Categorical Semantics of Time Travel and Its Paradoxes

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September 3, 2018

Submitted in partial fulfilment of the requirements for the degree of

MSc in Mathematics and the Foundations of Computer Science

Trinity 2018

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Acknowledgements

I would like to thank my supervisors Prof. Bob Coecke and Dr. Stefano Gogioso, without them I wouldn't have dared to touch a subject which although has always deeply fascinated my curiosity, was distant from my formal mathematical training and concealed under the abstruse esoterisms of physical jargon. The enchantment about the foundational aspects of quantum mechanics arised from the excellent course given by Prof. Coecke, where I came at the realisation that there are deep fundamental aspects about reality which get interpreted in the marvellous achievement of the human intellect represented by quantum physics.

Without Dr. Gogioso this dissertation would have never been written, I should like to thank him very warmly for his invaluable help and for the time taken in answering my questions, clearing my doubts. He made writing this dissertation an extremely rewarding experience and I will always remember our fruitful conversations.

Abstract

In this dissertation we use string diagrams and categorical quantum mechanics to study the models for quantum time travel developed by Deutsch (D-CTC) and Lloyd (P-CTC). We show that the P-CTC formalism can be described in a process theoretical fashion using the time symmetric formalism for quantum theory developed by Oreshkov and Cerf, and that among other properties has the drawback of allowing superluminal signalling. We provide a compositional extension of the D-CTC model and show that such a theory generates a symmetric monoidal category of processes **DMix**. In addition, we analyse some properties of **DMix** underlying that contrary to the P-CTCs it satisfies the causality principle thus preventing signalling trough space-like correlations. Finally we construct a framework that encodes the causal structure of a discretised version of spacetime and show that the D-CTCs have the potential to describe a specific class of cyclic causal diagrams while the P-CTCs allow more flexibility in the description of the allowed interactions with closed timelike curves (CTCs). In this thesis we also underline the parallelism between models explaining quantum time travel and abstract traces in symmetric monoidal categories.

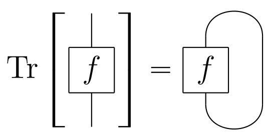
1 Introduction

"L'univers est vrai pour nous tous et dissemblable pour chacun."
Marcel Proust, *La Prisonnière*

According to general relativity spacetime is a dynamic variable dependent on the distribution of mass and energy. Quantum mechanics in its process theoretical presentation assumes a predefined direction of time and explains how preparations, quantum maps and effects relate to each other in order to assign probabilities for physical events. A fundamental contrast in the treatment of time between the two theories is therefore revealed. What happens in the process theoretical formulation if we loosen the rigidity implied by a predefined "arrow" of time?

In this dissertation we will try to provide a partial answer to this question by observing the implications of the assumption of the existence of localised *closed time-like curves* (CTCs); therefore to understand how is *chronology-violating* (CV) quantum information allowed to interact with a *chronology-respecting* (CR) region of spacetime which preserves the one-way global time asymmetry.

The genealogy of the ideas in this dissertation traces back to the talk about the Physics of Time-travel given by Seth Lloyd at the Foundations Discussions at Wolfson College about his work on CTCs (LMG^{+}11; LMGP^{+}11), it then appeared clear how the intuition related to his model is deeply connected to the graphical interpretation of quantum protocols; effectively one could speculatively compare quantum retrocausality with nonlocal correlations and entanglement. For example, in the compact dagger category **FdHilb**, the usual trace operation for matrices is graphically defined by:



If we take these diagrams to represent processes in the spacetime, the analogy with chronology violation is crystalline, an output of a process becomes at the same time its own input.

Our first thought was therefore that the ideas behind the Lloyd's model must, from a categorical perspective, be related to the introduction of a generalised concept of trace in the formalism of density matrices. In this work we want to discover to what extent is this intuition true and physically justified. There are two main alternative theories aimed at describing the interaction with a closed time-like curve; is it true that, using a categorical description of quantum mechanics, a structural fill rouge in both constructions can be given by *traced* monoidal categories? We will see however that from a process theoretical perspective there are some fundamental issues with Lloyd's model and mainly decided to narrow the scope of the thesis on the seminal work by Deutsch (Deu91) which culminated with the model known in the literature as D-CTC.

Chapter 2 provides a general and brief introduction to some of the main concepts and tools used throughout this dissertation. In Chapter 3, we formalise the description of the interaction with a CTC, we find that the best way to convey the essence of the problems associated with time travel is by appealing to the classical paradoxes which are well known to science fiction. All the models explaining the "time travel" of quantum information are in fact receipts to systematically resolve these troublesome conceptual obstacles. In Chapter 4 we present from an innovative diagrammatic perspective the peculiarities and consequences of the Deutsch model. We anticipate that trough the development of this dissertation we found the graphical approach to quantum circuits to be particularly useful in finding examples and in shedding light on the mechanisms at play in the often cumbersome calculations. Chapter 5 briefly discusses the solution exposed by Lloyd (known in the literature as P-CTC), where we show that this prescription can already be integrated in an entirely process theoretical framework developed by Oreshkov et al. (OC16; OC15). In Chapter 6 we design a process theoretical model for D-CTCs and in Chapter 6.4 discover that the analogy with traces can be recovered in Deutsch's formalism. In Chapter 7 we then provide a possible physical interpretation of this observation: we show that some properties of abstract traces have to be imposed if we want to preserve a relativistic covariance on the interpretations of cyclic graphs representing causal connections.

In the final part, the discussion focuses around the different types of CR-CV interactions that can arise from a causal graph and compare the D-CTCs and P-CTCs to observe a trade-off between the expressive flexibility of the model and the compliance to the principle of causality.

2 Background knowledge

In this section we will review some elementary concepts about density matrices, categorical quantum mechanics and process theories. It is not a comprehensive tutorial, its scope is mainly to set some of the basic definition of concepts. we invite the reader with a limited knowledge on the matter to address the comprehensive and complete account of the diagrammatic approach provided in (CK17). A discussion about the graphical calculus for monoidal categories can be found in (Sel11). We will start by explaining the conceptual ideas that relate to the mathematical notion of mixed states, the understanding of which plays a fundamental role in this dissertation.

2.1 The Density Matrix

This dissertation presupposes an elementary background knowledge in quantum mechanics, however we will review some elementary properties of the density matrix formalism. For an in-depth introduction to the subject we refer to (NC11) or to the entirely diagrammatic approach presented in (CK17). We can formulate quantum mechanics using the language of states and state vectors, rays of Hilbert spaces associated to the quantum states of a system. However, it soon becomes clear that this is insufficient to describe in a more general way quantum evolutions.

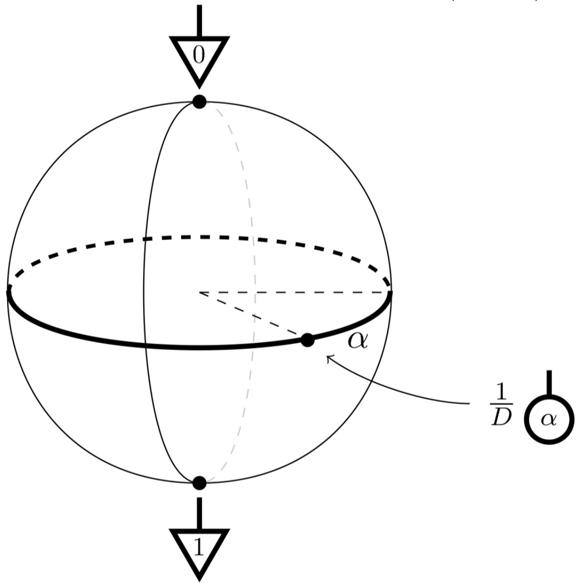
While Schrödinger equation imposes unitary evolution on closed systems, how can we characterise the evolution of subsystems and of open systems where the interaction with the environment plays a fundamental part? In particular, it becomes sometimes problematic to describe states of subsystems using the state-vector analogy. We now provide an example which will helpfully explain the importance of density matrices. Suppose that we have an ensemble of identically prepared entangled EPR states:

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

assume also that an observer in the location B doesn't have access to the state at A. How can B describe the state of the single particle which is to him accessible? He can answer this question by performing *state tomography*, i.e by trying to experimentally determine the state. If we only have a unique copy of the state ρ there is no hope to determine what the state of the qubit is; to overcome this problem suppose that the observer at B has an entire ensemble of identically prepared EPR states. By measuring multiple states he can therefore obtain statistics from which determine the global state describing the particles in the ensemble. Suppose now that the ensemble gets divided three equal parts in order to perform three different measurements. First we perform on the first third of the ensamble a measurement in the computational basis $\{|1\rangle, |0\rangle\}$. Assume that B observes that the measured qubits collapse to either $|1\rangle$ or $|0\rangle$ with equal probability. We already know a class of pure state which entails these probabilities: the equator of the Bloch sphere, (highlighted in black in Figure (1)), the observer may be therefore tempted to conclude that the quantum state which describes the ensemble is an equal *superposition* of the states $|0\rangle$ and $|1\rangle$:

$$|\Phi_A\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{\omega i\pi}}{\sqrt{2}}|1\rangle$$

Figure 1: Bloch sphere from (CK17)



To determine with more accuracy the vector on the Bloch sphere representing the state, the observer can now measure the second third of the ensemble with respect to the ONB:

$$\left\{ |+\rangle = \frac{|1\rangle + |0\rangle}{\sqrt{2}}, |-\rangle = \frac{|1\rangle - |0\rangle}{\sqrt{2}} \right\}$$

we still observe that half of the particles in the ensemble are in the state $|+\rangle$ and half in the state $|-\rangle$. With this knowledge the observer concludes that there are only two pure states that can describe the state of the spin of the particle (recall that each particle must be in the same quantum state as they are identically prepared), the so called Y basis, given by:

$$\left\{ |\psi_1\rangle = \frac{|1\rangle + i\,|0\rangle}{\sqrt{2}}, |\psi_2\rangle = \frac{|1\rangle - i\,|0\rangle}{\sqrt{2}} \right\}$$

Suppose B takes the third part of the ensemble and measures it with respect to the basis $\{|\psi_1\rangle, |\psi_2\rangle\}$ we find out that the statistics of either of the two states is again split in half, $|\psi_1\rangle$ and $|\psi_2\rangle$ are obtained with equal probability. How is it possible? If the state was represented by vector living on the surface of the Bloch sphere the outcomes of the experiment that we obtained would simply be inconsistent.

It turns out in fact that the best way to describe a single part of the EPR pair, a single subsystem while ignoring the other one is by taking a probabilistic mixture of the states $|0\rangle$ and $|1\rangle$. However we note that this is not a consequence of a lack of knowledge of the individual part of the system, it is just the *best* possible physical description of a class of particles which are subsystems of a bigger *identically* prepared physical system and is therefore not representing an ambiguity in the state derived from an ignorance of the local state. In order to describe those probabilistic mixtures we need to expand the vector-state notation:

Definition 2.1 (Density Matrix, (NC11))**.** Suppose a quantum system is in one of the number of states $\{|\psi_i\rangle\}_i$ with respective probabilities p_i, the *density operator* or *density* *matrix* for the system is defined by the equation

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle\psi_i| \,.$$

The density matrix associated to a pure state $|\psi\rangle$ is therefore given by the outer product

$$|\psi\rangle\langle\psi|$$

Suppose the state evolves according to a unitary evolution, then this can be represented in the language of density matrices

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| \overset{U}{\mapsto} \sum_i p_i U |\psi_i\rangle \langle\psi_i| U^\dagger = U\rho U^\dagger$$

we can also restate the measurement postulate in terms of density matrices:

Definition 2.2 (Measurement)**.** Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. The probability that the result m occurs is given by

$$p(m) = \mathrm{Tr}\left[M_m \rho M_m^\dagger\right]$$

and the state after the measurement is given by

$$\frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\left[M_m \rho M_m^\dagger\right]}$$

where the measurement operators satisfy the completness equation,

$$\sum_m M_m^\dagger M_m = \mathbb{1}$$

The quantum mechanics of closed system is characterised by the unitary evolution, what happens when we introduce the environment, how can we describe all the allowed transformation of a density matrix? The class of allowed *deterministic* evolutions for open systems is significantly larger and it is described by the *completely positive trace preserving maps*(CPTP maps).

Definition 2.3 (CPTP maps)**.** A general quantum map $\mathcal{E}$ is an operator taking density matrices to density matrices, it therefore has to satisfy the following properties:

- *Trace-preserving* For any density matrix ρ we have that $\mathrm{Tr}[\mathcal{E}(\rho)] = 1$.
- $\mathcal{E}$ is a *convex-linear* map on the set of density matrices.
- $\mathcal{E}$ is a completely positive maps, i.e it maps density operators to density operators.

