_0{:}0, \mathrm{b}_0{:}0) \ovee t(\mathrm{a}_0{:}0, \mathrm{b}_0{:}1) = t(\mathrm{a}_0{:}0, 1) = t(\mathrm{a}_0{:}0, \mathrm{b}_1{:}0) \ovee t(\mathrm{a}_0{:}0, \mathrm{b}_1{:}1)$$

since $\mathrm{b}_0{:}0 \ovee \mathrm{b}_0{:}1 = 1 = \mathrm{b}_1{:}0 \ovee \mathrm{b}_1{:}1$.

Second, suppose τ is a table satisfying the two conditions of Definition 8.1.1. We extend it to a bimorphism $t : E_\mathrm{A}, E_\mathrm{B} \to X$ as follows:

$$\begin{aligned}
t(\mathrm{a}_i{:}k, \mathrm{b}_j{:}l) &= \tau(\mathrm{a}_i{:}k, \mathrm{b}_j{:}l) & t(\mathrm{a}_i{:}k, 1) &= \tau(\mathrm{a}_i{:}k, \mathrm{b}_0{:}0) \ovee \tau(\mathrm{a}_i{:}k, \mathrm{b}_0{:}1) \\
t(x, 0) &= 0 & t(1, \mathrm{b}_j{:}l) &= \tau(\mathrm{a}_0{:}0, \mathrm{b}_j{:}l) \ovee \tau(\mathrm{a}_0{:}1, \mathrm{b}_j{:}l) \\
t(0, y) &= 0 & t(1,1) &= 1
\end{aligned}$$

For $i, j, k, l \in \{0,1\}$, $x \in E_A$ and $y \in E_B$. $\square$

It thus follows that the Bell table (6.2) extends to a bimorphism

$$E_\mathrm{A}, E_\mathrm{B} \to [0,1]$$

Now bimorphisms are not morphisms of effect algebras and therefore do not exist within the category of effect algebras. We can remedy this by introducing *tensor products*. These allow us to consider bimorphisms $A, B \to X$ as actual morphisms $A \otimes B \to X$. In particular, a table $t : E_\mathrm{A}, E_\mathrm{B} \to [0,1]$, and more specifically, the Bell table, can be considered as a morphism $E_\mathrm{A} \otimes E_\mathrm{B} \to [0,1]$.

Definition 8.2.3. A *tensor product* of two PPCMs E, E' is given by a PPCM $E \otimes E'$ and a bimorphism $i : E, E' \to E \otimes E'$, such that for every bimorphism $f : E, E' \to X$ there is a unique morphism $g : E \otimes E' \to X$ such that $f = g \circ i$.

$$\begin{tikzcd}
E, E' \arrow[r, "f"] \arrow[d, "i"'] & X \\
E \otimes E' \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

So tensor products give a bijective correspondence between morphisms $E \otimes E' \to F$ and bimorphisms $E, E' \to F$. In fact, all tensor products of effect algebras exist:

Proposition 8.2.4. Let E, E' be effect algebras. The tensor product $E \otimes E'$ exists and is itself an effect algebra.

Proof. For the proof, see [56]. $\square$

It should be noted that in general, the tensor product of two effect algebras can be trivial, i.e., the algebra where $0 = 1$. An example of this occurrence is given in [46]. Note that there it is claimed that the tensor product need not always exist, however, these authors do not take the trivial effect algebra to be an effect algebra.

We now give concrete descriptions of the algebras $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$ and $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$.

Proposition 8.2.5.

- The tensor product of Boolean algebras, $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$, is the free Boolean algebra on the four elements $\{\mathrm{a}_0{:}0, \mathrm{a}_1{:}0, \mathrm{b}_0 : 0, \mathrm{b}_1{:}0\}$.

- The tensor product of effect algebras, $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$, is the effect algebra with 16 atoms of the form $\mathrm{a}_i{:}k \wedge \mathrm{b}_j{:}l$ for $i, j, k, l \in \{0, 1\}$. The tests in which all elements are atoms are of the form

$$(a \wedge b, a \wedge b^{\perp}, a^{\perp} \wedge \tilde{b}, a^{\perp} \wedge \tilde{b}^{\perp}) \tag{8.10}$$

or

$$(a \wedge b, a^{\perp} \wedge b, \tilde{a} \wedge b^{\perp}, \tilde{a}^{\perp} \wedge b^{\perp}) \tag{8.11}$$

where $a, \tilde{a}$ and $b, \tilde{b}$ are atoms in E_{A} and E_{B}, respectively. This then also gives all non-trivial sums in $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$.

Proof. In the Boolean case it is a general result that for finite Boolean algebras the tensor product is equal to the coproduct of Boolean algebras:

$$2^n \otimes 2^m \cong 2^{n\times m} \cong 2^n \oplus 2^m \tag{8.12}$$

Indeed, for Boolean algebras, a pair of morphisms (f, g) corresponds to a bimorphism $t_{f,g}$, defined by $t_{f,g}(a, b) = f(a) \wedge g(b)$, so that the universal property of the tensor product is precisely that of the coproduct.

For the second statement. Let E be the effect algebra as in the proposition. First we note that for any effect algebra X, any elements $a, \tilde{a} \in E_{\mathrm{A}}$ $b, \tilde{b} \in E_{\mathrm{B}}$, and bimorphism $f : E_{\mathrm{A}}, E_{\mathrm{B}} \to X$, we have

$$\begin{aligned}
1 &= f(1,1) \\
&= f(a \ovee a^{\perp}, 1) \\
&= f(a,1) \ovee f(a^{\perp}, 1) \\
&= f(a,b) \ovee f(a,b^{\perp}) \ovee f(a^{\perp},\tilde{b}) \ovee f(a^{\perp},\tilde{b}^{\perp})
\end{aligned} \tag{8.13}$$

and similarly

$$\begin{aligned}
1 &= f(1,1) \\
&= f(1, b \ovee b^{\perp}) \\
&= f(1,b) \ovee f(1, b^{\perp}) \\
&= f(a,b) \ovee f(a^{\perp},b) \ovee f(\tilde{a},b^{\perp}) \ovee f(\tilde{a}^{\perp},b^{\perp})
\end{aligned} \tag{8.14}$$

Now let $g : E_{\mathrm{A}}, E_{\mathrm{B}} \to X$ be a bimorphism. We define $\tilde{g} : E \to X$ by extension of $\tilde{g}(\mathrm{a}_i{:}k \wedge \mathrm{b}_j{:}l) = g(\mathrm{a}_i{:}k, \mathrm{b}_j{:}l)$. We then need to show that $\tilde{g}(a \wedge b) \perp \tilde{g}(a' \wedge b')$ whenever $a \wedge b \perp a' \wedge b'$. This follows from (8.13). The map $g \mapsto \tilde{g}$ is invertible. Indeed, given $\tilde{g}$, define $g(a, b) = \tilde{g}(a \wedge b)$. We conclude the proof by noting that we now have bijections:

$E \to X$	effect algebra morphism
$E_{\mathrm{A}}, E_{\mathrm{B}} \to X$	bimorphism
$E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to X$	effect algebra morphism

Hence E is a tensor product and by Corollary 7.4.3 that we also have $E \cong E_{\mathrm{A}} \otimes E_{\mathrm{B}}$. we have $E \cong E_{\mathrm{A}} \otimes E_{\mathrm{B}}$. □

Note that each of the tests in (8.10) and (8.11) describe eight tests, however, four of these overlap, so the tensor product $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ has 12 tests with only atoms. A nice graphical representation of this algebra is given in Section 9.5 when we discuss test spaces.

8.3 Realizations

So far we have seen that tables describe no-signalling probability distributions over outcomes of joint measurement settings. Furthermore, such a table arises as a bimorphism, or morphism of the tensor product, of certain algebras describing these settings and outcomes. We also saw that the statistics in the Bell table can be obtained by performing quantum measurements, but we claimed that these statistics cannot be obtained from a local hidden variable.

In contrast, it is easy to see that a table such as

	0,0	0,1	1,0	1,1
a_0b_0	1	0	0	0
a_0b_1	0	1	0	0
a_1b_0	0	0	1	0
a_1b_1	0	0	0	1

(8.15)

can be realized both classically and quantum mechanically. It is also known that the following table, known as the *PR box* [12], cannot be realized either way.

	0,0	0,1	1,0	1,1
a_0b_0	1/2	0	0	1/2
a_0b_1	1/2	0	0	1/2
a_1b_0	1/2	0	0	1/2
a_1b_1	0	1/2	1/2	0

(8.16)

Classically, this can be seen in the same was as in the Bell table, which we show in Proposition 8.3.6. Quantum mechanically this follows from the fact that the PR box violates *Tsirelson's bound* [91]. In this section we will consider how general no-signalling tables of this kind might be realized. We consider quantum realizations and classical realizations. Quantum realizable tables are those tables which can occur by performing measurements on a quantum system whereas classical realizable tables are those tables which we can explain in a realistic setting.

We first consider quantum realizations. From a quantum theoretical point of view, one is interested whether a general table can be obtained as measurements on some quantum system. Such a system is modelled by a Hilbert space H and the bounded operators, $B(H)$, on this space. A yes-no question such as 'is the outcome of measuring a_0 equal to 1' is given by a projection on this Hilbert space and these projections form the effect algebra $Proj(H)$ (Example 7.2.5). Therefore we are interested in factorizations via the projections on some Hilbert space. Furthermore, since by locality the measuements of Alice cannot be influenced by Bob and *vice versa*, we ask for a factorization through a Hilbert space for each party.

Definition 8.3.1. A *quantum realization* for a distribution on joint measurements, i.e., a table $t : E_{\mathrm{A}}, E_{\mathrm{B}} \to [0,1]$, is given by finite dimensional Hilbert spaces $H_{\mathrm{A}}, H_{\mathrm{B}}$, two PPCM maps $r_{\mathrm{A}} : E_{\mathrm{A}} \to Proj(H_{\mathrm{A}})$ and $r_{\mathrm{B}} : E_{\mathrm{B}} \to Proj(H_{\mathrm{B}})$, and a bimorphism $p : Proj(H_{\mathrm{A}}), Proj(H_{\mathrm{B}}) \to [0,1]$, such that for all $a \in E_{\mathrm{A}}$ and $b \in E_{\mathrm{B}}$ we have $p(r_{\mathrm{A}}(a), r_{\mathrm{B}}(b)) = t(a,b)$.

Of course by the correspondence between bimorphisms and morphisms from the tensor product, we can rephrase this in in terms of tensors. Denote by $a \otimes b$ the element $i(a,b)$, where i is the bimorphism from Definition 8.2.3.

Definition 8.3.2. A *quantum realization* for a distribution on joint measurements, i.e., a table $t : E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to [0,1]$, is given by finite dimensional Hilbert spaces $H_{\mathrm{A}}, H_{\mathrm{B}}$, two PPCM maps $r_{\mathrm{A}} : E_{\mathrm{A}} \to Proj(H_{\mathrm{A}})$ and $r_{\mathrm{B}} : E_{\mathrm{B}} \to Proj(H_{\mathrm{B}})$, and an effect algebra morphism $p : Proj(H_{\mathrm{A}}) \otimes Proj(H_{\mathrm{B}}) \to [0,1]$, such that for all $a \in E_{\mathrm{A}}$ and $b \in E_{\mathrm{B}}$ we have $p(r_{\mathrm{A}}(a) \otimes r_{\mathrm{B}}(b)) = t(a,b)$.

$$\begin{tikzcd}
E_{\mathrm{A}} \arrow[d, "r_{\mathrm{A}}"'] & E_{\mathrm{B}} \arrow[d, "r_{\mathrm{B}}"] & E_{\mathrm{A}} \otimes E_{\mathrm{B}} \arrow[rr, "t"] \arrow[dr, "r_{\mathrm{A}} \otimes r_{\mathrm{B}}"'] & & {[0,1]} \\
Proj(H_{\mathrm{A}}) & Proj(H_{\mathrm{B}}) & & Proj(H_{\mathrm{A}}) \otimes Proj(H_{\mathrm{B}}) \arrow[ur, "p"'] &
\end{tikzcd} \tag{8.17}$$

Note that the map $r_{\mathrm{A}}, r_{\mathrm{B}}$ in the bimorphism formulation is itself not a bimorphism. As such, we cannot take an arbitrary embedding $E_{\mathrm{A}} \otimes E_{\mathrm{B}} \to Proj(H_{\mathrm{A}}) \otimes Proj(H_{\mathrm{B}})$ in the effect algebra formulation.

