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Bipartite unitary supermaps and their classification

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Abstract

The study of causal order in Quantum Theory has recently led to the consideration of physical scenarios in which the causal order between events is not well-defined in a classical way, a phenomenon called indefinite causal order. Yet, a proper understanding of the range and classification of the ensemble of such scenarios, as well as of their general features, is still lacking. In this report we present steps forward towards a better and more intuitive classification of these possible scenarios. We develop a new and more intuitive framework in which to present these scenarios based on the use of diagrammatic techniques, framing them as supermaps. We present some of the current definitions, theorems, examples and open questions related to them in this framework. We use this framework and techniques from the field of quantum causal structures to prove a theorem on the decomposition of bipartite unitary supermaps, which establishes a direct link between their classification and the classification of quantum channels obeying a specific condition. Based on this result, we present a conjecture about the general form of bipartite unitary supermaps and prove it in a special case.

Contents

1 Introduction

In recent years, the study of causality and causal structures in Quantum Theory has been a growing line of research in Quantum Information (QI). In particular, substantial interest has been generated by the idea, advocated in two seminal papers [1, 2], that quantum theory may allow for scenarios in which naive concepts of a pre-existing and well-defined causal structure break down in some way, and are replaced with a more exotic (yet arguably physical) causal structure featuring some typically quantum effects. Such structures have been grouped under the umbrella term of "indefinite causal order".

The need to study indefinite causal order appeared in two different research contexts. In the first one, illustrated by [1], the emphasis was put on the treatment of Quantum Theory as a theory of computation. Such a perspective entails the necessity to consider its higher-order transformations, which, in this case, are maps on quantum channels, called supermaps. A specific supermap (the quantum SWITCH) was then displayed as a paradigmatic example of the clash between the existence of such higher-order computations and the notion that every possible quantum computation can be realised in one of the most popular models of quantum computation, the quantum circuit model [3]. The other context, illustrated by [2], put the emphasis on an operational account of the correlations between local observers obeying the laws of quantum physics. It was argued that these correlations could, in their most general form, be described by a mathematical object, called a process matrix; and an example of a process matrix (the OCB process) was displayed as a paradigmatic example of the clash between the existence of process matrices and a naive notion of a definite causal order existing independently to the actions of local agents.

This diversity in the considerations having led to the study of indefinite causal order is in parallel with the diversity of the applications it has been considered for. One may want to consider it for its applications to quantum computation and quantum technologies: it has been shown that its use provides advantages over traditional quantum scenarios in some examples of computational [4, 5] and communicational [6–8] tasks. At the same time, one could also consider it for its fundamental implications, as a new physical phenomenon that quantum theory may give rise to. It has also been argued that indefinite causal order could be a central component in a theory of quantum gravity [9].

The physical status of indefinite causal order is still unclear. Whereas the quantum SWITCH, which has been shown to be an example of a weak form of indefinite causal order [10], has been experimentally implemented in photonic systems [11–14], more exotic scenarios like the OCB process still lack a physical interpretation, be it quantum gravitational or else. It is possible that both mathematical formalisms encompassing indefinite causal order (supermaps and process matrices) allow for scenarios which, although mathematically well-defined, are physically irrelevant.

More than that, even without considering physicality, the exact range of scenarios that these formalisms allow for in the first place is very poorly known; proving general and powerful theorems on the properties of all supermaps (or process matrices) - and therefore possibly on indefinite causal order - is made difficult by their abstruseness as mathematical objects, as well as the fact that they are still solely defined by "good-behaviour conditions", as opposed to intuitive structural properties.

This is why most of the research on indefinite causal order so far has focused on specific examples, such as the quantum SWITCH or the OCB process, acting as toy models.

Supermaps acting on one input channel are an example of a case in which a suitable general form has been found [15]. Our aim during this project has been to find the general form of unitary[1] bipartite supermaps. Proving such a form would be a major step forward in our understanding of unitary bipartite supermaps: it would give us an idea of the extent and nature of their non-classicality; it would allow us to prove general theorems about them more easily; and it could pave the way to a better understanding of their physical status.

To help us in this task, we used the powerful tools recently developed in a diagrammatic reformulation of Quantum Theory [16]. As we shall see, these diagrammatic techniques are particularly well adapted to the study of causal structures, in which the wiring between events is of crucial importance. Building on them, we provided a new formalism for supermaps, equivalent to both the usual formalism of supermaps and the process matrix formalism[2]; we found this new formalism to be better suited to the study of supermaps. We also made use of some of the tools provided by the recently developed field of Quantum Causal Models [17, 18].

Using these tools, we made significant progress towards finding a general form of bipartite unitary supermaps, by exhibiting some of their inner wirings, and reducing the problem of their classification to that of the classification of unitary channels satisfying a specific condition, which we call the no-bridge condition, and which is arguably easier to solve. However, we haven't reached yet the general form we have been looking for. In particular, we have not been able to prove yet our main conjecture: that all bipartite unitary supermaps are either causally-ordered, or a variation of the quantum SWITCH. We were however able to prove it in a specific case.

The structure of this report will be as follows. In Section 2, we will provide a short introduction to the objects and techniques of the diagrammatic framework for Quantum Theory, in order to lay down the tools we will use, and we will present the representation that it entails for quantum states and channels (which we call the double state representation) and its relations with two other formalisms for quantum theory: the density matrix representation, and the Choi-Jamiołkowski representation. In Section 3, we will very quickly present the language and tools of Quantum Causal Models. In Section 4, we will provide a definition of supermaps and present the main examples, theorems, applications and open questions which relate to them; this will also be the occasion to define all these results in a new formalism, based on the double state representation. Finally, in Section 5, we will lean on these developments to present the main results which we obtained during the course of this project, and discuss their significance.

[1] Unitary supermaps can be conceived as a generalisation of unitary quantum channels; their precise definition, as well as the motivation for considering them, will be presented later on.

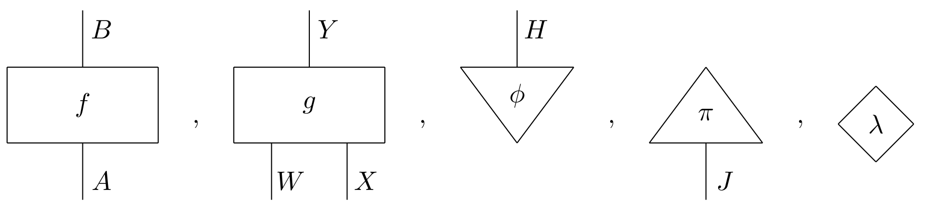
[2] We will not in general provide the proofs of the equivalence between our formalism and the two traditional ones; although straightforward, they are rather tedious and would bring this report well over its size limit.

2 An introduction to the diagrammatic framework for Quantum Theory

2.1 Linear maps

In this report, we will use the diagrammatic language recently developed by Bob Coecke, Aleks Kissinger and coworkers as a new framework for quantum theory. A standard reference on the subject is [16]. We will here provide the reader with a quick recap of its objects and techniques, which should be sufficient to make the rest of this report understandable.

This framework frames quantum theory as a *process theory*, in which the emphasis is put on the processes that compose the theory, rather than on its states. Its core elements will therefore be processes (represented by boxes), having a certain number of inputs (represented by wires connected to the lower side of the box) and of outputs (represented by wires connected to the upper side of the box). Each wire has a given *system-type*, corresponding to the kind of thing it conveys; it is represented by a name written next to the corresponding wire. Examples of processes are:

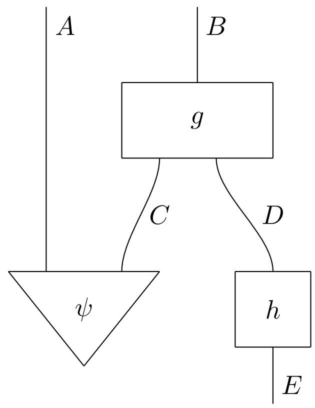


(1)

The three last examples showcase some important types of processes, which are graphically emphasised by their specific forms. Processes without any inputs (but with any number of outputs), such as ϕ, are called states. Processes without any outputs (but with any number of intputs), such as π, are called effects. Finally, processes without any outputs or inputs (i.e. processes which are both states and effects), such as λ, are called numbers.

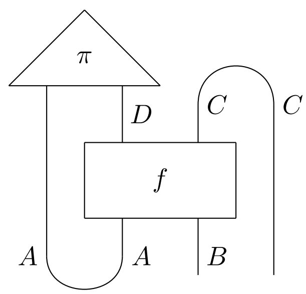
A process theory is to a large extent characterised by the way it allows to compose processes together. The process theories which will be of interest to us will be so-called Strict Symmetric Monoidal Categories (SSMC), which allow for sequential and parallel compositions between processes. Parallel composition $\otimes$[3] corresponds to our usual understanding of two things happening in parallel without interfering with each other; sequential composition $\circ$ corresponds to one thing happening after another. A process g can be sequentially composed with another process f only if the system-types of g's input and f's output match. SSMCs are graphically represented by circuit diagrams, in which two boxes being wired together corresponds to sequential composition (with information always going from the lower box to the upper one), and two boxes being placed next to each other horizontally corresponds to parallel composition. An example of a circuit diagram is:

[3] Note that, despite the same symbol being used, $\otimes$ here does not necessarily represent the usual tensor product.



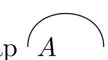
(2)

A feature of circuit diagrams is that they only allow for information to flow in a single direction: they do not feature wires being bent in the other direction. Adding this possibility leads to string diagrams, such as:

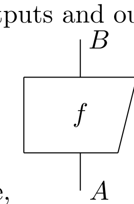


(3)

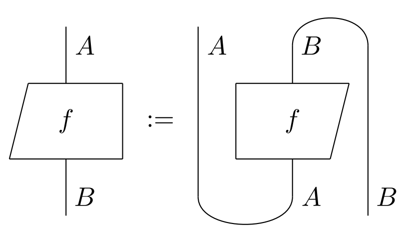
Process theories which make sense of string diagrams do so by introducing, for each system-type A, a bipartite state, the cup 

, and a bipartite effect, the cap 

A, which satisfy suitable axioms. The cup and the cap can then be used to turn inputs into outputs and outputs into inputs.

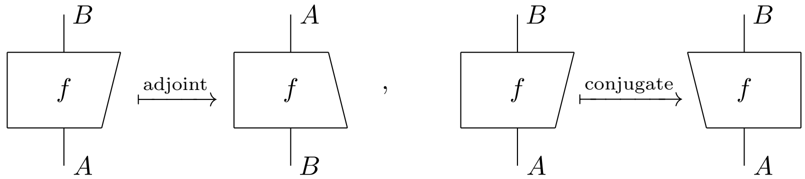
Let us now draw a process f from A to B with an uneven shape, 

. Process theories which admit string diagrams allow to define an involution among processes, called the transposition. The transposed process of f is a process from B to A, written:



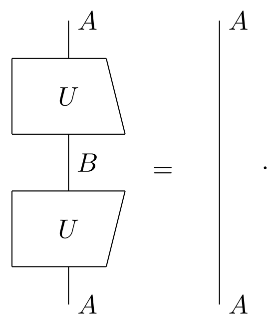
(4)

We see that graphically, transposition is represented by a 180° rotation. Moreover, one may also add another involution to the process theory , called the adjunction, represented by a vertical reflection. The composition of transposition and adjunction is then another involution, called the conjugation, represented by a horizontal reflection. The axioms satisfied by transposition, adjunction and conjugation ensure that graphical reasoning is natural for them.



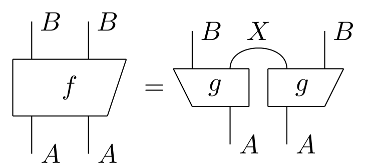
(5)

Dagger Compact Closed Categories, which are specific cases of SSMCs, are specific process theories which admit string diagrams as well as adjoints. From the introduction of adjoints, one can define isometries as the processes U which satisfy



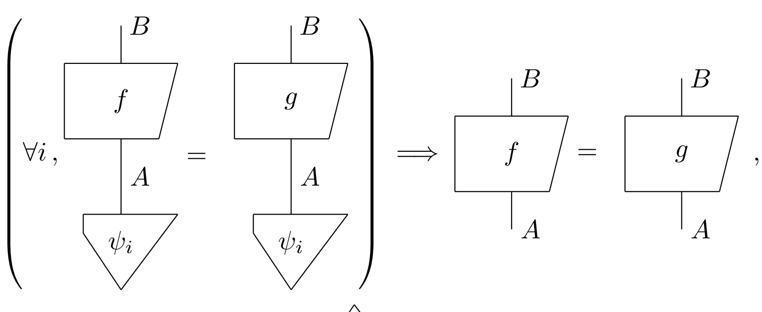
(6)

A unitary is then an isometry whose adjoint is an isometry as well. It will also be useful to define $\otimes$-positivity: a process f of type $A \otimes A \to B \otimes B$ is $\otimes$-positive if it can be decomposed as

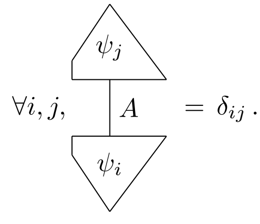


(7)

One can also define an orthonormal basis (ONB) for a given system-type A as a family $\{\psi_i\}_i$ of states of A satisfying:

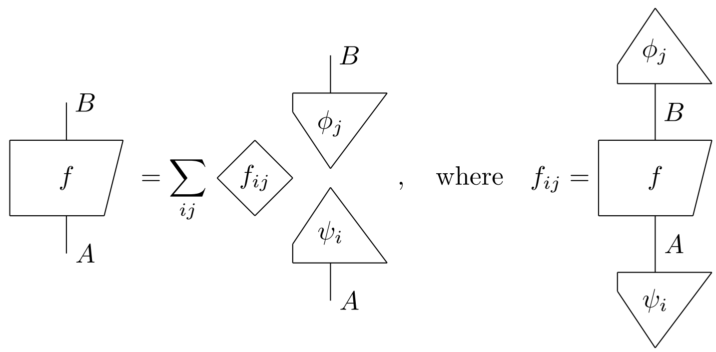


(8a)



(8b)

Finally, one can introduce sums of processes. Only processes with the same inputs' and outputs' system-types can be summed together. One requires of sums to be commutative, to distribute over the adjoint, and to distribute over diagrams (which essentially implies that all the processes considered are linear). Then, given ONBs $\{\psi_i\}_i$ and $\{\phi_j\}_j$ for system-types A and B respectively, any process f from A to B can be characterised by its matrix form in these bases:



(9)

In a process theory admitting sums, for a given system-type A, all ONBs have the same size, called the dimension of A. With all those ingredients, we can finally characterise a process theory for (finite-dimensional) linear maps on the complex numbers:

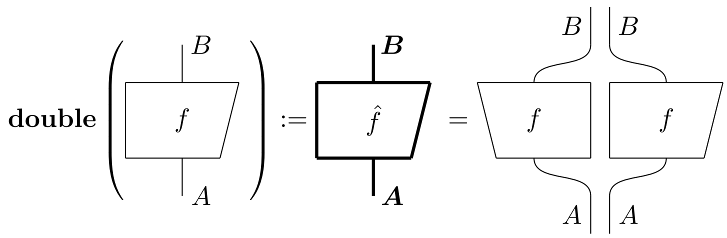
Definition 1 *The process theory of* ***linear maps*** *is the set of all processes described by string diagrams where:*

- *Each system-type has a finite ONB;*
- *There is at least one system-type of every dimension $D \in \mathbb{N}$;*
- *Processes of the same type admit sums;*
- *The numbers are the complex numbers $\mathbb{C}$.*

As we can see, we have recovered a theory of linear maps on $\mathbb{C}$ in a way quite different from the usual, set-theoretic one, which goes through the construction of Hilbert spaces. We can now build a theory of quantum processes by building on this new point of view on linear maps.