In the framework of categorical quantum mechanics these transformations have a particularly intuitive description. Before sketching the ideas behind categorical quantum mechanics it is useful to define the last concept which will be needed in defining D-CTCs, a generalisation of Shannon entropy for quantum states:

Definition 2.4 (Von Neumann Entropy, (NC11))**.** Let ρ be a density matrix associated to a quantum mechanical system, the Von Neumann entropy of the state is defined to be

$$S(\rho) \equiv -\sum_x \lambda_x \log \lambda_x$$

where λ_x are the eigenvalues of ρ.

It is clear by definition that the entropy of an operator in a d-dimensional system is between 0 and $\log(d)$. Pure states have entropy 0 and the maximally mixed state has entropy given by $\log d$.

2.2 Categorical Quantum Mechanics

In describing the quantum phenomena that arise in this dissertation we will make extensive use of the framework provided by *categorical quantum mechanics* (CQM). Initiated by Coecke and Abramsky (AC09; AC04), the approach describes the quantum phenomena and quantum processes as symmetric monoidal categories and allows us to deduce equational relationships between morphisms by performing graphical calculations. For a beautiful and self-contained survey on the graphical calculus we refer to (Sel11). We will only briefly define the object of interest which models physical theories in a process theoretical way, *symmetric monoidal categories* (SMC). Category theory brings the notion of processes to the forefront (CP11) and it can be used to provide a rigorous and universal language for defining the fundamental backbone of a physical theory, in (CP11) the authors provide a general example which captures the essential connections between physics and category theory, we reproduce the definition of this general setting provided in (CP11):

Example 2.5 (The category **PhysProc**)**.** The category of **PhysProc** can be defined as follows:

- All physical systems $A, B, C \ldots$ as objects
- Physical processes that take place between the physical systems of type A and of type B are considered to be the morphisms $A \to B$
- Sequential composition of the process as the composition of morphism
- The process Id_A which leaves the physical system A invariant

The authors in (CP11) then underline that the only additional axiom, associativity has in fact a straightforward physical intepretation. If we first apply the process f and then the processes g and h, it is irrelevant if we consider $g \circ f$ as a single process to whom we later apply h or if we instead consider $h \circ g$ as a single process applied to f.

In a physical theory it is however often necessary to distinguish between two types of compositions, the sequential composition which expresses a *time-like* separation of two processes and a *space-like* composition. The framework of a category accommodates for the first notion, to introduce a parallel, space-like composition we need the define on the category an additional structural layer:

Definition 2.6 (Monoidal Categories)**.** A *monoidal category* is a category **C** equipped with the following data:

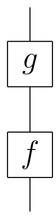
- a *tensor product* bifunctor $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$;
- a *unit object* I
- an *associator* natural isomoprhism $\alpha_{A,B,C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)$;
- a *left unitor* natural isomorphims $\lambda_A \colon I \otimes A \to A$;
- a *right unitor* natural isomoprhism $\rho_A \colon A \otimes I \to A$;

This data must satisfy the *triangle* and *pentagon* equations, for all objects A, B, C and D:

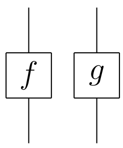
$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ & {\scriptstyle \rho_A \otimes \mathrm{Id}_B} \searrow \quad \swarrow {\scriptstyle \mathrm{Id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array}$$

$$\begin{array}{ccc} (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\ \uparrow {\scriptstyle \alpha_{A,B,C} \otimes_D} & & \downarrow {\scriptstyle \mathrm{Id}_A \otimes \alpha_{B,C,D}} \\ ((A \otimes B) \otimes C) \otimes D & & A \otimes (B \otimes (C \otimes D)) \\ & {\scriptstyle \alpha_{A \otimes B,C,D}} \searrow \quad \nearrow {\scriptstyle \alpha_{A,B,C \otimes D}} & \\ & (A \otimes B) \otimes (C \otimes D) & \end{array}$$

Any free morphism in a monoidal structure can be represented graphically, for morphisms $f \in \mathbf{C}(A, B), g \in \mathbf{C}(B, C)$ we can represent the composition $g \circ f$ as:



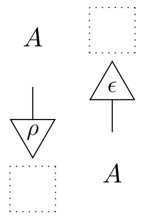
for $f \in \mathbf{C}(A, B), g \in \mathbf{C}(C, D)$, the monoidal $f \otimes g$ product of the morphisms is given by



The identities are just straight lines:

The unit object is defined by the empty diagram:

The states and effects are morphisms of the type $\rho : I \to A$ and $\epsilon : A \to I$, they can be graphically represented with upwards and downwards directed triangles



It is worth noticing that all the coherency conditions are implicit in the graphical definition of the morphisms, a SMC in fact satisfies the following powerful property:

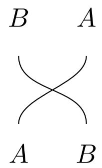
Theorem 2.7 (Correctness of the graphical calculus (HV12))**.** A well-typed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.

The categories that we will be interested on will all be symmetric monoidal categories, i.e. monoidal categories which encompass the idea of a symmetric braiding of objects:

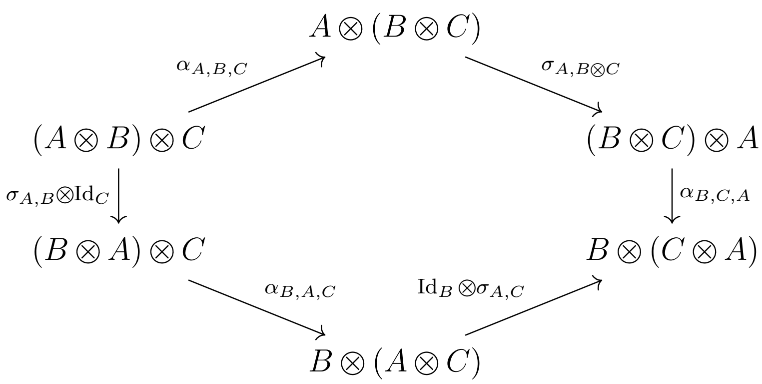
Definition 2.8 (Symmetric Monoidal Category)**.** A *symmetric monoidal category* is a monoidal category equipped with an additional natural isomorphism $\sigma_{A,B}$

$$\sigma_{A,B} \colon A \otimes B \to B \otimes A$$

which can be graphically denoted as



satisfying the following *hexagon identity*:



and such that $\sigma_{A,B} = \sigma^{-1}_{B,A}$.

Symmetric monoidal categories (SMC) have a sound and complete graphical calculus where the wires are therefore allowed to cross themselves as if the diagrams where objects living in a 4-dimensional space.

Theorem 2.9 (Correctness of the graphical calculus for symmetric monoidal categories (HV12))**.** A well-typed equation between morphisms in a symmetric monoidal category follows from the axioms is and only if it holds in the graphical calculus up to four-dimensional isotopy.

The leitmotif of this dissertation is certainly given by symmetric monoidal categories, we will impose on them a last additional structure which is an abstract generalisation of the usual concept of partial traces for linear maps. We now provide the definition of this categorical structure taken from (Sel11):

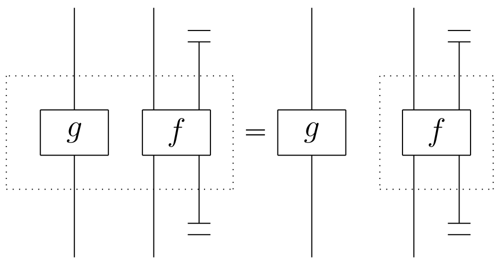
Definition 2.10 (Right Trace)**.** Let $\mathbf{C}$ be a monoisal category, a *right trace* is a family functions:

$$\mathrm{Tr}^{X}_{A,B} : \mathbf{C}(A \otimes X, B \otimes X) \to \mathbf{C}(A, B)$$

which are natural in A and B and dinatural in X. Satisfying the following three axioms:

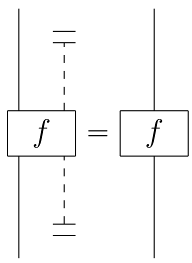
- **Strength**: For all $f : A \otimes X \to B \otimes X$ and $g : C \to D$ we have that

$$g \otimes \mathrm{Tr}^{X}_{A,B}(f) = \mathrm{Tr}^{X}_{A\otimes C, B\otimes D}(g \otimes f).$$



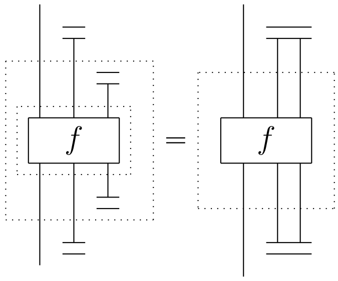
- **Vanishing I**: For all $f \colon A \otimes I \to B \otimes I$,

$$f = \mathrm{Tr}^{I}_{A,B}(f).$$



- **Vanishing II** For all $f : A \otimes X \otimes Y \to B \otimes X \otimes Y$,

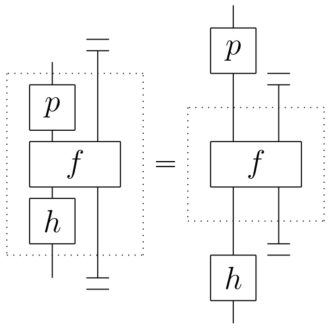
$$\mathrm{Tr}^{U}_{A,B}(\mathrm{Tr}^{V}_{A\otimes U, B\otimes U}(f)) = \mathrm{Tr}^{U\otimes V}_{A,B}(f).$$



We also explicitly state the meaning of naturality in A, B and dinaturality in X in order to stress their intuitive graphical interpretation

Definition 2.11 (Naturality in A, B)**.** For all $f : A \otimes U \to B \otimes U$, $h : C \to A$ and $p : B \to D$ we have that

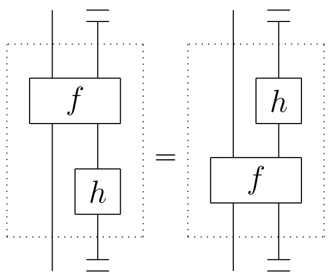
$$p \circ \mathrm{Tr}^{U}_{A,B}(f) \circ h = \mathrm{Tr}^{U}_{C,D}((p \otimes 1_U) \circ f \circ (h \otimes 1_U)) \tag{1}$$



(2)

Definition 2.12 (Dinaturality in X)**.** For all $f : A \otimes X \to B \otimes Y$, $h : X \to Y$ the following equation holds:

$$\mathrm{Tr}^{X}_{A,B}(f \circ (1_A \otimes h) = \mathrm{Tr}^{Y}_{A,B}((1_A \otimes h) \circ f) \tag{3}$$

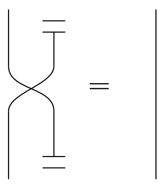


(4)

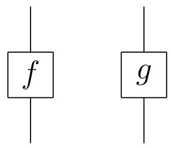
Definition 2.13 (symmetric traced category)**.** A *symmetric traced categroy* is a symmetric monoidal category with a right trace Tr satisfying the *symmetric yanking axiom*:

$$\mathrm{Tr}^{X}(\sigma_X, X) = \mathrm{Id}_X \tag{5}$$

or graphically:



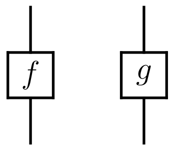
The category of finite-dimensional Hilbert spaces and linear maps will be denoted by **FdHilb**, the composition of morphisms is given by matrix multiplication, the tensor product is the usual tensor product of vector spaces. We will denote morphisms in this category using the undoubled boxes



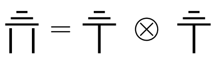
The symmetric monoidal category that we will treat as a framework for quantum theory is the category **Mix** (CL13), whose objects are finite dimensional Hilbert spaces and whose morphisms are completely positive maps:

$$\mathbf{Mix}(\mathcal{H}_A, \mathcal{H}_B) := \{ f \in \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B) \mid f \text{ is completely positive } \}$$

where the set of linear operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. The morphisms of this category will be denoted as *doubled* boxes. A justification for this use of notation is given in (CK17; CHK14)



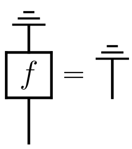
The category **Mix**, contrary to **FdHilb** allows us to define a family of*discarding* morphisms $\top_A \colon A \to I$ for every object A satisfying the following properties (KHC17):

$\top_{A \otimes B} =$  $= \top_A \otimes \top_B$

and

$\top_I =$  $= \mathrm{Id}_I$

Using this discarding map we can define the normalised subcategory of **Mix**, denoted by $\mathbf{Mix}_\top$. This is the category generated by processes obeying the causality principle:

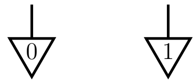


The principle is therefore a formal process theoretical version of the following concept: *"when the output of a process is discarded, then the process itself may also be discarded"* (KHC17) or alternatively: *"the only influence a causal process has is on its output"* (KS).