The Bell table (6.1) has a quantum realization, with $H_{\mathrm{A}} = H_{\mathrm{B}} = \mathbb{C}^2$ as shown in Proposition 6.1.1.

As an aside, the relation between (mixed) states and this notion of quantum realization is as follows. Let $H \otimes H'$ be the tensor product of Hilbert spaces. By Gleason's theorem [57] there is a bijection between morphisms $Proj(H \otimes H') \to [0,1]$ and density matrices on $H \otimes H'$ if $dim(H \otimes H') > 3$, which is certainly the case for $H = H' = \mathbb{C}^2$. The canonical map $Proj(H), Proj(H') \to Proj(H \otimes H')$ given by $p, q \mapsto p \otimes q$, where $p \otimes q(h \otimes h') = p(h) \otimes q(h')$, is a bimorphism. Therefore, any density matrix gives rise to a bimorphism $Proj(H), Proj(H') \to [0,1]$.

Next we consider classical realizations. As mentioned in Chapter 6, classically we consider outcomes for all measurement settings, i.e., sample spaces, and then consider probability distributions over these spaces to account for the fact that we might not know in which hidden variable the system is in. In the case of the Bell scenario, this sample space is the space of maps $\{0,1\}^{\{a_0,a_1,b_0,b_1\}}$. Let us again denote a map in this sample space as $a_0{:}i \wedge a_1{:}j \wedge b_0{:}k \wedge b_1{:}l$, with $i,j,k,l \in \{0,1\}$. A probability distribution over this sample space

$$p : \{0,1\}^{\{a_0,a_1,b_0,b_1\}} \to [0,1]$$

is now interesting because it induces a table by marginalization.

Proposition 8.3.3. Any probability distribution on the set $\{0,1\}^{\{a_0,a_1,b_0,b_1\}} \cong \{0,1\}^{\underline{4}}$, induces a table by summing over the settings that are not tested. For instance, the value of the table at $a_0{:}i, b_0{:}k$ is

$$t(a_0{:}i, b_0{:}k) = \sum_{j,l} p(a_0{:}i \wedge a_1{:}j \wedge b_0{:}k \wedge b_1{:}l) \tag{8.18}$$

Proof. Normalization is immediate, for example:

$$\sum_{i,k} t(a_0{:}i, b_0{:}k) = \sum_{i,j,k,l} p(a_0{:}i \wedge a_1{:}j \wedge b_0{:}k \wedge b_1{:}l) = 1$$

Marginalization is similar, for example:

$$\begin{aligned} t(a_0{:}i, b_0{:}0) + t(a_0{:}i, b_0{:}1) &= \sum_{j,k,l} p(a_0{:}i \wedge a_1{:}j \wedge b_0{:}k \wedge b_1{:}l) \\ &= t(a_0{:}i, b_1{:}0) + t(a_0{:}i, b_1{:}1) \end{aligned}$$

□

This leads us to classical realizable tables:

Definition 8.3.4. A table (or X-table) t is *classically realizable* if there is a function $p : \{0,1\}^4 \to [0,1]$ (or X) such that

- $\sum_{s \in \{0,1\}^4} p(s) = 1$,
- $t(a_0{:}i, b_0{:}k) = \sum_{j,l} p(a_0{:}i \wedge a_1{:}j \wedge b_0{:}k \wedge b_1{:}l)$.

and similar equations for the other settings.

It will be no surprise that classically realizable tables are related to Boolean algebras in the same way as general tables are related to effect algebras. Indeed, by Proposition 8.2.5, the elements $\mathrm{a}_0{:}i \wedge \mathrm{a}_1{:}j \wedge \mathrm{b}_0{:}k \wedge \mathrm{b}_1{:}l$ are precisely the atoms of $B_\mathrm{A} \otimes B_\mathrm{B}$. As such, a probability distribution $p : \{0,1\}^{\underline{4}}$ extends to the full Boolean algebra $B_\mathrm{A} \otimes B_\mathrm{B} \cong \mathcal{P}(\{0,1\}^{\underline{4}})$. Moreover, the second condition in Definition 8.3.4 corresponds precisely to the way to obtain the element $\mathrm{a}_0{:}i \wedge \mathrm{b}_0{:}k$ from the atoms $\mathrm{a}_0{:}i \wedge \mathrm{a}_1{:}j \wedge \mathrm{b}_0{:}k \wedge \mathrm{b}_1{:}l$, if we understand the sum as taking the join. This leads us to an equivalent definition of classically realizable tables. Since tables correspond to bimorphisms and hence to morphisms from the tensor product, we have the following two results:

Proposition 8.3.5. A *classical realization* for a bimorphism $t : E_\mathrm{A}, E_\mathrm{B} \to [0,1]$ is given by a factorization of t though $B_\mathrm{A}, B_\mathrm{B}$. That is, there are two effect algebra morphisms $r_\mathrm{A} : E_\mathrm{A} \to B_\mathrm{A}$, $r_\mathrm{B} : E_\mathrm{B} \to B_\mathrm{B}$ and a bimorphism $p : B_\mathrm{A}, B_\mathrm{B} \to [0,1]$ such that for all $a \in E_\mathrm{A}$ and $b \in E_\mathrm{B}$ we have $p(r_\mathrm{A}(a), r_\mathrm{B}(b)) = t(a,b)$.

To phrase this in terms of effect algebras we again note that the map $r_\mathrm{A}, r_\mathrm{B}$ is not a bimorphism and as such, we cannot take an arbitrary embedding $E_\mathrm{A} \otimes E_\mathrm{B} \to B_\mathrm{A} \otimes B_\mathrm{B}$.

Proposition 8.3.6. A *classical realization* for a morphism $t : E_\mathrm{A} \otimes E_\mathrm{B} \to [0,1]$ is given by two effect algebra morphisms $r_\mathrm{A} : E_\mathrm{A} \to B_\mathrm{A}$, $r_\mathrm{B} : E_\mathrm{B} \to B_\mathrm{B}$ and a morphism $p : B_\mathrm{A} \otimes B_\mathrm{B} \to [0,1]$ such that for all $a \in E_\mathrm{A}$ and $b \in E_\mathrm{B}$ we have $p(r_\mathrm{A}(a) \otimes r_\mathrm{B}(b)) = t(a \otimes b)$.

$$
\begin{tikzcd}
E_\mathrm{A} \arrow[d, "r_\mathrm{A}"'] & E_\mathrm{B} \arrow[d, "r_\mathrm{B}"] \\
B_\mathrm{A} & B_\mathrm{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
E_\mathrm{A} \otimes E_\mathrm{B} \arrow[rr, "t"] \arrow[dr, "r_\mathrm{A} \otimes r_\mathrm{B}"'] & & {[0,1]} \\
& B_\mathrm{A} \otimes B_\mathrm{B} \arrow[ur, "p"'] &
\end{tikzcd}
\tag{8.19}
$$

By Lemma 8.1.5, it follows that whenever there is a factorization through any Boolean algebras B and B', there automatically also is a factorization through B_A and B_B, since they are the free completions of E_A and E_B, respectively.

Proposition 8.3.3 shows that classical probability distributions give rise to tables. Interestingly enough, not all (no-signalling) tables arise from such a construction .

Proposition 8.3.7. The Bell table (6.2), $t : E_\mathrm{A} \otimes E_\mathrm{B} \to [0,1]$, does not factor through the canonical embedding $i : E_\mathrm{A} \otimes E_\mathrm{B} \to B_\mathrm{A} \otimes B_\mathrm{B}$. That is, it has no classical realization.

$$
\begin{tikzcd}
E_{\mathrm{A}} \otimes E_{\mathrm{B}} \arrow[rr, "t"] \arrow[dr, "i"'] & & {[0,1]} \\
& B_{\mathrm{A}} \otimes B_{\mathrm{B}} \arrow[ur] &
\end{tikzcd}
\tag{8.20}
$$

This proposition is well known and there are multiple proofs. See [2] for an example. Here we will give a similar proof, but in an algebraic setting, based on factorizations of morphisms.

Proof. Under the embedding $E_{\mathrm{A}} \otimes E_{\mathrm{B}} \hookrightarrow B_{\mathrm{A}} \otimes B_{\mathrm{B}}$, we view $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ as a subset of $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$. We can thus write:

$$
\begin{aligned}
\mathrm{a}_0{:}0 \wedge \mathrm{b}_0{:}0 =& (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
&\vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \qquad (\star) \\
\mathrm{a}_0{:}0 \wedge \mathrm{b}_1{:}1 =& (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \\
&\vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}1) \vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}1) \\
\mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 =& (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \vee (\mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
&\vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \vee (\mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \\
\mathrm{a}_1{:}0 \wedge \mathrm{b}_1{:}0 =& (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \vee (\mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
&\vee (\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}0) \vee (\mathrm{a}_0{:}1 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}1 \wedge \mathrm{b}_1{:}0)
\end{aligned}
$$

Now suppose there is a classical realization and let $\phi : B_{\mathrm{A}} \otimes B_{\mathrm{B}} \to [0,1]$ be the supposed probability distribution on $B_{\mathrm{A}} \otimes B_{\mathrm{B}}$. If we apply ϕ to both sides of the above equations we find in the Bell table (6.2), that the sum of the right hand sides must add up to $\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}$, respectively.

However, adding the last three equations we obtain

$$
\begin{aligned}
\frac{3}{8} =& \phi(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
&+\phi(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}0) \\
&+\phi(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}0 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \\
&+\phi(\mathrm{a}_0{:}0 \wedge \mathrm{a}_1{:}1 \wedge \mathrm{b}_0{:}0 \wedge \mathrm{b}_1{:}1) \\
&+\phi(\text{other terms}),
\end{aligned}
$$

but the first four terms already add up to $\frac{1}{2}$ by the first equation ($\star$) above, and since ϕ takes values in $[0,1]$, this cannot be. $\square$

This proof reveals more. Suppose we wish to know whether a table on $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ factors through some finite Boolean algebra B. Then, since the injection of the atoms in $E_{\mathrm{A}} \otimes E_{\mathrm{B}}$ can be written as a join of atoms in B, and the table dictates the value on this injection, the question of factorization reduces to a system of linear equations. The corresponding matrix to these equations is what is known as the *incidence matrix* in [2].

