Combinatorial Structures and Lambda Calculi

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Abstract

The bicategory of generalized species **Esp**, introduced by Fiore, is both a generalization of Joyal's combinatorial species and the co-Kleisli bicategory of a linear exponential pseudo-comonad on the bicategory of profunctors. As a result, the structure of **Esp** is closely related to the notion of combinatorial structure as well as to the differential λ-calculus. The precise connection between **Esp** and the differential λ-calculus has not before been investigated; such is the aim of this dissertation.

Here we define a model of the differential λ-calculus within **Esp**, using a new construction for interpreting differentiation. We then construct an alternative interpretation that is combinatorial in nature. Terms are viewed as functors which map lists of vertices equipped with information about their connectivity to sets of specifically constructed graphs on those vertices. Our main result is that these two interpretations are naturally isomorphic, so we can take a graph theoretic perspective on the differential λ-calculus. Several examples of interpreted λ-terms are presented, both in the categorical and the combinatorial interpretations. An method for determining cardinality of the graph sets arising from the combinatorial interpretation of βD-normal terms is given, and extensions of the model via syntactical additions are also considered.

Contents

Chapter 1

Introduction

Drawing connections between computation and category theory has been a useful tool in theoretical computer science, as developments like the Curry-Howard-Lambek correspondence provide a new angle from which to approach problems in either field. Due to the Curry-Howard-Lambek correspondence, we can use knowledge about the λ-calculus to learn about the theory of a Cartesian closed category (CCC), as well as use category theory to inform our understanding of computations in the λ-calculus. As new variants of computational calculi have arisen from different computing needs, similar correspondences between calculi and categorical models have been detailed. The linear λ-calculus, a variation introduced to manage computations involving finite resources, has been linked to symmetric monodical closed categories through a similar construction [1]. Several formulations have been put forth for creating models of Ehrhard's differential λ-calculus, which incorporates elements of linear logic to the full lambda calculus by introducing nondeterminism and linear application, inside differential categories [4, 6]. Process calculi have been introduced to formalize concurrent computation; these, too, have been analyzed from a categorical perspective [10].

The field of combinatorics, while home to a wide variety of mathematics, is often characterized by a focus on enumerative problems. Many combinatorial problems concern finding the size, or establishing the non-emptiness, of particular sets of structures given a collection of constraints. The underlying computational nature of this field suggests that combinatorics is another such area that could benefit from a correspondence with the theory of computation. In particular, being able to view sets of combinatorial structures as models of a calculus could spur new techniques for counting and transforming structures.

Fiore's introduction of generalized species of structure in [8] is an ideal starting point for an exploration of the connection between combinatorics and computation, and serves as the primary motivation for this project. The bicategory of generalized species **Esp** arises both as an encoding of combinatorial structure and also as the result of a construction which ensures the category will be a model of the differential λ-calculus. Though this fact is mentioned by Fiore and by others, the construction of a model of the differential λ-calculus inside **Esp** has not yet been made explicit. In this dissertation we provide the details of this model, and then reframe it from a combinatorial perspective as a model consisting of sets of graphs whose transformations mirror the behaviour of the differential λ-calculus. Finally, we provide a simplified method of counting these sets of graphs as well as provide further ideas for extending the correspondence through new syntactical constructions. Our main results are in Chapters 5-7:

- In Chapter 5, we interpret typing judgements $\Gamma \vdash t : A$ of differential λ-terms as species $[\![t]\!]_A^\Gamma : [\![\Gamma]\!] \mathrel{!\!\!\to} [\![A]\!]$. Methods in the literature of interpreting the differential λ-calculus in a general differential category rely on various formulations of an operator $D(-)$ satisfying certain axioms [4, 6]; as there is not one canonical choice and no such operator is specified in Fiore's treatment of **Esp**, we instead interpret differential terms by

$$[\![Ds \cdot t]\!]_{A\to B}^\Gamma = \mathrm{lev} \circ \langle D \circ [\![s]\!]_{A\to B}^\Gamma, [\![t]\!]_A^\Gamma \rangle$$

where lev and D are the new linear evaluation and differentiation operators we introduce in Chapter 2.3. In Proposition 5.2 we prove that this interpretation is sound, a step that is necessary because our choice of interpretation does not directly follow from a previous construction.

- In Chapter 6 we define a second interpretation of differential λ-terms: the typing judgement $\Gamma \vdash t : A$ is interpreted as a species $\langle t \rangle_A^\Gamma$ where $\langle t \rangle_A^\Gamma(G)(a)$ is a set of graphs on a fixed vertex set for each $G \in !\Gamma$ and $a \in \mathbb{A}$. The main result of this chapter is the following theorem.

  Theorem 6.5. *If $\Gamma \vdash t : A$ then the species $[\![t]\!]_A^\Gamma$ and $\langle t \rangle_A^\Gamma$ are naturally isomorphic.*

  This isomorphism between interpretations allows us to look at the model of differential λ-calculus within **Esp** from a graph-theoretic perspective.

- In Chapter 7 we show that for differential λ-terms in normal form, the definition of the term's interpreted species can be significantly simplified so as to accomodate easier calculation of the sizes of the graph sets that arise from the combinatorial interpretation.

  Proposition 7.1. *If $\Gamma \vdash t : A$ for a differential λ-term t in normal form and $G \in ! [\![\Gamma]\!]$, then there is an index set I and maps* `shape` $: I \to$ ***Set*** *and* `point` $: I \to \mathbb{A}$ *such that for all $a \in [\![A]\!]$,*

  $$[\![t]\!]_A^\Gamma (G)(a) \cong \sum_{i \in I} \texttt{shape}(i) \times \mathbb{A}(a, \texttt{point}(i))$$

In addition to being of interest to combinatorists for the purposes of studying structures via λ-calculi, these results may also be useful to computer scientists with an interest in better understanding the bicategory **Esp** as it relates to various models of computation.

The structure of the dissertation is as follows:

- Chapter 2 presents the definitions and operations of combinatorial species of structure, following [3] and [17]. The bicategory of generalized species of structure as defined in [8, 9] is then introduced. In addition to presenting the operations on species defined by Fiore, we also introduce two new operators to use in Chapter 5.

- Chapter 3 details how generalized species of structure arise not only as generalizations of combinatorial structures, but also as the co-Kleisli bicategory of a pseudo-comonad on the bicategory of profunctors. The fact that this construction gives the **Esp** the structure it needs to model the λ-calculus is explained by analogy to the category **Rel** as outlined in [6].

- In Chapter 4 the syntax of the differential λ-calculus is presented as an extension of the simply typed λ-calculus. The reduction and typing rules of the differential λ-calculus are also introduced.

- In Chapter 5, we define our interpretation of the differential λ-calculus into the bicategory **Esp** and prove the soundness of this interpretation. We also include several examples of interpreted differential λ-terms to motivate the combinatorial perspective of the next chapter.

- In Chapter 6 we define the alternative combinatorial description of the differential λ-calculus within **Esp**, and then prove that this description is naturally isomorphic to the categorical interpretation from Chapter 5. We illustrate this connection with several examples of representative graphs from interpreted differential λ-terms.

- Chapter 7 focuses specifically on terms in normal form, and gives another formulation of interpreted terms to enable easier computation of the cardinalities of the graph sets from the combinatorial interpretation in Chapter 6.

- Finally, Chapter 8 suggests further extensions to the syntax of the differential λ-calculus and proposes how those extensions would manifest in the combinatorial interpretation.

There are two appendices; the first presents lemmas relevant to carrying out the coend calculations upon which the contents of Chapters 5-7 depend, and the second contains details of various proofs presented in Chapters 5-7. Appendix A in particular will be useful to the reader wishing to follow along with the proof of soundness in Chapter 5.2.

This dissertation assumes a basic background in category theory up to the level of co/monads and co/ends. Treatment of these topics can be found in [14]. Chapter 3 assumes knowledge of linear logic, information about which can be found in [5]. In addition, we assume familiarity with the simply typed λ-calculus.

Chapter 2

Species of structure

In this chapter we will first introduce combinatorial species, a category theoretic framework for describing combinatorial structures, before generalizing these species according to a construction by Fiore [8]. The definitions in Chapter 2.1 are drawn from [3, 13, 17], while the definitions in Chapters 2.2-2.4, except where otherwise noted, are from [8].

2.1 Combinatorial species

Combinatorial structures, such as trees, permutations, and linear orderings, can be thought of as structures on sets of labels. For example, 'tree' describes all structures that could be created by taking a collection of labels (vertices) and connecting some of those labels via edges in a way that does not create any cycles. Thus we can define a map Tree which takes in a label set and returns the set of all trees on that label set.

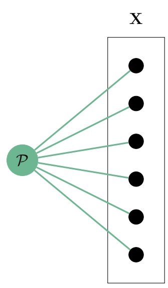
In 1981, Joyal [13] introduced combinatorial species as a way to formalize this notion: combinatorial species are functors $\mathcal{P} : \mathbf{B} \to \mathbf{Set}$, where $\mathbf{B}$ is the category of finite sets and bijections. For any finite label set X, $\mathcal{P}(X)$ is the set of all $\mathcal{P}$-structures on X. For a bijection $f : X \to Y$, $\mathcal{P}f : \mathcal{P}(X) \to \mathcal{P}(Y)$ is the function that takes a $\mathcal{P}$-structure over the label set X and maps it to the same structure where each label $x \in X$ is replaced by the label $f(x) \in Y$.

Example 2.1. For the linear order species LO,

$$\mathrm{LO}(\{A, B, C\}) = \begin{Bmatrix} A < B < C, & A < C < B, & B < A < C, \\ B < C < A, & C < A < B, & C < B < A \end{Bmatrix}$$

If $f : \{A, B, C\} \to \{1, 2, 3\}$ is the bijection defined by $f(A) = 1$, $f(B) = 2$, and $f(C) = 3$, then $LO(f)(A < C < B)$ is the linear order $1 < 3 < 2$.

In general, we will depict a $\mathcal{P}$-structure on X with the following diagram:



Combinatorial species are equipped with operations — namely addition, product, multiplication, composition and differentiation — which allow us to build up new structures from existing species. Diagrams in the style of the one above will be helpful in visualizing what each operation entails.

Addition

Adding two species together gives the coproduct, or disjoint union, of those species. That is, if $\mathcal{P}$ and $\mathcal{Q}$ are two combinatorial species, then for all label sets X we have

$$(\mathcal{P} + \mathcal{Q})(X) = \mathcal{P}(X) + \mathcal{Q}(X) = \mathcal{P}(X) \uplus \mathcal{Q}(X)$$

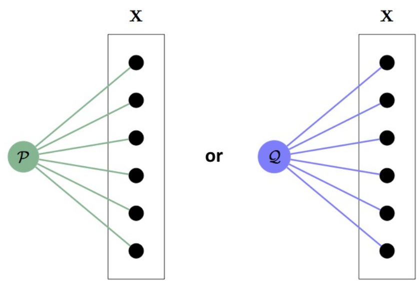


Figure 2.1: A $(\mathcal{P} + \mathcal{Q})$-structure.

This means that a $\mathcal{P} + \mathcal{Q}$ structure is either a $\mathcal{P}$-structure or a $\mathcal{Q}$-structure, along with an indicator of which type of structure it is.

Product

The product of species is also intuitive: for all X,

$$(\mathcal{P} \times \mathcal{Q})(X) = \mathcal{P}(X) \times \mathcal{Q}(X)$$

So a $(P \times \mathcal{Q})$-structure is a pair of a $\mathcal{P}$-structure and a $\mathcal{Q}$-structure.

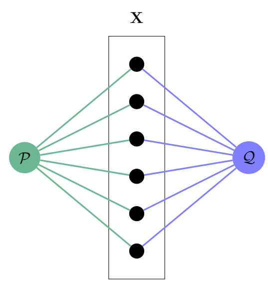


Figure 2.2: A $(\mathcal{P} \times \mathcal{Q})$-structure.

Multiplication

There is also a multiplication operation for species:

$$(\mathcal{P} \cdot \mathcal{Q})(X) = \sum_{\substack{(X_1, X_2) \\ \in \text{Part}(X)}} \mathcal{P}(X_1) \times \mathcal{Q}(X_2)$$

Here, $\text{Part}(X)$ denotes the set of partitions of X into two pieces. This multiplication operation amounts to partitioning the label set X into two sets X_1 and X_2, putting a $\mathcal{P}$-structure on X_1 and putting a $\mathcal{Q}$-structure on X_2.

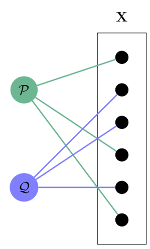


Figure 2.3: A $(\mathcal{P} \cdot \mathcal{Q})$-structure.

Composition

Species can be composed as well, which amounts to partitioning the input label set into pieces, putting a $\mathcal{P}$ structure on each piece, and then putting a $\mathcal{Q}$-structure on the set of new $\mathcal{P}$-structures we've created.

$$(Q \circ P)(X) = \sum_{U \in \text{Part}(X)} Q(U) \times \prod_{u \in U} P(u)$$

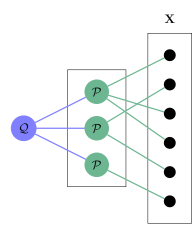


Figure 2.4: A $(\mathcal{Q} \circ \mathcal{P})$-structure.

Differentiation

The differentiation operator on combinatorial species takes a species $\mathcal{P}$ and creates a species that takes in a label set X and returns a $\mathcal{P}$-structure on $X \cup \{x\}$, where x is a fresh element not appearing in X:

$$\frac{\partial}{\partial x} P(X) = P(X \oplus \{x\})$$

.

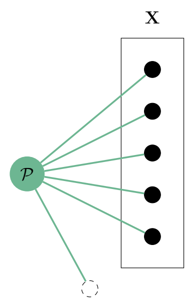


Figure 2.5: A $\frac{\partial}{\partial x}\mathcal{P}$-structure.

2.2 Generalized species

More recently, in [8] Fiore introduced *generalized species of structure*, which generalize combinatorial species from functors $\mathbf{B} \to \mathbf{Set}$ to profunctors of the form $!\mathbb{A} \times \mathbb{B}^{op} \to \mathbf{Set}$, where $\mathbb{A}$ and $\mathbb{B}$ are both small categories and $!\mathbb{A}$ is the free symmetric monoidal completion of $\mathbb{A}$. We can think of these functors as resembling combinatorial species in that they map 'bags' of $\mathbb{A}$ labels parameterized by a $\mathbb{B}$ label to sets of structures. These profunctors can also be seen as functors of the form $!\mathbb{A} \to \widehat{\mathbb{B}}$; we will use this notation to represent the action of species on objects and arrows. We will also sometimes denote an $(\mathbb{A}, \mathbb{B})$-species by $\mathbb{A} \mathrel{!\!\to} \mathbb{B}$.

Explicitly, the objects of $!\mathbb{A}$ are lists of objects from $\mathbb{A}$, possibly containing duplicates. For two lists $X = [x_1, ..., x_n]$ and $Y = [y_1, ..., y_m]$ in $!\mathbb{A}$, an arrow $X \to Y$ is a matching of the elements of the two lists, formed by arrows from $\mathbb{A}$:

$$\begin{array}{ccccccc} X = & [x_1, & x_2, & x_3, & \ldots, & x_n] \\ \downarrow f & \downarrow f_1 & \downarrow f_2 & \downarrow f_3 & & \downarrow f_n \\ Y = & [y_{\sigma(1)}, & y_{\sigma(2)}, & y_{\sigma(3)}, & \ldots, & y_{\sigma(n)}] \end{array} \qquad f_i \in \mathbb{A}(x_i, y_{\sigma(i)}),\ \sigma \in S_{|X|}$$

Note that this definition of hom-sets in $!\mathbb{A}$ immediately entails that $!\mathbb{A}(X, Y) = \emptyset$ if X and Y are lists of different lengths. We will use the notation $|A|$ to denote the length of a list A.

Though the objects of $!\mathbb{A}$ are lists and thus have internal ordering, it follows from the definition of arrows in $!\mathbb{A}$ that if two lists X and Y have the same contents in a different order, then for all $Z \in !\mathbb{A}$,

$$!\mathbb{A}(X, Z) \cong !\mathbb{A}(Y, Z) \quad \text{and} \quad !\mathbb{A}(Z, X) \cong !\mathbb{A}(Z, Y)$$

It will be useful to define equivalence of bags via their contents, as most functors we will see over the course of this dissertation will effectively disregard the ordering of input lists.

We can also apply ! to the disjoint union of categories. In this case, the objects of $!(\mathbb{A} \sqcap \mathbb{B})$ are lists of objects from $\mathbb{A}$ and $\mathbb{B}$, with each listed object indexed by which category — $\mathbb{A}$ or $\mathbb{B}$ — it is from. Arrows $X \to Y$ in $!(\mathbb{A} \sqcap \mathbb{B})$ are matchings of the objects in the two lists formed by arrows from $\mathbb{A}$ and $\mathbb{B}$. If A_1 and A_2 are both objects in $!\mathbb{A}$, then $A_1 \oplus A_2$ denotes the object of $!\mathbb{A}$ constructed by appending the list A_2 to the end of the list A_1. If $A \in !\mathbb{A}$ and $B \in !\mathbb{B}$, then

$A \otimes B$ denotes the object in $!(\mathbb{A} \sqcap \mathbb{B})$ constructed by considering each A and B as objects in $!(\mathbb{A} \sqcap \mathbb{B})$ and then appending B to the end of A.

Another fact we will utilize is that $!(\mathbb{A} \sqcap \mathbb{B})(X, Y) \cong !\mathbb{A}(X_{\mathbb{A}}, Y_{\mathbb{A}}) \times !\mathbb{B}(X_{\mathbb{B}}, Y_{\mathbb{B}})$, where $X_{\mathbb{A}}$ is the sublist of X whose objects are exactly those from $\mathbb{A}$. This is because the construction of an arrow in $!(\mathbb{A} \sqcap \mathbb{B})$ requires that objects from $\mathbb{A}$ can only be matched with other objects from $\mathbb{A}$, and likewise objects from $\mathbb{B}$ can only be matched with others from $\mathbb{B}$.

Remark. A specific case of note is when $\mathbb{A} = \mathbb{1}$, the category consisting of one object and the identity arrow. Each object in $!\mathbb{1}$ is of the form $[1, ..., 1]$. So for each n there is an object in $!\mathbb{1}$ consisting of a list of n copies of the object 1, and each arrow in $!\mathbb{1}$ is a permutation a list of 1's. Thus $!\mathbb{1}$ is equivalent to the category **B** of finite sets and bijections. This tells us that generalized species are indeed a generalization of combinatorial species, as a $(\mathbb{1}, \mathbb{1})$-species can be seen as a profunctor $\mathbf{B} \times \mathbb{1}^{op} \to \mathbf{Set}$, or more simply as a functor $\mathbf{B} \to \mathbf{Set}$.

2.3 Species operations

The operations on combinatorial species can also be generalized to operations on generalized species. In each case, the new operator bears resemblance to the same operation on combinatorial species.

Addition & Product

The definitions of addition and product for generalized species are the same as those for combinatorial species: for two species $\mathcal{P}, \mathcal{Q} : \mathbb{A} \;!\!\!\to \mathbb{B}$, the species $\mathcal{P} + \mathcal{Q} : \mathbb{A} \;!\!\!\to \mathbb{B}$ is defined by

$$(\mathcal{P} + \mathcal{Q})(A)(b) = \mathcal{P}(A)(b) + \mathcal{Q}(A)(b) \qquad \text{where } A \in !\mathbb{A}, b \in \mathbb{B}^{op}$$

and $\mathcal{P} \times \mathcal{Q} : \mathbb{A} \;!\!\!\to \mathbb{B}$ is defined by

$$(\mathcal{P} \times \mathcal{Q})(A)(b) = \mathcal{P}(A)(b) \times \mathcal{Q}(A)(b) \qquad \text{where } A \in !\mathbb{A}, b \in \mathbb{B}^{op}$$

Multiplication

The multiplication on generalized species is the Day tensor [8]. For species $\mathcal{P}, \mathcal{Q} : \mathbb{A} \;!\!\!\to \mathbb{B}$ and objects $A \in !\mathbb{A}, b \in \mathbb{B}^{op}$, the species $\mathcal{P} \cdot \mathcal{Q} : \mathbb{A} \;!\!\!\to \mathbb{B}$ is defined by

$$(\mathcal{P} \cdot \mathcal{Q})(A)(b) = \int^{A_1, A_2 \in !\mathbb{A}} \mathcal{P}(A_1)(b) \times \mathcal{Q}(A_2)(b) \times !\mathbb{A}(A_1 \oplus A_2, A)$$

Using properties of coends and the ! construction (see Appendix A), we can alter this definition to [1]

$$(\mathcal{P} \cdot \mathcal{Q})(A)(b) \cong \sum_{(A_1, A_2) \in \mathcal{D}(A)} \mathcal{P}(A_1)(b) \times \mathcal{Q}(A_2)(b)$$

where $\mathcal{D}(A)$ is the set of partitions of the list A into two sublists, preserving relative order of elements. As with combinatorial species, multiplying generalized species also involves separating

[1] Note that here we are redefining an operation up to isomorphism. This change is allowed because the set $(P \cdot Q)(A)(b)$ is originally defined as a coend, which is itself only unique up to isomorphism. Henceforth we will only be concerned with generalized species up to isomorphism.

the input bag into two pieces before creating a $\mathcal{P}$ structure on one and a $\mathcal{Q}$ structure on the other, each parameterized by the object b.

Composition

For $\mathcal{P} : \mathbb{B} \,!\!\rightarrow \mathbb{C}$ and $\mathcal{Q} : \mathbb{A} \,!\!\rightarrow \mathbb{B}$, the composition $\mathcal{P} \circ \mathcal{Q} : \mathbb{A} \,!\!\rightarrow \mathbb{C}$ is defined by

$$(\mathcal{P} \circ \mathcal{Q})(A)(c) = \int^{B \in !\mathbb{B}} \mathcal{P}(B)(c) \times \mathcal{Q}^{\#}(A)(B)$$

where $\mathcal{Q}^{\#}(A)(B) = \int^{X \in (!\mathbb{A})^{|B|}} \prod_{k \in |B|} \mathcal{Q}(X_k)(B_k) \times !\mathbb{A}(\bigoplus_{k \in |B|} X_k, A)$. Using techniques from Appendix A again, this definition simplifies to

$$(\mathcal{P} \circ \mathcal{Q})(A)(c) \cong \int^{B \in !\mathbb{B}} \sum_{\substack{(A_1, \ldots, A_{|B|}) \\ \in \mathcal{D}(A)}} \mathcal{P}(B)(c) \times \left[\prod_{k \in |B|} \mathcal{Q}(A_k)(B_k) \right]$$

Partial Differentiation

For a species $\mathcal{P} : \mathbb{A} \,!\!\rightarrow \mathbb{B}$ and an object $x \in \mathbb{A}$, the partial derivative $\frac{\partial}{\partial x}\mathcal{P} : \mathbb{A} \,!\!\rightarrow \mathbb{B}$ is defined by

$$\frac{\partial}{\partial x}\mathcal{P}(A)(b) = \mathcal{P}(A \oplus [x])(b)$$

In keeping with the earlier remark that a $(\mathbb{1}, \mathbb{1})$-species is a combinatorial species, these operations each reduce to the analagous operations on combinatorial species when $\mathbb{A} = \mathbb{B} = \mathbb{C} = \mathbb{1}$.

2.4 The bicategory of generalized species

From this definition of generalized species we can define the bicategory **Esp** whose 0-cells (objects) are small categories, 1-cells (arrows) are generalized species, and 2-cells are natural transformations [9]. Composition of arrows in **Esp** is defined by the composition operator given above. **Esp** has been shown to be a Cartesian closed bicategory as well as having symmetric monoidal structure [8]. The internal hom in **Esp** is defined by $\mathbb{A} \Rightarrow \mathbb{B} = !\mathbb{A}^{op} \times \mathbb{B}$ and the linear hom is $\mathbb{A} \multimap \mathbb{B} = \mathbb{A}^{op} \times \mathbb{B}$.

Projection & Pairing

The projection species $\pi_j : \sqcap_{i \in I} \mathbb{C}_i \,!\!\rightarrow \mathbb{C}_j$ is defined by

$$\pi_j(C)(c) = !(\sqcap_{i \in I} \mathbb{C}_i) \left(\left[\coprod_j (c) \right], C \right)$$

where $\coprod_j(z)$ is the inclusion of the object $z \in \mathbb{C}_j$ into the coproduct $\prod_{i \in I} \mathbb{C}_i$. If $P_i : \mathbb{C} \,!\!\rightarrow \mathbb{C}_i$, then the pairing $\langle \mathcal{P}_i \rangle_{i \in I} : \mathbb{C} \,!\!\rightarrow \sqcap_{i \in I} \mathbb{C}_i$ is defined by

$$\langle \mathcal{P}_i \rangle_{i \in I}(C)(c) = \sum_{j \in I} \left[\int^{z \in \mathbb{C}_j} \mathcal{P}_j(C)(z) \times \sqcap_{i \in I} \mathbb{C}_i(c, \coprod_j (z)) \right]$$

Note that if c's index in the product $\sqcap_{i\in I}\mathbb{C}_i$ is k, then $\sqcap_{i\in I}\mathbb{C}_i(c, \coprod_j(z)) = \emptyset$ whenever $j \neq k$. So for $c \in \mathbb{C}_k$,

$$\begin{aligned}\langle\mathcal{P}_i\rangle_{i\in I}(C)(\coprod_k(c)) &= \int^{z\in\mathbb{C}_k} \mathcal{P}_k(C)(z) \times \sqcap_{i\in I}\mathbb{C}_i(\coprod_k(c), \coprod_k(z)) \\ &\cong \int^{z\in\mathbb{C}_k} \mathcal{P}_k(C)(z) \times \mathbb{C}_k(c, z) \\ &\cong \mathcal{P}_k(C)(c) \quad \text{(by Lemma A.2)}\end{aligned}$$

Abstraction & Evaluation

For a species $\mathcal{P} : \Gamma \sqcap \mathbb{A} \,!\!\rightarrow \mathbb{B}$, the abstraction $\lambda_{\mathbb{A}}(\mathcal{P}) : \Gamma \,!\!\rightarrow (\mathbb{A} \Rightarrow \mathbb{B})$ is defined by

$$\lambda_{\mathbb{A}}(\mathcal{P})(X)(A, b) = \mathcal{P}(X \otimes A)(b)$$

For each $\mathbb{A}, \mathbb{B}$, the evaluation species $\mathrm{ev}_{\mathbb{A},\mathbb{B}} : (\mathbb{A} \Rightarrow \mathbb{B}) \sqcap \mathbb{A} \,!\!\rightarrow \mathbb{B}$ is defined by

$$\mathrm{ev}_{\mathbb{A},\mathbb{B}}(M)(b) = \iint^{F\in\,!(\mathbb{A}\Rightarrow\mathbb{B}),\, A\in\,!\mathbb{A}} !(\mathbb{A} \Rightarrow \mathbb{B})([(A, b)], F) \times !((\mathbb{A} \Rightarrow \mathbb{B}) \sqcap \mathbb{A})(F \oplus A, M)$$

Differentiation

The version of differentiation presented in [8] is an $(\mathbb{A} \Rightarrow \mathbb{B},\ \mathbb{A} \Rightarrow (\mathbb{A} \multimap \mathbb{B}))$-species. However, for reasons addressed in Chapter 4.2, we will slightly change this definition and instead use an $(\mathbb{A} \Rightarrow \mathbb{B},\ \mathbb{A} \multimap (\mathbb{A} \Rightarrow \mathbb{B}))$-species as the differentiation operator. For $F \in\, !(\mathbb{A} \Rightarrow \mathbb{B})$, $A \in\, !\mathbb{A}$, $a \in \mathbb{A}$, and $b \in \mathbb{B}^{op}$,

$$D_{\mathbb{A},\mathbb{B}}(F)(a, A, b) = !(\mathbb{A} \Rightarrow \mathbb{B})([(A \oplus [a], b)], F) \tag{1}$$

Note that in **Esp**, $\mathbb{A} \multimap (\mathbb{A} \Rightarrow \mathbb{B}) = \mathbb{A}^{op} \times !\mathbb{A}^{op} \times \mathbb{B} \;\cong\; !\mathbb{A}^{op} \times \mathbb{A}^{op} \times \mathbb{B} = \mathbb{A} \Rightarrow (\mathbb{A} \multimap \mathbb{B})$. So in fact all we have changed about the standard definition of differentiation is to swap the positions of a and A on the left-hand side of (1). In fact, for any categories $\mathbb{A}$ and $\mathbb{B}$, viewing differentiation as $\mathbb{A} \multimap (\mathbb{A} \Rightarrow \mathbb{B})$ rather than $\mathbb{A} \Rightarrow (\mathbb{A} \multimap \mathbb{B})$ turns out to be equivalent [15].