In (CD11) Coecke and Duncan proposed an alternative high-level language with a powerful graphical calculus: the ZX-calculus. The calculus is now a widely used technique to reason about qubits and quantum citcuits (CK17; CD13) and it has been shown not only to be a rigorous replacement for the Hilbert space formalism but to have the same deductive power (Bac16).

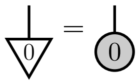
In the dissertation we are going to deal with models of time travel applied to finite dimensional quantum systems, those systems can be always simulated in terms of 2-dimensional qubits. We are therefore going to make use of the power of the ZX-calculus to construct counterexamples. It is in fact quite easier to construct a quantum circuit satisfying a certain property if you can keep in mind the graphical transformations that would be needed to prove it. We will not provide an introduction to the calculus, this can be found in (CK17), however we would like to clarify certain notational aspects.

We will associate to the computational Z-basis $\{|0\rangle, |1\rangle\}$ the white states

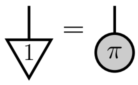


the X-basis states $\{|+\rangle, |-\rangle\}$ will be denoted in grey, we therefore have that

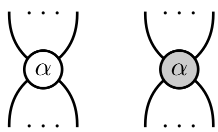
It is worth noticing that each state in one of the two basis can be written as a phase or "spider" of the other color, therefore we have:



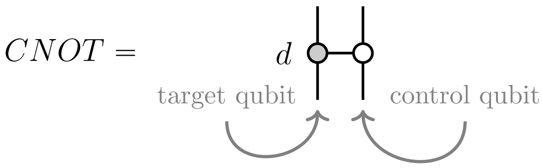
and:



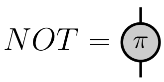
and similarly for the X basis states. We depict Z spiders and X spiders (CK17) as follows:



using this graphical language we can represent the CNOT gate as:

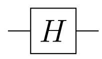


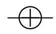
the NOT gate, or the Pauli-X gate as

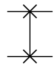


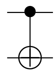
The gate above applies a rotation of the Bloch sphere by π around the X axis sending the state $|0\rangle$ to $|1\rangle$ and viceversa.

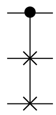
We will also make use of the classical diagram notation. Note that for formatting reasons we have decided to stick to the standard formalism, quantum circuits in the standard notation will be drawn from left to right. For ZX diagrams and other instances of the diagrammatic calculus we instead used the alternative convention of drawing the diagrams from bottom to top in line with the standard notation for spacetime diagrams in relativity. This is a summary of the notation used within standard quantum circuit:

Hadamard gate 

Pauli-X gate (not) 

Fredking gate (swap) 

controlled not 

control swap 

3 A Brief Introduction to Time Travel Paradoxes

When thinking about time travel, it immediately comes to mind as something disturbingly unphysical, often contradictory, meant to be exclusively relegated to the realm of science fiction. Too often in natural sciences and in mathematics, our intuitions leads to a misuse of the term *impossible*, labelling with it many phenomena on the ground of an apparent inconsistency with our conscious experience of the world.

Philosophy however has not been afraid to touch and study the subject, for many years metaphysics has been concerned with the notion of time, trying to reconcile it with the everyday conscious experience. In (Was18) Wasserman distinguishes between the notions of logical, technological, physical and metaphysical impossibility, reaching the conclusion that the paradoxes that populate science fiction books are related to the notion of a *metaphysical* impossibility as they imply a tension between the physical description of the word and the metaphysics of subjective experience. This doesn't make vain the study of the subject from an inherently *physical* or even computational perspective. The theory of general relativity itself is as widely accepted by the scientific community as it is indigestible for metaphysicians. In particular, the all encompassing nature of quantum mechanics has often allowed in the study of its foundations to break the adamant barriers between the domains of physics and metaphysics.

Looking at the time travel paradoxes from a macroscopical, human perspective they often invoke notions of *free will* or the problem of *identity*, it would be foolish and useless to dig in the depths of quantum theory in order to speak about those inherently philosophical questions. If we take a rigorous information theoretical approach many of those puzzling metaphysical paradoxes cease to exist or to be well defined problem at all. The entire literature on quantum solution to time travel paradoxes focuses on two main categories which continue to reveal formal inconsistencies between the evolution of quantum states in a chronology-violating setting and standard quantum theory. In this chapter we describe and analyse those two classes and describe the solution provided by David Deutsch in 1991 (Deu91) and Seth Lloyd in 2010 (LMG$^+$11; LMGP$^+$11).

We warn the reader that all those paradoxes can potentially all be described by charming science fiction scenarios, but to appreciate their fundamentality and generality we will approach them in their simplified circuit form. Following the conventions described in (Deu91) we will consider circuits in spacetime, bounded by an initial S_i and final S_f Cauchy hypersurface. Moreover, in accordance to (Deu91) we will denote the region of the spacetime containing the closed time-like curve *chronology-violating (CV)* and its complement will be denoted as a *chronology-respecting (CR)* region. The requirement on the existence of a chronology-respecting region assumes that we can talk about an ambient spacetime, containing the CTCs and we can therefore define an asymptotic notion of *unambiguous past* and *unambiguous future* (Deu91). Note that a paradox may arise only if there is the possibility of an interaction between a chronology-respecting and a chronology-violating region, a CR-CV interaction.

During the development and analysis of Einstein's theory of general relativity, there have been discoveries of solutions to its equations containing closed time-like curves (G4̈9; vS38). As such CTC's are not in contrast with the rules of general relativity. However, many physicists still reject the idea of the physical possibility of time travel. Most notoriously, Stephen Hawkings (Haw92) is a firm believer on the *chronology-respecting conjecture*, i.e that the law of physics must act in order to prevent the possibility of existence of chronology-violating regions. In this dissertation we will assume the existence of such regions and assume that the CV-CR interaction is allowed. We will not discuss the physicality of this assumption, we therefore warn the reader that this is an open physical debate.

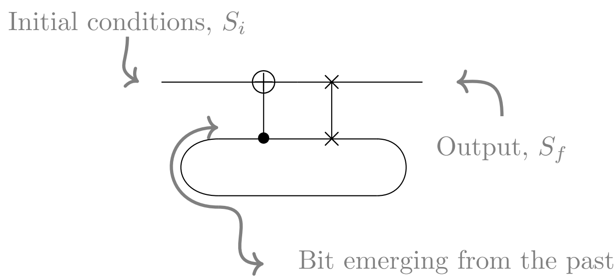
3.1 The Grandfather Paradox

Often journeys into the depths of time travel start with the grandfather paradox. The standard description of the paradox is the following:

> **Example 3.1** (The Grandfather Paradox)**.** A time traveller enters a wormhole at a location of the spacetime A, pops out at the other mouth of the wormhole in a spacetime region associated with its own grandfather's past and somehow menages to kill his grandfather before the conception of his father or mother and thus preventing his own birth.

We have clearly invoked notions that are unsuitable for a rigorous analysis, if we forget about grandchildren and grandfather, the paradox can be rephrased using physical system that are conceptually simpler and more amenable to an information theoretical analysis: bits, gates and circuits.

Consider the following *classical* circuit

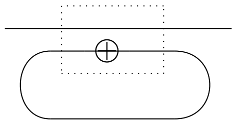


(6)

the initial condition is given by the value assigned to the bit on the open end on the left side of the circuit. Assume the initial bit is set to be equal to 1, it then interacts via a CNOT with a control bit emerging from the future, afterward the two bits exchange their values before the chronology-violating bit "returns" to the CTC. If we follow the closed path made by the information, we see that it makes a closed loop in spacetime.

The bit initially set to be equal to 1 is therefore sent in the past to flip its own value by activating the CNOT. This is not exactly the grandfather paradox but there is a clear logical analogy. The situation becomes paradoxical if we impose the requirement that the state of a bit entering the CTC must be equal to the state of the bit leaving the CTC, if the classical bit emerges from the CTC in the state 1 then it menages to kill itself, flipping the value of its past version to 0, if the classical bit emerges in the state 0 it doesn't change the value of the initial bit, which therefore enters the CTC in the state 1. Both cases are in contradiction with the consistency condition mentioned above. However we note that not all the initial values are contradictory, if we start with a qubit in the state 0, independently on the value of the classical bit emerging from the future we get a consistent event. We will address this phenomenon in the next subsection.

Of course there are many diagrams which present the same pathological behaviour. A contradiction is also obtained when considering the following simpler diagram, in particular this is the circuit that Lloyd (LMGP$^+$11) uses to describe the grandfather paradox.



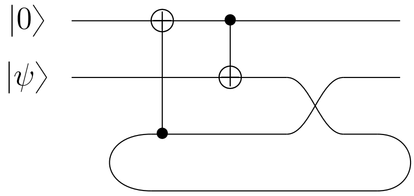
(7)

The main difference here is that the diagram leads to a contradictory evolution independently on the state of the chronology-respecting bit, any of the classically admissible states $\{0, 1\}$ entails a contradiction.

3.2 The Unwritten Poem

Consider again the Circuit (6), we have shown that if the bit enters in the state 1 it always leads to a contradictory timeline. Feeding the circuit with the input state $\rho_A = 0$ is however to certain extents even more worrying; when the bit initialised at 0 meets it's future self ρ_B travelling to the past, both values $\rho_B = 0$ and $\rho_B = 1$ are allowed and entirely consistent. The situation therefore may appear not to be paradoxical at all, as such there is no logical inconsistency. What is however the principle that made nature decide in which state to set the bit emerging from the mouth of the CTC? Any solution would seem to emerge arbitrarily and independently from the initial data defined on the Cauchy surface, we are witnessing a case of what Deutsch (Deu91) describes as a "knowledge creation paradox".

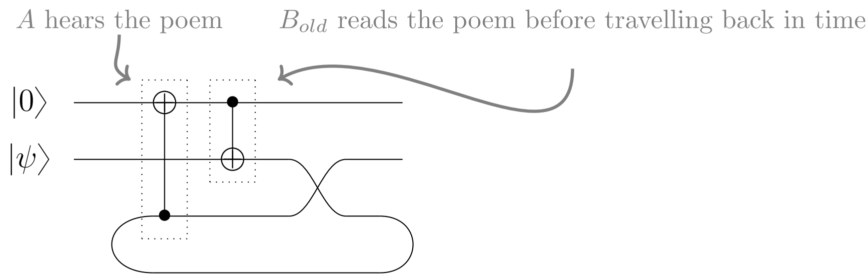
Usually in the literature the unwritten paradox is explained using the following circuit (LMG+11; All14):



(8)

The diagram faithfully reproduces the following story, which may help us to grasp the "paradoxical" nature of those logically consistent solutions:

> **Example 3.2** (The Unwritten Poem)**.** Suppose that A meets a charming time-traveller B who uninhibited by too much liquor loudly declaims a marvellous poem. Unaware of the origin of the mysterious stranger, A becames impressed by his lyrical virtues and decides to anonymously publish the poem. At a later time, the past version of B – B_{old} – reads the poem on a book written by A containing the poem just before entering the mouth of the CTC.



The story is therefore logically consistent, what has happened is that the interaction with a CTC created a cyclic causal relationship, there is information (in our case the poem) which arises from nowhere, without a well defined beginning, a creation which is confined to a circular and closed chain of events causally connected.

The information flowing in closed time-like curve without an apparent beginning has been extensively described in the work by Novikov and Lossev, in (LN92) they describe this phenomenon denoting it as a *Jinn of the second type*. In (Deu91) the author is particularly worried by the case described by our story defining it as violation of the *evolutionary principle*, which he informally states in the following statement: *"Knowledge comes into existence only by evolutionary processes"* (Deu91).

A satisfactory theory describing the way information behaves if we are allowing interaction with closed time-like regions of the spacetime would have to provide an explanation which minimises nature's ability to impose exotic and creative fixed points in such scenarios.