8.4 Generalization of the Bell 2,2,2 type

Finally, we wish to give some insight into more general settings than the *Bell* $(2,2,2)$ scenario where we have two observers, each with two measurement settings, which each have two outcomes. More generally we consider Bell (k,l,m) settings where we have k observers, called $A^1,\ldots,A^k$ (Alice is common name among scientists). Each A^i has l measurement settings $\mathrm{a}^i_j, j = 1,\ldots,l$ and each of these settings has m outcomes $\mathrm{a}^i_j{:}o, o = 1,\ldots m$. The assumptions that every observer has an equal amount of measurement settings is for notational convenience. The assumption on outcomes can always be fulfilled by adding outcomes which never occur. This way we can, without loss of generality, consider a fixed outcome set O. For each setting, the powerset $\mathcal{P}(O)$ is now an algebra describing that setting. For $X \subset O$ we obtain elements $\mathrm{a}^i_j{:}X := \bigvee_{x\in X} \mathrm{a}^i_j{:}x$, which we can think of as course grained outcomes. The coproduct over these algebras gives an algebra for A^i, $E_{\mathrm{A}^i} := \bigoplus \mathcal{P}(O)$. By the property of the coproduct, a morphism $E_{\mathrm{A}^i} \to [0,1]$ is equivalent to a probability distribution for each measurement setting. The tensor product over the algebras $E := \bigotimes_i E_{\mathrm{A}^i}$ then gives us our desired algebra.

An atom in this algebra is of the form $\mathrm{a}^1_\alpha{:}o^1_\alpha \wedge \ldots \mathrm{a}^k_\zeta{:}o^k_\zeta$ and we have an embedding into the boolean tensor product of the free Boolean completions of the E_{A^i} as before, giving classical realizations for a probability distribution $E \to [0,1]$.

8.5 Sheaf theoretic characterization

In the previous sections we have developed an effect algebraic framework to study scenarios in non-locality and contextuality. In Section 7.3 we showed that effect algebras embed in the presheaf category $[\mathbb{F}, \mathbf{Set}]$. Moreover, in a paper by Abramsky and Brandenburger ([2]) a framework for non-locality and contextuality based on presheaves over measurements is developed. Here we describe our effect algebraic framework in terms of presheaves. Then in Section 8.6 we will investigate the relation between these two approaches. To this end, we first introduce some categorical notions related to presheaves, which are best described in covariant form for our purposes.

Definition 8.5.1. Let c be an object of a category $\mathbb{C}$. A *sieve* on c is a set of morphisms with common domain, $S \subseteq \{f \mid f : c \to d\}$ that is closed under post-composition, i.e., $f \in S \implies gf \in S$.

Definition 8.5.2. Let S be a sieve on an object $c \in \mathbb{C}$ and let $F : \mathbb{C} \to \mathbf{Set}$ be a functor. A collection of elements $(x_f)_{f \in S}$, with $x_f \in F(d)$ whenever $f : c \to d$, is called a *matching family for S in F* if for $f, f' \in S$ and g, g' morphisms such that $g \circ f = g' \circ f'$, we have $F(g)(x_f) = F(g')(x_{f'})$. That is, if

$$\begin{tikzcd} & d \arrow[dr, "g"] & \\ c \arrow[ur, "f"] \arrow[dr, "f'"'] & & e \\ & d' \arrow[ur, "g'"'] & \end{tikzcd}$$

commutes, then

$$\begin{tikzcd} F(d) \ni x_f \arrow[dr, mapsto, "F(g)"] & \\ & F(e) \ni F(g)(x_f) = F(g')(x_{f'}) \\ F(d') \ni x_{f'} \arrow[ur, mapsto, "F(g')"'] & \end{tikzcd}$$

An *amalgamation* for a matching family $(x_f)_{f \in S}$ is an element $x \in F(c)$ such that $x_f = F(f)(x)$.

Sieves can equivalently be described as subfunctors of the Hom-functor. For a sieve S on an object $c \in \mathbb{C}$, define the functor $\overline{S} : \mathbb{C} \to \mathbf{Set}$ as follows: for an object d, let $\overline{S}(d) := \{f : c \to d \mid f \in S\} \subset Hom(c,d)$, be the subset of morphisms which are in the sieve S and for a morphism $g : a \to b$, let $\overline{S}(g) : \overline{S}(a) \to \overline{S}(b)$ be the map which post-composes with g, i.e., $\overline{S}(g)(h) = g \circ h$, which indeed is in $\overline{S}(b)$ because S is a sieve. In a similar vain, compatible families are natural transformations $\overline{S} \to F$. Indeed, the maps $\overline{S}(d) \ni f \mapsto x_f \in F(d)$ are the components of this transformation. For a morphisms $g : d \to e$ we then have $F(g)x_f = x_{gf}$, so that if $gf = g'f'$ we $F(g)x_f = x_{gf} = x_{g'f'} = F(g')x_{f'}$. Finally, a matching family has an amalgamation if the corresponding natural transformation factorizes via the Hom functor $\overline{S} \hookrightarrow Hom(c, -)$. The amalgamation is then the element corresponding to the identity on c.

Now any set of morphisms $\{f_i\}_{i \in I}$ with common domain c generates a sieve S by closure under post-composition. This allows us to make a connection between tables and matching families. Indeed, consider the family of functions $\pi_{i,j} : \{0,1\}^4 \to \{0,1\}^2$, where $i, j \in \{0,1\}$, given by

$$\pi_{i,j}(o_{\mathrm{a}_0}, o_{\mathrm{a}_1}, o_{\mathrm{b}_0}, o_{\mathrm{b}_1}) = (o_{\mathrm{a}_i}, o_{\mathrm{b}_j}) \tag{8.21}$$

We show that there is a correspondence between matching families for the sieve generated by (8.21) in $T(X)$ and X-tables. In particular, when $X = [0,1]$, we find that tables correspond to matching families in the distribution functor.

Proposition 8.5.3. Let X be a PPCM (e.g., $[0,1]$) and let $T(X) : \mathbb{F} \to \mathbf{Set}$ be the presheaf of tests on X. Then there is a bijective correspondence between X-tables τ and matching families $(d_{i,j})_{i,j \in \{0,1\}}$ in $T(X)(\{0,1\}^2)$ for $\{\pi_{i,j} \mid i,j \in \{0,1\}\}$, given by $\tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') = d_{i,j}(o,o')$.

Moreover, an X-table has a classical realization if and only if the corresponding matching family has an amalgamation.

Proof. Let $d_{i,j}$ be such a matching family and let $\tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') = d_{i,j}(o,o')$. Then, because $d_{i,j}$ is a test we have

$$\sum_{o,o'} \tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o') = \sum_{o,o'} d_{i,j}(o,o') = 1$$

To show no-signalling, we first consider $\pi_{0,0}$ and $\pi_{0,1}$. Define maps $g_{0,0}, g_{0,1} : \{0,1\}^2 \to \{0,1\}$ by

$$g_{0,0} = g_{0,1} : \begin{cases} (0,0),(0,1) \mapsto 0 \\ (1,0),(1,1) \mapsto 1 \end{cases}$$

Then $g_{0,0} \circ \pi_{0,0} = g_{0,1} \circ \pi_{0,1}$, so compatibility of the matching family implies

$$T(X)(g_{0,0})(d_{0,0}) = T(X)(g_{0,1})(d_{0,1})$$

Writing out these terms gives

$$\begin{aligned} T(X)(g_{0,0})(d_{0,0}) &= \left(\bigvee_{(o,o')\in g_{0,0}{}^{-1}(i)} d_{0,0}(o,o') \right)_{i=0,1} \\ &= (d_{0,0}(0,0) + d_{0,0}(0,1), d_{0,0}(1,0) + d_{0,0}(1,1)) \end{aligned} \tag{8.22}$$

and

$$\begin{aligned} T(X)(g_{0,1})(d_{0,1}) &= \left(\bigvee_{(o,o')\in g_{0,1}{}^{-1}(i)} d_{0,1}(o,o') \right)_{i=0,1} \\ &= (d_{0,1}(0,0) + d_{0,1}(0,1), d_{0,1}(1,0) + d_{0,1}(1,1)) \end{aligned}$$

Equality of the first entries of these tests then gives

$$\begin{aligned} \tau(\mathrm{a}_0{:}0, \mathrm{b}_0{:}0) + \tau(\mathrm{a}_0{:}0, \mathrm{b}_0{:}1) &= d_{0,0}(0,0) + d_{0,0}(0,1) \\ &= d_{0,1}(0,0) + d_{0,1}(0,1) \\ &= \tau(\mathrm{a}_0{:}0, \mathrm{b}_1{:}0) + \tau(\mathrm{a}_0{:}0, \mathrm{b}_1{:}1) \end{aligned}$$

The other no-signalling conditions follow similarly.

For the converse, let τ be an X-table and define $d_{i,j} \in T(X)(\{0,1\}^4)$ via $d_{i,j}(o,o') = \tau(\mathrm{a}_i{:}o, \mathrm{b}_j{:}o')$. Let $g, g' : \{0,1\}^2 \to \underline{n}$ be functions such that $g \circ \pi_{i,j} = g' \circ \pi_{k,l}$.

$$\begin{tikzcd} (o_{\mathrm{a}_i}, o_{\mathrm{b}_k}) \arrow[dr, mapsto, "g"'] & (o_{\mathrm{a}_0}, o_{\mathrm{a}_1}, o_{\mathrm{b}_0}, o_{\mathrm{b}_1}) \arrow[l, mapsto, "\pi_{i,j}"'] \arrow[r, mapsto, "\pi_{k,l}"] & (o_{\mathrm{a}_k}, o_{\mathrm{b}_l}) \arrow[dl, mapsto, "g'"] \\ & = & \end{tikzcd} \tag{8.23}$$

where $=$ is the corresponding image in $\underline{n}$.

We can then deduce, for example for $\pi_{0,0}$ and $\pi_{0,1}$, that

$$g(0,0) = g \circ \pi_{0,0}(0,0,0,1) = g' \circ \pi_{0,1}(0,0,0,1) = g'(0,1)$$

Doing the same for $(0,0,0,0)$ gives $g(0,0) = g'(0,0)$ and starting from $(0,0,1,0)$ we find that $g(0,1) = g'(0,0)$. This means that if $(0,0) \in g^{-1}(i)$, then also $(0,1) \in g^{-1}(i)$. Similarly we find $g(1,0) = g(1,1) = g'(1,0) = g'(1,1)$.

We wish to show that $T(X)(g)(d_{0,0)} = T(X)(g')(d_{0,1})$, but this is clear since the terms in these expressions always come two-by-two and by a calculation similar to (8.22) are precisely the no-signalling conditions that hold for τ. This works for every pair $(i,j),(k,l)$ where either $i = k$ or $j = l$. If $(i,j) = (k,l)$ the result is trivial and if $i \neq k$ and $j \neq l$ the result follows from the fact that if $h \circ \pi_{i,j} = h' \circ \pi_{k,l}$ then h and h' are constant.

Finally, note that a distribution on the classical sample space $\{0,1\}^4$ is an element of $(T(X))(\{0,1\}^4)$, so an amalgamation of a matching family corresponds to a classical realization. $\square$

Relating to the Bell table (6.2), we find it induces a matching family for the sieve S_π generated by the morphisms $(\pi_{i,j})_{i,j}$ in the distributions functor $D = T[0,1]$, which does not have an amalgamation.

$$\begin{tikzcd} \overline{S}_\pi \arrow[rr, "\tau"] \arrow[dr] & & D \\ & \mathbb{F}(\{0,1\}^4, -) \arrow[ur, "|" marking] & \end{tikzcd} \tag{8.24}$$

We can relate this non-factorization result to the result of Proposition 8.3.7. Applying the test functor $T : \mathbf{EA} \to [\mathbb{F}, \mathbf{Set}]$, which is full and faithful from effect algebras (Corollary 7.5.4), to diagram (8.20), we obtain the following non-factoring diagram

$$\begin{tikzcd} T(E_\mathrm{A} \otimes E_\mathrm{B}) \arrow[rr, "T\tau"] \arrow[dr] & & T([0,1]) \\ & T(B_\mathrm{A} \otimes B_\mathrm{B}) \arrow[ur, "|" marking] & \end{tikzcd} \tag{8.25}$$

We then find that in fact diagrams (8.24) and (8.25) are isomorphic. Indeed, $T([0,1]) \cong D$ and $T(B_\mathrm{A} \otimes B_\mathrm{B}) \cong T(\mathcal{P}(\{0,1\}^4)) \cong \mathbb{F}(\{0,1\}^4, -)$ by commutation of diagram (7.12). It remains to show that $T(E_\mathrm{A} \otimes E_\mathrm{B})$ is isomorphic to the sieve S_π generated by the maps $(\pi_{i,j})_{i,j}$. For this, we identify $\{0,1\}^4$ with the atoms of $B_\mathrm{A} \otimes B_\mathrm{B}$. The maps $\pi_{i,j}$ can then be seen as ways to make atoms of $E_\mathrm{A} \otimes E_\mathrm{B}$ from those of $B_\mathrm{A} \otimes B_\mathrm{B}$. A morphism in $S_\pi(\underline{n})$ is a composite $g \circ \pi_{i,j}$ for some $g : \{0,1\}^2 \to \underline{n}$ and hence corresponds to a test in $T(E_\mathrm{A} \otimes E_\mathrm{B})(\underline{n})$.

