Comparing intuitionistic quantum logics

From orthomodular lattices to frames

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Abstract

The aim of this dissertation is to explore two different forms of intuitionistic quantum logic, one due to Isham, Butterfield and Döring, and the second due to Coecke. The two schemes have very different philosophical and physical motivations, and we explore how this leads to differences in the lattice extensions that they introduce.

Isham, Butterfield and Döring suggested the so-called topos approach in which projections on a Hilbert space, or more generally in a von Neumann algebra, are mapped into a frame of subobjects of a particular presheaf by a map called daseinisation. Here, we present a factorization of the daseinisation embedding via an intermediate frame. The known properties of daseinisation are then recovered from this factorization. The relationship between our factorization of daseinisation and a free lattice construction is also considered. We also show that the frame generated by the codomain of daseinisation is in general strictly smaller than the frame of clopen subobjects considered to represent the quantum logic in this construction.

Central to the Coecke approach is the injective hull of a meet semilattice, originally described by Bruns and Lakser, and key to this construction are sets within a meet semilattice with distributive joins. The properties of these sets are analyzed in detail, primarily from a geometric perspective. We give a geometric characterization of sets with distributive joins in the lattice of projections on an arbitrary Hilbert space. By abstracting to an order theoretic viewpoint, this result is then extended to a characterization of such sets in a large class of complete lattices, in terms of their completely join irreducible elements.

Exploiting our factorization of daseinisation, we show the Coecke construction relates to the so called "inner daseinisation" of the topos approach, and symmetrically, daseinisation is related to the order theoretic dual of the Coecke construction. In the case of finite lattices, the topos construction is shown to be larger in size than the lattice of the Coecke approach. The question of universal properties for both schemes is also investigated, and a variety of adjunctions involving the two embeddings are described.

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Chapter 1

Introduction

This dissertation considers two different intuitionistic logics for describing propositions about systems governed by the laws of quantum physics. To introduce the concept of a logic of propositions for a general physical system, we consider some examples of the types of propositions we might wish to be able to model:

"Spin along the z-axis is up"

"Velocity is positive"

"Position is in the plane π"

In general these atomic propositions are statements about physical quantities taking values in certain sets of real numbers. Following [Döring and Isham, 2010], we represent the proposition "Physical quantity A has value in Δ" with the notation:

$$\text{“}A \,\epsilon\, \Delta\text{”} \tag{1.1}$$

A logic should provide ways to construct new propositions from old, by forming disjunctions:

"Position is in plane π" *or* "Velocity is positive"

conjunctions:

"Position is in plane π" *and* "Spin along the z-axis is up"

and negations:

not "Spin along the z-axis is up"

in some natural way. The objective of such a logic is to find suitable mathematical representatives of these propositions, consistent with the physical laws considered appropriate.

1.1 A logic of propositions for systems in classical physics

To construct a logic for systems governed by the the laws of classical physics, a mapping $[\![\_]\!]_{cl}$ taking propositions to their mathematical representatives is required. A system is considered to have a set of possible states, Σ, and each physical quantity A is represented by a function $\phi_A : \Sigma \to \mathbb{R}$, mapping each state of the system to the value of the physical quantity taken in that state.

To represent an atomic proposition about physical quantity A of the system, the idea is to choose the set of states that lead to values in the desired (measurable) set, by considering the inverse image:

$$[\![A \,\epsilon\, \Delta]\!]_{cl} := \phi_A^{-1}(\Delta) \tag{1.2}$$

To represent conjunctions of propositions, intuitively the appropriate states are those in which both conjuncts are satisfied. If p and q are propositions:

$$[\![p \;\; and \;\; q]\!]_{cl} := [\![p]\!]_{cl} \cap [\![q]\!]_{cl} \tag{1.3}$$

Dually, to represent disjunctions of propositions, the appropriate states are those in which either disjunct is satisfied. If p and q are propositions:

$$[\![p \;\; or \;\; q]\!]_{cl} := [\![p]\!]_{cl} \cup [\![q]\!]_{cl} \tag{1.4}$$

Finally we can represent the negation of a proposition p, we require the set of all states in which p does not hold, so we use the set theoretic complement:

$$[\![not \;\; p]\!]_{cl} := \Sigma \setminus [\![p]\!]_{cl} \tag{1.5}$$

In this way, we have constructed a logic of propositions about the classical system. If we order the sets with the obvious inclusion order, we have a Boolean σ-algebra of subsets of Σ, and so we get a classical logic with which to reason about the system.

1.2 Standard quantum logic

If the same approach is to be used to construct a logic for quantum systems, some modifications will be needed. Now a mapping $[\![\_]\!]_{qu}$ taking propositions to mathematical representatives consistent with the standard Hilbert space model of quantum systems is required. The Hilbert space model of quantum theory is fundamentally linear in nature, so a logic for a system governed by the laws of quantum physics cannot adopt arbitrary sets as the representatives of propositions. Instead, the natural representatives are closed subspaces of some Hilbert space, as described in the classic paper [Birkhoff and von Neumann, 1936]. Therefore, for atomic propositions we have:

$$[\![A \,\epsilon\, \Delta]\!]_{qu} := \sigma \tag{1.6}$$

where σ is some closed subspace that is obtained via the spectral theorem from the self-adjoint operator $\hat{A}$ representing the physical quantity A. The conjunction of two propositions p and q is given as in the classical case by:

$$[\![p \;\; and \;\; q]\!]_{qu} := [\![p]\!]_{qu} \cap [\![q]\!]_{qu} \tag{1.7}$$

This is a well defined operation as the intersection of two closed subspaces is another closed subspace. The negation of a proposition p is equally straightforward, the orthogonal complement of the closed subspace corresponding to p is the natural candidate:

$$[\![not \;\; p]\!]_{qu} := [\![p]\!]'_{qu} \tag{1.8}$$

Note that the orthogonal complement in a Hilbert space is not equivalent to the set-theoretic complement. Disjunction is more problematic, for the union of two closed subspaces is not in general another closed subspace. Therefore, as described by Birkhoff and von Neumann, we can take the representative of the disjunction of p and q as the least closed subspace containing the subspaces of the two disjuncts, given by the closure of their linear span:

$$[\![p \;\; or \;\; q]\!]_{qu} := \mathrm{cl}(\mathrm{lin}([\![p]\!]_{qu}, [\![q]\!]_{qu})) \tag{1.9}$$

If we order these subspaces by inclusion, again we have constructed a lattice of representatives of propositions about a physical system, but this lattice is not a Boolean algebra, in fact it is an orthomodular lattice. This structure will be referred to as *standard quantum logic*. As each closed subspace has a corresponding projection and vice versa, instead of the lattice of closed subspaces, we can equivalently consider the lattice of projection operators. This is generally more convenient, and is the approach adopted in this dissertation.

Whereas a Boolean algebra corresponds to classical logic, an orthomodular lattice does not have particularly good properties as a logic. For example, consider the distributive law:

$$\forall x, y, z.[x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)] \tag{1.10}$$

This law holds in a Boolean algebra, but is not true in general for an orthomodular lattice, which leads to awkward logical characteristics. As standard quantum logic has poor properties as a logical system, many attempts have been made to find other mathematical structures to provide a more satisfactory form of quantum logic. A detailed discussion of the characteristics and weaknesses of standard quantum logic can be found in [Chiara and Giuntini, 2002].

1.3 Intuitionistic quantum logics

1.3.1 Intuitionistic logic

The key idea of intuitionistic logic is to consider proofs as the central concept. Atomic propositions are associated with an abstract "proof object" establishing their correctness. If p and q are propositions, the conjunction:

$$p \wedge q \tag{1.11}$$

is associated with a pair, containing both a proof of p and a proof of q. Similarly, for p and q again propositions, the disjunction:

$$p \vee q \tag{1.12}$$

is associated with a discriminated union, containing either a labelled proof of p or a labelled proof of q. For p and q as before, the implication:

$$p \Rightarrow q \tag{1.13}$$

is represented by a function mapping proofs of p to proofs of q. The contradiction $\perp$ is the proposition with no proof. With $\perp$, and implication, we can encode the negation of proposition p as:

$$\neg p := p \Rightarrow \perp \tag{1.14}$$

In classical logic we can reason by contradiction, using the *law of the excluded middle*. For proposition p:

$$\vdash p \vee \neg p \tag{1.15}$$

Clearly this law cannot hold in the intuitionistic setting, as it would require a method for manufacturing, for arbitrary proposition p, a proof of either p or its negation.

Despite, or possibly because of, the absence of the law of the excluded middle, intuitionistic logic has excellent properties as a logical system, as witnessed by applications such as the well known Curry Howard isomorphism.

1.3.2 The logics under consideration

This dissertation will consider two different approaches to constructing an intuitionistic quantum logic. The first scheme, due to Isham, Butterfield and Döring, is described in [Döring, 2010b], [Isham, 2010] and [Döring and Isham, 2010], and will be referred to as *the topos approach*. The second scheme, due to Coecke, is described in [Coecke, 2002], and will be referred to as *the Coecke construction*.

$$L \xrightarrow{\quad \eta \quad} X(L)$$

Figure 1.1: Abstract form of the constructions

The two approaches have different physical and philosophical motivations, leading to very different mathematical properties. Abstractly though, both have the same simple form, as illustrated in figure 1.1. They both take some lattice L, representing properties of a physical system, and embed this lattice into an extension $X(L)$ which is a complete Heyting algebra. The focus of this dissertation will be:

- To explore some mathematical properties of the two approaches individually. This will involve analysis of the properties of the extension, and the associated embedding function.
- To relate and contrast the two approaches to each other. Here the extensions will be compared in terms of size and structure, and the associated embeddings described in relation to each other.

In particular the focus will be upon the lattice structures involved, and the embeddings into them. Both schemes have features outside of this scope that will not be considered in detail. The aim will be to provide answers to questions such as:

- Do the two approaches lead to the same lattice structure?
- Do the two approaches produce lattices that are dual in some sense?
- Which of the extensions is larger?
- Which morphisms between projection lattices lift to morphisms between the extensions?

1.4 Outline of the rest of the dissertation

This section describes the structure of the rest of the dissertation. The main contribution is given in the mathematical results of chapters 3 to 5. These are new results developed by the author during the project, except where otherwise indicated.

- Chapter 2 describes standard mathematical background required for later chapters. The material in this section is all standard, the purpose of this chapter is primarily to fix notation and definitions, and describe some examples that will be explored further in later sections.
- Chapter 3 explores the topos approach in detail. A factorization of the embedding used in this scheme is explored in some detail. The known properties of the embedding are then recovered from this factorization. The chapter then explores questions relating to the image of the embedding, and related frames.
- Chapter 4 explores the Coecke construction in detail. Much of the effort in this chapter is invested in exploring the key issue of distributive joins in projection lattices. The question of lifting morphisms between projection lattices to morphisms between their extensions is also investigated.
- Chapter 5 considers the two constructions in relation to each other, drawing on, and adding to results in the previous sections. The relative sizes of the two constructions are described. Then relationships between the various lattices involved are explored, and a variety of adjunctions between them are shown.

- Chapter 6 outlines some conclusions, and identifies some topics of interest for further study.

Chapter 2

Mathematical Background

2.1 Order Theory

This section contains an outline of the order theoretic material required in later sections, primarily with the intention of fixing notation and terminology. For a more detailed account of this material, there are many good references, such as [Davey and Priestley, 2002] and [Birkhoff, 1967]. Much of the notation used in this dissertation will parallel that adopted in [Davey and Priestley, 2002]. The naming conventions relating to locales and frames are influenced by the presentation in in [Johnstone, 1982].

2.1.1 Posets

Definition 2.1.1. If P is a set, a **partial order** on P is a reflexive, transitive, antisymmetric binary relation on P.

Definition 2.1.2. Let P be a set, and $\leq$ a partial order on P. Then the pair $(P, \leq)$ is a **partially ordered set** or **poset**. If the partial order can be inferred from the context, then we will often denote the poset simply as P.

Definition 2.1.3. Let $(P, \leq)$ and $(Q, \sqsubseteq)$ be partial orders. A function $f : P \to Q$ is said to be **monotone** if for all $x, y \in P$:

$$x \leq y \Rightarrow f(x) \sqsubseteq f(y) \tag{2.1}$$

Definition 2.1.4. Let $(P, \leq)$ be a poset. Two elements $x, y \in P$ are **uncomparable**, written $x \parallel y$, if $x \nleq y$ and $y \nleq x$.

Definition 2.1.5. For a poset $(P, \leq)$, we say that $y \in P$ **covers** $x \in P$, written $x \prec y$ if for all z:

$$x < y \ \textit{ and } \ x \leq z < y \Rightarrow x = z \tag{2.2}$$

Remark 2.1.6. The intuition for the covers relation is that if $x \prec y$, then y is "immediately above" x in the partial ordering.

Definition 2.1.7. Let $(P, \leq)$ be a poset. A set $A \subseteq P$ is an **upper set** if $x \in A$ and $x \leq y$ implies $y \in A$. Dually, a set A is a **lower set** if $x \in A$ and $y \leq x$ implies $y \in A$.

Definition 2.1.8. For a poset $(P, \leq)$, we define the sets:

$$\downarrow x := \{y \in P \mid y \leq x\} \tag{2.3}$$

$$\uparrow x := \{y \in P \mid y \geq x\} \tag{2.4}$$

A set of the form $\downarrow x$ is referred to as a **principal ideal**, and a set of the form $\uparrow x$ is referred to as a **principal filter**.

Definition 2.1.9. For a poset P the poset of **upper sets** of P with the inclusion order is denoted $\mathcal{U}P$ and the poset of **lower sets** of P with the inclusion order is denoted $\mathcal{D}P$.

The following straightforward result about upper sets will be used in later sections.

Lemma 2.1.10. *Any upper set can be written as a union of principal filters. Any lower set can be written as a union of principal ideals.*

Proof. If U is upward closed, we immediately have:

$$U = \bigcup_{u \in U} \uparrow u \tag{2.5}$$

The result for lower sets follows dually. □

Remark 2.1.11. Clearly, in general an upper (lower) set can be written as a union of principal filters (ideals) in many different ways.

Definition 2.1.12. For a poset P we denote the poset with the opposite order as P^{op}. i.e. $x \leq y$ in P^{op} if and only if $y \leq x$ in P.

Definition 2.1.13. For a poset $(P, \leq)$, for $A \subseteq P$ we define:

- $x \in P$ is an **upper bound** of A if for all $a \in A$, $x \geq a$
- $x \in P$ is a **lower bound** of A if for all $a \in A$, $x \leq a$
- A^u denotes the set of upper bounds of A in P
- A^l denotes the set of lower bounds of A in P

Definition 2.1.14. Let $(P, \leq)$ be a poset, $A \subseteq P$, and $a \in A$:

- $a \in A$ is a **minimal element** of A if $\forall b \in A. a \leq b$.
- $a \in A$ is a **maximal element** of A if $\forall b \in A. b \leq a$.
- A **top** for P, written $\top$, if it is exists, is a maximal element of P.
- A **bottom** for P, written $\bot$, if it exists, is a minimal element of P.
- A poset is **bounded** if it has both $\top$ and $\bot$.

2.1.2 Lattices

Definition 2.1.15. Let $(P, \leq)$ be a poset. For a set $A \subseteq P$, if the set A^u has a minimal element, this element is referred to as the **least upper bound** or **join** of A, written $\bigvee A$. Dually, if A^l has a maximal element, this element is referred to as the **greatest lower bound** or **meet** of A, written $\bigwedge A$.

Remark 2.1.16. If the poset P in which the join of some set S is being evaluated is potentially unclear, it will be indicated as follows:

$$\bigvee^{P} S \tag{2.6}$$

Similar notation will be used for meets if required.

Definition 2.1.17. Let $(P, \leq)$ be a poset, and $x, y \in P$. If $\bigvee\{x, y\}$ exists, it is referred to as the (binary) **join** of x and y, written $x \vee y$. Dually, if $\bigwedge\{x, y\}$ exists, it is referred to as the (binary) **meet** of x and y, written $x \wedge y$.

Remark 2.1.18. To avoid confusion between lattice operations and logical operators, a conjunction will be written:

$$x \;\; and \;\; y \tag{2.7}$$

and a disjunction as:

$$x \;\; or \;\; y \tag{2.8}$$

Other logical operators will be written in a similar manner.

Definition 2.1.19. Let $(L, \leq)$ be a poset. If every pair of elements in L has a join, and L has a bottom, L is referred to as a **join semilattice**. Dually, if every pair of elements in L has a binary meet, and L has a top, L is referred to as a **meet semilattice**.

Remark 2.1.20. The literature varies in whether a join/meet semilattice is required to have a bottom/top element. The lattices considered in this dissertation are invariably bounded, and so the convention described is most natural.

Definition 2.1.21. A **lattice** is a poset that is both a meet semilattice and a join semilattice.

Definition 2.1.22. Let L be a lattice. L is said to be **distributive** if all $x, y, z \in L$ satisfy the **distributive law**:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{2.9}$$

Definition 2.1.23. Let L be a lattice. L is said to be **modular** if all $x, y, x \in L$ satisfy the **modular law**:

$$x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z \tag{2.10}$$

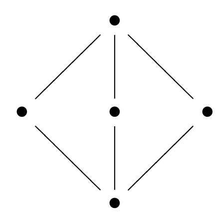

Figure 2.1: Lattice $\mathbf{M_3}$

Example 2.1.24. The lattice $\mathbf{M_3}$, with Hasse diagram shown in figure 2.1, is a lattice that is modular but not distributive. This lattice will be used in later examples.

Definition 2.1.25. Let L be a bounded lattice. An **orthocomplementation** is an operation $\neg : L \to L$ satisfying for all $x, y \in L$:

$$x \vee \neg x = \top \tag{2.11}$$

$$x \wedge \neg x = \bot \tag{2.12}$$

$$x \leq y \Rightarrow \neg y \leq \neg x \tag{2.13}$$

$$\neg\neg x = x \tag{2.14}$$

A bounded lattice equipped with an orthocomplement is said to be an **ortholattice**.

Definition 2.1.26. Let L be a lattice. If L is both a modular lattice and an ortholattice, L is said to be an **orthomodular lattice**.

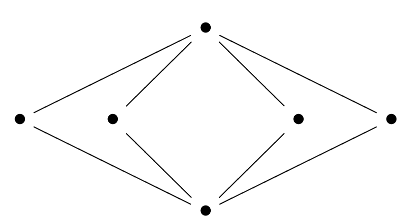

Figure 2.2: Lattice $\mathbf{M_4}$

Example 2.1.27. The lattice $\mathbf{M_3}$ is not an orthomodular lattice as it does not have a well defined orthocomplement, in particular there is no orthocomplement satisfying $\neg\neg x = x$ for all $x \in \mathbf{M_3}$. The lattice $\mathbf{M_4}$, with Hasse diagram as shown in 2.2, does have (several different ways of defining) a valid orthocomplement and is an example of an orthomodular lattice that is not distributive. This example will be used again in later sections.

Definition 2.1.28. A **complete join semilattice** is a poset in which every subset has a join. Dually, a **complete meet semilattice** is a poset in which every subset has a meet.

Definition 2.1.29. A **complete lattice** is a poset which is both a complete join semilattice and a complete meet semilattice.

Remark 2.1.30. It is well known that if a poset has arbitrary meets, it follows that it has arbitrary joins, and vice versa (proposition 3.5.2 in [Vickers, 1989]). The terms complete lattice, complete join semilattice and complete meet semilattice primarily serve to identify the correct morphisms to be considered.

Definition 2.1.31. A **Boolean algebra** is a distributive, orthocomplemented lattice.

Definition 2.1.32. Let L, M be join semilattices. A function $f : L \to M$ is said to **preserve all finite joins** if for all $x, y \in L$:

$$f(x \vee y) = f(x) \vee f(y) \tag{2.15}$$

and

$$f(\bot) = \bot \tag{2.16}$$

Preservation of all finite meets is defined dually.

Definition 2.1.33. Let L, M be complete join semilattices. A function $f : L \to M$ is said to **preserve arbitrary joins** if for all $A \subseteq L$:

$$f(\bigvee A) = \bigvee\{f(a) \mid a \in A\} \tag{2.17}$$

Preservation of arbitrary meets is defined dually.

Definition 2.1.34. For a poset P with $\bot$, we say $a \in P$ is an **atom** if $\bot \prec a$. The set of all atoms is denoted $\mathcal{A}(P)$. P is **atomic** if for all $b \in P$, $b \neq \bot$, there exists $a \in \mathcal{A}(P)$ such that $a \leq b$.

Definition 2.1.35. Lattice L is said to be **atomistic** if every element can be written as a join of elements from $\mathcal{A}(L)$.

Definition 2.1.36. Let L be a complete lattice, and $A \subseteq L$. A is said to be **join dense** in L if every element of L is the join of some subset of A. Dually, A is said to be **meet dense** in L if every element of L can be written as the meet of some subset of A.

Definition 2.1.37. Let P and Q be posets. If function $f : P \to Q$ is such that:

$$x \leq y \text{ iff } f(x) \leq f(y) \tag{2.18}$$

it is said to be an **ordering embedding**.

Definition 2.1.38. If P is a poset, L a complete lattice, and $\phi : P \to L$ an order embedding, then L is a **completion** of P.