Linear evaluation

Since our new definition of differentiation introduces the linear hom, we will need a new operator for linear evaluation. For each $\mathbb{A}, \mathbb{B}$, define the linear evaluation species

$$\mathrm{lev}_{\mathbb{A},\mathbb{B}} : (\mathbb{A} \multimap \mathbb{B}) \sqcap \mathbb{A} \,!\!\rightarrow \mathbb{B}$$

by

$$\mathrm{lev}_{\mathbb{A},\mathbb{B}}(M)(b) = \iint^{F\in!(\mathbb{A}\multimap\mathbb{B}),\, a\in\mathbb{A}} !(\mathbb{A} \multimap \mathbb{B})([(a, b)], F) \times !((\mathbb{A} \multimap \mathbb{B}) \sqcap \mathbb{A})(F \otimes [a], M)$$

The following lemmas pertain to calculating two common combinations of these operators that will be of use when we interpret λ-terms as species in Chapter 5. The proofs of these lemmas uses several of the techniques for calculating coends presented in Appendix A.

Lemma 2.2. *For species* $\mathcal{P} : \Gamma \,!\!\to \mathbb{A} \Rightarrow \mathbb{B}$ *and* $\mathcal{Q} : \Gamma \,!\!\to \mathbb{A}$*, and* $X \in !\Gamma$*,* $b \in \mathbb{B}$*,*

$$ev \circ \langle \mathcal{P}, \mathcal{Q}\rangle(X)(b) \cong \iiint^{\substack{A\in !\mathbb{A}, H\in !\Gamma,\\ N\in !(\Gamma)^{|A|}}} \mathcal{P}(H)(A,b) \times \left[\prod_{k\in|A|} \mathcal{Q}(N_k)(A_k)\right] \times !\Gamma\left(H \oplus \bigoplus_{k\in|A|} N_k,\ X\right)$$

Proof. $\mathrm{ev} \circ \langle \mathcal{P}, \mathcal{Q}\rangle(X)(b)$

$$\begin{aligned}
&= \int^{F\in !((\mathbb{A}\Rightarrow\mathbb{B})\sqcap\mathbb{A})} \mathrm{ev}(F)(b) \times \langle \mathcal{P}, \mathcal{Q}\rangle^{\#}(X)(F)\\
&\cong \iint^{F\in !((\mathbb{A}\Rightarrow\mathbb{B})\sqcap\mathbb{A}),\, A\in !\mathbb{A}} !((\mathbb{A} \Rightarrow \mathbb{B}) \sqcap \mathbb{A})([(A,b)] \otimes A, F) \times \langle \mathcal{P}, \mathcal{Q}\rangle^{\#}(X)(F)\\
&\cong \int^{A\in !\mathbb{A}} \langle \mathcal{P}, \mathcal{Q}\rangle^{\#}(X)([(A,b)] \otimes A)\\
&\cong \iiint^{\substack{A\in !\mathbb{A}, H\in !\Gamma,\\ N\in (!\Gamma)^{|A|}}} \langle \mathcal{P}, \mathcal{Q}\rangle(H)(A,b) \times \left[\prod_{k\in|A|} \langle \mathcal{P}, \mathcal{Q}\rangle(N_k)(A_k)\right] \times !\Gamma\left(H \oplus \bigoplus_{k\in|A|} N_k,\ X\right)\\
&\cong \iiint^{\substack{A\in !\mathbb{A}, H\in !\Gamma,\\ N\in (!\Gamma)^{|A|}}} \mathcal{P}(H)(A,b) \times \left[\prod_{k\in|A|} \mathcal{Q}(N_k)(A_k)\right] \times !\Gamma\left(H \oplus \bigoplus_{k\in|A|} N_k,\ X\right)
\end{aligned}$$

□

Lemma 2.3. *For species* $\mathcal{P} : \Gamma \,!\!\to (\mathbb{A} \Rightarrow \mathbb{B})$ *and* $\mathcal{Q} : \Gamma \,!\!\to \mathbb{A}$*, and* $X \in !\Gamma$*,* $A \in !\mathbb{A}$*,* $b \in \mathbb{B}$*:*

$$lev \circ \langle D' \circ \mathcal{P}, \mathcal{Q}\rangle(X)(A,b) \cong \iint^{\substack{a\in\mathbb{A},\\ F_1,F_2\in !\Gamma}} \mathcal{P}(F_1)(A \oplus [a], b) \times \mathcal{Q}(F_2)(a) \times !\Gamma\,(F_1 \oplus F_2, X)$$

Proof. $\mathrm{lev} \circ \langle D' \circ \mathcal{P}, \mathcal{Q}\rangle(X)(A,b)$

$$\begin{aligned}
&= \int^{F\in !(\underline{lin}(A,(\mathbb{A}\Rightarrow\mathbb{B}))\sqcap\mathbb{A})} \mathrm{lev}(F)(A,b) \times \langle D' \circ \mathcal{P}, \mathcal{Q}\rangle^{\#}(X)(F)\\
&\cong \iint^{F\in !((\mathbb{A}\Rightarrow\mathbb{B})\sqcap\mathbb{A}),\ a\in\mathbb{A}} !((A \multimap (\mathbb{A} \Rightarrow \mathbb{B})) \sqcap \mathbb{A})([(a,A,b),a], F) \times \langle \mathcal{P}, \mathcal{Q}\rangle^{\#}(X)(F)\\
&\cong \int^{a\in\mathbb{A}} \langle \mathcal{P}, \mathcal{Q}\rangle^{\#}(X)([(a,A,b),a])\\
&\cong \iint^{\substack{a\in\mathbb{A},\\ F_1,F_2\in !\Gamma}} \langle D' \circ \mathcal{P}, \mathcal{Q}\rangle(F_1)(a,A,b) \times \langle D' \circ \mathcal{P}, \mathcal{Q}\rangle(F_2)(a) \times !\Gamma\,(\mathcal{F}_1 \oplus F_2,\ X)
\end{aligned}$$

$$\cong \iint^{\substack{a\in\mathbb{A},\\ F_1,F_2\in\,!\Gamma}} (D'\circ\mathcal{P})(F_1)(a,A,b)\times\mathcal{Q}(F_2)(a)\times\,!\Gamma\,(\mathcal{F}_1\oplus F_2,\ X)$$

$$\cong \iint^{\substack{a\in\mathbb{A},\\ F_1,F_2\in\,!\Gamma}} \mathcal{P}(F_1)(A\oplus[a],b)\times\mathcal{Q}(F_2)(a)\times\,!\Gamma\,(\mathcal{F}_1\oplus F_2,\ X)$$

□

Chapter 3

Esp as a model of linear logic

We've noted that Fiore's species of structure are indeed generalizations of Joyal's combinatorial species, and also that **Esp** is a Cartesian closed bicategory, making it a model of the simply typed λ-calculus. To see why generalizing combinatorial species should yield a category with a connection to computation, we can take a different perspective on **Esp** and compare it by analogy to the construction of **MRel**, the category of finite multisets and relations. Both **Esp** and **MRel** are formed via a construction that makes them models of linear logic, and as a result, Cartesian closed [12]. In fact, this construction guarantees that both **MRel** and **Esp** will be models of the differential λ-calculus, which will be presented in Chapter 4 [15]. This chapter assumes knowledge of linear logic; a good preliminary source is [5].

3.1 Relations and profunctors

In the category **Rel**, objects are sets and arrows are relations between sets, though we can view a relation $R : X \nrightarrow Y$ as a function $r : X \times Y \to \{0, 1\}$ where $r(x, y) = 1$ if xRy and $r(x, y) = 0$ otherwise. The function r indicates only whether two objects are related, and holds no other information about the nature of their relatedness.

Prof is the category whose objects are small categories and whose arrows are profunctors, or functors of the form $F : \mathcal{C} \times \mathcal{D}^{op} \to \mathbf{Set}$. We can think of a profunctor as a relation holding more information about the related objects: if objects c and d are not related then $F(c, d) = \emptyset$, and if c and d are related then the set $F(c, d)$ holds 'proofs' of their relatedness. The similarity of composition in **Rel** and **Prof** also lends to this analogy: the elements a and c are related by $A \xrightarrow{R} B \xrightarrow{S} C$ in **Rel** iff there is some $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$, while $(S \circ R)(a, c) = \int^{b \in \mathcal{B}} R(a, b) \times S(b, c)$ for profunctors $R : \mathcal{A} \times \mathcal{B}^{op} \to \mathbf{Set}$ and $S : \mathcal{B} \times \mathcal{C}^{op} \to \mathbf{Set}$.

This similarity arises from the fact that **Rel** is the Kleisli category of the powerset monad and **Prof** is, roughly speaking, the Kleisli bicategory of a presheaf 2-monad. (For more information on why this is not quite a monad, yet results in a similar structure, see [12].) As a result of these constructions, **Rel** and **Prof** each have the necessary structure to be extended into models of the full linear logic. This is done through another Kleisli construction, using a linear exponential comonad.

3.2 Making Rel and Prof into models of linear logic

To create a new category **MRel**, take the free commutative monoid monad, or the multiset monad, on **Rel**. By the duality of **Rel**, this monad is also a comonad. The multiset comonad maps a set X to $\mathcal{M}_f(X)$, the set of all finite multisets of X. A multiset (or 'bag') $m \in \mathcal{M}_f(X)$ will be represented as an unordered list $[x_1, ..., x_k]$ with all $x_i \in X$. The co-Kleisli category of this comonad is **MRel**, the category whose objects are sets and whose arrows $X \to Y$ are relations $\mathcal{M}_f(X) \nrightarrow Y$. Composition in **MRel** is given by

$$(m, c) \in R \circ S \iff \exists b_1, ..., b_k \in B \text{ and a partition of } m \text{ into } m_1, ..., m_k \text{ such that}$$
$$([b_1, ..., b_k], c) \in R \text{ and } (m_i, b_i) \in S \text{ for all } i, \text{ where } R : B \nrightarrow C \text{ and } S : A \nrightarrow B$$

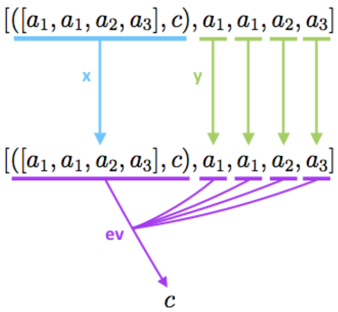
MRel is Cartesian closed, with the exponential $X \Rightarrow Y$ defined as $\mathcal{M}_f(X) \times Y$, the abstraction operator defined by

$$\Lambda(R) = \{(x, (y, z)) : (x \uplus y, z) \in R\} \text{ for } R \text{ a relation } X \times Y \nrightarrow Z$$

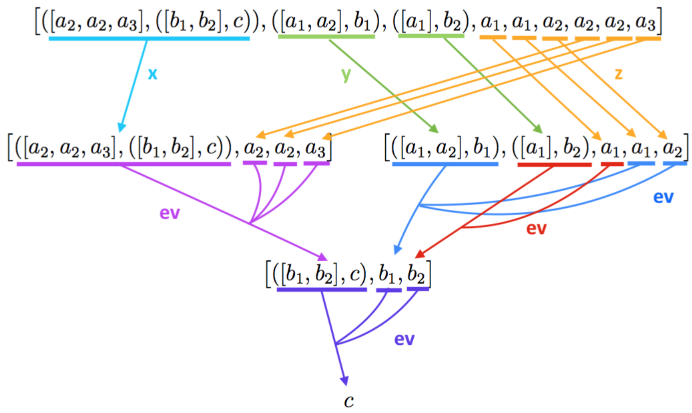
and the evaluation relation $\mathrm{ev}_{X,Y} : (X \Rightarrow Y) \times X \nrightarrow Y$ defined by

$$\mathrm{ev}_{X,Y} = \{((([(m, y)] \uplus m), y) : m \in \mathcal{M}_f(X), y \in Y\}.$$

This model resembles a system of resource trading: $(r, m) \in (\mathcal{M}_f(X) \Rightarrow Y) \,\&\, \mathcal{M}_f(X)$ is related to $b \in Y$ by application iff r is the 'rule' (m, b) encoding the ability to trade all the elements we have (consisting of the multiset m) for the element we want ('b'). Figure 3.1 illustrates how a bag of elements is related to an element c by the relations in **MRel** corresponding to the λ-terms xy and $xy(zy)$, respectively. The multiple instances of application in the latter term correspond to multiple trades taking place in order to obtain c.



(a) A bag and an element related by $[\![xy]\!]$.



(b) A bag and an element related by $[\![xz(yz)]\!]$.

Figure 3.1: In each diagram, the bag of elements at the top is related to the element at the bottom because there is a way to first assign each element a variable of an appropriate type, and then make a trade (or series of trades) according to the variable assignments and using rules inside the bag.

Just as we started with **Rel** and used a co-Kleisli construction to create a new category, we can use a similar (pseudo)-comonad on the bicategory **Prof** to obtain the bicategory **Esp** [9]. This time, the linear exponential comonad is the free symmetric monoidal completion pseudo-comonad ! defined in Chapter 2.2. The objects in the co-Kleisli bicategory of the ! pseudo-comonad are objects of **Prof** (small categories), and arrows $\mathbb{A} \to \mathbb{B}$ are arrows $!\mathbb{A} \nrightarrow \mathbb{B}$ in **Prof**. That is, arrows are profunctors of the form $!\mathbb{A} \times \mathbb{B}^{op} \to$ **Set**. Composition of arrows in the bicategory is also as defined in Chapter 2.3. Thus generalized species arise not only as general forms of combinatorial species of structure, but also as the co-Kleisli bicategory of the ! pseudo-comonad, analagous to the **MRel** construction that extends **Rel** to a model of linear logic. The analogy extends to the resource perspective of **MRel** illustrated above: **Esp** has also been studied for its connection to resource calculi [19].

Chapter 4

The differential λ-calculus

The λ-calculus is a notational system in which 'terms' representing computation can be formed through three constructions: variables, abstractions, and applications. Abstracting a variable from a term creates a function in that variable, while applying one term to another corresponds to feeding an argument into a function. However, the pure λ-calculus does not include restrictions on the use of resources, which in practice may be limited. Ehrhard and Regnier [7] developed the differential λ-calculus, an extension of the simply typed λ-calculus which integrates resource-awareness by introducing the ability to both linearly and fully apply terms. Linear application is important to managing finite resources, as it prevents an argument from duplicating when applied to an abstraction whose leading bound variable appears free as a subterm multiple times. In the differential λ-calculus, linear application is achieved by means of syntax for nondeterminism and differentiation, where differentiation is viewed as linear substitution into a function. We will alter the terminology of [7] slightly in presenting the specifics of the differential λ-calculus: while Ehrhard and Regnier originally formulated linear combinations of λ-terms as having coefficients from an arbitrary semiring, we follow the convention here of taking that semiring to be $\mathbb{N}$.

4.1 Syntax + reduction rules

The differential λ-calculus extends the syntax of the λ-calculus by allowing linear combinations of terms as well as adding a differentiation operator:

$$s, t := 0 \mid x \mid \lambda x.t \mid s\,t \mid s + t \mid Dt \cdot s$$

The two reduction rules in the differential λ-calculus are

$$(\lambda x.t)\, s \to t\,[s/x] \quad (\beta\text{-rule}) \qquad D(\lambda x.t) \cdot\, s \to \lambda x. \frac{\partial t}{\partial x} \cdot s \quad (D\text{-rule})$$

where $\frac{\partial t}{\partial x} \cdot s$ is defined inductively:

$$\frac{\partial y}{\partial x} \cdot t = \begin{cases} t & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \qquad \frac{\partial (s+u)}{\partial x} \cdot t = \frac{\partial s}{\partial x} \cdot t + \frac{\partial u}{\partial x} \cdot t$$

$$\frac{\partial 0}{\partial x} \cdot t = 0 \qquad \frac{\partial (s\,u)}{\partial x} \cdot t = \left(\frac{\partial s}{\partial x} \cdot t\right) u + \left(Ds \cdot \left(\frac{\partial u}{\partial x} \cdot t\right)\right) u$$

$$\frac{\partial (\lambda y.s)}{\partial x} \cdot t = \lambda y. \frac{\partial s}{\partial x} \cdot t \qquad \frac{\partial (Ds \cdot u)}{\partial x} \cdot t = D\left(\frac{\partial s}{\partial x} \cdot t\right) \cdot u + Ds \cdot \left(\frac{\partial u}{\partial x} \cdot t\right)$$

The new term 0 represents the empty sum, or the empty linear combination of terms. Linear combinations of terms roughly correspond to nondeterministic choice: thinking of terms s and t as short computer programs, the term $s + t$ represents a program that either computes s or computes t. However, these terms do not embody the choice itself, i.e. there is no reduction rule in the differential λ-calculus that mimics choice by reducing a term $s + t$ to s or to t. In keeping with this notion of nondeterministic choice, we would like to be able to identify certain terms containing sums. For instance, the term $\lambda x.s + t$ should act the same as $\lambda x.s + \lambda x.t$, and $(t + u)\, v$ should act like $t\, v + u\, v$. Later, when we interpret the differential λ-calculus in the bicategory of generalized species, we will see that these identifications indeed hold in the model.

The other new syntactical construction is the differentiation operator $Ds \cdot t$. Whereas the term st represents the application of a function s to an argument t, the new term $Ds \cdot t$ represents the *linear* application of s to t. In particular, if s is a function of x (i.e. $s \equiv \lambda x.u$), then $Ds \cdot t$ represents the substitution of t for *one* instance of x within the term u. The choice of which x to replace is what brings about the need for linear combinations of terms. The D-reduction rule implements the linear substitution of s for x into the term t; note the resemblance between the inductive definition of $\frac{\partial t}{\partial x} \cdot s$ and the rules governing derivatives in calculus.

4.2 Typing system

The typing rules for the differential λ-calculus are as follows:

$$\frac{\Gamma(x) = A}{\Gamma \vdash x : A} \qquad \frac{}{\Gamma \vdash 0 : A}$$

$$\frac{\Gamma \vdash s : A \text{ and } \Gamma \vdash t : A}{\Gamma \vdash s + t : A} \qquad \frac{\Gamma \vdash s : A \to B \text{ and } \Gamma \vdash t : A}{\Gamma \vdash s\, t : B}$$

$$\frac{\Gamma \vdash s : A \to B \text{ and } \Gamma \vdash t : A}{\Gamma \vdash Ds \cdot t : A \to B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}$$

In the typing system we will also allow for reordering of the context Γ, as well as typing a term in a context containing extraneous variables not appearing free in the term:

$$\frac{\Gamma, x : A, y : B \vdash t : C}{\Gamma, y : B, x : A \vdash t : C} \quad \text{(Exchange)}$$

$$\frac{\Gamma \vdash t : A \text{ and } x \notin \Gamma}{\Gamma, x : B \vdash t : A} \quad \text{(Weakening)}$$

In Chapter 2.3 we defined differentiation in **Esp** as an $((\mathbb{A} \Rightarrow \mathbb{B}), (\mathbb{A} \multimap (\mathbb{A} \Rightarrow \mathbb{B})))$-species. This is motivated by the behaviour of the differentiation operator in the differential λ-calculus. For any function $t : A \to B$, we can think of differentiation in the differential λ-calculus as an operator $Ds \cdot \_\_$ which takes in an argument t of type A and returns another function of type $A \to B$. It does this by linearly substituting t into the function s, so $Ds \cdot \_\_$ is in effect a function of type $A \multimap (A \to B)$. As such, we would like the species embodying differentiation, or linear substitution, to behave like the operator $D\_\_ \cdot \_\_ : (A \to B) \to (A \multimap (A \to B))$.

Chapter 5

Interpreting the differential λ-calculus in Esp

It is known that **Esp** is a model of the differential λ-calculus [8, 18], but the details of constructing such a model have not been expressly spelled out. Multiple methods have been provided for modeling the differential λ-calculus in differential categories using a derivative operator on morphisms [6, 4], but this construction is not immediately compatible with the differential structure presented in [8]. Here we use the newly introduced operator $\mathrm{lev}_{\mathbb{A},\mathbb{B}}$ and the altered definition of $D_{\mathbb{A},\mathbb{B}}$ to interpret the differential λ-calculus into **Esp**.

To demonstrate that the bicategory of generalized species forms a model of the simply typed differential λ-calculus, we will interpret each λ-term as a morphism in **Esp**, and each type and typing context as an object of **Esp**. The typing judgement of each term will be represented in the endpoints of the interpreted species: if context Γ gives the term t type A, then the interpretation of this typing judgement, denoted $[\![t]\!]^{\Gamma}_{A}$, will be a $([\![\Gamma]\!], [\![A]\!])$-species. For ease of notation, we will refer to $[\![\Gamma]\!]$ simply as Γ, and we will denote $[\![A]\!] := \mathbb{A}, [\![B]\!] := \mathbb{B}$, etc. In addition, if the type or typing context of t is unambiguous, we may omit the subscript and/or superscript of $[\![t]\!]^{\Gamma}_{A}$.

5.1 Interpreting types, contexts, and terms

We define interpretations of type inductively, beginning with atomic types:

- For an atomic type A, $[\![A]\!] = \mathbb{A}$ where $\mathbb{A}$ is some small category,
- $[\![A \to B]\!] = [\![A]\!] \Rightarrow [\![B]\!] = \,![\![A]\!]^{\mathrm{op}} \times [\![B]\!]$.

Next, we interpret typing contexts, beginning with the empty context:

- $[\![\emptyset]\!] = \mathbf{0}$, the empty category with no objects or arrows,
- $[\![\Gamma, x : A]\!] = [\![\Gamma]\!] \sqcap [\![A]\!]$, the disjoint union of the categories $[\![\Gamma]\!]$ and $[\![A]\!]$.

It may be that multiple variables in a context Γ are assigned the same type; for example, when $\Gamma = \{x : A, y : A, z : B\}$. To avoid confusion in these cases we will label each category with the variable from which it arose, so that the interpretation of Γ is denoted $[\![\Gamma]\!] = \mathbf{0} \sqcap \mathbb{A}_x \sqcap \mathbb{A}_y \sqcap \mathbb{B}_z$.

Finally, we interpret typing judgements of λ-terms. Interpretations of terms are defined inductively as follows:

- $[\![0]\!]^{\Gamma}_{A} = \mathcal{O} : [\![\Gamma]\!] \,!\!\to [\![A]\!]$, defined by $\mathcal{O}(X)(a) = \emptyset$ for all $X \in ! [\![\Gamma]\!]$ and $a \in [\![A]\!]$
- $[\![x]\!]^{\Gamma}_{A} = \pi_{n+1} : [\![\Gamma]\!] \;!\!\to\; [\![A]\!]$, where n is the index of the type assignment $x : A$ in the context Γ. (Note that the correct interpretation is π_{n+1} rather than π_n because the 1-indexed component of $[\![\Gamma]\!]$ is always the empty category $\mathbf{0}$.)

- $[\![\lambda x.t]\!]^{\Gamma}_{A\Rightarrow B} = \Lambda_{[\![A]\!]}([\![t]\!]^{\Gamma,x:A}_{B}) : [\![\Gamma]\!] \,!\!\to ([\![A]\!] \Rightarrow [\![B]\!])$
- $[\![s\,t]\!]^{\Gamma}_{B} = \mathrm{ev} \circ \langle [\![s]\!]^{\Gamma}_{A\to B}, [\![t]\!]^{\Gamma}_{A} \rangle : [\![\Gamma]\!] \,!\!\to [\![B]\!]$
- $[\![s+t]\!]^{\Gamma}_{A} = [\![s]\!]^{\Gamma}_{A} + [\![t]\!]^{\Gamma}_{A} : [\![\Gamma]\!] \,!\!\to [\![A]\!]$
- $[\![Dt \cdot s]\!]^{\Gamma}_{A\to B} = \mathrm{lev} \circ \langle D \circ [\![t]\!]^{\Gamma}_{A\to B}, [\![s]\!]^{\Gamma}_{A} \rangle : [\![\Gamma]\!] \,!\!\to ([\![A]\!] \Rightarrow [\![B]\!])$

For the purposes of working with our interpretation, it will be necessary to permute elements of a context, and to extend the context typing a term t to include variables that do not appear free in t. Suppose Γ has size k, and G is a bag in $!(\Gamma \sqcap \mathbb{B} \sqcap \mathbb{A})$. Let $G' \in !(\Gamma \sqcap \mathbb{A} \sqcap \mathbb{B})$ be the bag created by replacing all objects in G of the form $(b, k+1)$ with $(b, k+2)$, and replacing all objects of the form $(a, k+2) \in G$ with $(a, k+1)$. Then for all $c \in \mathbb{C}$, define $[\![t]\!]^{\Gamma,y:B,x:A}_{C}$ by

$$[\![t]\!]^{\Gamma,y:B,x:A}_{C}(G)(c) = [\![t]\!]^{\Gamma,x:A,y:B}_{C}(G')(c)$$

Finally, if $\Gamma \vdash t : B$ and $x \notin \mathrm{FV}(t)$, then

$$[\![t]\!]^{\Gamma,x:A}_{B} = [\![t]\!]^{\Gamma}_{B} \circ \pi_1 : [\![\Gamma]\!] \sqcap [\![A]\!] \,!\!\to [\![B]\!]$$

5.2 Soundness

This interpretation matches the standard interpretation of the simply typed λ-calculus into a CCC on ordinary λ-terms (those containing no linear or differential subterms), but as we have devised a new method of interpreting differential terms, we will need to show that this choice still gives a sound interpretation. In order for the interpretation to be sound, all statements that are true in the theory of the differential λ-calculus must also hold of the interpreted terms. That is, if $\Gamma \vdash t : A$ and $\Gamma \vdash s : A$ and $t =_{\beta D} s$, then $[\![t]\!]^{\Gamma}_{A}(X)(a) \cong [\![s]\!]^{\Gamma}_{A}(X)(a)$ for all $X \in !\Gamma$ and $a \in \mathbb{A}$.

Proposition 5.1. *For two differential λ-terms t and s, if $\Gamma \vdash t : A \to B$ and $\Gamma \vdash s : A$ then*

$$[\![(\lambda x.t)\,s]\!]^{\Gamma}_{B}(X)(b) \cong [\![t\,[s/x]]\!]^{\Gamma}_{B}(X)(b) \tag{$*$}$$

and

$$[\![D(\lambda x.t) \cdot s]\!]^{\Gamma}_{A\to B}(X)(A,b) \cong \left[\!\!\left[\lambda x.\frac{\partial t}{\partial x} \cdot s\right]\!\!\right]^{\Gamma}_{B}(X)(A,b) \tag{$\dagger$}$$

for all $X \in !\Gamma$, $A \in !\mathbb{A}$, and $b \in \mathbb{B}$.

To prove Proposition 5.1 we will first need several lemmas. The following two corollaries arise from Lemmas 2.2 and 2.3, respectively:

Corollary 5.2.

$$[\![ts]\!]^{\Gamma}_{B}(X)(b) \cong \iiint^{A\in !\mathbb{A},\, H\in !\Gamma,\, N\in !(\Gamma)^{|A|}} [\![t]\!]^{\Gamma}_{A\to B}(H)(A,b) \times \left[\prod_{k\in|A|} [\![s]\!]^{\Gamma}_{A}(N_k)(A_k)\right] \times !\Gamma(H \otimes \bigoplus_{k\in|A|} N_k,\, X)$$

Corollary 5.3.

$$\llbracket Dt \cdot s \rrbracket^{\Gamma}_{A\to B}(X)(A,b) \cong \iint^{a\in\mathbb{A},\ F_1,F_2\in !\Gamma} \llbracket t \rrbracket (F_1)(A\oplus[a],b) \times \llbracket s \rrbracket (F_2)(a) \times !\Gamma(F_1\oplus F_2, X)$$

Recall that we can define equivalence of bags for $G, G' \in !\Gamma$ by

$$G \equiv G' \text{ iff } G \text{ and } G' \text{ have the same contents.}$$

Lemma 5.4. *If $G \equiv G' \in !\Gamma$ and $\Gamma \vdash t : A$, then $\llbracket t \rrbracket^{\Gamma}_{A}(G)(a) \cong \llbracket t \rrbracket^{\Gamma}_{A}(G')(a)$ for all $a \in \mathbb{A}$.*

Lemma 5.5. *Let $\Gamma \vdash t : A$, and suppose that the typing context Γ contains an assignment $x : B$ such that x does not appear free in any subterm of t. If the bag $G \in ! \llbracket \Gamma \rrbracket$ contains any objects whose index in $\llbracket \Gamma \rrbracket$ is the index of x in Γ, then $\llbracket t \rrbracket^{\Gamma}_{A}(G)(a) = \emptyset$ for all $a \in \mathbb{A}$.*

Proofs of Lemmas 5.4 and 5.5, both by structural induction on t, are presented in Appendix B.