8.6 Relation to the work of Abramsky and Brandenburger

Abramsky and Brandenburger [2] phrase Bell's paradox in terms of compatible local sections which have no global section. In order to relate our statement with theirs, we start by recalling their framework and some general notation.

Fix a set X, and consider the presheaf category $[\mathcal{P}(X)^{op}, \mathbf{Set}]$, where $\mathcal{P}(X)^{op}$ is ordered by reverse inclusion, i.e., there is a (unique) morphism $U \to V$ if and only if $V \subset U$.

Definition 8.6.1. For a presheaf $F : \mathcal{P}(X)^{op} \to \mathbf{Set}$, an element $s \in F(U)$ is called a *section* for F over U. A section over X is called a *global section*.

If $s \in F(U)$ is a section and $V \subset U$, we obtain a section by restriction:

$$s|_V := F(V \subset U)s \tag{8.26}$$

Two section s_1 over U_1 and s_2 over U_2 are *compatible* if they coincide on their overlap, that is, when

$$s_1|_{U_1 \cap U_2} = s_2|_{u_1 \cap U_2} \tag{8.27}$$

We say that a compatible family of sections s_i over U_i has a global section if there exists a global section s which restricts to s_i for all inclusions of the u_i in X.

Abramsky and Brandenburger work over the set X of measurements. They also fix a set O of outcomes and a family $\mathcal{M}$ of subsets of X, called a *measurement cover*, subject to the following conditions:

- $\mathcal{M}$ covers X, i.e., for all $x \in X$, there exists $C \in \mathcal{M}$ with $x \in C$.
- $\mathcal{M}$ is an anti-chain, i.e., for $C, C' \in \mathcal{M}$, if $C \subset C'$ then $C = C'$.

The measurement cover represents the set of maximal compatible measurements. For example, in the Bell scenario we have $X = \{a_0, a_1, b_0, b_1\}$, $O = \{0, 1\}$ and $\mathcal{M} = \{\{a_i, b_j\} \mid i, j \in \{0, 1\}\}$.

Now if $U \subset X$ is a set of measurements, then an *event* is an assignment of outcomes to the measurements in U. That is, an elements $s \in O^U$. Such an element is therefore a section of the *event presheaf* $O^{(-)} : \mathcal{P}(X)^{op} \to \mathbf{Set}$,

$U \mapsto O^U$. Interest now lies in distributions over these sections, so consider the presheaf $D(O^{(-)})$. While compatible sections over the event presheaf always have global section, Bell's paradox says we can find compatible local sections s_C over $C \in \mathcal{M}$ for the distributions over events, $D(O^{(-)})$, which do not have a global section.

The first step in relating this framework to our framework is to rephrase the sheaf theoretic analysis of Abramsky and Brandenburger in terms of a non-factorization in the presheaf category $[\mathcal{P}(X)^{\mathrm{op}}, \mathbf{Set}]$. The inclusion of the measurement cover $\mathcal{M}$ in X gives a set of morphisms $X \to C$ in $\mathcal{P}(X)^{op}$, which generate a sieve $S_{\mathcal{M}}$. A matching family for $S_{\mathcal{M}}$ in $D(O^{(-)})$ then corresponds to a compatible family of sections of distributions over $\mathcal{M}$ and the existence of a global section corresponds to the existence of an amalgamation.

Following the discussion below Definition 8.5.2, we obtain a preseaf for the measurement cover $\overline{S}_{\mathcal{M}}$, which satisfies $\overline{S}_{\mathcal{M}}(U) = \{*\}$ if $U \subset C \in \mathcal{M}$ and $\overline{S}_{\mathcal{M}}(U) = \emptyset$ otherwise. It is a subfunctor of $1 := Hom_{\mathcal{P}(X)^{op}}(X, -)$, satisfying $1(U) = \{*\}$ for all subsets $U \subset X$. Here $\{*\}$ is a one element set representing the unique morphism corresponding to an inclusion. Moreover, the matching family, or compatible sections, becomes a natural transformation $\overline{S}_{\mathcal{M}} \to D(O^{(-)})$. Hence we may represent the non-existence of a global section as the non-factorization of the following diagram:

$$\begin{tikzcd} \overline{S}_{\mathcal{M}} \arrow[rr] \arrow[dr] & & D(O^{(-)}) \\ & 1 \arrow[ur, "|" marking] & \end{tikzcd} \tag{8.28}$$

8.6.1 Transferring the paradox to other categories via adjunctions

We now have several non-factoring diagrams in different categories, which describe some local behaviour which is incompatible from a global point of view. Our next step is to relate these diagrams. Adjunctions allow us to transport non-factoring diagrams to other categories (Lemma 8.6.2). We construct an adjunction between **EA** and $[\mathcal{P}(X)^{op}, \mathbf{Set}]$ which will link the diagrams (8.28) and (8.20).

Lemma 8.6.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $R : \mathcal{D} \to \mathcal{C}$ be a functor with a left adjoint $L : \mathcal{C} \to \mathcal{D}$. Let $j : X \to Y$ be a morphism in $\mathcal{C}$, and let $f : L(X) \to A$ be a morphism in $\mathcal{D}$ and denote by $f^\sharp : X \to R(A)$ the transpose of f. Then f factors through $L(j)$ if and only if $f^\sharp$ factors through j.

$$
\begin{tikzcd}
L(X) \arrow[rr, "f"] \arrow[dr, "L(j)"'] & & A \\
& L(Y) \arrow[ur, "|" marking] &
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
X \arrow[rr, "f^\sharp"] \arrow[dr, "j"'] & & R(A) \\
& Y \arrow[ur, "|" marking] &
\end{tikzcd}
\tag{8.29}
$$

Proof. First suppose that there exists $g : L(Y) \to A$ such that f factors through $L(j)$, i.e., $f = g \circ L(j)$. Then by naturality of the isomorphism $Hom(L(Y), A) \cong Hom(Y, R(A))$, we have $(g \circ L(j))^\sharp = g^\sharp \circ j$. So then $f^\sharp = (g \circ L(j))^\sharp = g^\sharp \circ j$, so $f^\sharp$ factors though j.

Conversely, suppose $h : Y \to R(A)$ is such that $f^\sharp = h \circ j$. Then, again by naturality, $(h \circ j)_\sharp = h_\sharp \circ L(j)$, where $(-)_\sharp$ is the inverse of $(-)^\sharp$. So now $f = f^\sharp_\sharp = (h \circ j)_\sharp = h_\sharp \circ L(j)$. □

A special case of this lemma, applied to $R = T$, the test functor, with its left adjoint L (Theorem 7.5.2), allows us to transfers the Bell paradox between effect algebras and the presheaf category $[\mathbb{F}, \mathbf{Set}]$. Using Proposition 7.5.5, we may rewrite diagram (8.20) and use the transportation result:

$$
\begin{tikzcd}
LT(E_A \otimes E_B) \arrow[r, "LTt"] \arrow[d, "LT(i)"'] & {[0,1]} \\
LT(B_A \otimes B_B) \arrow[ur, "|" marking] &
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
T(E_A \otimes E_B) \arrow[r, "(LTt)^\sharp"] \arrow[d, "Ti"'] & T([0,1]) \\
T(B_A \otimes B_B) \arrow[ur, "|" marking] &
\end{tikzcd}
\tag{8.30}
$$

To fully relate the effect algebraic framework to that of Abramsky and Brandenberger, we continue by creating an adjunction between the presheaf categories $[\mathbb{F}, \mathbf{Set}]$ and $[\mathcal{P}(X)^{op}, \mathbf{Set}]$. For this, we consider the slice category:

Definition 8.6.3. Let Y be an object of a category $\mathcal{C}$, then the *slice category* $\mathcal{C}/Y$ has as objects pairs (C, f) where C is an object of $\mathcal{C}$ and $f : C \to Y$ is a morphism in $\mathcal{C}$. A morphism $h : (C, f) \to (D, g)$ is a morphism $h : C \to D$ in $\mathcal{C}$ such that $f = g \circ h$.

We recall the following facts about slice categories:

Lemma 8.6.4.
- The slice category $\mathcal{C}/Y$ always has a terminal object, (Y, id_Y).
- If $\mathcal{C}$ has products, then the projection map $\Sigma_Y : \mathcal{C}/Y \to \mathcal{C}$, with $\Sigma_Y(C, f) = C$, has a right adjoint $\Delta_Y : \mathcal{C} \to \mathcal{C}/Y$ with $\Delta_Y(C) = (C \times Y, \pi_2 : C \times Y \to Y)$.

Proof. Explicitly, the isomorphism $Hom(\Sigma(C, f), D) \cong Hom((C, f), \Delta D)$ is given by $g : C \to D$ corresponds to $\langle g, f \rangle : (C, f) \to (D \times Y, \pi_2)$. The rest is straightforward. □

Now suppose some measurement scenario $(X, O, \mathcal{M})$ is given. Using the adjunction $\Sigma_{Hom_{\mathbb{F}}(O^X,-)} \dashv \Delta_{Hom_{\mathbb{F}}(O^X,-)}$ we can rewrite diagram (8.25) in the slice category $[\mathbb{F}, \mathbf{Set}]/Hom(O^X, -)$ as:

$$\begin{tikzcd}
(T(E_A \otimes E_B), Ti) \arrow[rr, "\langle Tt, Ti \rangle"] \arrow[dr] & & (D \times Hom(O^X, -), \pi_2) \\
& (Hom(O^X, -), id) \arrow[ur, "|" marking] &
\end{tikzcd} \tag{8.31}$$

Since $(Hom(O^X, -), id)$ is terminal, we can phrase Bell's paradox as "*the local sections* $\langle Tt, Ti \rangle : (T(E_A \otimes E_B), Ti) \to (D \times Hom(O^X, -), \pi_2)$ *have no global section*".

The following is a general result about slices by *representable* presheaves, i.e., Hom-functors:

Lemma 8.6.5. Let $\mathcal{C}$ be a category and c an object in $\mathcal{C}$. Then there is an equivalence $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]/Hom_{\mathcal{C}}(-, d) \simeq [(\mathcal{C}/d)^{\mathrm{op}}, \mathbf{Set}]$.

Proof. See for example [59, Prop. A.1.1.7, Lemma C2.2.17] □

In particular, this implies an equivalence

$$[\mathbb{F}, \mathbf{Set}]/Hom(O^X, -) \simeq [(\mathbb{F}^{op}/O^X)^{op}, \mathbf{Set}] \tag{8.32}$$

We complete the chain of adjunctions by defining a final adjunction between $[\mathbb{F}, \mathbf{Set}]$ and $[\mathcal{P}(X)^{op}, \mathbf{Set}]$ by the following general result:

Lemma 8.6.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories. Then composition with F defines a functor $F^* : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$. Furthermore, F^* has a left adjoint $F_!$.