2.2 Quantum processes

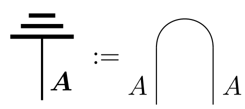
The next step in the way to quantum theory is to introduce doubling. This is a step equivalent to the transition from vectors to density matrices in the usual construction of quantum theory; the difference is that rather than turning a ket $|\psi\rangle$ into an operator formed by $|\psi\rangle$ and its adjoint bra, $|\psi\rangle\langle\psi|$, it rather turns it into a bipartite ket formed by $|\psi\rangle$ and its conjugate ket, $|\psi^*\rangle|\psi\rangle$. Doubling allows to go from a process theory admitting string diagrams and adjoints to another process theory admitting string diagrams and adjoints. It does so by taking as system-types of the doubled theory the $\boldsymbol{X} := X \otimes X$, where X is a system-type in the original theory, and by defining the double of any process f in the original theory as:



(10)

As we see, a bold wire in the doubled theory can be understood as representing two single wires in the original theory; and a bold, hatted process in the doubled theory represents the associated single process parallely composed with its conjugate process in the original theory. One can show that the doubled theory then naturally inherits the string diagrams structure, as well as adjoints, - but not summation or ONBs -, from the original theory. The doubled theory one can get from **linear maps** is called **pure quantum maps**. The latter still lacks our usual requirement that states should be normalised and processes unitary, and it doesn't allow yet for impure states.

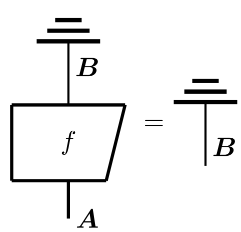
The step which will bring us there is the definition of the discarding effect, which can be thought of physically as the act of throwing away some subsystem and not caring about it anymore. The discarding effect for any system-type $\boldsymbol{A}$ is defined at the single level as:



(11)

We see that discarding corresponds to the action of tracing-out in the usual density matrix picture. The theory of **quantum maps** is the process theory obtained by adding discarding to **pure quantum maps**; it admits string diagrams, adjoints, and sums. Discarding also serves to introduce the concept of causality in **quantum maps** in the following way.

Definition 2 *A process f from $\boldsymbol{A}$ to $\boldsymbol{B}$ in **quantum maps** is called causal if:*



(12)

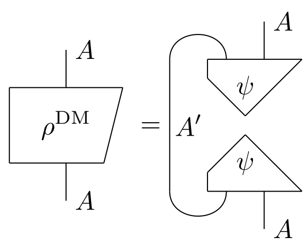
In **pure quantum maps**, causal states are the equivalent of normalised states in the usual presentation of quantum theory, and causal processes are the equivalent of isometries. In **pure quantum maps**, causal states are the equivalent of process matrices, and causal processes are the equivalent of CPTP maps. Thus, the introduction of doubling, discarding and causality allowed us to recover a new presentation of quantum theory.

Given that this report will be dealing with causal order, it is worth noting that causality, as its name suggests, is the core element in any consideration on causal order in quantum theory. Indeed, it can be understood as the requirement that there is no signalling from the future to the past. Imagine that at some point in a quantum process, a subsystem K is produced which will not later be measured or made to interact with other subsystems (i.e., K is essentially discarded). Causality then ensures that whatever action is realised on K after it has ceased interacting with the other subsystems will have no influence on those other subsystems.

2.3 Relations between some representations for states and channels in quantum theory

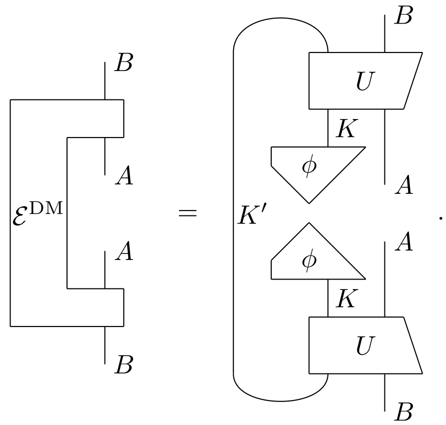
As we have seen, the representation we have introduced, which we shall call the doubled state (DS) representation, is equivalent to the usual representation, which we shall call the density matrix (DM) representation; furthermore, as we shall see, there exists a third representation for maps, which stems from the Choi-Jamiołkowski isomorphism, and which we shall therefore call the Choi-Jamiołkowski (CJ) representation. All these representations essentially differ in what they deem to be inputs and outputs at the tensorial level, and although the transformations between them might look trivial, one can easily get confused when incautiously jumping from one to another. As most of our study will involve dealing with higher-order maps, such as maps on maps, it will be all the more important for us to properly deal with those different representations. Fortunately, the diagrammatic language for tensor calculus and adjoints provides us with a highly natural way to express those representations and understand how they relate to one another. Here, the representation in which a quantity is expressed will be denoted by the superscript DS, DM or CJ.

Let us consider a system A, with an associated Hilbert space $\mathcal{H}_A$ of finite dimension. In the DM representation, a state of A is represented by its density matrix $\rho^{\mathrm{DM}} \in \mathrm{Lin}\left(\mathcal{H}_A\right)$; ρ^{DM} is positive with trace 1, which is equivalent to saying that it can be purified, i.e. that there exists an auxiliary system A' and a normalised vector $\psi \in \mathcal{H}_{A'} \otimes \mathcal{H}_A$ such that



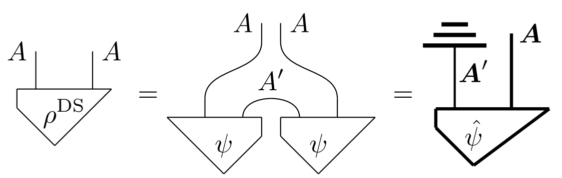
. (13)

We see that, even though it physically represents a state, ρ^{DM} is mathematically an operator. In turn, in the DM representation, a physical channel from A to B will be represented by a (linear) *super-operator* $\mathcal{E}^{\mathrm{DM}} \in \mathrm{Lin}\left(\mathrm{Lin}\left(\mathcal{H}_A\right) \to \mathrm{Lin}\left(\mathcal{H}_B\right)\right)$. $\mathcal{E}^{\mathrm{DM}}$ is a Completely Positive Trace-Preserving (CPTP) map, which is equivalent to saying that it possesses a Stinespring dilation, i.e. that there exist auxiliary systems K and K', a normalised vector $\phi \in \mathcal{H}_K$ and a unitary $U \in \mathrm{Unit}\left(\mathcal{H}_K \otimes \mathcal{H}_A \to \mathcal{H}_{K'} \otimes \mathcal{H}_B\right)$ such that



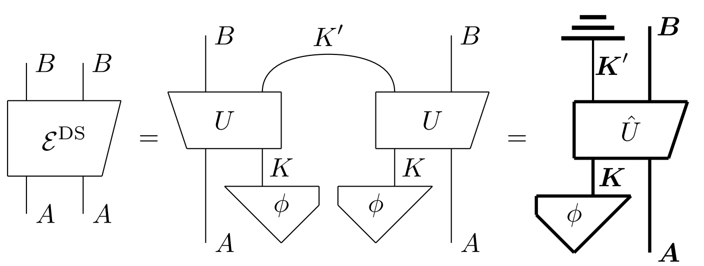
(14)

In contrast, in the DS picture, the same physical state as in (13) is represented not by an operator, but by a (doubled) state $\rho^{\mathrm{DS}} \in \mathrm{Lin}\left(\mathcal{H}_A^{\otimes 2}\right)$. ρ^{DS} is $\otimes$-positive and causal, which is equivalent to saying that there exist an auxiliary system A' and a normalised vector $\psi \in \mathcal{H}_{A'} \otimes \mathcal{H}_A$ (which can be chosen to be the same as in (13)) such that



(15)

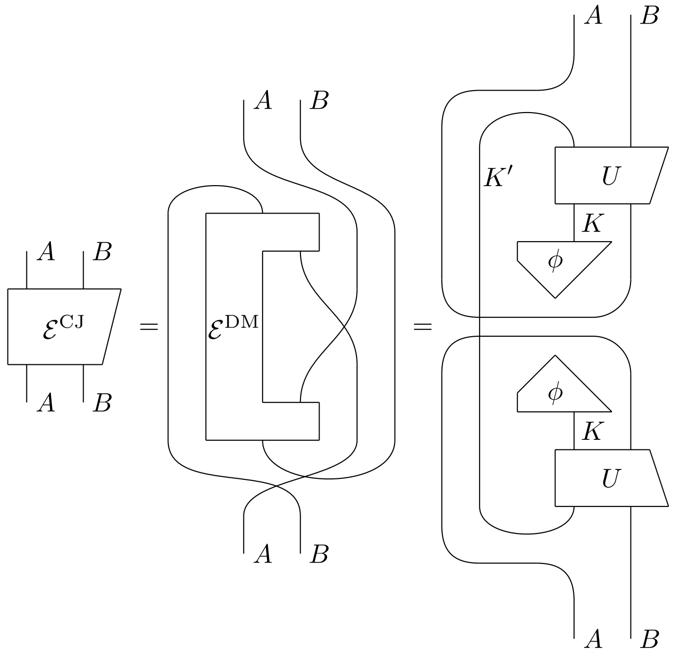
where the last expression makes use of doubling. Consequently, a channel is represented by an operator $\mathcal{E}^{\mathrm{DS}} \in \mathrm{Lin}\left(\mathcal{H}_A^{\otimes 2} \to \mathcal{H}_B^{\otimes 2}\right)$; $\mathcal{E}^{\mathrm{DS}}$ is $\otimes$-positive and causal, which is equivalent to saying it has a Stinespring dilation, i.e. that there exist auxiliary systems K and K', a normalised vector $\phi \in \mathcal{H}_K$ and a unitary $U \in \mathrm{Unit}\left(\mathcal{H}_K \otimes \mathcal{H}_A \to \mathcal{H}_{K'} \otimes \mathcal{H}_B\right)$ (all of which can be chosen to be the same as in (14)) such that



(16)

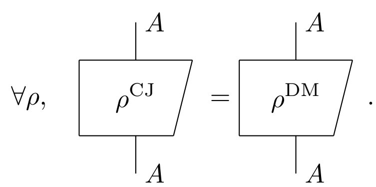
Finally, there exists a third representation of a channel $\mathcal{E}$, which can be recovered from its DM representation $\mathcal{E}^{\mathrm{DM}}$ via the Choi-Jamiołkowski (CJ) isomorphism [19, 20]. The CJ isomorphism relates completely positive maps in $\mathrm{Lin}\left(\mathrm{Lin}\left(\mathcal{H}_A\right) \to \mathrm{Lin}\left(\mathcal{H}_B\right)\right)$ to positive semi-definite operators in $\mathrm{Lin}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$. In particular it allows us to go from $\mathcal{E}^{\mathrm{DM}}$, as defined in (14), to a representation $\mathcal{E}^{\mathrm{CJ}} \in \mathrm{Lin}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$ given by[4]

[4] Here we shall follow the definition of the CJ isomorphism used in [2], which differs from the original one by a transpose.



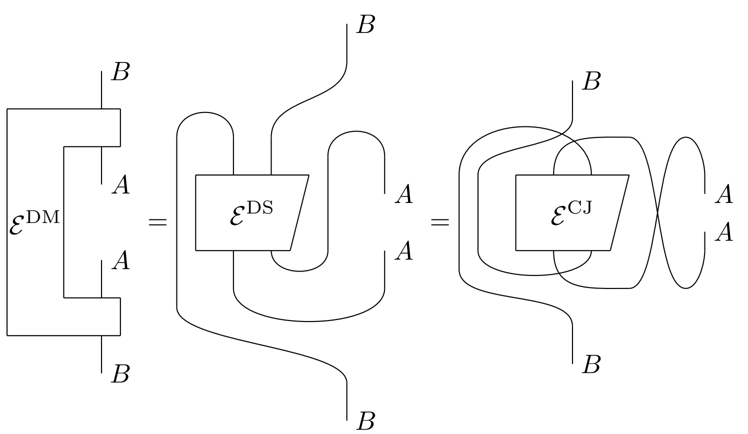
(17)

It is straightforward to see that $\mathcal{E}^{\mathrm{CJ}}$ corresponds to a (non normalised) state in the DM representation; this fact is known as the *channel-state duality*[5]. One may note that the CJ representation of a physical state (which can be deduced from (17) by taking A to be a trivial system) is the same as its DM representation,



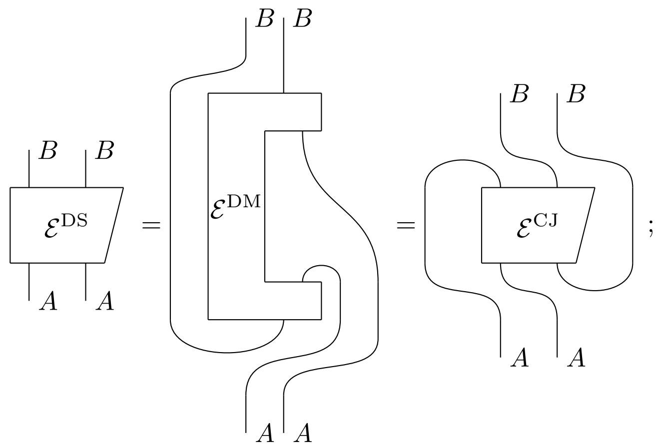
(18)

We can thus derive a set of formulas relating the different representations of a channel $\mathcal{E}$:

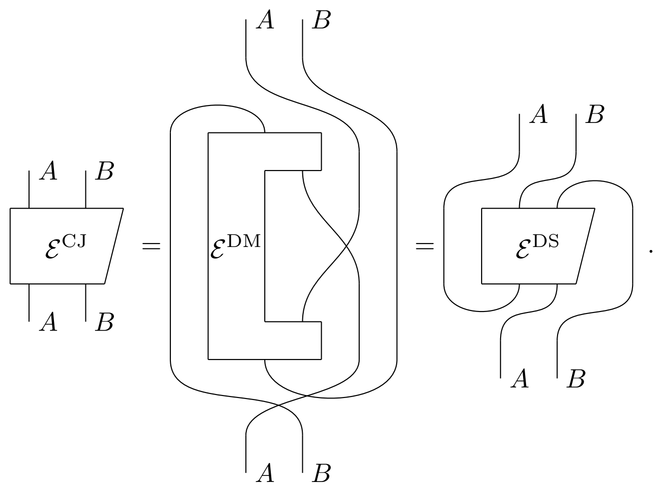


(19)

[5]It is worth noting that channel-state duality can be obtained in the DS representation as well: indeed, composing (16) with a doubled cup yields an unnormalised state in the DS representation.



(20)



(21)

3 Quantum causal models

We will briefly present here a subject of active research whose methods and results will be useful to us. The study of quantum causal models [17] aims to study the constraints imposed on multipartite unitary channels by the assumption that they are compatible with a causal structure. Roughly speaking, a causal structure can be understood as defining which inputs can have an effect on which outputs through the unitary evolution. What will matter to us here are the consequences of the requirement that a multipartite unitary channel satisfies one or more *no-influence relations*, defined as the requirement that a specific input cannot have an effect on a specific output. In the DS formalism, a no-influence relation on a multipartite unitary channel can be defined as follows.