Definition 2.1.39. A complete lattice L is said to satisfy the **infinite distributive law** if for all $a \in L$ and $B \subseteq L$:

$$a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b) \tag{2.19}$$

Definition 2.1.40. A complete lattice L which satisfies the infinite distributive law is called a **frame** or **locale**. There is a distinction in terms of the appropriate morphisms as will be described below.

Definition 2.1.41. A lattice L is a **Heyting algebra** if it is such that and for every map $a \wedge \_ : L \to L$, where $a \in L$, there exists a map $a \Rightarrow \_ : L \to L$, called the **Heyting implication**, such that for all $x, y \in L$:

$$a \wedge x \leq y \iff x \leq (a \Rightarrow y) \tag{2.20}$$

Remark 2.1.42. Categorically, the condition above is equivalent to requiring each functor $a \wedge \_ : L \to L$ to have a right adjoint.

Remark 2.1.43. It is straightforward to show L is a frame/locale if and only if it is a complete Heyting algebra. (See for example proposition 1.3.2 in [Borceux, 1994]). The different names, frame, locale and complete Heyting algebra, serve to identify the appropriate type of morphisms to be considered.

Definition 2.1.44. Let L be a complete lattice. L is said to satisfy the **complete distributive law** if for any doubly indexed family of elements in L, $(x_{i,j})_{i \in I, j \in J}$ we have:

$$\bigwedge_{i \in I} \bigvee_{j \in J} x_{i,j} = \bigvee_{f : I \to J} \bigwedge_{i \in I} x_{i, f(i)} \tag{2.21}$$

L is then said to be **completely distributive**.

Remark 2.1.45. The complete distributive law is a stronger form of the previous types of distributivity encountered. In particular the distributive law and the infinite distributive law are special cases of complete distributivity. The functions of type $I \to J$ are simply a technical expression of the need to "pick all possible combinations" of indices when calculating the meets and joins.

Definition 2.1.46. A lattice L is called a **coframe** if L^{op} is a frame.

Definition 2.1.47. Let L, M be Heyting algebras. A function $f : L \to M$ is said to **preserve Heyting implications** if for all $x, y \in L$:

$$f(x \Rightarrow y) = f(x) \Rightarrow f(y) \tag{2.22}$$

2.2 Category Theory

This dissertation will use some basic category theory, the reader will ideally be familiar with categories, functors, natural transformations and adjunctions. In particular, the notion of a free object will be used frequently. The standard reference is [MacLane, 1971], and many introductory texts exist, suitable for a variety of different backgrounds.

2.2.1 Definitions of relevant categories

Definition 2.2.1. We define the following categories of order structures:

- **Set**, the category with objects sets, and morphisms total functions.
- **Frm**, the category with objects frames, and morphisms functions that preserve arbitrary joins including the empty join, which is $\perp$, and finite meets.
- **CoFrm**, the category with objects coframes, and morphisms functions that preserve arbitrary meets including the empty meet, which is $\top$, and finite joins.
- **Loc** is the category with objects locales. A morphism in **Loc** corresponds to a morphism in the opposite direction in **Frm**, therefore $\mathbf{Loc} = \mathbf{Frm}^{op}$.
- **cHa**, the category with objects complete Heyting algebras, and morphisms frame morphisms that also preserve the Heyting implication.
- **MSLat**, the category with objects meet semilattices, and morphisms functions that preserve finite meets.
- **JSLat**, the category with objects join semilattices, and morphisms functions that preserve finite joins.
- **cMSLat**, the category with objects complete meet semilattices, and morphisms functions that preserve arbitrary meets.
- **cJSLat**, the category with objects complete join semilattices, and morphisms functions that preserve arbitrary joins.

2.3 Operators and projections

In our motivating discussion in the introduction, we had focussed on the lattice of projections on some Hilbert space $\mathcal{H}$. In the remainder of this dissertation we will consider the more general setting of the projections of some von Neumann algebra. For details related to von Neumann algebras and their projection lattices, a standard reference is both volumes of [Kadison and Ringrose, 1997a] and [Kadison and Ringrose, 1997b]. The following definitions will be required in later sections.

Definition 2.3.1. Let $\mathcal{H}$ be a Hilbert space. $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$.

By convention, all Hilbert spaces that we use here are complex.

Definition 2.3.2. If N is a von Neumann algebra, then $\mathcal{P}(N)$ is the set of all projections in N. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{P}(\mathcal{H})$ denotes the set of all projections on $\mathcal{H}$.

Remark 2.3.3. $\mathcal{P}(N)$ is a generalization of the orthomodular projection lattice of standard Birkhoff and von Neumann quantum logic.

Remark 2.3.4. Every von Neumann algebra N is a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ which is closed in the weak-operator topology, see [Kadison and Ringrose, 1997a]).

Chapter 3

The topos approach

3.1 Introduction

3.1.1 The topos approach

The topos approach to theoretical physics is a very general framework for describing physical theories, initiated in the series of papers [Isham, 1997], [Isham and Butterfield, 1998], [Isham and Butterfield, 1999], [Isham and Butterfield, 2000] and [Isham and Butterfield, 2002]. A comprehensive description of the scheme is given in [Döring and Isham, 2010].

This is the only section of this dissertation that may require some very basic understanding of topos theory, although little technical detail is required. A straightforward introduction with minimal prerequisites is given in [Lawvere and Rosebrugh, 2003]. Standard references include [MacLane and Moerdijk, 1992] and the comprehensive [Johnstone, 2002a], [Johnstone, 2002b]. In subsequent sections we will move to a simplified, but for our purposes equivalent, setting that does not require the machinery of topos theory.

Central to the topos approach are 3 concepts:

1. Some mathematical object Σ, the *state object*. This object models representations of the states of the system under consideration.
2. Some mathematical object $\mathcal{R}$, the *value object*. This object models the values that physical quantities can take.
3. Morphisms of the form $f : \Sigma \to \mathcal{R}$. These morphisms model the physical quantities of the sytem, capturing the idea of taking states to values.

The precise choice of these objects depends on the physical system being described. In order to enable the description of as broad a range of physical theories as possible, both [Isham, 2010] and [Döring and Isham, 2010] argue for a high level of freedom in the choice of suitable objects. For example, outside the world of classical physics, it is not necessarily reasonable to assume that physical quantities can take a continuum of values, and so the value object need not necessarily be the set of real numbers $\mathbb{R}$. In fact, as hinted at by the language above, neither Σ or $\mathcal{R}$ need to be sets at all, and may need to be described by some more general objects, and so the corresponding morphisms need not be set theoretic functions.

Amongst the many different perspectives that can be taken upon topos theory, it can be viewed as a "generalized set theory", the correspondence is outlined in table 3.1. It is this generalization that allows the topos approach to describe a great variety of physical theories.

Set theory	**Topos theory**
Sets	Objects
Cartesian Products	Products
Disjoint unions	Coproducts
Functions	Morphisms
Function Spaces	Exponentials
Subsets	Subobjects
Elements	Global elements

Table 3.1: Generalized set theory

In the topos approach, one then selects an appropriate topos to describe the physical system in question. For example, a system in classical physics will simply be modelled using the topos **Set**, with Σ some set of states, and $\mathcal{R}$ given by the set of real numbers, $\mathbb{R}$, and physical quantities then represented as functions between the state and value objects.

A different topos is chosen to describe the structure of quantum physics.

Definition 3.1.1. Let C be a category. A **presheaf**, with **base category** C, is a contravariant functor from C to **Set**. $\mathbf{Set}^{C^{op}}$ is the functor category with objects presheaves with base category C, and morphisms natural transformations.

Remark 3.1.2. Presheaves will be indicated by underlining, for example $\underline{F}$.

In one standard formulation of quantum theory, a quantum system is modelled by a von Neumann algebra, with physical quantities corresponding to the self adjoint operators in the algebra. For a general quantum system, it is well known that, under some natural conditions, it is impossible to assign values to all physical quantities at the same time. This is shown by the Kochen-Specker theorem, described in the context of von Neumann algebras in [Döring, 2005]. Certain families of physical quantities can be assigned values simultaneously, precisely those corresponding to self adjoint operators which commute with each other; these families provide different *classical perspectives* on the system. Rather than favour a particular perspective, the topos approach uses a category of presheaves to model all classical perspectives together.

Definition 3.1.3. Let N be a von Neumann algebra, then $\mathcal{V}(N)$ is the poset of all abelian von Neumann subalgebras, or **classical perspectives**, of N that contain the identity operator $\hat{1}$, except the trivial algebra $\mathbb{C}\hat{1}$. $\mathcal{V}(N)$ has the obvious inclusion order.

Remark 3.1.4. If $\mathcal{H}$ is a Hilbert space, $\mathcal{V}(\mathcal{H})$ will be used as shorthand for $\mathcal{V}(\mathcal{B}(\mathcal{H}))$.

A quantum system will be modelled in the category $\mathbf{Set}^{\mathcal{V}(N)^{op}}$, and can be understood as a category of sets fibred over the classical perspectives on the system. Interestingly the value object is not the real number object within the chosen topos. As the value object is not particularly relevant to later sections, its details are omitted, a straightforward description can be found in [Döring, 2010b].

Definition 3.1.5. Let V be an abelian C*-algebra. A **multiplicative state** is a positive linear functional of norm 1, $\lambda : V \to \mathbb{C}$, such that:

$$\forall \hat{A}, \hat{B} \in V : \lambda(\hat{A})\lambda(\hat{B}) = \lambda(\hat{A}\hat{B}) \tag{3.1}$$

The **Gel'fand spectrum** of N, Σ_V is defined as:

$$\Sigma_V := \{\lambda : N \to \mathbb{C} \mid \lambda \text{ a multiplicative state}\} \tag{3.2}$$

The Gel'fand spectrum of an abelian algebra can be viewed as a state space. In order to provide a state object capturing all classical contexts, these individual state spaces need to be combined in some coherent manner.

Remark 3.1.6. Every abelian von Neumann algebra also is an abelian C^*-algebra and therefore has a Gel'fand spectrum.

Definition 3.1.7. Let N be a von Neumann algebra. The **spectral presheaf**, $\underline{\Sigma} : \mathcal{V}(N)^{op} \to \mathbf{Set}$ is defined on objects as:

$$V \mapsto \Sigma_V \tag{3.3}$$

The (contravariant) action on arrows is for $f : V \to V'$ in $\mathcal{V}(N)^{op}$:

$$f_{V,V'} \mapsto (\lambda \mapsto \lambda|_{V'}) \tag{3.4}$$

The spectral presheaf, $\underline{\Sigma}$, is chosen as the state object Σ for a quantum system.

3.1.2 Daseinisation into a topos

Daseinisation is a key feature of the topos approach. It provides a method for mapping the usual projection lattice of a quantum system into the topos constructed to model the system. This can be seen as finding the mathematical representatives of propositions about the quantum system within the topos. These representatives will be subobjects of the spectral presheaf.

Definition 3.1.8. Let C be a small category, and $T : C^{op} \to \mathbf{Set}$ a contravariant functor. A functor $T' : C^{op} \to \mathbf{Set}$ is said to be a **subfunctor** of T if for each object c of C, $T'_c \subseteq T_c$, and for each $f : A \to B$ in C, $T'(f)$ is a restriction of $T(f)$.

Remark 3.1.9. The subfunctors in a presheaf category correspond to the subobjects, in the topos theoretic sense, as described in [MacLane and Moerdijk, 1992].

Attention will be restricted to a particular class of subobjects.

Definition 3.1.10. Let S be a topological space. Let $\mathcal{C}l(S)$ denote the set of all clopen (closed and open) subsets of S.

Definition 3.1.11. Let V be an abelian von Neumann algebra. As described in [Döring, 2010b], it can be shown that for every projection $\hat{P} \in \mathcal{P}(V)$ the set $\{\lambda \in \Sigma_V \mid \lambda(\hat{P}) = 1\}$ is a clopen (in the weak* topology) subset of Σ_V. Define the function α_V:

$$\alpha_V : \mathcal{P}(V) \to \mathcal{C}l(\Sigma_V) \tag{3.5}$$

$$\hat{P} \mapsto \{\lambda \in \Sigma_V \mid \lambda(\hat{P}) = 1\} \tag{3.6}$$

Remark 3.1.12. It is well-known that for an abelian von Neumann algebra V, α_V gives an isomorphism between the lattice of projections $\mathcal{P}(V)$ and the lattice of clopen subsets of Σ_V, $\mathcal{C}l(\Sigma_V)$ (see for example [Döring, 2010b]).

Definition 3.1.13. Let N be a von Neumann algebra. A subobject T of $\underline{\Sigma}$ is said to be a **clopen subobject** if for every $V \in \mathcal{V}(N)$, T_V is a clopen subset of $\underline{\Sigma}_V$. The lattice of all clopen subobjects of $\underline{\Sigma}$ is denoted $Sub_{cl}(\underline{\Sigma})$.

Remark 3.1.14. $Sub_{cl}(\underline{\Sigma})$ is a frame, with meets and joins defined locally at each subalgebra, and is the lattice that provides the intuitionistic quantum logic of the topos approach.

With these definitions in place, daseinisation can now be defined.

Definition 3.1.15. Let N be a von Neumann algebra. The **daseinisation** function $\delta : \mathcal{P}(N) \to Sub_{cl}(\underline{\Sigma})$ is defined as:

$$\delta : \mathcal{P}(N) \to Sub_{cl}(\underline{\Sigma}) \tag{3.7}$$

$$\hat{P} \mapsto \underline{\delta(\hat{P})} = (\underline{\delta(\hat{P})_V})_{V \in \mathcal{V}(N)}, \tag{3.8}$$

where, for each abelian subalgebra $V \in \mathcal{V}(N)$:

$$\underline{\delta(\hat{P})}_V := \alpha_V\left(\bigwedge(\uparrow \hat{P} \cap \mathcal{P}(V))\right) \tag{3.9}$$

It is straightforward to check that $\underline{\delta(\hat{P})}$ indeed is a presheaf, and by construction, it is a clopen subobject of $\underline{\Sigma}$.

Remark 3.1.16. The daseinisation function δ maps each projection (proposition) in the projection lattice of standard quantum logic into the frame of clopen subobjects of the spectral presheaf. It implements the idea of "approximation from above", finding the nearest weaker projection to the desired projection in each $V \in \mathcal{V}(N)$. As the result of daseinisation is a presheaf, it also has an action on arrows. As a subobject, it must map the inclusion morphism between abelian subalgebras $V \subset V' \in \mathcal{V}(N)$, $i_{V,V'} : V \to V'$ to the appropriate domain restriction of $\underline{\Sigma}(i_{V,V'})$. As these morphisms are completely defined by the action on objects, their description is often left implicit.

3.1.3 Daseinisation outside a topos

Working within a topos leads to undue complexities when considering daseinisation and the related intuitionistic logic. The topos theoretic formulation given in the previous section was introduced in order to provide a connection with the existing literature, and to motivate our interest in the lattice $Sub_{cl}(\underline{\Sigma})$. We observe that $Sub_{cl}(\underline{\Sigma})$ already is a set, namely the set of clopen subobjects of the spectral presheaf. In topos-theoretic terms $Sub_{cl}(\underline{\Sigma})$ is a topos-external object. In this section, we define in direct set-theoretic terms a lattice isomorphic to $Sub_{cl}(\underline{\Sigma})$, avoiding topos-theoretical concepts altogether. Daseinisation will then be rephrased in terms of this new lattice, decoupling the analysis of daseinisation from the details of topoi. This is the formulation that will then be used in all later sections.

Definition 3.1.17. Let N be a von Neumann algebra. Then the set:

$$\{f : \mathcal{V}(N)^{op} \to \mathcal{P}(N) \mid f \text{ monotone } \textit{and } \forall V \in \mathcal{V}(N).f_V \in \mathcal{P}(V)\} \tag{3.10}$$

will be referred to as the set of **well typed (contravariant)** monotone functions of type $\mathcal{V}(N)^{op} \to \mathcal{P}(N)$. $\mathcal{W}(N)$ will refer to the poset of well typed monotone functions, with the pointwise order. The term well typed refers to the restriction that each $V \in \mathcal{V}(N)$ is mapped to a projection in the corresponding $\mathcal{P}(V)$.

Remark 3.1.18. Following the convention adopted for presheaves, the component of a well typed function f at a given $V \in \mathcal{V}(N)$ will be written f_V to avoid excessive brackets in the notation.

Lemma 3.1.19. *Let N be a von Neumann algebra, then:*

$$\mathcal{W}(N) \cong Sub_{cl}(\underline{\Sigma}) \tag{3.11}$$

Proof. Define ϕ as the function:

$$\phi : \mathcal{W}(N) \to Sub_{cl}(\underline{\Sigma}) \tag{3.12}$$

$$\forall V \in \mathcal{V}(N).[\phi(f)_V := \alpha_V(f_V)] \tag{3.13}$$

Define ψ as the function:

$$\psi : Sub_{cl}(\underline{\Sigma}) \to \mathcal{W}(N) \tag{3.14}$$

$$\forall V \in \mathcal{V}(N).[\psi(\underline{F})_V := \alpha_V^{-1}(\underline{F}_V)] \tag{3.15}$$

ψ and ϕ are clearly monotone as α is an order isomorphism. To show $\psi \circ \phi = 1_{\mathcal{W}(N)}$, for each $V \in \mathcal{V}(N)$:

$$\begin{aligned}
(\psi \circ \phi)(f)_V &= \psi(\phi(f))_V && (3.16)\\
&= \alpha_V^{-1}(\phi(f)_V) \quad \text{definition} && (3.17)\\
&= \alpha_V^{-1}(\alpha_V(f_V)) \quad \text{definition} && (3.18)\\
&= f_V && (3.19)
\end{aligned}$$

To show $\phi \circ \psi = 1_{Sub_{cl}(\underline{\Sigma})}$, for each $V \in \mathcal{V}(N)$:

$$\begin{aligned}
(\phi \circ \psi)(\underline{F})_V &= \phi(\psi(\underline{F}))_V && (3.20)\\
&= \alpha_V(\psi(\underline{F})_V) \quad \text{definition} && (3.21)\\
&= \alpha_V(\alpha_V^{-1}(\underline{F}_V)) \quad \text{definition} && (3.22)\\
&= F_V && (3.23)
\end{aligned}$$

Therefore, ψ and ϕ witness an order isomorphism between $Sub_{cl}(\underline{\Sigma})$ and $\mathcal{W}(N)$ as required. □

It is not in general true that for a frame L, L^{op} is also a frame, but the next lemma shows this is the case for $\mathcal{W}(N)$, and therefore $Sub_{cl}(\underline{\Sigma})$.

Lemma 3.1.20. *For an arbitrary von Neumann algebra N, $\mathcal{W}(N)$ is a coframe.*

Proof. $\mathcal{W}(N)$ is a complete lattice as it is a frame. Meets and joins are calculated locally at each $V \in \mathcal{V}(N)$, and each $\mathcal{P}(V)$ is a complete Boolean algebra. Therefore, $\mathcal{P}(V)^{op}$ is a frame and so finite joins distribute over arbitrary meets in each $\mathcal{P}(V)$. It follows that finite joins distribute over arbitrary meets in $\mathcal{W}(N)$. □

Definition 3.1.21. Let N be a von Neumann algebra. The function $\delta : \mathcal{P}(N) \to \mathcal{W}(N)$ is defined locally at each $V \in \mathcal{V}(N)$ as:

$$\delta(\hat{P})_V := \bigwedge(\uparrow \hat{P} \cap \mathcal{P}(V)) \tag{3.24}$$

Remark 3.1.22. This function is (up to isomorphism) equivalent to the original form of daseinisation, and hence the symbol δ is still used. It has the advantages that:

- It exists outside of the topos.
- It avoids the complexities of the Gel'fand spectrum by working directly with the corresponding projections, which generally turns out to be more convenient.

3.2 Factoring daseinisation

In this section we analyze the factorization of the daseinisation construction via an intermediate frame. One possibility would be to factor via the free frame generated by a complete join semilattice. Unfortunately although this free frame is known to exist, see for example [Johnstone, 2002b], a concrete representation is not known. [1]

Instead, we will consider another natural factorization, via the frame of upward closed sets. The first half of the factorization is a relatively simple embedding of the original lattice into the upper set lattice. The second component is a more complex function that handles the approximation part of daseinisation.

A deliberate effort is made in this section not to use the known properties of daseinisation in the proofs presented. The motivation behind this decision is to improve understanding of where the properties of the functions come from in elementary terms, and then to recover the properties of daseinisation from its components, as will be done in a later section.

Definition 3.2.1. Let N be a von Neumann algebra. We define the function:

$$F : \mathcal{P}(N) \to (\mathcal{UP}(N))^{op} \tag{3.25}$$

$$\hat{P} \mapsto \uparrow \hat{P} \tag{3.26}$$

Remark 3.2.2. If we regard daseinisation as consisting of building a consistent family of approximations to an element in the original projection lattice, the function F can be seen as building the collection of all possible approximants, from above, to the chosen projection.