Lemma 5.6. *Let $P : \Gamma \,!\!\to \mathbb{A}$ and $Q : \Gamma \,!\!\to \mathbb{B}$ be generalized species and let $X \in !\Gamma$, $a \in \mathbb{A}$, and $b \in \mathbb{B}$. Then*

$$\begin{aligned}
&\int^{X_1,X_2\in !\Gamma} P(X_1\oplus[x])(a) \times Q(X_2)(b) \times !\Gamma(X_1\oplus X_2, X) \\
&+ \int^{X_1,X_2\in !\Gamma} P(X_1)(a) \times Q(X_2\oplus[x])(b) \times !\Gamma(X_1\oplus X_2, X) \\
&\cong \int^{X_1,X_2\in !\Gamma} P(X_1)(a) \times Q(X_2)(b) \times !\Gamma(X_1\oplus X_2, X\oplus[x])
\end{aligned}$$

Proof. We prove this lemma using Lemma A.4 twice: once in the forward direction and once backward.

$$\begin{aligned}
&\int^{X_1,X_2\in !\Gamma} P(X_1\oplus[x])(a) \times Q(X_2)(b) \times !\Gamma(X_1\oplus X_2, X) \\
&\qquad + \int^{X_1,X_2\in !\Gamma} P(X_1)(a) \times Q(X_2\oplus[x])(b) \times !\Gamma(X_1\oplus X_2, X) \\
&\cong \left[\sum_{\substack{(X_1,X_2)\\ \in\mathcal{D}(X)}} P(X_1\oplus[x])(a) \times Q(X_2)(b) \right] + \left[\sum_{\substack{(X_1,X_2)\\ \in\mathcal{D}(X)}} P(X_1)(a) \times Q(X_2\oplus[x])(b) \right] \\
&\cong \sum_{\substack{(X_1,X_2)\\ \in\mathcal{D}(X\oplus[x])}} P(X_1)(a) \times Q(X_2)(b) \\
&\cong \int^{X_1,X_2\in !\Gamma} P(X_1)(a) \times Q(X_2)(b) \times !\Gamma(X_1\oplus X_2, X\oplus[x])
\end{aligned}$$

□

Lemma 5.7. *If* $\Gamma, x : A \vdash t : B$ *and* $\Gamma \vdash s : A$ *then*

$$[\![(\lambda x.t)\, s]\!]_B^\Gamma (X)(b) \;\cong\; ([\![t]\!]_B^{\Gamma,x:A} \circ \langle Id_\Gamma, [\![s]\!]_A^\Gamma\rangle)(X)(b)$$

for all $X \in !\Gamma$ *and* $b \in \mathbb{B}$.

Proof.

$$\begin{aligned}
[\![(\lambda x.t)\, s]\!]_B^\Gamma (X)(b) &= (\mathrm{ev} \circ \langle [\![\lambda x.t]\!]_{A\to B}^\Gamma , [\![s]\!]_A^\Gamma\rangle)(X)(b) \\
&\cong (\mathrm{ev} \circ \langle \Lambda_{\mathbb{A}}([\![t]\!]_B^{\Gamma,x:A}) \circ \mathrm{Id}_\Gamma, \mathrm{Id}_{\mathbb{A}} \circ [\![s]\!]_A^\Gamma\rangle)(X)(b) \\
&\cong (\mathrm{ev} \circ \langle \Lambda_{\mathbb{A}}([\![t]\!]_B^{\Gamma,x:A}), \mathrm{Id}_{\mathbb{A}}\rangle \circ \langle \mathrm{Id}_\Gamma, [\![s]\!]_A^\Gamma\rangle)(X)(b) \\
&\cong ([\![t]\!]_B^{\Gamma,x:A} \circ \langle \mathrm{Id}_\Gamma, [\![s]\!]_A^\Gamma\rangle)(X)(b)\langle \mathrm{Id}_\Gamma, [\![s]\!]_A^\Gamma\rangle
\end{aligned}$$

□

We are now ready to prove that the interpretation is sound.

Proof of Proposition 5.1. By structural induction on the term t. When restricted to ordinary λ-terms, this interpretation is the standard interpretation of the simply typed λ-calculus into a CCC, so we can infer that $(*)$ holds when t is an ordinary λ-term. What remains is to show that $(*)$ holds for differential terms t, and that $(\dagger)$ holds for all terms t and s. First, we prove the inductive cases of $(*)$ when t is 0, a linear combination or a differentiation term.
case $t \equiv 0$:

$$\begin{aligned}
[\![0\,[s/x]]\!]_B^\Gamma (X)(b) &= [\![0]\!]_B^\Gamma (X)(b) \\
&= \emptyset \\
&= ([\![0]\!]_B^{\Gamma,x:A} \circ \langle \mathrm{Id}_\Gamma, [\![s]\!]_A^\Gamma\rangle)(X)(b) \\
&\cong [\![(\lambda x.0)\, s]\!]_B^\Gamma (X)(b)
\end{aligned}$$

case $t \equiv u + v$:

$$\begin{aligned}
[\![(u+v)[s/x]]\!]_B^\Gamma (X)(b) &\cong [\![u[s/x]]\!]_B^\Gamma (X)(b) + [\![v[s/x]]\!]_B^\Gamma (X)(b) \\
&\cong [\![(\lambda x.u)\, s]\!]_B^\Gamma (X)(b) + [\![(\lambda x.v)\, s]\!]_B^\Gamma (X)(b) \\
&\cong [\![((\lambda x.u) + (\lambda x.v))\, s]\!]_B^\Gamma (X)(b) \\
&\cong [\![(\lambda x.u+v)\, s]\!]_B^\Gamma (X)(b)
\end{aligned}$$

case $t \equiv Du \cdot v$:

$$\begin{aligned}
[\![(Du\cdot v)[s/x]]\!]_{B\to C}^\Gamma &= \mathrm{lev} \circ \langle D \circ [\![u[s/x]]\!]_{B\to C}^\Gamma , [\![v[s/x]]\!]_B^\Gamma\rangle \\
&\cong \mathrm{lev} \circ \langle D \circ [\![u]\!]_{B\to C}^{\Gamma,x:A} \circ \langle \mathrm{Id}_\Gamma, [\![s]\!]_A^\Gamma\rangle, [\![v]\!]_B^{\Gamma,x:A} \circ \langle \mathrm{Id}_\Gamma, [\![s]\!]_A^\Gamma\rangle\rangle \\
&\cong \mathrm{lev} \circ \langle D \circ [\![u]\!]_{B\to C}^{\Gamma,x:A} , [\![v]\!]_B^{\Gamma,x:A}\rangle \circ \langle \mathrm{Id}_\Gamma, [\![s]\!]_A^\Gamma\rangle \\
&= [\![Du\cdot v]\!]_{B\to C}^{\Gamma,x:A} \circ \langle \mathrm{Id}_\Gamma, [\![s]\!]_A^\Gamma\rangle \\
&\cong [\![(\lambda x.Du\cdot v)\, s]\!]_{B\to C}^\Gamma
\end{aligned}$$

Next, we need to show that $(\dagger)$ holds for all terms t and s, by structural induction on t. A full proof of $(\dagger)$ can be found in Appendix B; here, we will only present two cases to demonstrate the technique.

case $t \equiv x$: Let $|\Gamma| = n-1$, so that the index of x in $\Gamma, x : A$ is n. (Or, in the case that x is already typed in Γ, let the index of x in Γ be n.)

$$\llbracket D(\lambda x.x) \cdot s \rrbracket^{\Gamma}_{A \to A}(X)(A, a)$$

$$\cong \iint^{a^* \in \mathbb{A}, M_1, M_2 \in !\Gamma} \llbracket x \rrbracket^{\Gamma, x:A}_{A}(M_1 \otimes A \otimes [\coprod_n (a^*)])(a) \times \llbracket s \rrbracket^{\Gamma}_{A}(M_2)(a^*) \times !\Gamma(M_1 \oplus M_2, X) \tag{1}$$

$$\cong \iint^{a^* \in \mathbb{A}, M_1, M_2 \in !\Gamma} \pi_n(M_1 \otimes A \otimes [\coprod_n (a^*)])(a) \times \llbracket s \rrbracket^{\Gamma}_{A}(M_2)(a^*) \times !\Gamma(M_1 \oplus M_2, X) \tag{2}$$

$$\cong \iint^{a^* \in \mathbb{A}, M_1, M_2 \in !\Gamma} !(\Gamma \sqcap \mathbb{A})([\coprod_n (a)], M_1 \otimes A \otimes [\coprod_n (a^*)]) \times \llbracket s \rrbracket^{\Gamma}_{A}(M_2)(a^*) \times !\Gamma(M_1 \oplus M_2, X) \tag{3}$$

$$\cong \iint^{a^* \in \mathbb{A}, M_2 \in !\Gamma} !\mathbb{A}([a], A \otimes [a^*]) \times \llbracket s \rrbracket^{\Gamma}_{A}(M_2)(a^*) \times !\Gamma(M_2, X) \tag{4}$$

$$\cong \int^{a^* \in \mathbb{A}} !\mathbb{A}([a], A \otimes [a^*]) \times \llbracket s \rrbracket^{\Gamma}_{A}(X)(a^*) \tag{5}$$

$$\cong \begin{cases} \int^{a^* \in \mathbb{A}} \mathbb{A}(a, a^*) \times \llbracket s \rrbracket^{\Gamma}_{A}(X)(a^*) & \text{if } A = [\,] \\ \emptyset & \text{if } A \neq [\,] \end{cases} \tag{6}$$

$$\cong \begin{cases} \llbracket s \rrbracket^{\Gamma}_{A}(X)(a) = \llbracket s \rrbracket^{\Gamma, x:A}_{A}(X \otimes A)(a) & \text{if } A = [\,] \\ \emptyset & \text{if } A \neq [\,] \end{cases} \tag{7}$$

$$\cong \begin{cases} \lambda(\llbracket s \rrbracket^{\Gamma, x:A}_{A})(X)(A, a) & \text{if } A = [\,] \\ \emptyset & \text{if } A \neq [\,] \end{cases} \tag{8}$$

$$\cong \llbracket \lambda x.s \rrbracket^{\Gamma}_{A \to A}(X)(A, a) \tag{9}$$

$$\cong \left\llbracket \lambda x. \frac{\partial x}{\partial x} \cdot s \right\rrbracket^{\Gamma}_{A \Rightarrow A}(X)(A, a) \tag{10}$$

(1) Corollary 5.3; (2) Definition of $\llbracket x \rrbracket^{\Gamma, x:A}_{A}$; (3) Definition of π_n; (4) Lemma A.6; (5) and (7) Lemma A.2; (8)-(9) Lemma 5.5; (10) Definition of $\frac{\partial x}{\partial x} \cdot s$.

case $t \equiv Du \cdot v$:

$$\left[\!\left[\lambda x.\frac{\partial(Du\cdot v)}{\partial x}\cdot s\right]\!\right]^{\Gamma}_{A\to B\to C}(X)(A,B,c)$$

$$= \left[\!\left[\lambda x.D\left(\frac{\partial u}{\partial x}\cdot s\right)\cdot v + Du\cdot\left(\frac{\partial v}{\partial x}\cdot s\right)\right]\!\right]^{\Gamma}(X)(A,B,c) \tag{1}$$

$$\cong \left[\!\left[\lambda x.D\left(\frac{\partial u}{\partial x}\cdot s\right)\cdot v\right]\!\right]^{\Gamma}(X)(A,B,c) + \left[\!\left[\lambda x.Du\cdot\left(\frac{\partial v}{\partial x}\cdot s\right)\right]\!\right]^{\Gamma}(X)(A,B,c) \tag{2}$$

$$= \left[\!\left[D\left(\frac{\partial u}{\partial x}\cdot s\right)\cdot v\right]\!\right]^{\Gamma,x:A}_{B\to C}(X\otimes A)(B,c) + \left[\!\left[Du\cdot\left(\frac{\partial v}{\partial x}\cdot s\right)\right]\!\right]^{\Gamma,x:A}_{B\to C}(X\otimes A)(B,c) \tag{3}$$

$$\cong \iint^{\substack{b\in\mathbb{B},\\ M_1,M_2\in\,!(\Gamma\sqcap\mathbb{A})}} \left[\!\left[\frac{\partial u}{\partial x}\cdot s\right]\!\right]^{\Gamma,x:A}_{B\to C}(M_1)(B\oplus[b],c)\times[\![v]\!]^{\Gamma,x:A}_{B}(M_2)(b)\times\,!(\Gamma\sqcap\mathbb{A})(M_1\oplus M_2,X\otimes A)$$
$$+\iint^{\substack{b\in\mathbb{B},\\ M_1,M_2\in\,!(\Gamma\sqcap\mathbb{A})}} [\![u]\!]^{\Gamma,x:A}_{B\to C}(M_1)(B\oplus[b],c)\times\left[\!\left[\frac{\partial v}{\partial x}\cdot s\right]\!\right]^{\Gamma,x:A}_{B}(M_2)(b)\times!(\Gamma\sqcap\mathbb{A})(M_1\oplus M_2,X\otimes A) \tag{4}$$

$$\cong \sum_{\substack{(X_1,X_2)\in\mathcal{D}(X)\\(A_1,A_2)\in\mathcal{D}(A)}}\int^{b\in\mathbb{B}} \left[\!\left[\frac{\partial u}{\partial x}\cdot s\right]\!\right]^{\Gamma,x:A}_{B\to C}(X_1\otimes A_1)(B\oplus[b],c)\times[\![v]\!]^{\Gamma,x:A}_{B}(X_2\otimes A_2)(b)$$
$$+\sum_{\substack{X_1,X_2\in\mathcal{D}(X)\\A_1,A_2\in\mathcal{D}(A)}}\int^{b\in\mathbb{B}} [\![u]\!]^{\Gamma,x:A}_{B\to C}(X_1\otimes A_1)(B\oplus[b],c)\times\left[\!\left[\frac{\partial v}{\partial x}\cdot s\right]\!\right]^{\Gamma,x:A}_{B}(X_2\otimes A_2)(b) \tag{5}$$

$$\cong \sum_{\substack{(X_1,X_2)\in\mathcal{D}(X)\\(A_1,A_2)\in\mathcal{D}(A)}}\int^{b\in\mathbb{B}} [\![D(\lambda x.u)\cdot s]\!]^{\Gamma,x:A}_{B\to C}(X_1)(A_1,B\oplus[b],c)\times[\![v]\!]^{\Gamma,x:A}_{B}(X_2\otimes A_2)(b)$$
$$+\sum_{\substack{(X_1,X_2)\in\mathcal{D}(X)\\(A_1,A_2)\in\mathcal{D}(A)}}\int^{b\in\mathbb{B}} [\![u]\!]^{\Gamma,x:A}_{B\to C}(X_1\otimes A_1)(B\oplus[b],c)\times[\![D(\lambda x.v)\cdot s]\!]^{\Gamma,x:A}_{B}(X_2)(A_2,b) \tag{6}$$

$$\cong \sum_{\substack{(X_1,X_2)\in\mathcal{D}(X)\\(A_1,A_2)\in\mathcal{D}(A)\\(Y_1,Y_2)\in\mathcal{D}(X_1)}}\iint^{b\in\mathbb{B},\,a\in\mathbb{A}} [\![u]\!]^{\Gamma,x:A}_{B\to C}(Y_1\otimes A_1\otimes[a])(B\oplus[b],c)\times[\![s]\!]^{\Gamma,x:A}_{A}(Y_2)(a)\times[\![v]\!]^{\Gamma,x:A}_{B}(X_2\otimes A_2)(b)$$
$$+\sum_{\substack{(X_1,X_2)\in\mathcal{D}(X)\\(A_1,A_2)\in\mathcal{D}(A)\\(Y_1,Y_2)\in\mathcal{D}(X_1)}}\iint^{b\in\mathbb{B},\,a\in\mathbb{A}} [\![u]\!]^{\Gamma,x:A}_{B\to C}(X_1\otimes A_1)(B\oplus[b],c)\times[\![s]\!]^{\Gamma,x:A}_{A}(Y_1)(a)\times[\![v]\!]^{\Gamma,x:A}_{B}(Y_2\otimes A_2\otimes[a])(b) \tag{7}$$

$$\cong \sum_{\substack{(X_1,X_2,X_3)\\ \in\mathcal{D}(X\otimes A)}} \iint^{b\in\mathbb{B},\, a\in\mathbb{A}} [\![u]\!]^{\Gamma,x:A}_{B\to C}(X_1\otimes[a])(B\oplus[b],c)\times[\![s]\!]^{\Gamma,x:A}_{A}(X_2)(a)\times[\![v]\!]^{\Gamma,x:A}_{B}(X_3)(b)$$

$$+\sum_{\substack{(X_1,X_2,X_3)\\ \in\mathcal{D}(X\otimes A)}} \iint^{b\in\mathbb{B},\, a\in\mathbb{A}} [\![u]\!]^{\Gamma,x:A}_{B\to C}(X_1)(B\oplus[b],c)\times[\![s]\!]^{\Gamma,x:A}_{A}(X_2)(a)\times[\![v]\!]^{\Gamma,x:A}_{B}(X_3\otimes[a])(b) \quad (8)$$

$$\cong \iint^{a\in\mathbb{A},\, b\in\mathbb{B}} \sum_{\substack{(X_1,X_2,X_3)\\ \in\mathcal{D}(X\otimes A\otimes[a])}} [\![u]\!]^{\Gamma,x:A}_{B\to C}(X_1)(B\oplus[b],c)\times[\![s]\!]^{\Gamma,x:A}_{A}(X_2)(a)\times[\![v]\!]^{\Gamma,x:A}_{B}(X_3)(b) \quad (9)$$

$$\cong \sum_{(X_1,X_2)\in\mathcal{D}(X)} \int^{a\in\mathbb{A}} [\![Du\cdot v]\!]^{\Gamma,x:A}_{B\to C}(X_1\otimes A\otimes[a])(B\oplus[b],c)\times[\![s]\!]^{\Gamma,x:A}_{A}(X_2)(a) \quad (10)$$

$$\cong [\![D(\lambda x.Du\cdot v)\cdot s]\!]^{\Gamma}_{A\to B\to C}(X)(A,B,c) \quad (11)$$

(1) Definition of $\frac{\partial(Du\cdot v)}{\partial x}\cdot s$; (2)-(3) by the interpretation; (4) Corollary 5.3; (5) Lemma A.4; (6) Inductive hypothesis; (7) Corollary 5.3; (8) Rewriting sums and Lemma 5.4; (9) Lemma 5.6; (10)-(11) Corollary 5.3 and Lemma A.4.

□

5.3 Examples

Now that we have a sound interpretation of the differential λ-calculus in **Esp**, we can calculate some examples of interpreted terms to motivate a combinatorial description of the interpretation. The results of these sample calculations are unions and products of hom-sets, and as such are not particularly enlightening at first glance, but we will revisit them in a more visually intuitive form after we have established the combinatorial interpretation in Chapter 6.

Example 5.8 (Variable)**.** Suppose $\Gamma\vdash x : A$ such that the index of x in Γ is n, and let $G\in !\Gamma$ and $a\in\mathbb{A}$.

$$\begin{aligned}[\![x]\!]^{\Gamma}_{A}(G)(a) &= \pi_n(G)(a)\\ &= !\Gamma([\coprod_n(a)],G)\\ &\cong \begin{cases}\mathbb{A}(a,a^*) & \text{if } G=[a^*] \text{ where the index of } a^* \text{ in } \Gamma \text{ is } n\\ \emptyset & \text{otherwise}\end{cases}\end{aligned}$$

Example 5.9 (Application)**.** Let $\Gamma=\Gamma', x : A\to B, y : A$, and let $G\in !\Gamma$ and $b\in\mathbb{B}$. By Lemma 5.5, $[\![xy]\!]^{\Gamma}_{B}(G)(b)=\emptyset$ if G contains any objects from $[\![\Gamma']\!]$. So we can say $G\equiv G_x\otimes G_y$, where G_x is the sublist of G containing exactly the objects from $(\mathbb{A} \Rightarrow \mathbb{B})$ and G_y is the sublist of G containing exactly the objects from $\mathbb{A}$. Then

$$\begin{aligned}
\llbracket xy \rrbracket_B^\Gamma (G)(b) &\cong \iiint^{A\in !\mathbb{A}, H \in !\Gamma, N\in !\Gamma^{|A|}} \llbracket x \rrbracket_{A\to B}^\Gamma (H)(A,b) \times \left[\prod_{k\in|A|} \llbracket y \rrbracket_A^\Gamma (N_k)(A_k)\right] \times !\Gamma(H \otimes \bigoplus_{k\in|A|} N_k, G) \\
&\cong \iiint^{A\in !\mathbb{A}, H \in !\Gamma, N\in !\Gamma^{|A|}} !\Gamma([(A,b)], H) \times \left[\prod_{k\in|A|} !\Gamma(A_k, N_k)\right] \times !\Gamma(H \otimes \bigoplus_{k\in|A|} N_k, G) \\
&\cong \int^{A\in !\mathbb{A}} !\Gamma([(A,b)] \otimes \bigoplus_{k\in|A|} [A_k], G) \\
&\cong \int^{A\in !\mathbb{A}} !\Gamma([(A,b)] \otimes A, G) \\
&\cong \int^{A\in !\mathbb{A}} !(\mathbb{A} \Rightarrow \mathbb{B})([(A,b)], G_x) \times !\mathbb{A}(A, G_y) \\
&\cong !(\mathbb{A} \Rightarrow \mathbb{B})([(G_y, b)], G_x)
\end{aligned}$$

From this calculation we can observe a few things: first, that $\llbracket xy \rrbracket_B^\Gamma (G)(b) = \emptyset$ unless $|G_x| = 1$. So we can limit our calculations specifically to the instance

$$\llbracket xy \rrbracket_B^\Gamma ([(A^*, b^*)] \otimes G_y)(b)$$

for some $\mathbb{A}^* \in !\mathbb{A}^{op}$ and $b^* \in \mathbb{B}$:

$$\begin{aligned}
\llbracket xy \rrbracket_B^\Gamma ([(A^*, b^*)] \otimes G_y)(b) &\cong !(\mathbb{A} \Rightarrow \mathbb{B})([(G_y, b)], [(A^*, b^*)]) \\
&\cong !\mathbb{A}(A^*, G_y) \times \mathbb{B}(b, b^*)
\end{aligned}$$

Example 5.10 (Differentiation)**.** Again,let $\Gamma = \Gamma', x : A \to B, y : A$ and $G \in !\Gamma$ and $b \in \mathbb{B}$. Then

$$\begin{aligned}
\llbracket Dx \cdot y \rrbracket_{A\to B}^\Gamma (G)(A,b) &\cong \iint^{F_1, F_2 \in !\Gamma, a\in\mathbb{A}} \llbracket x \rrbracket_{A\to B}^\Gamma (F_1)(A \oplus [a], b) \times \llbracket y \rrbracket_A^\Gamma (F_2)(a) \times !\Gamma(F_1 \oplus F_2, G) \\
&\cong \iint^{F_1, F_2 \in !\Gamma, a\in\mathbb{A}} !\Gamma([(A \oplus [a], b)], F_1) \times !\Gamma([a], F_2) \times !\Gamma(F_1 \oplus F_2, G) \\
&\cong \int^{a\in\mathbb{A}} !\Gamma([(A \oplus [a], b), a], G) \\
&\cong \int^{a\in\mathbb{A}} !(\mathbb{A} \Rightarrow \mathbb{B})([(A \oplus [a], b)], G_x) \times !\mathbb{A}([a], G_y)
\end{aligned}$$

This calculation tells us that $[\![Dx \cdot y]\!]^{\Gamma}_{A \to B}(G)(A, b) = \emptyset$ unless $|G_x| = |G_y| = 1$. Limiting ourselves to the scenario where $G_y = [a^*]$ and $G_x = [(A^*, b^*)]$ for $a^* \in \mathbb{A}$, $A^* \in !\mathbb{A}$, and $b^* \in \mathbb{B}$, we get

$$\begin{aligned}
[\![Dx \cdot y]\!]^{\Gamma}_{A \to B}(G)(A, b) &\cong \int^{a \in \mathbb{A}} !(\mathbb{A} \Rightarrow \mathbb{B})([(A \oplus [a], b)], [(A^*, b^*)]) \times !\mathbb{A}([a], [a^*]) \\
&\cong \int^{a \in \mathbb{A}} !(\mathbb{A} \Rightarrow \mathbb{B})([(A \oplus [a], b)], [(A^*, b^*)]) \times \mathbb{A}(a, a^*) \\
&\cong !(\mathbb{A} \Rightarrow \mathbb{B})([(A \oplus [a^*], b)], [(A^*, b^*)]) \\
&\cong !\mathbb{A}(A^*, A \oplus [a^*]) \times \mathbb{B}(b, b^*)
\end{aligned}$$

Example 5.11. Let $\Gamma = \{x : (A \to A) \to B\}$, $G \in !\Gamma$ and $b \in \mathbb{B}$.
$[\![x(\lambda y.y)]\!]^{\Gamma}_{B}(G)(b)$

$$\begin{aligned}
&\cong \iiint^{F \in !(\mathbb{A} \Rightarrow \mathbb{A}), H \in !\Gamma, N \in !\Gamma^{|F|}} [\![x]\!]^{\Gamma}(H)(F, b) \times \left[\prod_{k \in |F|} [\![\lambda y.y]\!]^{\Gamma}(N_k)(F_k)\right] \times !\Gamma(H \otimes \bigoplus_{k \in |F|} N_k, G) \\
&\cong \iint^{F \in !(\mathbb{A} \Rightarrow \mathbb{A}), N \in !\Gamma^{|F|}} \left[\prod_{k \in |F|} [\![y]\!]^{\Gamma, y:A}(N_k \otimes A_k)(a_k)\right] \times !\Gamma([(F, b)] \otimes \bigoplus_{k \in |F|} N_k, G) \quad \text{where } F_k = (A_k, a_k) \\
&\cong \iint^{F \in !(\mathbb{A} \Rightarrow \mathbb{A}), N \in !\Gamma^{|F|}} \sum_{(F^*, b^*) \in G} \sum_{\substack{(G_1, \ldots, G_{|F|}) \\ \in \mathcal{D}(G - (F^*, b^*))}} \left[\prod_{k \in |F|} !\Gamma([a_k], N_k \otimes A_k) \times !\Gamma(N_k, G_k)\right] \times \Gamma((F, b), (F^*, b^*)) \\
&\cong \sum_{(F^*, b^*) \in G} \sum_{\substack{(G_1, \ldots, G_{|F^*|}) \\ \in \mathcal{D}(G - (F^*, b^*))}} \left[\prod_{k \in |F^*|} !\Gamma([a^*_k], G_k \otimes A^*_k)\right] \times \mathbb{B}(b, b^*) \quad \text{where } F^*_k = (A^*_k, a^*_k)
\end{aligned}$$

In Examples 5.8-5.11 an interpreted λ-term maps two inputs to a disjoint union of products of hom-sets, where each component of each input appears exactly once in the expression (though perhaps split up, as in the object (F^*, b^*) being split into the parts A^*_k, a^*_k, and b^* in Example 5.11). This pattern suggests that interpreted terms can be visualized as sets of structures built up out of all the pieces of the input objects, connected together by arrows. Rather, looking at objects as vertices and arrows as edges between them, we can view these sets as sets of graphs.