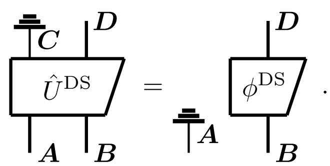
Definition 3 *[No-influence relation]*

Let $U \in \mathrm{Unit}\left((\mathcal{H}_A \otimes \mathcal{H}_B) \to (\mathcal{H}_C \otimes \mathcal{H}_D)\right)$ *be a bipartite unitary*[6]*, whose associated unitary channel is* $\hat{U}$*. Then* $\hat{U}$ *is* ***compatible with the no-influence relation***

[6]Note that the Hilbert spaces here could be further decomposed into factor Hilbert spaces, which means that this definition also holds for no-influence relations on multipartite channels.

$$A \nrightarrow D \tag{22}$$

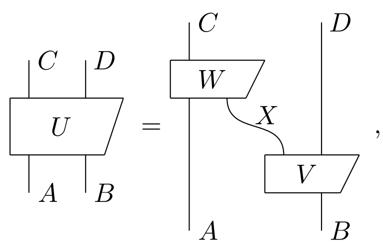
if there exists a quantum channel $\phi \in \mathrm{Lin}\left(\mathcal{H}_C^{\otimes 2} \to \mathcal{H}_D^{\otimes 2}\right)$ *such that*



(23)

The above constraint can be interpreted in the following way: if the output C is discarded, then the input A may as well have been discarded right away, as it does not have any influence on the remaining output, D. As shown in [21], this is equivalent to the fact that it is impossible to communicate to D by varying the input state at A.

One of the central questions in the study of no-influence relations is whether the compatibility of a unitary channel with one or several of them has consequences in terms of its possible compositional decomposition. For example, it turns out that this is the case for a unitary compatible with a single no-influence relation; it is proven in [22] that for the unitary quantum channel $\hat{U}$ used in Definition 3, to be compatible with $A \nrightarrow D$ is equivalent to the fact that U can be decomposed as



(24)

where X is an auxiliary type and V and W are unitary operators. We see here that $\hat{U}$ featuring no influence from A to D is equivalent to U displaying a compositional structure in which there is no path from A to D. Such a theorem is extremely appealing in that it allows to translate the causal structure of a unitary channel into a straightforward diagrammatic form which it is easy and natural to work with. Thus, one would like to have similar results for unitary channels satisfying several constraints, that is, that such channels can be decomposed in a compositional form on which the no-influence relations can directly be read.

It turns out that such a decomposition theorem exists in a few simple cases. In others, a more intricate kind of decomposition is possible using the so-called dot formalism [17]; it is currently unknown whether the dot formalism can accommodate all situations. An upcoming paper [18] provides decompositions for a collection of simple cases.

4 Quantum supermaps

While quantum *channels* are physical operations which map quantum states to other quantum states, quantum *supermaps* are physical operations which map quantum channels to other quantum channels.

As such, they are prime examples of quantum higher-order transformations; more precisely, they constitute second-order transformations. Understanding the full extent of what supermaps can be, the kinds of advantages they would allow for, and whether they are all physically implementable, has been an active subject of research in the past decade. Let us first present how supermaps can be defined and some of the main results and examples which have been recently found in the field. This presentation will also be an opportunity to introduce a new representation for supermaps (which is just an extension of the DS representation introduced in section 2 for states and channels) and to re-express these results in this representation.

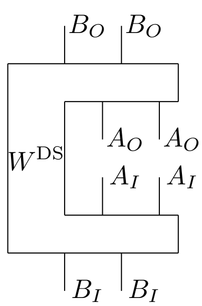
Here we shall focus our study on deterministic supermaps.

4.1 Supermaps on a single operation

Supermaps which take as an input a single channel, and output a single channel, were the first to be studied and are the best understood, with a 2008 paper [15] which both defined them, found their general form, and proved their physical realisability. Let us present a definition of supermaps in the DS picture; it is immediately equivalent to the definition in the DM picture given in [15].

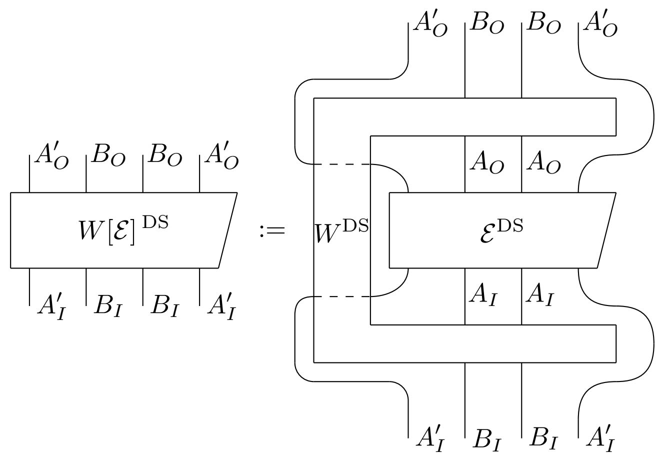
Supermaps are defined to be linear, in order to remain consistent with a probabilistic interpretation of quantum theory. Moreover, remember that a quantum channel from X to Y is defined to be a suitable map not only on states of type X, but on states of type $X \otimes X'$ for any auxiliary system X'; here we will proceed in the same way, asking for a supermap to yield a suitable transformation also on channels possessing any ancillary input and output systems. Here we will use subscripts I and O to respectively denote input and output types of channels, and ' will be used to denote auxiliary types.

Definition 4 *A **monopartite supermap** of type $(A_I \to A_O) \to (B_I \to B_O)$ is a superoperator*



$$\in \mathrm{Lin}\left(\mathrm{Lin}\left(\mathcal{H}_{A_I}^{\otimes 2} \to \mathcal{H}_{A_O}^{\otimes 2}\right) \to \mathrm{Lin}\left(\mathcal{H}_{B_I}^{\otimes 2} \to \mathcal{H}_{B_O}^{\otimes 2}\right)\right) \tag{25}$$

such that, for any auxiliary systems A'_I, A'_O and for any $\otimes$-positive and causal operator $\mathcal{E}^{\mathrm{DS}} \in \mathrm{Lin}\left(\left(\mathcal{H}_{A_I} \otimes \mathcal{H}_{A'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{A_O} \otimes \mathcal{H}_{A'_O}\right)^{\otimes 2}\right)$,

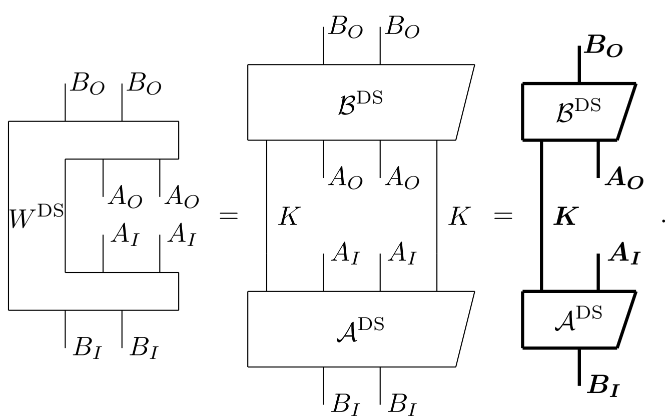


(26)

is a $\otimes$-positive and causal operator in $\mathrm{Lin}\left(\left(\mathcal{H}_{B_I} \otimes \mathcal{H}_{A'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{B_O} \otimes \mathcal{H}_{A'_O}\right)^{\otimes 2}\right)$. [7]

[15] finds a suitable general form for such supermaps: they all arise from the application of a pre- and a post-processing to the input channel. In an interesting contrast to what will come next, the following theorem can be understood as a proof that all monopartite supermaps can be obtained by inserting the input channel inside a quantum circuit.

Theorem 1 *For any monopartite supermap W^{DS} as defined in Definition 4, there exist a type K and $\otimes$-positive causal operators $\mathcal{A}^{\mathrm{DS}} \in \mathrm{Lin}\left(\mathcal{H}_{B_I}^{\otimes 2} \to (\mathcal{H}_K \otimes \mathcal{H}_{A_I})^{\otimes 2}\right)$, $\mathcal{B}^{\mathrm{DS}} \in \mathrm{Lin}\left((\mathcal{H}_K \otimes \mathcal{H}_{A_O})^{\otimes 2} \to \mathcal{H}_{B_O}^{\otimes 2}\right)$ such that*



(27)

Theorem 1 also provides a proof of the fact that any monopartite supermap is physically realisable, as the form (27) just consists in pre- and post-processing. [23] shows a generalisation of this result in the case of quantum combs.

[7] Note that the dashed lines in the diagram stand to denote that a wire is just passing through the box representing an operator, without entering it.

4.2 Multipartite supermaps

A more interesting and much more intricate case is that of supermaps which take several channels as inputs, while still outputting a single channel. This case has been introduced twice in the literature, from slightly different perspectives and with slightly different motivations.

In [1], bipartite supermaps were introduced[8] from a computational perspective. One objective was to ponder the existence of computational advantages provided by higher-order quantum transformations; another was to provide an example of a transformation which cannot be embedded in the quantum circuit framework. The representation used to study those supermaps was as superoperators acting on the CJ representations of input channels.

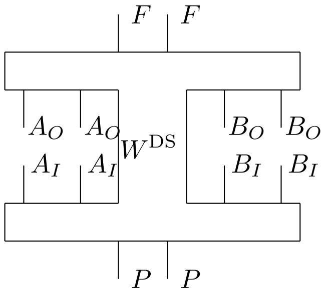
In [2], those same bipartite supermaps were introduced from an operational perspective. The objective was to study the most general kinds of correlations which might exist between two observers for which quantum theory is assumed to hold only locally, without any assumptions as to the existence of a global causal order; this was in particular motivated by questions about the correlations which could arise in a theory of quantum gravity. In line with this objective, [2] actually only studied supermaps with a trivial output, that is, mapping two channels to scalars, as this was sufficient to consider the correlations between the probabilities of given outcomes if those channels are taken to be measurement operations; yet, the formalism developed for that purpose readily generalises to more general supermaps - this was done in [24]. This formalism is based on so-called *process matrices*[9], which belong to the Hilbert-Schmidt dual of the operator space of the CJ representations of input channels.

As we did in the case of monopartite supermaps, we will present here a third, equivalent representation of bipartite supermaps, based on the DS representation of channels. Although we restrict ourselves to two parties for simplicity, this definition readily generalises to supermaps acting on any number of channels. Those two channels will be represented by the letters A and B, while the output channel will be written as a channel from an input P (which can be thought of as a global past) to an output F (global future). Here again, we will define supermaps to be suitable maps on channels featuring ancillary systems.

Definition 5 *A **bipartite supermap** of type* $((A_I \to A_O) \otimes (B_I \to B_O)) \to (P \to F)$ *is a superoperator*

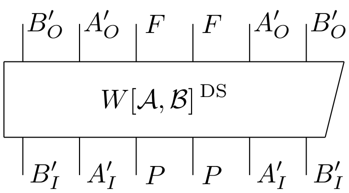
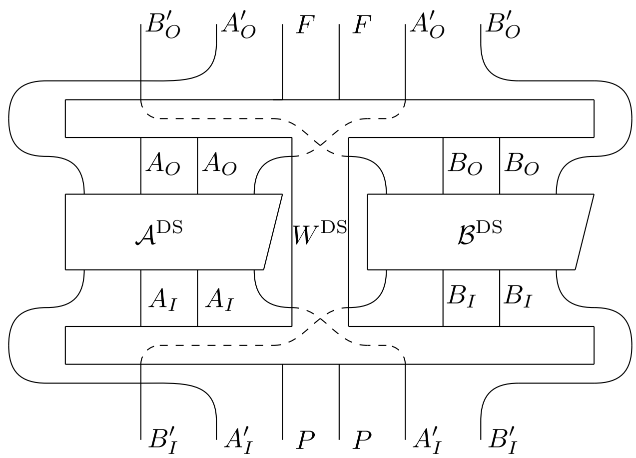
[8]Strictly speaking, [1] considered the seemingly more general case of supermaps defined on non-signalling bipartite channels; but, as proven by Theorem 2 in [1], the two kinds of supermaps are actually equivalent.

[9]This use of the word "process" as a synonym to "second-order transformation" should not be confused with its use in the context of process theories, as introduced in Section 2. To avoid this confusion, we adopt the "supermap" terminology in this report.

 $\in \mathrm{Lin}\left(\mathrm{Lin}\left(\left(\mathcal{H}_{A_I}^{\otimes 2} \otimes \mathcal{H}_{B_I}^{\otimes 2}\right) \to \left(\mathcal{H}_{A_O}^{\otimes 2} \otimes \mathcal{H}_{B_O}^{\otimes 2}\right)\right) \to \mathrm{Lin}\left(\mathcal{H}_P^{\otimes 2} \to \mathcal{H}_F^{\otimes 2}\right)\right)$

(28)

such that, for any auxiliary systems A'_I, A'_O, B'_I, B'_O and for any $\otimes$-positive and causal operators $\mathcal{A}^{\mathrm{DS}} \in \mathrm{Lin}\left(\left(\mathcal{H}_{A_I} \otimes \mathcal{H}_{A'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{A_O} \otimes \mathcal{H}_{A'_O}\right)^{\otimes 2}\right)$, $\mathcal{B}^{\mathrm{DS}} \in \mathrm{Lin}\left(\left(\mathcal{H}_{B_I} \otimes \mathcal{H}_{B'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{B_O} \otimes \mathcal{H}_{B'_O}\right)^{\otimes 2}\right)$,

 $:=$  (29)

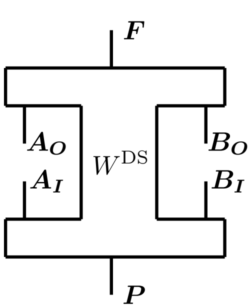
is a $\otimes$-positive and causal operator in $\mathrm{Lin}\left(\left(\mathcal{H}_P \otimes \mathcal{H}_{A'_I} \otimes \mathcal{H}_{B'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_F \otimes \mathcal{H}_{A'_O} \otimes \mathcal{H}_{B'_O}\right)^{\otimes 2}\right)$.

A first important result on bipartite supermaps, given by Theorem 1 in [1] and equation (4) in [2], is that (in the DS picture) they are $\otimes$-positive:

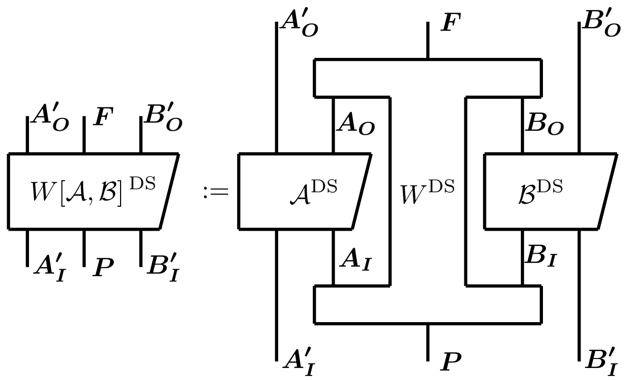
Theorem 2 *If W^{DS} is a bipartite supermap, then it is $\otimes$-positive.*

This means that we can re-express everything in the doubled process theory, which will make our diagrams way less cumbersome.