Definition 3.2.3. Let N be a von Neumann algebra. We define the function $S : (\mathcal{UP}(N))^{op} \to \mathcal{W}(N)$. S is defined locally at each $V \in \mathcal{V}(N)$:

$$S(U)_V := \bigwedge (U \cap \mathcal{P}(V)) \tag{3.27}$$

Remark 3.2.4. We can see the function as S as taking the collection of approximants built by F, and then finding the best approximation for each abelian von Neumann subalgebra of N.

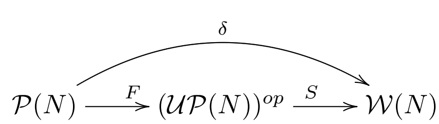

Figure 3.1: Factoring daseinisation

Lemma 3.2.5. $\delta = S \circ F$

[1] Peter Johnstone, private communication.

Proof. Let N be a von Neumann algebra. We reason locally at an arbitrary $V \in \mathcal{V}(N)$:

$$\delta(\hat{P})_V = \bigwedge(\uparrow\hat{P} \cap \mathcal{P}(V)) \tag{3.28}$$

$$= \bigwedge(F(\hat{P}) \cap \mathcal{P}(V)) \tag{3.29}$$

$$= S(F(\hat{P}))_V \tag{3.30}$$

□

Remark 3.2.6. The factorization of δ through $(\mathcal{UP}(N))^{op}$ is illustrated in figure 3.1.

So the function δ factors through $(\mathcal{UP}(N))^{op}$. Which properties the components of the factorization preserve will be examined in the following sections.

3.2.1 Properties of the function F

The next few lemmas capture some basic properties of F. F is a fairly standard construction, with little that is specific to daseinisation in it apart from its domain type, but some understanding of its properties will be required to fully understand the whole factorization.

Lemma 3.2.7. *For any von Neumann algebra N, F is monotone.*

Proof. Immediate as if $\hat{P} \leq \hat{Q}$ then $\uparrow\hat{Q} \subseteq\uparrow\hat{P}$, and $(\mathcal{UP}(N))^{op}$ has the reverse inclusion order. □

Lemma 3.2.8. *F is injective.*

Proof. Let N be a a von Neumann algebra. For arbitrary $\hat{P}, \hat{Q} \in \mathcal{P}(N)$:

$$\uparrow\hat{P} = \uparrow\hat{Q} \;\; \textit{iff} \;\; \hat{P} = \hat{Q} \tag{3.31}$$

□

Lemma 3.2.9. *F is not surjective.*

Proof. Let N be a von Neumann algebra, and let $\hat{P}, \hat{Q} \in \mathcal{P}(N)$ be such that $\hat{P} \parallel \hat{Q}$, then $\uparrow\hat{P} \cup \uparrow\hat{Q}$ is not of the form $\uparrow\hat{R}$ for some $\hat{R} \in \mathcal{P}(N)$. □

Lemma 3.2.10. *Let P be a poset. $(\mathcal{U}P)^{op}$ is a completely distributive lattice.*

Proof. $\mathcal{U}P$ is a complete lattice of sets, with join and meet given by set theoretic union and intersection, and is therefore completely distributive. Complete distributivity is a self dual concept, so $(\mathcal{U}P)^{op}$ is also completely distributive. □

Corollary 3.2.11. *$(\mathcal{U}P)^{op}$ is a frame and a coframe.*

Proof. This follows as complete distributivity is a self dual concept, and the infinite distributivity follows from complete distributivity. □

Lemma 3.2.12. *F preserves arbitrary joins.*

Proof. We note:

$$\bigvee_i \uparrow \hat{P}_i = \bigcap_i \uparrow \hat{P}_i = \uparrow \bigvee_i \hat{P}_i \tag{3.32}$$

Therefore:

$$F(\bigvee_i \hat{P}_i) = \uparrow \bigvee_i \hat{P}_i \qquad \text{definition} \tag{3.33}$$

$$= \bigvee_i \uparrow \hat{P}_i \qquad \text{observation above} \tag{3.34}$$

$$= \bigvee_i F(\hat{P}_i) \qquad \text{definition} \tag{3.35}$$

□

Corollary 3.2.13. *F preserves $\perp$.*

Lemma 3.2.14. *Let N be a von Neumann algebra, then for all $x, y \in \mathcal{P}(N)$:*

$$F(x \wedge y) \leq F(x) \wedge F(y) \tag{3.36}$$

Proof. As $x \wedge y$ is less than x and y, we have:

$$\uparrow x \subseteq \uparrow (x \wedge y) \;\; \textit{and} \;\; \uparrow y \subseteq \uparrow (x \wedge y) \tag{3.37}$$

From the definition of F:

$$F(x) \subseteq F(x \wedge y) \;\; \textit{and} \;\; F(y) \subseteq F(x \wedge y) \tag{3.38}$$

As the order is the reverse of inclusion order:

$$F(x \wedge y) \leq F(x) \wedge F(y) \tag{3.39}$$

□

Lemma 3.2.15. *Let N be a von Neumann algebra. Then:*

$$F(\top) \neq \top \tag{3.40}$$

Proof.

$$F(\top) = F(\hat{1}) = \{\hat{1}\} \neq \emptyset = \top \tag{3.41}$$

□

Lemmas (3.2.14) and (3.2.15) show that F does not behave well with respect to finite meets in the original projection lattice, preserving neither binary meets, nor the empty meet.

3.2.2 Structure preserved by S

Lemma 3.2.16. *S is monotone.*

Proof. Let N be a von Neumann algebra. We reason locally at an arbitrary subalgebra $V \in \mathcal{V}(N)$. Let $U_1, U_2 \in (\mathcal{UP}(N))^{op}$ be such that $U_1 \subseteq U_2$, in the normal inclusion order. Then $U_1 \cap \mathcal{P}(V) \subseteq U_2 \cap \mathcal{P}(V)$, and so, as meet is antitone in the size of the set, $S(U_2)_V \leq S(U_1)_V$. □

Lemma 3.2.17. *S preserves $\top$ and $\bot$.*

Proof. Let N be a von Neumann algebra, and $V \in \mathcal{V}(N)$. For $\top$:

$$S(\top)_V = \bigwedge(\emptyset \cap \mathcal{P}(V)) \tag{3.42}$$
$$= \bigwedge \emptyset \tag{3.43}$$
$$= \top_V \tag{3.44}$$

For $\bot$:

$$S(\bot)_V = \bigwedge(\uparrow \hat{0} \cap \mathcal{P}(V)) \tag{3.45}$$
$$= \hat{0} \tag{3.46}$$
$$= \bot_V \tag{3.47}$$

□

Lemma 3.2.18. *S preserves arbitrary meets.*

Proof. Let N be a von Neumann algebra , and let $(U_i)_{i \in I}$ be a family of upper sets. We reason locally: for every $V \in \mathcal{V}(N)$:

$$S(\bigwedge_i U_i)_V = S(\bigcup_i U_i)_V \qquad \text{definition} \tag{3.48}$$
$$= \bigwedge((\bigcup_i U_i) \cap \mathcal{P}(V)) \qquad \text{definition} \tag{3.49}$$
$$= \bigwedge \bigcup_i (U_i \cap \mathcal{P}(V)) \tag{3.50}$$
$$= \bigwedge_i \bigwedge (U_i \cap \mathcal{P}(V)) \qquad \text{meet over union} \tag{3.51}$$
$$= \bigwedge_i S(U_i)_V \qquad \text{definition} \tag{3.52}$$

□

The following simple lemma is a well known result, it is presented here for convenience as it will be used in subsequent proofs.

Lemma 3.2.19. *Let L be a complete lattice, and $(u_i)_{i \in I}$ be a family of elements in L. Then for monotone G, $\bigvee G(u_i) \leq G(\bigvee u_i)$.*

Proof. For all i as G is monotone:

$$G(u_i) \leq G(\bigvee u_i) \tag{3.53}$$

From which, as $G(\bigvee u_i)$ is an upper bound on the $G(u_i)$:

$$\bigvee G(u_i) \leq G(\bigvee u_i) \tag{3.54}$$

□

The next result shows that arbitrary joins are preserved by S for a particular restriction of the domain. This lemma is both a technical lemma the for proofs that follow, and it can also be seen as crucial to the join preservation of δ as shall be discussed later.

Lemma 3.2.20. *Let N be a von Neumann algebra, and $(\hat{P}_i)_{i\in I}$ be a family of projections in $\mathcal{P}(N)$. Then:*

$$\bigvee S(\uparrow \hat{P}_i) \geq S(\bigvee \uparrow \hat{P}_i) \tag{3.55}$$

Proof. For each $V \in \mathcal{V}(N)$:

$$\forall i. S(\uparrow \hat{P}_i)_V \geq \hat{P}_i \Rightarrow \bigvee S(\uparrow \hat{P}_i)_V \geq \bigvee \hat{P}_i \tag{3.56}$$

$$\Leftrightarrow \bigvee S(\uparrow \hat{P}_i)_V \in \uparrow (\bigvee \hat{P}_i) \cap \mathcal{P}(V) \tag{3.57}$$

$$\Rightarrow \bigvee S(\uparrow \hat{P}_i)_V \geq \bigwedge (\uparrow (\bigvee \hat{P}_i) \cap \mathcal{P}(V)) \tag{3.58}$$

$$\Leftrightarrow \bigvee S(\uparrow \hat{P}_i)_V \geq S(\uparrow (\bigvee \hat{P}_i))_V \tag{3.59}$$

$$\Leftrightarrow \bigvee S(\uparrow \hat{P}_i)_V \geq S(\bigvee \uparrow \hat{P}_i)_V \tag{3.60}$$

□

Corollary 3.2.21.

$$\bigvee S(\uparrow \hat{P}_i) = S(\bigvee \uparrow \hat{P}_i) \tag{3.61}$$

Proof. Immediate from lemma (3.2.19). □

The question of whether S preserves arbitrary joins turns out not to be entirely straightforward. Therefore, preservation of binary joins is considered first, as this avoids some of the difficulties of the general case.

Proposition 3.2.22. *Let N be a von Neumann algebra. S preserves binary joins in $(\mathcal{UP}(N))^{op}$.*

Proof. For every $V \in \mathcal{V}(N)$:

$$\begin{aligned}
S(U_1 \vee U_2)_V &= S(\bigcup_i \uparrow \hat{P}_i \cap \bigcup_j \uparrow \hat{Q}_j)_V && \text{lemma (2.1.10)} \quad (3.62)\\
&= S(\bigcup_i \bigcup_j (\uparrow \hat{P}_i \cap \uparrow \hat{Q}_j))_V && \text{distributivity} \quad (3.63)\\
&= \bigwedge((\bigcup_i \bigcup_j (\uparrow \hat{P}_i \cap \uparrow \hat{Q}_j)) \cap \mathcal{P}(V)) && \text{definition} \quad (3.64)\\
&= \bigwedge \bigcup_i \bigcup_j (\uparrow \hat{P}_i \cap \uparrow \hat{Q}_j \cap \mathcal{P}(V)) && \text{distributivity} \quad (3.65)\\
&= \bigwedge_i \bigwedge_j \bigwedge (\uparrow \hat{P}_i \cap \uparrow \hat{Q}_j \cap \mathcal{P}(V)) && \text{meet over union} \quad (3.66)\\
&= \bigwedge_i \bigwedge_j S(\uparrow \hat{P}_i \vee \uparrow \hat{Q}_j)_V && \text{definition} \quad (3.67)\\
&= \bigwedge_i \bigwedge_j (S(\uparrow \hat{P}_i)_V \vee S(\uparrow \hat{Q}_j)_V) && \text{corollary (3.2.21)} \quad (3.68)\\
&= (\bigwedge_i S(\uparrow \hat{P}_i)_V) \vee (\bigwedge_j S(\uparrow \hat{Q}_j)_V) && \text{each } \mathcal{P}(V) \in \mathbf{CoFrm} \quad (3.69)\\
&= S(\bigwedge_i \uparrow \hat{P}_i)_V \vee S(\bigwedge_j \uparrow \hat{Q}_j)_V && \text{lemma (3.2.18)} \quad (3.70)\\
&= S(\bigcup_i \uparrow \hat{P}_i)_V \vee S(\bigcup_j \uparrow \hat{Q}_j)_V && \text{definition} \quad (3.71)\\
&= S(U_1)_V \vee S(U_2)_V && (3.72)
\end{aligned}$$

□

Corollary 3.2.23. *S preserves all finite joins.*

Whether the projection lattices of all the abelian subalgebras satisfy the complete distributive law will be a crucial property for join preservation. The following lemma shows that a large class of von Neumann algebras have this property.

Lemma 3.2.24. *Let $M_n(\mathbb{C})$ be an n-dimensional matrix algebra. Then for every $V \in \mathcal{V}(M_n(\mathbb{C}))$, $\mathcal{P}(V)$ is completely distributive.*

Proof. As n is finite, and V is abelian, $\mathcal{P}(V)$ can only contain finitely many commuting projections. Every projection lattice of an abelian von Neumann algebra is a Boolean algebra, therefore $\mathcal{P}(V)$ is a finite Boolean algebra, and therefore isomorphic to the powerset lattice of some finite set. It follows that $\mathcal{P}(V)$ is completely distributive. □

Lemma 3.2.25. *Let N be a von Neumann algebra. If $\mathcal{P}(V)$ is completely distributive for each $V \in \mathcal{V}(N)$, then S preserves arbitrary joins.*

Proof. Firstly, we note that $(\mathcal{UP}(N))^{op}$ is completely distributive as it is a complete lattice of sets. Let $(U_i)_{i\in I}$ be an arbitrary family of upper sets in $(\mathcal{UP}(N))^{op}$. By lemma (2.1.10), each U_i can be written as a union of principal filters:

$$U_i = \bigcup_j \uparrow \hat{P}_{i,j} \tag{3.73}$$

We have for each $V \in \mathcal{V}(N)$:

$$\begin{aligned}
S(\bigvee_i U_i)_V &= S(\bigcap_i U_i)_V && \text{definition} && (3.74)\\
&= S(\bigcap_i \bigcup_j \uparrow \hat{P}_{i,j})_V && \text{lemma (2.1.10)} && (3.75)\\
&= S(\bigcup_{f:I\to J} \bigcap_i \uparrow \hat{P}_{i,f(i)})_V && \mathcal{UP}(N)^{op} \text{ compl. dist.} && (3.76)\\
&= \bigwedge_{f:I\to J} S(\bigcap_i \uparrow \hat{P}_{i,f(i)})_V && \text{lemma (3.2.18)} && (3.77)\\
&= \bigwedge_{f:I\to J} \bigvee_i S(\uparrow \hat{P}_{i,f(i)})_V && \text{corollary (3.2.21)} && (3.78)\\
&= \bigvee_i \bigwedge_j S(\uparrow \hat{P}_{i,j})_V && \text{assumption (compl. dist.)} && (3.79)\\
&= \bigvee_i S(\bigcup_j \uparrow \hat{P}_{i,j})_V && \text{lemma (3.2.18)} && (3.80)\\
&= \bigvee_i S(U_i)_V && && (3.81)
\end{aligned}$$

□

Corollary 3.2.26. *If $N = M_n(\mathbb{C})$ is a matrix algebra, then S preserves arbitrary joins.*

The following is a very straightforward property of S that will be required later. It just clarifies the fact that no approximation is required in subalgebras that contain the original projection.

Lemma 3.2.27. *Let N be a von Neumann algebra, and $V \in \mathcal{V}(N)$. If $\hat{P} \in \mathcal{P}(V)$, then $S(\uparrow \hat{P})_V = \hat{P}$.*

Proof. Immediate from the definition of S. For the assumptions above:

$$S(\uparrow \hat{P})_V = \bigwedge(\uparrow \hat{P} \cap \mathcal{P}(V)) = \hat{P} \tag{3.82}$$

□

Now we consider the converse of lemma (3.2.25).

Lemma 3.2.28. *Let N be a von Neumann algebra. If S preserves arbitrary joins, then for each $V \in \mathcal{V}(N)$, $\mathcal{P}(V)$ is completely distributive.*

Proof. We consider an arbitrary family $(U_i)_{i\in I}$ of upper sets. By lemma (2.1.10), each U_i can be written as a union of principal filters:

$$U_i = \bigcup_j \uparrow \hat{P}_{i,j} \tag{3.83}$$

First we manipulate $S(\bigvee_i U_i)_V$ into a suitable form. For all $V \in \mathcal{V}(N)$:

$$\begin{aligned}
S(\bigvee_i U_i)_V &= S(\bigcap_i U_i)_V && \text{definition} && (3.84)\\
&= S(\bigcap_i \bigcup_j \uparrow \hat{P}_{i,j})_V && U_i \text{ as components} && (3.85)\\
&= S(\bigcup_{f:I\to J} \bigcap_i \uparrow \hat{P}_{i,f(i)})_V && \mathcal{UP}(N)^{op} \text{ compl. dist.} && (3.86)\\
&= \bigwedge_{f:I\to J} S(\bigcap_i \uparrow \hat{P}_{i,f(i)})_V && \text{lemma (3.2.18)} && (3.87)\\
&= \bigwedge_{f:I\to J} \bigvee_i S(\uparrow \hat{P}_{i,f(i)})_V && \text{corollary (3.2.21)} && (3.88)
\end{aligned}$$

Now we manipulate $\bigvee_i S(U_i)_V$. Again for all $V \in \mathcal{V}(N)$:

$$\begin{aligned}
\bigvee_i S(U_i)_V &= \bigvee_i S(\bigcup_j \uparrow \hat{P}_{i,j})_V && U_i \text{ as components} && (3.89)\\
&= \bigvee_i S(\bigwedge_j \uparrow \hat{P}_{i,j})_V && \text{definition} && (3.90)\\
&= \bigvee_i \bigwedge_j S(\uparrow \hat{P}_{i,j})_V && \text{meet preservation, lemma (3.2.18)} && (3.91)
\end{aligned}$$

By assumption, from the above calculations, we can conclude that for all $V \in \mathcal{V}(N)$:

$$\bigwedge_{f:I\to J} \bigvee_i S(\uparrow \hat{P}_{i,f(i)})_V = \bigvee_i \bigwedge_j S(\uparrow \hat{P}_{i,j})_V \tag{3.92}$$

The above holds for an arbitrary family of upper sets, therefore we can fix some V and choose each U_i to be arbitrary unions of principal filters $\uparrow \hat{P}_{i,j}$ where each $\hat{P}_{i,j} \in \mathcal{P}(V)$. Applying lemma (3.2.27):

$$\bigwedge_{f:I\to J} \bigvee_i \hat{P}_{i,f(i)} = \bigvee_i \bigwedge_j \hat{P}_{i,j} \tag{3.93}$$

So each $\mathcal{P}(V)$ is completely distributive. ☐

Corollary 3.2.29. *S preserves arbitrary joins if and only if for each $V \in \mathcal{V}(N)$, $\mathcal{P}(V)$ is completely distributive.*

Corollary (3.2.29) shows that complete distributivity is critical for S to preserve arbitrary joins. It is interesting that S is not in general a morphism in **Frm**, and even when S does preserve arbitrary joins, it is significantly more complicated to show than the arbitrary meet preservation. This can be explained by taking a different perspective on the upper set lattice. It is well known [Johnstone, 2002b], [Johnstone, 1982] that the free frame generated by a meet semilattice is its down set lattice. δ is not a **MSLat** morphism, but it is a **cJSLat**, and therefore **JSLat** morphism. Combined with the observation that S is a morphism in **CoFrm**, it is therefore natural to consider the dual of the free frame generated by a meet semilattice.

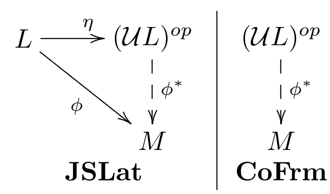

Figure 3.2: The free coframe for a join semilattice

Lemma 3.2.30. *The forgetful functor* **CoFrm** $\rightarrow$ **JSLat** *has a left adjoint whose action on objects is given by the mapping to the corresponding upper set lattice:*

$$L \mapsto (\mathcal{U}L)^{op} \tag{3.94}$$

The unit of the adjunction η is given by:

$$\eta_L : L \rightarrow (\mathcal{U}L)^{op} \tag{3.95}$$

$$l \mapsto \uparrow l \tag{3.96}$$

For a morphism $\phi : L \rightarrow M$ in **JSLat***, the unique extension to a morphism $\phi^* : (\mathcal{U}L)^{op} \rightarrow M$ in* **CoFrm** *is given by:*

$$\phi^* = \bigwedge\{\phi(u) | u \in U\} \tag{3.97}$$

Proof. Dual of the proof of the left adjoint of the forgetful functor of type **Frm** $\rightarrow$ **MSLat** given in lemma C1.1.3 of [Johnstone, 2002b]. $\square$

Remark 3.2.31. Phrased in terms of free objects, $(\mathcal{U}L)^{op}$ is the free coframe generated by the join semilattice L. Note that although η is a morphism in **JSLat**, it actually preserves arbitrary joins.

From lemma (3.1.20) we know that $\mathcal{W}(N)$ is a coframe. Therefore, we consider the extension of δ to the free coframe.