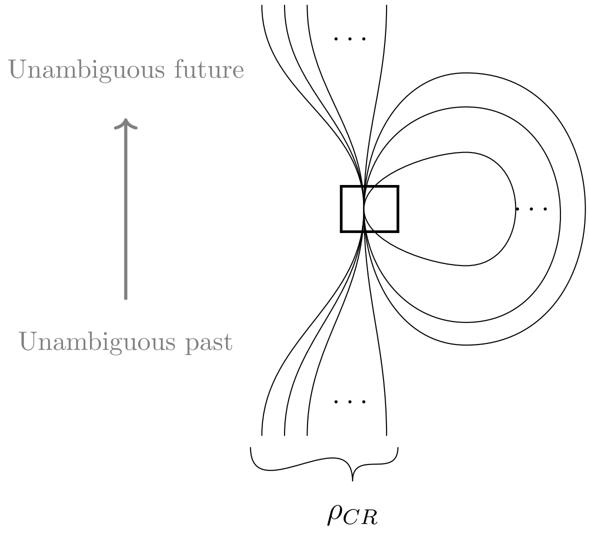
As David Deutsch explains in (Deu91), there is an extremely fascinating reason to believe that there are only those two distinct classes of "paradoxical behaviours". The existence of closed time-like curves in fact, creates regions of the spacetime which are both *under-specified* and *over-specified* by the initial conditions. Under-specified because for certain initial data there is still some nonitial *supplementary* data to be chosen, over-specified as certain initial solutions are retrospectively prohibited. In the next section we discuss a possible solution provided by Deutsch. We will see how shifting the domain from quantum to classical information gives more flexibility and allows us to find solutions which are always self-consistent.

4 Deutsch Closed Time-like Curves

4.1 Description of the D-CTC Model

Deutsch in the seminal paper (Deu91) started to use an informational theoretical approach to deal with the interaction with CTCs, this approach is clearly confined to the description of finite quantum systems which can always be simulated by qubits (NBD$^+$02). Deutsch starts by assuming that we are dealing only with the internal finite degrees of freedom of particles and that the carrier particles can be represented as localised wavepackets with well defined worldlines. Deutsch therefore neglects the dynamics of the particles which is assumed to be classical and given. It may seems limiting to consider situations involving this class of classical particles with finite internal degrees of freedom but Deutsch justifies this choice by saying that the class of such models can simulate and represent the behaviour of any finite quantum system.

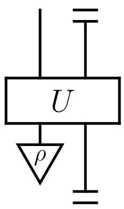
In (Deu91) the D-CTC protocol aims at describing situations where there is a single and localised interaction between chronology-respecting and chronology-violating qubits. We can always rewrite any physical situation in the following way: no particle actually enters in the closed time-like curve, information gets exchanged in a localised region of the spacetime. The network has therefore a well defined output and a well defined input lying two separate space-like hypersurfaces S_i and S_f. Moreover we can always enlarge the interaction such that the local evolution of the chronology-violating and chronology preserving qubits will by unitary. The situation can therefore be schematically described as follows, where the evolution is represented by a reversible unitary gate applied in the area denoted by the black rectangle:



(9)

In the digram above ρ_{CR} represents the state of the chronology-respecting qubits and the closed curves on the right represents the chronology-violating qubits ρ_{CV}. We assume that the unambiguous time flows from the bottom to the top. Finding a solution to the paradoxes is therefore equivalent to prescribing a recipe which assigns a consistent state to the chronology-violating qubits for any given unitary interaction and any initial state of the CR the qubits define an unambiguous dynamic for the inputs ρ_{CR}.

From here the generalisation to the process theoretical and informational approach to quantum theory is straightforward, we simply denote the region of interaction as a blackbox and the separate carriers are just representing the "wires" connecting those boxes. Such a digram is in fact no more than a quantum circuit and nothing prevents us to rewrite it using the diagrammatic calculus for quantum circuit described in (CD13), in fact we will see that finding fixed point and determining the evolution of the CR region, which is an essential part of the method that we are about to describe turns out to be particularly intuitive when rewritten in graphical terms. We will denote the graphical description of the interaction as follows:



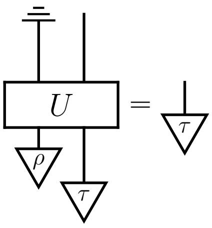
(10)

The construction starts by imposing a consistency that is a straightforward generalisation of the kinematical consistency condition on classical bits, i.e imposing that the state of the bit entering the CTC is equal to the state of the bit leaving the CTC. We want that the overall evolution leaves the state in the CTC unchanged, we therefore require a normalised fixed point such that:

$$\tau = Tr_{CR}[U^{\dagger}(\rho \otimes \tau)U]$$

Where the partial trace is taken over the subspaces of the chronology respecting systems.

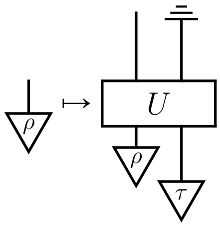
Graphically:



(11)

The evolution of the state ρ_{CR} is then given tracing out the content of the CTC:

$$\rho_{CR} \mapsto Tr_{CV}[U^{\dagger}(\rho_{CR} \otimes \tau)U] \tag{12}$$



(13)

The state τ is therefore a fixed point of the superoperator given by

$$S(\star) = Tr_{CR}[U^{\dagger}(\rho \otimes \star)U]$$

Deutsch proved analytically the existence of at least one fixed point for each such operators, we however underline that this is in fact a corollary of a well known result:

Theorem 4.1 (Existence of fixed points)**.** Let Ψ be a quantum operation, a completely positive trace preserving map acting on an N dimensional system, therefore there exists state such that $\Psi(\rho) = \rho$.

Proof. This is an immediate consequence of the Brouwer fixed point theorem,

Theorem 4.2 (Brouwer fixed-point theorem)**.** Let K be a convex compact subset of Euclidean space. Then every continuous function $f : K \to K$ has a fixed point.

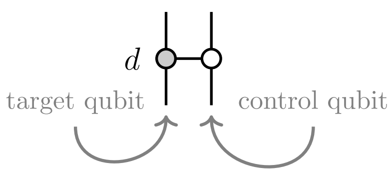
It is clearly that the set of density operators with trace equals to 1 is convex. A completely positive trace preserving map sends the set of normalised density operators to itself and must be continuous by definition. □

We will now see how this consistency condition provides a coherent quantum mechanical solution to the two different paradoxes that we have described above

4.1.1 The Quantum Grandfather

Let us reconsider the circuit (6) but now in its quantum version, the generalised CNOT gate is therefore given by the unitary

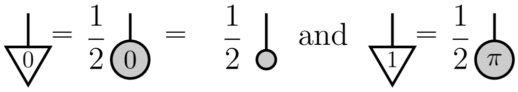
$CNOT =$



where the choice of basis ○ and ◉ form a complementary pair and d is the dimension of the Hilbert space (CK17, p. 612). In our case we are considering qubits, two dimensional quantum systems. The complementary basis are given by Z(○) and X(◉) and $d = 2$. In the CNOT gate above the Hilbert space of the control bit is on the right hand side and the target is on the left. Circuit (6) can therefore be reconstructed by setting the unitary map to be equal to $U = SWAP \circ CNOT$. Suppose that we are evaluating the circuit for the density matrix $\rho_{CR} = |1\rangle\langle 1|$; The normalised state $|1\rangle$ can be written in the vector-state formalism as a superposition of the elements of the X basis,

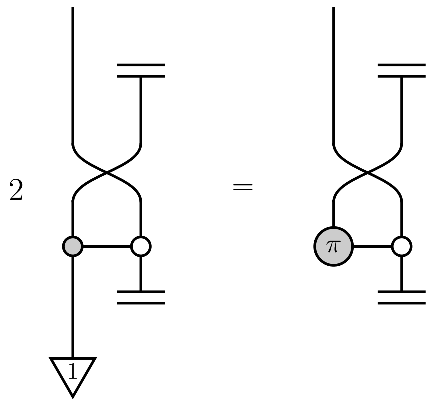
$$\begin{aligned} |1\rangle &= 1/\sqrt{2}\,|+\rangle - 1/\sqrt{2}\,|-\rangle \\ &= \frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\pi}\,|-\rangle\right) \end{aligned}$$

since we also have that $|0\rangle = 1/\sqrt{2}\,|+\rangle + 1/\sqrt{2}\,|-\rangle$, we can write

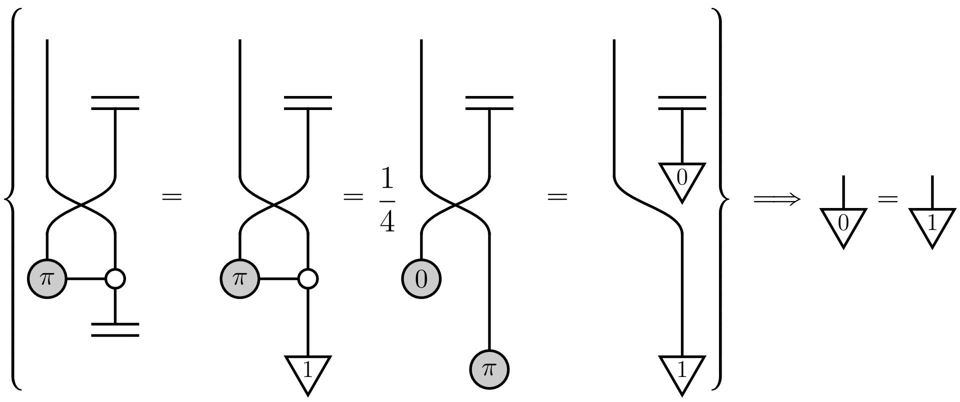


(14)

therefore we have that:

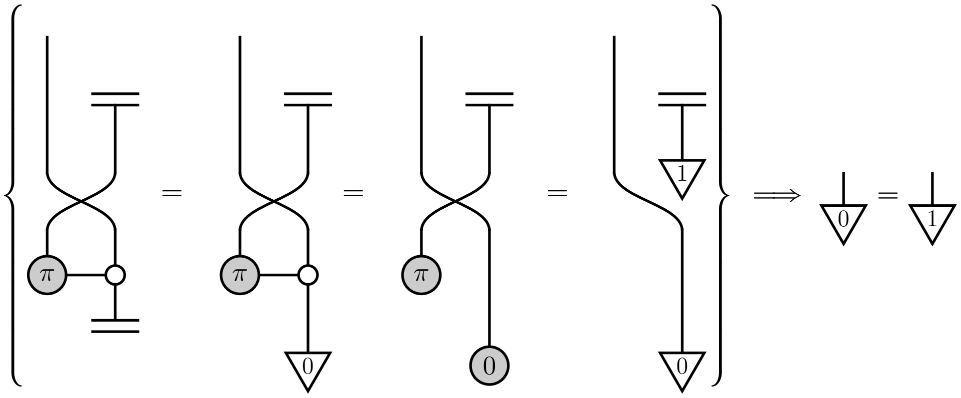


We can now recreate the contradiction for the classical paradox, assume that the density matrix entering the CTC needs to be equal to the density matrix leaving the CTC. Let $|1\rangle\langle 1|$ be the quantum state of the system travelling in the CTC then:



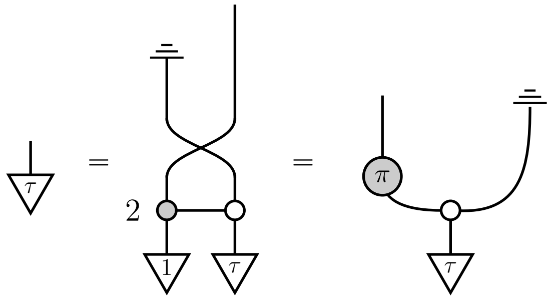
(15)

this is a contradiction as the qubit enters the CTC in the state represented by the density matrix $|0\rangle\langle 0|$ and it exits the CTC with density matrix $|1\rangle\langle 1|$. Similarly for $\tau = |0\rangle\langle 0|$



(16)

However the formalism of density matrices gives rise to possible consistent quantum stories. Let us apply the condition imposed by Deutsch, suppose we want to find the fixed point τ, we can perform the following diagrammatic calculations to show that it must satisfy:

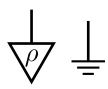


(17)

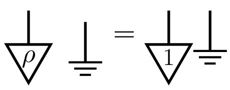
The state ρ has therefore to be invariant under decoherence and the subsequent action of the Pauli-X gate. The only state satisfying those conditions is given by the maximally mixed density matrix:

$$\rho = \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1|$$

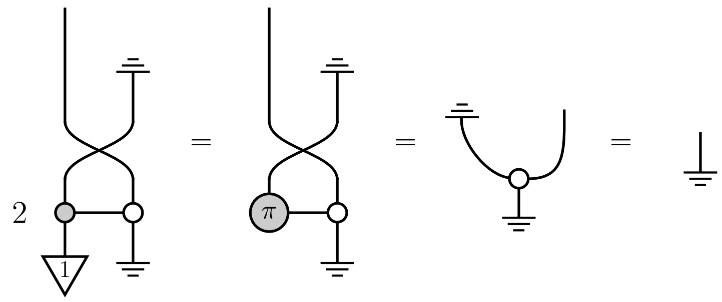
this is because decoherence would send any pure state which is not either $|1\rangle\langle 1|$ or $|0\rangle\langle 0|$ to a mixed state and the action of the Pauli-X gate sends $|1\rangle\langle 1|$ to $|0\rangle\langle 0|$ and viceversa, imposing that the state must be an equal mixture of the two. The solution is this case unique, to calculate the evolution of the chronology-respecting qubit we need to apply the unitary on the density matrix

$$\rho_{CR} \otimes (\frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1|) =$$ 

and trace out the Hilbert spaces of the chronology-violating qubit. To show the solution of the paradox let



then we have



note that we can therefore observe a first peculiarity of the model: pure states can be sent to mixed states and the evolution is therefore certainly not unitary. In standard Quantum Mechanics, systems that are initially in a pure state remain pure when evolving in isolation from everything else, moreover one can always find a larger pure system containing any mixed state as a subsystem (Deu91). Could it therefore be possible that the theory of D-CTC allows to transform a mixed state into a pure one and by this reduce the entropy of the Universe contradicting the laws of thermodynamics? An answer to this question has been provided by Deutsch (Deu91), we present the simple proof provided in that paper:

Theorem 4.3. Let ρ_{CR} be the input to the region which interacts with a CTC undergoing the evolution

$$\rho_{CR} \mapsto \rho'_{CR}$$

then $S(\rho_{CR}) \leqslant S(\rho'_{CR})$.