Definition 6 *[Equivalent to Definition 5] A **bipartite supermap** of type $((A_I \to A_O) \otimes (B_I \to B_O)) \to (P \to F)$ is a $\otimes$-positive superoperator*

 $\in \mathrm{Lin}\left(\mathrm{Lin}\left(\left(\mathcal{H}_{A_I}^{\otimes 2} \otimes \mathcal{H}_{B_I}^{\otimes 2}\right) \to \left(\mathcal{H}_{A_O}^{\otimes 2} \otimes \mathcal{H}_{B_O}^{\otimes 2}\right)\right) \to \mathrm{Lin}\left(\mathcal{H}_P^{\otimes 2} \to \mathcal{H}_F^{\otimes 2}\right)\right)$ (30)

such that, for any auxiliary systems A'_I, A'_O, B'_I, B'_O and for any $\otimes$-positive and causal operators $\mathcal{A}^{\mathrm{DS}} \in \mathrm{Lin}\left(\left(\mathcal{H}_{A'_I} \otimes \mathcal{H}_{A_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{A'_O} \otimes \mathcal{H}_{A_O}\right)^{\otimes 2}\right)$, $\mathcal{B}^{\mathrm{DS}} \in \mathrm{Lin}\left(\left(\mathcal{H}_{B_I} \otimes \mathcal{H}_{B'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{B_O} \otimes \mathcal{H}_{B'_O}\right)^{\otimes 2}\right)$,

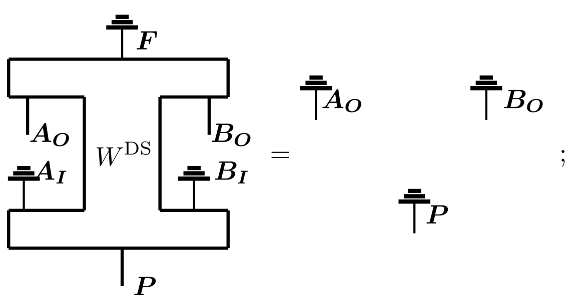


(31)

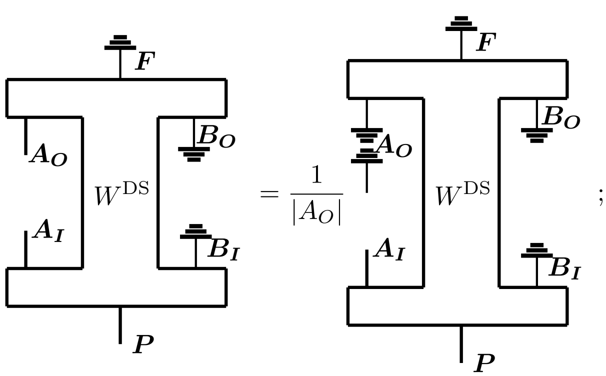
is a causal operator in $\mathrm{Lin}\left(\left(\mathcal{H}_{A'_I} \otimes \mathcal{H}_P \otimes \mathcal{H}_{B'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{A'_O} \otimes \mathcal{H}_F \otimes \mathcal{H}_{B'_O}\right)^{\otimes 2}\right)$.

The characterisation of bipartite supermaps given in Definition 6 is just a condition of good behaviour; it would be interesting to turn it into more practical conditions, which would pave the way towards a (still lacking) general form similar to (27). The Araùjo conditions, proven in [25] (equations 8 to 11), are a first step in this direction.

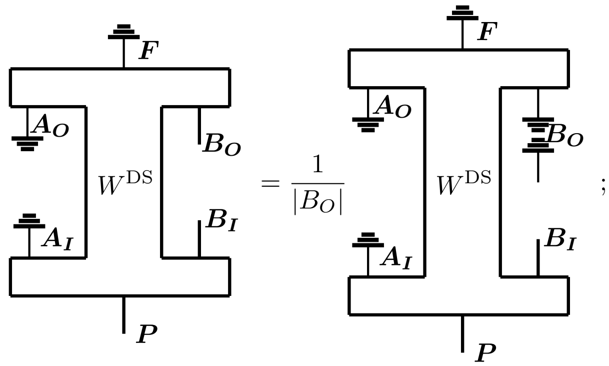
Theorem 3 (Araújo conditions) *Let W^{DS} be a $\otimes$-positive superoperator, as written in (30). Then W^{DS} is a bipartite supermap if and only if the four following conditions are satisfied:*



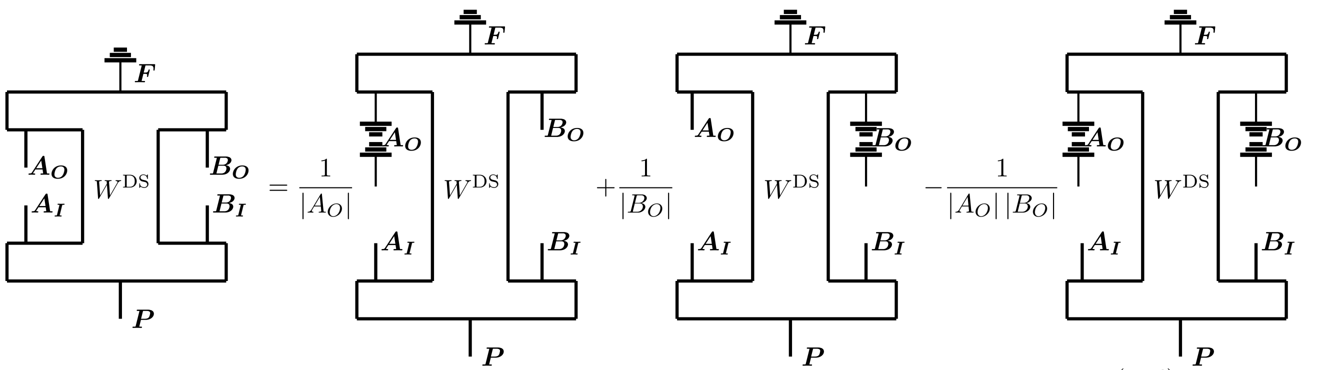
(32a)



(32b)



(32c)

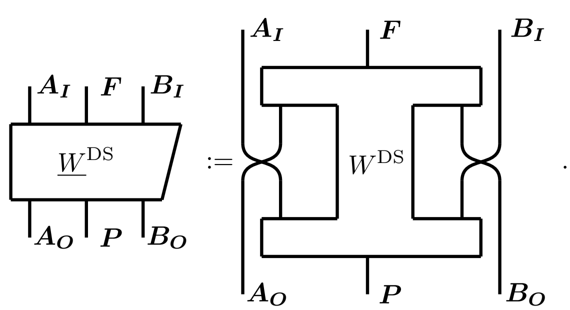


(32d)

The Araújo conditions can be interpreted in terms of the inner causal structure of W. (32a) is just global causality. (32b) and (32c) mean that W does not feature local loops, that is, a direct influence of A_O on A_I or of B_O on B_I. (32d) corresponds to the absence of global loops, that is, an influence of A_O on A_I going through the channel B, or reciprocally. We see that the condition for a supermap to map causal channels to other causal channels is, at least in the bipartite case, directly related to a simple physical requirement: that a given party's output can never have a causal influence on this party's input, i.e. that there is no backwards in time signalling.

It will also be useful to define the quantum channel associated to a supermap.

Definition 7 (Associated quantum channel) *If W^{DS} is a bipartite supermap, its* ***associated quantum channel*** $\underline{W}^{\mathrm{DS}}$ *is*



(33)

From Theorem 2 and (32a), we know that the associated quantum channel to any bipartite supermap is indeed a quantum channel[10]. This allows for the following definitions:

[10] Equivalently - and this is true for any number of parties - this can be seen as coming from the fact that the associated

Definition 8 (Isometric and unitary supermaps) *A bipartite supermap W^{DS} is called **unitary** (resp. **isometric**) if the quantum channel $\underline{W}^{\mathrm{DS}}$ is unitary (resp. an isometry).*

The following result, proven in [24] (Theorem 2), relates the unitarity of a supermap to the fact that it maps unitary channels to unitary channels.

Theorem 4 *If W^{DS} is a bipartite supermap, then W^{DS} is a unitary supermap if and only if, for any auxiliary systems A'_I, A'_O, B'_I, B'_O and for any unitary channels $\mathcal{A}^{\mathrm{DS}} \in \mathrm{Lin}\left(\left(\mathcal{H}_{A_I} \otimes \mathcal{H}_{A'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{A_O} \otimes \mathcal{H}_{A'_O}\right)^{\otimes 2}\right)$, $\mathcal{B}^{\mathrm{DS}} \in \mathrm{Lin}\left(\left(\mathcal{H}_{B_I} \otimes \mathcal{H}_{B'_I}\right)^{\otimes 2} \to \left(\mathcal{H}_{B_O} \otimes \mathcal{H}_{B'_O}\right)^{\otimes 2}\right)$, $W[\mathcal{A}, \mathcal{B}]$ is a unitary channel.*

Moreover, if a supermap W^{DS} is unitary, then one can talk about the single version of W^{DS}, as W^{DS} is then just the parallel composition of a single superoperator with its conjugate. In this case, in a slight abuse of notation, we will write W to denote the single superoperator, and $\hat{W}^{\mathrm{DS}}$ to denote the quantum supermap obtained by doubling W.

4.3 Two paradigmatic examples of supermaps

As we saw, in the case of monopartite supermaps, Theorem 1 is a proof that all supermaps are essentially trivial, in the sense that they do not defy any form of common sense as to what such supermaps may look like. More specifically:

- They can be embedded in the quantum circuit framework;
- They do not challenge any notion of a partial order in the Universe, existing independently to local agents' operations.

These two features are precisely those which are questioned by some cases of multipartite supermaps. In particular, [1] and [2] are each centred around the introduction of an example of a bipartite supermap which challenges those expectations. These two examples, called the quantum SWITCH and the OCB process, grew to be paradigmatic in the study of supermaps; presenting them will be an occasion to introduce some of the non-trivial features of such supermaps, as well as the challenges which still limit our understanding of those.

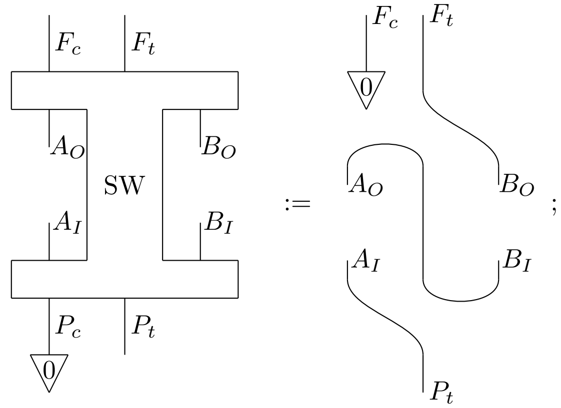
4.3.1 The quantum SWITCH

The quantum SWITCH is an exemple of a bipartite supermap introduced in [1], which essentially involves mapping input channels $\mathcal{A}$ and $\mathcal{B}$ to either $\mathcal{A} \circ \mathcal{B}$ or $\mathcal{B} \circ \mathcal{A}$, with the choice between these alternatives being coherently controlled by the state of an auxiliary qubit. The quantum SWITCH can be rigorously defined in the following way:

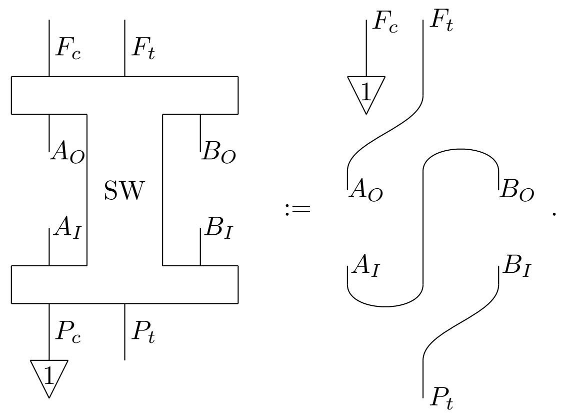
Definition 9 (Quantum SWITCH) *For a given $p \in \mathbb{N}$, the **quantum SWITCH** in dimension p is a unitary supermap $\hat{\mathrm{SW}}^{\mathrm{DS}}$ of type $((A_I \to A_O) \otimes (B_I \to B_O)) \to ((P_c \otimes P_t) \to (F_c \otimes F_t))$, where*

channel is just the image of swap channels by the supermap.

$|P_c| = |F_c| = 2$ *and* $|P_t| = |F_t| = |A_I| = |A_O| = |B_I| = |B_O| = p$*, whose associated single superoperator* SW *is defined on a basis* $\{|0\rangle, |1\rangle\}$ *of* $\mathcal{H}_{P_c}$ *by:*

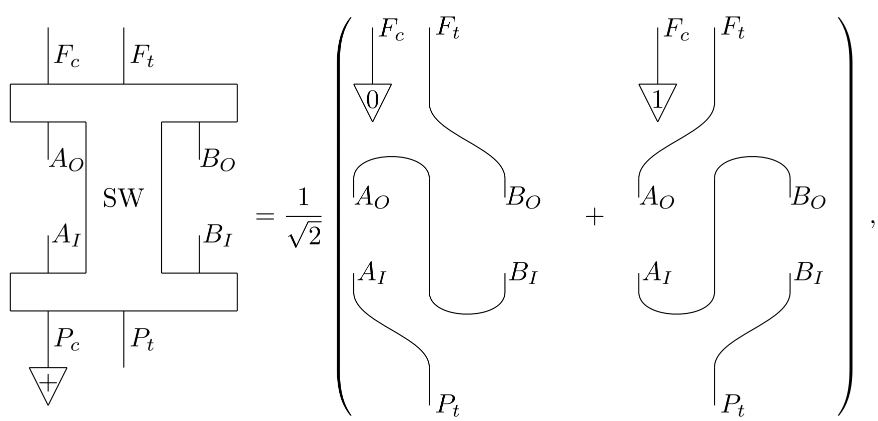


(34a)



(34b)

It is easy to see that this is indeed a unitary supermap. As we can see, the quantum SWITCH maps channels $\mathcal{A}$ and $\mathcal{B}$ to a channel in which the state of the control qubit P_c decides the order in which the target system P_t will undergo channels $\mathcal{A}$ and $\mathcal{B}$. Crucially, this control is coherent, which means that for a control qubit prepared in the state $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the supermap becomes



(35)

which can be interpreted as a situation in which the state of the control qubit is entangled with the causal order between channels $\mathcal{A}$ and $\mathcal{B}$. As discussed in [1], such a situation cannot be embedded in the quantum circuit model [3]: if the input operations $\mathcal{A}$ and $\mathcal{B}$ are considered to be physical black boxes, then $\hat{SW}[\mathcal{A}, \mathcal{B}]$ cannot be simulated by a quantum circuit using only one query to $\mathcal{A}$ and $\mathcal{B}$. In a broader sense, [10] shows that the quantum SWITCH is not causally separable, that is, it cannot be decomposed as a mixture of supermaps featuring a definite causal order. The quantum SWITCH has thus been considered as one of the prime examples of *indefinite causal order*.