Lemma 3.2.32. *Let N be a von Neumann algebra. Viewing δ as a* **JSLat** *morphism, and with the notation as in lemma* (3.2.30)*:*

$$\delta^* = S \tag{3.98}$$

Proof. Let $U \in (\mathcal{UP}(N))^{op}$. As meets are calculated componentwise in δ, for each $V \in \mathcal{V}(N)$:

$$\delta^*(U)_V = \bigwedge\{\delta_V(u) \mid u \in U\} \quad \text{definition} \tag{3.99}$$
$$= \bigwedge_{u\in U} \bigwedge(\uparrow u \cap \mathcal{P}(V)) \quad \text{definition} \tag{3.100}$$
$$= \bigwedge \bigcup_{u\in U} (\uparrow u \cap \mathcal{P}(V)) \quad \text{meet over union} \tag{3.101}$$
$$= \bigwedge((\bigcup_{u\in U} \uparrow u) \cap \mathcal{P}(V)) \quad \text{distributivity} \tag{3.102}$$
$$= \bigwedge(U \cap \mathcal{P}(V)) \quad U \text{ an upper set} \tag{3.103}$$
$$= S(U)_V \quad \text{definition} \tag{3.104}$$

□

Lemmas (3.2.30) and (3.2.32) show that the factorization $\delta = S \circ F$ can be seen as factoring a join semilattice morphism via the free coframe. F is the unit of the adjunction, and S is the unique extension of δ to the free coframe, and could have been calculated directly from this definition. Many of the properties of F and S derived previously are direct consequences of this perspective on the construction.

We now move on to consider whether S is a morphism in **cHa**. Firstly a particular element is identified, with a property that is useful in proving later claims.

Definition 3.2.33. For a von Neumann algebra N, define:

$$\Omega := \mathcal{P}(N) \setminus \{\hat{0}\} \tag{3.105}$$

Remark 3.2.34. Ω is upward closed, as there are no projections beneath $\hat{0}$.

Lemma 3.2.35. $S(\Omega) = \bot$.

Proof. Let N be a von Neumann algebra. As by convention $\mathcal{V}(N)$ does not contain the trivial subalgebra $\mathbb{C}\hat{1}$, every $V \in \mathcal{V}(N)$ contains at least one projection not equal to $\hat{0}$ or $\hat{1}$, and its orthogonal complement. As Ω contains all projections except $\hat{0}$, it follows that for all $V \in \mathcal{V}(N)$:

$$S(\Omega)_V = \bigwedge(\Omega \cap \mathcal{P}(V)) = \bot \tag{3.106}$$

and the claim follows. □

The following lemma shows that negation is essentially trivial in the upper set frame.

Lemma 3.2.36. *Let N be a von Neumann algebra, and let $U \in (\mathcal{UP}(N))^{op}$, with $U \neq \bot$. Then $\neg U = \bot$.*

Proof. $\neg U$ is the greatest element such that:

$$U \wedge \neg U = \perp \tag{3.107}$$

Which by definition is equivalent to:

$$U \cup \neg U = \mathcal{UP}(N)^{op} \tag{3.108}$$

As $\perp$ is the only element containing $\hat{0}$, and $\hat{0} \notin U$, it follows that $\neg U = \perp$. □

We now exploit the element Ω to demonstrate that S is not a morphism in **cHa**.

Lemma 3.2.37. *S does not preserve negation.*

Proof.

$$\begin{aligned}
\neg S(\Omega) &= S(\Omega) \Rightarrow \perp && \text{definition} && (3.109)\\
&= \perp \Rightarrow \perp && \text{lemma (3.2.35)} && (3.110)\\
&= \top && && (3.111)\\
&\neq \perp && && (3.112)\\
&= S(\perp) && \text{lemma (3.2.17)} && (3.113)\\
&= S(\neg\Omega) && \text{lemma (3.2.36)} && (3.114)
\end{aligned}$$

□

Corollary 3.2.38. *S does not preserve Heyting implication.*

Proof. Follows as from lemma (3.2.17):

$$\neg S(\Omega) = (S(\Omega) \Rightarrow \perp) = (S(\Omega) \Rightarrow S(\perp)) \tag{3.115}$$

□

3.2.3 The relationship between $\mathcal{W}(N)$ and two free constructions

This short section contains two negative results, showing that the lattice of well typed function is not in general isomorphic to two relevant free frame constructions.

In the case of the free frame of a complete join semilattice, a concrete representation is not known, so we must show $\mathcal{W}(N)$ does not have the required universal property.

Lemma 3.2.39. *For a von Neumann algebra N, $\mathcal{W}(N)$ is not in general isomorphic to the free frame generated by $\mathcal{P}(N)$ as a complete join semilattice.*

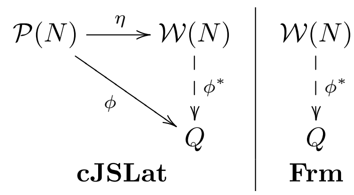

Figure 3.3: Assumption $\mathcal{W}(N)$ is the free frame

Proof. Let N be the von Neumann algebra $\{\hat{1}, \hat{P}\}''$ for some projection $\hat{P} \notin \{\hat{0}, \hat{1}\}$. In this case $\mathcal{P}(N)$ has four elements $\{\hat{0}, \hat{P}, \hat{1} - \hat{P}, \hat{1}\}$, and $\mathcal{W}(N)$ is isomorphic to $\mathcal{P}(N)$. We assume $\mathcal{W}(N)$ is the free frame generated by $\mathcal{P}(N)$ as a complete join semilattice, as illustrated in figure 3.3, and aim to show a contradiction. If we consider factoring the identity function through $\mathcal{W}(N)$, we see the unit of the corresponding adjunction must then be the obvious embedding of $\mathcal{P}(N)$ into $\mathcal{W}(N)$. Let $\phi : \mathcal{P}(N) \to \mathbf{2}$ be the **cJSLat** morphism mapping every element in $\mathcal{P}(N)$ to $\bot$. Now consider factoring ϕ via $\mathcal{W}(N)$. There can be no lifting of ϕ to a **Frm** morphism ϕ^* satisfying the universal property of the free frame, as any such ϕ^* must also map every element of $\mathcal{W}(N)$ to $\bot$, and therefore does not preserve $\top$. Therefore we have a contradiction. □

A concrete representation of the free frame of a meet semilattice is well known, and so in this case the argument can be more direct.

Lemma 3.2.40. *For a von Neumann algebra N, $\mathcal{W}(N)$ is not in general isomorphic to the free frame generated by $\mathcal{P}(N)$ as a meet semilattice.*

Proof. Let N be the von Neumann algebra $\{\hat{1}, \hat{P}\}''$ for some projection $\hat{P} \notin \{\hat{0}, \hat{1}\}$. In this case $\mathcal{P}(N)$ has four elements $\{\hat{0}, \hat{P}, \hat{1} - \hat{P}, \hat{1}\}$, and $\mathcal{W}(N)$ is isomorphic to $\mathcal{P}(N)$, and hence has 4 element. The free frame of a meet semilattice is isomorphic to the corresponding lower set lattice(see for example [Johnstone, 2002b]). This lattice has 6 element, and therefore cannot be isomorphic to $\mathcal{W}(N)$. □

Remark 3.2.41. The counterexamples in the previous two lemmas are for a particular minimal case. Such small cases are sometimes not representative, for example the important Gleason and Kochen-Specker theorems do not apply in 2-dimensions. It is possible that an isomorphism may still hold if some pathological cases are excluded.

3.2.4 Injectivity, surjectivity and related questions

This section considers questions of the injectivity and surjectivity of S. It then moves into related questions regarding the codomain of S.

It is straightforward to show that S is not fully injective, as an immediate corollary of some of the results from section 3.2.2.

Lemma 3.2.42. *S is not injective.*

Proof. From lemma (3.2.35) and lemma (3.2.17):

$$S(\top) = \top = S(\Omega) \tag{3.116}$$

□

The unusual element Ω is mapped to $\perp$ by S; the next lemma shows that there are no other elements except $\perp$ that are mapped onto $\perp$.

Lemma 3.2.43. *Let N be a von Neumann algebra and $U \in (\mathcal{UP}(N))^{op}$. If there exists projection $\hat{P} \in \mathcal{P}(N)$ such that $\hat{P} \neq \hat{0}$ and $\hat{P} \notin U$, then $S(U) \neq \perp$.*

Proof. Assume $U \in (\mathcal{UP}(N))^{op}$, $S(U) = \perp$ and there exists projection $\hat{P} \in \mathcal{P}(N)$ such that $\hat{P} \neq \hat{0}$ and $\hat{P} \notin U$. As $S(\emptyset) = \top$, U cannot be the empty set. As $\hat{1}$ must be in every non empty upper set, $\hat{P} \neq \hat{1}$. $\hat{0} \notin U$ as otherwise U must contain every projection. Therefore, we can consider the subalgebra $\{\hat{1}, \hat{P}\}''$, as $\hat{0} \notin U$ and $\hat{P} \notin U$, we have $S(U)_{\{\hat{1},\hat{P}\}''} \neq \hat{0}$, and so $S(U) \neq \perp$, a contradiction. □

The singleton set $\{\hat{1}\}$ is not the top element of the upper set lattice, as $\emptyset$ is also an upper set. It provides another counterexample to injectivity. Clearly the previous counterexample was sufficient to disprove injectivity, but understanding exactly where injectivity breaks down will provide insight into "how close" S is to being injective.

Lemma 3.2.44. *For a von Neumann algebra N,*

$$S(\{\hat{1}\}) = \top \tag{3.117}$$

Proof. For every $V \in \mathcal{V}(N)$:

$$S(\{\hat{1}\})_V = \bigwedge(\{\hat{1}\} \cap \mathcal{P}(V)) = \bigwedge\{\hat{1}\} = \top_V \tag{3.118}$$

□

$\{\hat{1}\}$ is mapped to $\top$ by S. The next lemma shows that there are no other elements except $\top$ that are mapped onto $\top$.

Lemma 3.2.45. *Let N be a von Neumann algebra and $U \in (\mathcal{UP}(N))^{op}$. If $S(U) = \top$ then $U \in \{\emptyset, \{\hat{1}\}\}$.*

Proof. Let $U \in (\mathcal{UP}(N))^{op}$. Assume $S(U) = \top$ and $U \notin \{\emptyset, \{\hat{1}\}\}$. Then there exists $\hat{P} \in U$ with $\hat{P} \neq \hat{1}$. If $\hat{P} = \hat{0}$ then $U = \perp$ and $S(\perp) \neq \top$. So $\hat{P} \notin \{\hat{0}, \hat{1}\}$. Consider the abelian subalgebra $V = \{\hat{1}, P\}''$, we must have:

$$S(U)_V \in \{\hat{0}, \hat{P}\} \tag{3.119}$$

and so $S(U) \neq \top$, a contradiction. □

The following proposition shows that Ω and $\{\hat{1}\}$ are actually the only elements that break injectivity. If the trivial subalgebra $\{\hat{1}\}'' = \mathbb{C}\hat{1}$ was added to $\mathcal{V}(N)$, Ω and $\perp$ could be distinguished, but $\top$ and $\{\hat{1}\}$ are both still mapped to $\top$. So lack of injectivity is not an artifact of not having the trivial subalgebra in $\mathcal{V}(N)$.

Proposition 3.2.46. *$S|_{(\mathcal{UP}(N))^{op}\setminus\{\top,\perp\}}$ is injective.*

Proof. Let U_1 and U_2 be upper sets, $U_1, U_2 \in (\mathcal{UP}(N))^{op} \setminus \{\top, \perp\}$. Assume $U_1 \neq U_2$. Without loss of generality, we can assume there exists projection $\hat{P}$ such that:

$$\uparrow\hat{P} \subseteq U_1 \ \ \textit{and} \ \ \uparrow\hat{P} \not\subseteq U_2 \tag{3.120}$$

As $\hat{1}$ is in every upward closed set except $\emptyset = \top$, and the only upward closed set containing $\hat{0}$ is $\mathcal{P}(N) = \perp$, and $\{\top, \perp\}$ are removed by the restriction, we have:

$$\hat{P} \neq \hat{0} \quad \textit{and} \quad \hat{P} \neq \hat{1} \tag{3.121}$$

We consider $V_0 \in \mathcal{V}(N)$ with $\mathcal{P}(V_0) = \{\hat{0}, \hat{P}, \hat{1} - \hat{P}, \hat{1}\}$. Then:

$$U_1 \cap \mathcal{P}(V_0) = \{\hat{P}, \hat{1}\} \ \textit{or} \ \{\hat{P}, \hat{1} - \hat{P}, \hat{1}\} \tag{3.122}$$

and:

$$U_2 \cap \mathcal{P}(V_0) = \{\hat{1} - \hat{P}, \hat{1}\} \ \textit{or} \ \{\hat{1}\} \tag{3.123}$$

In each case $\bigwedge(U_1 \cap \mathcal{P}(V_0)) \neq \bigwedge(U_2 \cap \mathcal{P}(V_0))$ and so $S(U_1)_{V_0} \neq S(U_2)_{V_0}$ from which:

$$S(U_1) \neq S(U_2) \tag{3.124}$$

as required. □

Corollary 3.2.47. *For a von Neumann algebra N, $S|_{F(\mathcal{P}(N))}$ is injective.*

Proposition 3.2.48. *Let N be a von Neumann algebra, $\hat{P} \in \mathcal{P}(N)$ a fixed projection such that $\hat{P} \notin \{\hat{0}, \hat{1}\}$. Also let $F : \mathcal{V}(N)^{op} \to \mathcal{P}(N)$ be the well typed function such that:*

$$F_V = \begin{cases} \hat{P} & \textit{if } V = \{\hat{1}, \hat{P}\}'' \\ \hat{0} & \textit{otherwise} \end{cases} \tag{3.125}$$

Then $F \in \mathcal{W}(N)$, and:

$$F \in \text{codomain}(S) \iff (\hat{1} - \hat{P}) \in \mathcal{A}(\mathcal{P}(N)) \tag{3.126}$$

Remark 3.2.49. The condition says that F is in codomain(S) if and only if the orthocomplement of $\hat{P}$ is an atom. We note that it may be the case in some von Neumann algebras that $\mathcal{P}(N)$ has no atoms.

Proof. F is clearly a well typed function. Assume there exists an upper set U such that $S(U) = F$. Then as $F_{\{\hat{1},\hat{P}\}''} = \hat{P}$, $\hat{P} \in U$ and $\hat{1} - \hat{P} \notin U$. As for all $\hat{R} \neq \hat{P}, \hat{1} - \hat{P}$, we have $F_{\{\hat{1},\hat{R}\}''} = \hat{0}$, $\hat{R} \in U$ and $\hat{1} - \hat{R} \in U$. Any larger abelian subalgebra will be mapped to $\hat{0}$ as it must contain some projection $\hat{R} \neq \hat{P}$ and hence also its complement $\hat{1} - \hat{R}$, and U contains R and its complement. Therefore, if $\hat{1} - \hat{P} \in \mathcal{A}(\mathcal{P}(N))$, U is an upward closed set, otherwise we have a contradiction, since each $\hat{R} < \hat{1} - \hat{P}$ would be in the upper set U, while $\hat{1} - \hat{P}$ is not. □

Corollary 3.2.50. *If the von Neumann algebra N is such that:*

$$\mathcal{P}(N) \setminus (\{\hat{0}, \hat{1}\} \cup \mathcal{A}(N)) \neq \emptyset \tag{3.127}$$

then S is not surjective. (In fact, $\mathcal{P}(N) \setminus (\{\hat{0}, \hat{1}\} \cup \mathcal{A}(N)) = \emptyset$ if and only if $N = M_2(\mathbb{C}) = \mathcal{B}(\mathbb{C}^2)$.)

Remark 3.2.51. Informally, the above condition simply requires the existence of a single non trivial projection in the projection lattice, that is not an atom.

Now that the question of surjectivity has been addressed, we consider the frames generated by arbitrary joins of finite meets of elements from the codomains of δ and S.

Definition 3.2.52. Define $Frm(\delta)$ as the frame generated by arbitrary joins of finite meets of elements in codomain(δ). Also define $Frm(S)$ as the frame generated by elements of codomain(S) in the same way.

Remark 3.2.53. The frame $Frm(\delta)$ is of interest as it contains all the logical combinations (infinite conjunctions are not admitted for the usual reason that they require infinite effort to affirm them) of elements inserted by daseinisation into $\mathcal{W}(N)$. $Frm(S)$ is of interest primarily as an upper bound for $Frm(\delta)$, as codomain $\delta \subseteq$ codomain(S).

Proposition 3.2.54. *Let N be a von Neumann algebra, $\hat{P} \in \mathcal{P}(N)$ a fixed projection such that $\hat{P} \notin \{\hat{0}, \hat{1}\}$, and $\hat{1} - \hat{P} \notin \mathcal{A}(\mathcal{P}(N))$. Also let $F : \mathcal{V}(N)^{op} \to \mathcal{P}(N)$ be the well typed function such that:*

$$F_V = \begin{cases} \hat{P} & \text{if } V = \{\hat{1}, \hat{P}\}'' \\ \hat{0} & \text{otherwise} \end{cases} \tag{3.128}$$

Then F cannot be written as an arbitrary join of finite meets of elements in the codomain of S.

Proof. As S preserves meets, it is sufficient to show F cannot be written as a join of elements in codomain(S). Assume $T \subseteq$ codomain(S) is such that

$$\bigvee T = F \tag{3.129}$$

Then as $F_{\{\hat{1},\hat{P}\}''} = \hat{P}$, for every $G \in T$, $G_{\{\hat{1},\hat{P}\}''} \leq \hat{P}$ i.e. $G_{\{\hat{1},\hat{P}\}''} \in \{\hat{P}, \hat{0}\}$. Also, as for every other subalgebra $V \in \mathcal{V}(N)$, $F_V = \hat{0}$, we have for every $G \in T$, $G_V = \hat{0}$. It follows that every $G \in T$ is either $S(\perp)$ or F. If T only contains elements equal to $S(\perp)$ then the join would not equal F. Therefore, T must contain an element equal to F, but by proposition (3.2.48), $F \notin \text{codomain}(S)$, a contradiction. □

Remark 3.2.55. The above lemma shows that $Frm(S)$ is not in general equal to $\mathcal{W}(N)$.

Proposition 3.2.56. *Let N be a von Neumann algebra. If $\mathcal{P}(V)$ is completely distributive for each $V \in \mathcal{V}(N)$, then* codomain(S) *is a complete sublattice of $\mathcal{W}(N)$. If N is such that:*

$$\mathcal{P}(N) \setminus (\{\hat{0}, \hat{1}\} \cup \mathcal{A}(N)) \neq \emptyset \tag{3.130}$$

i.e., if $N \neq M_2(\mathbb{C})$, then the inclusion is strict.

Proof. If each $\mathcal{P}(V)$ is completely distributive, S preserves arbitrary meets by lemma (3.2.18), and by lemma (3.2.25) it also preserves arbitrary joins. Therefore, codomain(S) is closed under arbitrary meets and joins, and is therefore a sublattice of $\mathcal{W}(N)$. If N is such that:

$$\mathcal{P}(N) \setminus (\{\hat{0}, \hat{1}\} \cup \mathcal{A}(N)) \neq \emptyset \tag{3.131}$$

then by corollary (3.2.50) there are elements in $\mathcal{W}(N)$ which are not in codomain(S), and therefore the latter is a strict complete sublattice. □

Theorem 3.2.57. *Let $N = \mathcal{B}(\mathcal{H})$ for some Hilbert space with $dim(\mathcal{H}) \geq 2$. Then there exists an element of* codomain(S) *that cannot be written as an arbitrary join of finite meets of elements in* codomain(δ).

Proof. Let $\hat{P}_\psi$ be a projection onto a ray, and let $U = \mathcal{P}(N) \setminus \{\hat{0}, \hat{P}_\psi\}$. Clearly, U is an upper set. Assume there exists a family of finite subsets of codomain(δ), $(M_i)_{i\in I}$ such that:

$$S(U) = \bigvee_{i\in I} \bigwedge M_i \tag{3.132}$$

There must exist some M_i, say M^*, such that:

$$\bigwedge M^* \neq \perp \tag{3.133}$$

Then $\perp \notin M^*$. For any projection onto a ray $\hat{Q} \in \mathcal{P}(N)$, $\hat{Q} \neq \hat{P}_\psi$, we have:

$$S(U)_{\{\hat{1},Q\}''} = \hat{0} \tag{3.134}$$

and therefore we must have:

$$(\bigwedge M^*)_{\{\hat{1},Q\}''} = \hat{0} \tag{3.135}$$

As $\perp \notin M^*$, this can only be true if there exist $F, G \in M^*$ such that:

$$F_{\{\hat{1},Q\}''} = \hat{Q} \ \ and \ \ G_{\{\hat{1},Q\}''} = \hat{1} - \hat{Q} \tag{3.136}$$

As, by assumption, Q is a projection onto a ray, $\hat{Q}$ is the only projection such that $\delta(Q)_{\{\hat{1},Q\}''} = \hat{Q}$, and so:

$$\delta(\hat{Q}) \in M^* \tag{3.137}$$

There are infinitely many such projections onto rays not equal to $\hat{P}_\psi$ for which this argument can be repeated, and so M^* cannot have finitely many elements, a contradiction. □

Remark 3.2.58. The theorem above shows that in general $Frm(\delta) \neq Frm(S)$.