Proof. This will guarantee that for closed systems the entropy always increases or stays constant. To prove this we will need two well known lemmas about entropy (AL70):

Lemma 4.4. Let U be a unitary transformation, therefore:

$$S(U\rho U^{\dagger}) = S(\rho)$$

Lemma 4.5 (Subadditivity of entropy, (AL70))**.** Let ρ_{AB} be the density matrix of a system with two subsystems with density matrices ρ_A and ρ_B and $S(\rho_A)$ be the von Neumann entropy of the quantum mechanical system ρ_A, then the entropy satisfies subadditivity:

$$S(\rho_{AB}) \leqslant S(\rho_A) + S(\rho_B) \tag{18}$$

if $\rho_{AB} = \rho_A \otimes \rho_B$ then

$$S(\rho_{AB}) = S(\rho_A) + S(\rho_B)$$

Consider the case when the chronology-respecting qubit ρ_{CR} evolves interacting with a CTC into the density matrix ρ'_{CR}.

$$\rho_{CR} \mapsto \rho'_{CR}$$

The consistency condition and the definition of the evolution imply that

$$Tr_{CV}[U(\rho_{CR} \otimes \tau)U^{\dagger}] = \rho'_{CR}$$

and

$$Tr_{CR}[U(\rho_{CR} \otimes \tau)U^{\dagger}] = \tau$$

According to Lemma (4.5) we get that

$$S(U(\rho_{CR} \otimes \tau)U^\dagger) \leqslant S(\rho'_{CR}) + S(\tau). \tag{19}$$

However, before interacting with U the two systems are independent by assumption and the unitary evolution does not increase entropy, Lemma (4.5) and Lemma (4.4) imply that

$$S(U(\rho_{CR} \otimes \tau)U^\dagger) = S(\rho_{CR} \otimes \tau) = S(\rho_{CR}) + S(\tau) \tag{20}$$

therefore from (19) and (20) we obtain that

$$S(\rho_{CR}) + S(\tau) \leqslant S(\rho'_{CR}) + S(\tau)$$

$$S(\rho_{CR}) \leqslant S(\rho'_{CR})$$

□

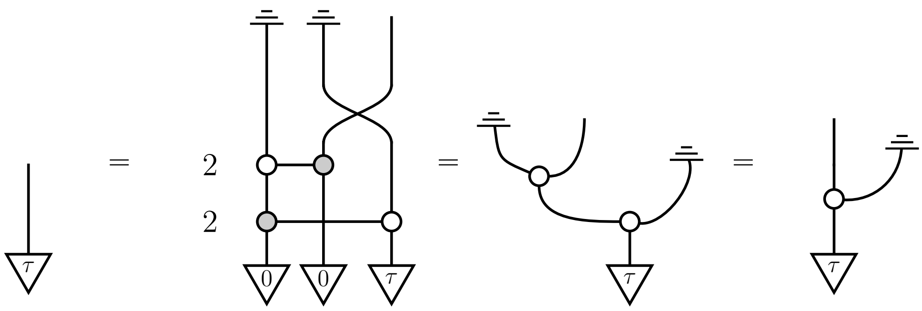
We derive the following observation as a simple corollary to the theorem:

Corollary 4.6. The evolution described by the D-CTC model cannot map mixed state into pure states.

Proof. Follows directly from the Theorem (4.3) and the fact that ρ is pure if and only if $S(\rho) = 0$. □

4.1.2 The Unwritten Poem of Maximal Entropy

If we set the initial value to be equal to $|0\rangle\langle 0| \otimes |0\rangle\langle 0|$, the fixed point must satisfy the equation:



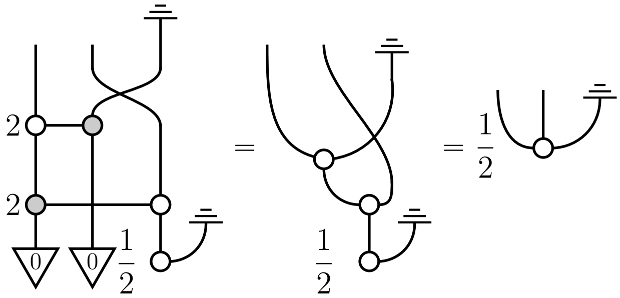
we are therefore looking to a fixed point which is invariant under decoherence with respect to the computational basis, this is given by the mixed states with density matrix

$$\tau = \alpha\,|0\rangle\langle 0| + (1-\alpha)\,|1\rangle\langle 1|)$$

For $0 \leqslant \alpha \leqslant 1$, we see that there is an entire continuous spectrum of possible fixed points; in order to define a deterministic dynamic we need to introduce a criterion that allows to uniquely select the CV state that leads to the evolution. Deutsch introduces the following *maximum entropy rule*:

The state of the supplementary data (i.e., data required elsewhere than at the past boundary of spacetime for fixing a global solution of the dynamical equations) is the state of greatest entropy compatible with the initial data.

Going back to the "unwritten poem circuit", the state of maximal entropy is given by setting $\alpha = (1/2)\mathbb{1}$ and the evolution of the chronology-respecting qubit is then expressed as:



(21)

We have seen that two classes of paradoxes can arise, the physical process can to be both underdetermined –unwritten poem– and overdetermined –grandfather paradox– by the initial data. In the case of the grandfather paradox the initial state $\rho = |1\rangle\langle 1|$ excluded any classical solutions but allowed the solution to be a quantum mechanical 'ontic mixture'. In the second paradox, the physical process necessitates additional data, we therefore require nature itself to provide this missing data in a way that is as much compatible as possible with an exclusive knowledge of the initial conditions. We have shown how the D-CTC model works, now we present some of its peculiarities and drawbacks.

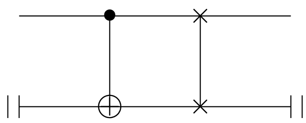
4.2 Properties of the D-CTC model

The first question that one may want to answer is wether the new maps can be define inside the already existent theory, i.e. if there exists trace preserving completely positive maps that can substitute the effect of a chronology-violating region on ρ_{CR}. We now see that in fact this is in fact far from being the case, in particular we provide examples showing that the Deutsch's model can produce maps which are non-linear and discontinuous.

4.2.1 Nonlinearity and Discontinuity

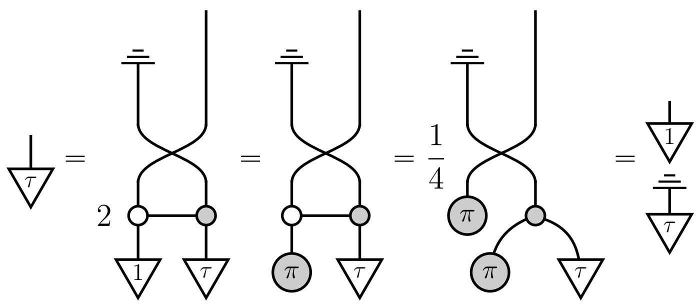
The non linearity of the Deutsch's model is hinted at from its definition, we see that for every chronology-respecting input ρ_{CR} the receipt produces a completely positive trace preserving operator $D(\rho_{CR})$ that gets applied to ρ_{CR}, the output of the interaction, given by Equation (12) clearly depends on the input ρ_{CR} but there is also another degree of dependency given by the influence of the fixed point $\tau(\rho_{CR})$.

We now provide an example of graphical calculations witnessing nonlinearity. Consider the quantum circuit



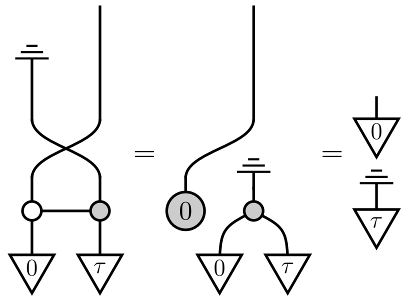
(22)

To find the fixed point:



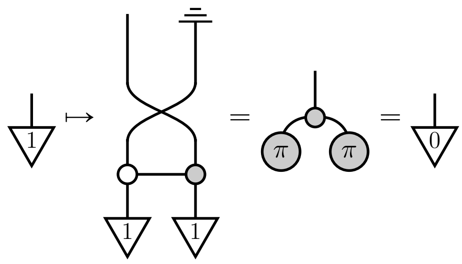
(23)

and the fixed point is therefore given by $\tau = |1\rangle\langle 1|$. Similarly for $\rho = |0\rangle\langle 0|$:

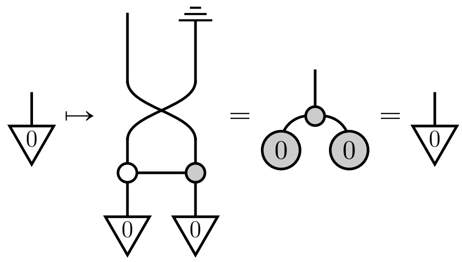


(24)

Therefore the respective evolutions are given by



(25)



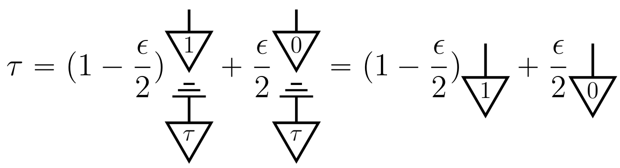
(26)

(27)

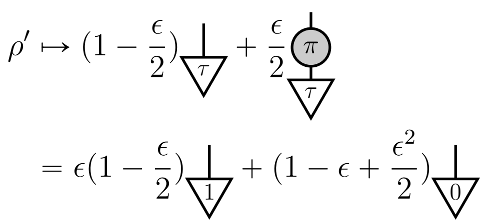
however if we consider adding an arbitrary small amount of noise to the input state $\rho = |0\rangle\langle 0|$

$$\rho' = \frac{\epsilon}{2}\mathbb{1} + (1-\epsilon)\,|0\rangle\langle 0| = (1-\frac{\epsilon}{2})\,|0\rangle\langle 0| + \frac{\epsilon}{2}\,|1\rangle\langle 1|$$

the fixed point has to satisfy the equation:



the evolution of ρ' is the equal to:



We see that the state ρ' evolves into

$$\epsilon(1-\frac{\epsilon}{2})\,|1\rangle\langle 1| + (1-\epsilon+\frac{\epsilon^2}{2})\,|0\rangle\langle 0| \neq |0\rangle\langle 0|$$

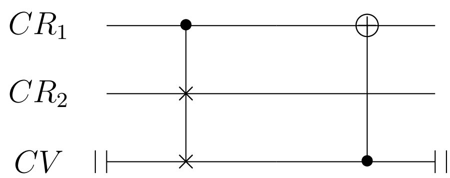
for all $\epsilon > 0$ and the map is therefore nonlinear. What about continuity? In this case we have that for $\epsilon \to 0$

$$\epsilon(1-\frac{\epsilon}{2})\,|1\rangle\langle 1| + (1-\epsilon+\frac{\epsilon^2}{2})\,|0\rangle\langle 0| \to |0\rangle\langle 0|$$

and for $\epsilon \to 1$

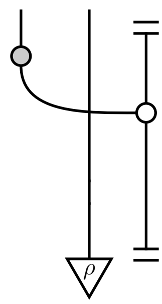
$$\epsilon(1-\frac{\epsilon}{2})\,|1\rangle\langle 1| + (1-\epsilon+\frac{\epsilon^2}{2})\,|0\rangle\langle 0| \to \frac{\mathbb{1}}{2}$$

in this case this is the same behaviour expected from a continuous map. In order to demonstrate discontinuity we therefore have to appeal to a different example. The existence of discontinuous solutions has already been shown in (DFI10), we provide the example of a different circuit which is witnessing the discontinuity.