As such, it has been a subject of thorough investigation to understand whether the use of the quantum SWITCH provides computational and communicational advantages over the class of scenarios displaying definite causal order, in a similar way to the fact that use of quantum resources and channels provides computational and communicational advantages over the class of classical scenarios. Many such advantages of the use of the quantum SWITCH have been found:

- it allows for perfect discrimination of some channels with a single query to these channels [4];
- a generalised, n-partite version of it allows to design an algorithm which solves a specific problem with only $\mathcal{O}(n)$ queries to a black box, whereas the best known quantum algorithm with definite causal order for this problem requires $\mathcal{O}(n^2)$ queries [5];
- In the context of quantum Shannon theory, the quantum SWITCH can be used to turn two copies of a completely depolarising channel into a classical channel with non-zero classical capacity [6], to enhance the quantum capacity of a quantum channel [7], and even to turn two instances of a channel with zero quantum capacity into a a channel with perfect quantum capacity [8]. The attribution of these advantages to indefinite causal order as opposed to usual, causally-ordered coherent control of quantum channels, however, has been challenged [26, 27].

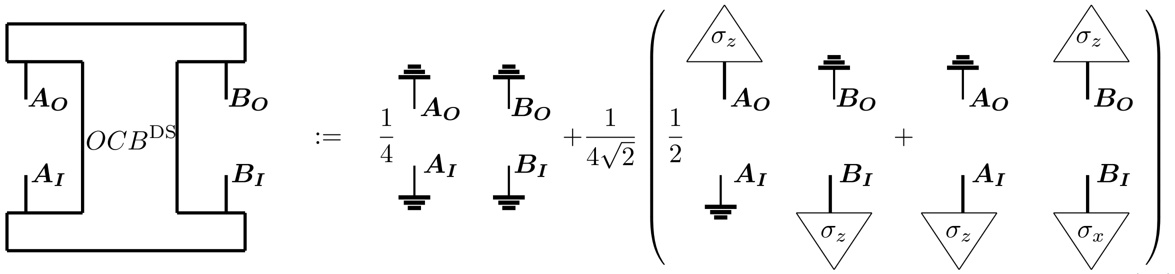
In parallel, experimental realisations of the quantum SWITCH in photonic systems have been developed and shown to satisfy some definitions of indefinite causal order, with increasing degrees of precision and sophistication [11–14].

4.3.2 The OCB process

The other paradigmatic example of a non-trivial bipartite supermap is the Oreshkov-Costa-Brukner (OCB) process, introduced in [2], defined in the following way.

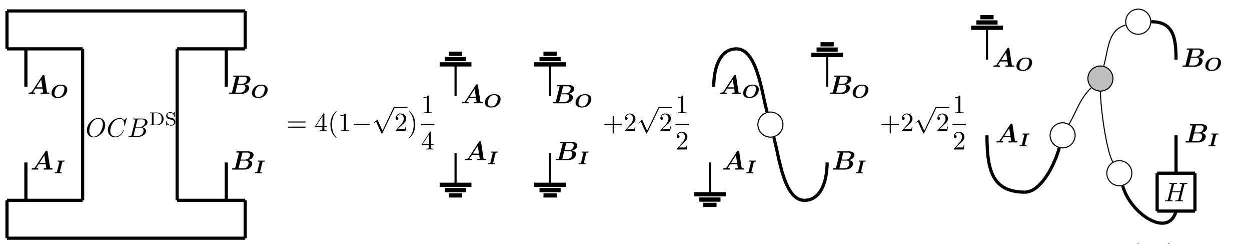
Definition 10 (OCB process) *The* ***OCB process*** *is a supermap* $\mathrm{OCB}^{\mathrm{DS}}$ *of type* $((A_I \to A_O) \otimes (B_I \to B_O)) \to \mathbb{C}$*, where* $|A_I| = |A_O| = |B_I| = |B_O| = 2$*, defined by*[11]

[11] Note that the following sum does not in any way correspond to a mixture of supermaps, as the two last terms are not proportional to supermaps.



(36)

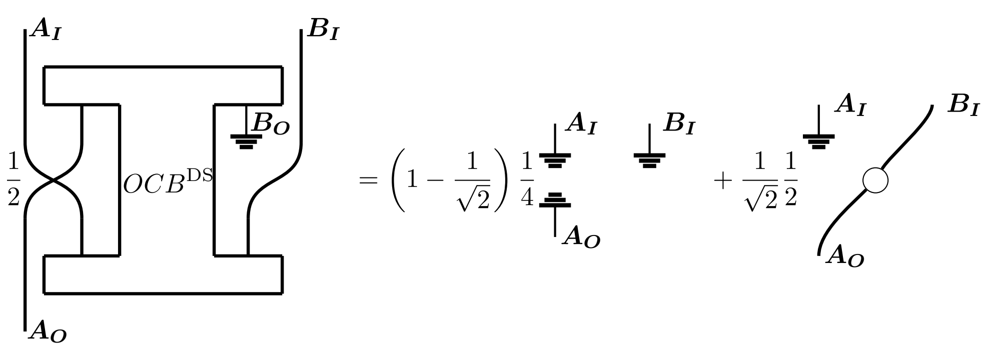
During the course of this project, another, more explicit form of the OCB process has been discovered, which makes use of the ZX-calculus formalism described in [16]:



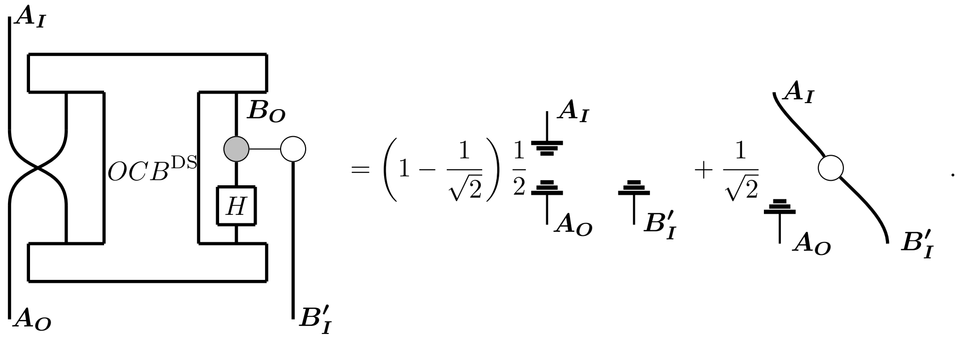
(37)

The form (37) and the rules of ZX-calculus make it easy to check that the OCB process satisfies the Araújo conditions (32). One can also check with Matlab that it is ⊗-positive; it is thus indeed a supermap.

ZX calculus also helps to easily compute from (37) the equations which illustrate the OCB process' queer properties. Let us in particular consider how the OCB process maps two specific sets of channels:



(38a)



(38b)

Let us comment on the physical significance on these two equations, by adopting the interpretation according to which the input maps $\mathcal{A}$ and $\mathcal{B}$ can be related to free choices of channel implementation by local agents, Alice and Bob. In both scenarios, Alice implements the same channel, a swap; but Bob chooses either to carry on his input and feed noise to his output, or to perform a Hadamard gate followed by a CNOT gate controlled by another qubit system he possesses. In the first case, the result is a noisy decoherent channel from Alice to Bob, with classical capacity $\frac{1}{\sqrt{2}}$; in the second case, an equivalent channel is obtained, with the difference that it is directed from Bob to Alice.

Suppose now that the causal structure between Alice and Bob is fixed, irrespectively of any choices of channel implementation on their part, in a mixture of causally definite relations between them. Then, Alice lies in the causal past of Bob with probability p, Alice and Bob are spacelike separated with probability p', and Bob lies in the causal past of Alice with probability $1-p-p'$. The sum of the probability for a channel from Alice to Bob to be implementable with the probability for a channel from Bob to Alice to be implementable is therefore

$$P(A \preceq B) + P(B \preceq A) = 1 - p' \leq 1\,; \tag{39}$$

yet, as we saw, the OCB process yields

$$P(A \preceq B) + P(B \preceq A) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 1\,. \tag{40}$$

In that sense, the OCB process is incompatible with the existence of a fixed mixture of definite causal structures existing between Alice and Bob independently of their choices of channel implementations. This was the core finding of [2], which made it even more striking by showing that the existence of such a causal structure as the OCB process would allow some agents to perform better at a given game than they would ever be able to if the Universe only featured (mixtures of) definite causal order.

The discrepancy between (39) and (40), called a violation of a causal inequality, shares striking features with the violation of realist locality by quantum theory, in both its concepts and its quantities. It has been argued that, in the same way that violation of locality is an exclusive feature of quantum effects, violation of causality could be an exclusive feature of quantum gravitational effects and that an experimental finding of such a violation could be considered as an unmistakable signature of quantum gravity.

Following the terminology and definitions of [10], the OCB process is *non-causal*. It is interesting to note that, in this same terminology, the quantum SWITCH is causal, but not causally separable. In that sense, the OCB process features a stronger form of non-classicality than the quantum SWITCH, and the latter can be thought of as an "intermediate step" on the road to indefinite causal order; for instance, it has been proven that it cannot be used to violate any causal inequalities [25]. This intermediate status essentially stems from the fact that the quantum SWITCH becomes causally separable once its control qubit is discarded.

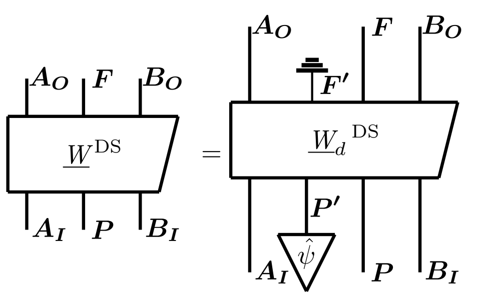
The status of the OCB process as featuring a "strong" violation of causality makes it a matter of enquiry whether specific physical models (e.g. quantum gravitational ones) could be devised in which such a process would arise. No such models have yet been proposed, either for the OCB process or for any other non-causal supermaps (in the sense of [10]).

4.4 The purification postulate

Clarifying this uncertain physical status of non-causal supermaps (exemplified by the OCB process) has been the motivation of a significant portion of subsequent work on indefinite causal order. Some considered paths towards a clarified view on the matter have been to study the dynamics of supermaps [28], their compositional rules [29, 30], or their mathematical structure [10, 25].

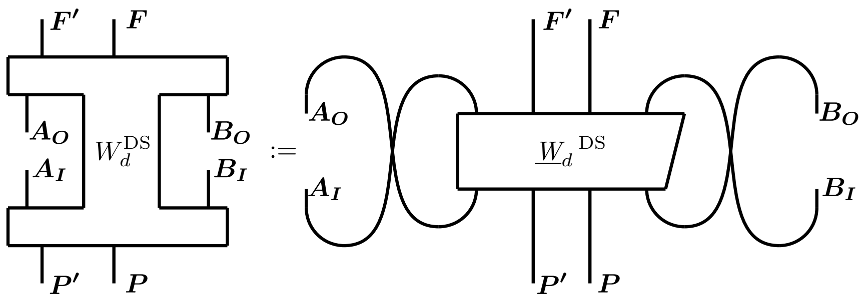
We will follow more thoroughly another path, introduced in [24]: considering the purifiability of supermaps. We will introduce it here in a slightly different, yet equivalent way. Let us first define the purification of a supermap from its associated quantum channel.

Definition 11 (Purification of a supermap) *Let W^{DS} be a bipartite supermap of type $((A_I \to A_O) \otimes (B_I \to B_O)) \to (P \to F)$. Its associated quantum channel $\underline{W}^{\mathrm{DS}}$ then admits a minimal Stinespring dilation, that is, auxiliary systems P', F', a unitary quantum channel $\underline{W_d}^{\mathrm{DS}} \in \mathrm{Lin}\left((\mathcal{H}_{A_I} \otimes \mathcal{H}_P \otimes \mathcal{H}_{P'} \otimes \mathcal{H}_{B_I})^{\otimes 2} \to (\mathcal{H}_{A_O} \otimes \mathcal{H}_F \otimes \mathcal{H}_{F'} \otimes \mathcal{H}_{B_O})^{\otimes 2}\right)$ and a pure quantum state $\hat{\psi} \in \mathcal{H}_{P'}^{\otimes 2}$ such that*



(41)

*The **purification** of W is then the $\otimes$-positive superoperator W_d^{DS} defined by*



. (42)

The crucial point here is that, even though the purification of a supermap is a $\otimes$-positive superoperator, it is not necessarily a supermap, i.e. it does not necessarily satisfy the Araújo conditions (32). This motivates the following definition.

Definition 12 (Purifiable supermaps) *A supermap W is* ***purifiable*** *if its purification W_d is itself a supermap. In this case, W_d is a unitary supermap.*

Not all supermaps are purifiable, and the OCB process again proves to be an example of this, with the following theorem, proven in [24].

Theorem 5 *The OCB process is not a purifiable supermap.*

This is an interesting fact in itself. In the case of quantum channels, asking that they map quantum states (possibly with ancillas) to quantum states is a sufficient requirement for them to be purifiable - as proven by the existence of a Stinespring dilation for any quantum channel. We find here that the same does not apply to supermaps[12].

[24] argues that the physical existence of non-purifiable supermaps would be highly surprising. If they were to exist, Theorem 4 would entail that, even under taking any dilation of the systems considered, unitary channels can be mapped to a non-unitary channel. In particular, this induces a form of intrinsic non-reversibilty in physics, which is in stark contrast with the reversible status of all currently admitted fundamental physical theories. Moreover, reversibility has been a central axiom in all recent informational reconstructions of quantum theory.

These considerations lead to the central postulate proposed in [24] to constrain physical supermaps.

Postulate 1 (Purification Postulate) *A supermap is physical if and only if it is purifiable.*

Note that any causally separable supermap is purifiable by Stinespring dilation. In addition, the quantum SWITCH, being a unitary supermap, is trivially purifiable. It turns out that no examples have yet been found of purifiable bipartite supermaps which are non-causal (in the sense of [10]), which may indicate that, in the bipartite case, the purification postulate rules out any violation of causal inequalities. Yet there exists a non-causal tripartite supermap, the Baumeler-Wolf process [31], which is purifiable [24].

[12] An interesting possibility would be that a generalised Stinespring dilation theorem still holds for supermaps, but that the purification of a bipartite supermap is to be defined as an n-partite unitary supermap which would reduce to the bipartite supermap when it takes suitable unitary channels as all but two of its inputs.

5 Towards a complete classification of bipartite unitary supermaps

5.1 Motivation and main conjecture

As can be seen in Section 4, most research on bipartite supermaps has been conducted by looking at specific examples. This essentially comes from the fact that supermaps are defined by the fact that they map channels to channels: this is a condition that tells us nothing about the general form of supermaps which satisfy it. This makes the determination of the general properties of such channels hard and dependent on the exhibition of specific counter-examples[13]. In contrast, finding the general form of bipartite supermaps, in the same way that the general form of monopartite supermaps has been found to be that of Theorem 1, would represent a substantial step forward, as:

- it would allow to prove general theorems about supermaps;
- it would allow to determine in all generality what the things are which can and cannot be achieved with supermaps;
- it would provide a basis on which to define suitable classes of supermaps;
- it would help building a physical interpretation of supermaps.

The objective of this project is to move forward towards such a general form. Its results and state of advancement on this path shall be described in this section.

We will here restrict ourselves to finding the general form of *unitary* bipartite supermaps. This is motivated by two reasons:

- if one is to subscribe to the Purification Postulate, then it is sufficient to work on unitary supermaps, as the form of any physical supermap can then be directly deduced from the form of its purification, which is a unitary supermap;
- as we shall see, some of the crucial tools which we will use, such as quantum causal models or the determination of relevant factor algebras, are only defined within unitary theories. In contrast, it is considerably harder to draw results without being able to lean on the powerful structural features of unitarity.