To summarize the previous results, for von Neumann algebra N we have seen the following inclusions:

$$Frm(\delta) \subseteq Frm(S) \subseteq \mathcal{W}(N) \tag{3.138}$$

As theorem (3.2.57) and proposition (3.2.54) show, the inclusions are in general strict.

3.3 Recovering the known properties of daseinisation

In this section we look at how we can derive all the known properties of daseinisation given in [Döring, 2010b], from the properties of F and S. In general the proofs go through very straightforwardly, requiring little other than application of results derived previously.

Lemma 3.3.1. *Daseinisation preserves arbitrary joins.*

Proof. Let N be a von Neumann algebra, and $(\hat{P}_i)_{i\in I}$ be an arbitrary family of projections in $\mathcal{P}(N)$.

$$\delta(\bigvee_i \hat{P}_i) = SF(\bigvee_i \hat{P}_i) \quad \text{lemma (3.2.5)} \tag{3.139}$$

$$= S(\bigvee_i F(\hat{P}_i)) \quad \text{lemma (3.2.12)} \tag{3.140}$$

$$= \bigvee_i (SF(\hat{P}_i)) \quad \text{corollary (3.2.21)} \tag{3.141}$$

$$= \bigvee_i \delta(\hat{P}_i) \quad \text{lemma (3.2.5)} \tag{3.142}$$

□

Remark 3.3.2. In this proof we see that although arbitrary joins are not preserved in general by S, the preservation of joins of the special form in corollary (3.2.21) allows arbitrary joins in the projection lattice to pass through to $\mathcal{W}(N)$.

Corollary 3.3.3. *Daseinisation preserves* $\bot$.

Lemma 3.3.4.
$$\delta(\top) = \top \tag{3.143}$$

Proof.

$$\begin{aligned}
\delta(\top) &= SF(\top) && \text{lemma (3.2.5)} && (3.144)\\
&= S(\{\hat{1}\}) && \text{definitions} && (3.145)\\
&= \top && \text{lemma (3.2.44)} && (3.146)
\end{aligned}$$

□

Remark 3.3.5. Recovering preservation of $\top$ in this way is slightly awkward, due to the unusual path taken by the top element, which is not mapped to $\top$ in $(\mathcal{U}N)^{op}$, as shown in lemma (3.2.15).

Lemma 3.3.6. *Let N be a von Neumann algebra, $V \in \mathcal{V}(N)$, and $x, y \in \mathcal{P}(N)$. Then:*
$$\delta(x \wedge y) \leq \delta(x) \wedge \delta(y) \tag{3.147}$$

Proof. By lemma (3.2.14), for arbitrary $x, y \in \mathcal{P}(N)$:

$$F(x \wedge y) \leq F(x) \wedge F(y) \tag{3.148}$$

By monotonicity of S (lemma (3.2.16)), for all $V \in \mathcal{V}(N)$:

$$SF(x \wedge y)_V \leq S(F(x) \wedge F(y))_V \tag{3.149}$$

As S preserves arbitrary meets (lemma (3.2.18)):

$$SF(x \wedge y)_V \leq SF(x)_V \wedge SF(y)_V \tag{3.150}$$

Finally by lemma (3.2.5):

$$\delta(x \wedge y)_V \leq \delta(x)_V \wedge \delta(y)_V \tag{3.151}$$

□

Remark 3.3.7. Although we have recovered this result from the properties of the factorization, it is certainly easier in this case to show this result directly from the fact δ is monotone.

Lemma 3.3.8. *Daseinisation is injective.*

Proof. Follows directly from lemma (3.2.8) and corollary (3.2.47). □

Lemma 3.3.9. *Let N be a von Neumann algebra. For all $\hat{P}, \hat{Q} \in \mathcal{P}(N)$:*

$$\hat{P} < \hat{Q} \Rightarrow \delta(P) < \delta(Q) \tag{3.152}$$

Proof. That δ is monotone follows from lemma (3.2.7) and lemma (3.2.16) and the fact that composition preserves monotonicity. The strictness of the inequality then follows from lemma (3.3.8). $\square$

Lemma 3.3.10. *Daseinisation is not surjective.*

Proof. Follows immediately from corollary (3.2.50) $\square$

Chapter 4

Properties of the Coecke construction

4.1 Background

4.1.1 Introduction

This section analyzes another approach to the development of an intuitionistic quantum logic, as described in the paper [Coecke, 2002]. This paper addresses quantum logic from an operationalist perspective, leading to a very different construction to that seen in the topos approach. The operational methodology views a quantum system from the perspective of the experiments that can be performed on the system, and the outcomes of those experiments. Therefore, formal descriptions of experiments and their outcomes are required.

Definition 4.1.1. An **experimental project** is a specification of:

- An exact experimental procedure that can be performed on a physical system.
- The precise conditions constituting a positive result for the experiment.

A experimental project is said to be **certain** in a particular state of the system if a positive outcome is sure to occur in that state.

Remark 4.1.2. [Coecke, 2002] uses the slightly longer term "definite experimental project".

Remark 4.1.3. In this model of experiments, each experiment only has two possible outcomes, success or failure. An experiment can be seen as a test whether some criteria are satisfied.

Definition 4.1.4. A partial ordering on experimental projects can be defined. For projects α and β, $\alpha \leq \beta$ if whenever α is certain, β is also certain.

Definition 4.1.5. **Properties** of the physical system are identified with the equivalence classes of definite experiment projects induced by the partial ordering, denoted $[\alpha]$.

Definition 4.1.6. Let $(\alpha_i)_{i\in I}$ be a family of experimental projects. A **product experimental project** $\Pi_{i\in I}\alpha_i$ is an experimental project in which any one of the individual α_i may be chosen and performed. The property defined by $[\Pi_i\alpha_i]$ is actual if and only if each of the individual α_i are actual.

It is clear that equivalence classes of product experimental projects give the order theoretic meet with respect to the ordering on experimental projects:

$$\bigwedge_{i\in I}[\alpha_i] = [\Pi_{i\in I}\alpha_i] \tag{4.1}$$

A standard logical conjunction is true only if each of it's components is true. Similarly, given the above definition of meets, the property defined by the meet of experimental projects is actual if and only if each individual experimental project is actual. From an operational perspective this suggests meets be considered as conjunctions, as it done in [Coecke, 2002].

Remark 4.1.7. Product experiments of arbitrary families of experimental projects are used to define meets. Note that this does not require the implementation of some "infinite experimental procedure" as only one arbitrarily chosen experiment in the family is chosen to be performed.

As noted previously, if a lattice has arbitrary meets, it also has arbitrary joins, so the poset of properties is a complete lattice. If L is the lattice of properties, and $([\alpha_i])_{i\in I}$ a family of properties, join is defined in terms of the meet as:

$$\bigvee_{i\in I}[\alpha_i] = \bigwedge\{[\beta]\in L \mid \forall i\in I.[\alpha_i]\leq[\beta]\} \tag{4.2}$$

So the joins in this lattice can be seen as an artifact inherited from the construction of meets from physical and logical considerations. The question of which joins are desirable from the physical and logical perspective of operationalism is now considered.

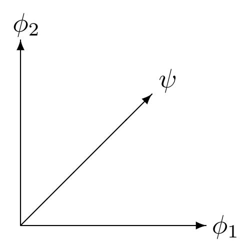

Figure 4.1: Superpositions

In the standard Hilbert space model, as described in the introduction, properties of a quantum system are identified with closed subspaces of a Hilbert space. The join of two properties is given by the least closed subspace containing the subspaces of the individual properties. Consider two properties with corresponding (distinct) subspaces the rays ϕ_1 and ϕ_2. The subspace corresponding to their disjunction is then the plane containing ϕ_1 and ϕ_2. If the system is in some state ψ in this plane, but not in ϕ_1 or ϕ_2, as shown in figure 4.1, then $\phi_1 \vee \phi_2$ is actual, without either ϕ_1 or ϕ_2 being actual. To consider the implications of this phenomenon for a logic of propositions two related concepts are introduced:

Definition 4.1.8. Let L be the complete lattice of properties, and $A \subseteq L$. $\bigvee A$ is said to introduce **superposition states** if there exists a state such that $\bigvee A$ is actual, but no $a \in A$ is actual.

Definition 4.1.9. Let L be the complete lattice of properties, and $A \subseteq L$. Let $c \in L$ and $c \leq \bigvee A$. c is said to be a **superposition property** introduced by $\bigvee A$ if there exists a state such that c is actual, but no $a \in A$ is actual.

As shown in [Coecke, 2002], if a set A has superposition properties then it also introduces superposition states. In general the converse does not hold, so an additional concept is introduced:

Definition 4.1.10. Let L be a property lattice. L is said to be **superpositionally faithful** if for each $A \subseteq L$, in every superposition state introduced by $\bigvee A$ there exists a superposition property introduced by $\bigvee A$, that is actual in that state.

Definition 4.1.11. Let L be a meet semilattice, and $A \subseteq L$. A has a **distributive join** if $\bigvee A$ exists and for every $x \in L$:

$$x \wedge \bigvee A = \bigvee_{a \in A} (x \wedge a) \tag{4.3}$$

Proposition 3 of [Coecke, 2002] then shows that a set of properties introduces superposition properties if and only if it does not have a distributive join.

From the operationalist perspective taken in the Coecke scheme, a proper disjunction is actual if and only if one of its components is actual. In the light of Proposition 3 from the paper, the join of a set of properties represents a proper disjunction if and only if the set has a distributive join. Via this argument, based upon physics, and philosophical consideration about the correct nature of conjunction and disjunction, a specification has been generated. The aim is then to find a suitable extension of the property lattice, with the following attributes:

- The embedding preserves meets as these have properties compatible with this perspective on quantum logic.
- The embedding preserves distributive joins.
- The resulting lattice has new joins added making it a complete Heyting algebra (frame), so that all joins are distributive, and an intuitionistic logic results.

Possibly surprisingly, this physically motivated specification is satisfied by a construction originally defined from a purely mathematical perspective. The appropriate construction is the injective hull of a meet semilattice, as described in [Bruns and Lakser, 1970]. The Coecke scheme is very general, applying to any complete lattice. In parts of this dissertation we will assume the conventional Hilbert space model of quantum physics, in which case the lattices of properties can be identified with the usual projection lattice.

Remark 4.1.12. The Coecke approach also comprises components beyond the core lattice extension, including an "operational resolution" for mapping propositions back to the level of properties. As the focus of attention in this dissertation is the lattice extensions, these features will not be explored further.

4.1.2 The injective hull of a meet semilattice

The mathematical entity central to the Coecke construction is the injective hull of a meet semilattice, first described in [Bruns and Lakser, 1970].

It is simpler to introduce the basic definitions in the more abstract setting of arbitrary algebras and their homomorphisms, and then specialize to the particular lattices of interest with these ideas in place.

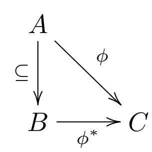

Figure 4.2: Injective extension

Definition 4.1.13. Let A, B, C be algebras, with A a subalgebra of B. Algebra C is said to be **injective** if and only if every homomorphism $\phi : A \to C$ has an extension to a homomorphism $\phi^* : B \to C$, as shown in figure 4.2.

Remark 4.1.14. In contrast with many categorical definitions of this type, there is no uniqueness requirement on ϕ^* for a given ϕ.

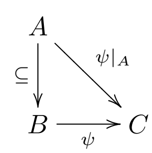

Figure 4.3: Essential extension

Definition 4.1.15. Let A, B be algebras, with A a subalgebra of B. B is said to be an **essential extension** of A if for any third algebra C and homomorphism $\psi : B \to C$, if ψ restricted to A is injective, then ψ is injective, as shown in 4.3.

Definition 4.1.16. Let A, B be algebras, A a subalgebra of B. B is said to be an **injective hull** of A if it is an essential injective extension of A.

Crucially, Theorem 1 of [Bruns and Lakser, 1970] shows that any injective extension of a meet semilattice will be a frame, so considering injective extensions of meet semilattices will lead to intuitionistic logics. The paper describes how every meet semilattice has an injective hull, unique up to isomorphism, and gives an explicit construction.

Definition 4.1.17. Let L be a meet semilattice, and $S \subseteq L$. S is an **admissible set** if it has a distributive join.

Definition 4.1.18. Let L be a meet semilattice, and S a lower set of L. Then S is a **D-ideal** if for every $U \subseteq S$, if U is an admissible set, $\bigvee U \in S$. The set of all D-ideals of L is denoted $DI(L)$.

Definition 4.1.19. Let L be a meet semilattice. Then we define:

$$B : L \to DI(L) \tag{4.4}$$

$$x \mapsto \downarrow x \tag{4.5}$$

As shown in [Bruns and Lakser, 1970], the injective hull of a meet semilattice is isomorphic to the lattice of D-ideals. The function B injects the original semilattice into its injective hull.

4.1.3 The MacNeille completion of a poset

For a poset P, the MacNeille completion is the (unique up to isomorphism) completion of P such that P is both join dense and meet dense in L. A categorical presentation of the MacNeille completion is described in [Banaschewski and Bruns, 1967].

Remark 4.1.20. Although the other constructions in this section apply only to particular types of (semi)lattices, the MacNeille completion is more general in that it applies to any poset.

Concretely, the MacNeille completion of a poset P is given by the lattice of all $A \subseteq P$ such that $A^{ul} = A$, i.e., the sets A which are equal to the collection of lower bounds of their upper bounds, with the usual inclusion order. P is embedded into this lattice by the map:

$$x \mapsto \downarrow x \tag{4.6}$$

Remark 4.1.21. Clearly, all sets $A \subseteq P$ such that $A^{ul} = A$ will be lower sets.

4.1.4 The free frame generated by a meet semilattice

As every frame is a meet semilattice, and every **Frm** morphism is an **MSLat** morphism, there is a forgetful functor **Frm** $\to$ **MSLat**. This functor has a left adjoint as described for example in [Joyal and Tierney, 1984]. The action of the left adjoint on objects is:

$$L \mapsto \mathcal{D}L \tag{4.7}$$

The unit of the adjunction $\eta : 1 \to \mathcal{D}L$ is given for an arbitrary meet semilattice L by:

$$\eta_L : L \to \mathcal{D}L \tag{4.8}$$

$$x \mapsto \downarrow x \tag{4.9}$$

η is clearly an order embedding of L into $\mathcal{D}L$.

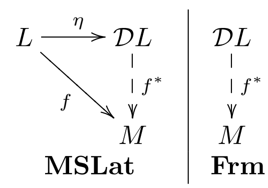

Figure 4.4: The free frame generated by a meet semilattice

Definition 4.1.22. For a meet semilattice L, the **free frame** generated by L is given by $\mathcal{D}L$, and has the universal property that for any other frame M, and morphism f in **MSLat**, there exists a unique morphism $f^* : \mathcal{D}L \to M$ in **Frm** such that:

$$f = f^* \circ \eta_L \tag{4.10}$$

as shown in figure 4.4.

Remark 4.1.23. Free objects in category theory are relative to objects and their respective morphisms in some other category. For example, the free frame we have defined here in **Frm** is relative to a meet semilattice in **MSLat**. There are other free frame constructions, for example the free frame over a complete join semilattice in **cJSLat**. Occasionally, this dissertation will simply refer to "the free frame" when the categories involved are clear from the context.

4.2 Small concrete examples of the Bruns-Lakser construction

To develop some intuition for the behaviour of the Bruns-Lakser construction, we consider the injective hull of some small meet semilattices, and compare this to the corresponding MacNeille completion and free frame constructions for those lattices. The lattices encountered in the Coecke scheme will all be complete, but we start in this slightly more general setting so that we can contrast the behaviour of the Bruns-Lakser construction with the MacNeille completion. The lattices considered will be small enough that we can visualize them using Hasse diagrams. An introduction to Hasse diagrams, if required, is given in 1.15 of [Davey and Priestley, 2002].

Example 4.2.1. Ideally, we would consider a small meet semilattice to show how joins are added by the injective hull construction. Unfortunately, a finite meet semilattice always has all meets, therefore all joins, and therefore is a join semilattice as well. So we first consider a poset with binary meets, but no top element, as shown in figure 4.5, and examine the D-ideal structure:

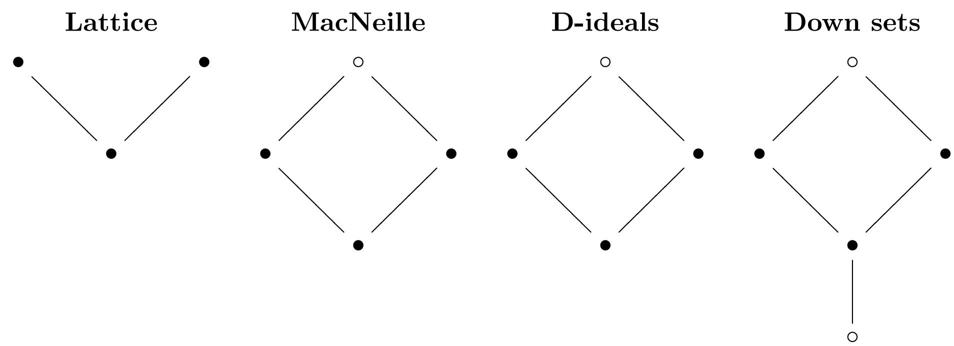

Figure 4.5: Extending a meet semilattice

We see that the D-ideal lattice very efficiently adds only one element to transform the original lattice to a frame. The down set construction adds an additional element; generally the free frame will be larger than the injective hull of a meet semilattice. The MacNeille completion is in this case the same as the Bruns-Lakser construction. In general we will find that the MacNeille lattice embeds into the the D-ideal lattice, as will be made mathematically precise in later sections.

Example 4.2.2. We consider the injective hull of a frame, as shown in figure 4.6.

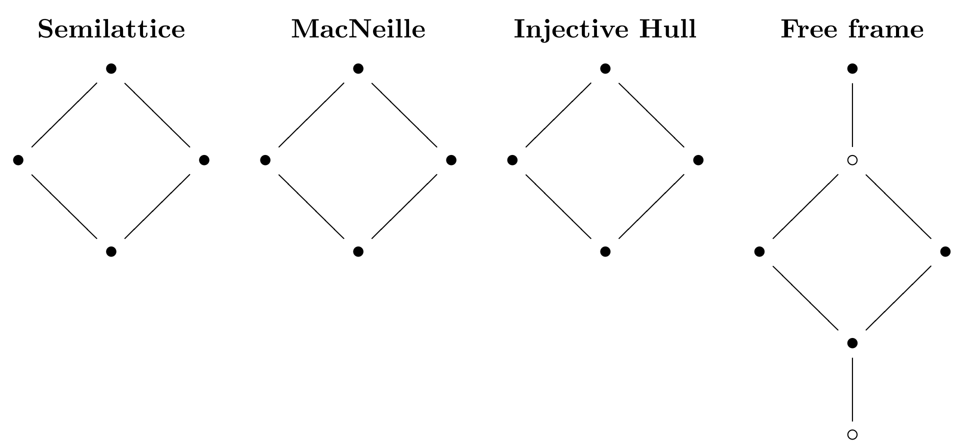

Figure 4.6: Extending a frame

Both the MacNeille completion and injective hull are isomorphic to the original lattice in this case. Intuitively, both constructions do not need to do any work as the source lattice is already a frame, and therefore is complete and satisfies the infinite distributive law. By comparison, the free frame of a frame viewed as a meet semilattice is always strictly larger than the original. Informally, we can see this is necessarily so, as any frame is an object in **MSLat**. It is easy to find **MSLat** morphisms between frames that are not frame morphisms, for example mapping every element to $\top$, and so the free frame requires more elements to provide the "extra flexibility" needed to define the **Frm** morphism satisfying the universal property.

Example 4.2.3. Now we illustrate the action of the various constructions on the $\mathbf{M_3}$ lattice, which is complete, modular, but not distributive.

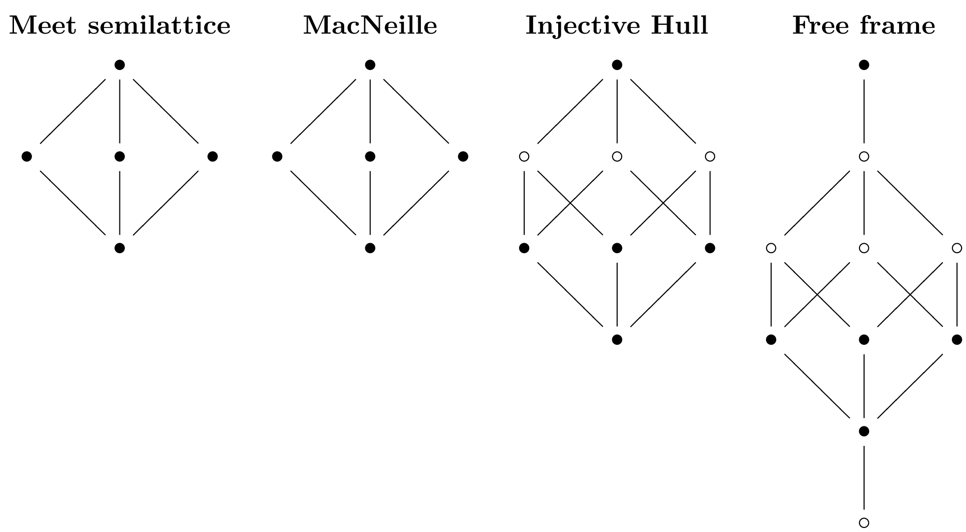

Figure 4.7: Extending non distributive lattice $\mathbf{M_3}$

In this case the Bruns-Lakser extension needs to "repair distributivity" in the lattice. The lattice is already complete, so we can see this as meaning the MacNeille completion has nothing to do. As with the previous examples the injective hull does this by inserting less elements than the free frame construction.