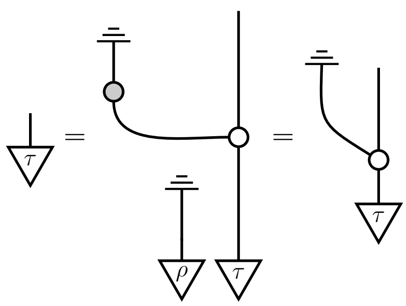
(28)

The circuit above has three carriers and we denote the gate on the left to be a controlled SWAP, a quantum gate swapping the qubits on CR_2 and CV if the control qubit CR_1 is in the state $|1\rangle\langle 1|$. Consider the first chronology-respecting qubit CR_1 to be initially in the state $|0\rangle\langle 0|$ while the second qubit CR_2 is allowed to be in any state ρ. Rewriting the circuit using the graphical calculus we see that it can be simplified to:



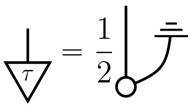
(29)

the fixed point therefore satisfies the following equation

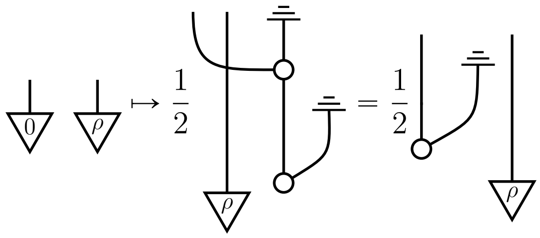


(30)

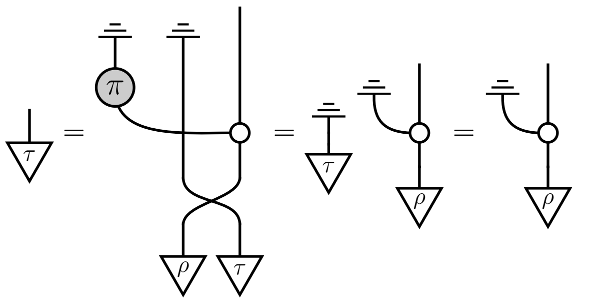
and its therefore invariant under decoherence, to get the state with maximal entropy, we set it to be the maximally mixed state:


(31)

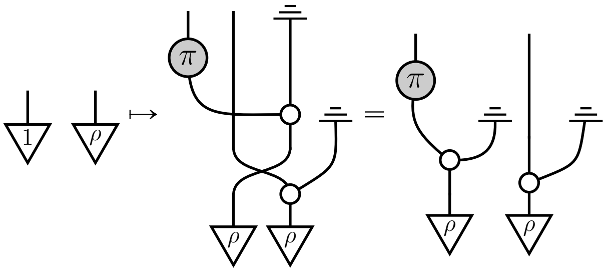
the evolution of $\rho_{CR} = |0\rangle\langle 0| \otimes \rho$ is therefore given by:


(32)

We now calculate the evolution and the fixed point given by using the state $\rho_{CR} = |1\rangle\langle 1| \otimes \rho$, in this case the fixed point is always equal to the decoherence of ρ:

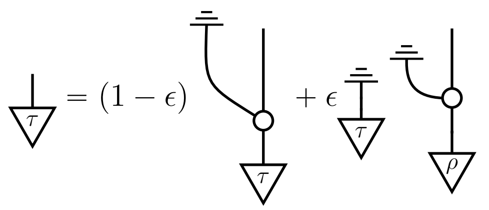

(33)

therefore:

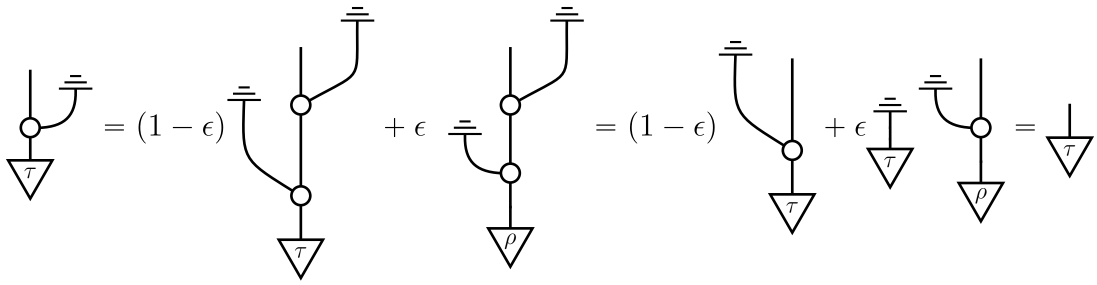

(34)

we now consider the evolution of the state

$$\rho_{CR} = [(1-\epsilon)\,|0\rangle\langle 0| + \epsilon\,|1\rangle\langle 1|] \otimes \rho$$

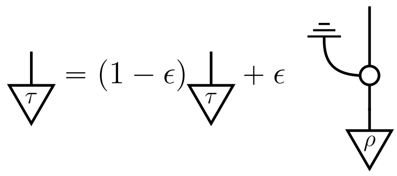

(35)

To solve this equation we first notice that such a fixed point would be invariant under decoherence:

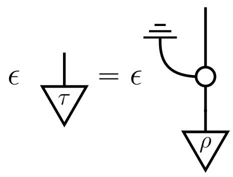


(36)

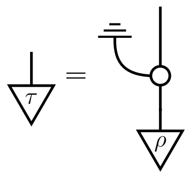
We can rewrite Equation (35) considering the invariance under decoherence to get:



(37)

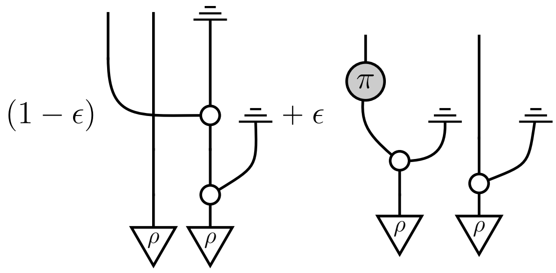


(38)



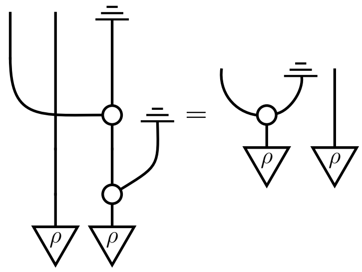
(39)

The first hint of a discontinuous behaviour is in the fact that an arbitrary small influence of the state $|1\rangle\langle 1| \otimes \rho$ forces us to consider a completely different τ. With this fixed point we can calculate the evolution of the state ρ_{CR} for ϵ in the range $0 < \epsilon < 1$ which is given by:



(40)

If we let ϵ go to 0 we see that the limiting value of the evolution is therefore:



which, for a density matrix representing a state which is not mutually unbiased to $\circ$, is different from the evolution of the limiting state $\rho_{CR} \to |0\rangle\langle 0| \otimes \rho$ given by Equation

(32).

4.2.2 Breaking entanglement

Deutsch model is originally defined on the unitary evolution of system, it is therefore defined for closed systems. If we consider the scenario where we have two entangled qubits A and B forming a Bell state:

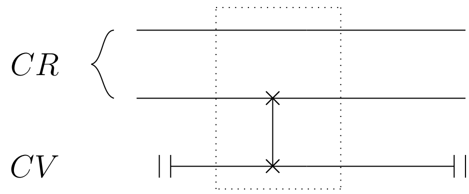
$$\rho_{AB} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| + \frac{1}{2}|11\rangle\langle 00| + \frac{1}{2}|00\rangle\langle 11|$$

the states of the single qubit B (see the discussion at the beginning of Section (2.1)) is given by

$$Tr_A\left(\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| + \frac{1}{2}|11\rangle\langle 00| + \frac{1}{2}|00\rangle\langle 11|\right) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

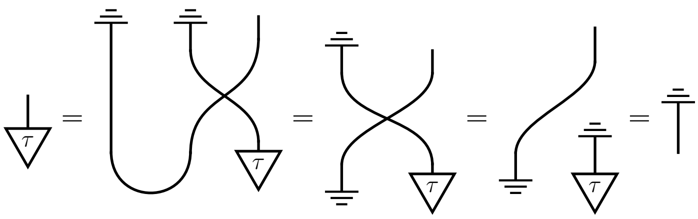
The single qubits are therefore mixed states, the evolution of one of the entangled subsystems is a marginalisation of a global unitary evolution.

Suppose that ρ_{AB} is the input to the following quantum circuit:

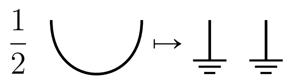


(41)

The gate on which we apply the consistency condition is given by the dotted rectangle. Using the graphical calculus, we can find the fixed point by marginalising both chronology preserving qubits:

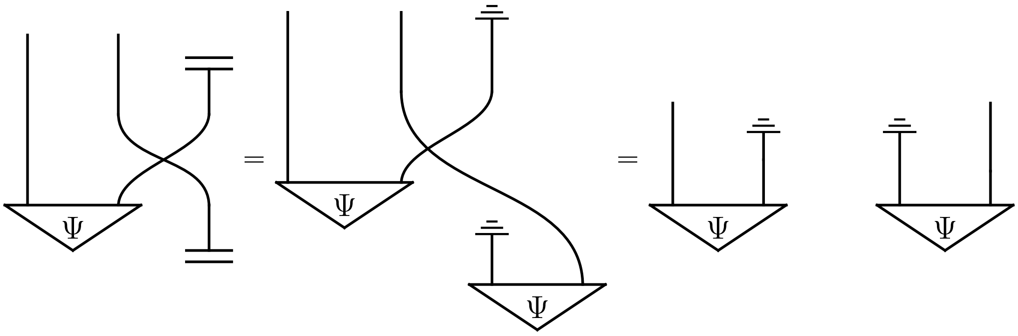


The fixed point is therefore given by the maximally mixed $\tau = (1/2)\mathbb{1}$. In this case the evolution of the Bell state breaks the entanglement.



(42)

We can more generally show that in this way entanglement is broken for all bipartite states:

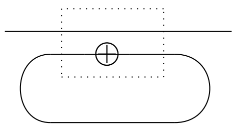


In the presence of CTC we can therefore realise the nonlinear map entanglement breaking map:

$$\rho_{AB} \mapsto Tr_A(\rho_{AB}) \otimes Tr_B(\rho_{AB})$$

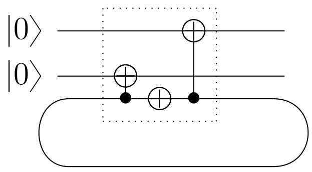
The D-CTCs allow to send a pure state into a mixed one considering the entire evolution of the *closed* system; we have already presented the proof in (Deu91) to show that the opposite unphysical behaviour is in fact impossible. However this picture still runs, as noticed by Bennet (BLSS09) in contradiction with the principle of the "church of the larger Hilbert space", the idea that it is always possible to "purify" a mixed state in the universe. In our case we produce mixed states which are not subsystems of pure states. This an upsetting consequence, however Bennet et al. in (BLSS09) explain how one can recover the principle by invoking, Deutsch' ontological belief on the Everettian multiverse.