Proof. See Example A.4.1.4 in [59]. The idea is that F is equivalently given by a functor $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$. The composite $y \circ F^{op} : \mathcal{C}^{op} \to [\mathcal{D}, \mathbf{Set}]$ is now a functor into a cocomplete category, so $F_!$ given by the left Kan extension as in Section 7.4. □

In our case, we obtain an adjoint pair $I_! \dashv I^*$, where the functor $I^* : [(\mathbb{F}^{op}/O^X)^{op}, \mathbf{Set}] \to [\mathcal{P}(X)^{op}, \mathbf{Set}]$ is induced by precomposing with the functor $I : \mathcal{P}(X) \to \mathbb{F}^{op}/O^X$. This functor I takes a subset $U \subseteq X$ to the pair $(O^U, O^{i_U} : O^X \to O^U)$ where $i_U : U \to X$ is the set inclusion function. We can now relate diagram (8.28) to our diagram (8.20) by using an adjunction between $\mathbf{EA}$ and $\mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}$. We construct this adjunction as the following composite:

$$\begin{aligned} \mathbf{EA} \underset{L}{\overset{T}{\underset{\top}{\rightleftarrows}}} [\mathbb{F}, \mathbf{Set}] &\underset{\Sigma_{Hom(O^X,-)}}{\overset{\Delta_{Hom(O^X,-)}}{\underset{\top}{\rightleftarrows}}} [\mathbb{F}, \mathbf{Set}]/Hom(O^X, -) \\ &\simeq [(\mathbb{F}^{op}/O^X)^{op}, \mathbf{Set}] \underset{I_!}{\overset{I^*}{\underset{\top}{\rightleftarrows}}} [\mathcal{P}(X)^{op}, \mathbf{Set}] \end{aligned} \tag{8.33}$$

Proposition 8.6.7. The right adjoint in (8.33) takes the effect algebra $[0,1]$ to the presheaf $D(O^{(-)}) : \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}$. The left adjoint in (8.33) takes the measurement cover $\overline{S}_{\mathcal{M}} \subset 1$ to the effect algebra $E_A \otimes E_B \subset B_A \otimes B_B$.

Proof. Denote the left adjoint of the chain of adjunctions by L and the right adjoint by R. Reading the chain of adjunctions from left to right, starting with an effect algebra A gives the presheaf $RA = T(A)(O^{(-)}) : \mathcal{P}(X)^{\mathrm{op}} \to \mathbf{Set}$. A special case gives $R[0,1] = D(O^{(-)})$.

For any effect algebra X, we then have

$$\mathbf{EA}(L\overline{S}_{\mathcal{M}}, X) \cong \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(\overline{S}_{\mathcal{M}}, RX) = \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(\overline{S}_{\mathcal{M}}, T(X)(O^{(-)}))$$

If we can now show $\mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(\overline{S}_{\mathcal{M}}, T(X)(O^{(-)})) \cong \mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(E_A \otimes E_B, X)$, natural in X, we can conclude that $L\overline{S}_{\mathcal{M}} \cong E_A \otimes E_B$, by uniqueness of left adjoints.

By Proposition 8.2.2, it suffices to show that $\mathbf{Set}^{\mathcal{P}(X)^{\mathrm{op}}}(\overline{S}_{\mathcal{M}}, T(X)(O^{(-)}))$ is in natural bijection with X-tables. In other words, it suffices that a matching family $(d_{a_i,b_j})_{i,j}$ for $\overline{S}_{\mathcal{M}}$ in $(T(X))(O^{(-)})$ is the same thing as an X-table. This follows from expanding the definition of matching family. Note that each d_{a_i,b_j} is by definition a four-tuple $(x_{a_i:0,b_j:0}, x_{a_i:0,b_j:1}, x_{a_i:1,b_j:0}, x_{a_i:1,b_j:1})$

such that $\bigovee_{o,o'} x_{a_i:o,b_j:o'} = 1$. We define a table by $t(a_i{:}o, b_j{:}o') = (x_{a_i:o,b_j:o'})$. The first condition on tables amounts to requiring that each 4-tuple is a test, and the second condition on tables amounts to the compatibility condition for matching families. $\square$

Corollary 8.6.8. The adjunction (8.33) relates the effect algebra formulation of Bell's paradox (diagram (8.20)), with the formulation of Abramsky and Brandenburger (8.28).

Chapter 9

Other paradoxes

We have extensively covered the Bell paradox in effect algebras and the relation to formulations in other frameworks. We now see how other paradoxes in contextuality can be explained by effect algebras and study the relation to other frameworks.

9.1 Different values in tables: paradoxes of possibility

In this section we move from probability to possibility. This originated from [42]. Let $\underline{n}$ be a finite set, considered as a sample space; a *possibility distribution* on $\underline{n}$ is a non-empty subset S of $\underline{n}$. The elements of S are the events of $\underline{n}$ which are 'possible'. Equivalently, a possibility distribution is a function $p : \underline{n} \to \{0,1\}$ such that at least one of the values is assigned 1, meaning that it is possible. We can move away from the classical situation by replacing the set $\underline{n}$ by an effect algebra E. We say that a possibility distribution on an effect algebra E is a morphism of PPCMs $p : E \to (\{0,1\}, \vee, 0, 1)$ into the pointed monoid (Definition 7.2.1).

Just as in the probabilistic case, we can make a second generalization (Section 7.6) by using the Yoneda lemma to conclude that a possibility distribution on $\underline{n}$ is a natural transformation $\mathbb{F}(\underline{n}, -) \to \mathcal{P}^+$, where $\mathcal{P}^+ : \mathbb{F} \to \mathbf{Set}$ is the non-empty powerset functor. We can therefore say that a possibility distribution on a functor $F : \mathbb{F} \to \mathbf{Set}$ is a natural transformation

$F \to \mathcal{P}^+$. These two approaches are again related by the test funcor (Definition 7.5.1) because $T(\{0,1\},\vee,0,1) \cong \mathcal{P}^+$.

Possibilities are related to probabilities by the PPCM morphism $s : ([0,1],+,0,1) \to (\{0,1\},\vee,0,1)$ given by $s(0) = 0, s(x) = 1$ for $x \neq 0$. This takes a probability distribution to its support, and by composing this with a probability distribution we get a possibility distribution.

Following the work of Hardy [49], we consider the following possibilistic table

$$\tau : \{\mathrm{a}_0{:}0, \mathrm{a}_0{:}1, \mathrm{a}_1{:}0, \mathrm{a}_1{:}1\} \times \{\mathrm{b}_0{:}0, \mathrm{b}_0{:}1, \mathrm{b}_1{:}0, \mathrm{b}_1{:}1\} \to (\{0,1\},\vee,0,1)$$

	0,0	0,1	1,0	1,1
$\mathrm{a}_0\mathrm{b}_0$	1	1	1	1
$\mathrm{a}_0\mathrm{b}_1$	0	1	1	1
$\mathrm{a}_1\mathrm{b}_0$	0	1	1	1
$\mathrm{a}_1\mathrm{b}_1$	1	1	1	0

(9.1)

This has a quantum realization but no classical realization. A quantum realization is found by letting $\mathrm{a}_0, \mathrm{b}_0$ be measurements in the $|+\rangle, |-\rangle$ basis and letting $\mathrm{a}_1, \mathrm{b}_1$ be measurements in the $|0\rangle, |1\rangle$ basis on the state $|\psi\rangle = \frac{1}{\sqrt{3}}(|01\rangle + |10\rangle + |00\rangle)$. The non-existence of a classical realization follows form the fact that $\mathrm{a}_0{:}0 \wedge \mathrm{b}_0{:}0$ is possible. Then since $\mathrm{a}_0{:}0 \wedge \mathrm{b}_1{:}0$ and $\mathrm{a}_1{:}0 \wedge \mathrm{b}{:}1$ are impossible, it must follow that $\mathrm{a}_1{:} \wedge \mathrm{b}_1{:}1$ is possible, which is a contradiction. That is to say, the Boolean atoms comprising $\mathrm{a}_0{:}0 \wedge \mathrm{b}_0{:}0$ are all contained in the ones of the impossible events.

We can again relate our effect algebraic formulation of this paradox with the analysis of Abramsky and Brandenburger [2], by using the chain of adjunctions in (8.33).

Corollary 9.1.1. The right adjoint of the composite adjunction (8.33) takes the effect algebra $(\{0,1\},\vee,0,1)$ to the presheaf $\mathcal{P}^+(O^{(-)}) : \mathcal{P}(X)^{op} \to \mathbf{Set}$. Thus the adjunction (8.33) relates the effect algebra formulation of Hardy's paradox with the formulation of Abramsky and Brandenburger.

9.1.1 The Kochen-Specker theorem

In Section 6.2 we explained the Kochen-Specker theorem by showing there are measurements which contradict a non-contextual deterministic theory of quantum mechanics. Here we will see how this statement can be put in an effect algebraic framework and how it can be related to a sheaf theoretic approach by Hamilton, Isham and Butterfield [48].

Recall first that the projections, $Proj(H)$, on a Hilbert space H form an effect algebra (Example 7.2.5). The operators $P_i := 1 - S_i^2$ from Section 6.2 are projections in $Proj(\mathbb{C}^3)$ which add to unity for any basis. That is, for any basis x, y, z the set $\{P_x, P_y, P_z\}$ is a test. Now any effect algebra morphism from $Proj(\mathbb{C}^3)$ to the effect algebra $(\{0,1\}, \otimes, 0, 1)$ would, by restriction, precisely give an assignment as in the Kochen-Specker theorem and thus does not exist. Hence we find:

$$\text{There is no effect algebra morphism } Proj(\mathbb{C}^3) \to \{0,1\}.$$

From this we can obtain the full statement of the Kochen-Specker theorem, using effect algebras:

Corollary 9.1.2 (Kochen-Specker)**.** There is no effect algebra morphism $Proj(H) \to \{0,1\}$ if $\dim H \geq 3$.

Proof. Let H be any Hilbert space with $\dim H \geq 3$ and suppose there is a morphism $v : Proj(H) \to \{0,1\}$. Let p be a 1-dimensional projection with $v(p) = 1$. Consider two 1-dimensional projections q, q' such that $p + q + q'$ is the identity on some 3-dimensional subspace H'. Now let p_1, p_2, p_3 be any set of pairwise orthogonal projections in H'. Then $v(p_1) + v(p_2) + v(p_3) = v(p_1 + p_2 + p_3) = v(1_{H'}) = 1$, so one of p_1, p_2, p_3 is assigned value 1. Hence v restricts to a morphism $Proj(H') \to \{0,1\}$ which is a contradiction. $\square$

A *Kochen-Specker system* such as in [92] can then be given as a sub-effect algebra E of $Proj(H)$, such that there is no effect algebra morphism $E \to \{0,1\}$.