4.3 Characterization of admissible sets and D-ideals

This section concentrates on the characterization of sets which have distributive joins in various types of lattices. Also included are some results that directly characterize which sets are D-ideals.

4.3.1 Basic results

This section covers some basic results that provide straightforward tests for distributive joins and D-ideals. Firstly, the empty set is never a D-ideal in the situations relevant to the Coecke construction.

Lemma 4.3.1. *For any meet semilattice L with bottom element, the empty set is an admissible set.*

Proof. For an arbitrary $x \in L$:

$$x \wedge \bigvee \emptyset = x \wedge \bot = \bot = \bigvee_{y \in \emptyset} (x \wedge y) \tag{4.11}$$

□

Corollary 4.3.2. *Let L be a meet semilattice with a bottom element. Then $\emptyset$ is not a D-ideal.*

The next result shows that the "MacNeille completion component" of the D-ideal lattice of a complete lattice will be isomorphic to the original lattice. As was observed in the earlier examples, the MacNeille completion "does no work" if a lattice is already complete.

Lemma 4.3.3. *Let L be a complete lattice and $A \subseteq L$. Then $A^{ul} = A$ if and only if A is a principal ideal.*

Proof. If $A^{ul} = A$, then $\bigvee A \in A$, and so A is a principal ideal. Now we assume A is a principal ideal. In a complete lattice, we have:

$$\bigwedge A^u \in A^{ul} \tag{4.12}$$

Also in a complete lattice:

$$\bigvee A = \bigwedge A^u = \bigvee A^{ul} \tag{4.13}$$

Therefore as A^{ul} is clearly downward closed, and $\bigvee A^{ul} \in A^{ul}$, it follows that A^{ul} is a principal ideal. As A is also a principal ideal by assumption, and $\bigvee A = \bigvee A^{ul}$, it must be the case that $A = A^{ul}$ as required. □

Corollary 4.3.4. *The MacNeille completion of a complete lattice is isomorphic to the original lattice.*

Proof. The claim follows as any poset is isomorphic to its poset of principal ideals, with the inclusion order. □

Remark 4.3.5. All the lattices relevant to the Coecke construction will be complete lattices, the above lemma shows that all the subsets encountered such that $A^{ul} = A$ will in fact be principal ideals.

For a set to be a D-ideal, two criteria must be evaluated:

1. Is the set downward closed?
2. Does the join of any admissible set "jump outside" of the set? This is the more difficult criterion to check directly, as potentially a large number of sets need to have their admissibility and joins assessed.

The following lemma, and its proof, make explicit why all principal ideals are D-ideals. This clearly must be true for the Bruns-Lakser construction to work as specified. From the point of view of the criteria above, the second criterion becomes vacuous as every join that exists is contained within the original set, so we need not concern ourselves about admissibility.

Lemma 4.3.6. *If L is a meet semilattice, and $A \subseteq L$ a principal ideal, then for every $B \subseteq A$, either $\bigvee B$ does not exist or $\bigvee B \in A$.*

Proof. Let A be a principal ideal, and $B \subseteq A$. Then as A is a principal ideal, $\bigvee A$ exists and is an upper bound on B, and $\bigvee A \in A$. As A is downward closed, and an upper bound for B is in A, then if $\bigvee B$ exists, it is in A. □

Corollary 4.3.7. *Every principal ideal of a meet semilattice is a D-ideal.*

4.3.2 General results about admissible sets

This subsection concentrates on some technical lemmas that reduce the effort required to identify sets with distributive joins. Firstly, we introduce a small lemma with a corollary that means we can effectively ignore top and bottom elements when checking if a set is admissible.

Lemma 4.3.8. *Let P be a poset and $S \subseteq P$ such that $\bigvee S$ exists. Then for all $a \in S^u \cup S^l$:*

$$a \wedge \bigvee S = \bigvee_{s \in S} (s \wedge a) \tag{4.14}$$

Proof. For $a \in S^u$:

$$\begin{aligned} a \wedge \bigvee S &= \bigvee S && a \in S^u && (4.15)\\ &= \bigvee_{s\in S} s && && (4.16)\\ &= \bigvee_{s\in S} (s \wedge a) && a \in S^u && (4.17)\end{aligned}$$

For $b \in S^l$:

$$\begin{aligned} b \wedge \bigvee S &= b && b \in S^l && (4.18)\\ &= \bigvee_{s\in S} b && && (4.19)\\ &= \bigvee_{s\in S} (b \wedge s) && b \in S^l && (4.20)\end{aligned}$$

□

Corollary 4.3.9. *In a bounded lattice, $\bot$ and $\top$ always distribute over arbitrary joins.*

The next lemma shows a very large, but somewhat trivial class of distributive sets exists. This particular class has no impact upon the nature of D-ideals as the join of a singleton set is its single element, and clearly this will be included in the original set, so we cannot "jump out" of a set using the join of a singleton set.

Lemma 4.3.10. *For any meet semilattice every singleton set is an admissible set.*

Proof. Immediate from definition, as for arbitrary x and y:

$$x \wedge \bigvee\{y\} = x \wedge y = \bigvee_{y \in \{y\}} x \wedge y \tag{4.21}$$

□

Another trivial example of a set with a distributive join is the entire lattice. Again this has no impact on the nature of D-ideals as the entire lattice must contain its own join.

Lemma 4.3.11. *For any meet semilattice L, L is an admissible set*

Proof. We note L has a top element as it is a meet semilattice. For $x \in L$:

$$x \wedge \bigvee L = x \wedge \top = x = \bigvee_{y \in L} (x \wedge y) \tag{4.22}$$

where the last equality holds as clearly $x \in L$, and for all $y \in L$, $x \wedge y \leq x$. □

Having seen two trivial examples of admissible sets, the question is: are there actually many interesting admissible sets in lattices that might be of interest for quantum systems? As shall be seen in later sections, the answer is yes.

The next lemma is particularly useful as given an admissible set, it enables the construction of large numbers of additional admissible sets, by adding additional elements beneath those already contained in the set. Conversely the lemma can be seen as a mechanism for simplifying a set when checking it for admissibility, by stripping out elements that are beneath others within the set.

Proposition 4.3.12. *For a complete lattice L, $A \subseteq L$ and $B \subseteq \downarrow A$:*

- *A is an admissible set if and only if $A \cup B$ is an admissible set.*
- *$\bigvee A = \bigvee A \cup B$.*

Proof. Assuming A admissible:

$$\begin{aligned}
c \wedge \bigvee A \cup B &= c \wedge (\bigvee A \vee \bigvee B) && (4.23)\\
&= c \wedge \bigvee A && (4.24)\\
&= \bigvee_{a \in A} (c \wedge a) \qquad A \text{ admissible} && (4.25)\\
&= \bigvee_{a \in A} (c \wedge a) \vee \bigvee_{b \in B} (c \wedge b) && (4.26)\\
&= \bigvee_{a \in A \cup B} (c \wedge a) && (4.27)
\end{aligned}$$

and so A admissible implies $A \cup B$ is admissible. Now if we assume $A \cup B$ is admissible:

$$\begin{aligned}
c \wedge \bigvee A &= c \wedge (\bigvee A \vee \bigvee B) && (4.28)\\
&= c \wedge \bigvee (A \cup B) && (4.29)\\
&= \bigvee_{a \in A \cup B} (c \wedge a) \qquad A \cup B \text{ admissible} && (4.30)\\
&= \bigvee_{a \in A} (c \wedge a) \vee \bigvee_{b \in B} (c \wedge b) && (4.31)\\
&= \bigvee_{a \in A} (c \wedge a) && (4.32)
\end{aligned}$$

and so $A \cup B$ admissible implies A admissible as required. For the second part, as $\bigvee A$ must be an upper bound on B, we have:

$$\bigvee A = \bigvee A \vee \bigvee B = \bigvee (A \cup B) \tag{4.33}$$

□

In general, admissible sets are not closed under union, and so this does not provide a convenient mechanism for finding new admissible sets from existing ones. As the following lemma shows with a suitable constraint on the joins of the sets involved, admissible sets can be built by taking unions.

Lemma 4.3.13. *Let L be a complete lattice, let $(P_i)_{i \in I}$ be a family of admissible subsets of L, and assume there exists $P \in (P_i)_{i \in I}$ such that for all P_i, $\bigvee P_i \leq \bigvee P$, then:*

- *$\bigcup_i P_i$ is admissible.*

Proof. Let $S = \bigcup_i P_i$, then:

$$\begin{aligned}
\bigvee_{s \in S} (r \wedge s) &= \bigvee_i \bigvee_{p \in P_i} (r \wedge p) && (4.34)\\
&= \bigvee_i (r \wedge \bigvee P_i) && \text{admissibility of each } P_i \quad (4.35)\\
&= r \wedge \bigvee P && \text{by assumption} \quad (4.36)\\
&= r \wedge \bigvee_i \bigvee P_i && \bigvee_i \bigvee P_i = \bigvee P \text{ from assumption} \quad (4.37)\\
&= r \wedge \bigvee \bigcup_i P_i && \text{join over union} \quad (4.38)\\
&= r \wedge \bigvee S && (4.39)
\end{aligned}$$

□

The next proposition is a special case where admissible sets can be extended arbitrarily if their join is the top element. It is not an immediate corollary of the previous result, as it does not require admissibility of the set of elements to be added.

Proposition 4.3.14. *Let L be a complete lattice. If $A \subseteq L$ is an admissible set with $\bigvee A = \top$, and $A \subseteq B$, then B is an admissible set.*

Proof. For arbitrary $x \in L$:

$$\begin{aligned}
x \wedge \bigvee (A \cup B) &= x \wedge (\bigvee A \vee \bigvee B) && \text{join over union} \quad (4.40)\\
&= x \wedge (\top \vee \bigvee B) && \text{assumption} \quad (4.41)\\
&= x \wedge \top && (4.42)\\
&= x && (4.43)\\
&= x \vee \bigvee_{b \in B} (b \wedge x) && x \text{ an u.b. on each } b \wedge x \quad (4.44)\\
&= (x \wedge \bigvee A) \vee \bigvee_{b \in B} (b \wedge x) && \bigvee A = \top \quad (4.45)\\
&= \bigvee_{a \in A} (a \wedge x) \vee \bigvee_{b \in B} (b \wedge x) && A \text{ admissible} \quad (4.46)\\
&= \bigvee_{y \in A \cup B} (y \wedge x) && \text{join over union} \quad (4.47)
\end{aligned}$$

□

4.4 Admissible sets in projection lattices

In this section we explore in some detail the nature of admissible sets, and therefore distributive joins in the projection lattices of various Hilbert spaces.

The approach taken is essentially geometric, with the characterization of various sets given in terms of the geometric nature of their members.

4.4.1 Projections lattices in arbitrary dimensional Hilbert space

The following lemmas show two different examples of admissible sets. Both are still relatively simple in structure, but they do have an impact on which downward closed sets are valid D-ideals, as they do not contain their own join.

Lemma 4.4.1. *Let $\mathcal{H}$ be an arbitrary Hilbert space. The set of all rank 1 projections in $\mathcal{H}$ is an admissible set.*

Proof. Let S be the set of all rank 1 projections in some Hilbert space. Then for an arbitrary projection $\hat{P}$:

$$\hat{P} \wedge \bigvee S = \hat{P} \wedge \top = \hat{P} \tag{4.48}$$

As S contains all projections onto rays, $\{\hat{S} \wedge \hat{P} \mid \hat{S} \in S\}$ contains all the projections onto rays in the subspace projected onto by P. Therefore:

$$\hat{P} = \bigvee_{\hat{Q} \in S} (\hat{Q} \wedge \hat{P}) \tag{4.49}$$

□

Lemma 4.4.2. *Let $\mathcal{H}$ be an N-dimensional Hilbert space, where N is some natural number. The set of all rank $N-1$ projections in $\mathcal{P}(\mathcal{H})$ is an admissible set.*

Proof. Let S be the set of all rank $N-1$ projections, and $\hat{P}$ an arbitrary projection not equal to $\hat{1}$. Then:

$$\hat{P} \wedge \bigvee S = \hat{P} \wedge \top = \hat{P} = \bigvee_{\hat{S} \in S} \hat{P} \tag{4.50}$$

As S contains all rank $N-1$ projections, there must be a projection $\hat{S} \in S$ such that $\hat{P} \wedge \hat{S} = \hat{P}$, and so:

$$\bigvee_{\hat{S} \in S} \hat{P} = \bigvee_{\hat{S} \in S} (\hat{P} \wedge \hat{S}) \tag{4.51}$$

By corollary (4.3.9), $\hat{1}$ also distributes over S, and so S is admissible. □

The admissible sets in the previous lemmas have very similar form. As will be seen in later sections, these lemmas can be generalized greatly.

4.4.2 Admissible sets in low dimensional Hilbert spaces

The following lemmas further characterize admissible sets, in some low dimensional Hilbert spaces. Both positive and negative results are presented, for a variety of different forms of set.

Lemma 4.4.3. *Let $\mathcal{H}$ be a 2-dimensional Hilbert space. Let S be a strict subset of all the rank 1 projections in $\mathcal{P}(\mathcal{H})$, containing two or more projections. Then S is not an admissible set.*

Proof. Let $\hat{P}$ be a projection onto an arbitrary ray, $\hat{P} \notin S$.

$$\hat{P} \wedge \bigvee S = \hat{P} \wedge \top \tag{4.52}$$
$$= \hat{P} \tag{4.53}$$
$$\neq \bot \tag{4.54}$$
$$= \bigvee_{\hat{Q} \in S} \bot \tag{4.55}$$
$$= \bigvee_{\hat{Q} \in S} (\hat{Q} \wedge \hat{P}) \quad \text{as } \hat{P} \notin S \tag{4.56}$$

□

Lemma 4.4.4. *Let $\mathcal{H}$ be a 3-dimensional Hilbert space. Let S be a strict subset of all the rank 1 projections in $\mathcal{P}(\mathcal{H})$ into some plane π, and also let S contain at least two projections. Then S is not an admissible set.*

Proof. Let S_π be the set of all rank 1 projections into the plane π, and $\hat{P} \in S_\pi / S$. Let P_π be the rank 2 projection onto π.

$$\hat{P} \wedge \bigvee S = \hat{P} \wedge \hat{P}_\pi \tag{4.57}$$
$$= \hat{P} \tag{4.58}$$
$$\neq \bot \tag{4.59}$$
$$= \bigvee_{\hat{S} \in S} \bot \tag{4.60}$$
$$= \bigvee_{\hat{S} \in S} (\hat{S} \wedge \hat{P}) \tag{4.61}$$

□

Lemma 4.4.5. *Let $\mathcal{H}$ be a 3-dimensional Hilbert space. The set of all rank 1 projections in $\mathcal{P}(\mathcal{H})$ into a fixed plane is an admissible set.*

Proof. Let S_π be the set of all rank 1 projections into plane π, and $\hat{P}_\pi$ the corresponding projection onto the plane. For an arbitrary rank 1 projection $\hat{P}$ in π:

$$\hat{P} \wedge \bigvee S_\pi = \hat{P} \wedge \hat{P}_\pi = \hat{P} = \bigvee_{\hat{Q} \in S_\pi} (\hat{Q} \wedge \hat{P}) \tag{4.62}$$

For an arbitrary rank 1 projection $\hat{P}$ not in π:

$$\hat{P} \wedge \bigvee S_\pi = \hat{P} \wedge \hat{P}_\pi = \perp = \bigvee_{\hat{Q} \in S_\pi} (\hat{Q} \wedge \hat{P}) \tag{4.63}$$

The rank 2 projection onto π is an upper bounds for S_π and so distributes over $\bigvee S_\pi$ by lemma (4.3.8). For a rank 2 projection $\hat{P}$ not in π:

$$\hat{P} \wedge \bigvee S_\pi = \hat{P} \wedge \hat{P}_\pi = \hat{P}_\psi \tag{4.64}$$

where $\hat{P}_\psi$ is the projection onto the ray in the planes of both projections. Also, as $\hat{P}_\psi$ must be in S_π:

$$\bigvee_{\hat{Q} \in S_\pi} (\hat{Q} \wedge \hat{P}) = \hat{P}_\psi \vee \bigvee_{\hat{Q} \in S_\pi / \{\hat{P}_\psi\}} (\hat{Q} \wedge \hat{P}) = \hat{P} \vee \perp = \hat{P}_\psi \tag{4.65}$$

□

The admissible sets described in lemma (4.4.5) have richer structure than the previous examples, they are more complex than a singleton set, but do not contain all the projections of a particular rank. Even in just 3-dimensional Hilbert space, the admissible sets will turn out to have a relatively rich structure.

Lemma 4.4.6. *Let $\mathcal{H}$ be 3 dimensional Hilbert space. Let S_ψ be the set of all rank 2 projections in $\mathcal{P}(\mathcal{H})$ onto planes through the ray ψ. S_ψ is an admissible set.*

Proof. For projection onto an arbitrary plane π, $\hat{P}_\pi$:

$$\hat{P}_\pi \wedge \bigvee S_\psi = \hat{P}_\pi \wedge \top \tag{4.66}$$

$$= \hat{P}_\pi \tag{4.67}$$

$$= \bigvee_{\hat{Q} \in S_\psi} (\hat{Q} \wedge \hat{P}_\pi) \tag{4.68}$$

where the last line follows as either $\hat{P}_\pi \in S_\psi$ or the plane it projects onto intersects each plane in S_ψ at a different ray, and so their join is the whole plane. For a projection onto an arbitrary ray ϕ, $\hat{P}_\phi$:

$$\hat{P}_\phi \wedge \bigvee S_\psi = \hat{P}_\phi \wedge \top \tag{4.69}$$

$$= \hat{P}_\phi \tag{4.70}$$

$$= \bigvee_{\hat{Q} \in S_\psi} (\hat{Q} \wedge \hat{P}_\phi) \tag{4.71}$$

where the last line follows as the ray projected onto by $\hat{P}_\phi$ must lie in the plane projected onto by some $\hat{P} \in S_\psi$. By corollary (4.3.9) this is sufficient to show admissibility. □

Lemma 4.4.7. *Let $\mathcal{H}$ be a 3-dimensional Hilbert space. Let S be a strict subset of all the rank 2 projections in $\mathcal{P}(\mathcal{H})$ containing some ray ψ, and let S contain two or more projections. Then S is not an admissible set.*

Proof. Let S_ψ be the set of all rank 2 projections containing the ray ψ. We can choose a rank 2 projection $\hat{P} \in S_\psi / S$. Let $\hat{P}_\psi$ be the rank 1 projection onto ray ψ. Then:

$$\hat{P} \wedge \bigvee S = \hat{P} \wedge \top \tag{4.72}$$

$$= \hat{P} \tag{4.73}$$

$$\neq \hat{P}_\psi \tag{4.74}$$

$$= \bigvee_{\hat{S} \in S} (\hat{S} \wedge \hat{P}) \tag{4.75}$$

□

Some patterns have emerged in the forms of the admissible sets in the previous lemmas, they are "complete" in some sense, containing all the projections in some ray, plane, etc. Similarly, the sets described in the negative results all seem "incomplete", there are always projections missing from these sets. This suggests some general rule can be found describing this "completeness" condition, as will be discussed in the next section.

4.5 A general geometric result about distributive joins in projection lattices

The lemmas given in the previous sections capture whether many different forms of sets of projections have a distributive join. Interestingly, even in 3 dimensions, there are cases still to be considered, as the following example shows.

Example 4.5.1. Let $\mathcal{H}$ be 3 dimensional Hilbert space. Let ψ be a ray, and π a plane containing ψ, with corresponding projection $\hat{P}_\pi$. Let S be a set of projections containing only:

- All the rank 2 projections onto planes containing ψ except $\hat{P}_\pi$.
- All the rank 1 projections onto rays in π.

The set S has a distributive join.

The proof of the claim in the example is omitted, it is similar to lemma (4.4.6). The example demonstrates that rather complicated sets with distributive joins can be constructed, even in just 3 dimensions. We need to find some more abstract property of a set of projections that captures the requirements of distributivity. An informal algorithmic description of assembling the set S above is:

1. Start with the set of all rank 2 projections containing some ray.
2. Remove an arbitrary projection onto a plane.
3. "Fill the gap" left by the missing projection by rank 1 projections.

The idea of "filling the gap" is key to a more general result. In order to construct a set with a distributive join, we need to completely cover a particular subspace with projections, so there is no ray in the subspace that is not covered by some projection. This intuition leads to the following proposition which gives a complete characterization of which sets of projections have distributive joins, for an arbitrary Hilbert space.