Chapter 6

A combinatorial view of the interpretation

To visualize the interpretation of differential λ-terms in **Esp** from Chapter 5.1, we will view $[\![t]\!]_A^\Gamma(G)(a)$ as a set of rooted, directed graphs, each of which has the same vertex set $V(G,a)$ as well as a 'shape' corresponding to the term t. The edges of each graph will be arrows in the category $[\![\Gamma]\!]$. To establish that this is an accurate description, we will first define a new translation of terms $\langle t\rangle_A^\Gamma : [\![\Gamma]\!] \,!\!\rightarrow [\![\mathbb{A}]\!]$ (where types and contexts are interpreted in the same way as in the categorical interpretation) and then prove that $\langle t\rangle_A^\Gamma$ and $[\![t]\!]_A^\Gamma$ are naturally isomorphic whenever $\Gamma \vdash t : A$.

6.1 Objects of $!\Gamma$ as vertices

In the graphs we will be defining for the combinatorial interpretation, objects of the input list G are vertices. Recall that $[\![\Gamma]\!] = \sqcap_{i\in|\Gamma|}[\![T_i]\!]$ where $\Gamma = \{x_1 : T_1, ..., x_{|\Gamma|} : T_{|\Gamma|}\}$. This means that each object in the list G arises from $[\![T_i]\!]$ for some i. For organizational purposes, we will associate each vertex in the tree with a variable and an index. The variable associated with $g \in G$ is the aforementioned x_i such that g arises from the category $[\![T_i]\!]$. The index of a vertex is its position in the list G.

Definition 6.1. Suppose $T_1, ..., T_n$ are types, and T_n is atomic. For an object

$$a = (A_1, ..., A_{n-1}, a_n) \in [\![T_1 \to T_2 \to ... \to T_n]\!],$$

the vertex set $V(G,a)$ is the multiset underlying the list $G \otimes A_1 \otimes \ldots \otimes A_{n-1} \otimes [a_n]$. The vertex a_n will be the root of each graph in $\langle t\rangle_A^\Gamma(G)(a)$.

Remark. Any type T can be written in a unique way as $T_1 \to T_2 \to \ldots \to T_n$ where T_n is atomic (though $T_1, ..., T_{n-1}$ need not be atomic types), so $V(G,a)$ is well-defined. Also note that $V(G,a)$ is not dependent upon which term t is being interpreted.

Since every object in the input bag G will appear as a vertex in every graph in $\langle t\rangle_A^\Gamma(G)(a)$, we need a method of linking together objects via edges. In order to describe how vertices may be connected by edges, it will be helpful to visualize each vertex as containing an *internal tree* whose root and leaves are ports by which it can be connected to other vertices. To distinguish between internal trees and external edges when drawing an element of $\langle t\rangle_A^\Gamma(G)(a)$, we will delineate the vertices from G by containing them in circles.

The internal structure of a vertex in G is dependent on the type of its associated variable. For an atomic type A, an object $a \in [\![A]\!]$ will be represented by the tree consisting of the single node a, while an object $(A,b) \in (\mathbb{A} \Rightarrow \mathbb{B})$ can either be represented as a tree with a single node

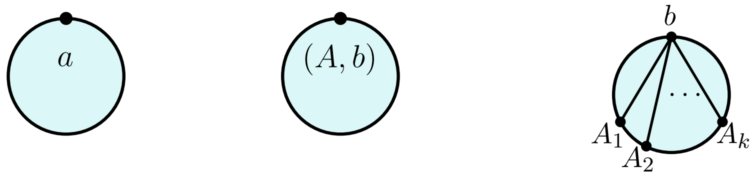


Figure 6.1: Nodes representing the objects $a \in \mathbb{A}$ (left) and $(A, b) \in \mathbb{A} \Rightarrow \mathbb{B}$ (middle and right).

(A, b) or as a tree consisting of the root node b whose children are the members of the list A (Fig. 6.1).

The choice of how to represent a vertex of type $\mathbb{A} \Rightarrow \mathbb{B}$ relates to the mechanism for connecting vertices. A vertex (A, b) can link to a vertex a via an arrow $f : A_i \to a$ (Fig. 6.2a), or it can connect to a vertex whose leaves have type $\mathbb{A} \Rightarrow \mathbb{B}$ (Fig. 6.2b).

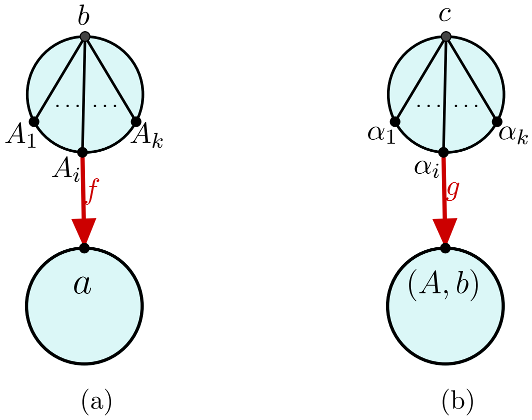


Figure 6.2: Two ways to connect the node (A, b) to another node. In (a), (A, b) is the parent vertex linked to child node a by the arrow $f : A_i \to a$. In (b), (A, b) is the child of the vertex $(\alpha, c) \in (\mathbb{A} \Rightarrow \mathbb{B}) \Rightarrow \mathbb{C}$, connected by the arrow $g : \alpha_i \to (A, b)$.

Note that while we may draw the children $A_1, ..., A_k$ of a root node b in a different order than that of the list A, we will keep track of each child's index in the list A, as it is a relevant part of the vertex's structure and will affect the action of morphisms on the graph. Drawing the children in a different order than index order, however, does not change the underlying combinatorial object.

The two methods of considering the internal tree of the vertex (A, b) seem like distinct cases, but they are simply different ways of visualizing the fact that every vertex v is connected to the rest of the graph by a single arrow of the form $f : v^* \to v$. The one exception to this rule is the root node r, in which case the arrow will be of the form $f : r \to r^*$. In the case of Fig. 6.2b the vertex (A, b) is connected to the graph by the arrow $g : \alpha_i \to (A, b)$. Figure 6.3 extends the graph in Figure 6.2a by connecting all open ports in the (A, b) node to other vertices. In this diagram, the vertex (A, b) is connected to the rest of the graph by the arrow $g : ([a_1, ..., a_k], b^*) \to (A, b)$ which consists of $h : b^* \to b$ and a collection of arrows $f_i : A_i \to a_{\sigma_i}$ for $i \in k = |A|$ and some permutation $\sigma \in S_k$.

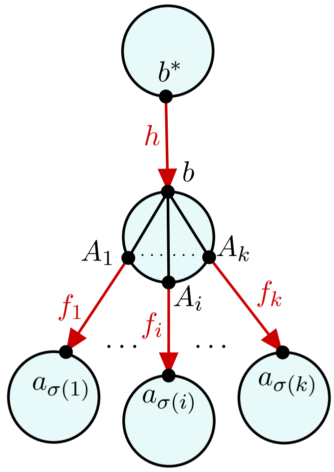


Figure 6.3

Similarly, in the case that an object $g \in G$ has an associated variable with type $\mathbb{A} \Rightarrow \mathbb{B} \Rightarrow \mathbb{C}$, that object can be portrayed by any of the following vertices:

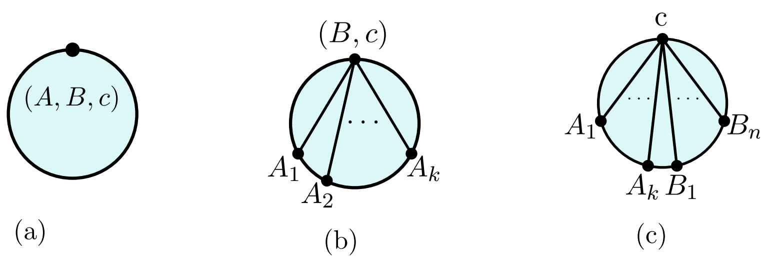


Figure 6.4: Three vertices representing the object $(A, B, c) \in (\mathbb{A} \Rightarrow \mathbb{B} \Rightarrow \mathbb{C})$

Finally, we will impose rules on the construction of the graphs in $\langle t \rangle_A^\Gamma(G)(a)$ built up out of these vertices:

Definition 6.2. A graph on the vertex set $V(G, a)$ is *well-formed* if it is connected and every port of each vertex is linked to a port of another vertex by an arrow in $!\Gamma$. That is, the degree of each vertex is one more than the number of leaves in its internal tree, and all edges connect two ports of the same type.

6.2 Grafting + the action of morphisms in $!\Gamma$ and $\mathbb{A}$ on graphs

An operation we will be using to combine and transform graphs is *grafting*, which consists of removing a vertex in a graph and replacing that vertex with another graph. Figure 6.5 demonstrates a simple example of grafting a tree S into a tree T at the vertex v. Note that the number of children of the vertex being replaced must match the number of leaves in the tree being inserted.

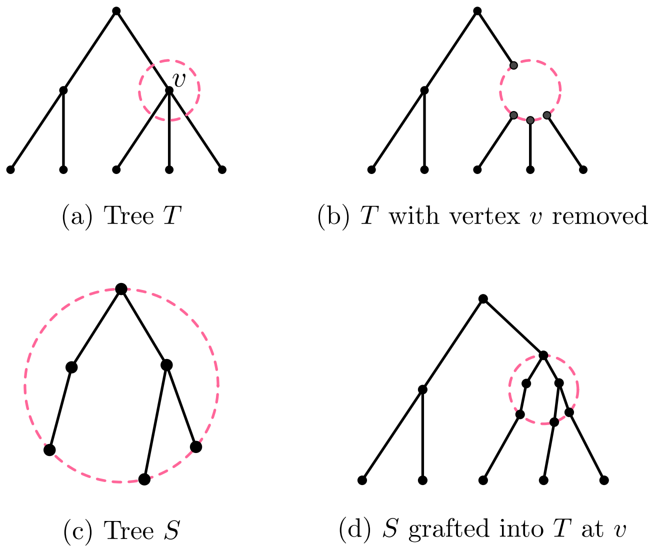


Figure 6.5: The process of grafting a tree S into the tree T at vertex v.

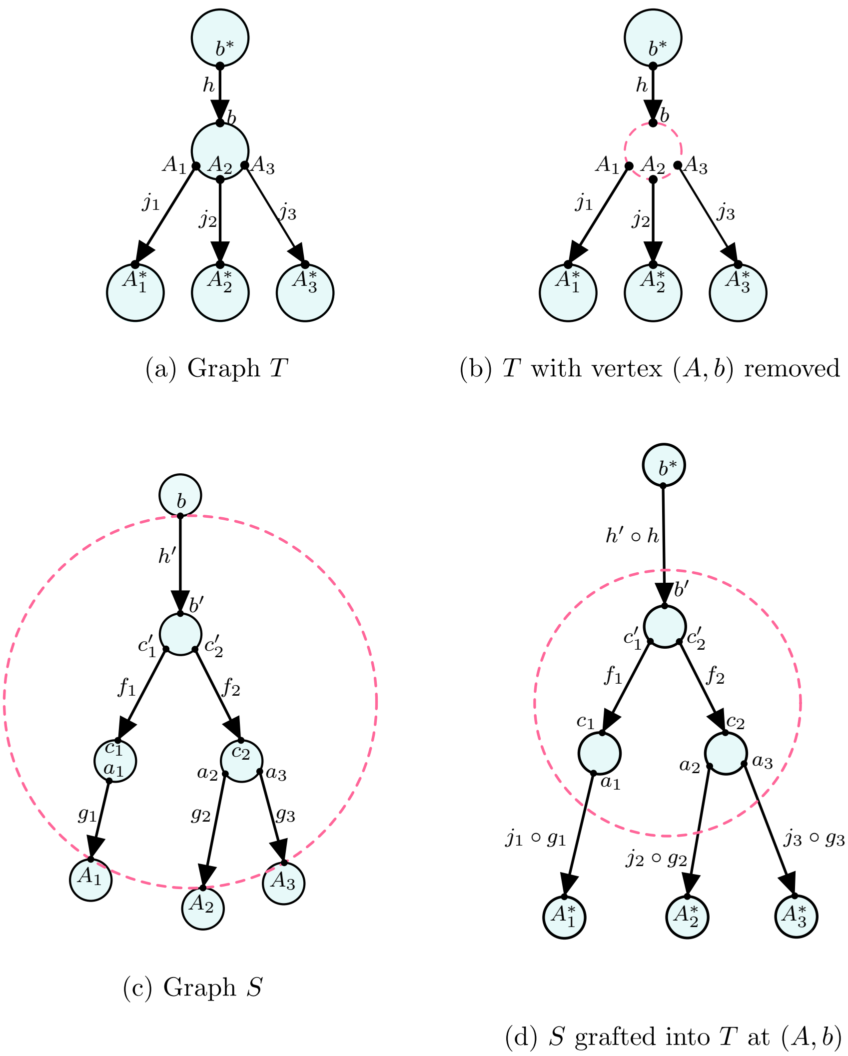
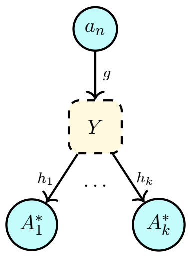


Figure 6.6: Grafting S into T.

For the graphs in $\langle t\rangle_A^\Gamma(G)(a)$, whose edges are arrows in Γ, the grafting process requires composition of arrows. Figure 6.6 restates the grafting example from Figure 6.5 in terms of graphs made up of objects from Γ.

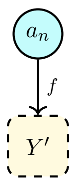
Since the elements of $\langle t\rangle_A^\Gamma(G)(a)$ will be graphs, we will need to understand how morphisms in $!\Gamma$ and $\mathbb{A}$ act on these graphs. To do so, we combine the notion of objects as vertices and the concept of grafting described above. A property of $\langle t\rangle_A^\Gamma(G)(a)$ is that all graphs in the set include an arrow from a into the graph, and for all $g \in \mathcal{G}$ each graph includes an arrow from the rest of the graph to g.

Proposition 6.3. *Suppose $G \in !\Gamma$ and $a = (A_1, ..., A_{n-1}, a_n) \in \mathbb{A}_1 \Rightarrow \ldots \Rightarrow \mathbb{A}_n$. Then for any well-formed graph T on the vertex set $V(G, a)$ with a_n as the root, there is a graph Y such that T can be written as*



where $A^* = A_1 \otimes \ldots \otimes A_{n-1}$ and $k = |A^*|$.

Proof. Because a_n is the root node of the graph, T can be written as



where Y' is the rest of the graph. This means that all of the objects in the list A^* are vertices inside Y'. (Note: here we are using the fact that T is well-formed, so all ports of the vertex A_i^* are connected to other nodes in the graph.) Suppose the variable associated with vertex A_i^* has type $B_1 \to \ldots \to B_n$. Recall that $\mathbb{B}_1 \Rightarrow \ldots \mathbb{B}_m = !\mathbb{B}_1^{op} \times \ldots \times !\mathbb{B}_{m-1}^{op} \times \mathbb{B}_m$, so an arrow

$$j : (B'_1, ..., B'_{m-1}, b'_m) \to (B_1, ..., B_{m-1}, b_m)$$

consists of an arrow $g : b'_m \to b_m$ and an arrow $h_i : B_i \to B'_i$ for each $i \in m-1$. Breaking down the latter arrows further, each h_i consists of a permutation $\sigma_i : |B_i| \to |B_i|$ and a collection of arrows $(h_i)_k : (B_i)_{\sigma(k)} \to (B'_i)_k$. Take all the edges (arrows) connecting the A_i^* vertex to the rest of the graph and consolidate them into a single arrow $j : \alpha \to A_i^*$ (Figure 6.7).

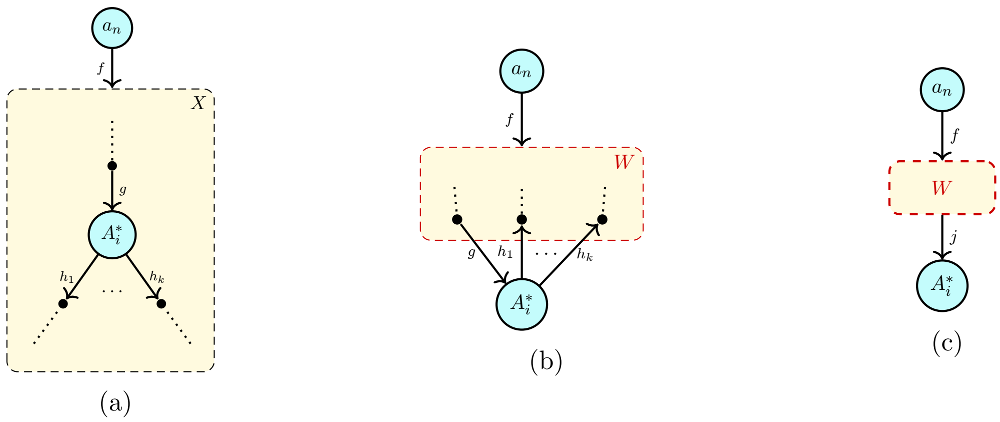
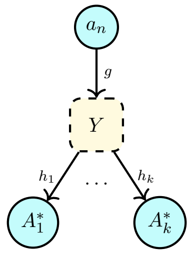


Figure 6.7: 'Extracting' a vertex A_i^* from the interior of the graph. (a) Isolate the vertex within the graph, (b) Rearrange the graph, (c) Consolidate the arrows linking A_i^* into the graph.

Repeating this process with A_i^* for each $i \in |A^*|$ results in a shape of the form



Note: in the case that any of the vertices A_i^* and A_j^* are adjacent to each other, or to the root node a_n in the graph Y, we must modify the graph slightly before extracting each vertex. Figure 6.8 demonstrates the process of modifying a graph so that the nodes A_i^* can be extracted.

□

Suppose $a = (A_1, ..., A_{n-1}, a_n)$ and $a' = (A'_1, ..., A'_{n-1}, a'_n)$, and let $\alpha = A_1 \otimes \ldots \otimes A_{n-1}$ and $\alpha' = A'_1 \otimes \ldots \otimes A'_{n-1}$. Given an arrow $f : a \to a'$, f acts on a graph $S_{a'} \in \langle s \rangle_A^\Gamma(G)(a')$ to create a new graph $S_a \in \langle s \rangle_A^\Gamma(G)(a)$. By Proposition 6.3 there is a graph Y such that $S_{a'}$ can be written as the graph in Figure 6.9(a). Transform $S_{a'}$ to S_a by composing this graph with f. The arrow $f : a \to a'$ consists of an arrow $g : a_n \to a'_n$ and a family of arrows $h_i : \alpha'_{\sigma(i)} \to \alpha_i$ where σ is a permutation of $|\alpha|$. In this case, transforming $S_{a'}$ to S_a results in Figure 6.9(b).

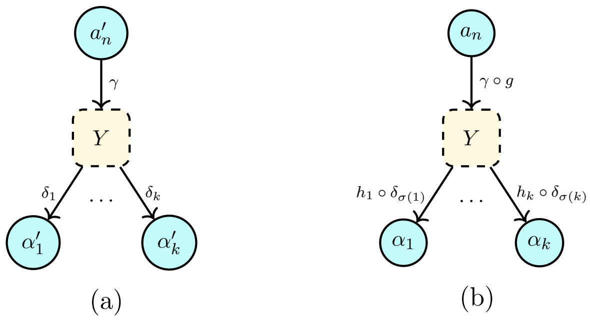


Figure 6.9

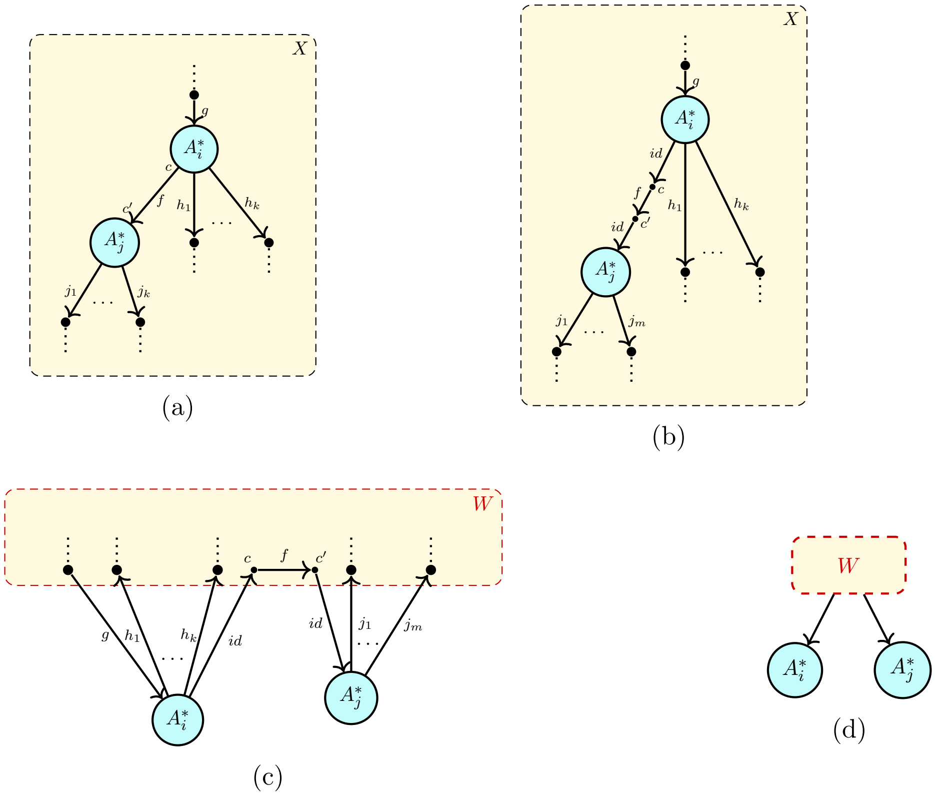


Figure 6.8: Extracting adjacent nodes A_i^* and A_j^*. (a) A_i^* and A_j^* are connected by an arrow $f : c \to c'$; (b) Add placeholder nodes c and c' linked by f so that the vertices A_i^* and A_j^* are now connected to c and c', respectively, by id arrows; (c) Rearrange the graph; (d) Consolidate the arrows linking A_i^* and A_j^* to the new shape W. When this graph is grafted into another graph, the placeholder nodes will be removed and the edges composed.

Similarly, given any arrow $f : a \to a'$ and a graph T containing the vertex a in any position other than the root, f acts on T by transforming it into a graph T' containing a' in the place of a. This happens by first isolating the node a inside the graph, then replacing it with the graph constructed by composing the vertex a' with the arrow f (Fig. 6.10(b)). Call this new vertex $f(a)$ and graft it into T at the vertex a.

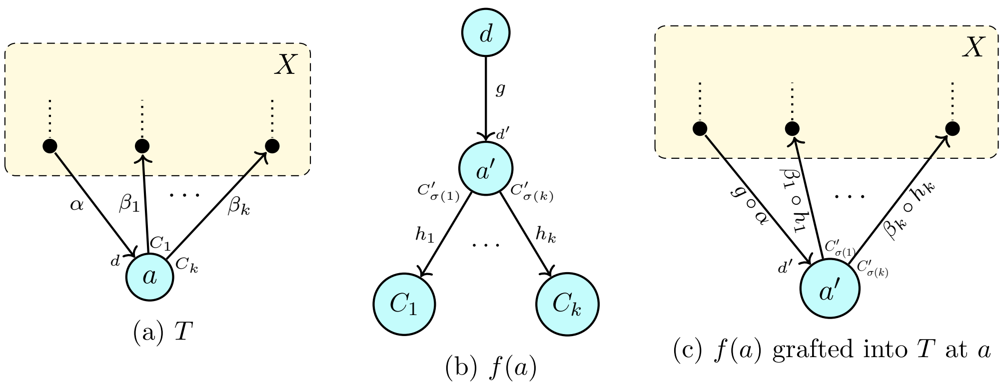


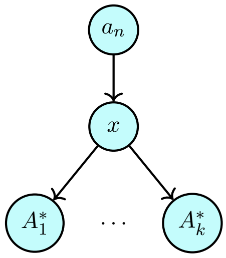
Figure 6.10

Likewise, an arrow $f : G \to G'$ in $!\Gamma$ is a permutation $\sigma : |G| \to |G|$ and a family of arrows $f_i : G_i \to G'_{\sigma(i)}$, so f acts on a graph in $\langle t \rangle_A^\Gamma(G)(a)$ containing the objects of G as interior vertices by iteratively grafting $f_i(G_i)$ into the graph in place of G_i for each i.

6.3 Reinterpreting terms as sets of graphs

Definition 6.4. The combinatorial interpretation of λ-terms is defined inductively:

- $\langle 0\rangle_A^\Gamma(G)(a)$ is the set of empty graphs that can be made on the vertex set $V(G,a)$.
- $\langle x\rangle_A^\Gamma(G)(a)$ is the set of trees of the shape

  that can be made out of the vertex set $V(G,a)$, where $a = (A_1, A_2, ..., A_{n-1}, a_n)$ and $A^* = A_1 \otimes A_2 \otimes \ldots \otimes A_{n-1}$ and $|A^*| = k$.
- $\langle s+t\rangle_A^\Gamma(G)(a)$ is the disjoint union of the sets $\langle s\rangle_A^\Gamma(G)(a)$ and $\langle t\rangle_A^\Gamma(G)(a)$.
- $\langle \lambda x.t\rangle_{A\to B}^\Gamma(G)(A,b)$ is defined as the set $\langle t\rangle_B^{\Gamma,x:A}(G\otimes A)(b)$.
- $\langle Ds\cdot t\rangle_{A\to B}^\Gamma(G)(A,b)$ is the set of graphs on the vertex set $V(G,(A,b))$ that can be constructed by the following process:

  1. Partition the list G into G_1 and G_2,
  2. Choose an object $a \in \mathbb{A}$,
  3. Pick a graph T from $\langle t\rangle_{A\to B}^\Gamma(G_1)(A\oplus[a],b)$,
  4. Pick a graph S from $\langle s\rangle_A^\Gamma(G_2)(a)$,
  5. Graft S into T at the vertex a.

- $\langle st\rangle_B^\Gamma(G)(b)$ is the set of graphs on the vertex set $V(G,b)$ that can be constructed by the following process:

  1. Split the vertex set G into G_1 and G_2,
  2. Choose a list of objects $A \in !\mathbb{A}$,
  3. Split the vertex set G_2 into $H_1, ..., H_{|A|}$,
  4. Pick a graph T from $\langle t\rangle_{A\to B}^\Gamma(G_1)(A,b)$,
  5. For each $i \in |A|$, pick a graph S_i from $\langle s\rangle_A^\Gamma(H_i)(A_i)$,
  6. For each $i \in |A|$, graft S_i into T at the vertex A_i.

We can see from this definition that the root of every graph in $\langle t\rangle_A^\Gamma(G)(a)$ is a_n (as defined in Definition 6.1), and for each $g \in G$, every graph in $\langle t\rangle_A^\Gamma(G)(a)$ contains g as a non-root vertex. The latter is because in the application and differentiation constructions, the vertices eliminated by the grafting step are exactly the new vertices introduced from outside the vertex set $V(G,a)$ in Step 2.

Theorem 6.5. *If* $\Gamma \vdash t : A$*, then* $\langle t \rangle^\Gamma_A \cong [\![t]\!]^\Gamma_A$.

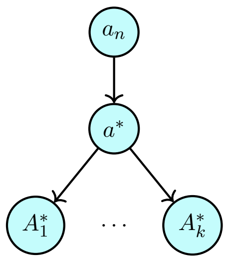
Proof. By induction on t.

- case $t \equiv 0$: The vertex set $V(G, a)$ is necessarily nonempty, so there are no empty graphs that can be constructed out of those vertices. Accordingly,

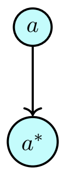
$$\langle 0 \rangle^\Gamma_A(G)(a) = \emptyset = [\![0]\!]^\Gamma_A(G)(a)$$

for all $G \in !\Gamma$ and $a \in \mathbb{A}$. Since both $\langle 0 \rangle^\Gamma_A$ and $[\![0]\!]^\Gamma_A$ map all objects to the empty set, they both must map all arrows to the empty function $\emptyset \to \emptyset$. So in fact $\langle 0 \rangle^\Gamma_A$ and $[\![0]\!]^\Gamma_A$ are the same functor.