We now consider the Kochen-Specker theorem in another light, in terms of presheaves on the poset $\mathcal{C}(B(H))$ of commutative sub-algebras of $B(H)$ [48]. For this, we first need some background on operator algebras. Let $\mathbf{CC}^*_\mathrm{f}$ be the category of finite dimensional commutative C^*-algebras and $*$-homomorphisms. The *spectrum* of a an algebra $A \in \mathbf{CC}^*_\mathrm{f}$ is the space of non-zero $*$-homomorphisms $A \to \mathbb{C}$, also called *characters*. That is

$$\mathrm{Spec}(A) := \{\phi \in Hom(A, \mathbb{C}) \mid \phi \neq 0\} \tag{9.2}$$

Usually, the spectrum of an algebra comes equipped with a topology, however, in the finite dimensional case it is discrete, so we consider the spectrum as a set. Moreover, when $A \cong \mathbb{C}^n$, $\mathrm{Spec}(A) \cong \underline{n}$. The characters are then precisely the point valuations $\phi_i(a_1, \ldots, a_n) \mapsto a_i$. We extend Spec to a functor $\mathrm{Spec} : \mathbf{CC}^*_{\mathrm{f}} \to \mathbb{F}^{op}$, by $\mathrm{Spec}(f : A \to B)(\phi \in \mathrm{Spec}(B)) = \phi \circ f \in \mathrm{Spec}(A)$. This functor has a left adjoint $\mathbb{C}^{(-)} \dashv \mathrm{Spec}$ which sends a finite set $\underline{n}$ to the space of functions on $\underline{n}$, $\mathbb{C}^{\underline{n}} := \{\phi : \underline{n} \to \mathbb{C}\}$. For a a map $f : \underline{n} \to \underline{m}$ we obtain the $*$-homomorphism $\mathbb{C}^f : \mathbb{C}^m \to \mathbb{C}^n$ sending ϕ to $\phi \circ f$. By restricted Gelfand duality we have:

Proposition 9.1.3. The functors $\mathbb{C}^{(-)} \dashv \mathrm{Spec}$ form an adjunction and constitute an equivalence of categories $\mathbf{CC}^*_{\mathrm{f}} \simeq \mathbb{F}^{op}$.

Now let $B(H)$ be the (non-commutative) C^*-algebra of bounded operators on a finite dimensional Hilbert space H, it generally has many commutative sub-algebras. Let $\mathcal{C}(B(H))$ be the set of commutative sub-algebras of A. This becomes a poset, and therefore a category, under inclusion. We consider the the spectrum of each commutative sub-algebra of $B(H)$. Whenever $A \subset B$ are commutative sub-algebras of $B(H)$, and ϕ a character on B, we obtain a character $\phi|_A$ on A by restriction. Hence we obtain the *spectral presheaf*, which we also call $\mathrm{Spec} \in [\mathcal{C}(B(H))^{op}, \mathbf{Set}]$,

Now if p is a projection in some commutative sub-algebra A of $B(H)$ and ϕ is a character in $\mathrm{Spec}(A)$, we have $\phi(p) = \phi(pp) = \phi(p)\phi(p)$. Hence any projection is assigned a value in $\{0, 1\}$ by a character.

A natural transformation $1 \to \mathrm{Spec}$, from the terminal presheaf 1, which assigns a singleton set to every commutative sub-algebra, to the spectral presheaf then assigns to every commutative sub-algebra a particular character. Moreover, naturality implies this assignment is independent of the surrounding sub-algebra. Hence a natural transformation $1 \to \mathrm{Spec}$ is a global assignment of outcomes for projections, which therefore does not exist.

We can thus phrase the Kochen-Specker theorem as follows ([48]):

Theorem 9.1.4. There is no natural transformation $1 \to \mathrm{Spec}$ in $\mathcal{C}(B(H))$ if $\dim H \geq 3$.

9.1.2 Transporting the paradox to different categories

As with the Bell paradox, we can use adjunctions to transport the statement of the Kochen-Specker theorem to other categories. First of all, by

Corollary 7.5.4, we can write the Kochen-Specker theorem in the presheaf category $[\mathbb{F}, \mathbf{Set}]$ as:

There is no natural transformation $T(Proj(H)) \to T(\{0,1\})$ if $\dim H \geq 3$.

By Gelfand duality (Proposition 9.1.3), we obtain an functor $\mathrm{Spec}^* : [\mathbb{F}, \mathbf{Set}] \to [\mathbf{CC}_{\mathrm{f}}^{*op}, \mathbf{Set}]$. Now an n-test in the effect algebra $Proj(H)$ can be identified with a $*$-homomorphism $\mathbb{C}^n \to B(H)$. Indeed, the elements $\chi_i : \underline{n} \to \mathbb{C}$, $\chi_i(j) = \delta_{i,j}$ are projections and sum to unity, hence their image under a $*$-homomorphism is a test in $Proj(H)$. Similarly, if $(p_1, \ldots, p_n)$ is an n-test, define a map ϕ as extension of $\phi(\chi_i) = p_i$. We thus find

Lemma 9.1.5. $T(Proj(H)) \cong \mathbf{C}^*(\mathbb{C}^-, B(H))$.

On the other hand, $T(\{0,1\})(\underline{n}) \cong \underline{n}$. Under the equivalence of Proposition 9.1.3 we have presheaves $T(Proj(H)), T(\{0,1\}) \in \mathbf{Set}^{\mathbf{CC}_{\mathrm{f}}^{*\mathrm{op}}}$ with

$$T(Proj(H))(A) = \mathbf{C}^*(A, B(H)) \qquad T(\{0,1\})(A) = \mathrm{Spec}(A)$$

Thus the Kochen-Specker paradox can be stated as:

$$\text{There is no natural transformation } \mathbf{C}^*(-, B(H)) \to \mathrm{Spec} \text{ in } \mathbf{Set}^{\mathbf{CC}_{\mathrm{f}}^{*\mathrm{op}}}. \tag{9.3}$$

(See also [78], Theorem 1.2.)

If a functor $R : \mathbf{Set}^{\mathbf{CC}_{\mathrm{f}}^{*\mathrm{op}}} \to \mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathbf{Set}^{\mathbf{CC}_{\mathrm{f}}^{*\mathrm{op}}}$ and $L(X) = \mathbf{C}^*(-, B(H))$ then the paradox says there is no morphism $X \to R(\mathrm{Spec})$ in $\mathcal{C}$. We transport this using the following composite adjunction, similar to (8.33).

$$\begin{aligned} &\mathbf{Set}^{\mathbf{CC}_{\mathrm{f}}^{*op}} \underset{\Sigma_{\mathbf{C}^*(-,B(H))}}{\overset{\Delta_{\mathbf{C}^*(-,B(H))}}{\rightleftarrows}} \mathbf{Set}^{\mathbf{CC}_{\mathrm{f}}^{*\mathrm{op}}}/\mathbf{C}^*(-, B(H)) \\ &\simeq \mathbf{Set}^{(\mathbf{CC}_{\mathrm{f}}^{*}\downarrow B(H))^{\mathrm{op}}} \underset{J_!}{\overset{J^*}{\rightleftarrows}} \mathbf{Set}^{C(B(H))^{op}} \end{aligned} \tag{9.4}$$

The first adjunction between slice categories is as in Section 8.6.1. The middle equivalence is standard (e.g. [59, Prop. A.1.1.7]); here $(\mathbf{CC}_{\mathrm{f}}^* \downarrow B(H))$ is the category whose objects are pairs $(A, f : A \to B(H))$ where A is a finite-dimensional commutative C*-algebra and f is a *-homomorphism. The adjunction $J_! \dashv J^*$ is induced by the evident embedding $J : C(B(H)) \to (\mathbf{CC}_{\mathrm{f}}^* \downarrow B(H))$, where a commutative C^*-sub-algebra A of $B(H)$ is mapped to $(A \hookrightarrow B(H))$. We then have

Proposition 9.1.6. The right adjoint of (9.4) takes the spectral presheaf on $\mathbf{CC}^*_\mathrm{f}$ to the spectral presheaf on $\mathcal{C}(B(H))$. The left adjoint of the composite (9.4) takes the terminal presheaf on $C(B(H))$ to the presheaf $\mathbf{C}^*(-, B(H))$ on $\mathbf{CC}^*_\mathrm{f}$.

Proof. Let $K : C(B(H)) \to \mathbf{CC}^*_\mathrm{f}$ be the inclusion functor. Reading the adjunction from left to right sends a presheaf F on $\mathbf{CC}^*_\mathrm{f}$ to $F \circ K$ on $C(B(H))$. In particular, the spectral presheaf gets mapped to the spectral presheaf.

To show the second half of the statement we show, similar to Proposition 8.6.7, that natural transformations $\sigma : 1 \to G \circ K$ are in natural correspondence to natural transformations $\alpha : \mathbf{C}^*(-, B(H)) \to G$. This bijection is given as follows: given $\sigma : 1 \to G \circ K$, define $\alpha : \mathbf{C}^*(-, B(H)) \to G$ as $\alpha_A(f) = \sigma_{f(A)}(*)$ and given $\alpha : \mathbf{C}^*(-, B(H)) \to G$, define $\sigma : 1 \to G \circ K$ as $\sigma_A(*) = \alpha_A(i_A)$ where $i_A : A \hookrightarrow B(H)$. □

Corollary 9.1.7. The paradox (9.3) is equivalent to the statement of [48]: the spectral presheaf has no global section.

9.2 Test spaces

In this section we want to consider another approach to non-locality and contextuality via *test spaces*. However, test spaces come in different guises in the literature. Here we want to present a small overview of these different approaches to test spaces and make a link with effect algebras. Currently, the following definition, which comes from [35], is probably the most common.

Definition 9.2.1 (Test space)**.** A test space (X, Σ) consists of a set X together with a set of subsets $\Sigma \subset 2^X$ such that the members of Σ cover X, i.e., $\bigcup_{T \in \Sigma} T = X$. A probability measure on a test space (X, Σ) is a function $\mu : X \to \mathbb{R}_{\geq 0}$ such that $\sum_{x \in T} \mu(x) = 1$ for every test $T \in \Sigma$.

More details about these test spaces can be found in [35]. The version of test spaces we will use is more general in the sense that tests can include multiple instances of the same element. This will be comparable to how it might be possible to add an element in an effect algebra to itself. The following definitions are from [44] and [45].

Definition 9.2.2. An *effect test space* $(X, \mathcal{T})$ consists of a set X and a collection $\mathcal{T} \subset \mathbb{N}^X$ such that

- For any $x \in X$ there exists some $t \in \mathcal{T}$ such that $t(x) \neq 0$.
- If $s, t \in \mathcal{T}$ with $s(x) \leq t(x)$ for all $x \in X$, then $s = t$.

An important notion in the theory of test spaces is that of *perspectivity*.

Definition 9.2.3. Given an effect test space $(X, \mathcal{T})$, any function $f \leq t \in \mathcal{T}$ is called an *event*. Two events f, g are *orthogonal* if $f + g$ is again an event and *complementary* if $f + g \in \mathcal{T}$. Two events f, g are *perspective* if there exists an event h such that f, h and g, h are complementary. We write $f \approx g$ if f and g are perspective.

Definition 9.2.4. An effect test space $(X, \mathcal{T})$ is *algebraic* if every $t \in \mathcal{T}$ has finite support and if for events f, g, h, if $f \approx g$ and $h + f \in \mathcal{T}$ then $h + g \in \mathcal{T}$. That is to say, if two events share a complement, they share all complements.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be algebraic effect test spaces. Any (partial) function $\psi : X \to Y$ defines a function $\hat{\psi} : \{f \in \mathbb{N}^X \mid f \text{ has finite support}\} \to \mathbb{N}^Y$ by $\hat{\psi}(f)(y) = \sum\{f(x) \mid \phi(x) = y\}$. We understand the empty sum to be zero. In particular, if $\mathbf{1}_x$ is the characteristic function of $x \in X$, then $\hat{\psi}(\mathbf{1}_x)(y) = \sum\{\mathbf{1}_x(x') \mid \psi(x') = y\}$, which is 1 only if $x' = x$, that is, $y = \psi(x)$, so $\hat{\psi}(\mathbf{1}_x) = \mathbf{1}_{\psi(x)}$.