Proposition 4.5.2. *Let $\mathcal{H}$ be a Hilbert space. Let σ be a closed subspace of $\mathcal{H}$, with corresponding projection $\hat{P}_\sigma$. Let S be a set of projections with $\bigvee S = \hat{P}_\sigma$. Then S has a distributive join in $\mathcal{P}(\mathcal{H})$ if and only if for every rank 1 projection $\hat{P}_\psi$, if $\hat{P}_\psi \wedge \hat{P}_\sigma = \hat{P}_\psi$, there exists $\hat{P} \in S$ such that $\hat{P}_\psi \wedge \hat{P} = \hat{P}_\psi$.*

Proof. First we assume there exists P_ψ such that $\hat{P}_\sigma \wedge \hat{P}_\psi = \hat{P}_\psi$, and for all $\hat{S} \in S$, $\hat{S} \wedge \hat{P}_\psi = \perp$. Then:

$$\hat{P}_\psi \wedge \bigvee S = \hat{P}_\psi \wedge \hat{P}_\sigma = \hat{P}_\psi \tag{4.76}$$

and:

$$\bigvee_{\hat{S} \in S} (\hat{S} \wedge \hat{P}_\psi) = \bigvee_{\hat{S} \in S} \perp = \perp \tag{4.77}$$

and so S does not have a distributive join.

Now we assume that for every projection onto a ray P_ψ such that $\hat{P}_\sigma \wedge \hat{P}_\psi = \hat{P}_\psi$, there exists $\hat{S} \in S$ such that $\hat{S} \wedge \hat{P}_\psi = \hat{P}_\psi$. Let P_τ be a projection onto an arbitrary subspace τ. Then:

$$\hat{P}_\tau \wedge \bigvee S = \hat{P}_\tau \wedge \hat{P}_\sigma = \hat{P}_{\tau \cap \sigma} \tag{4.78}$$

For every $\hat{S} \in S$:

$$\hat{S} \wedge \hat{P}_\tau \leq \bigvee S = \hat{P}_\sigma \tag{4.79}$$

Also, for every $\hat{S} \in S$:

$$\hat{S} \wedge \hat{P}_\tau \leq \hat{P}_\tau \tag{4.80}$$

It follows that

$$\bigvee_{\hat{S} \in S} (\hat{S} \wedge \hat{P}_\tau) \leq \hat{P}_{\tau \cap \sigma} \tag{4.81}$$

Next, for any ray ψ in $\sigma \cap \tau$, there exists $\hat{S} \in S$ such that:

$$\hat{S} \wedge \hat{P}_\tau \geq \hat{P}_\psi \tag{4.82}$$

Therefore, we have:

$$\bigvee_{\hat{S} \in S} (\hat{S} \wedge \hat{P}_\tau) \geq \hat{P}_{\tau \cap \sigma} \tag{4.83}$$

And so:

$$\bigvee_{\hat{S} \in S} (\hat{S} \wedge \hat{P}_\tau) = \hat{P}_{\tau \cap \sigma} \tag{4.84}$$

Therefore, S has a distributive join. □

The geometric results in the previous sections can all be seen as corollaries of this result. The complicated set in example (4.5.1) can also now easily be seen to be an admissible set.

4.6 A general result about distributive joins in complete lattices

The geometric approach adopted so far has lead to a characterization of distributive joins in the projection lattice of an arbitrary Hilbert space. Although proposition (4.5.2) nicely captures all the situtations considered so far, it does not describe distributive joins in any simple finite lattices, or any restrictions of the full set of projections into a Hilbert space.

In order to extend proposition (4.5.2), the geometric details of the problem are abstracted away, and we consider the problem from a purely order theoretic perspective. To do this, note that:

- For a Hilbert space $\mathcal{H}$, $\mathcal{P}(\mathcal{H})$ is an atomistic lattice.
- The rank 1 projections are the atoms of $\mathcal{P}(\mathcal{H})$.

It is tempting to focus attention on atomistic lattices based on these observations. Instead, we note that in an atomistic lattice, the completely join irreducible elements are exactly the atoms and the atoms form a join dense subset of the original lattice. So the geometric characterization of proposition (4.5.2) can be seen as a characterization in terms of the completely join irreducible elements of the projection lattice. Now we explore the extension of this point of view to a characterization of distributive joins in complete lattices where the set of completely join irreducible elements form a join dense subset.

As atoms and completely join irreducible elements are going to be important in what follows, we define the following notation, following [Coecke, 2002].

Definition 4.6.1. Let L be a lattice, and $x \in L$. Define:

$$\mu(a) := \downarrow a \cap \mathcal{A}(L) \tag{4.85}$$

For $A \subseteq L$, define:

$$\mu(A) := \bigcup_{a \in A} \mu(a) \tag{4.86}$$

Definition 4.6.2. Let L be a complete lattice and $x \in L$. Define:

$$\gamma(x) := \{y \in \downarrow x \mid y \text{ completely join irreducible}\} \tag{4.87}$$

For $A \subseteq L$, define:

$$\gamma(A) := \bigcup_{a \in A} \gamma(a) \tag{4.88}$$

Firstly a couple of technical lemmas capturing basic properties of completely join irreducible elements will be needed.

Lemma 4.6.3. *Let L be a complete lattice in which the completely join irreducible elements are join dense. Let $x \in L$. Then:*

$$x = \bigvee \gamma(x) \tag{4.89}$$

Proof. x can be written as a join of some $A \subseteq \gamma(x)$, as the completely join irreducible elements are join dense in L by assumption. As x is an upper bound on $\gamma(x)$, and it is the least upper bound on some subset of $\gamma(x)$, it must be the least upper bound on $\gamma(x)$ as required. □

Remark 4.6.4. For an example of a complete lattice in which the completely join irreducible elements are not join dense, consider the real interval $[0, 1]$, with the usual ordering. For this interval the only completely join irreducible element is 0.

Lemma 4.6.5. *Let L be a complete lattice in which the completely join irreducible elements are join dense. Let $x, y \in L$. Then:*

$$x \wedge y = \bigvee \gamma(x) \cap \gamma(y) \tag{4.90}$$

Proof.

$$x \wedge y = \bigvee \gamma(x \wedge y) \qquad \text{lemma (4.6.3)} \tag{4.91}$$

$$= \bigvee \gamma(x) \cap \gamma(y) \tag{4.92}$$

□

With the basic definitions and lemmas in place, we can prove our main result:

Theorem 4.6.6. *Let L be a complete lattice in which the completely join irreducible elements are join dense. Also let $A \subseteq L$. Then A has a distributive join if and only if:*

$$\gamma(A) = \gamma(\bigvee A) \tag{4.93}$$

Proof. Let $j \in L$. By lemma (4.6.5):

$$j \wedge \bigvee A = \bigvee \gamma(j) \cap \gamma(\bigvee A) \tag{4.94}$$

Also:

$$\bigvee_{a\in A}(j\wedge a) = \bigvee_{a\in A}\bigvee(\gamma(j)\cap\gamma(a)) \qquad \text{lemma (4.6.5)} \tag{4.95}$$

$$= \bigvee\bigcup_{a\in A}(\gamma(j)\cap\gamma(a)) \qquad \text{join over unions} \tag{4.96}$$

$$= \bigvee\gamma(j)\cap\bigcup_{a\in A}\gamma(a) \qquad \text{distributivity} \tag{4.97}$$

$$= \bigvee\gamma(j)\cap\gamma(A) \qquad \text{definition} \tag{4.98}$$

Therefore, if $\gamma(A) = \gamma(\bigvee A)$ then:

$$j\wedge\bigvee A = \bigvee_{a\in A}(j\wedge a) \tag{4.99}$$

It is clear that $\gamma(A)\subseteq\gamma(\bigvee A)$. Now we assume $\gamma(A)\subsetneq\gamma(\bigvee A)$ and that A has a distributive join, and aim to show a contradiction. Let $j\in\gamma(\bigvee A)\setminus\gamma(A)$, then as $j\in\gamma(\bigvee A)$:

$$j\wedge\bigvee A = \bigvee(\gamma(j)\cap\gamma(\bigvee A)) \qquad \text{lemma (4.6.5)} \tag{4.100}$$

$$= \bigvee\gamma(j) \qquad j\in\gamma(\bigvee A) \tag{4.101}$$

$$= j \qquad \text{lemma (4.6.3)} \tag{4.102}$$

As A is assumed to have a distributive join, this implies:

$$j = j\wedge\bigvee A = \bigvee_{a\in A}(j\wedge a) = \bigvee\gamma(j)\cap\gamma(A) \tag{4.103}$$

Then j can be written as the join of a subset not containing j itself, as by assumption $j\notin\gamma(A)$, but this contradicts the complete join irreducibility of j, therefore A is not completely distributive. □

Corollary 4.6.7. *Let L be a complete atomistic lattice, and $A\subseteq L$. Then A has a distributive join if and only if:*

$$\mu(A) = \mu(\bigvee A) \tag{4.104}$$

Proof. By definition, for an atomistic lattice L, $\mathcal{A}(L)$ are the completely join irreducible elements, and they form a join dense subsets of L. □

Corollary 4.6.8. *Let L be a complete lattice in which the completely join irreducible elements form a join dense subset. Then the following are equivalent:*

- *For all $A\subseteq L$, $\gamma(A) = \gamma(\bigvee A)$.*
- *L is a frame.*

Remark 4.6.9. Corollary (4.6.8) gives an interesting characterization of a large class of frames, without any mention of the infinite distributivity law, or in fact any direct reference to meets at all.

Corollary 4.6.10. *Let L be a finite lattice, and $A \subseteq L$. Then A has a distributive join if and only if:*

$$\gamma(A) = \gamma(\bigvee A) \tag{4.105}$$

Proof. Every finite lattice is complete. The join irreducible elements of a finite lattice form a join dense subset (5.1 in [Davey and Priestley, 2002]). □

The geometric proposition (4.5.2) follows as a corollary of this much stronger order theoretic formulation. The simple characterization for all atomistic lattices covers all complete atomic orthomodular lattices, as they are well known to be atomistic (see for example [Kalmbach, 1983]). For finite lattices, checking a subset A for admissibility consists of performing the following simple steps:

1. Find all the join irreducible elements beneath $\bigvee A$. (Complete join irreducibility reduces to join irreducibility for the finite case)

2. If every join irreducible element found above is beneath an element in A, then A is admissible, otherwise A is not admissible.

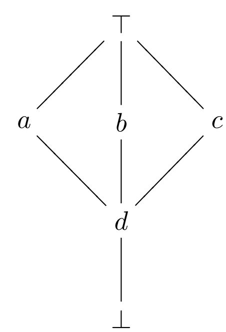

Figure 4.8: A non atomistic lattice

The generalization beyond atomistic lattices to lattices where the completely join irreducible elements form a join dense subset allows the result to be applied to lattices such as that shown in figure 4.8, which is not atomistic.

4.7 Properties of the injective hull mapping

From [Coecke, 2002] and [Stubbe, 2005], we have the following properties of the embedding B described in definition (4.1.19):

- B is monotone.

- B is injective.
- B is not surjective.
- B preserves arbitrary meets.
- B preserves distributive joins.

As atoms and atomicity proved to be important in the characterization of distributive ideals, the following two lemmas consider questions of the action of B on atoms.

Lemma 4.7.1. *Let L be a complete lattice. Then:*

$$B(\mathcal{A}(L)) = \mathcal{A}(DI(L)) \tag{4.106}$$

Proof. Assume $a \in \mathcal{A}(L)$. Then $B(a) = \{a, \bot\}$. Then the only element of $DI(L)$ less than $B(a)$ is $\{\bot\}$, as $\emptyset$ is not a D-ideal for a complete lattice, therefore $B(a)$ is an atom if a is an atom. If $b \in DI(L)$ is an atom, then b must be a principal ideal of the form $\{c, \bot\}$, and therefore the image of some atom c. □

Lemma 4.7.2. *Let L be a complete atomistic lattice. Then $DI(L)$ is a complete atomistic lattice.*

Proof. $DI(L)$ is complete as it is a frame. By Theorem 2 in [Bruns and Lakser, 1970], $B(L)$ is join dense in $DI(L)$. Every element in $B(L)$ can be written as $B(l)$ for some $l \in L$. As L is atomistic we have:

$$l = \bigvee \mu(l) \tag{4.107}$$

By corollary (4.6.7), $\mu(l)$ has a distributive join, and therefore:

$$B(l) = B(\bigvee \mu(l)) = \bigvee \{B(a) \mid a \in \mu(l)\} \tag{4.108}$$

and from lemma (4.7.1), each $B(a)$ is an atom in $DI(L)$. It follows that every element in $DI(L)$ can be written as a join of elements of $\mathcal{A}(DI(L))$. □

Now we consider the action of B on orthocomplemented elements of its domain.

Lemma 4.7.3. *B does not preserve negation.*

Proof. Consider the orthomodular lattice $\mathbf{M_4}$. The injective hull of this lattice is a Boolean algebra with 4 atoms. In $\mathbf{M_4}$, a is an atom if and only if $\neg a$ is an atom. For $a \in \mathcal{A}(\mathbf{M_4})$, by lemma (4.7.1), $B(\neg a)$ is an atom. Therefore, as the injective hull is a Boolean algebra, $\neg B(a)$ is not an atom, and so:

$$\neg B(a) \neq B(\neg a) \tag{4.109}$$

□

As described in [Johnstone, 2002b], the free frame for a meet semilattice L is $\mathcal{D}L$, the lattice of lower sets in L, therefore we can factor B through this free frame. This is convenient to do as it relates the injective hull construction to $\mathcal{D}L$, the dual of $\mathcal{U}L$ that we have previously related the topos construction to.

Definition 4.7.4. For meet semilattice L, define the function F^i:

$$F^i : L \to \mathcal{D}L \tag{4.110}$$

$$x \mapsto \downarrow x \tag{4.111}$$

Lemma 4.7.5. *Let L be a meet semilattice, then F^i is monotone.*

Proof. Let $x, y \in L$, $x \leq y$. Then $\downarrow x \subseteq \downarrow y$. □

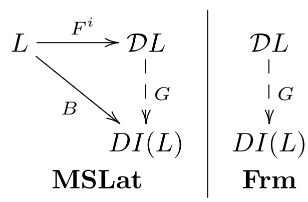

Figure 4.9: Factoring B through the free frame

Definition 4.7.6. For a meet semilattice L, define the function G:

$$G : \mathcal{D}L \to DI(L) \tag{4.112}$$

$$U \mapsto \bigvee^{DI(L)} \{\downarrow x \mid x \in U\} \tag{4.113}$$

Remark 4.7.7. The form of G is a special case of the construction given for the proof of Theorem II.1.2 in [Johnstone, 1982]. It follows from that theorem that G is a morphism in **Frm**, and $B = G \circ F^i$, as illustrated in figure 4.9.

As G is a morphism in **Frm**, it preserves arbitrary joins, and therefore has a right adjoint, which we now investigate.

Definition 4.7.8. Let L be a meet semilattice. Define the function $I : DI(L) \to \mathcal{D}L$ to be the obvious injection.

Lemma 4.7.9.

$$G \dashv I \tag{4.114}$$

Proof. Let L be a meet semilattice. I is clearly monotone, and G is monotone as described in remark (4.7.7). Therefore, it is sufficient to show:

$$1_{\mathcal{D}L} \leq I \circ G \;\; and \;\; G \circ I \leq 1_{DI(L)} \tag{4.115}$$

For arbitrary $U \in \mathcal{D}L$:

$$(I \circ G)(U) = \bigvee^{DI(L)} \{\downarrow x \mid x \in U\} \quad \text{definitions} \tag{4.116}$$
$$\supseteq U \quad U \text{ may not be a D-ideal} \tag{4.117}$$

For arbitrary $U \in DI(L)$:

$$(G \circ I)(U) = \bigvee^{DI(L)} \{\downarrow x \mid x \in U\} \quad \text{definitions} \tag{4.118}$$
$$= U \quad \text{All D-ideals are lower sets} \tag{4.119}$$

□

So the right adjoint of G is the obvious embedding of the D-ideals into the down set lattice.

4.8 Extending meet semilattice morphisms to frame morphisms

For meet semilattices L, M it is natural to ask for suitable types of morphisms $f : L \to M$: is there some canonical morphism $f^* : DI(L) \to DI(M)$? First a small technical lemma is required. This lemma makes concrete the intuition that the Bruns-Lakser construction "does no work" if a meet semilattice is already a frame.

Lemma 4.8.1. *Any frame F considered as a meet semilattice is equal to its injective hull.*

Proof. As F is a frame, it is an injective extension of itself by Theorem 1 in [Bruns and Lakser, 1970]. It is also obviously join-dense in itself, and distributive joins are trivially preserved by the identity function, therefore it is an essential extension by Theorem 2 in [Bruns and Lakser, 1970]. □

Now the question of whether two different classes of morphisms can be lifted to operations between the corresponding injective hulls is investigated.

Lemma 4.8.2. *There exist morphisms in* **MSLat** *that cannot be lifted to a morphism in* **Frm** *between the corresponding pair of injective hulls.*

Proof. We consider the following **MSLat** morphism ϕ:

$$\phi : \mathbf{2} \to \mathbf{2} \tag{4.120}$$
$$x \mapsto \top \tag{4.121}$$

As $\mathbf{2}$ is a frame, by lemma (4.8.1) it is isomorphic to its own injective hull. Therefore, there can be no $\phi^* : \mathbf{2} \to \mathbf{2}$ in **Frm** as ϕ is not a frame morphism. □

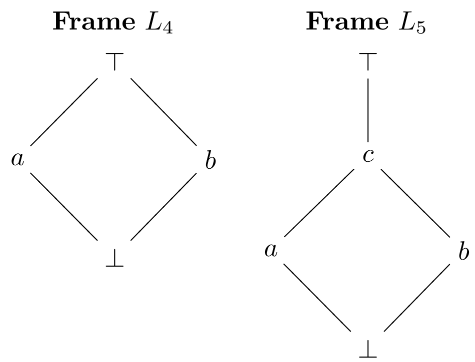

Figure 4.10: Meet semilattices L_4 and L_5

Lemma 4.8.3. *There exist morphisms in* **MSLat** *that preserve* $\perp$ *that cannot be lifted to a morphism in* **Frm** *between the corresponding pair of injective hulls.*

Proof. Consider the frames L_4 and L_5 shown in figure 4.10. The obvious embedding of L_4 into L_5 is an **MSLat** morphism that preserves $\perp$, but it is not a frame morphism, as the join of $\{a, b\}$ is not preserved. As both lattices are frames, they are isomorphic to their injective hulls, and so no lifting can exist. □

The two negative results in lemma (4.8.2) and lemma (4.8.3) suggest we need a category of meet semilattices with a more restricted collection of morphisms.

Definition 4.8.4. The category $\mathbf{MSLat_{dis}}$ has meet semilattices as objects, and morphisms meet semilattice homomorphisms that also preserve distributive joins.

Proposition 2 of [Stubbe, 2005] shows that **Frm** is a full monoreflective subcategory of $\mathbf{MSLat_{dis}}$. This is sufficient for there to exist a lifting of any $\mathbf{MSLat_{dis}}$ morphism to a morphism between the corresponding injective hulls.

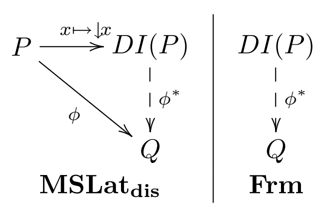

Figure 4.11: **Frm** is a full monoreflective subcategory of $\mathbf{MSLat_{dis}}$

Lemma 4.8.5. *Let* $\phi : P \to Q$ *be a morphism in* $\mathbf{MSLat_{dis}}$*, then there exists a unique* $\phi^* : DI(P) \to DI(Q)$ *in* **Frm** *such that:*

$$\phi^*|_P = \phi \tag{4.122}$$

Proof. Let F be the left adjoint of the injection of **Frm** into $\mathbf{MSLat_{dis}}$, and η the unit of the adjunction. Then the following diagram commutes by the naturality of η:

$$\begin{array}{ccc} P & \xrightarrow{\eta_P} & DI(P) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle F(\phi)} \\ Q & \xrightarrow{\eta_Q} & DI(Q) \end{array}$$

By the universal property of the adjunction, $F(\phi)$ is the unique such morphism such that the diagram commutes. As η is an injection, it follows that:

$$\phi^*|_P := \phi^*(\eta_P(P)) = \phi \tag{4.123}$$

□

4.9 Final examples

This section applies the results proved previously to explore two examples inspired by the literature.

Example 4.9.1. In an example in [Coecke, 2002] it is shown that for any complete atomistic lattice L, $DI(L)$ is isomorphic to the powerset of $\mathcal{A}(L)$. A proof is given in the paper, but as this possibly a surprising result, that B produces a Boolean algebra for such a large class of lattices, we explore why this happens using the tools that have been developed in this dissertation. Firstly we show every atomistic frame is a complete Boolean algebra.