- case $t \equiv x$: The only way to assemble all the vertices in $V(G, a)$ into a single graph of the desired shape is if the bag G in fact only contains a single object, which must be associated with the variable x. Then $\langle x \rangle^\Gamma_A(G)(a) = \emptyset$ if $|G| \neq 1$ or if G contains any objects associated with a different variable. (Note the connection between this fact and Lemma 5.5, which tells us that $[\![t]\!](G)(a)$ is always empty if G contains any objects arising from variables that do not appear in t.) If $G = [a^*]$ for some $a^* \in \mathbb{A}_x$ and $A^* = A_1 \otimes ... \otimes A_{n-1}$, then $\langle x \rangle^\Gamma_A(G)(a)$ is the set of graphs of the form



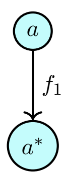
or equivalently



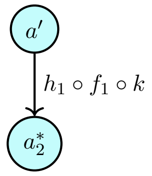
The number of ways to make such a graph is $|\mathbb{A}(a, a^*)|$. So

$$\begin{aligned}\langle x \rangle^\Gamma_A(G)(a) &\cong \begin{cases} \mathbb{A}(a, a^*) & \text{if } |G| = [a^*] \text{ for any } a^* \in \mathbb{A} \\ \emptyset & \text{otherwise} \end{cases} \\ &\cong !\mathbb{A}([a], G) \\ &= [\![x]\!]^\Gamma_A(G)(a)\end{aligned}$$

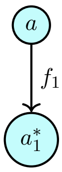
Recall that an arrow $f : !\mathbb{A}([a_1], [a_2])$ consists of a single arrow $f_1 : a_1 \to a_2$. For each $G \in !\Gamma$ and $a \in \mathbb{A}$, define an isomorphism $\eta_{G,a}\, [\![x]\!]^\Gamma_A(G)(a) \to \langle x \rangle^\Gamma_A(G)(a)$ so that for $f \in !\mathbb{A}([a], G)$, $\eta_{G,a}(f)$ is the graph



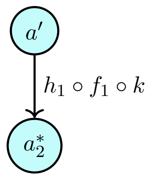
Suppose we have an arrow $k : a' \to a$ and an arrow $h : G \to G'$, implying that $|G| = |G'|$. Both $\langle x\rangle^\Gamma_A(G)(a)$ and $[\![x]\!]^\Gamma_A(G)(a)$ are empty if $|G| \neq 0$, so in that case $\eta_{G',a'} \circ [\![x]\!]^\Gamma_A(h)(k) = \langle x\rangle^\Gamma_A(h)(k) \circ \eta_{G,a}$ because both are the empty map. If $G = [a_1^*]$ and $G' = [a_2^*]$ then $\eta_{G',a'} \circ [\![x]\!]^\Gamma_A(h)(k)$ is the map that takes an arrow $f \in !\mathbb{A}([a], G)$, composes it with h and $[k]$ to get $h \circ f \circ [k] \in !\mathbb{A}([a'], G')$, and then maps this new arrow to the graph



While the morphism $\langle x\rangle^\Gamma_A(h)(k) \circ \eta_{G,a}$ first maps f to the graph



and then composes that graph with h_1 and k to get



Since these arrows commute, η is a natural isomorphism $[\![x]\!]^\Gamma_A \Rightarrow \langle x\rangle^\Gamma_A$.

- case $t \equiv u + v$: By the inductive hypothesis, there exist two natural isomorphisms $U : \langle u\rangle^\Gamma_A \Rightarrow [\![u]\!]^\Gamma_A$ and $V : \langle v\rangle^\Gamma_A \Rightarrow [\![v]\!]^\Gamma_A$. By the functoriality of colimits, the profunctors $\langle u\rangle^\Gamma_A + \langle v\rangle^\Gamma_A$ and $[\![u]\!]^\Gamma_A + [\![v]\!]^\Gamma_A$ are also naturally isomorphic. So $\langle u + v\rangle^\Gamma_A \cong [\![u + v]\!]^\Gamma_A$.

- case $t \equiv \lambda x.s$: By induction, there is a natural isomorphism $S : [\![s]\!]^{\Gamma,x:A}_B \Rightarrow \langle s\rangle^{\Gamma,x:A}_B$. Use S to define a new natural isomorphism $\eta : [\![\lambda x.s]\!]^\Gamma_{A\to B} \Rightarrow \langle \lambda x.s\rangle^\Gamma_{A\to B}$ by setting

$$\eta_{G,(A,b)} = S_{G\otimes A,b}$$

For differentiation and application of terms, we require a more involved strategy to show that the functor $\langle t\rangle^\Gamma_A$ is naturally isomorphic to the categorical interpretation. Recall that the interpretation of a differential term $Dt \cdot s$ was defined by

$$[\![Dt \cdot s]\!]^\Gamma_{A\to B}(G)(A, b) \cong \iint^{a\in\mathbb{A},\ F_1,F_2\in !\Gamma} [\![t]\!](F_1)(A \oplus [a], b) \times [\![s]\!](F_2)(a) \times !\Gamma(F_1 \oplus F_2, G)$$

Or, simplifying this expression with Lemma A.2,

$$[\![Dt \cdot s]\!]^\Gamma_{A\to B}(G)(A, b) \cong \sum_{\substack{(G_1,G_2)\\ \in\mathcal{D}(G)}} \int^{a\in\mathbb{A}} [\![t]\!](G_1)(A \oplus [a], b) \times [\![s]\!](G_2)(a)$$

We can also state this definition as

$$[\![Dt \cdot s]\!]^{\Gamma}_{A \to B}(-)(-,-) \cong \sum_{\substack{(G_1,G_2) \\ \in \mathcal{D}(-)}} \int^{a \in \mathbb{A}} [\![t]\!](G_1)(- \oplus [a], -) \times [\![s]\!](G_2)(a)$$

By the inductive hypothesis there are natural isomorphisms $T : [\![t]\!] \Rightarrow \langle t \rangle$ and $S : [\![s]\!] \Rightarrow \langle s \rangle$. This means for any fixed G_1 and G_2 there is also a natural isomorphism

$$[\![t]\!](G_1)(- \oplus [a], -) \times [\![s]\!](G_2)(a) \Rightarrow \langle t \rangle (G_1)(- \oplus [a], -) \times \langle s \rangle (G_2)(a)$$

Furthermore, by the functoriality of colimits, and therefore coends, there is a natural isomorphism

$$\sum_{\substack{(G_1,G_2) \\ \in \mathcal{D}(-)}} \int^{a \in \mathbb{A}} [\![t]\!](G_1)(- \oplus [a], -) \times [\![s]\!](G_2)(a) \Rightarrow \sum_{\substack{(G_1,G_2) \\ \in \mathcal{D}(-)}} \int^{a \in \mathbb{A}} \langle t \rangle (G_1)(- \oplus [a], -) \times \langle s \rangle (G_2)(a)$$

If we can show that the functor $\langle Dt \cdot s \rangle^{\Gamma}_{A \to B}$ is in fact the same as the functor

$$\sum_{\substack{(G_1,G_2) \\ \in \mathcal{D}(-)}} \int^{a \in \mathbb{A}} \langle t \rangle (G_1)(- \oplus [a], -) \times \langle s \rangle (G_2)(a),$$

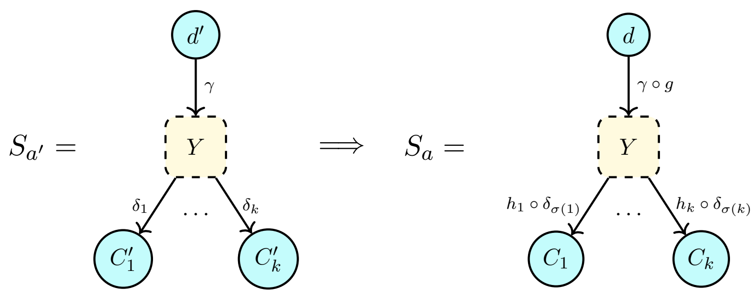
we will have the desired result that $[\![Dt \cdot s]\!]^{\Gamma}_{A \to B} \cong \langle Dt \cdot s \rangle^{\Gamma}_{A \to B}$. To accomplish this, we will use the coequalizer definition of coends. Fix a partition of the bag $G \in !\Gamma$ into G_1 and G_2, and let

$$F(a, a') = \langle t \rangle^{\Gamma}_{A \to B}(G_1)(A \oplus [a], b) \times \langle s \rangle^{\Gamma}_{A}(G_2)(a')$$

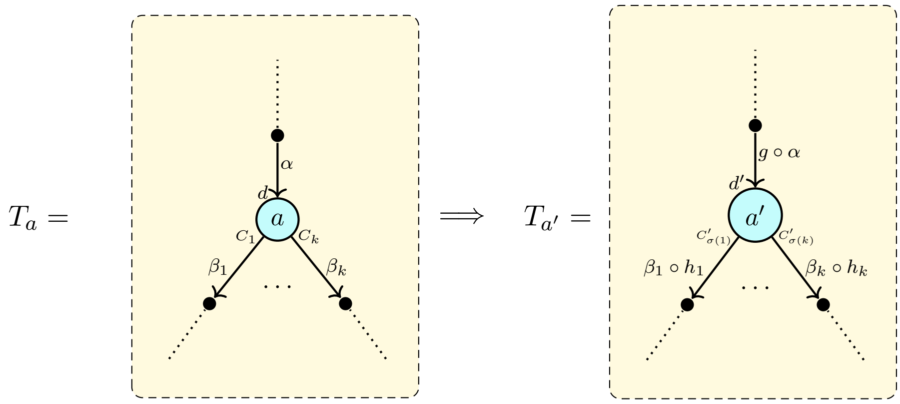
Then $\coprod_{f : a \to a'} F(a, a') = \coprod_{a, a' \in \mathbb{A}} \mathbb{A}(a, a') \times F(a, a')$ is the set of tuples $(f, T_a, S_{a'})$ where

- $f : a \to a'$ is an arrow in $\mathbb{A}$,
- T_a is a graph from $\langle t \rangle^{\Gamma}_{A \to B}(G_1)(A \oplus [a], b)$, and
- $S_{a'}$ is a graph from $\langle s \rangle^{\Gamma}_{A}(G_2)(a')$.

The set $\coprod_{a \in \mathbb{A}} F(a, a)$ is the set of tuples (T, S) where T is a graph from $\langle t \rangle^{\Gamma}_{A \to B}(G_1)(A \oplus [a], b)$ and S is a graph from $\langle s \rangle^{\Gamma}_{A}(G_2)(a)$. Recall from Chapter 6.2 that given an arrow $f : a \to a'$, we can alter a graph $S_{a'} \in \langle s \rangle^{\Gamma}_{A}(G_2)(a')$ to get a new graph $S_a \in \langle s \rangle^{\Gamma}_{A}(G_2)(a)$ by composing the graph with the arrow f.



We can also use f to make a graph $T_a \in \langle t \rangle^{\Gamma}_{A \to B}(G_1)(A \oplus [a], b)$ into a graph $T_{a'}$ from $\langle t \rangle^{\Gamma}_{A \to B}(G_1)(A \oplus [a'], b)$ by replacing the vertex a with the modified vertex $f(a)$.



The coend $\int^{a \in \mathbb{A}} F(a,a)$ is the coequalizer of

$$\coprod_{f: a \to a'} F(a,a') \underset{F_*}{\overset{F^*}{\rightrightarrows}} \coprod_{a \in \mathbb{A}} F(a,a)$$

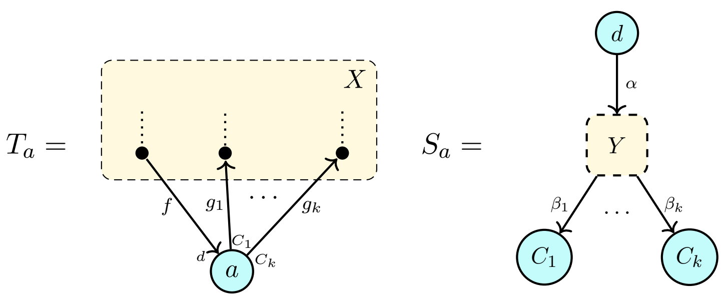
where F^* maps $(f, T_a, S_{a'})$ to (T_a, S_a), and F_* maps $(f, T_a, S_{a'})$ to $(T_{a'}, S_{a'})$. Since we are working in the category **Set**, the coequalizer of these arrows is formed by quotienting $\coprod_{a \in \mathbb{A}} F(a,a)$ by the smallest equivalence relation such that for all $x \in \coprod_{f: a \to a'} F(a,a')$, $F^*(x) \equiv F_*(x)$. Our strategy for explicitly defining this coend will be as follows:

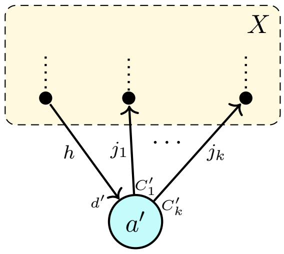
1) Define an equivalence relation on $\coprod_{a \in \mathbb{A}} F(a,a)$.

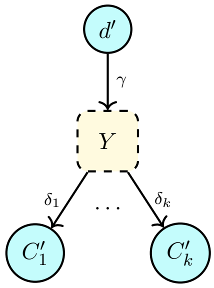
2) Show that $F^*(f, T_a, S_{a'}) \equiv F_*(f, T_a, S_{a'})$ for all tuples $(f, T_a, S_{a'}) \in \coprod_{f: a \to a'} F(a,a')$.

3) Show that if $(T_a, S_a) \equiv (T_{a'}, S_{a'})$ in this equivalence relation, then it must be true that $(T_a, S_a) \equiv (T_{a'}, S_{a'})$ in any equivalence relation satisfying $F^*(x) = F_*(x)$ for all x in $\coprod_{f: a \to a'} F(a,a')$.

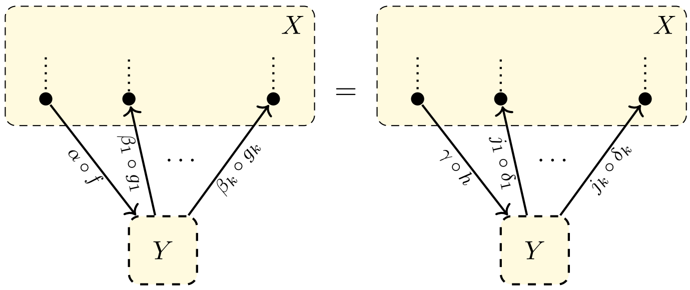
Say that $(T_a, S_a) \equiv (T_{a'}, S_{a'})$ if there exist graphs X and Y such that



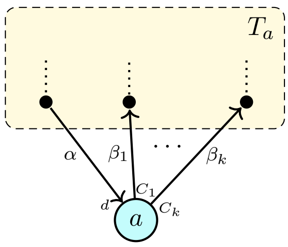
$T_{a'} =$ 

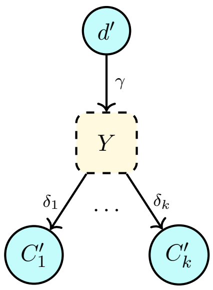
$S_{a'} =$ 

where $\gamma \circ h = \alpha \circ f$ and $j_i \circ \delta_i = \beta_i \circ g_i$ for all i in $|C|$. (Note that in order for (T_a, S_a) and $(T_{a'}, S_{a'})$ to be equivalent, $|C|$ must equal $|C'|$.) The second condition is akin to saying that grafting S_a into T_a at a gives the same graph as grafting $S_{a'}$ into $T_{a'}$ at a':

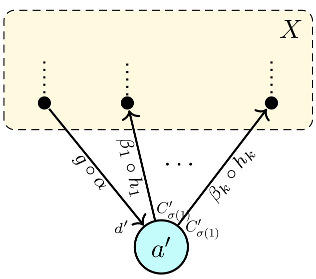


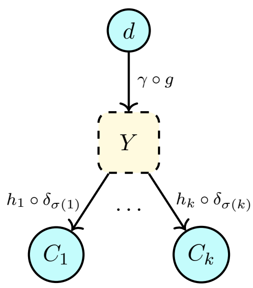
For step 2), take a tuple $(f : a \to a', T_a, S_{a'})$ where f consists of arrows $g : d \to d'$ and $h_i : C'_i \to C_i$, and

$T_a =$ 

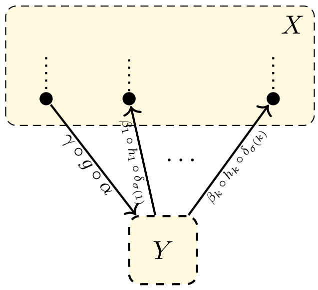
$S_{a'} =$ 

Then $F^*(f, T_a, S_{a'}) = (T_a, S_a)$ and $F_*(f, T_a, S_{a'}) = (T_{a'}, S_{a'})$, where

$T_{a'} =$ 

$S_a =$ 

These pairs satisfy the first condition of the equivalence relation, and grafting each pair results in the graph:



So indeed $(T_a, S_a) \equiv (T_{a'}, S_{a'})$, as desired. Next, we need to show that this is the *smallest* equivalence relation on $\coprod_{a\in\mathbb{A}} F(a,a)$ satisfying this constraint. Suppose $(T_a, S_a) \equiv (T_{a'}, S_{a'})$. Then there are graphs X and Y such that we can express $T_a, S_a, T_{a'}$, and $S_{a'}$ with the diagrams in Fig. 6.11, and such that $\gamma \circ h = \alpha \circ f$ and $j_i \circ \delta_i = g_i \circ \beta_i$ for all i in $|C| = k$.

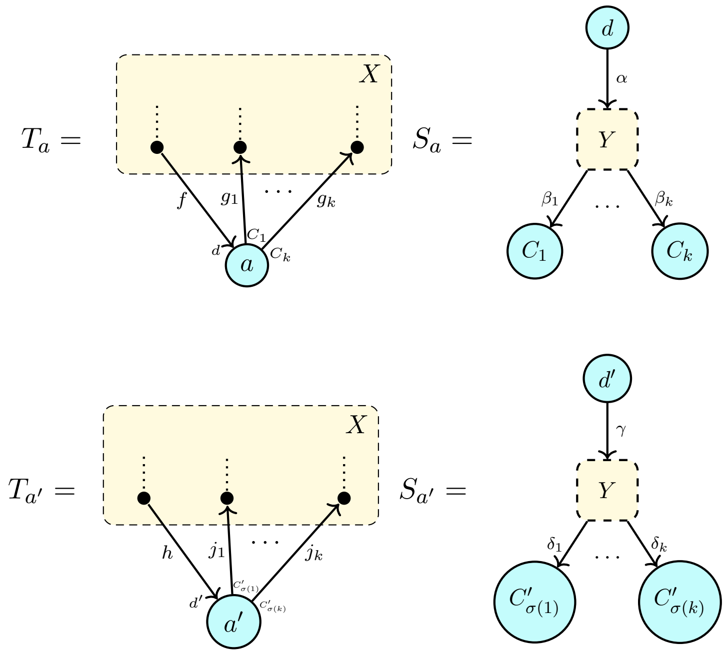
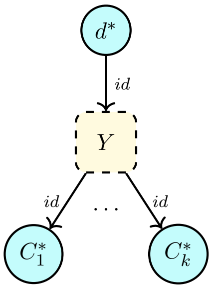


Figure 6.11

There is some object $a^* = (C^*, d^*)$ and permutation $\sigma : k \to k$ such that $\alpha : d \to d^*$ and $\gamma : d' \to d^*$, and $\beta_i : C_i^* \to C_i$ and $\delta_i : C_{\sigma(i)}^* \to C_i'$ for $i \in k$. We can then define $S_{a^*} \in \langle s \rangle_A^\Gamma(G_2)(a^*)$:



Define an arrow $p : a \to a^*$ by $u : d \to d^*$ and $v : C^* \to C$ where $u = \alpha$ and $v_i = \beta_i$. Then $F^*(p, T_a, S_{a^*}) = (T_a, S_a)$ and $F_*(p, T_a, S_{a^*}) = (T_{a^*}, S_{a^*})$ where T_{a^*} is the graph in Fig. 6.12.

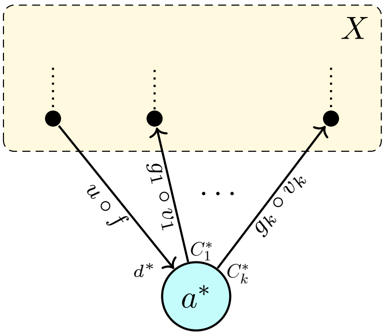


Figure 6.12: T_{a^*}

Now define the arrow $p' : a' \to a^*$ by $u' : d' \to d^*$ and $v' : C^* \to C'$ where $u' = \gamma$ and $v' = \delta$. Then $F^*(p', T_{a'}, S_{a^*}) = (T_{a'}, S_{a'})$ and $F_*(p', T_{a'}, S_{a^*}) = (T_{a^*}, S_{a^*})$ (Fig. 6.13).

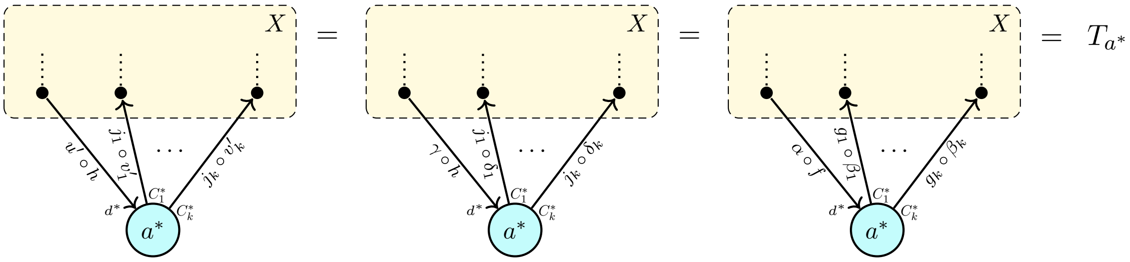


Figure 6.13

By these two constructions, $(T_a, S_a) \equiv (T_{a^*}, S_{a^*})$ and $(T_{a'}, S_{a'}) \equiv (T_{a^*}, S_{a^*})$, so by transitivity $(T_a, S_a) \equiv (T_{a'}, S_{a'})$. Since $(T_a, S_a) \equiv (T_{a'}, S_{a'})$ implies that (T_a, S_a) and $(T_{a'}, S_{a'})$ are necessarily equivalent in any equivalence relation where $F^*(x) \equiv F_*(x)$, our chosen relation is the smallest equivalence relation satisfying this property and therefore the set of equivalence classes of this relation is the coend $\int^{a \in \mathbb{A}} F(a, a)$. So $\sum_{\substack{(G_1, G_2) \\ \in \mathcal{D}(G)}} \int^{a \in \mathbb{A}} \langle t \rangle (G_1)(A \oplus [a], b) \times \langle s \rangle (G_2)(a)$ is the set of graphs with vertices $V(G, (A, b))$ that can be made by separating G into two pieces, choosing a new object $a \in \mathbb{A}$, picking a graph T from $\langle t \rangle^\Gamma_{A \to B}(G_1)(A \oplus [a], b)$, and grafting an graph S from $\langle s \rangle^\Gamma_A (G_2)(a)$ into T at the vertex a. This is precisely how we defined $\langle Dt \cdot s \rangle^\Gamma_{A \to B}(G)(A, b)$, so

$$\langle Dt \cdot s \rangle^\Gamma_{A \to B} = \sum_{\substack{(G_1, G_2) \\ \in \mathcal{D}(-)}} \int^{a \in \mathbb{A}} \langle t \rangle (G_1)(- \oplus [a], -) \times \langle s \rangle (G_2)(a)$$

and therefore

$$[\![Dt \cdot s]\!]^\Gamma_{A \to B} \cong \langle Dt \cdot s \rangle^\Gamma_{A \to B}$$

By a similar construction, we can define the interpretation $[\![ts]\!]^{\Gamma}_{B}(G)(b)$ as a set of graphs whose vertices are the input objects. Once again, begin with the definition

$$[\![ts]\!]^{\Gamma}_{B}(G)(b) \cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(G)}} \int^{A\in !\mathbb{A}} \sum_{\substack{(G_1,\dots,G_{|A|})\\ \in\mathcal{D}(H_2)}} [\![t]\!]^{\Gamma}_{A\to B}(H_1)(A,b) \times \left[\prod_{k\in|A|} [\![s]\!]^{\Gamma}_{A}(G_k)(A_k)\right]$$
$$\cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(G)}} \int^{A\in !\mathbb{A}} \sum_{\substack{(G_1,\dots,G_{|A|})\\ \in\mathcal{D}(H_2)}} \langle t\rangle^{\Gamma}_{A\to B}(H_1)(A,b) \times \left[\prod_{k\in|A|} \langle s\rangle^{\Gamma}_{A}(G_k)(A_k)\right]$$

By induction there are natural isomorphisms $[\![t]\!]^{\Gamma}_{A\to B} \Rightarrow \langle t\rangle^{\Gamma}_{A\to B}$ and $[\![s]\!]^{\Gamma}_{A} \Rightarrow \langle s\rangle^{\Gamma}_{A}$, so by functoriality of colimits there is a natural isomorphism

$$[\![ts]\!]^{\Gamma}_{B} \Rightarrow \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(-)}} \int^{A\in !\mathbb{A}} \sum_{\substack{(G_1,\dots,G_{|A|})\\ \in\mathcal{D}(H_2)}} \langle t\rangle^{\Gamma}_{A\to B}(H_1)(A,-) \times \left[\prod_{k\in|A|} \langle s\rangle^{\Gamma}_{A}(G_k)(A_k)\right]$$

As in the differentiation case, we will prove that $[\![ts]\!]^{\Gamma}_{B} \cong \langle ts\rangle^{\Gamma}_{B}$ by solving for the coend above to show that it is the same functor as $\langle ts\rangle^{\Gamma}_{B}$. Fix a partition $(H_1,H_2)\in\mathcal{D}(G)$ and let

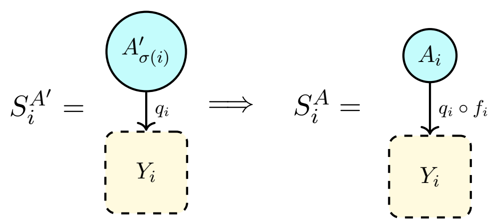
$$F(A,A') = \langle t\rangle^{\Gamma}_{A\to B}(H_1)(A,b) \times \sum_{\substack{(G_1,\dots,G_{|A'|})\\ \in\mathcal{D}(H_2)}} \left[\prod_{k\in|A'|} \langle s\rangle^{\Gamma}_{A}(G_k)(A'_k)\right]$$

Then $\coprod_{f:A\to A'} F(A,A') = \coprod_{A,A'\in\mathbb{A}} !\mathbb{A}(A,A')\times F(A,A')$ is the set of tuples $(f, T_A, \left(S_i^{A'}\right)_{i\in|A'|})$ where

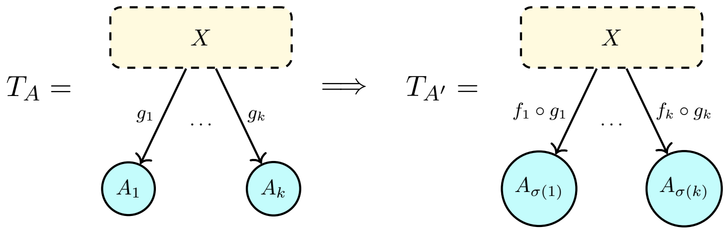
- $f : A \to A'$ is an arrow in $!\mathbb{A}$,
- T_A is a graph from $\langle t\rangle^{\Gamma}_{A\to B}(H_1)(A,b)$, and
- There is a partition of H_2 into $G_1,\dots,G_k$, where $k=|A'|$, such that for each $i\in|A'|$, $S_i^{A'}$ is a graph from $\langle s\rangle^{\Gamma}_{A}(G_i)(A'_i)$.

The set $\coprod_{A\in !\mathbb{A}} F(A,A)$ is the set of tuples $(T, \left(S_i^A\right)_{i\in|A|})$ where $T\in\langle t\rangle^{\Gamma}_{A\to B}(H_1)(A,b)$ and there is a partition of H_2 into $k=|A|$ lists $G_1,\dots,G_k$ such that $S_i^A\in\langle s\rangle^{\Gamma}_{A}(G_i)(A_i)$ for each $i\in k$.

