New reasoning techniques for monoidal algebra

Aleks Kissinger

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Quantum Group
Department of Computer Science
University of Oxford

Algebra and rewriting

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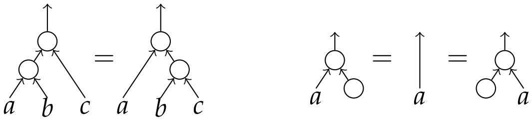
- It is also possible to write these equations as trees:



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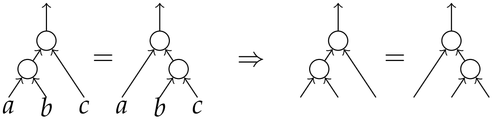
- Since these equations are (left- and right-) linear in the free variables, we can drop them:



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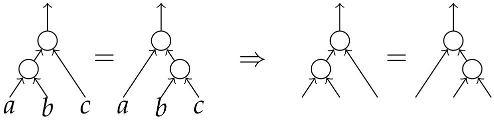

- The role of variables is replaced by the notion that the LHS and RHS have a *shared boundary*

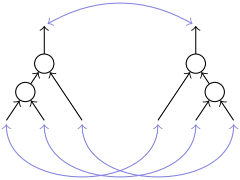

Diagram substitution

- One could apply the rule "$(a \cdot b) \cdot c \to a \cdot (b \cdot c)$" using the usual "instantiate, match, replace" style:

$$w \cdot ((x \cdot (y \cdot e)) \cdot z) \quad \longrightarrow \quad w \cdot (x \cdot ((y \cdot e) \cdot z))$$

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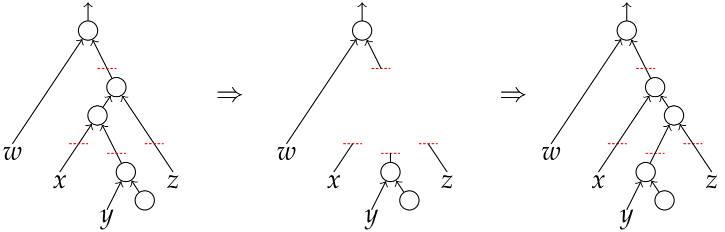

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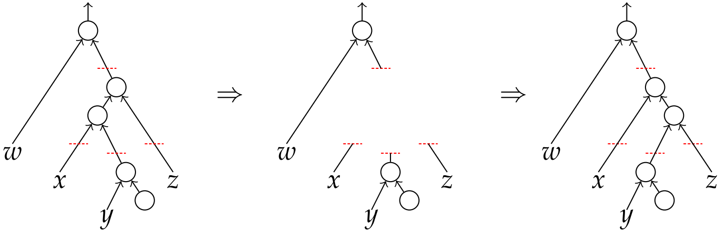

- This treats inputs and outputs symmetrically

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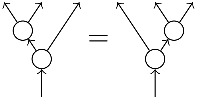

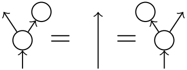

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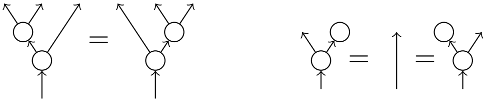
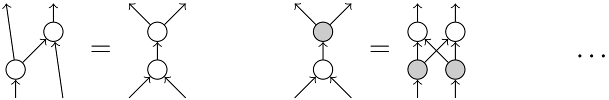


- Algebra and coalgebra can interact in many interesting ways:



Equational reasoning with diagram substitution

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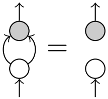

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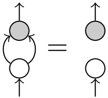

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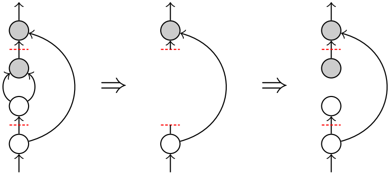

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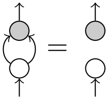

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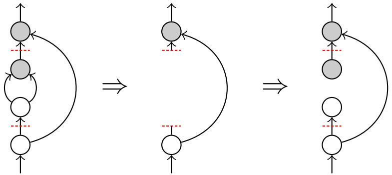
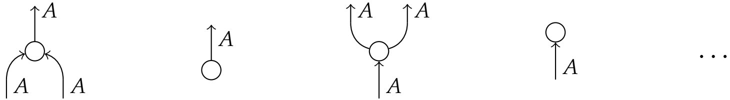


- This style of rewriting works for any (co)algebraic structure in a *monoidal category*, a.k.a. *monoidal algebras*.