We will also restrict ourselves to supermaps where $|A_I| = |A_O| = |B_I| = |B_O|$ and $|P| = |F|$. The form of more general supermaps should be easily deducible from them.

[13] The Araújo conditions (32) are already a progress, because they can be interpreted as conditions on the quantum channel associated to the supermap rather than on the supermap itself, and because they come with a simple circuital interpretation (the absence of loops, i.e. no backwards in time signalling) which lends itself well to intuition. Yet the form of the last and most important one, (32d), remains obscure and makes it hard to work with. This is directly related to the fact that, whereas local loops are easy to spot, ruling out global loops proves to be trickier, as they could only arise for a specific choice in one of the input channels. For example, to rule out a global loop from Alice to herself, one has to check that it does not arise whatever the channel is that Bob implements.

Moreover, our goal, which we have not reached yet, shall be to prove a specific conjecture about this general form of bipartite unitary supermaps. Loosely speaking, this conjecture can be phrased as follows.

Conjecture 1 (Main Conjecture) *If W is a bipartite unitary supermap, then W is essentially either trivial, or the quantum SWITCH.*

Here and in what follows, by "trivial" we shall mean "causally ordered"[14]. A fully rigorous statement of this conjecture will be presented later on.

This conjecture is motivated mainly by heuristic reasons, among which the impossibility to provide any counter-example to it. We believe that even if this conjecture were to actually be false, this line of research would still be of interest in order to come up with the specific counter-examples to it, and to investigate the general structure of supermaps.

If this conjecture were to be proven true, it would directly follow that there are no non-causal purifiable bipartite supermaps. It would also follow that there are essentially no other non-trivial examples of unitary bipartite supermaps than the quantum SWITCH, and that the study of computational and communicational advantages provided by such supermaps can be reduced to the study of the quantum SWITCH.

During the course of this project, we learned (without more details) that Ognyan Oreshkov apparently found a proof of this result [32], even though it has not appeared yet in pre-publication.

5.2 A first step: inner wirings of a supermap

Our main objective should be to "open the blackbox" that a supermap constitutes, and reveal its inner wirings, with the help of our diagrammatic formalism. Our first step will do precisely this, and allow us to prove that the problem of the form of suitable supermaps is equivalent to an arguably more graspable problem on quantum channels.

An advantage of working with unitary supermaps will be that they are composed of a single superoperator parallely composed with its conjugate, that is, any bipartite unitary supermap can be written as

$$\hat{W}^{\mathrm{DS}} = (W)^* \otimes W \,. \tag{43}$$

where W is a single superoperator, that is, a map on single operators. It will thus be sufficient to work in the single picture; we will limit our use of the double picture to the cases where we have to introduce discarding.

Let us state the result of this first step[15].

[14] Note that in the case of unitary supermaps, the class of causally separable supermaps is irrelevant as there is no concept of mixture present here.

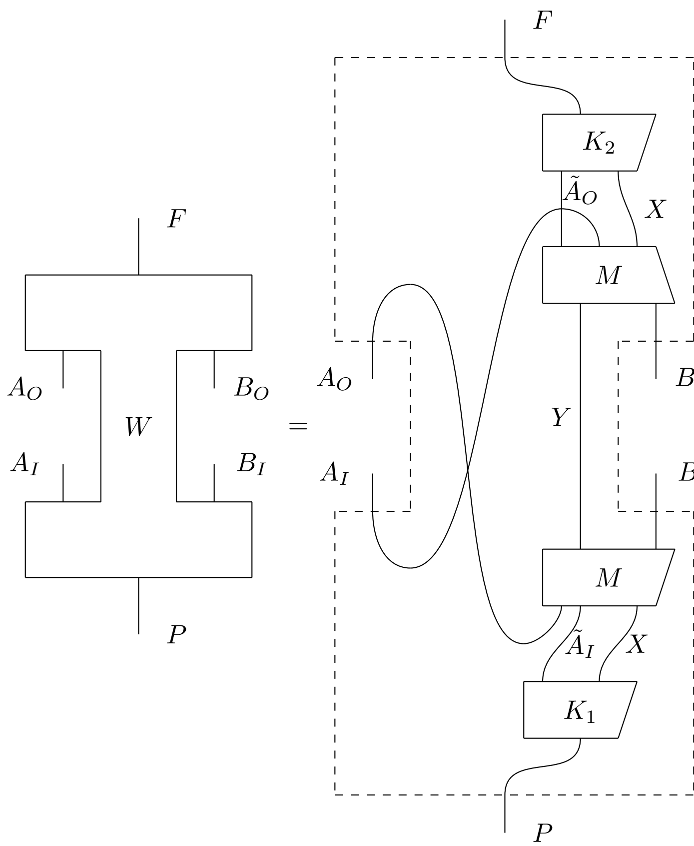
[15] Note that in the following, there will be slight abuses of notation on the labelling of income and outcome types of some operators. This abuse is rendered harmless by the fact that the substituted types always have the same dimension, and hence are isomorphic. Explicitly spelling out the corresponding isomorphisms in the diagrams would create unnecessary clutter.

Theorem 6 *Let $\hat{W}^{\mathrm{DS}}$ be a bipartite superoperator of type $((A_I \to A_O) \otimes (B_I \to B_O)) \to (P \to F))$, with $|A_I| = |A_O| = |B_I| = |B_O| = m$ and $|P| = |F| = n$.*

$\hat{W}^{\mathrm{DS}}$ is a unitary supermap if and only if there exist

- *types X, $\tilde{A}_I$, $\tilde{A}_O$ and Y, with $\left|\tilde{A}_I\right| = \left|\tilde{A}_O\right| = m$, $|X| = n - m$ and $|Y| = n$,*
- *unitaries $K_1 \in \mathrm{Unit}\left(\mathcal{H}_P \to \left(\mathcal{H}_{\tilde{A}_I} \otimes \mathcal{H}_X\right)\right)$, $K_2 \in \mathrm{Unit}\left(\left(\mathcal{H}_{\tilde{A}_O} \otimes \mathcal{H}_X\right) \to \mathcal{H}_F\right)$, $M \in \mathrm{Unit}(\left(\mathcal{H}_{A_O} \otimes \mathcal{H}_{\tilde{A}_I} \otimes \mathcal{H}_X\right) \to (\mathcal{H}_Y \otimes \mathcal{H}_{B_I}),$*

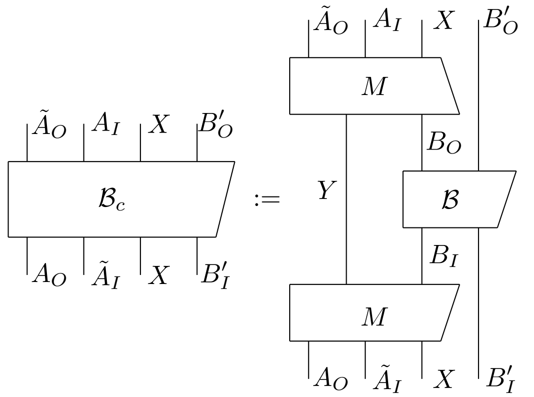
such that W, the single version of $\hat{W}$, can be written as



$$, \tag{44}$$

*and such that M satisfies the **no-bridge condition** presented next.*

Definition 13 (no-bridge condition) *Let M be a unitary operator as described in Theorem 6. By conjugation, for any auxiliary types B_I', B_O', it maps any unitary operator $\mathcal{B} \in \mathrm{Unit}\left(\left(\mathcal{H}_{B_I} \otimes \mathcal{H}_{B_I'}\right) \to \left(\mathcal{H}_{B_O} \otimes \mathcal{H}_{B_O'}\right)\right)$ to a unitary operator $\mathcal{B}_c \in \mathrm{Unit}(\left(\mathcal{H}_{A_O} \otimes \mathcal{H}_{\tilde{A}_I} \otimes \mathcal{H}_X \otimes \mathcal{H}_{B_I'}\right) \to \left(\mathcal{H}_{\tilde{A}_O} \otimes \mathcal{H}_{A_I} \otimes \mathcal{H}_X \otimes \mathcal{H}_{B_O'}\right))$ given by*



(45)

The ***no-bridge condition*** *is then stated as:*

$$\forall \mathcal{B}, \mathcal{B}_c \textit{ is compatible with the no-influence relation } A_O \nrightarrow A_I\,. \tag{46}$$

We present a proof of this theorem in Appendix A.

Let us comment on the general form (44) that this theorem provides for any supermap; although seemingly obscure, it is in fact of considerable interest in its ability to pinpoint the crucial structure in a supermap. In particular, we will elaborate on the specific meanings and significances of the unitaries K_1, K_2 and M.

Note first that if the second input channel $\mathcal{B}$ is simply chosen to be the identity channel (i.e. "if Bob does nothing"), then M and $M^\dagger$ cancel each other out, and we are left with a supermap on the first input channel only, with precisely the canonical form (27), where K_1 and K_2 respectively are the pre- and post- processings. In particular, K_1 then defines the factorisation of the past type P into two factor types, $\tilde{A}_I$ and X, with $\tilde{A}_I$ being the factor affected by Alice's operations (which maps it to $\tilde{A}_O$), and X being the factor on which Alice's operations have no influence; K_2 then maps $\tilde{A}_O$ and X back together to the future F. To put it more simply, K_1 and K_2 define which parts of P and of F Alice is acting on. Thus, K_1 and K_2 are to be understood as describing *the localisation of Alice's operation* in the absence of any operation on Bob's part.

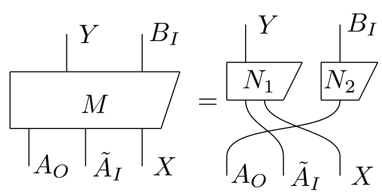
The conjugation $M^\dagger \cdot M$ provided by the unitary M will then describe, on top of that, *the causal localisation of Bob with respect to Alice.* It provides the mapping from the tensor product of types A_I (Alice's input), A_O, (Alice's output), and X (what is not affected at all by Alice), to the tensor product of types B_I, on which Bob will act, and Y, on which Bob has no influence. In other words, M tells us where Bob's operations are located among Alice's past, future, and elsewhere. It is therefore the central element in W, the one which will determine its causal structure.

With this interpretation in mind, the no-bridge condition that we obtain on M gets a precise meaning: *the causal localisation of Bob's actions must be such that, whatever channel he implements, he cannot wire Alice's future into her past.* In other words, Bob cannot build a bridge back in time for Alice. As we see, this is now the only constraint left for us to work on. Moreover, solving this condition (i.e. finding all the unitaries M which satisfy it) is now a problem on the causal structure of unitaries, rather than on that of superoperators.

Let us also comment on the relation between the problem of finding solutions to the no-bridge condition, and the study of no-influence relations in the literature, as described in Section 3. The currently known theorems about quantum causal structure are results about the form of a *given* operator satisfying some no-influence relations. In contrast, solving the no-bridge condition is about finding the form of a factor Von Neumann subalgebra[16] whose unitary elements all satisfy some no-influence relations.

More generally, we want to emphasise that, irrespective of the technicalities involved, the general philosophy of the research program presented here can be defined in a straightforward way: we want to locate the central Von Neumann subalgebra corresponding to Bob's operations. This is precisely what the form (44), and in particular the definition of the unitary M, achieve. This program was strongly inspired by the conceptual clarifications provided by two recent papers [33, 34]. We think that Theorem 6 is a strong testimony to the power of this research program in the search for a classification of bipartite unitary supermaps, as it can be understood as proving that the causal structure of a bipartite unitary supermap can be characterised by the localisation of its "second-input subalgebra", provided by M. Moreover, the question left for us to solve - finding the solutions to the no-bridge condition - can be broadly phrased as: "given a central Von Neumann subalgebra (Alice's operations), what are the central Von Neumann subalgebras which do not mess up its input/output structure?".

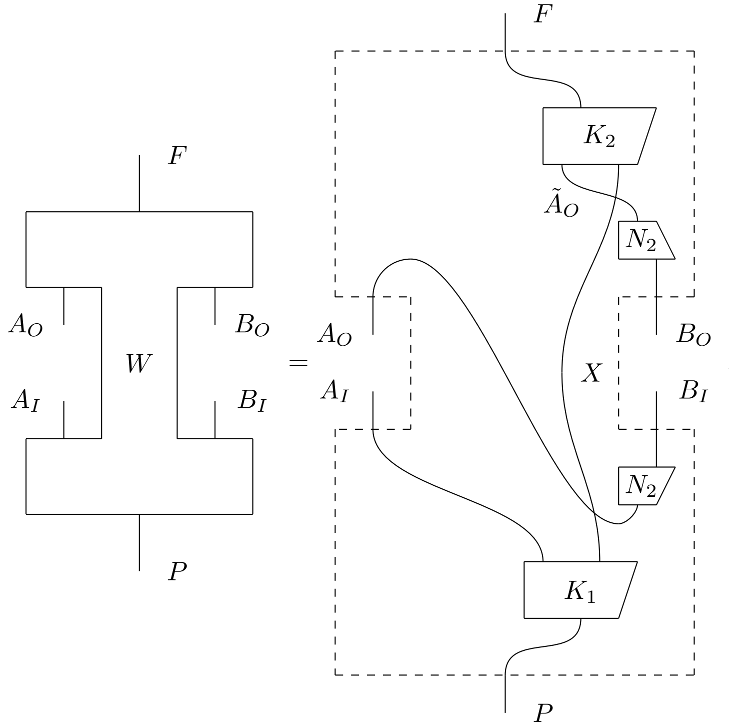
Let us look at some examples of unitaries M satisfying the no-bridge condition and at the causal structure they entail, in order to improve our understanding of the physical significance of M. A first possibility is that it wires A_O into B_I,



, (47)

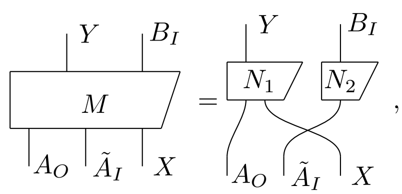
where N_1 and N_2 are unitaries. It is easy to see that such an M satisfies the no-bridge condition. Moreover, one can then work out the form (44) of W and find that it reduces to

[16] i.e., a subalgebra on the algebra of operators which is stable under adjoints, and whose center is trivial.



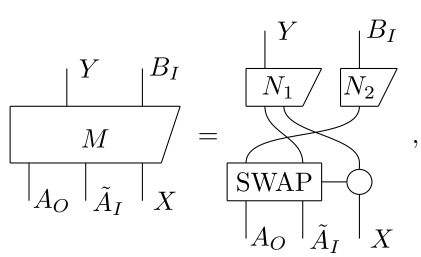
(48)

which allows us to conclude that M being of the form (47) corresponds to Bob being located in Alice's future. In a symmetric way, it is easy to see that the form



(49)

which also satisfies the no-bridge condition, corresponds to Bob being located in Alice's past. Finally, one can get a combination of the two above scenarios controlled by an auxiliary state, as in[17]



(50)

which also satisfies the no-bridge condition, and can be seen to yield precisely the quantum SWITCH.

Our objective is now to prove that those three possibilities are essentially the only solutions to the no-bridge condition; this would immediately prove our main conjecture.

[17]In this example we take $|X| = 2$.

Before we consider this, we will first prove an important theorem about the no-bridge condition: it also forbids Bob's operations to create a communication from $\tilde{A}_I$ to $\tilde{A}_O$. This can be interpreted as the fact that Bob's operations cannot be used to bypass Alice's operations.