Let $f : A\to A'$ such that the component arrows are $f_i : A_i \to A'_{\sigma(i)}$ for a permutation $\sigma : k\to k$ where $|A|=|A'|=k$. Using the action of the arrow f on the tuple of graphs $\left(S_i^{A'}\right)_{i\in k}$, $F^*(f, T_A, \left(S_i^{A'}\right)) = (T_A, \left(S_i^{A}\right))$ where each $S_i^A\in\langle s\rangle^{\Gamma}_{A}(G_i)(A'_{\sigma(i)}$ is formed by composing $S_i^{A'}$ with the arrow f_i:



Next, $F_*(f, T, \left(S_i^{A'}\right)) = (T_{A'}, \left(S_i^{A'}\right))$ where $T_{A'}$ is created by finding and extracting all nodes A_i inside T_A via the method of Proposition 6.3 and composing the arrows linking them into the graph with the arrows f_i:

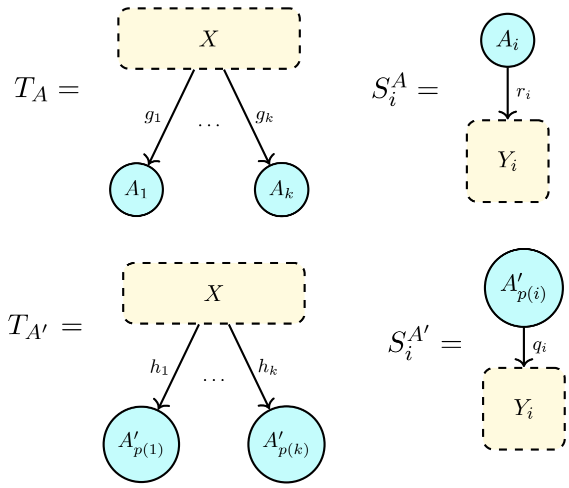


This time, the equivalence relation on the set $\coprod_{A \in !\mathbb{A}} F(A, A)$ is defined by

$$(T_A, (S_i^A)_{i \in |A|}) \equiv (T_{A'}, (S_i^{A'})_{i \in |A'|}) \text{ iff}$$

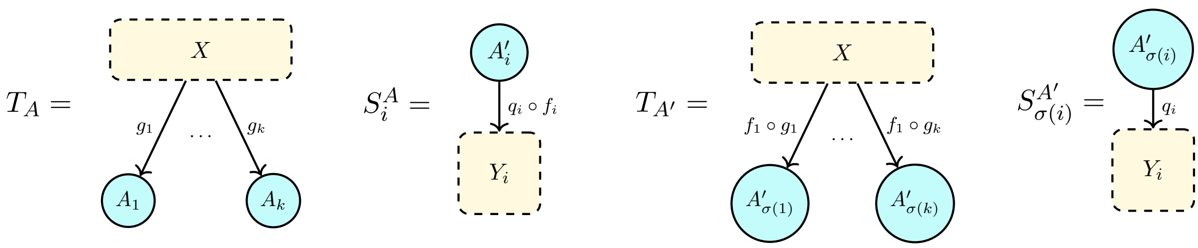
(1) $|A| = |A'| = k$,

(2) There is a permutation $p : k \to k$ and graphs $X, Y_1, ..., Y_k$ such that $T_A, T_{A'}, S_i^A$, and $S_i^{A'}$ can be written as



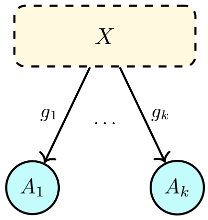
(3) $r_i \circ g_i = q_i \circ h_i$ for all $i \in k$.

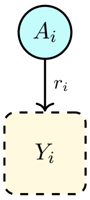
Let $f : A \to A'$ consist of the collection of arrows $f_i : A_i \to A'_{\sigma(i)}$. For the tuple $(f, T_A, \left(S_i^{A'}\right)_{i \in k})$ in $\coprod_{f : A \to A'} F(A, A')$, we have $F^*(f, T_A, \left(S_i^{A'}\right)_{i \in k}) = (T_A, (S_i^A)_{i \in k})$ and $F_*(f, T_A, \left(S_i^{A'}\right)_{i \in k}) = (T_{A'}, \left(S_i^{A'}\right)_{i \in k})$, where

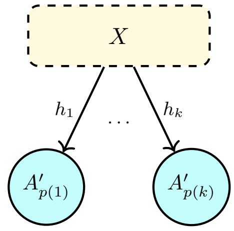


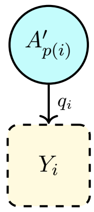
These diagrams confirm that $F^*(f, T, \left(S_i^{A'}\right)_{i\in k}) \equiv F_*(f, T, \left(S_i^{A'}\right)_{i\in k})$.

Now suppose that $(T_A, \left(S_i^A\right)_{i\in k}) \equiv (T_{A'}, \left(S_i^{A'}\right)_{i\in k})$. Then $|A| = |A'| = k$ and there are shapes $X, Y_1, ..., Y_k$ and a permutation $p : k \to k$ such that using the graphs can be represented by the following diagrams, where $r_i \circ g_i = q_i \circ h_i$ for all $i \in k$.

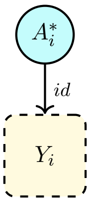
$T_A =$ 

$S_i^A =$ 

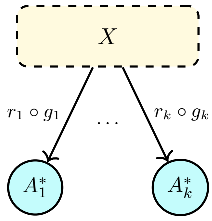
$T_A =$ 

$S_{p(i)}^{A'} =$ 

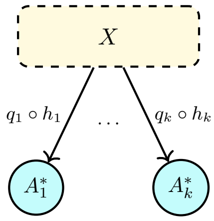
For each $i \in k$ there is an object $\alpha_i \in \mathbb{A}$ such that $r_i : A_i \to \alpha_i$ and $q_i : A'_{p(i)} \to \alpha_i$; define a list $A^* \in !\mathbb{A}$ so that $A_i^* = \alpha_i$. Let $S_i^{A^*}$ be the graph

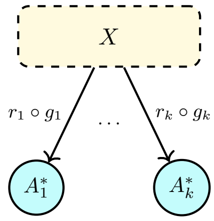
$S_i^{A^*} =$ 

Then $F^*(r, T_A, (S_i^{A^*})_{i\in k}) = (T_A, S_i^A)$ and $F_*(r, T_A, (S_i^{A^*})_{i\in k}) = (T_{A^*}, S_i^{A^*})$, where

$T_{A^*} =$ 

Also, $F^*(r, T_A, (S_i^{A^*})_{i\in k}) = (T_A, (S_i^A)_{i\in k})$ and $F_*(r, T_A, (S_i^{A^*})_{i\in k}) = (T_{A^*}, (S_i^{A^*})_{i\in k})$, where

$T_{A^*} =$ 

$=$ 

So $(T_A, \left(S_i^A\right)_{i\in k}) \equiv (T_{A^*}, \left(S_i^{A^*}\right)_{i\in k}) \equiv (T_{A'}, \left(S_i^{A'}\right)_{i\in k})$ in any equivalence relation on $\coprod_{A\in !\mathbb{A}} F(A, A)$ that satisfies $F^*(x) \equiv F_*(x)$ for all x. Therefore

$$\sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(G)}} \int^{A\in !\mathbb{A}} \sum_{\substack{(G_1,\ldots,G_{|A|})\\ \in\mathcal{D}(H_2)}} \langle t\rangle^{\Gamma}_{A\to B}(H_1)(A, b) \times \left[\prod_{k\in|A|} \langle s\rangle^{\Gamma}_{A}(G_k)(A_k)\right]$$

is the set of graphs that can be created via the procedure:

1. Partition G into H_1 and H_2,
2. Choose a list $A \in !\mathbb{A}$,
3. Form a $\langle t\rangle^{\Gamma}_{A\to B}(H_1)(A, b)$ graph T,
4. Partition H_2 into $G_1, ..., G_{|A|}$,
5. Form an $\langle s\rangle^{\Gamma}_{A\to B}(G_i)(A_i)$ graph S_i for each $i \in |A|$, and
6. Graft S_i into T at the vertex A_i for each $i \in |A|$.

This is how we defined the functor $\langle ts\rangle^{\Gamma}_{B}$, so $[\![ts]\!]^{\Gamma}_{B} \cong \langle ts\rangle^{\Gamma}_{B}$ as desired.

□

6.4 Examples

In this section, example members of four interpreted term sets $\langle t\rangle^{\Gamma}_{A}(G)(a)$ are constructed. Examples 6.6 and 6.9 recalculate the terms from Examples 5.9 and 5.11, respectively, in the combinatorial interpretation.

Example 6.6 (Application of variables)**.** For this example, we will choose an input list G so that $\langle xy\rangle^{\Gamma}_{B}(G)(b)$ is nonempty and then construct a member of $\langle xy\rangle^{\Gamma}_{B}(G)(b)$. Suppose $\Gamma = \{x : A \to B, y : A\}$. Let $G = [([a^x_1, a^x_2], b^x), a^y_1, a^y_2]$, where $([a^x_1, a^x_2], b^x)$ has index 1, and a^y_1 and.a^y_2 have index 2. To construct a sample graph from $\langle xy\rangle^{\Gamma}_{B}(G)(b)$, follow the steps from Definition 6.4 (Figure 6.14).

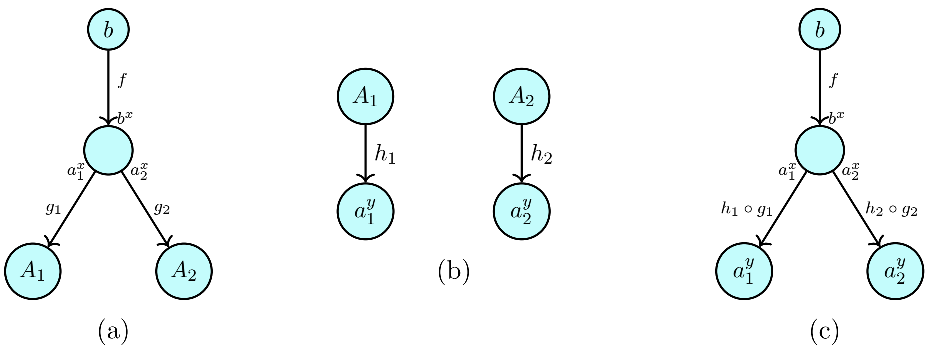
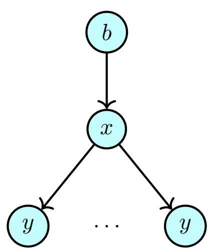


Figure 6.14: (a) Take the sublist $H = [([a^x_1, a^x_2], b^x)] \subset G$ and choose a list $A = [A_1, A_2] \in !\mathbb{A}$. Construct a graph from the set $\langle x\rangle^{\Gamma}_{A\to B}(H)(A, b)$. (b) Split the remainder of G into two lists $G_1 = [a^y_1]$ and $G_2 = [a^y_2]$ and construct a $\langle y\rangle^{\Gamma}_{A}(G_1)(A_1)$ graph as well as a $\langle y\rangle^{\Gamma}_{A}(G_2)(A_2)$ graph.[1] (c) Graft the two $\langle y\rangle$ graphs into the $\langle x\rangle$ graph.

Depending on our choices of G_1, G_2, and the arrows in the graph we could have obtained a different member of $\langle xy\rangle^{\Gamma}_{B}(G)(b)$. However, all members will be of the shape



Example 6.7 (β-reduction)**.** This time we will construct a member of $\langle(\lambda z.zy)x\rangle^{\Gamma}_{B}(G)(b)$. Again, let $G = [([a^x_1, a^x_2], b^x), a^y_1, a^y_2]$. First, take the sublist $H = [a^y_1, a^y_2] \subset G$ and choose a list $F \in !(\mathbb{A} \Rightarrow \mathbb{B})$. In this case we will choose $F = [([a^F_1, a^F_2], b^F)]$. Using the result of Example 6.6, make a graph from $\langle\lambda z.zy\rangle^{\Gamma}_{(A\to B)\to B}(H)(F, b) = \langle zy\rangle^{\Gamma}_{B}(H \otimes F)(b)$. Next, use the object from G to make a member of the set

$$\langle x\rangle^{\Gamma}_{A\to B}([([a^x_1, a^x_2], b^x)])(F_1) = \langle x\rangle^{\Gamma}_{A\to B}([([a^x_1, a^x_2], b^x)])([a^z_1, a^z_2], b^F)$$

Finally, graft the x graph into the $\lambda z.zy$ graph at the vertex F_1. Figure 6.15 illustrates this 3-step process.

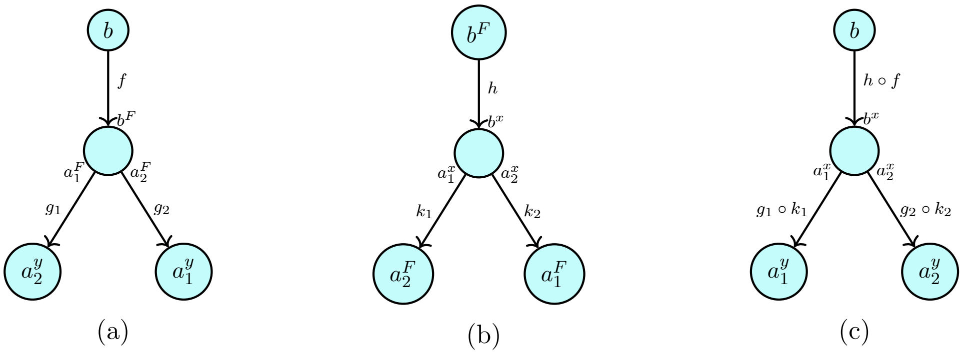


Figure 6.15: (a) A graph from $\langle zy\rangle^{\Gamma}_{B}(H \otimes F)(b)$; (b) A graph from $\langle x\rangle^{\Gamma}_{A\to B}([([a^x_1, a^x_2], b^x)])([a^z_1, a^z_2], b^F)$; (c) Graph (a) grafted into graph (b).

Notice that in the grafting step of Example 6.7 a graph with the shape of Fig. 6.16a is transformed into a graph with the shape of Fig. 6.16b.

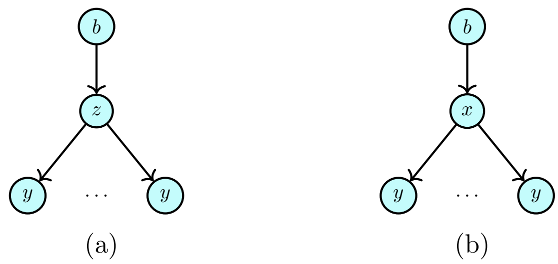


Figure 6.16

This transformation embodies the β-reduction $(\lambda z.zy)x \to_{\beta} xy$, which replaces all free instances of z in the original tree with x.

Example 6.8 (D-reduction)**.** Figure 6.17 shows the process of constructing an element of $\langle D(\lambda x.yx) \cdot z\rangle^{\Gamma}_{A\to B}(G)(A, b)$ where $\Gamma = \{y : A \to B, z : A\}$ and $G = [(A^y, b^y), a^z]$. Note that taking a D-reduction step on this term yields the term $\lambda x.(Dy \cdot w)x$. Accordingly, the graph in Fig. 6.17(c) could have instead been created by following the procedure for making an element of $\langle\lambda x.(Dy \cdot z)x\rangle^{\Gamma}_{B}(G)(A, b) = \langle(Dy \cdot z)x\rangle^{\Gamma,x:A}_{B}(G \otimes A)(b)$ (Figure 6.18).

Example 6.9. Let $\Gamma = \{x : (A \to A) \to B\}$ and $G = [(F^*, b^*)]$ where $F^* = [([A^*_1], a_1), ([A^*_2], a_2)]$. Figure 6.19 illustrates the construction of a graph from $\langle x(\lambda y.y)\rangle^{\Gamma}_{B}(G)(b)$.

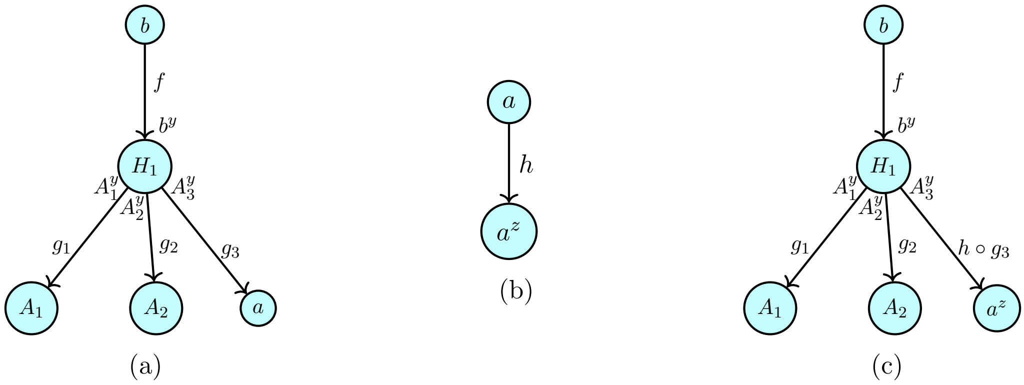


Figure 6.17: Construction of an element of $\langle D(\lambda x.yx) \cdot z\rangle^{\Gamma}_{A\to B}(G)(A, b)$. (a) Choose a subset $H = [(A^y, b^y)]$ of G, as well as some object $a \in \mathbb{A}$, and make a graph from $\langle xy\rangle^{\Gamma}_{B}(H \otimes A \otimes [a])(b)$. (b) Using the remaining element a^z of G, make a $\langle z\rangle^{\Gamma}_{A}(a^z)(a)$ graph. (c) Graft the $\langle z\rangle$ graph into the $\langle \lambda x.yx\rangle$ graph at the vertex a.

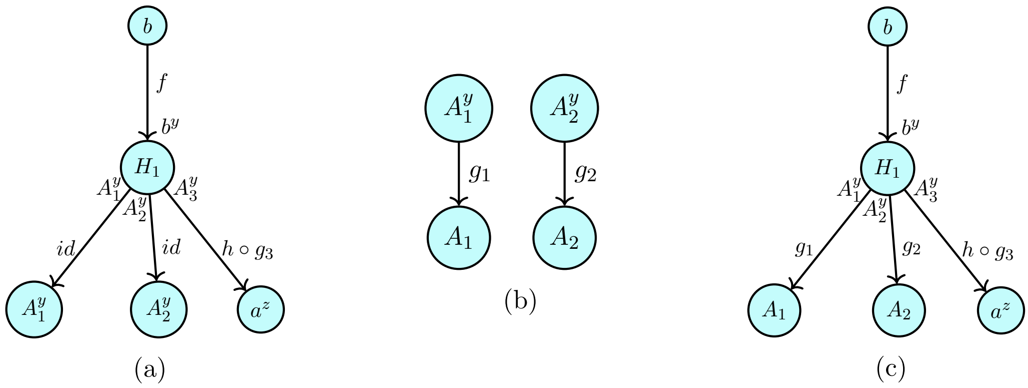


Figure 6.18: Construction of an element of $\langle \lambda x.(Dy \cdot z)x\rangle^{\Gamma}_{B}(G)(A, b)$. (a) Choose the list $A^* = [A_1^y, A_2^y]$ and make an element of $\langle Dy \cdot z\rangle^{\Gamma, x:A}_{A\to B}(G)(A^*, b)$. (b) For $i = 1, 2$, make a graph from $\langle x\rangle^{\Gamma, x:A}_{A}([A_i])(A_i^y)$. (c) Graft the graphs in (b) into the graph in (a) at the appropriate vertices.

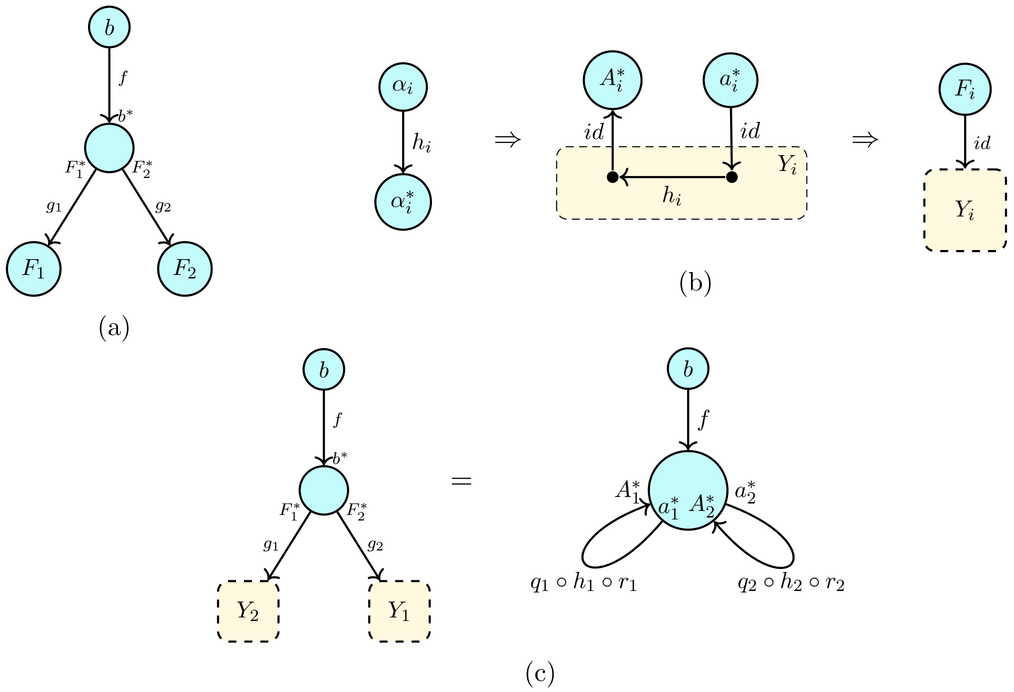


Figure 6.19: Construction of an element of $\langle x(\lambda y.y)\rangle^{\Gamma}_{B}(G)(b)$. (a) A graph from $\langle x\rangle(G)(F, b)$ where $F_i = ([\alpha_1^*], \alpha_1)$ and each arrow g_i consists of an arrow $r_i : a_i^* \to \alpha_i$ and an arrow $q_i : \alpha_i^* \to A_i^*$. (b) A graph from $\langle \lambda y.y\rangle([\,])(F_i) = \langle y\rangle([\alpha_i^*])(\alpha_i)$. (c) The graphs from (b) grafted into the graph from (a).

Chapter 7

Counting graphs in the interpretation

Now we have a combinatorial description of the model of differential λ-calculus within **Esp**, in which each term and pair of inputs is associated with a set of graphs, and inductively constructing terms corresponds to transforming and combining these graphs. The next step we might want to take is to count explicitly the number of graphs associated with each interpreted term $\langle t \rangle^{\Gamma}_{A}$ for inputs G and a. However, reasoning by graph transformations alone is not the most computationally straightforward method of determining cardinality. The following proposition rewrites the sets $[\![t]\!]^{\Gamma}_{A}(G)(a)$ in a form whose size will be easier to calculate.

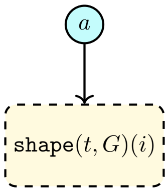
Proposition 7.1. *For any bag $G \in !\Gamma$ and any typing judgement $\Gamma \vdash t : A$ where t is in normal form and t is not an abstraction, there is*

1. *an index set* $\mathtt{I}(t, \Gamma, G)$
2. *a map* $\mathtt{shape}(t, \Gamma, G) : I(t, \Gamma, G) \to$ ***Set****, and*
3. *a map* $\mathtt{point}(t, \Gamma, G) : I(t, \Gamma, G) \to \mathbb{A}$

such that for all $a \in \mathbb{A}$,

$$[\![t]\!]^{\Gamma}_{A}(G)(a) \cong \sum_{i \in \mathtt{I}(t,\Gamma,G)} \mathtt{shape}(t, \Gamma, G)(i) \times \mathbb{A}(a, \mathtt{point}(t, \Gamma, G)(i))$$

The notation in Proposition 7.1 is informed by the combinatorial description. Each member of a term's associated set of graphs can be thought of as having two layers: its unlabeled shape, and the choice of arrow for each edge within that shape. The `point` component represents our ability to extract the root of the graph when undertaking a grafting step:



Remark. Though this proposition only covers non-abstracted normal forms, which may include abstracted subterms, it will be useful in simplifying the interpretation of abstracted terms as well. Any term in normal form which is an abstraction must be of the form

$$t \equiv \lambda \vec{x}.s$$

where s is some non-abstracted normal term. Say $\Gamma \vdash t : C$ where $C \equiv A_1 \to \ldots \to A_n \to B$, and $\Gamma \vdash s : B$. Then by the proposition,

$$[\![t]\!]^{\Gamma}_{C}(G)(A_1, ..., A_n, b) = [\![s]\!]^{\Gamma}_{B}(G \otimes A_1 \otimes \ldots \otimes A_n)(b)$$

$$\cong \sum_{i \in \mathtt{I}(s,\Gamma',G\otimes A_1\otimes\ldots\otimes A_n)} \mathtt{shape}(s,\Gamma',G\otimes A_1\otimes\ldots\otimes A_n)(i)\times\mathbb{B}(b,\mathtt{point}(s,\Gamma',G\otimes A_1\otimes\ldots\otimes A_n)(i))$$

where $\Gamma' = \Gamma, x_1 : A_1, ..., x_n : A_n$.

Though Proposition 7.1 still expresses terms as sets, we can use this result to define a numerical expression for the size of each set.

Definition 7.2. If $\Gamma \vdash t : A$, then for $G \in !\Gamma$ and $a \in \mathbb{A}$ define

$$\#t(G)(A) = \sum_{i \in \mathtt{I}(t,\Gamma,G)} |\mathtt{shape}(t,\Gamma,G)(i)| \times |\mathbb{A}(a,\mathtt{point}(t,\Gamma,G)(i))|$$

where in this case the $\times$ and $\sum$ symbols refer to numerical operations on integers rather than the categorical product and coproduct of sets.

Proposition 7.1 only holds in the case that t is in normal form – that is, there are no β or D-reductions that can be done on t – so we will use the fact that the simply typed differential λ-calculus is strongly normalizing [7]. To calculate the cardinality of $\langle t\rangle^{\Gamma}_{A}(G)(a)$, first reduce t fully to get a term s in normal form, and then recursively calculate $\#s(G)(a)$.

Let $\mathcal{D}(G,n) = \{(G_1, ..., G_n) : G_1, ..., G_n$ form a partition of the contents of $G\}$. We will need to enumerate the partitions of G: let $G^i_1, ..., G^i_n$ denote the ith partition of G into n bags. The sets `I`, `shape`, and `point` are defined inductively as follows. As shorthand, we will omit Γ and simply write $\mathtt{I}(t,G)$ when Γ is unambiguous.

Definition 7.3. For the empty sum term 0 of type A, let $\mathtt{I}(0,G) = \mathtt{shape}(0,G) = \emptyset$ for all $G \in !\Gamma$. Choose an arbitrary object $a \in \mathbb{A}$ and let $\mathtt{point}(0,G) = a$ for all G. (We can assume that $\mathbb{A}$ is nonempty, as otherwise any $(\Gamma,\mathbb{A})$-species is the empty functor $\mathbf{0} \to \mathbf{Set}$.)