Lemma 4.9.2. *Let L be an atomistic distributive complete lattice. Then L is a complete Boolean algebra.*

Proof. Define for arbitrary $x \in L$:

$$\neg x := \bigvee \mathcal{A}(L) \setminus \mu(x) \tag{4.124}$$

This is a valid orthocomplement, and as L is complete and distributive lattice with an orthocomplement, it is a complete Boolean algebra. □

Corollary 4.9.3. *Every atomistic frame is a complete Boolean algebra.*

Now if L is a complete atomistic lattice, by lemma (4.7.2), $DI(L)$ is also a complete atomistic lattice. It is a frame by [Bruns and Lakser, 1970], and therefore a complete Boolean algebra by lemma (4.9.2). As every atomistic boolean algebra is isomorphic to the powerset of its atoms, $DI(L)$ is isomorphic to the powerset of $\mathcal{A}(DI(L))$. By lemma (4.7.1) and the injectivity of B, it follows that $DI(L)$ is isomorphic to the powerset of $\mathcal{A}(L)$, as shown in the original paper.

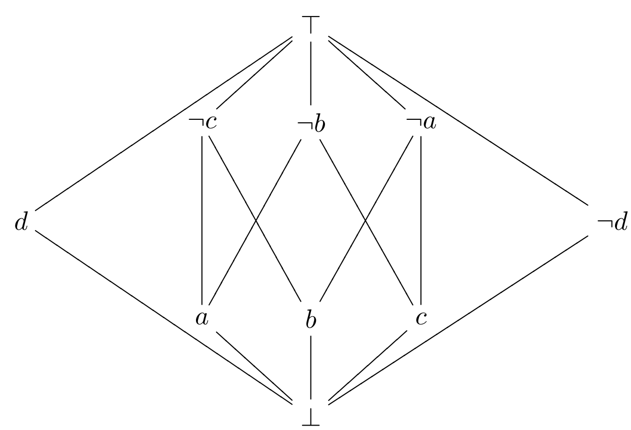

Figure 4.12: Complete atomistic lattice

Example 4.9.4. In this section we consider a more complex example, the injective hull of the meet semilattice L shown in figure 4.12. The example was originally shown in [Heunen et al., 2010], but unfortunately the paper contains several errors and constructs the incorrect injective hull. By observation, the lattice is clearly atomistic, applying the result of example (4.9.1), the injective hull of the lattice will be isomorphic to the powerset of a 5 atom set, and will therefore have $2^5 = 32$ elements, not the 72 claimed in [Heunen et al., 2010]. The result of example (4.9.1) is particularly useful in that it provides a simple checksum for the number of elements in small examples. $DI(L)$ is characterized up to isomorphism by the previous example, we now explore its concrete representation in terms of D-ideals, as the lattice is sufficiently complex that this requires some care.

$\downarrow\{d, \neg d\}$		
$\downarrow\{d, a\}$	$\downarrow\{d, b\}$	$\downarrow\{d, c\}$
$\downarrow\{\neg d, a\}$	$\downarrow\{\neg d, b\}$	$\downarrow\{\neg d, c\}$
$\downarrow\{d, \neg d, a\}$	$\downarrow\{d, \neg d, b\}$	$\downarrow\{d, \neg d, c\}$
$\downarrow\{d, \neg a\}$	$\downarrow\{d, \neg b\}$	$\downarrow\{d, \neg c\}$
$\downarrow\{\neg d, \neg a\}$	$\downarrow\{\neg d, \neg b\}$	$\downarrow\{\neg d, \neg c\}$
$\downarrow\{d, \neg d, \neg a\}$	$\downarrow\{d, \neg d, \neg b\}$	$\downarrow\{d, \neg d, \neg c\}$
$\downarrow\{\neg a, \neg b, \neg c\}$		
$\downarrow\{d, \neg a, \neg b, \neg c\}$		
$\downarrow\{\neg d, \neg a, \neg b, \neg c\}$		

Table 4.1: Non principal D-ideals of L

The 10 principal ideals of L are D-ideals by corollary (4.3.7). There are very many downward closed sets, but many are not D-ideals as they contain an admissible set and not its corresponding join. It is impossible to give an exhaustive discussion of the issues involved, as this would require excessive space, so only an outline of key points is given.

Theorem (4.6.6) makes reading off admissible sets from the diagram of L straightforward. The set $\{a, b\}$ is admissible. It follows that any D-ideal containing a and b must contain $\neg c$. More subtly, any D-ideal containing $\neg a$ and $\neg b$ will have $\{a, b\}$ as a subset, and must therefore also contain $\neg c$. Similarly any D-ideal containing $\neg a$ and a must also contain $\neg c$ as again $\{a, b\}$ will be a subset. There are other symmetrical versions of these ideas, from the symmetry of the Hasse diagram. There are also more straightforward admissible sets, such as $\{d, a, b, c, \neg d\}$ and $\{d, \neg a, \neg b, \neg c, \neg d\}$ The remaining 22 D-ideals are given in table 4.1.

Chapter 5

A comparison of the two approaches

5.1 Properties preserved by the constructions

The two constructions preserve very different properties of the original projection lattice. This can be seen as a trade off, something has to be changed in order to produce a new lattice with different properties from an old one, so choices have to made based on the particular aims of the scheme in question. Daseinisation preserves arbitrary joins, and does not preserve even finite meets, where as the injective hull construction preserves arbitrary meets and distributive joins. These differences originate in the differing motivations of the two approaches. Preservation of joins relates to superpositions, a physically observed phenomenon, and so is desirable to retain from the realist perspective of the topos approach. It also follows automatically from the "approximation from above" used in the daseinisation mapping. On the other hand, from an operationalist perspective, where certainty of of experimental outcomes is the guiding concept, meets are well behaved, as are distributive joins, but superposition like behaviour is abandoned as undesirable.

We also see that B preserves the atoms of the original lattice, where as it is easy to see that daseinisation does not in general preserve atoms, as it is straightforward to construct an atomic well typed function that is not the result of daseinisation (consider a well typed function equal to some projection $\hat{P}$ in the subalgebra $\{\hat{1}, \hat{P}\}''$ and $\hat{0}$ everywhere else). The preservation of atoms by B can probably be seen as a secondary effect, not part of the physically motivated construction, but an artifact of the mathematics that results.

Both constructions are injective as they are embeddings, and they are inevitably not surjective as "extra elements" are introduced in the resulting lattices to generate the desired complete Heyting algebra.

5.2 The relative sizes of the constructions

This section considers some bounds on the sizes of the two constructions, for von Neumann algebras with finite projection lattices. If a von Neumann algebra only has finitely many projections, it follows that it is abelian [Kadison and Ringrose, 1997b], so this restriction leads to a greatly simplified situation, as the projection lattices must be finite Boolean algebras.

Definition 5.2.1. The number of elements in finite set or lattice A will be denoted $\#A$.

Lemma 5.2.2. *Let N be a von Neumann algebra with 2^n projections, $n \geq 1$. Then:*

$$\#DI(\mathcal{P}(N)) = 2^n \tag{5.1}$$

and $DI(\mathcal{P}(N))$ is isomorphic to $\mathcal{P}(N)$.

Proof. From lemma (4.8.1), as $\mathcal{P}(N)$ is a Boolean algebra, and therefore a frame:

$$DI(\mathcal{P}(N)) \cong \mathcal{P}(N) \tag{5.2}$$

and the size claim follows directly. □

Lemma 5.2.3. *Let N be a von Neumann algebra with 2^n projections, $n \geq 1$. Then:*

$$\#\mathcal{W}(N) \geq 2(2^n - 2) \tag{5.3}$$

Proof. As $\mathcal{P}(N)$ is a Boolean algebra with 2^n elements, it has $\frac{2^n-2}{2}$ 4 element Boolean subalgebras. If a well typed function sets all larger algebras to $\hat{0}$, the values taken at each of these 4 element subalgebras can be chosen independently. Therefore, we have at least:

$$4\frac{(2^n - 2)}{2} = 2(2^n - 2) \tag{5.4}$$

different well typed functions in $\mathcal{W}(N)$. □

If we dismiss the trival algebra $\{\hat{0}, \hat{1}\}$, the previous two lemmas show that for finite von Neumann algebras, the injective hull is always smaller than or equal in size to the lattice of well typed functions. Equality occurs only for the 4 element projection lattice, i.e., for abelian von Neumann algebras of the form $\mathbb{C}\hat{P} + \mathbb{C}\hat{1}$ for some non-trivial projection P. Such a von Neumann algebra has projections $\hat{0}, \hat{P}, \hat{1} - \hat{P}$ and $\hat{1}$. As the projection lattice gets larger, the bound in lemma (5.2.3) becomes weaker, as more Boolean subalgebras will exist, and the lemma only accounts for the simplest type. For these finite cases, we see that daseinisation is significantly less efficient than the Bruns-Lakser construction, this result contradicts a conclusion reached for an equivalent situation in [Heunen et al., 2010], due to errors in the example considered in that paper.

5.3 Relationships between the various lattices

We begin with a straightforward relationship between the up and down set lattices. This can be viewed as a relationship between the lattices of open set and closed sets of the Alexandrov topology.

Lemma 5.3.1. *Let P be a poset.*

$$\mathcal{D}P \cong (\mathcal{U}P)^{op} \tag{5.5}$$

Proof. Define the function ϕ:

$$\phi : \mathcal{D}P \to (\mathcal{U}P)^{op} \tag{5.6}$$
$$U \mapsto P \setminus U \tag{5.7}$$

If $U \in \mathcal{D}P$ then $P \setminus U$ is an upper set as if $x \in P$ and $x \notin U$ then as U is a lower set, every $y \in P$ such that $y \leq x$ implies $y \notin P \setminus U$. So ϕ is well defined. ϕ is monotone as for any $U_1, U_2 \in \mathcal{D}P$,

$$U_1 \subseteq U_2 \to (P \setminus U_1) \supseteq (P \setminus U_2) \tag{5.8}$$

ϕ is also clearly injective as set theoretic complement is injective. Also define the function ψ:

$$\psi : (\mathcal{U}P)^{op} \to \mathcal{D}P \tag{5.9}$$
$$U \mapsto P \setminus U \tag{5.10}$$

ψ is a well defined injective monotone function by the dual of the prior argument for ϕ. For arbitrary $U \in \mathcal{D}P$:

$$(\psi \circ \phi)(U) = \psi(P \setminus U) \quad \text{definition} \tag{5.11}$$
$$= P \setminus (P \setminus U) \quad \text{definition} \tag{5.12}$$
$$= U \quad \text{properties of complements} \tag{5.13}$$

For arbitrary $U \in (\mathcal{U}P)^{op}$:

$$(\phi \circ \psi)(U) = \phi(P \setminus U) \quad \text{definition} \tag{5.14}$$
$$= P \setminus (P \setminus U) \quad \text{definition} \tag{5.15}$$
$$= U \quad \text{properties of complements} \tag{5.16}$$

Therefore, ϕ and ψ witness an isomorphism between $\mathcal{D}P$ and $(\mathcal{U}P)^{op}$ as required. □

The following definitions are duals of objects and morphisms already encountered in the discussion of the two different constructions. They will be required to provide some symmetry in the relationship between the two constructions.

Definition 5.3.2. Let N be a von Neumann algebra. Then the set:

$$\{f : \mathcal{V}(N) \to \mathcal{P}(N) \mid f \text{ monotone } \textit{and } \forall V \in \mathcal{V}(N). f_V \in \mathcal{P}(V)\} \tag{5.17}$$

will be referred to as the set of **well typed (covariant)** monotone functions of type $\mathcal{V}(N) \to \mathcal{P}(N)$. $\mathcal{W}^i(N)$ will refer to the poset of well typed covariant monotone functions, with the pointwise order.

Definition 5.3.3. Let N be a von Neumann algebra. The function $\delta^i : \mathcal{P}(N) \to \mathcal{W}^i(N)$ is defined locally at each $V \in \mathcal{V}(N)$ as:

$$\delta^i(\hat{P})_V := \bigvee(\downarrow \hat{P} \cap \mathcal{P}(V)) \tag{5.18}$$

Remark 5.3.4. The function δ^i is considered in the topos scheme, it is referred to as **inner daseinisation**. It encodes the unusual idea of "approximation from below", finding a more specific approximant to a proposition in each subalgebra. The full title of the function δ encountered earlier is **outer daseinisation**, it is more natural from the contravariant point of view, since it is the mathematical realisation of the physical concept of coarse-graining. The adjective "outer" is often dropped. When introducing duals of objects that naturally exist in the two schemes, the superscripts "i" and "o" will be used to indicated new inner or outer versions of objects.

Definition 5.3.5. Let N be a von Neumann algebra. We define the function $S^i : \mathcal{DP}(N) \to \mathcal{W}^i(N)$. S is defined locally at each $V \in \mathcal{V}(N)$:

$$S^i(U)_V := \bigvee(U \cap \mathcal{P}(V)) \tag{5.19}$$

Lemma 5.3.6.

$$\delta^i = S^i \circ F^i \tag{5.20}$$

Proof. Dual of lemma (3.2.5). □

Definition 5.3.7. Let L be a join semilattice. Call $A \subseteq L$ a **U-filter** if it is a D-ideal in L^{op}. Let $UF(L)$ be the lattice of U-filters. By duality, $(UF(L))^{op}$ is the injective hull of L (as a join semilattice).

Remark 5.3.8. The name U-filter has no particular signficance except it follows D-ideals in form, and provides some symmetry to the names of the lattices under consideration.

Definition 5.3.9. Let L be a join semilattice. Define the function B^o:

$$B^o : L \to (UF(L))^{op} \tag{5.21}$$

$$x \mapsto \uparrow x \tag{5.22}$$

Remark 5.3.10. B^o is just the dual of the embedding B discussed previously.

With these dual definitions in place, the relationship between the Coecke construction and the topos approach can now be explored. Firstly we observe that δ can be factored through the injective hull of a join semilattice, and dually, inner daseinisation can be factored through the injective hull of a meet semilattice. This follows directly from their definitions.

Lemma 5.3.11.

$$\delta = S \circ I^o \circ B^o \tag{5.23}$$

$$\delta^i = S^i \circ I \circ B \tag{5.24}$$

Proof. Let N be a von Neumann algebra, and $\hat{P} \in \mathcal{P}(N)$, then for every $V \in \mathcal{V}(N)$:

$$S(I^o(B^o(\hat{P})))_V = S(I^o(\uparrow \hat{P}))_V \quad \text{definition} \tag{5.25}$$

$$= S(\uparrow \hat{P})_V \quad \text{definition} \tag{5.26}$$

$$= \bigwedge(\uparrow \hat{P} \cap \mathcal{P}(V))_V \quad \text{definition} \tag{5.27}$$

$$= \delta(\hat{P})_V \quad \text{definition} \tag{5.28}$$

The factorization of δ^i follows by duality. □

The following proposition shows a variety of adjunctions that exist between the various components of the two schemes, and their duals. The symbol $\dashv$ to the left/right of a map indicates that this map has a left/right adjoint. For details, see the following proposition.

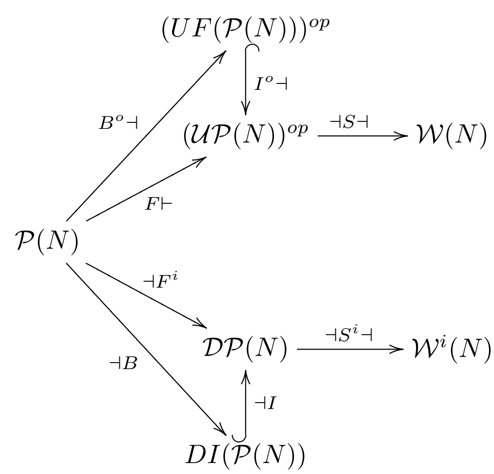

Figure 5.1: Daseinisation and Bruns-Lakser

Proposition 5.3.12. *Let N be a von Neumann algebra. We have the following adjunctions, as shown in figure 5.1.*

- *B has a left adjoint, B^o has a right adjoint.*
- *I has a left adjoint, I^o has a right adjoint.*
- *F has a right adjoint, F^i has a left adjoint.*
- *S has a left adjoint, S^i has a right adjoint.*
- *If and only if for every $V \in \mathcal{V}(N)$, $\mathcal{P}(V)$ is completely distributive then S has a right adjoint and S^i has a left adjoint.*

Proof. By O-3.3 in [Giertz et al., 2003], if a functor between complete lattices has a left adjoint, it preserves arbitrary meets, and dually. By O-3.4 in [Giertz et al., 2003], if a functor between complete lattices preserves arbitrary meets, it has a left adjoint, and dually.

To confirm the claimed adjunctions, we have:

- B preserves arbitrary meets from [Bruns and Lakser, 1970], B^o preserves arbitrary joins by duality.
- I has a left adjoint by lemma (4.7.9), I^o has a right adjoint by duality.
- F preserves arbitrary joins by lemma (3.2.12), F^i preserves arbitrary meets by duality.
- S preserves arbitrary meets by lemma (3.2.18), S^i preserves arbitrary joins by duality.
- If for every $V \in \mathcal{V}(N)$, $\mathcal{P}(V)$ is completely distributive, then S preserves arbitrary joins by corollary (3.2.29), S^i preserves arbitrary meets under those conditions by duality.

□

Remark 5.3.13. A large family of related adjunctions for the topos approach were shown in [Lal, 2008]. For daseinisation of projections, this result can be outlined as follows. For von Neumann algebra M, and (not necessarily abelian) subalgebra $N \subseteq M$, the injection $\mathcal{P}(N) \to \mathcal{P}(M)$ has both left and right adjoints. The left adjoint is given by:

$$\delta(\_)^o_{M,N} : \mathcal{P}(M) \to \mathcal{P}(N) \tag{5.29}$$

$$\delta(\hat{P})^o_{M,N} := \bigwedge \uparrow \hat{P} \cap \mathcal{P}(N) \tag{5.30}$$

a generalized form of outer daseinisation. Dually, the right adjoint gives a generalized form of inner daseinistation:

$$\delta(\_)^i_{M,N} : \mathcal{P}(M) \to \mathcal{P}(N) \tag{5.31}$$

$$\delta(\hat{P})^i_{M,N} := \bigvee \downarrow \hat{P} \cap \mathcal{P}(N) \tag{5.32}$$

5.4 Universal properties

As $\mathbf{MSLat_{dis}}$ is a full monoreflective subcategory of **Frm** [Stubbe, 2005], the corresponding adjunction gives a universal property for the Coecke construction, as discussed in section 4.8.

For the topos approach, ideally, some class of morphisms between von Neumann algebras could be lifted canonically to morphisms between the corresponding lattices of clopen subobjects. It is not yet known if the topos approach comes equiped with such as universal property. Progress in this area is described in [Döring, 2010a], where the perspective taken is to see the spectral presheaf as a candidate for a noncommutative extension of the Gel'fand spectrum. This approach uses a modified base category, removing all subalgebras of the centre of the given von Neumann algebra such that the construction reduces to ordinary Gel'fand duality for commutative algebras.

Chapter 6

Conclusion

6.1 Summary

In this dissertation two alternative approaches to the construction of a quantum logic were analyzed, the topos approach due to Isham, Butterfield and Döring, and the Coecke approach.

We analyzed the structure of the topos construction by considering a natural factorization of the daseinisation function. This factorization and the properties preserved by its components were investigated in detail, many of these properties were shown to be a result of it being the factorization via the free coframe of a join semilattice. The analysis of the factorization was then shown to be sufficient to recover the known properties of daseinisation. We then moved on to consider the relationships between various frames related to the codomain of daseinisation, showing that the frame generated by the codomain of daseinisation is in general a strict subframe of the lattice of clopen subobjects of the spectral presheaf.

The Coecke construction was then considered. Attention was focused on the nature of sets with distributive joins, as these are key to the underlying Bruns-Lakser injective hull construction. A geometric characterization of distributive joins in the lattice of projections into an arbitrary Hilbert space was presented. This was then extended to a characterization of distributive joins in a large class of complete lattices, in terms of their completely join irreducible elements.

Finally, the two approaches were related to each other. For finite cases, the topos approach lattice of clopen subobjects was shown to always be larger than the injective hull. The prior factorization of daseinisation was exploited to relate the two constructions. By considering the dual of each embedding, a family of adjunctions between the various lattices involved was exhibited. It was shown that the Coecke scheme is actually more closely related to inner daseinisation than the standard (outer) daseinisation that is considered more physically natural in the topos approach. Universal properties of the two constructions were also investigated, the Bruns-Lakser lattice was seen to have a universal property. Currently the question of a universal property for the topos approach is unresolved.

6.2 Future work

The work in this dissertation has suggested that the following areas may be of interest for further investigation:

- Can a universal property for the topos approach be found, or a demonstration that no reasonable property will exist?
- The standard form of the topos approach is constructed for von Neumann algebra N by using $\mathcal{V}(N)$ as the base category. Various other base categories are worthy of consideration, for example including the trivial subalgebra, or removing all subalgebras contained within the centre of N. What is the impact of such decisions on the properties of daseinisation? Which properties are robust to a wide range of decisions, and which are sensitive to the precise form of the base category?
- Can a concrete characterization of the frame generated by the codomain of daseinisation be found? Which is the "correct" frame for the quantum logic of the topos approach? Do the additional elements in $Sub_{cl}(\underline{\Sigma})$ describe addititional physics, or are they just artifacts of the mathematical construction?
- The characterization of distributive joins given in theorem (4.6.6) is for a restricted class of complete lattices. Can this be generalized to arbitrary complete lattices, or even to meet semilattices, without the completeness condition?

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