Theorem 7 *A unitary M providing a mapping $\mathcal{B} \to \mathcal{B}_c$, as defined earlier, satisfies the no-bridge condition if and only if*

$$\forall \mathcal{B}, \mathcal{B}_c \textit{ is compatible with } \tilde{A}_I \nrightarrow \tilde{A}_O \,. \tag{51}$$

The proof is presented in Appendix B; it essentially comes from the fact that the subalgebra of Bob's operations is closed under adjoints.

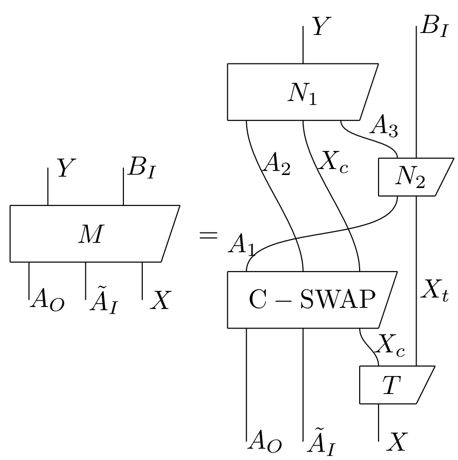
5.3 A proof of our conjecture in a special case

Our conjecture is that a unitary bipartite supermap can either have a definite causal structure, or the causal structure of the quantum SWITCH. Finding a completely general statement of it is challenging, especially given the fact that the second-input subalgebra could be acting on elements of X as well. A tentative formulation would be the following.

Conjecture 2 (Solutions to the no-bridge condition) *Taking the previous definitions, M satisfies the no-bridge condition if and only if there exist*

- *types X_c, X_t, A_1, A_2, A_3, with $|A_1| = |A_2| = m$, $|X_c| + |X_t| = n - m$, $|A_3| = |X_t|$;*
- *unitary operators $T \in \mathrm{Unit}\left(\mathcal{H}_X \to (\mathcal{H}_{X_c} \otimes \mathcal{H}_{X_t})\right)$, $\mathrm{C-SWAP} \in \mathrm{Unit}((\mathcal{H}_{A_O} \otimes \mathcal{H}_{\tilde{A}_I} \otimes \mathcal{H}_{X_c}) \to (\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{X_c}))$, $N_1 \in \mathrm{Unit}((\mathcal{H}_{A_2} \otimes \mathcal{H}_{X_c} \to \mathcal{H}_{A_3}) \to \mathcal{H}_Y$, $N_2 \in \mathrm{Unit}((\mathcal{H}_{A_1} \otimes \mathcal{H}_{X_t}) \to (\mathcal{H}_{A_3} \otimes \mathcal{H}_{B_I}))$;*

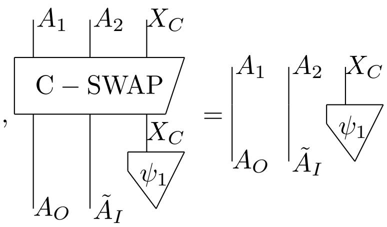
such that



, (52)

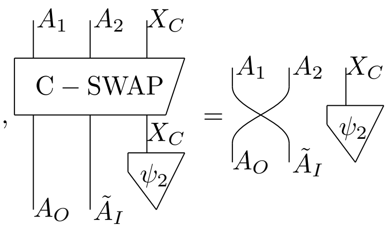
and with the C − SWAP *denoting a generalised controlled SWAP; i.e., there exists a decomposition of $\mathcal{H}_{X_c}$ into orthogonal subspaces, $\mathcal{H}_{X_c} = \mathcal{H}_{X_c1} \oplus \mathcal{H}_{X_c2}$, such that*

$\forall \psi_1 \in \mathcal{H}_{X_c 1}$,



; (53a)

$\forall \psi_2 \in \mathcal{H}_{X_c 2}$,



. (53b)

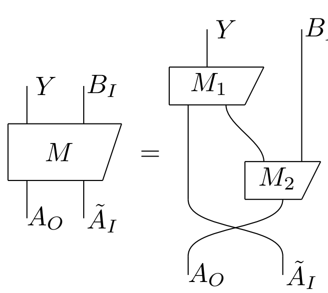
If this conjecture were true, we see that the most general possibility would be that the second-input subalgebra acts on a part of $A_2 \otimes X_t$, where X_t is a part of X, and A_2 is either A_I or A_O - the choice between the two possibly being under coherent control by the other part of X (X_c). It is easy to see that the most general form for a bipartite unitary supermap would then just be a generalisation of the quantum SWITCH. Moreover, this form also embeds causally ordered scenarios, which correspond to the cases when $\mathcal{H}_{X_c} = \mathcal{H}_{X_c 1}$ or $\mathcal{H}_{X_c} = \mathcal{H}_{X_c 2}$. It is therefore a rigorous formulation of the main conjecture we presented in broader terms at the beginning of this section.

It is easy to see that the form (52) indeed satisfies the no-bridge condition. We have not yet been able to prove the crucial reverse implication, i.e. that all solutions of the no-bridge condition can be decomposed in the form (52). Yet, we were able to provide a first step in this direction, by proving it in the case when X is trivial. In this case, we can actually even prove it in a slightly more general way, in which one does not necessarily have $|A_I| = |B_I|$. Note that this scenario does not involve any quantum SWITCH, as there is no auxiliary system to control it.

Theorem 8 *Let $M \in \mathrm{Unit}((\mathcal{H}_{A_O} \otimes \mathcal{H}_{\tilde{A}_I}) \to (\mathcal{H}_Y \otimes \mathcal{H}_{B_I}))$, where $|A_I| = |A_O|$ and $|B_I| = |B_O|$.*

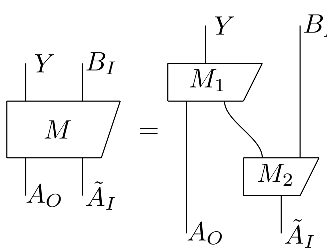
Then M satisfies the no-bridge condition if and only if:

- *either*



,

- *or*



,

where M_1 and M_2 are unitary operators.footnoteNote that this theorem entails that if $|A_I| >$ $absB_I$, then there are no solutions to the no-bridge condition.

A proof of this theorem is presented in Appendix C.

Per this theorem, we see that if there is no environment X, the only possible causal localisations for Bob's operations are to be either before or after Alice.

Unfortunately, we have so far been unable to generalise this result to the case where X is non-trivial. In particular, a crucial step in the proof of Theorem 8 is to make use of the decomposition (67), the existence of which is proven by [18] to be implied by the existence of a set of no-influence relations. Unfortunately, if we try to apply the same strategy to the case with non-trivial X, we obtain a set of no-influence relations for which no corresponding decomposition has been found in the literature yet.

6 Conclusion

During the course of this project, we developed a new framework for the study of supermaps, based on diagrammatic techniques. This framework helped us to re-express the major current notions, theorems and examples in the study of supermaps and indefinite causal order in a more manageable and intuitive way. Moreover, using techniques coming from the recent research both on diagrammatic quantum theory, quantum causal structures, and supermaps, we were able to provide steps forward in the way to a general classification of bipartite unitary supermaps, and to a proof of our conjecture that these all essentially come down to causally-ordered structures or to variations of the quantum SWITCH, although some progress remains to be made in this direction.

Let us emphasise that it is well possible that this conjecture turns out to be false, and that there actually exist bipartite unitary supermaps which entail a different causal structure. However, we think working with this conjecture in mind would, in the case that it is false, still constitute the best opportunity to find a counterexample to it, and therefore also make significant progress in our understanding of supermaps.

The logical continuation of this project would be to further study the solutions of the no-bridge condition in order to find a general form of bipartite unitary supermaps. One can think of three possible ways to reach this objective:

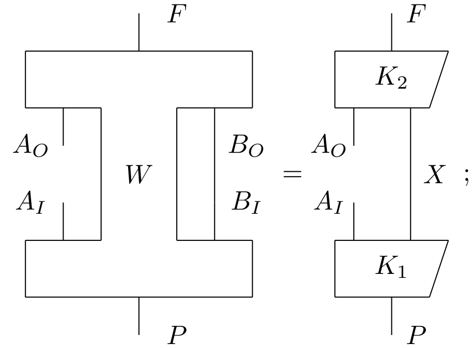
- finding a more general version of the proof we found for Theorem 8;
- reducing the study of the no-bridge condition in general to that of the no-bridge condition with a trivial X, in order to make use of the results of Theorem 8;
- going a completely different way.

Once a general form of bipartite unitary supermaps is found, this would open the way to a significant amount of follow-up research. One could then make use of this form to prove general results about bipartite supermaps, such as the extent of their computational and communicational powers, their physical status and implementability, and the definition of a rigorous operational framework in which to embed them, for example a generalisation of the quantum circuit framework. Another direction would be to try to lean on this result to find the form of non-unitary purifiable bipartite supermaps, and of non-purifiable bipartite supermaps. Finally, this form could also be used to work towards a general form of tripartite unitary supermaps, in the same way that the classification of monopartite supermaps by Theorem 1 has been a core component in our proofs concerning the inner structure of bipartite supermaps. Hopefully, such results could then provide the relevant intuition and techniques to tackle the problem of general n-partite supermaps.

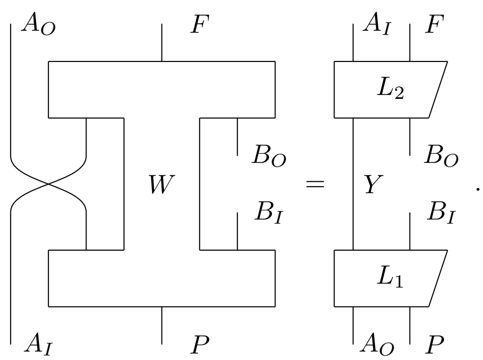
Appendix A Proof of Theorem 6

Let us start with direct implication, and take a bipartite unitary supermap $\hat{W}$ as defined in Theorem 6.

By Theorem 1, we can find decompositions of the form (27) for two monopartite supermaps obtained from $\hat{W}$:[18]

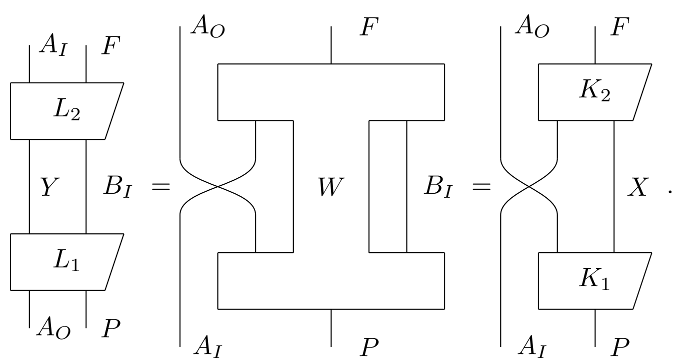


(54a)



(54b)

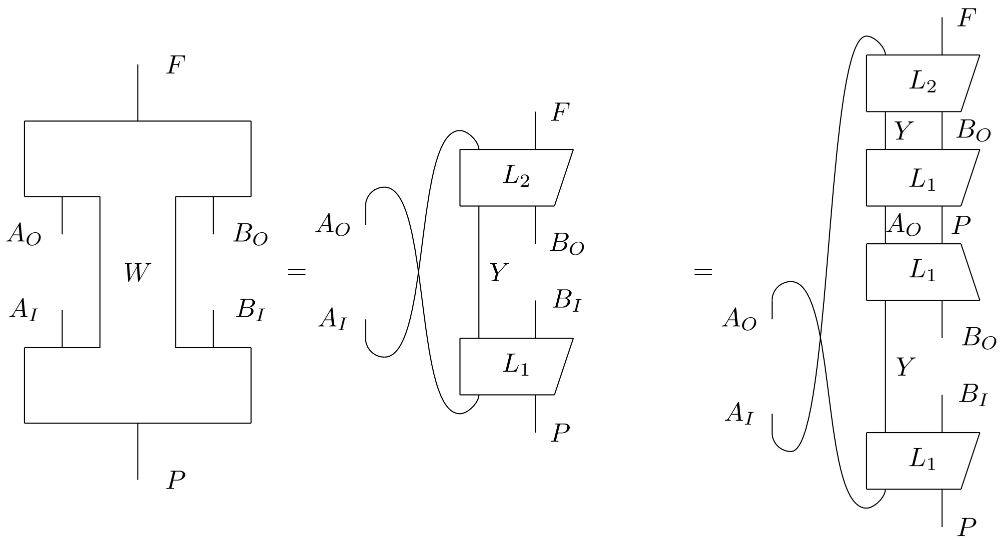
These decompositions are related by



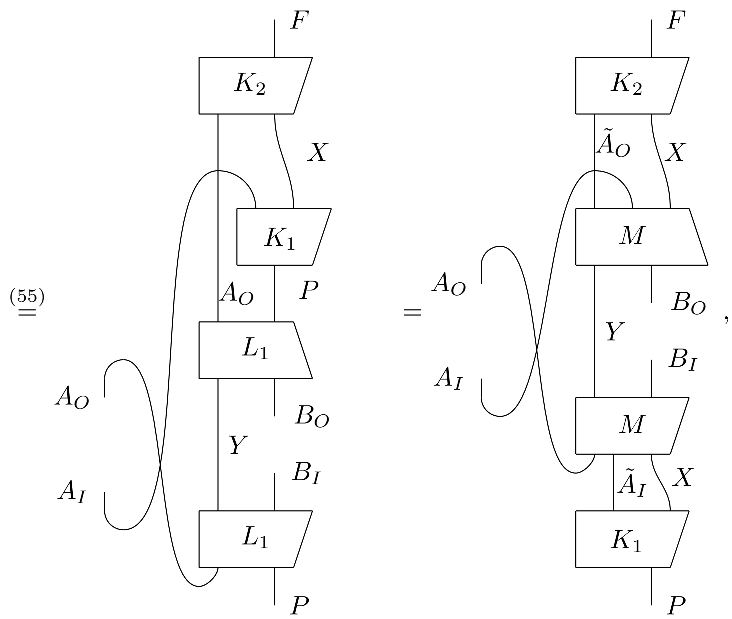
(55)

One then has

[18]Note that as the monopartite supermaps considered are unitary, all the channels in their decomposition can be taken to be unitary as well. This moreover allows us to work at the single level.