For a variable x, context Γ such that $\Gamma \vdash x : A$, and $G \in !\Gamma$,

- $\mathtt{I}(x,G) = \begin{cases} \{1\}, & \text{if } G = [a] \text{ for some } a \in A \\ \emptyset & \text{otherwise} \end{cases}$
- $\mathtt{shape}(x,G)(1) = \{1\}$
- $\mathtt{point}(x,G)(1) = \begin{cases} a, & \text{if } G = [a] \text{ for } a \in A \\ a^* & \text{otherwise, for arbitrary } a^* \in \mathbb{A} \end{cases}$

If t and s are non-abstracted terms in normal form such that $\Gamma \vdash t + s : A$, then for $G \in !\Gamma$,

- $\mathtt{I}(t+s,G) = \mathtt{I}(t,G) \uplus \mathtt{I}(s,G) = \{(i,1) : i \in \mathtt{I}(t,G)\} \cup \{(i,2) : i \in \mathtt{I}(s,G)\}$
- $\mathtt{shape}(t+s,G)(i,j) = \begin{cases} \mathtt{shape}(t,G)(i) & \text{if } j = 1 \\ \mathtt{shape}(s,G)(i) & \text{if } j = 2 \end{cases}$
- $\mathtt{point}(t+s,G)(i,j) = \begin{cases} \mathtt{point}(t,G)(i) & \text{if } j = 1 \\ \mathtt{point}(s,G)(i) & \text{if } j = 2 \end{cases}$

If t and s are both in normal form such that $\Gamma \vdash t : A \to B$ and $\Gamma \vdash s : A$, and t is not an abstraction, then for $G \in !\Gamma$,

- $\mathtt{I}(ts, G) = \{(i, j, k) : i \in |\mathcal{D}(G, 2)|, j \in \mathtt{I}(t, G_1^i), \text{ and } k \in |\mathcal{D}(G_2^i, |A|)| \text{ where } (A, b) = \mathtt{point}(t, G_1^i)(j)\}$
- $\mathtt{shape}(ts, G)(i, j, k) = \mathtt{shape}(t, G_1^i) \times \prod_{m \in |A|} [\![s]\!]_A^\Gamma ((G_2^i)_m^k)(A_m)$, where $(A, b) = \mathtt{point}(t, G_1^i)(j)$
- $\mathtt{point}(ts, G)(i, j, k) = b$, where $(A, b) = \mathtt{point}(t, G_1^i)(j)$

Note that because s is a subterm of ts, it must also be in normal form. The $[\![s]\!]_A^\Gamma ((G_2^i)_m^k)(A_m)$ component of the expression for $\mathtt{shape}(ts, G)(i, j, k)$ can therefore be rewritten in $\mathtt{I}, \mathtt{shape}, \mathtt{point}$ form as well.

If t and s are both in normal form such that $\Gamma \vdash t : A \to B$ and $\Gamma \vdash s : A$, and t is not an abstraction, then for $G \in !\Gamma$,

- $\mathtt{I}(Dt \cdot s, G) = \{(i, j, k) : i \in |\mathcal{D}(G, 2)|, j \in \mathtt{I}(t, G_1^i), \text{ and } k \in |A|, \text{where } (A, b) = \mathtt{point}(t, G_1^i)(j)\}$
- $\mathtt{shape}(Dt \cdot s, G)(i, j, k) = \mathtt{shape}(t, G_1^i)(j) \times [\![s]\!]_A^\Gamma (G_2^i)(A_k)$, where $(A, b) = \mathtt{point}(t, G_1^i)(j)$
- $\mathtt{point}(Dt \cdot s, G)(i, j, k) = (A - [A_k], b)$, where $(A, b) = \mathtt{point}(t, G_1^i)(j)$

Proof of Proposition 7.1. By induction on t. The full proof is presented in Appendix B. ☐

Chapter 8

Further Work

Beyond the syntax of the differential λ-calculus, there is plenty of remaining structure in both **Esp** and the graph-theoretic model left unexplored. An interesting further direction of this work would be to extend the correspondence by introducing new syntax to the differential λ-calculus that possesses a meaningful interpretation in both models.

One operation introduced in [8] but not incorporated into the interpretation is the Day tensor, which we gave as the multiplication of species in Chapter 2.3 by

$$(\mathcal{P} \cdot \mathcal{Q})(A)(b) = \int^{A_1, A_2 \in !\mathbb{A}} \mathcal{P}(A_1)(b) \times \mathcal{Q}(A_2)(b) \times !\mathbb{A}(A_1 \oplus A_2, A)$$

or more simply,

$$(\mathcal{P} \cdot \mathcal{Q})(A)(b) = \sum_{(A_1, A_2) \in \mathcal{D}(A)} \mathcal{P}(A_1)(b) \times \mathcal{Q}(A_2)(b)$$

To incorporate the Day tensor into the categorical and combinatorial interpretations is fairly straightforward:

- Extend the differential λ-calculus with the syntax $s \bullet t$ and the typing rule

$$\frac{\Gamma \vdash s : A \text{ and } \Gamma \vdash t : A}{\Gamma \vdash s \bullet t : A}$$

- Define $(s \bullet t)[u/x] = s[u/x] \bullet t[u/x]$ and $\frac{\partial(s \bullet t)}{\partial x} \cdot u = (\frac{\partial s}{\partial x} \cdot u) \bullet t + s \bullet (\frac{\partial t}{\partial x} \cdot u)$.
- Interpret $[\![s \bullet t]\!]^\Gamma_A (G)(a) = \sum_{(G_1, G_2) \in \mathcal{D}(G)} [\![s]\!]^\Gamma_A (G_1)(a) \times [\![t]\!]^\Gamma_A (G_2)(a)$.
- Interpret $\langle s \bullet t \rangle^\Gamma_A(G)(a)$ as the set of graphs that can be constructed by first partitioning the list G into G_1 and G_2 and then choosing both a $\langle s \rangle^\Gamma_A(G_1)(a)$ graph and a $\langle t \rangle^\Gamma_A(G_2)(a)$ graph. (We can think of this pair of chosen graphs as being connected by a common root node.)

The relevant proofs of soundness and isomorphism between interpretations can be extended to include the Day tensor using the above definitions. A potential source of interest in this addition is that it adds new shapes of graph to the model. In the sets $\langle t \rangle^\Gamma_A(G)(a)$, all graphs share the property that there is a root node of degree 1 whose single adjacent edge is outgoing. However, introducing the Day tensor creates graphs within the model whose roots have outgoing degree > 1.

Another construction left to implement is a fixed point operator. This would take the form of a term or an operator f in the syntax such that

$$[\![t(ft)]\!]^\Gamma_A (G)(a) \cong [\![ft]\!]^\Gamma_A (G)(a) \;\; \forall\, G \in !\Gamma,\, a \in \mathbb{A}$$

From the perspective of our combinatorial model, an operator fulfilling this description would need to satisfy a condition akin to closure under grafting: there must be a 1-1 correspondence

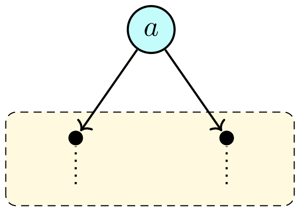


Figure 8.1: The shape of a graph in $\langle s \bullet t \rangle^{\Gamma}_{A}(G)(a)$, where s and t are terms of the differential λ-calculus.

between graphs in $[\![ft]\!]^{\Gamma}_{A}(G)(a)$ and graphs that result from grafting an $[\![ft]\!]^{\Gamma}_{A}(G_i)(A_i)$ graph into a $[\![t]\!]^{\Gamma}_{A \to A}(H)(A, a)$ for some choice of $A \in !\mathbb{A}$ and partition $(H, G_1, ..., G_{|A|}) \in \mathcal{D}(G)$. Constructions of fixed points by iterative processes as in [2] may provide the behaviour of fixed point operators in the categorical and combinatorial models.

The combinatorial description and accompanying counting method introduced in this dissertation may have future uses for studying the existence and number of graphs satisfying certain constraints. When viewed combinatorially, the profunctors in this model appear to take an input bag of vertices along with information about how each vertex may connect to each other, and map that bag to the set of graphs that can be 'properly' constructed out of them. In the case where atomic types are interpreted as the terminal category $\mathbb{1}$, the information included in each vertex in Γ is its degree; when atomic types are interpreted as categories with only identity arrows (though potentially multiple objects), the internal information can be configured to encode what type, or color, each vertex can be connected to via each edge. Investigating the connection of this model to graph coloring and other such problems is an opportunity for further work.

Appendix A

Coends

The proofs of various lemmas and theorems in this dissertation require calculating (up to isomorphism) several coends; the following lemmas allow us to manipulate and simplify coends towards this goal.

Theorem A.1 (Fubini's theorem for coends [16]). *Given a functor $F : \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{D}$,*

$$\int^{(c,e)} F(c,c,e,e) \cong \int^{c}\int^{e} F(c,c,e,e) \cong \int^{e}\int^{c} F(c,c,e,e)$$

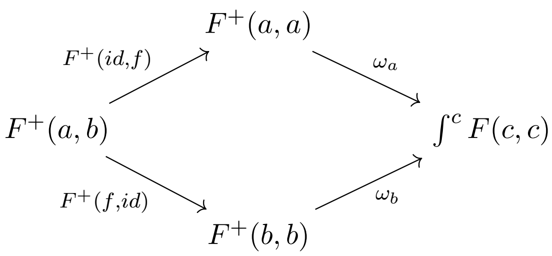
Lemma A.2 (Ninja Yoneda Lemma [16]). *Let $F : \mathcal{C}^{op} \to$ **Set** and $G : \mathcal{C} \to$ **Set** be functors. Then*

1. $\displaystyle\int^{c} F(c) \times \mathcal{C}(-, c) \cong F$
2. $\displaystyle\int^{c} G(c) \times \mathcal{C}(c, -) \cong G$

Lemma A.3 (Empty Set Lemma). *Let $\mathcal{C}$ be a category and $F : \mathcal{C} \times \mathcal{C}^{op} \to$ **Set** a profunctor. Suppose $\mathcal{C}_\emptyset$ is a subcategory of $\mathcal{C}$ such that for all $c \in \mathcal{C}_\emptyset$, $F(c,c) = \emptyset$. Let $\mathcal{C}^+$ be the subcategory of $\mathcal{C}$ consisting of exactly the objects and arrows that are not in $\mathcal{C}_\emptyset$, and let $F^+ : \mathcal{C}^+ \times (\mathcal{C}^+)^{op} \to$ **Set** be the profunctor F restricted to the domain $\mathcal{C}^+$. Then*

$$\int^{c\in\mathcal{C}} F(c,c) \cong \int^{c\in\mathcal{C}^+} F^+(c,c)$$

Proof. First, let $(\omega_a)_{a\in\mathcal{C}}$ be the projection maps of the coend $\int^c F^+(c,c)$ and $(\omega^+_a)_{a\in\mathcal{C}^+}$ the projection maps of the coend $\int^c F^+(c,c)$. By the properties of $\int^c F(c,c)$ the following diagram commutes for all arrows $f : a \to b$ in $\mathcal{C}^+$:



There is a unique morphism $h^+ : \int^c F^+(c,c) \to \int^c F(c,c)$ such that $h^+ \circ \omega^+_a = \omega_a$ for all $a \in \mathcal{C}^+$. Now define ω^+_b to be the empty map $\emptyset \to \int^c F^+(c,c)$ for each $b \in \mathcal{C}_\emptyset$. Since $F(b,b) = \emptyset$, it must be the case that ω_b is the empty map $\emptyset \to \int^c F(c,c)$. So $h^+ \circ \omega^+_b = \omega_b$ on the values $b \in \mathcal{C}_\emptyset$ as well.

Also, the following diagram commutes for all $f : a \to b$ in $\mathcal{C}$:

$$
\begin{array}{ccccc}
 & & F(a,a) & & \\
 & \overset{F(id,f)}{\nearrow} & & \overset{\omega_a^+}{\searrow} & \\
F(a,b) & & & & \int^c F^+(c,c) \\
 & \underset{F(f,id)}{\searrow} & & \underset{\omega_b^+}{\nearrow} & \\
 & & F(b,b) & &
\end{array}
$$

This is in part because if $F(a,a) = \emptyset$ then $F(b,a)$ and $F(a,b)$ must also be empty. If any arrow $f : a \to b$, then the functions $F(id, f)$ and $F(f, id)$ induced by f can only be the empty function, which makes the above diagram commute trivially.

Then there is a unique morphism $h : \int^c F(c,c) \to \int^c F^+(c,c)$ such that $h \circ \omega_a = \omega_a^+$ for all $a \in \mathcal{C}$. But then we have $h \circ h^+ \circ \omega_a^+ = \omega_a^+$ for all $a \in \mathcal{C}$, which implies that $h \circ h^+ = id$ due to the fact that a coend is unique up to unique isomorphism. Thus $\int^c F^+(c,c) \cong \int^c F(c,c)$ as desired.

□

Lemma A.4. *Let $P : \Gamma \,!\!\to \mathbb{A}$ and $Q : \Gamma \,!\!\to \mathbb{B}$ be generalized species and let $Y \in !\Gamma$, $a \in \mathbb{A}$, and $b \in \mathbb{B}$. Then*

$$\int^{X_1,X_2 \in !\Gamma} P(X_1)(a) \times Q(X_2)(b) \times !\Gamma(X_1 \oplus X_2, Y) \cong \sum_{(Y_1,Y_2)\in\mathcal{D}(Y)} P(Y_1)(a) \times Q(Y_2)(b)$$

Proof.

$$
\begin{aligned}
&\int^{X_1,X_2 \in !\Gamma} P(X_1)(a) \times Q(X_2)(b) \times !\Gamma(X_1 \oplus X_2, Y) \\
&\cong \int^{X_1,X_2 \in !\Gamma} P(X_1)(a) \times Q(X_2)(b) \times \left[\sum_{(Y_1,Y_2)\in\mathcal{D}(Y)} !\Gamma(X_1,Y_1) \times !\Gamma(X_2,Y_2) \right] \\
&\cong \int^{X_1,X_2 \in !\Gamma} \sum_{(Y_1,Y_2)\in\mathcal{D}(Y)} P(X_1)(a) \times Q(X_2)(b) \times !\Gamma(X_1,Y_1) \times !\Gamma(X_2,Y_2) \\
&\cong \sum_{(Y_1,Y_2)\in\mathcal{D}(Y)} \int^{X_1,X_2 \in !\Gamma} P(X_1)(a) \times Q(X_2)(b) \times !\Gamma(X_1,Y_1) \times !\Gamma(X_2,Y_2) \\
&\cong \sum_{(Y_1,Y_2)\in\mathcal{D}(Y)} P(Y_1)(a) \times Q(Y_2)(b)
\end{aligned}
$$

□

The following lemma and corollary follow from the functoriality of coends and Lemma A.3.

Lemma A.5. *If $F : \mathcal{C} \times \mathcal{C}^{op} \to$ **Set** is a profunctor and X is a set, then*

$$\int^c F(c,c) \times X \cong X \times \int^c F(c,c)$$

Corollary A.6. *If $F : !\mathbb{A} \times !\mathbb{A}^{op} \to$ **Set** is a profunctor and $F(A,A) = \emptyset$ for all $A \in !\mathbb{A}$ except for the empty list $[\,]$, then*

$$\int^A F(A,A) \cong F([\,],[\,])$$

Appendix B

Assorted proof details

This appendix contains proofs of several lemmas stated in Sections 5-7.

Lemma 5.5. *Let $\Gamma \vdash t : A$, and suppose that the typing context Γ contains an assignment $x : B$ such that x does not appear free in any subterm of t. If the bag $G \in ! \llbracket \Gamma \rrbracket$ contains any objects whose index in $\llbracket \Gamma \rrbracket$ is the index of x in Γ, then $\llbracket t \rrbracket_A^\Gamma (G)(a) = \emptyset$ for all $a \in \mathbb{A}$.*

Proof. By induction on t. Let $\Gamma = \{x_1 : A_1, ..., x_{|\Gamma|} : A_{|\Gamma|}\}$. Suppose the variable x_n does not appear free in any subterm of t, and the list $G \in ! \llbracket \Gamma \rrbracket$ contains some object $g \in G$ with index n within $\llbracket \Gamma \rrbracket = \prod_{i \in |\Gamma|} \llbracket x_i : A_i \rrbracket$.

- case $t \equiv 0$: $\llbracket 0 \rrbracket_A^\Gamma (G)(a) = \emptyset$, so the property holds trivially.
- case $t \equiv x_i$ where $i \neq n$: $\llbracket x_i \rrbracket_A^\Gamma (G)(a) = !\Gamma([\coprod_i (a)], G) = \emptyset$ because there is no object in $[\coprod_i (a)]$ that can be matched to the object g with index n in G.
- case $t \equiv u + v$:

$$\begin{aligned} \llbracket u + v \rrbracket_A^\Gamma (G)(a) &= \llbracket u \rrbracket_A^\Gamma (G)(a) + \llbracket v \rrbracket_A^\Gamma (G)(a) \\ &= \emptyset + \emptyset \quad \text{by inductive hypothesis} \\ &= \emptyset \end{aligned}$$

- case $t \equiv \lambda x_i.s$:

$$\begin{aligned} \llbracket \lambda x_i.s \rrbracket_{A \to B}^\Gamma (G)(A, b) &= \llbracket s \rrbracket_B^\Gamma (G \otimes A)(b) \\ &= \emptyset \quad \text{by inductive hypothesis} \end{aligned}$$

- case $t \equiv Du \cdot v$:

$$\llbracket Du \cdot v \rrbracket_{A \to B}^\Gamma (G)(A, b) \cong \sum_{\substack{(G_1, G_2) \\ \in \mathcal{D}(G)}} \int^{a \in \mathbb{A}} \llbracket u \rrbracket (G_1)(A \oplus [a], b) \times \llbracket v \rrbracket (G_2)(a)$$

  For each partition $(G_1, G_2) \in \mathcal{D}(G)$, g appears either in G_1 or G_2. If $g \in G_1$ then $\llbracket u \rrbracket (G_1)(A \oplus [a], b) = \emptyset$, and if $g \in G_2$ then $\llbracket v \rrbracket (G_2)(a) = \emptyset$. So

$$\begin{aligned} \sum_{\substack{(G_1, G_2) \\ \in \mathcal{D}(G)}} \int^{a \in \mathbb{A}} \llbracket u \rrbracket (G_1)(A \oplus [a], b) \times \llbracket v \rrbracket (G_2)(a) &\cong \sum_{\substack{(G_1, G_2) \\ \in \mathcal{D}(G)}} \int^{a \in \mathbb{A}} \emptyset \\ &= \emptyset \end{aligned}$$

- case $t \equiv uv$:

$$\llbracket uv \rrbracket^{\Gamma}_{B}(G)(b) \cong \sum_{\substack{(H_1,H_2)\\ \in \mathcal{D}(G)}} \int^{A \in !\mathbb{A}} \sum_{\substack{(G_1,\ldots,G_{|A|})\\ \in \mathcal{D}(H_2)}} \llbracket u \rrbracket^{\Gamma}_{A \to B}(H_1)(A,b) \times \left[\prod_{k \in |A|} \llbracket v \rrbracket^{\Gamma}_{A}(G_k)(A_k) \right]$$

For each partition $(H_1, H_2) \in \mathcal{D}(G)$, g appears either in H_1 or H_2. If $g \in H_1$ then $\llbracket u \rrbracket (H_1)(A, b) = \emptyset$ for any choice of A. If $g \in H_2$ then it will be put into one of the sets in the partition $(G_1, \ldots, G_{|A|}) \in \mathcal{D}(H_2)$. For each $i \in |A|$, if $g \in G_i$ then $\llbracket v \rrbracket (G_i)(A_i) = \emptyset$. So

$$\begin{aligned}\sum_{\substack{(H_1,H_2)\\ \in \mathcal{D}(G)}} \int^{A \in !\mathbb{A}} \sum_{\substack{(G_1,\ldots,G_{|A|})\\ \in \mathcal{D}(H_2)}} \llbracket u \rrbracket (H_1)(A,b) \times \left[\prod_{k \in |A|} \llbracket v \rrbracket (G_k)(A_k) \right] &\cong \sum_{\substack{(H_1,H_2)\\ \in \mathcal{D}(G)}} \int^{A \in !\mathbb{A}} \sum_{\substack{(G_1,\ldots,G_{|A|})\\ \in \mathcal{D}(H_2)}} \emptyset \\ &= \emptyset\end{aligned}$$

□

Lemma 5.4. *If $G \equiv G' \in !\Gamma$ and $\Gamma \vdash t : A$, then $\llbracket t \rrbracket^{\Gamma}_{A}(G)(a) \cong \llbracket t \rrbracket^{\Gamma}_{A}(G')(a)$ for all $a \in \mathbb{A}$.*

Proof. By induction on t.

- case $t \equiv 0$:

$$\llbracket 0 \rrbracket^{\Gamma}_{A}(G)(a) = \emptyset = \llbracket 0 \rrbracket^{\Gamma}_{A}(G')(a)$$

- case $t \equiv x$:

$$\llbracket x \rrbracket^{\Gamma}_{A}(G)(a) = !\Gamma([a], G) \cong !\Gamma([a], G') = \llbracket x \rrbracket^{\Gamma}_{A}(G')(a)$$

- case $t \equiv u + v$:

$$\begin{aligned}\llbracket u+v \rrbracket^{\Gamma}_{A}(G)(a) &= \llbracket u \rrbracket^{\Gamma}_{A}(G)(a) + \llbracket v \rrbracket^{\Gamma}_{A}(G)(a) \\ &\cong \llbracket u \rrbracket^{\Gamma}_{A}(G')(a) + \llbracket v \rrbracket^{\Gamma}_{A}(G')(a) \quad \text{by inductive hypothesis} \\ &= \llbracket u+v \rrbracket^{\Gamma}_{A}(G')(a)\end{aligned}$$

- case $t \equiv \lambda x.s$:

$$\begin{aligned}\llbracket \lambda x.s \rrbracket^{\Gamma}_{A \to B}(G)(A,b) &= \llbracket s \rrbracket^{\Gamma}_{B}(G \otimes A)(b) \\ &\cong \llbracket s \rrbracket^{\Gamma}_{B}(G' \otimes A)(b) \quad \text{by inductive hypothesis} \\ &= \llbracket \lambda x.s \rrbracket^{\Gamma}_{A \to B}(G)(A,b)\end{aligned}$$

- case $t \equiv Du \cdot v$:

$$\begin{aligned}&\llbracket Du \cdot v \rrbracket^{\Gamma}_{A \to B}(G)(A,b) \\ &= \sum_{\substack{(G_1,G_2)\\ \in \mathcal{D}(G)}} \int^{a \in \mathbb{A}} \llbracket u \rrbracket (G_1)(A \oplus [a], b) \times \llbracket v \rrbracket (G_2)(a) \\ &\cong \sum_{\substack{(G_1,G_2)\\ \in \mathcal{D}(G')}} \int^{a \in \mathbb{A}} \llbracket u \rrbracket (G_1)(A \oplus [a], b) \times \llbracket v \rrbracket (G_2)(a) \quad \text{because } \mathcal{D}(G) \cong \mathcal{D}(G') \\ &= \llbracket Du \cdot v \rrbracket^{\Gamma}_{A \to B}(G')(A,b)\end{aligned}$$

- case $t \equiv Du \cdot v$:

$$\begin{aligned}
\llbracket uv \rrbracket^{\Gamma}_{B}(G)(b) &= \sum_{\substack{(H_1,H_2)\\ \in \mathcal{D}(G)}} \int^{A \in !\mathbb{A}} \sum_{\substack{(G_1,\ldots,G_{|A|})\\ \in \mathcal{D}(H_2)}} \llbracket u \rrbracket^{\Gamma}_{A \to B}(H_1)(A,b) \times \left[\prod_{k \in |A|} \llbracket v \rrbracket^{\Gamma}_{A}(G_k)(A_k) \right] \\
&\cong \sum_{\substack{(H_1,H_2)\\ \in \mathcal{D}(G')}} \int^{A \in !\mathbb{A}} \sum_{\substack{(G_1,\ldots,G_{|A|})\\ \in \mathcal{D}(H_2)}} \llbracket u \rrbracket^{\Gamma}_{A \to B}(H_1)(A,b) \times \left[\prod_{k \in |A|} \llbracket v \rrbracket^{\Gamma}_{A}(G_k)(A_k) \right] \\
&= \llbracket uv \rrbracket^{\Gamma}_{B}(G')(b)
\end{aligned}$$

□

Proposition 5.1. In Section 5.2 we gave two cases of the inductive proof that the equation

$$\llbracket D(\lambda x.t) \cdot s \rrbracket^{\Gamma}_{A \to B}(X)(A,b) \cong \left\llbracket \lambda x.\frac{\partial t}{\partial x} \cdot s \right\rrbracket^{\Gamma}_{B}(X)(A,b) \qquad (\dagger)$$

holds for all s and t. The remaining cases are presented here.

case $t \equiv 0$:

$\llbracket D(\lambda x.0) \cdot s \rrbracket^{\Gamma}_{A \to B}(X)(A,b)$

$$\begin{aligned}
&\cong \iint^{a \in \mathbb{A},\, M_1, M_2 \in !\Gamma} \llbracket 0 \rrbracket^{\Gamma, x:A}_{B}(M_1 \otimes A \otimes [a])(b) \times \llbracket s \rrbracket^{\Gamma}_{A}(M_2)(a) \times !\Gamma(M_1 \oplus M_2, X) && (1)\\
&\cong \iint^{a \in \mathbb{A},\, M_1, M_2 \in !\Gamma} \emptyset && (2)\\
&= \emptyset && (3)\\
&\cong \llbracket 0 \rrbracket^{\Gamma, x:A}_{B}(X \otimes A)(b) && (4)\\
&\cong \llbracket \lambda x.0 \rrbracket^{\Gamma}_{A \to B}(X)(A,b) && (5)\\
&\cong \left\llbracket \lambda x.\frac{\partial 0}{\partial x} \cdot s \right\rrbracket^{\Gamma}_{A \Rightarrow B}(X)(A,b) && (6)
\end{aligned}$$

(1) Corollary 5.3 and the definition of $\llbracket \lambda x.0 \rrbracket$; (2) and (4) Definition of $\llbracket 0 \rrbracket$; (5) Definition of $\llbracket \lambda x.0 \rrbracket$; (6) Definition of $\frac{\partial 0}{\partial x} \cdot s$.

case $t \equiv y$:

$\llbracket D(\lambda x.y) \cdot s \rrbracket^{\Gamma}_{A \to B}(X)(A,b)$

$$\begin{aligned}
&\cong \iint^{\substack{a \in \mathbb{A},\\ M_1, M_2 \in !\Gamma}} \llbracket y \rrbracket^{\Gamma, x:a}_{B}(M_1 \otimes A \otimes [a])(b) \times \llbracket s \rrbracket^{\Gamma}_{A}(M_2)(a) \times !\Gamma(M_1 \oplus M_2, X) && (1)\\
&\cong \iint^{\substack{a \in \mathbb{A},\\ M_1, M_2 \in !\Gamma}} \emptyset \times \llbracket s \rrbracket^{\Gamma}_{A}(M_2)(a) \times !\Gamma(M_1 \oplus M_2, X) && (2)
\end{aligned}$$

$$
\begin{aligned}
&\cong \quad \emptyset && (3)\\
&\cong \quad [\![0]\!]^{\Gamma,x:A}_{B}(X\otimes A)(b) && (4)\\
&\cong \quad [\![\lambda x.0]\!]^{\Gamma}_{A\to B}(X)(A,b) && (5)\\
&\cong \quad \left[\!\!\left[\lambda x.\frac{\partial y}{\partial x}\cdot s\right]\!\!\right]^{\Gamma}_{A\to B}(X)(A,b) && (6)
\end{aligned}
$$

(1) Corollary 5.3; (2) Lemma 5.5; (4) Definition of $[\![0]\!]$; (5) Definition of $[\![\lambda x.0]\!]$; (6) Definition of $\frac{\partial y}{\partial x}\cdot s$.