Algebraic structures in SMCs

- A (single-sorted) monoidal algebra $\mathcal{A}$ consists of an object A and a set of morphisms whose inputs/outputs have type A:



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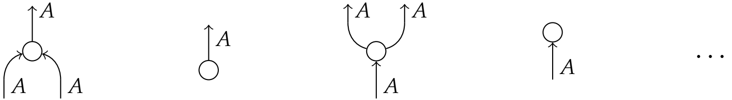

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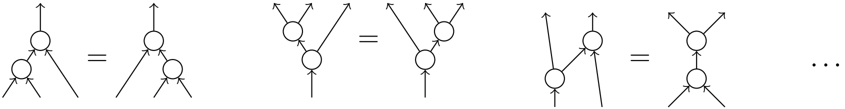
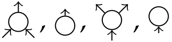
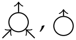
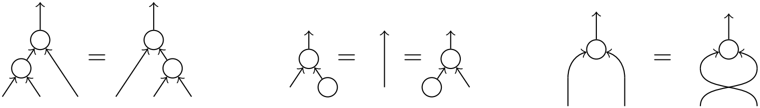
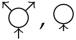
- and some equations:

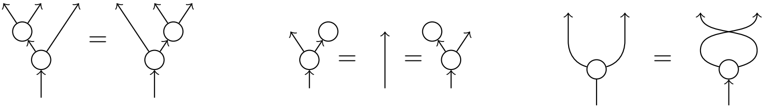


Example: Frobenius algebras

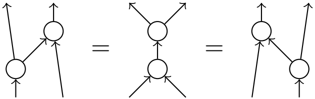
- A *commutative Frobenius algebra* consists of a tuple (A, ) such that:
  - (A, ) forms a commutative monoid:



  - (A, ) forms a commutative comonoid:



  - The *Frobenius law* is satisfied:



Example: Bialgebras

- A *(bi)commutative bialgebra* consists of a tuple $(A,$ , , , $)$ such that:
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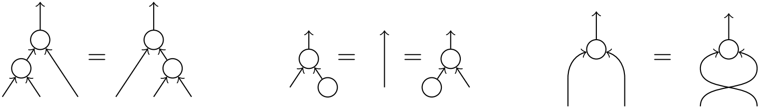

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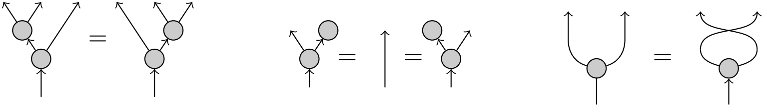
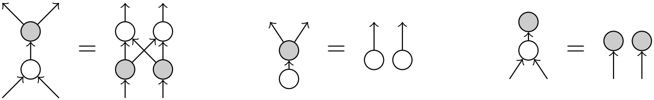


  - The *bialgebra laws* are satisfied:



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  2. *Semantic* PROPs have morphisms with a concrete description (functions, relations, finite matrices, etc.). Equality is usually (easily) decidable.

Example: Commutative monoids are functions

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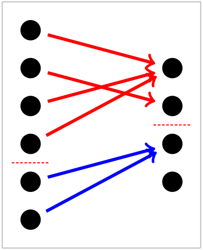

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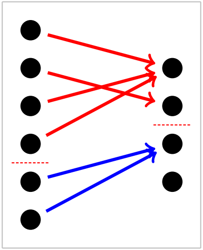
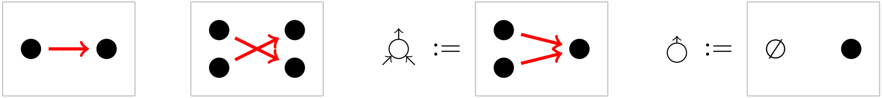


- This whole category is generated by identities, swaps, and a single commutative monoid:



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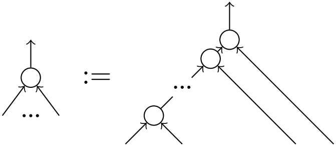

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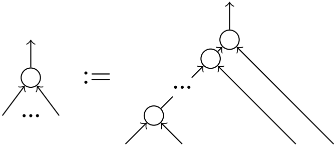
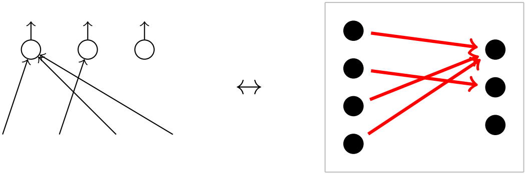


- Then, any diagram of  and  can be put in normal form, and those normal forms are classified by functions:



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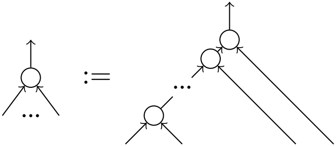

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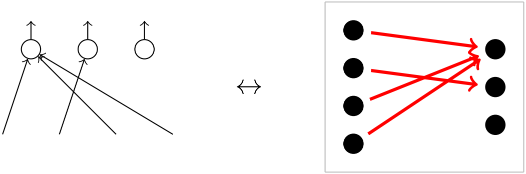

- Similarly, $\mathbb{F}^{\mathrm{op}}$ is the PROP for cocommutative comonoids.

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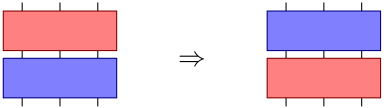
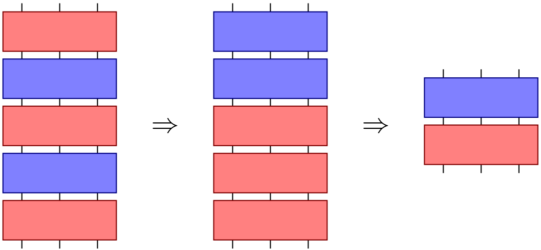
- This is a distributive law of monads in the bicategory of monoids in spans of categories ...or something like that...

Distributive laws

- More concretely, give us the means to move two pieces of structure past each other:



- So, normal forms for each of the individual theories become normal forms for the composed theory:



Example: Bialgebras are matrices

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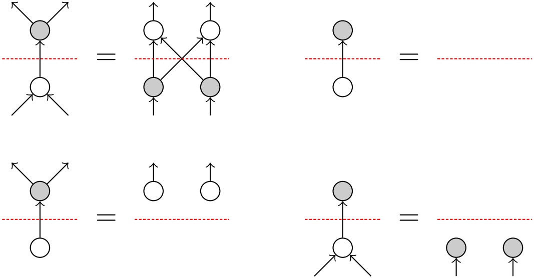

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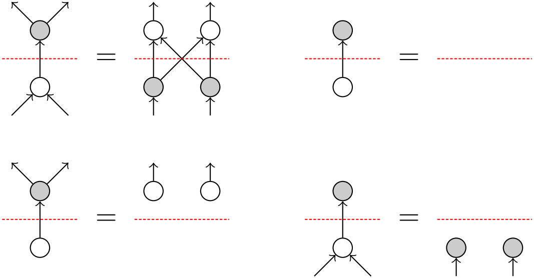
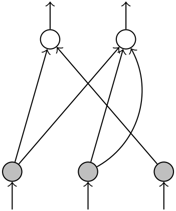


- So, normal forms look like this:



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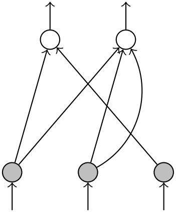

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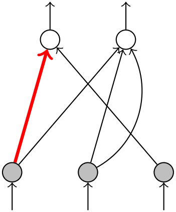

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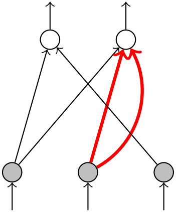