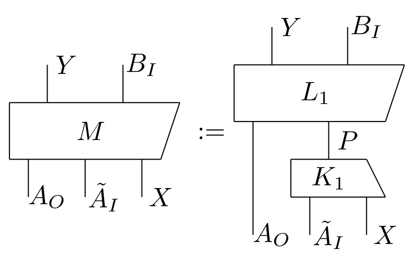


(56)



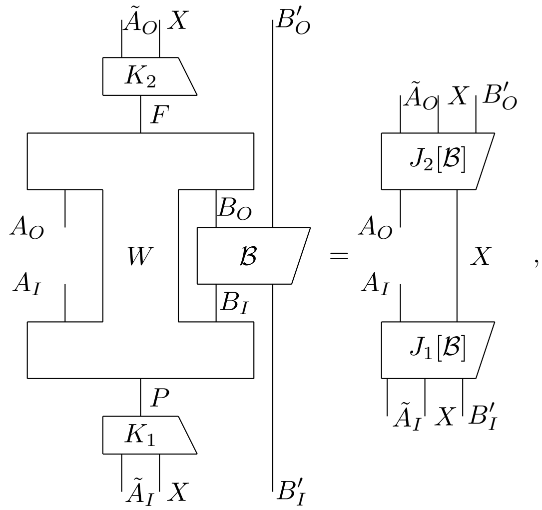
(57)

where in the last step we defined



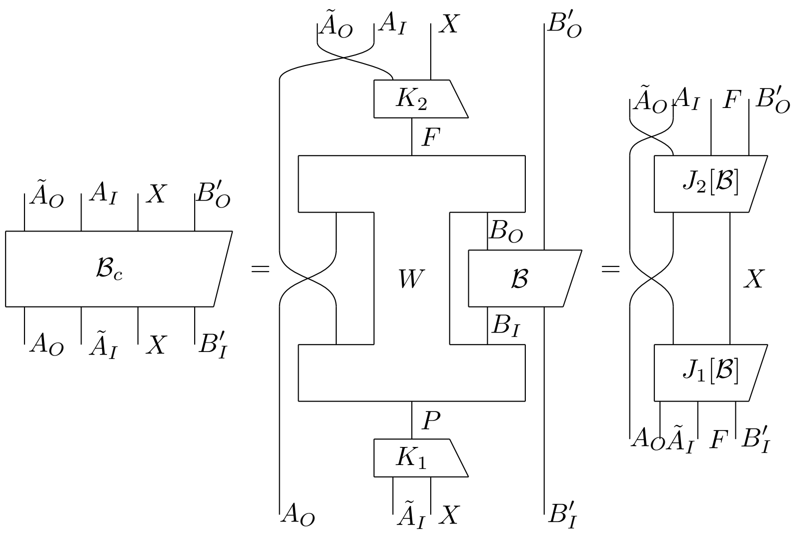
(58)

Moreover, for a given unitary operation $\mathcal{B}$ as introduced in Definition 13, one has, once again by Theorem 1,



(59)

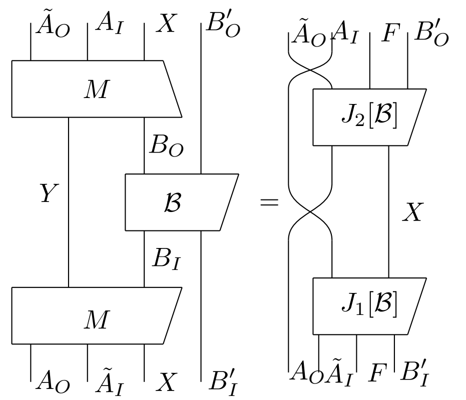
which implies that, for $\mathcal{B}_c$ defined as in (45),



(60)

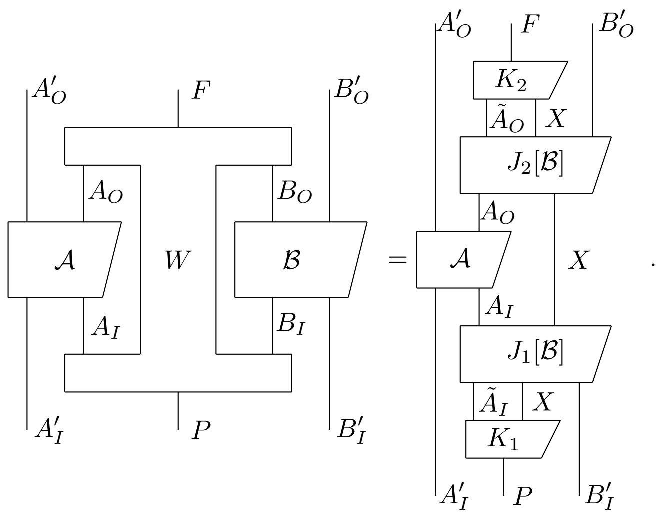
which directly implies the no-bridge condition.

Conversely, let us consider a superoperator $\hat{W}$ defined as in (44), with M satisfying the no-bridge condition. Then for any $\mathcal{B}$, the no-bridge condition implies that $\mathcal{B}_c$ is of the form (24), i.e.



. (61)

Therefore, for any $\mathcal{A}$ one has



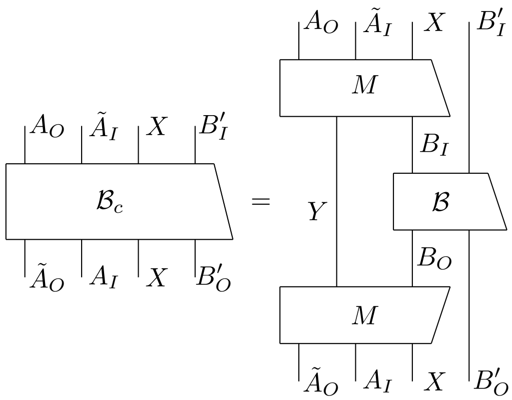
. (62)

By doubling this, we see that any pair of unitary channels are indeed mapped to unitary channels. One can see that general quantum channels are also mapped to quantum channels, by considering their Stinespring dilations. Therefore, $\hat{W}$ is indeed a unitary supermap, which ends the proof.

Appendix B Proof of Theorem 7

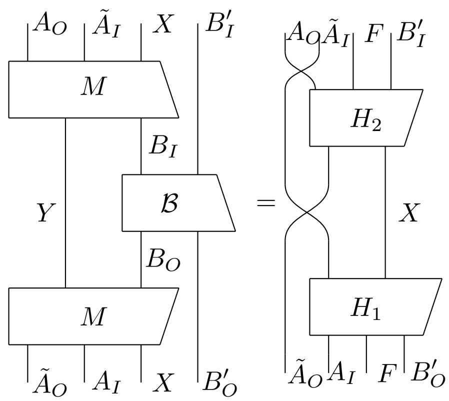
Let us consider a unitary M satisfying the no-bridge condition and a unitary $\mathcal{B} \in \text{Unit}\left((\mathcal{H}_{B_I} \otimes \mathcal{H}_{B'_I}) \to (\mathcal{H}_{B_O} \otimes \mathcal{H}_{B'_O})\right)$. By applying the no-bridge condition to its adjoint $\mathcal{B}^\dagger$ and renaming some types[19], we get that

[19]This is possible as long as we do replace a type by a type with the same dimension, which is the case here. This could be formalised by introducing some isomorphisms, which we do not write here in order to avoid clutter.



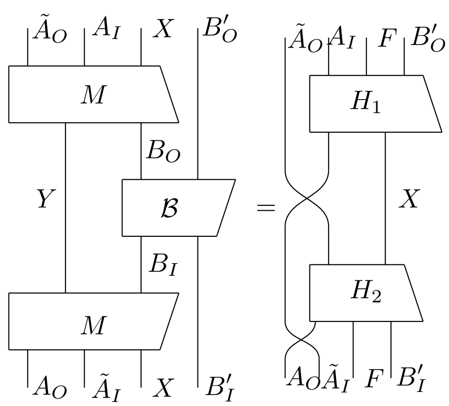
is compatible with $\tilde{A}_O \not\twoheadrightarrow \tilde{A}_I$. (63)

Thus, it has the form (24), i.e.



; (64)

taking the adjoint, this equation becomes

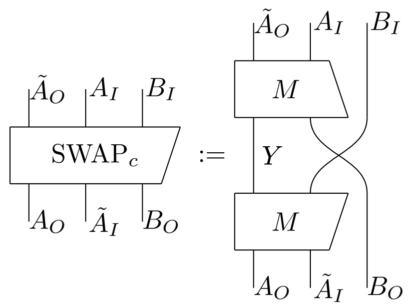


; (65)

which implies that $\mathcal{B}_c$ is compatible with $\tilde{A}_I \not\twoheadrightarrow \tilde{A}_O$. The reverse implication is proven in a completely symmetric way.

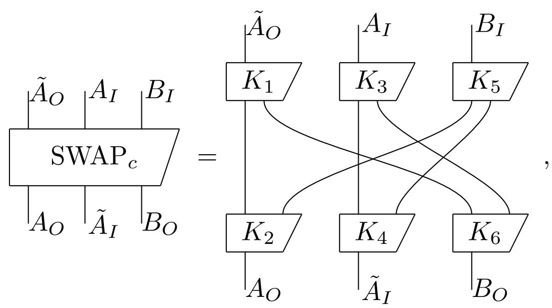
Appendix C Proof of Theorem 8

Let us take such a unitary M, and define



(66)

Then SWAP$_c$ is compatible with $A_O \nrightarrow A_I$. Moreover, by Theorem 7[20], it is also compatible with $\tilde{A}_I \nrightarrow \tilde{A}_O$. Finally, one can see from its form that it is compatible with $B_O \nrightarrow B_I$. [18] gives a form for an operator compatible with this set of no-influence relations: there exists a decomposition

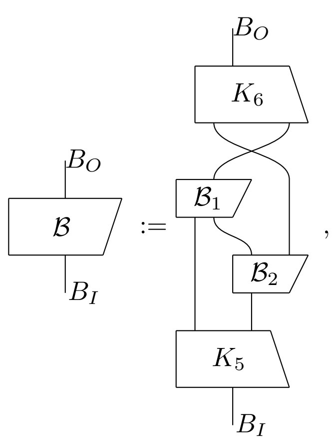


(67)

in which all operators are unitary. We will denote the types of intermediary wires by the path they provide: for example, the type of the wire between K_2 and K_1 will be named "$A_O \to \tilde{A}_O$", as it is the wire connecting A_O to $\tilde{A}_O$, and so on. We do not writes these types on the diagrams to avoid clutter. Considerations of dimensionality imply that $\left|B_O \to \tilde{A}_O\right| = |A_O \to B_I|$ and $|B_O \to A_I| = \left|\tilde{A}_I \to B_I\right|$.

Let us now suppose that $|A_O \to B_I| > 1$ and $\left|\tilde{A}_I \to B_I\right| > 1$, and reach a contradiction.

Let us suppose that $|A_O \to B_I| \leq \left|\tilde{A}_I \to B_I\right|$; we can then define the unitary operator[21]

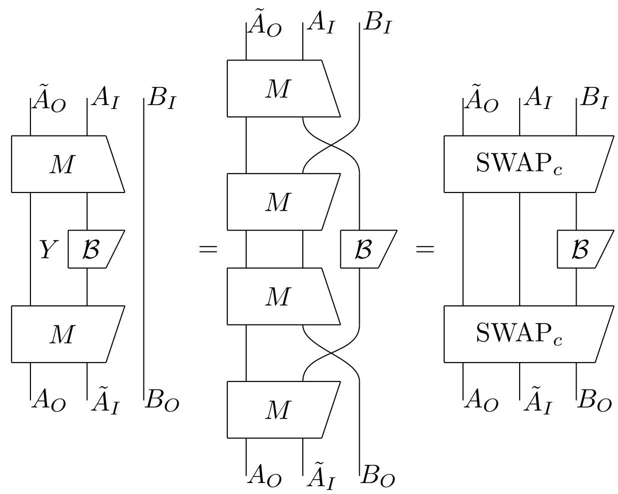


(68)

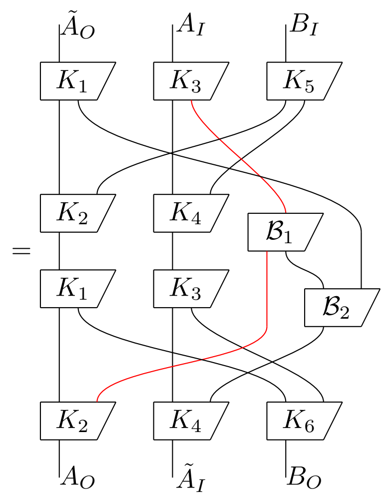
which yields

[20] Note that, as its proof does not depend on any consideration on dimension, this theorem is also valid when $|A_I| \neq |B_I|$.

[21] Here $\mathcal{B}_1$ and $\mathcal{B}_2$ are arbitrary unitaries.



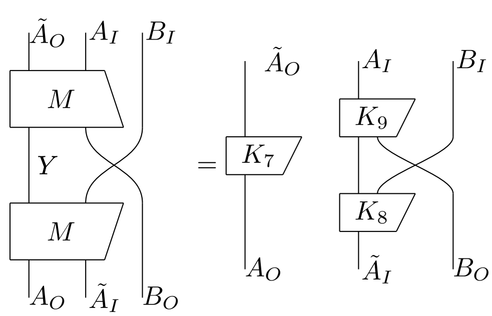
(69)



, (70)

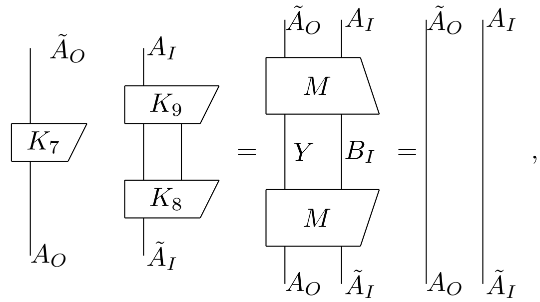
in which the red path is a non-trivial (as $|A_O \to B_I| > 1$ and $\left|\tilde{A}_I \to B_I\right| > 1$) path from A_O to A_I, which contradicts the no-bridge condition. If we suppose that $|A_O \to B_I| \geq \left|\tilde{A}_I \to B_I\right|$, then we can obtain a non-trivial path from $\tilde{A}_I$ to $\tilde{A}_O$, and thus a contradiction, in the same way.

Therefore we have $|A_O \to B_I| = 1$ or $\left|\tilde{A}_I \to B_I\right| = 1$. Let us take the case $|A_O \to B_I| = 1$; then the form (67) becomes



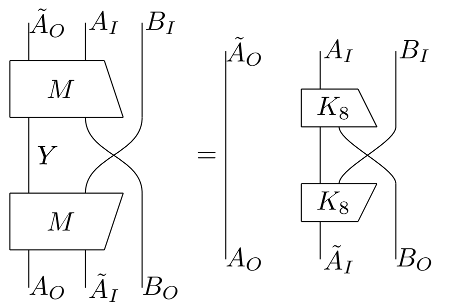
. (71)

As we have



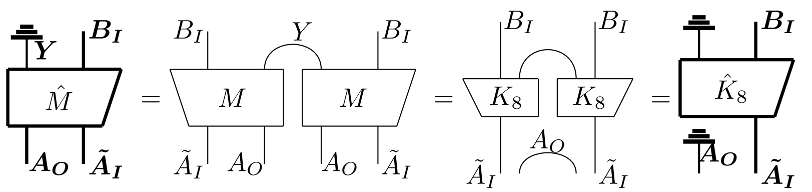
, (72)

we get $K_7 = \mathbb{1}$ and $K_9 = K_8^\dagger$ and this further simplifies to



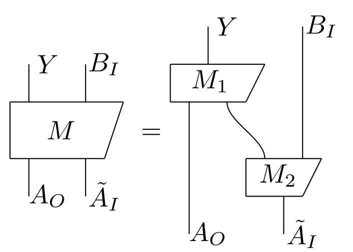
. (73)

Bending some wires gives



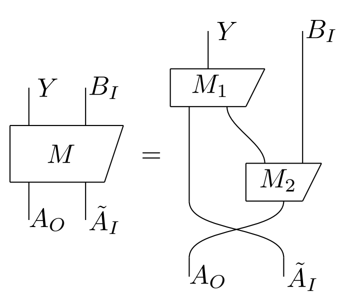
, (74)

i.e. M is compatible with $A_O \nrightarrow B_I$, which implies the form



. (75)

In the case $\left|\tilde{A}_I \to B_I\right| = 1$, the same reasoning gives



. (76)

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