<u>case $t \equiv \lambda y.u$:</u>

$$
\begin{aligned}
&\left[\!\!\left[\lambda x.\frac{\partial(\lambda y.u)}{\partial x}\cdot s\right]\!\!\right]^{\Gamma}_{A\to B\to C}(X)(A,B,c) = \left[\!\!\left[\lambda xy.\frac{\partial u}{\partial x}\cdot s\right]\!\!\right]^{\Gamma}_{A\to B\to C}(X)(A,B,c) && (1)\\
&= \left[\!\!\left[\frac{\partial u}{\partial x}\cdot s\right]\!\!\right]^{\Gamma,x:A,y:B}_{C}(X\otimes A\otimes B)(c) && (2)\\
&\cong \left[\!\!\left[\frac{\partial u}{\partial x}\cdot s\right]\!\!\right]^{\Gamma,x:A,y:B}_{C}(X\otimes B\otimes A)(c) && (3)\\
&= \left[\!\!\left[\lambda x.\frac{\partial u}{\partial x}\cdot s\right]\!\!\right]^{\Gamma,y:B}_{A\to C}(X\otimes B)(A,c) && (4)\\
&\cong [\![D(\lambda x.u)\cdot s]\!]^{\Gamma,y:B}_{A\to C}(X\otimes B)(A,c) && (5)\\
&\cong \iint^{\substack{a\in\mathbb{A},\\ M_1,M_2\in\,!(\Gamma\sqcap\mathbb{B})}} [\![u]\!]^{\Gamma,x:A,y:B}_{C}(M_1\otimes A\otimes[a])(c)\times[\![s]\!]^{\Gamma,y:B}_{A}(M_2)(a)\times\,!(\Gamma\sqcap\mathbb{B})(M_1\oplus M_2, X\otimes B) && (6)\\
&\cong \sum_{\substack{(X_1,X_2)\\ \in\mathcal{D}(X\otimes B)}}\int^{a\in\mathbb{A}} [\![u]\!]^{\Gamma,x:A,y:B}_{C}(X_1\otimes A\otimes[a])(c)\times[\![s]\!]^{\Gamma,y:B}_{A}(X_2)(a) && (7)\\
&\cong \sum_{\substack{(X_1,X_2)\\ \in\mathcal{D}(X)}}\int^{a\in\mathbb{A}} [\![u]\!]^{\Gamma,x:A,y:B}_{C}(X_1\otimes B\otimes A\otimes[a])(c)\times[\![s]\!]^{\Gamma,y:B}_{A}(X_2)(a) && (8)\\
&\cong \iint^{a\in\mathbb{A},\ M_1,M_2\in\,!\Gamma} [\![u]\!]^{\Gamma,x:A,y:B}_{C}(M_1\otimes B\otimes A\otimes[a])(c)\times[\![s]\!]^{\Gamma,y:B}_{A}(M_2)(a)\times\,!\Gamma(M_1\oplus M_2, X) && (9)\\
&\cong [\![D(\lambda xy.u)\cdot s]\!]^{\Gamma}_{A\to B\to C}(X)(A,B,c) && (10)
\end{aligned}
$$

(1) Definition of $\frac{\partial(\lambda y.u)}{\partial x}$; (2) Interpretation of abstraction; (3) Lemma 5.4; (4) Interpretation of abstraction; (5) Inductive hypothesis; (6) Corollary 5.3; (7) Lemma A.4; (8) Lemma 5.5 (because s contains no instances of y); (9) Lemma A.4; (10) Corollary 5.3.

case $t \equiv u + v$:

$$\left[\!\left[\lambda x.\tfrac{\partial(u+v)}{\partial x}\cdot s\right]\!\right]^{\Gamma}_{A\to B}(X)(A,b)$$

$$= \left[\!\left[\lambda x.\frac{\partial u}{\partial x}\cdot s + \frac{\partial v}{\partial x}\cdot s\right]\!\right]^{\Gamma}_{A\to B}(X)(A,b) \quad (1)$$

$$\cong \left[\!\left[\lambda x.\frac{\partial u}{\partial x}\cdot s\right]\!\right]^{\Gamma}_{A\to B}(X)(A,b) + \left[\!\left[\lambda x.\frac{\partial v}{\partial x}\cdot s\right]\!\right]^{\Gamma}_{A\to B}(X)(A,b) \quad (2)$$

$$\cong [\![D(\lambda x.u)\cdot s]\!]^{\Gamma}_{A\to B}(X)(A,b) + [\![D(\lambda x.v)\cdot s]\!]^{\Gamma}_{A\to B}(X)(A,b) \quad (3)$$

$$\cong \iint^{a\in\mathbb{A},\, M_1,M_2\in !\Gamma} \left[[\![u]\!]^{\Gamma,x:A}_{B}(M_1\otimes A\otimes[a])(b)\times[\![s]\!]^{\Gamma}_{A}(M_2)(a)\times !\Gamma(M_1\oplus M_2,X)\right] + \left[[\![v]\!]^{\Gamma,x:A}_{B}(M_1\otimes A\otimes[a])(b)\times[\![s]\!]^{\Gamma}_{A}(M_2)(a)\times !\Gamma(M_1\oplus M_2,X)\right] \quad (4)$$

$$\cong \iint^{a\in\mathbb{A},\, M_1,M_2\in !\Gamma} \left[[\![u]\!]^{\Gamma,x:A}_{B}(M_1\otimes A\otimes[a])(b) + [\![v]\!]^{\Gamma,x:A}_{B}(M_1\otimes A\otimes[a])(b)\right] \times[\![s]\!]^{\Gamma}_{A}(M_2)(a)\times !\Gamma(M_1\oplus M_2,X) \quad (5)$$

$$\cong \iint^{a\in\mathbb{A},\, M_1,M_2\in !\Gamma} [\![u+v]\!]^{\Gamma,x:A}_{B}(M_1\otimes A\otimes[a])(b)\times[\![s]\!]^{\Gamma}_{A}(M_2)(a)\times !\Gamma(M_1\oplus M_2,X) \quad (6)$$

$$\cong [\![D(\lambda x.u+v)\cdot s]\!]^{\Gamma}_{A\to B}(X)(A,b) \quad (7)$$

Figure B.1: (1) Definition of $\frac{\partial u+v}{\partial x}\cdot s$; (3) Inductive hypothesis; (4) Corollary 5.3 and the fact that coends commute with coproducts; (5) Distributivity of product over coproduct; (6) Definition of $[\![u+v]\!]$; (7) Corollary 5.3.

case $t \equiv u\, v$: First, note that

$$\left[\!\left[\lambda x.\frac{\partial(uv)}{\partial x}\cdot s\right]\!\right]^{\Gamma}(X)(A,b) \cong \left[\!\left[\left(\frac{\partial u}{\partial x}\cdot s\right)v\right]\!\right]^{\Gamma,x:A}(X\otimes A)(b) + \left[\!\left[\left(Du\cdot\left(\frac{\partial v}{\partial x}\cdot s\right)\right)v\right]\!\right]^{\Gamma,x:A}(X\otimes A)(b)$$

We will calculate the two summands separately.

$$\left[\!\left[\left(\tfrac{\partial u}{\partial x}\cdot s\right)v\right]\!\right]^{\Gamma,x:A}_{B}(X\otimes A)(b)$$

$$\cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(X\otimes A)}} \int^{C\in !\mathbb{C}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_2)}} \left[\!\left[\frac{\partial u}{\partial x}\cdot s\right]\!\right]^{\Gamma,x:A}(H_1)(C,b)\times\left[\prod_{k\in|C|}[\![v]\!]^{\Gamma,x:A}(X_k)(C_k)\right] \quad (1)$$

$$\cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(X)}} \sum_{\substack{(A_1,A_2)\\ \in\mathcal{D}(A)}} \int^{C\in !\mathbb{C}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_2\otimes A_2)}} \left[\!\left[\frac{\partial u}{\partial x}\cdot s\right]\!\right]^{\Gamma,x:A}(H_1\otimes A_1)(C,b)\times\left[\prod_{k\in|C|}[\![v]\!]^{\Gamma,x:A}(X_k)(C_k)\right] \quad (2)$$

$$\cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(X)}} \sum_{\substack{(A_1,A_2)\\ \in\mathcal{D}(A)}} \int^{C\in !\mathbb{C}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_2\otimes A_2)}} [\![D(\lambda x.u)\cdot s]\!]^{\Gamma}\ (H_1)(A_1,C,b) \times \left[\prod_{k\in|C|} [\![v]\!]^{\Gamma,x:A} (X_k)(C_k)\right] \tag{3}$$

$$\cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(X)}} \sum_{\substack{(Y_1,Y_2)\\ \in\mathcal{D}(H_1)}} \sum_{\substack{(A_1,A_2)\\ \in\mathcal{D}(A)}} \iint^{C\in !\mathbb{C},\ a\in\mathbb{A}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_2\otimes A_2)}} [\![u]\!]^{\Gamma,x:A}\ (Y_1\otimes A_1\otimes[a])(C,b) \times [\![s]\!]^{\Gamma,x:A} (Y_2)(a) \times \left[\prod_{k\in|C|} [\![v]\!]^{\Gamma,x:A} (X_k)(C_k)\right] \tag{4}$$

$$\cong \sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X)}} \sum_{\substack{(A_1,A_2)\\ \in\mathcal{D}(A)}} \iint^{C\in !\mathbb{C},\ a\in\mathbb{A}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3\otimes A_2)}} [\![u]\!]^{\Gamma,x:A}\ (H_1\otimes A_1\otimes[a])(C,b) \times [\![s]\!]^{\Gamma,x:A} (H_2)(a) \times \left[\prod_{k\in|C|} [\![v]\!]^{\Gamma,x:A} (X_k)(C_k)\right] \tag{5}$$

$$\cong \sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X\otimes A)}} \iint^{C\in !\mathbb{C},\ a\in\mathbb{A}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3)}} [\![u]\!]^{\Gamma,x:A}\ (H_1\otimes[a])(C,b) \times [\![s]\!]^{\Gamma,x:A} (H_2)(a) \times \left[\prod_{k\in|C|} [\![v]\!]^{\Gamma,x:A} (X_k)(C_k)\right] \tag{6}$$

(1) Lemmas 5.3 and A.4; (2) Lemma 5.4; (3) Inductive hypothesis; (4) Lemma 5.3; (5)-(6) By consolidating the limits of summation and Lemma 5.4.

Next, $[\![\big(Du\cdot\big(\frac{\partial v}{\partial x}\cdot s\big)\big)\ v]\!]\ (X\otimes A)(b)$

$$\cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(X)}} \sum_{\substack{(A_1,A_2)\\ \in\mathcal{D}(A)}} \int^{C\in !\mathbb{C}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_2\otimes A_2)}} \left[\!\!\left[Du\cdot\left(\frac{\partial v}{\partial x}\cdot s\right)\right]\!\!\right] (H_1\otimes A_1)(C,b) \times \left[\prod_{k\in|C|} [\![v]\!]\ (X_k)(C_k)\right] \tag{7}$$

$$\cong \sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X)}} \sum_{\substack{(A_1,A_2,A_3)\\ \in\mathcal{D}(A)}} \iint^{C\in !\mathbb{C},\ c\in\mathbb{C}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3\otimes A_3)}} [\![u]\!]\ (H_1\otimes A_1)(C\oplus[c],b) \times \left[\!\!\left[\frac{\partial v}{\partial x}\cdot s\right]\!\!\right] (H_2\otimes A_2)(c) \times \left[\prod_{k\in|C|} [\![v]\!]\ (X_k)(C_k)\right] \tag{8}$$

$$\cong \sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X)}} \sum_{\substack{(A_1,A_2,A_3)\\ \in\mathcal{D}(A)}} \iint^{C\in !\mathbb{C},\ c\in\mathbb{C}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3\otimes A_3)}} [\![u]\!]\ (H_1\otimes A_1)(C\oplus[c],b) \times [\![D(\lambda x.v)\cdot s]\!]\ (H_2)(A_2,c) \times \left[\prod_{k\in|C|} [\![v]\!]\ (X_k)(C_k)\right] \tag{9}$$

$$\cong \sum_{\substack{(H_1,H_2,H_3,H_4)\\ \in\mathcal{D}(X)}} \sum_{\substack{(A_1,A_2,A_3)\\ \in\mathcal{D}(A)}} \iiint^{\substack{C\in !\mathbb{C},\, c\in\mathbb{C},\\ a\in\mathbb{A}}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_4\otimes A_3)}} [\![u]\!]\,(H_1\otimes A_1)(C\oplus[c],b)\times[\![v]\!]\,(H_2\otimes A_2\otimes[a])(c)$$

$$\times[\![s]\!]\,(H_3)(a)\times\left[\prod_{k\in|C|}[\![v]\!]\,(X_k)(C_k)\right] \tag{10}$$

$$\cong \sum_{\substack{(H_1,H_2,H_3,H_4)\\ \in\mathcal{D}(X\otimes A)}} \iiint^{\substack{C\in !\mathbb{C},\, c\in\mathbb{C},\\ a\in\mathbb{A}}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_4)}} [\![u]\!]\,(H_1)(C\oplus[c],b)\times[\![v]\!]\,(H_2\otimes[a])(c)\times[\![s]\!]\,(H_3)(a)$$

$$\times\left[\prod_{k\in|C|}[\![v]\!]\,(X_k)(C_k)\right] \tag{11}$$

$$\cong \sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X\otimes A)}} \iiint^{\substack{C\in !\mathbb{C},\, c\in\mathbb{C},\\ a\in\mathbb{A}}} \sum_{\substack{(X_1,\dots,X_{|C|+1})\\ \in\mathcal{D}(H_3\otimes[a])}} [\![u]\!]\,(H_1)(C\oplus[c],b)\times[\![s]\!]\,(H_2)(a)\times\left[\prod_{k\in|C|+1}[\![v]\!]\,(X_k)((C\oplus[c])_k)\right] \tag{12}$$

$$\cong \sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X\otimes A)}} \iint^{C\in !\mathbb{C},\, a\in\mathbb{A}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3\otimes[a])}} [\![u]\!]\,(H_1)(C,b)\times[\![s]\!]\,(H_2)(a)\times\left[\prod_{k\in|C|+1}[\![v]\!]\,(X_k)(C_k)\right] \tag{13}$$

(7)-(8) Corollary 5.3 and Lemma A.4; (9) Inductive hypothesis; (10) Corollary 5.3 and Lemma A.4; (11)-(13) By consolidating the limits of summation and Lemma 5.4.

Finally, $[\![\left(\frac{\partial u}{\partial x}\cdot s\right) v]\!]^{\Gamma,x:A}\,(X\otimes A)(b)+[\![\left(Du\cdot\left(\frac{\partial v}{\partial x}\cdot s\right)\right) v]\!]^{\Gamma,x:A}\,(X\otimes A)(b)$

$$\cong \sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X\otimes A)}} \iint^{C\in !\mathbb{C},\, a\in\mathbb{A}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3)}} [\![u]\!]\,\,(H_1\otimes[a])(C,b)\times[\![s]\!]\,(H_2)(a)\times\left[\prod_{k\in|C|}[\![v]\!]\,(X_k)(C_k)\right]$$

$$+\sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X\otimes A)}} \iint^{C\in !\mathbb{C},\, a\in\mathbb{A}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3\otimes[a])}} [\![u]\!]\,(H_1)(C,b)\times[\![s]\!]\,(H_2)(a)\times\left[\prod_{k\in|C|+1}[\![v]\!]\,(X_k)(C_k)\right] \tag{14}$$

$$\cong \iint^{a\in\mathbb{A},\, C\in !\mathbb{C}} \sum_{\substack{(H_1,H_2,H_3)\\ \in\mathcal{D}(X\otimes A\otimes[a])}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3)}} [\![u]\!]\,(H_1)(C,b)\times[\![s]\!]\,(H_2)(a)\times\left[\prod_{k\in|C|}[\![v]\!]\,(X_k)(C_k)\right] \tag{15}$$

$$\cong \iint^{a\in\mathbb{A},\, C\in !\mathbb{C}} \sum_{\substack{(H_1,H_2,H_3)\in\mathcal{D}(X)\\ (A_1,A_2)\in\mathcal{D}(A\oplus[a])}} \sum_{\substack{(X_1,\dots,X_{|C|})\\ \in\mathcal{D}(H_3\otimes A_2)}} [\![u]\!]\,(H_1\otimes A_1)(C,b)\times[\![s]\!]\,(H_2)(a)\times\left[\prod_{k\in|C|}[\![v]\!]\,(X_k)(C_k)\right] \tag{16}$$

$$\cong \sum_{(H_1,H_2)\in\mathcal{D}(X)} \int^{a\in\mathbb{A}} [\![uv]\!]\,(H_1 \otimes A \otimes [a])(b) \times [\![s]\!]\,(H_2)(a) \tag{17}$$

$$\cong [\![D(\lambda x.uv) \cdot s]\!]^{\Gamma}_{A\to B}\,(X)(A,b) \tag{18}$$

(15) Lemma 5.6; (16) Rearranging the limits of summation and Lemma 5.4; (17) Corollary 5.2 and Lemma A.4; (18) Corollary 5.3 and Lemma A.4.

□

Proof of Proposition 7.1

- case $t \equiv 0$: for $G \in !\Gamma$ and $a \in \mathbb{A}$,

$$\begin{aligned}\sum_{i\in\mathtt{I}(0,G)} \mathtt{shape}(0,G)(i) \times \mathbb{A}(a,\mathtt{point}(0,G)(i)) &= \sum_{i\in\emptyset} \emptyset \times \mathbb{A}(a,a^*)\\ &= \emptyset\\ &= [\![0]\!]^{\Gamma}_{A}\,(G)(a)\end{aligned}$$

- case $t \equiv x$: Suppose $\Gamma \vdash x : A$ and $a \in \mathbb{A}$. If $G = [a^*]$ for some $a^* \in \mathbb{A}$, then

$$\begin{aligned}&\sum_{i\in\mathtt{I}(x,G)} \mathtt{shape}(x,G)(i) \times \mathbb{A}(a,\mathtt{point}(x,G)(i))\\ &\quad= \sum_{i\in\{1\}} \mathtt{shape}(x,G)(i) \times \mathbb{A}(a,\mathtt{point}(x,G)(i))\\ &\quad= \mathtt{shape}(x,G)(1) \times \mathbb{A}(a,\mathtt{point}(x,G)(1))\\ &\quad= \{1\} \times \mathbb{A}(a,a^*)\\ &\quad\cong \mathbb{A}(a,a^*)\\ &\quad\cong !\mathbb{A}([a],[a^*])\\ &\quad= !\mathbb{A}([a],G)\\ &\quad\cong !\Gamma([a],G)\\ &\quad= [\![x]\!]^{\Gamma}_{A}\,(G)(a)\end{aligned}$$

On the other hand, if G is not of the form $[a^*]$ for any $a^* \in \mathbb{A}$, then

$$\begin{aligned}&\sum_{i\in\mathtt{I}(x,G)} \mathtt{shape}(x,G)(i) \times \mathbb{A}(a,\mathtt{point}(x,G)(i))\\ &\quad= \sum_{i\in\emptyset} \mathtt{shape}(x,G)(i) \times \mathbb{A}(a,\mathtt{point}(x,G)(i))\\ &\quad= \emptyset\\ &\quad= !\Gamma([a],G)\\ &\quad= [\![x]\!]^{\Gamma}_{A}\,(G)(a)\end{aligned}$$

- <u>case $t \equiv u+v$</u>: Suppose $\Gamma \vdash u : A$ and $\Gamma \vdash v : A$. Then for any $G \in !\Gamma$,

$$\sum_{(i,j)\in \mathtt{I}(u+v,G)} \mathtt{shape}(u+v,G)(i,j) \times \mathbb{A}(a, \mathtt{point}(u+v,G)(i,j))$$

$$= \left[\sum_{(i,1)\in \mathtt{I}(u+v,G)} \mathtt{shape}(u+v,G)(i,1) \times \mathbb{A}(a, \mathtt{point}(u+v,G)(i,1))\right] + \left[\sum_{(i,2)\in \mathtt{I}(u+v,G)} \mathtt{shape}(u+v,G)(i,2) \times \mathbb{A}(a, \mathtt{point}(u+v,G)(i,2))\right]$$

$$= \left[\sum_{i\in \mathtt{I}(u,G)} \mathtt{shape}(u,G)(i) \times \mathbb{A}(a, \mathtt{point}(u,G)(i))\right] + \left[\sum_{i\in \mathtt{I}(v,G)} \mathtt{shape}(v,G)(i) \times \mathbb{A}(a, \mathtt{point}(v,G)(i))\right]$$

$$\cong [\![u]\!]^\Gamma_A (G)(a) + [\![v]\!]^\Gamma_A (G)(a)$$

$$= [\![u+v]\!]^\Gamma_A (G)(a)$$

- <u>case $t \equiv Du \cdot v$</u>: Suppose $\Gamma \vdash u : A \to B$ and $\Gamma \vdash v : A$. Since t is in normal form, both u and v must be in normal form, and u must not be an abstraction. Then the induction hypothesis applies to u. For any $G \in !\Gamma$,

$$[\![Du \cdot v]\!]^\Gamma_B (G)(A, b)$$

$$\cong \int^{a\in\mathbb{A}} \int^{F_1,F_2\in !\Gamma} [\![u]\!]^\Gamma_{A\to B} (F_1)(A \oplus [a], b) \times [\![v]\!]^\Gamma_A (F_2)(a) \times !\Gamma(F_1 \oplus F_2, G)$$

$$\cong \sum_{(G_1,G_2)\in \mathcal{D}(G)} \int^{a\in\mathbb{A}} [\![u]\!]^\Gamma_{A\to B} (G_1)(A \oplus [a], b) \times [\![v]\!]^\Gamma_A (G_2)(a)$$

$$\cong \sum_{i\in|\mathcal{D}(G)|} \int^{a\in\mathbb{A}} [\![u]\!]^\Gamma_{A\to B} (G_1^i)(A \oplus [a], b) \times [\![v]\!]^\Gamma_A (G_2^i)(a)$$

$$\cong \sum_{i\in|\mathcal{D}(G)|} \int^{a\in\mathbb{A}} \left[\sum_{j\in \mathtt{I}(u,G_1^i)} \mathtt{shape}(u,G_1^i)(j) \times (\mathbb{A} \Rightarrow \mathbb{B})((A \oplus [a], b), \mathtt{point}(u,G_1^i)(j))\right] \times [\![v]\!]^\Gamma_A (G_2^i)(a)$$

$$\cong \sum_{i\in|\mathcal{D}(G)|} \sum_{j\in \mathtt{I}(u,G_1^i)} \int^{a\in\mathbb{A}} \mathtt{shape}(u,G_1^i)(j) \times !\mathbb{A}(A^*, A \oplus [a]) \times \mathbb{B}(b, b^*) \times [\![v]\!]^\Gamma_A (G_2^i)(a) \quad \text{where } (A^*, b^*) = \mathtt{point}(u,G_1^i)(j)$$

$$\cong \sum_{i\in|\mathcal{D}(G)|} \sum_{j\in \mathtt{I}(u,G_1^i)} \int^{a\in\mathbb{A}} \mathtt{shape}(u,G_1^i)(j) \times \mathbb{B}(b, b^*) \times [\![v]\!]^\Gamma_A (G_2^i)(a) \times \sum_{a^*\in A^*} [\,!\mathbb{A}(A^* - [a^*], A) \times \mathbb{A}(a^*, a)]$$

$$\cong \sum_{i\in|\mathcal{D}(G)|}\sum_{j\in\mathtt{I}(u,G_1^i)}\sum_{a^*\in A^*}\int^{a\in\mathbb{A}} \mathtt{shape}(u,G_1^i)(j)\times\mathbb{B}(b,b^*)\times[\![v]\!]_A^\Gamma(G_2^i)(a) \times !\mathbb{A}(A^*-[a^*],A)\times\mathbb{A}(a,a^*)$$

$$\cong \sum_{i\in|\mathcal{D}(G)|}\sum_{j\in\mathtt{I}(u,G_1)}\sum_{a^*\in A^*} \mathtt{shape}(u,G_1^i)(j)\times\mathbb{B}(b,b^*)\times[\![v]\!]_A^\Gamma(G_2^i)(a^*)\times !\mathbb{A}(A^*-[a^*],A)$$

$$\cong \sum_{i\in|\mathcal{D}(G)|}\sum_{j\in\mathtt{I}(u,G_1^i)}\sum_{a^*\in A^*} \mathtt{shape}(u,G_1^i)(j)\times[\![v]\!]_A^\Gamma(G_2^i)(a^*)\times(\mathbb{A}\Rightarrow\mathbb{B})((A,b),(A^*-[a^*],b^*))$$

$$\cong \sum_{i\in|\mathcal{D}(G)|}\sum_{j\in\mathtt{I}(u,G_1^i)}\sum_{k\in|A^*|} \mathtt{shape}(u,G_1^i)(j)\times[\![v]\!]_A^\Gamma(G_2^i)(A_k)\times(\mathbb{A}\Rightarrow\mathbb{B})((A,b),(A^*-[A_k^*],b^*))$$

$$= \sum_{(i,j,k)\in\mathtt{I}(Du\cdot v,G)} \mathtt{shape}(u,G_1^i)(j)\times[\![v]\!]_A^\Gamma(G_2^i)(A_k)\times(\mathbb{A}\Rightarrow\mathbb{B})((A,b),(A^*-[A_k^*],b^*)) \quad \text{where } (A^*,b^*)=\mathtt{point}(u,G_1^*)(j)$$

$$= \sum_{(i,j,k)\in\mathtt{I}(Du\cdot v,G)} \mathtt{shape}(Du\cdot v,G)(i,j,k)\times\mathtt{point}(Du\cdot v,G)(i,j,k)$$

- <u>$t\equiv uv$ case</u>: Suppose $\Gamma\vdash u : A\to B$ and $\Gamma\vdash v : A$. Then for any $G\in !\Gamma$,

$[\![uv]\!]_B^\Gamma(G)(b)$

$$\cong \iiint^{A\in !\mathbb{A},H\in !\Gamma,\ N\in(!\Gamma)^{|A|}} [\![u]\!]_{A\to B}^\Gamma(H)(A,b)\times\left[\prod_{k\in|A|}[\![v]\!]_A^\Gamma(N_k)(A_k)\right]\times !\Gamma(H\otimes\bigoplus_{k\in|A|}N_k,G)$$

$$\cong \sum_{(H_1,H_2)\in\mathcal{D}(G,2)}\iint^{A\in !\mathbb{A},\ N\in(!\Gamma)^{|A|}} [\![u]\!]_{A\to B}^\Gamma(H_1)(A,b)\times\left[\prod_{k\in|A|}[\![v]\!]_A^\Gamma(N_k)(A_k)\right]\times !\Gamma(\bigoplus_{k\in|A|}N_k,H_2)$$

$$\cong \sum_{(H_1,H_2)\in\mathcal{D}(G,2)}\iint^{A\in !\mathbb{A},\ N\in(!\Gamma)^{|A|}} \left[\sum_{i\in\mathtt{I}(u,H_1)}\mathtt{shape}(u,H_1)(j)\times(\mathbb{A}\Rightarrow\mathbb{B})((A,b),\mathtt{point}(u,H_1)(j))\right] \times\left[\prod_{k\in|A|}[\![v]\!]_A^\Gamma(N_k)(A_k)\right]\times !\Gamma(\bigoplus_{k\in|A|}N_k,H_2)$$

$$\cong \sum_{(H_1,H_2)\in\mathcal{D}(G,2)}\sum_{j\in\mathtt{I}(u,H_1)}\iint^{A\in !\mathbb{A},\ N\in(!\Gamma)^{|A|}} \mathtt{shape}(u,H_1)(j)\times\left[\prod_{k\in|A|}[\![v]\!]_A^\Gamma(N_k)(A_k)\right]\times\mathbb{B}(b,b^*)\times !\mathbb{A}(A^*,A) \times !\Gamma(\bigoplus_{k\in|A|}N_k,H_2),\ \text{where } (A^*,b^*)=\mathtt{point}(u,H_1)(i)$$

$$\cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(G,2)}} \sum_{j\in\mathtt{I}(u,H_1)} \int^{N\in(!\Gamma)^{|A^*|}} \mathtt{shape}(u,H_1)(j) \times \left[\prod_{k\in|A^*|} [\![v]\!]^{\Gamma}_{A}(N_k)(A^*_k)\right] \times \mathbb{B}(b,b^*) \times !\Gamma(\bigoplus_{k\in|A^*|} N_k, H_2)$$

$$\cong \sum_{\substack{(H_1,H_2)\\ \in\mathcal{D}(G,2)}} \sum_{j\in\mathtt{I}(u,H_1)} \sum_{\substack{(G_1,\ldots,G_{|A^*|})\\ \in\mathcal{D}(H_2,|A^*|)}} \mathtt{shape}(u,H_1)(j) \times \left[\prod_{k\in|A^*|} [\![v]\!]^{\Gamma}_{A}(G_k)(A^*_k)\right] \times \mathbb{B}(b,b^*)$$

$$\cong \sum_{i\in|\mathcal{D}(G,2)|} \sum_{j\in\mathtt{I}(u,H_1)} \sum_{k\in|\mathcal{D}(G^i_2,|A^*|)|} \mathtt{shape}(u,G^i_1)(j) \times \left[\prod_{m\in|A^*|} [\![v]\!]^{\Gamma}_{A}((G^i_2)^k_m)(A^*_m)\right] \times \mathbb{B}(b,b^*)$$

$$\text{where } (A^*,b^*) = \mathtt{point}(u,G^i_1)(j)$$

$$= \sum_{(i,j,k)\in\mathtt{I}(uv,G)} \mathtt{shape}(uv,G)(i,j,k) \times \mathtt{point}(uv,G)(i,j,k)$$

□

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