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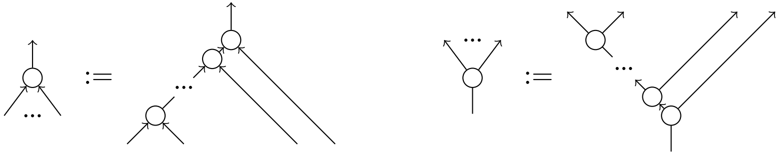
Diagrams with repetition

- Many of these theorems have something in common: the deal with repreated structures, like **trees** and **cotrees**:



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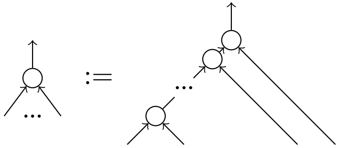
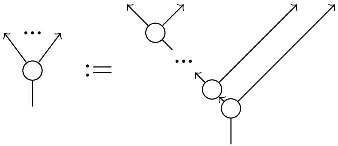
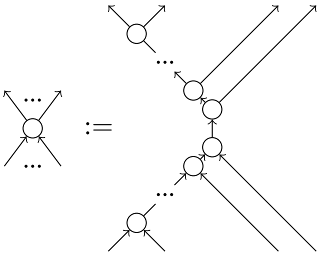




- ...and tree/cotrees, a.k.a. **spiders**:



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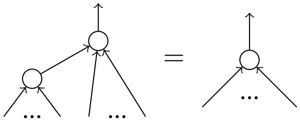

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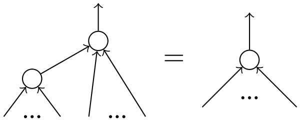
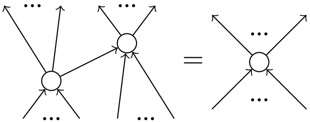


- Similarly, the rules of commutative Frobenius algebras are expressed by letting spiders fuse:



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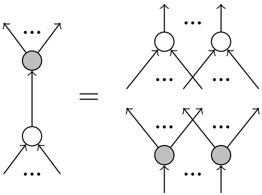

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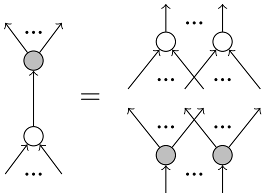

where the RHS is a connected bipartite graph.

- These three examples have something in common: they rely on your brain, and some “blah blah” to fill in the “$\cdots$”

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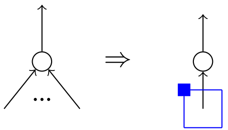
- One answer is the *!-box langauge*

!-boxes

- We can formalise families of diagrams (with variable-arity generators) using some graphical syntax:



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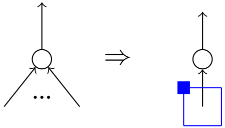
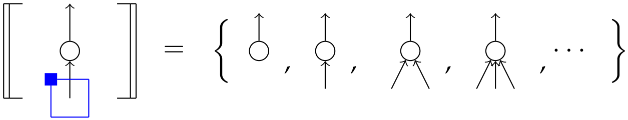
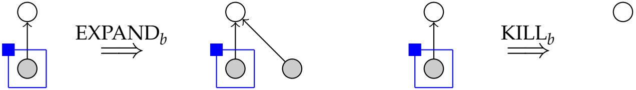


- The blue boxes are called !-boxes. A graph with !-boxes is called a !-graph. Can be interpreted as a set of concrete graphs:



!-boxes

- The diagrams represented by a !-graph are all those obtained by performing EXPAND and KILL operations on !-boxes



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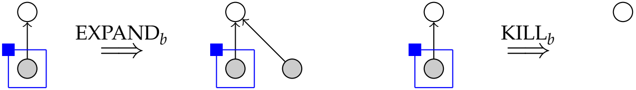
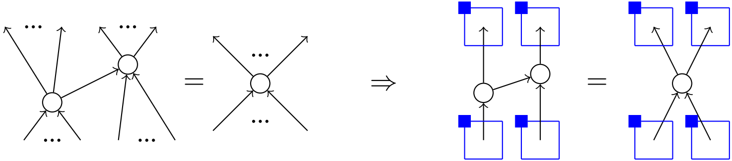
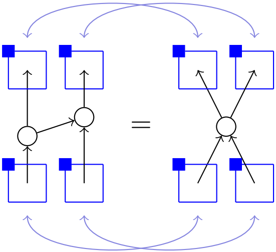