We obtain a category **AEtest** of algebraic effect test spaces whose morphisms $(X, \mathcal{T}) \to (Y, \mathcal{S})$ are partial functions $\psi : X \to Y$ such that $\hat{\psi}(t) \in \mathcal{S}$ if $t \in \mathcal{T}$. The reason to consider partial functions becomes clear when we consider $E = 0$, the terminal effect algebra, in the adjunction below.

Example 9.2.5. Let $\mathcal{I} = \big((0,1], \{f : (0,1] \to \mathbb{N} \mid \mathrm{supp}(f) \text{ is finite, } \sum_x (f(x)) \cdot x = 1\}\big)$. Then $\mathcal{I}$ is an algebraic effect test space describing the unit interval. Here $(0,1]$ is the half-open unit interval, which we need because of the second point of Definition 9.2.2. Let $(X, \mathcal{T})$ be any algebraic effect test space. A morphism $\mu : (X, \mathcal{T}) \to \mathcal{I}$ corresponds to a probability measure $\tilde{\mu}$ on X, where $\tilde{\mu}(x) = 0$ if $\mu(x)$ is undefined.

By slight modification of the ideas in [45] we now distil an adjunction between algebraic effect test spaces and effect algebras, which will allow us to transpose paradoxes. (We note that Jacobs and Mandemaker [56, §3] also extracted a similar adjunction from [45], but for a modified notion of test space called 'test perspective'; we contend that our use of bona fide effect test spaces and partial maps stands more closely to test space literature.)

Let f be an event in an algebraic effect space $(X, \mathcal{T})$. Denote by $\pi(f)$ the set $\pi(f) = \{g \mid g \approx f\}$ and let $\Pi(X) = \{\pi(f) \mid f \text{ an event}\}$. In [45] it is shown that that $\Pi(X)$ can be given the structure of an effect algebra in a straightforward way. That is, $\pi(f) \ovee \pi(g) = \pi(f+g)$ whenever this makes sense and $\pi(f)^{\perp} = \pi(h)$ if h is a complement of f. We extend Π to a functor $\Pi : \mathbf{AEtest} \to \mathbf{EA}$ by $\Pi(\psi)(f) = \pi(\hat{\psi}(f))$.
There is also a functor in the other direction, which we denote by S. Let E be an effect algebra. We obtain an algebraic test space $S(E) = (X, \mathcal{T})$ where $X = E \backslash \{0\}$ and $\mathcal{T} = \{f : X \to \mathbb{N} \mid \text{supp}(f) \text{ is finite}, \bigovee_x (f(x)) \cdot x = 1\}$. If $\varphi : E \to A$ is an effect algebra morphism, we obtain an **AEtest** morphism $S(\varphi)$ by restricting to $E \backslash \{0\}$. Note that $\mathcal{I} = S([0,1])$ from Example 9.2.5.

Lemma 9.2.6. Let E be an effect algebra. The map $\phi : E \to \Pi S(E)$, $e \mapsto \pi(\mathbf{1}_e)$ is an isomorphism.

Proof. An inverse to ϕ is given as follows: let f be an event in $\Pi S(E)$, then $f = \sum_e f(e)\mathbf{1}_e$. Now $\phi^{-1}(\pi(f)) = \bigovee_e f(e)e$. See [45] for details. □

Proposition 9.2.7. The functors $\Pi : \mathbf{AEtest} \to \mathbf{EA}$ and $S : \mathbf{EA} \to \mathbf{AEtest}$ form an adjoint pair $\Pi \dashv S$. The map ϕ^{-1} is the counit of this adjunction.

Proof. We want to show $Hom(\Pi(X, \mathcal{T}), E) \cong Hom((X, \mathcal{T}), S(E))$. Given $\varphi : \Pi(X, \mathcal{T}) \to E$, define $\bar{\varphi} : (X, \mathcal{T}) \to S(E)$ by $\bar{\varphi}(x) = \varphi(\pi(\mathbf{1}_x))$. Given $\psi : (X, \mathcal{T}) \to S(E)$ define $\bar{\psi} : \Pi(X, \mathcal{T}) \to E$ by $\bar{\psi}(\pi(f)) = \phi^{-1}(\pi(\hat{\psi}(f)))$. Notice that any event $f : X \to \mathbb{N}$ can be written as $f = \sum_x f(x)\mathbf{1}_x$. We then have

$$\begin{aligned}
\bar{\bar{\varphi}}(\pi(f)) &= \phi^{-1}(\pi(\hat{\bar{\varphi}}(f))) \\
&= \phi^{-1}(\pi(\hat{\bar{\varphi}}(\sum_x f(x)\mathbf{1}_x))) \\
&= \phi^{-1}(\pi(\sum_x f(x)\mathbf{1}_{\bar{\varphi}(x)})) \\
&= \sum_x f(x)\bar{\varphi}(x) \\
&= \sum_x f(x)\varphi(\pi(\mathbf{1}_x)) \\
&= \varphi(\pi(f)),
\end{aligned}$$

and

$$\begin{aligned}
\bar{\bar{\psi}}(x) &= \bar{\psi}(\pi(\mathbf{1}_x)) \\
&= \phi^{-1}(\pi(\hat{\psi}(\mathbf{1}_x))) \\
&= \phi^{-1}(\pi(\mathbf{1}_{\psi(x)})) \\
&= \psi(x).
\end{aligned}$$

□

In order to show the relation between test spaces and effect algebras, we take a look at two non-locality scenarios.

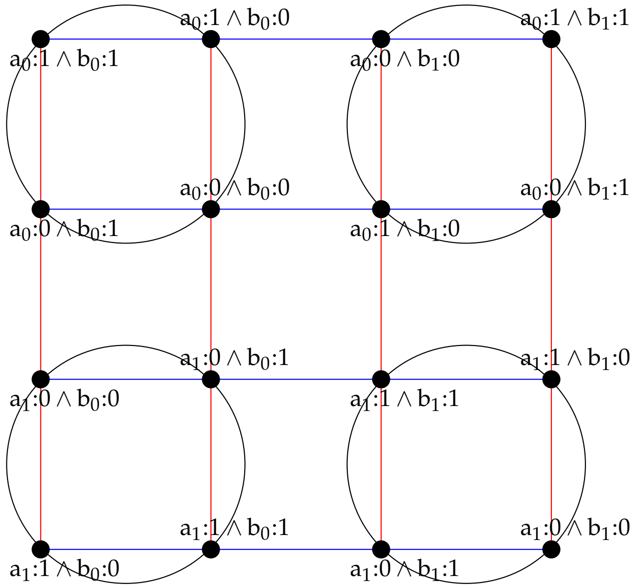
9.2.1 Bell scenario

We shall take a look on how to transfer the Bell paradox to test spaces. It follows from Lemma 9.2.6 and Proposition 9.2.7 that S is fully faithful. Hence we can easily transfer the Bell paradox by applying S to the non-factoring triangle as follows:

$$\begin{tikzcd}
S(E_A \otimes E_B) \arrow[rr] \arrow[dr] & & S([0,1]) \\
& S(B_A \otimes B_B) \arrow[ur, "|" marking] &
\end{tikzcd}$$

In fact, whenever we have a morphism $\phi : (X, \mathcal{T}) \to (Y, \mathcal{S})$ such that $\Pi(\phi) : \Pi(X, \mathcal{T}) \to \Pi(Y, \mathcal{S})$ is the inclusion $i : E_A \otimes E_B \to B_A \otimes B_B$, the method of diagram (8.29) allows us to transfer the paradox to **AETest**. This might be relevant as the space $S(E_A \otimes E_B)$ is quite involved and we could find a smaller space.

Example 9.2.8. Consider figure (9.5) below. This is also depicted in [6, Fig. 7], where it is called a hyper-graph, but we understand it as a test space by identifying vertices and hyper-edges of a graph with points and tests of a test space. Let Z be the set of points in it and for every line or circle define a function from Z to $\mathbb{N}$, which sends the points on this line or circle to 1 and the rest to 0. We call the set of these functions $\mathcal{Q}$. Then $(Z, \mathcal{Q})$ is an algebraic effect test space.



(9.5)

We have conveniently labelled the points of this space. In the terminology of non-locality, the circles correspond to fixed measurement settings and the lines correspond to the no-signalling conditions. Hence we see that applying the functor Π to this test space gives an effect algebra isomorphic to $E_A \otimes E_B$. The Bell table thus describes a distribution $(Z, \mathcal{Q}) \to \mathcal{I}$ which does not factor though the canonical map $(Z, \mathcal{Q}) \to S(B_A \otimes B_B)$.

N.B. Since the functor Π is not full, we cannot just take any test space $(X, \mathcal{T})$ for which $\Pi(X, \mathcal{T}) \cong B_A \otimes B_B$ in order to find a non-factorization of the Bell scenario. In particular there is no map $(Z, \mathcal{Q}) \to (16, \{f\})$ where 16 is a 16 element set and $f : 16 \to \mathbb{N}$ is the map $f(i) = 1 \, \forall i \in 16$.

9.2.2 GHZ scenario

The second example we will look at is the GHZ scenario [42]. We will use the adjunction to explore the scenario from the perspective of both test spaces and effect algebras. Like the Bell scenario, the GHZ scenario involves separate observers, each with measurement settings and possible outcomes. But like the Kochen-Specker scenario the 'paradox' here is absolute and not probabilistic.

There are three separate observers, Alice, Bob and Charlie, each with two measurement settings (x and y) and two possible outcomes (-1 and $+1$). The quantum realization of the scenario is as follows. Alice, Bob and Charlie share a quantum state of the form $\Psi_{GHZ} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow> - |\downarrow\downarrow\downarrow>)$. They each have the choice to perform the Pauli-x operator, σ_x, which sends $\uparrow$ to $\downarrow$ and $\downarrow$ to $\uparrow$ or Pauli-y operator, σ_y, which sends $\uparrow$ to $i\cdot\downarrow$ and $\downarrow$ to $-i\cdot\uparrow$. The crux is that certain combinations of Pauli operators have the GHZ state as an eigenvector. Indeed,

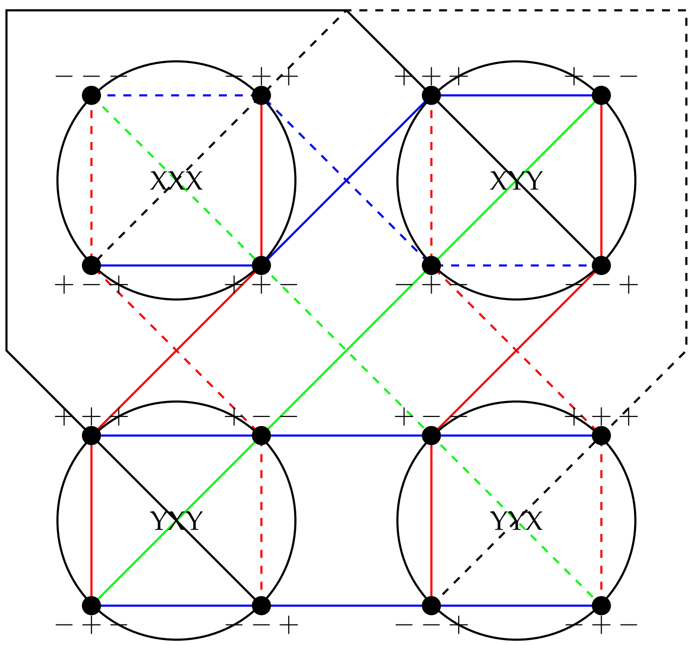
$$\sigma_x\sigma_x\sigma_x\Psi_{GHZ} = -\Psi_{GHZ}, \tag{9.6}$$

$$\sigma_x\sigma_y\sigma_y\Psi_{GHZ} = \Psi_{GHZ}, \tag{9.7}$$

$$\sigma_y\sigma_x\sigma_y\Psi_{GHZ} = \Psi_{GHZ}, \tag{9.8}$$

$$\sigma_y\sigma_y\sigma_x\Psi_{GHZ} = \Psi_{GHZ}. \tag{9.9}$$

Now in a local non-contextual setting we should be able to assign eigenvalues, $+1$ or -1 to the Pauli operators in such a way that it respects the above products, but this is impossible as we can see from a parity argument: every Pauli operator occurs twice on the left hand sides, hence the total product is $+1$, while the product of the right hand side is -1. By the methods of [6] we can write down a test space, $(X_{GHZ}, \mathcal{T}_{GHZ})$, for this scenario (figure (9.10)). The vertices on the circles correspond to the outcome of measurements whose settings are written inside the circle. The remaining lines correspond to tests coming from no-signalling.