- We can also introduce equations involving !-boxes:



!-boxes: matching

- !-boxes on the LHS are in 1-to-1 correspondence with RHS



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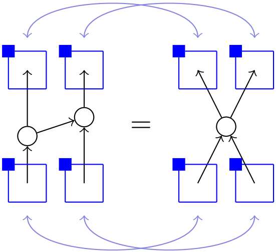
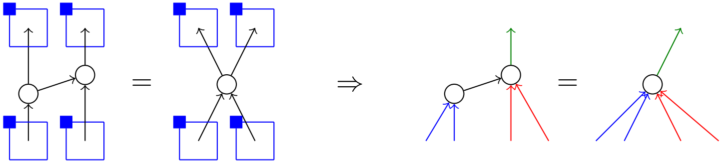


- EXPAND and KILL operations applied to both sides simultaneously to instantiate a rule.

!-graph to concrete graph rewriting

- Rewriting concrete diagrams: find an instantiation of the rule such that the LHS matches the diagram:



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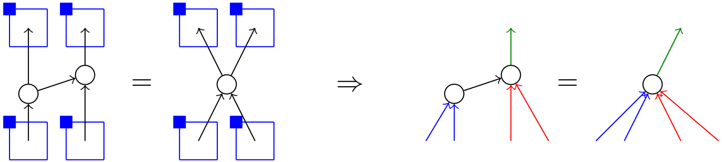

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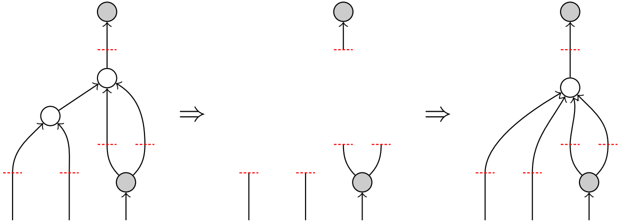

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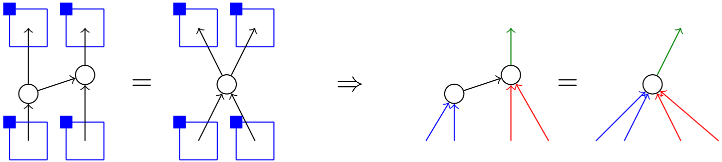

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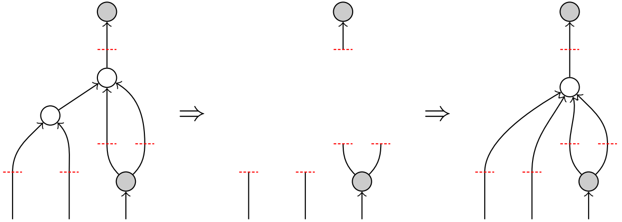

- Sound and complete, in the absence of “wild” !-boxes

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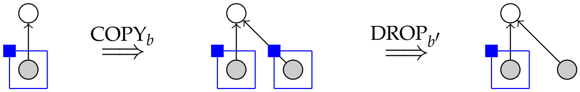

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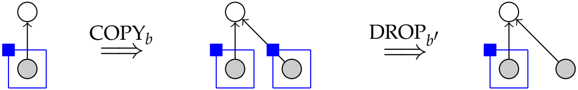

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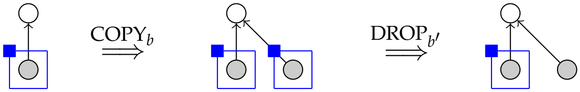

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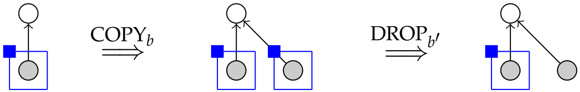

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- If any part of an edge is in a !-box, **we must cut through it.**

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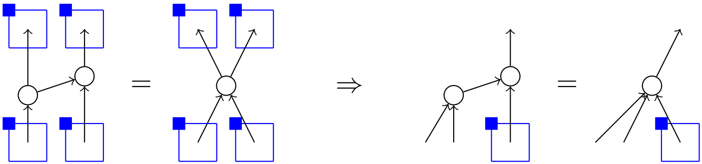
- !-graph rewriting: first instantiate:



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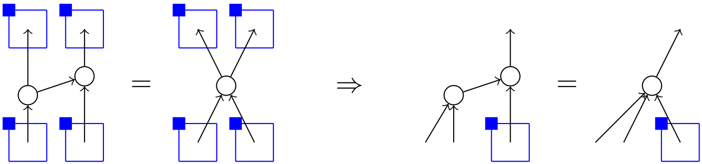

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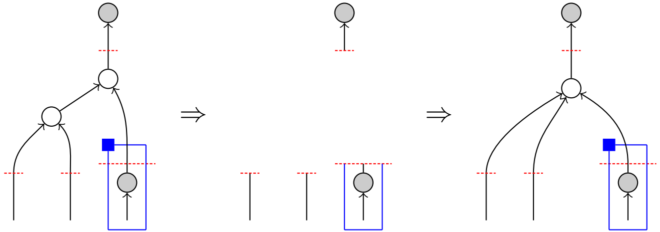

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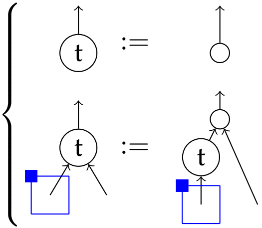

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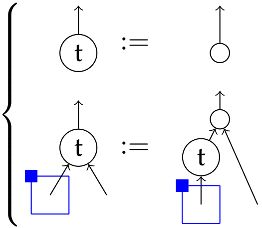

- And, as usual, recursive definition goes hand-in-hand with inductive proof...

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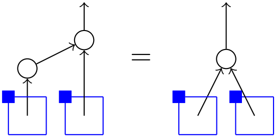
- By (normal) induction over proofs involving concrete graphs, we can prove admissibility.

Induction principle for !-graphs

- Using !-box induction, we can now prove standard things like:



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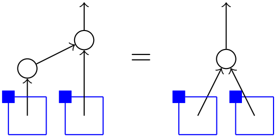
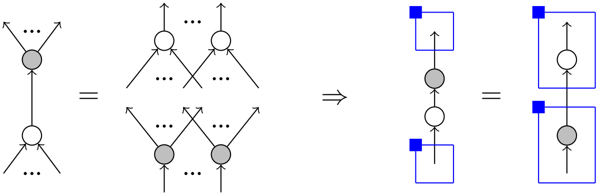


- But this just looks like something in term-land. We can actually prove much more interesting things like:



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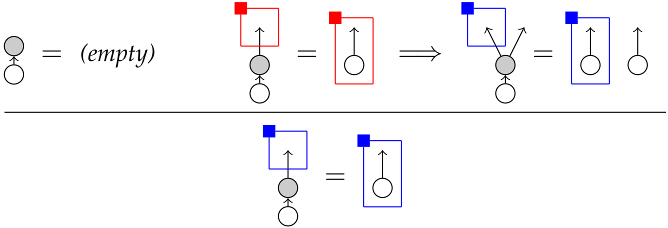

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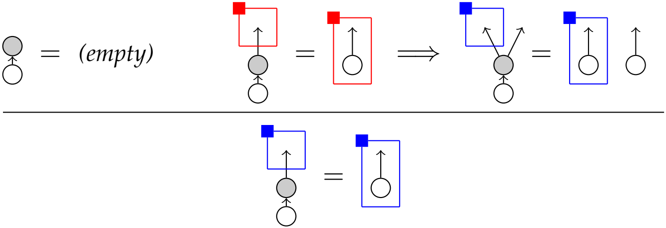
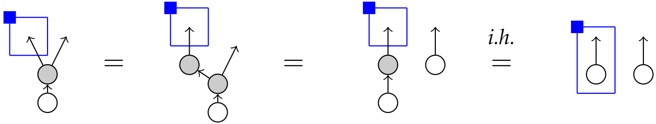
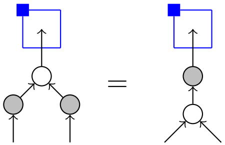


- The base case is an assumption, step case by rewriting:



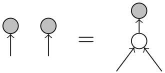
Induction Example

Lemma



Proof.

Base:



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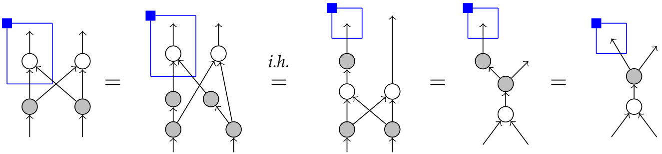
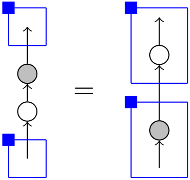


Induction Example

Theorem



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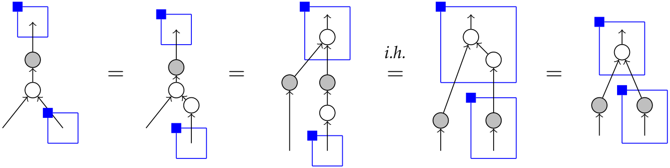

□

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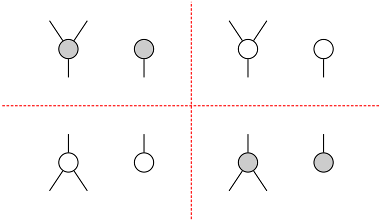

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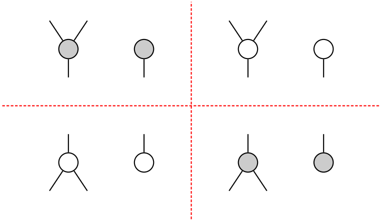

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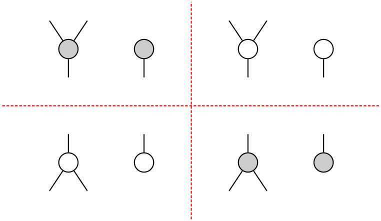

- This theory is known as $\mathbb{IB}$, or the phase-free fragment of the ZX-calculus.
- Its pops up all over the place: signal-flow networks, Petri nets with boundaries, quantum circuits...

Interacting bialgebras

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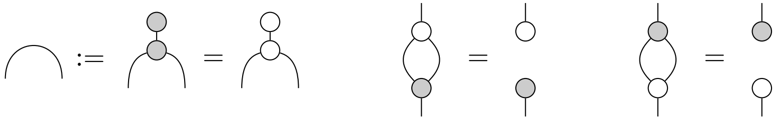

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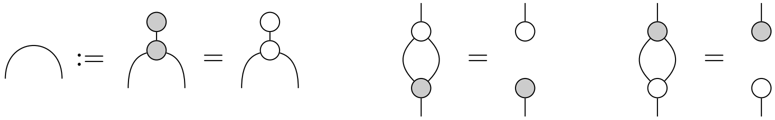

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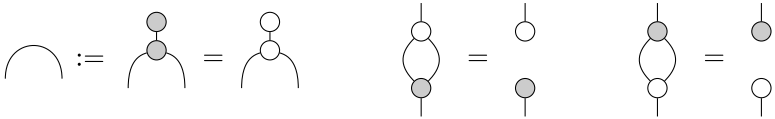

- The first essentially means we can ignore directions in diagrams, and the second means these *bialgebras* are actually *Hopf algebras*, with trivial antipode.
- Last year, Sobocinski and Bonchi showed (using non-rewriting techniques) that the PROP for this thing is $\mathrm{VecRel}_{\mathbb{Z}_2}$, the category of *linear relations*.

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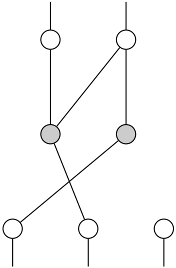
  - **grey dots** are vectors spanning the subspace

Lets see how this works...

- Subspaces can be represented as:



$$\leftrightarrow \quad \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ \hline 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \hline 0 \\ 1 \end{pmatrix} \right\rangle$$

- The 1′s indicate where edges appear for each vector.