(9.10)

The statement of the paradox is now that there are no **AETest** morphisms from $(X_{GHZ}, \mathcal{T}_{GHZ})$ to the test space $(\{*\}, \{!\})$ with one point and one test $! : * \mapsto 1$. Translated to effect algebras this statement becomes: there is no effect algebra map $\Pi(X_{GHZ}, \mathcal{T}_{GHZ})$ to $(\{0,1\}, \ovee)$, which is exactly a Kochen-Specker type theorem.

Chapter 10

Conclusion and future work

We have shown how effect algebras give a natural setting to study non-locality and contextuality. We used this setting to consider the paradoxes of Bell, Kochen-Specker, GHZ and Hardy. The latter involves possibilities instead of probabilities and requires a more general notion of partial monoids. Using an embedding of effect algebras in a presheaf category, we gave two generalizations of probability theory. Using chains of adjunctions we linked our effect algebraic approach to the established presheaf approach and test space approach.

There are still many open questions regarding non-locality and causality, varying from how it can be applied to obtain computational speed up ([51], [16]) to fundamental questions as to why quantum mechanics allows for violations of the Bell inequalities, but not as much as they could be violated ([91]). Hopefully, this effect algebraic approach can shine some light on these questions. More explicitly related to this work it is possible to consider more examples of non-locality and contextuality using effect algebras, although this will probably be more of a fun little exercise than give fundamental new insight. A more interesting line would be to consider which effect algebras give rise to contextuality scenarios. While we took effect algebras as generalized probability spaces, it is fair to say this category is too large. A direct restriction would be to consider only effect algebras with $a \perp a \Rightarrow a = 0$, as it does not make much sense for a measurement setting to be in a context with itself. Likewise, it is interesting to consider which generalizations of the interval $[0, 1]$ can be used as generalizations of probabilities. Another interesting line is to consider further the relation of effect algebraic cohomology with the presheaf cohomology.

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Lekensamenvatting

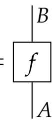
Wetenschap wordt over het algemeen (terecht) betaald uit publieke gelden. Daar hoort dan ook tegenover te staan dat de behaalde resultaten voor datzelfde publiek toegankelijk moeten zijn. Vandaar wil ik hier graag een kleine samenvatting van het werk in deze thesis geven. Deze thesis bestaat uit twee delen waar we in ieder deel een aspect binnen de quantumtheorie bekijken met behulp van categorietheorie. Dit is een deel van de wiskunde waar we niet zo zeer geïnteresseerd zijn in een specifiek wiskundig object (zoals een groep, verzameling of algebra), maar meer in hoe deze objecten met elkaar samenhangen. Een categorie bestaat dan ook uit een collectie objecten met voor ieder paar objecten een collectie pijlen (morfismen) tussen deze objecten die we kunnen samenstellen. Over het algemeen zien we een object als een systeem wat we voor handen hebben en een morfisme als een transformatie van dit systeem in een (mogelijk) ander systeem.

Causaliteit

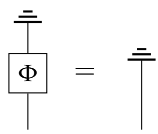
Het eerste deel van deze thesis gaat over causaliteit. Dit is het gegeven dat een oorzaak voorafgaat aan een gevolg. Iets wat in de toekomst plaatsvind kan geen invloed hebben op wat in het verleden is gebeurd. We willen dit bekijken in *proces theorieën*, bepaalde categorieën (symmetrisch monoidiale) waar we ook systemen kunnen samenstellen om zo composiete systemen te bestuderen. Helemaal mooi is dat we deze compositionele structuur grafisch kunnen weergeven. We tekenen in CQM een systeem A als een lijn

$$\textit{systeem:} \qquad A := id_A = \Big|_A$$

en een proces als een box tussen dit soort lijnen

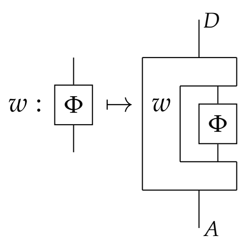
proces: $f : A \to B :=$ 

Composiete systemen kunnen we weergeven door lijnen naast elkaar te tekenen. Het blijkt dat als we nu causaliteit willen vangen in deze categorische taal, we een manier moeten hebben om een systeem 'weg te gooien', of te vergeten. We geven dit aan met dit 'discard' symbool: ⏚. We zeggen dan dat een proces causaal is als het weggooien van de uitkomst van een dergelijk proces gelijk is aan het weggooien van de input:



Dus als we de directe invloed van een causaal proces vergeten, is het alsof het proces nooit heeft plaatsgevonden en kan het dus ook geen invloed op het verleden hebben.

De volgende stap is om te kijken naar processen die als input andere processen hebben, zogenaamde *hogere orde processen*. In onderstaand plaatje is w zo'n proces. Het neemt een (eerste order) afbeelding Φ en maakt er een nieuwe afbeelding van, die mogelijk tussen andere systeem werkt. We kunnen dit grafisch als volgt weergeven:



Zo kunnen we ook kijken naar hogere orde afbeeldingen waar de input bestaat uit meerdere processen. Dit is belangrijk als we een (quantum) protocol willen uitvoeren. Een ander belangrijk voorbeeld hier is de *quantum switch*. Deze neemt als input twee processen Φ en Ψ en een qubit en geeft als output de samenstelling waar Ψ voor Φ gebeurd of waar Φ voor Ψ gebeurd, afhankelijk van of de qubit de waarde 0 of 1 heeft. Omdat een qubit ook een combinatie van deze waarden kan hebben, kan ook de uitkomst een mix van de twee causale ordeningen zijn. Dit heeft weer interessante informatietheoretische consequenties.

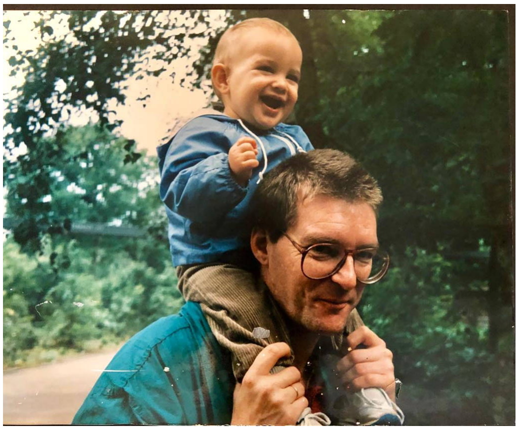
We willen dat deze hogere order afbeeldingen causaliteit bewaren en dit blijkt niet het geval wanneer we de 'voordehandliggende' aanpak gebruiken. In deze thesis lossen we dit probleem op door te beginnen we met een klasse van categorieën, die we *precausaal* noemen, waarin hogere orde processen op een bepaalde manier factoriseren. Zowel de categorie die gebruikt wordt voor quantum berekeningen als de categorie die gebruikt wordt voor kansrekening zijn precausaal. Er wordt dan een constructie gegeven die van een precausale categorie een nieuwe categorie maakt waar de objecten extra informatie dragen over de causale verbanden tussen de inputs en outputs van processen. Hierdoor kunnen we bepaalde klassen van processen classificeren, inclusief de hogere order processen die causaliteit bewaren.

Contextualiteit

In het tweede deel van deze thesis gebruiken we de categorie van effect algebras om contextualiteit te beschrijven. Op een (quantum) systeem kunnen we vaak verschillende metingen doen. Sommige van deze metingen kunnen we gelijktijdig uitvoeren (ze zijn compatibel) en andere metingen zijn dat niet. Een maximale verzameling compatibele metingen noemen we een context. Contextualiteit - of beter gezegd niet-contextualiteit - is nou de vraag of we meetuitkomsten, of kansverdelingen daarover, kunnen verklaren zonder dat deze afhangen van de context waarin zo'n meting plaatsvind. Ter illustratie bekijken we het *Bell scenario*. Twee observatoren, Alice en Bob, hebben ieder een meetapparaat met twee standen. A_1, A_2 voor Alice en B_1, B_2 voor Bob. Zij kunnen ieder een meting doen aan een deel van een samengesteld systeem (zoals een verstrengeld deeltjespaar), maar kunnen slechts een van hun metingen kiezen. De contexten zijn dus $\{A_1, B_1\}$, $\{A_1, B_2\}$, $\{A_2, B_1\}$ en $\{A_2, B_2\}$. Wanneer Alice en Bob een meetstand kiezen en een meting doen krijgen ze, met een bepaalde kans, een uitkomst. Dit levert voor iedere context een kansverdeling over de uitkomsten op. Het scenario is nu contextueel als er *geen* kansverdeling over alle metingen bestaat die marginaliseert tot de gevonden kansverdelingen over de contexten. Interessant genoeg komen dit soort kansverdelingen daadwerkelijk voor in quantum mechanica en dit is de grondslag voor de zogenaamde *Bell ongelijkheden*.

Effect algebras zijn een bepaald soort algebra waar de optelling partieel is. Dat wil zeggen dat we niet altijd twee elementen kunnen optellen. Bekijk bijvoorbeeld het interval $[0,1]$. We kunnen twee elementen hier alleen optellen als hun som niet groter is dan 1. Het feit dat sommige metingen wel en andere niet compatibel zijn blijkt precies door deze partiële structuur gevangen te worden. Dit maakt effect algebras zeer nuttig om dit soort scenario's te beschrijven. Bovendien blijken de kansverdelingen over contexten precies gevangen te worden door effect algebra morfismen naar het interval. We kunnen contextualiteit dan precies vangen als het niet bestaan van bepaalde factorisaties van morfismen. We werken dit uit en bekijken expliciet een aantal belangrijke voorbeelden uit de literatuur. Tenslotte laten we zien dat we de effect algebra aanpak als het ware kunnen vertalen naar andere aanpakken door middel van de categorische techniek van *adjuncties*.

About the author



Sander Uijlen started his studies in 2005 with the bachelor Physics and Astronomy at Radboud University in Nijmegen. Soon he discovered that this math thing was true and started a second bachelor Mathematics. These were completed cum laude in 2011 and 2012, respectively. Sander then proceeded to finish the master Mathematics, which was finished cum laude 2013. During his studies, he became interested in category theory, quantum theory and operator algebras, which amalgamated in the master thesis *Categorical Aspects of von Neumann Algebras and AW*-algebras* supervised by prof. Klaas Landsman. This, together with the fact that life sometimes is a string of (fortunate) coincidences and prof. Bart Jacobs just received an ERC advanced grant, made it that Sander got a position on the Quantum Computation, Logic and Security project at the Digital Security group at Radboud, of which this thesis is the final result. After his time as a PhD student (but before actually finishing his PhD), Sander worked at the University of Oxford with prof. Bob Coecke on quantum causal structures. Sander is currently eager to start a new position at the University of Tulene on quantum programming languages.