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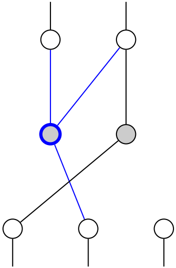

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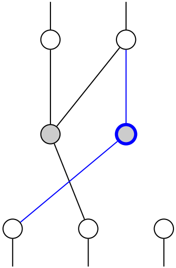

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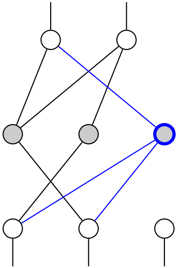
- However, this is not unique. We can always add or remove a vector that is the sum of two other spanning vectors and get the same space:



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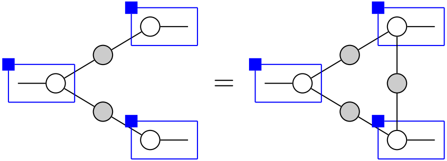
Addition is a !-box rule

- This 'addition' operation can be written as a !-box rule:



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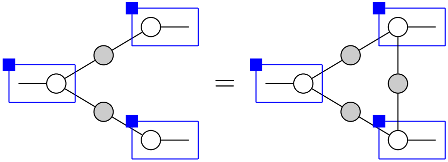
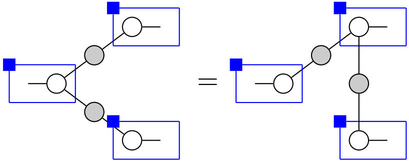


- We can also apply this forward then backward to get a ‘rotation’ rule:



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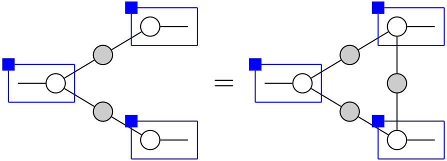

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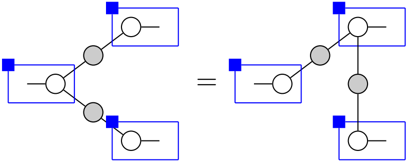

- Note this rule decreases the arity of the white dot on the left by 1.

A reduction strategy...

- This gives a reduction strategy for **IB**-diagrams.

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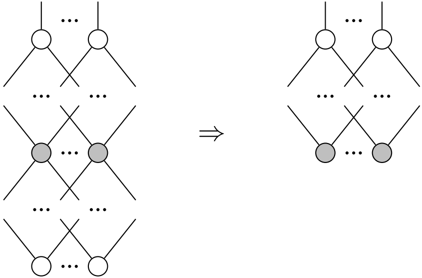
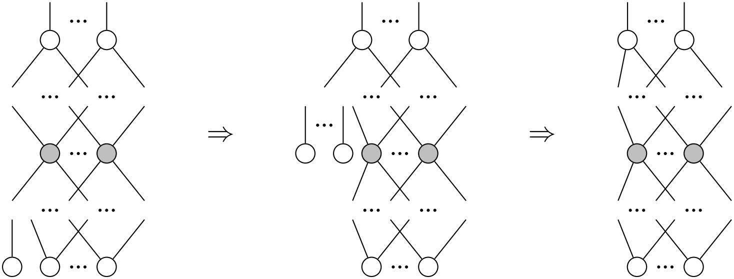
- First, write diagram as a layer of **interior white** dots, then **interior grey** dots, then **boundary white** dots.
- To get to pseudo-normal form, we just need to get rid of the interior white dots:



A reduction strategy...

- We do this by applying a rule to reduce the arity of a single white dot, until the arity is 1, then copy through:



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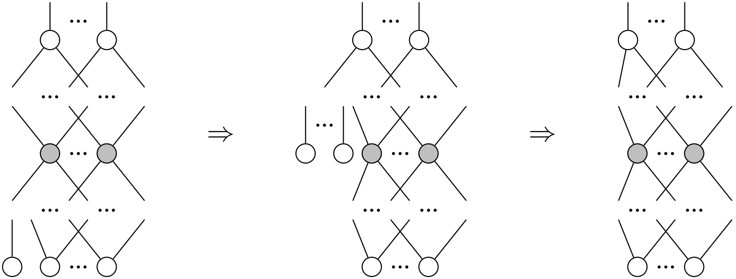

- **Time to fire up Quantomatic!**

Thanks!

- Joint work with Lucas Dixon, Alex Merry, Ross Duncan, Vladimir Zamdzhiev, David Quick, and others
- See: `quantomatic.github.io`