To understand Bennet's argument we have to present Deutsch's description in terms of multiple universes, according to Deutsch the mixed states represents ensembles of identically prepared states across different branches of the multiverse. Let us see what are the implication of this ontological assumption, consider the following circuit:



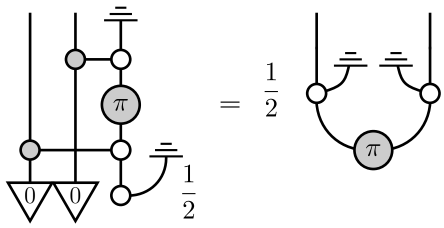
(43)

It rapresents a qubit travelling back in time to negate itself, Deutsch's consistency condition resolves the contradiction by prescribing the CV qubit to be in the maximally mixed state. In the multiverse prescription, does the qubit manage to negate itself? To answer this question we can try to actually observe the value of the chronology-violating qubit before and after the action of the CNOT. We modify the circuit accordingly to obtain



(44)

The presence of the CNOTs do not change the value of the fixed point as the addition of the new gates only add additional decohering maps in the process of finding the fixed point and therefore the CNOT gates do not significantly alter the picture; the chronology-violating qubit is still in the maximally mixed state. We now calculate the state of the qubits which measure the effect of the gate using the graphical calculus:



(45)

The obtained quantum state is therefore the mixed state

$$\rho'_{CR} = \frac{1}{2}\,|01\rangle\langle 01| + \frac{1}{2}\,|10\rangle\langle 10|$$

The two measurements are anticorrelated, if measure one qubit to be $|0\rangle$ we will be in the state $|1\rangle$ and viceversa. It appears that the qubit always manages to negate itself, what happend therefore to the contradiction? A possible interpretation can be derived using the multiverse description given by Deutsch, in this case the maximally mixed state represents the fact that in half of the universes the qubit has value $|1\rangle$ and in the other half it has value $|0\rangle$. The consistency condition is therefore applied to the global state of this ensemble over *all* universes What happens in the example of Diagram (43) is that the CV

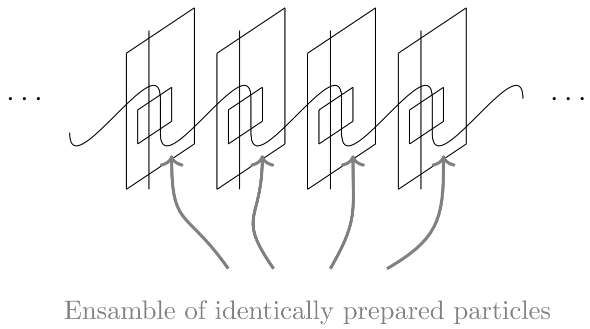


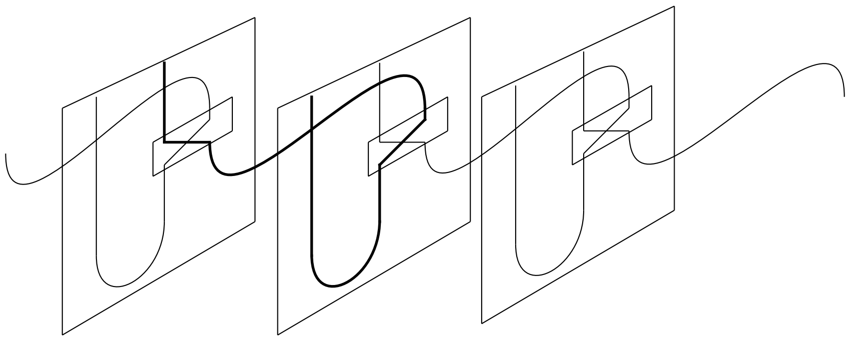
Figure 2: Closed time-like curves allow different branches of the multiverse to interact.

qubit is not emerging from the future of its current universe, it emerges from the future of a different branch of the multiverse, menages to negate itself disappears into another branch. The circular movement of a closed time-like curve gets therefore interpreted to represent a way to communicate between different universes. In Deutsch' words, "closed time-like lines would provide gateways between Everett universes" (Deu91).

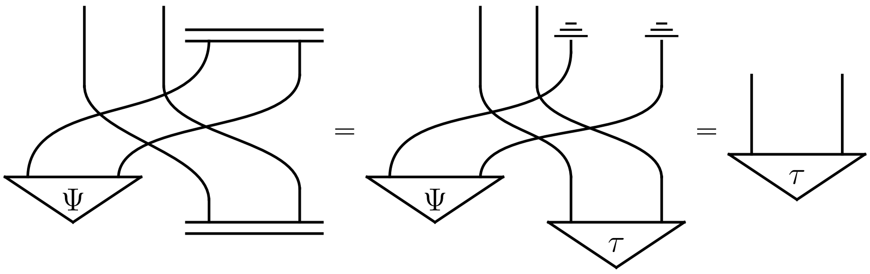
After the interaction has occurred, the chronology-violating system remains in the same overall state but the particular state in each universe gets always swapped by the NOT gate before disappearing by entering the future mouth of the CTC.

We underline that Figure (2) can be misleading as Deutsch also specifies that the time travelling information never encounters barriers between the different universes, those distinct realities "form part of a larger object which has yet to be given a proper geometrical description but which, according to quantum theory, is the real arena in which things happen". It is an essential simplification to depict the multiple branches of the universe to be two-dimensional surfaces topologically disconnected.

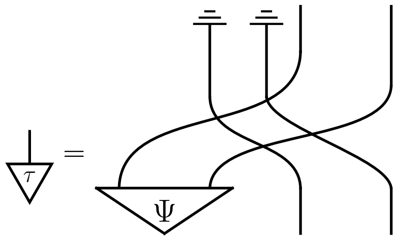
According to this interpretation, what happens if we make part of an entangled EPR pair interact with a CTC? If we stick to the Deutsch multiple universes interpretation each half of the EPR state stays entangled with its other half but accesses a different branch of the multiverse. In each separated universe a pure state evolves into a maximally mixed state because we only have access to a part of the entangled pair. The purification can be preserved if we take enlarge the system *across* the multiple universes and not within a single universe.



A CTC can break the correlations with the enviroment, what happens if we send a pure bipartite state which is uncorrelated with the environment inside a CTC?



Recall that the find the fixed point τ we have to marginalise the chronology-respecting region:



and the fixed point must therefore be the bipartite state itself. We observe that entanglement is preserved inside the CTC but it is broken between the system entering the CTC and the rest of the universe.

4.2.3 Violation of No-Cloning?

No-cloning is one of the most fundamental results in quantum information theory, it points at a very profound difference between the behaviour of classical and quantum information. In works by Abramsky and Coecke (Abr09; CK17) it is shown that an equivalent form of the theorem generalises for a wide class of process theories, in (Abr09) the author explains that no-cloning is in fact incompatible with basic structural features of quantum entanglement.

The classical formulation of the theorem states that it is impossible to provide a unitary operation that would take unknown pure quantum states to copies of themselves.

This can be eventually generalised to mixed states and CPTP maps entailing a stronger result, the *no-broadcasting theorem* which according to (CK17) discriminates between classical probabilistic theories and quantum theories.

Definition 4.7 (broadcasting)**.** A *broadcaster* is a map β_A that takes all states of the form $\rho\colon \mathbb{C} \to A$ to a state $\rho'\colon \mathbb{C} \to A \otimes A'$ such that $\mathrm{Tr}_A(\rho') = \rho$ and $\mathrm{Tr}_{A'}(\rho') = \rho$.

Clearly the non-broadcasting theorem implies no-cloning. Even though the classical proof of the no-cloning theorem provided in textbooks is an elementary result about Hilbert spaces, the categorical perspective on quantum mechanics shed light on the the physical meaning of the theorem. In (Abr09) the author provides a more general no-cloning principle which underlines the impossibility of combining basic properties of entanglement with the notion of cloning, cloning itself is prohibited by a deeper structural property of entanglement. Does the D-CTC model allow for the possibility of cloning arbitrary states? A standard proof of the no-cloning theorem would clearly not apply in this case as we are not confined to linear maps. In fact we will see that D-CTCs give us some more flexibility regarding the ability of cloning.

The no-cloning theorem is not a peculiarity of quantum data, it can be applied also to a theory with the states representing finite dimensional probability distribution and maps between states given by finite stochastic matrices. To avoid confusions with the ket notation we introduce a way to describe probability distributions of classical states

Definition 4.8. Let $\{\delta_i\}_i$ a set of complete classical states, then a probability distribution over the states is denoted as

$$\sum_j p_j \delta_j$$

where

$$\sum_j p_j = 1$$

Consider the following gedankenexperiment: suppose that we have a bag of unknown envelopes containing either two blue δ_b or two red cards δ_r. We randomly pick one, the general state of an envelope can now by denoted by

$$\rho_E = p_b \delta_b + p_r \delta_r$$

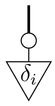
where p_b and p_r represent respectively the ration of the envelopes containing blue cards and the envelopes containing red cards. This means that the best description of the contents of an envelope is described by a statistical mixture of *pure* states. The act of preparing a state is therefore equivalent to randomly select an envelope from the bag.

We can open the envelope, take one of the two card without looking at its colour, reseal it and create another envelope $\rho_{E'} = p_b \delta_b + p_r \delta_r$ which has the same statistics of ρ_E however the joint state of $\rho_{EE'} \neq \rho_E \otimes \rho_{E'}$. In fact

$$\rho_{EE'} = p_b \delta_b \delta_b + p_r \delta_r \delta_r \neq (p_b \delta_b + p_r \delta_r)(p_b \delta_b + p_r \delta_r) = \rho_E \otimes \rho_{E'}$$

The system of the two envelopes, a statistical mixture describing the possible colours of the cards contained in the two envelopes is now "entangled", the knowledge of the content of one of the envelopes implies knowledge about the colour of the other one. We heaven't therefore managed to copy the state of the envelope, we haven't created two different envelopes with the same independent statistic but we managed to broadcast the state of the system. Destroying either of the envelops would leave us with a statistically identical copy of the original one.

A quantum mixed state is a quantum encoding of a probability distribution, such a state is obtained by performing a quantum encoding, with respect to a particular ONB, of a probabilistic distribution, in accordance to the graphical calculus we denote such a state as:



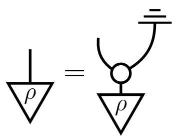
(46)

Where the classical states are now described by the thin-wire notation in accordance to (CK17):

$\delta_i =$ 

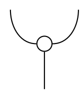
For example, the maximally mixed state is obtained encoding the discrete uniform distribution on a qubit using an arbitrary basis. We will prove that the D-CTC model allows to copy those unknown probabilistic distributions encoded as quantum states provided that we know which was the ONB used to encode it. The result is an unavoidable consequence of the fact that such states can be first broadcasted and then, using the entanglement breaking map, we can break the correlation:

Theorem 4.9. We say that ρ is invariant under $\bigcirc$-decoherence if for a fixed choice of basis $\bigcirc$:



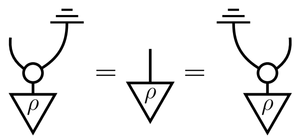
There exists a a map $\Delta_{\bigcirc}$ which uniformly clones all the states ρ that are invariant under decoherence.

Proof. First we note that there exists a map

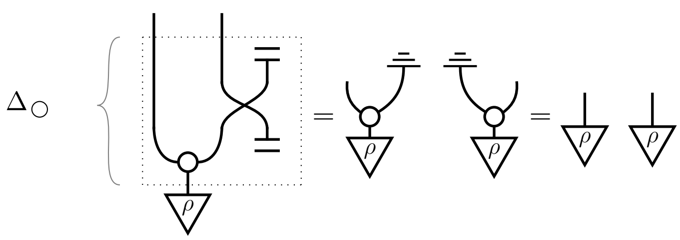
$\beta =$ 

$$\beta = \sum_i |i\rangle |i\rangle \langle i|$$

which is a broadcasting map for the state ρ:



If we then apply the entanglement breaking map:



(47)

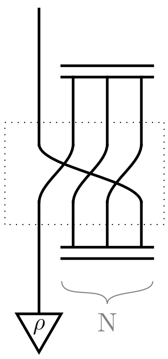
□

So there exists maps cloning entire families of states which cannot be normally uniformly cloned in quantum theories and in theories representing classical probabilistic mixing. What about the cloning of an arbitrary state? In the literature there is a general confusion in assessing the cloning properties of the Deutsch model. To clear the fog we want a cloner which provides two copies of a given state ρ in the chronology-respecting part of the spacetime, unconstrained by additional consistency condition and that can e freely reused and manipulated. We provide the standard definition of cloning for pure quantum states:

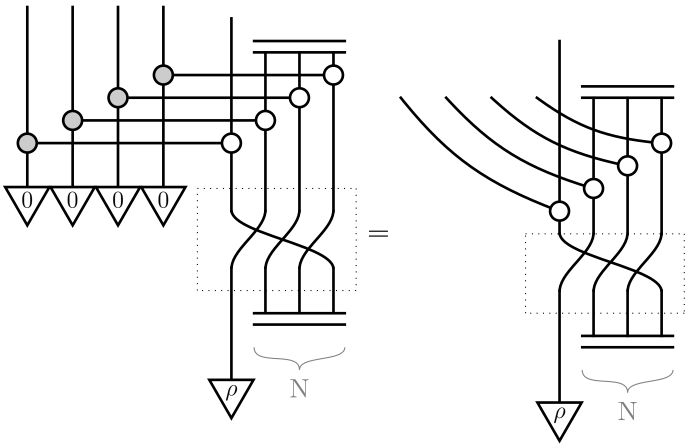
Definition 4.10 (Cloning map)**.** In (NC11, p. 532) cloning is defined to be the a unitary evolution which overwrites an ancillary state $|s\rangle$ with a perfect copy of the state pure state $|\psi\rangle$:

$$|\psi\rangle \otimes |s\rangle \mapsto |\psi\rangle \otimes |\psi\rangle \tag{48}$$

In the paper by A. Brun et al. (BWW13) it is argued that the fact itself of sending a state inside an open CTC can be seen as a cloner, however they also specify that "the N in the CTC systems are not available after this system enter in the future mouth of the wormhole". One could for example naively consider the following circuit to be a cloner:



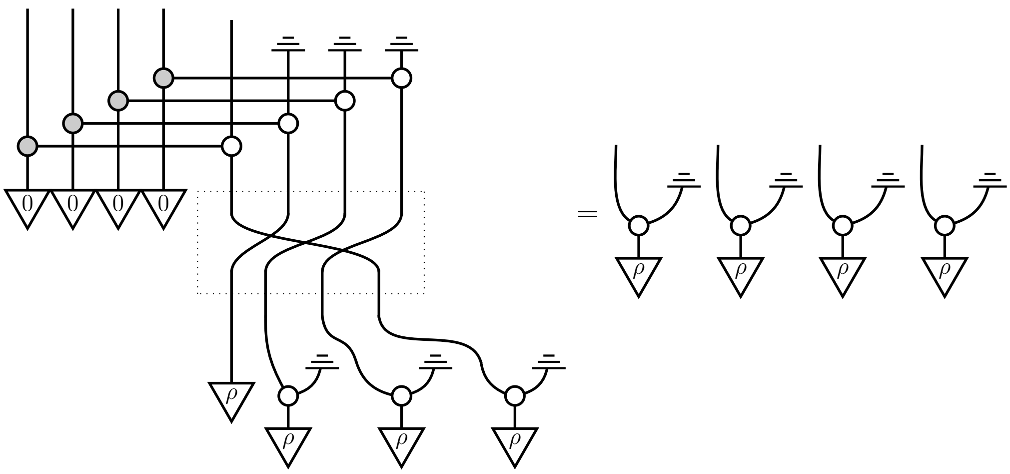
The fixed point is uniquely given by $\rho^{\otimes N}$, the circuit therefore creates N virtual copies of the state ρ, however they are merely virtual, they disappear after the interaction has taken place and this is not in accordance to definition of cloning. In the same paper however, they point at a solution which generalises the method to clone discrete probability distribution that we have described earlier. Another way to copy a quantum state is to perform a complete state tomography, i.e to completely identify a state by performing measurements (NC11, p. 389), is it possible to access and extract some information about the *virtual* clones? We know that the fixed point itself depends on the applied map, we therefore have to be careful to prepare the interaction in a way that doesn't disturb too much the chronology-violating qubits. One could for example try to perform multiple measurements of ρ by simply performing a series of CNOTs:



However the CNOTS affect the selection of the fixed point, which is now given by $\rho'^{\otimes N}$ where ρ' is

$$\sum_j \mathrm{Tr}[|j\rangle\langle j|\,\rho\,|j\rangle\langle j|]\,|j\rangle\langle j| = \sum_j \langle j|\,\rho\,|j\rangle\,|j\rangle\langle j|$$

where in the case of qubits $\{|j\rangle\}_j$ is the Z basis, $\{|0\rangle, |1\rangle\}$. The state gets decohered with respect to this ONB. Substituting ρ':



The ancillary bits became copies of the decohered version of ρ', they can be used to tomograph the state. However in reading off the state of a decohered version we still loose a lot of information about ρ. In the work by Brun et al. (BWW13) they appealing to informationally complete measurements (BWW13; FSC05), those are measurements for which "the probability of the outcomes are in one-to-one correspondence with a classical density operator description of a quantum state"(BWW13).

This is a CPTP map which maps the state ρ in a mixture of possible outcomes of the informationally complete measurement.

$$\rho \mapsto \sum_{x=0}^{d-1} \mathrm{Tr}[M_x \rho]\,|x\rangle\langle x| := \rho'$$

This state is clearly invariant to decoherence with respect to the basis $\{|x\rangle\langle x|\}_{x=0}^{d-1}$, we can therefore replace the CNOT used above with a d dimensional generalisation of such a gate and a large number of CV qubit to estimate arbitrarily the state ρ' and thus to completely determine ρ. Once we determine the statistics with arbitrary precision of ρ' –we suppose we can copy arbitrary many versions of the outcome using the CTC– we can reproduce as many *approximate* copies of ρ as we like.

While we can show that we can produce clones with arbitrary fidelity increasing the dimension of the chronology-violating space, the question whether there exists a a circuit rapresenting a CR-CV interaction which is an *exact* cloner has, to the best of our knowledge, not been answered. There is a published paper claiming that perfect cloning of an arbitrary state is possible with finite resources (KCP$^+$18), we however underline that their results are based on a misconception of the consistency condition, they assume that once the consistency condition determine the CTC states we have afterwards the freedom to apply additional unitary transformation on the CTC state as long as we "assume that the Deutsch condition has to be applied every time we consider a dynamics including a CTC." (KCP$^+$18). The consistency condition has to be applied considering the entire interaction between the CR and CV qubits only once. After that the "virtual" content of the CTC is not anymore available. It is useful to think about the content of the CTC as something that can only be instrumental in selecting the TPCP map leading to a consistent evolution of the input throughout the entire circuit.

5 Postselected Closed Time-like Curves

One of the main reactions to the various problems posed by the D-CTC model was the proposal of alternative descriptions of the CV-CR interaction. Seth Lloyd in (LMG+11) proposes a model which is based on postselection. The concept of postselection refers to the ability to deterministically force the outcome of a particular experiment.

A fundamental asymmetry of quantum theory is that preparations are deterministic while tests are of an inherently probabilistic nature, reversing the arrow of time trivialises any causal theory as shown in (CGS17). The idea of introducing post-selection is coeherent with the fact that closed time-like curves may require to break this fundamental asymmetry at least locally.

Even though the model is normally attributed to Lloyd, the idea of using a teleportation-like protocol to simulate circuits with backward-in-time connections should be attributed by Svetlichny (Sve11), inspired by the graphical treatment of quantum protocols provided by Coecke and Ambramsky in (AC04).

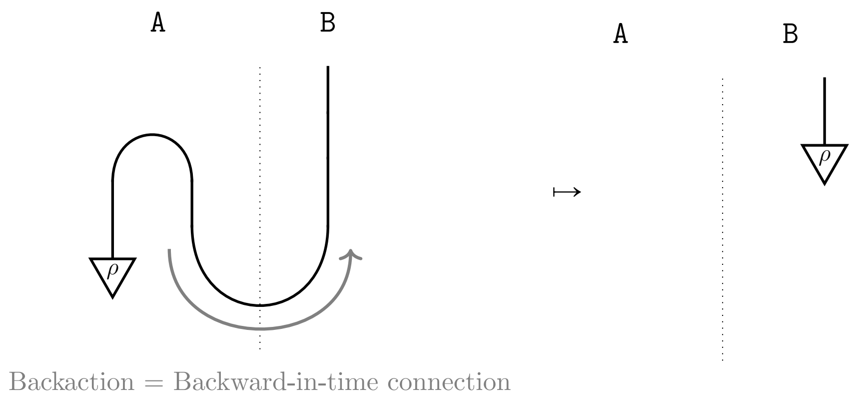
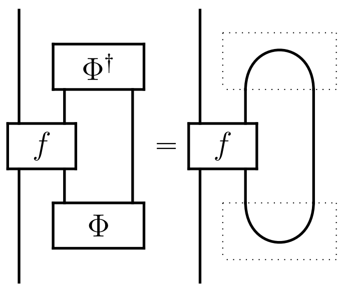


Figure 3: Graphical abstraction of quantum teleportation

Suppose that the interaction between the two regions, is given by the map $f : A \otimes C \to B \otimes C$, Lloyd proposes to obtain a map representing the interaction with a CTC as follows:

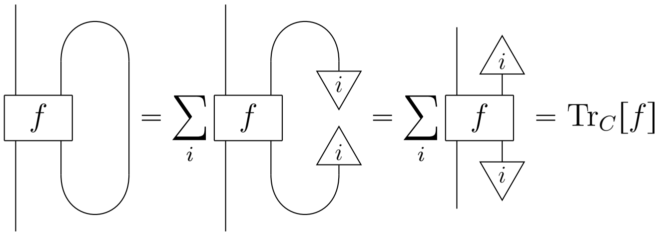


(49)

where $\Phi : \mathbb{C} \to A \otimes A$ is the completely positive map associated with the state:

$$\frac{1}{\sqrt{d}} \sum_{i}^{d} |i\rangle |i\rangle$$

In $\mathbf{Mix}_{\top}$, a unitary transformation acts on a density matrix by mapping $\rho \mapsto U \rho U^{\dagger}$, the undoubled version of the diagram, refers to the category **FdHilb** where a unitary matrix acts on pure states v by matrix multiplication, $v \mapsto Cv$. We can provide a description of the evolution implied by the model, by considering the diagram given by its undoubled version. Recall that the undoubled notation represents matrices in **FdHilb** where the composition is defined by matrix multiplication



(50)

Where the first equation is a consequence of the *resolution of the identity*: let $\{|i\rangle\}_i$ be an arbitrary orthonormal basis for the Hilbert space $\mathcal{H}_a$, therefore we have that:

$$\mathbb{1}_{\mathcal{H}_a} = \sum_i |i\rangle\langle i| \tag{51}$$

So the evolution in **FdHilb** sends pure states $|\Phi\rangle$ to $\mathrm{Tr}_C[f]\,|\Phi\rangle$, or in its doubled version acting on density matrices: $\rho \mapsto \mathrm{Tr}_{CV}[U]\rho\,\mathrm{Tr}_{CV}[U]^\dagger$.

The map $\Phi^\dagger$ in Diagram (49) is not a completely positive trace preserving map as it doesn't satisfy the causality principle, there is no guarantee that the map sends normalised states to normalised states, we therefore complete the definition of the action by applying a further nonlinear renormalisation:

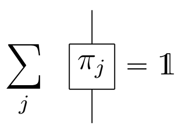
Definition 5.1 (P-CTC interaction (LMG$^+$11))**.** Let $f : A \otimes C \to B \otimes C$ be the morphism representing the interaction of the state ρ with a CTC to which we associate finite dimensional Hilbert space C. Therefore the evolution $\rho \mapsto \rho'$ is given by:

$$\rho' = \frac{E\rho E^\dagger}{\mathrm{Tr}[E\rho E^\dagger]} \tag{52}$$

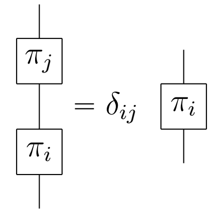
where $E := \mathrm{Tr}_{CV}[f]$ and $C \neq 0$. If E gives the zero process (we will denote it by 0) then we suppose that the evolution doesn't happen.

Lloyd's model therefore allows to post-select outcomes for certain experiments. The question that now naturally arises is whether we can use this feature to deterministically reproduce any postselected effect, i.e does there exists an interaction between CR and CV qubits which has the consequence of deterministically forcing the outcome of a measurement? We can show that this is in fact the case and a straightforward consequence of the definition:

Theorem 5.2. Let $\{\pi_j\}_j$ be the POVM elements associated to a projective measurement. We therefore require that in the **FdHilb** formalism:



and

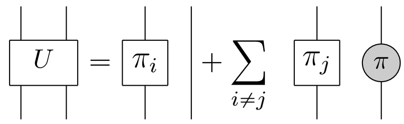


Assume thast we have access to an ancillary qubit travelling along a CTC, therefore Lloyd prescription for P-CTCs allows us to deterministically realise any projector π_i.

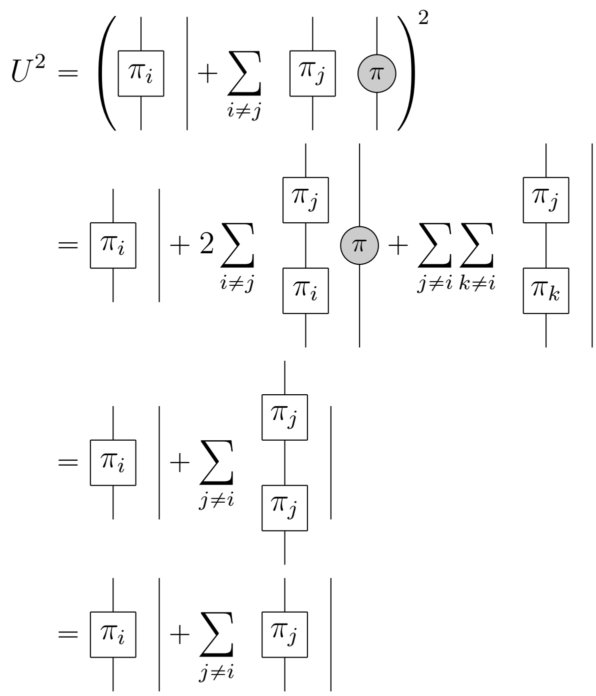
Proof. Consider the unitary evolution represented by the matrix

$$U = \pi_i \otimes \mathbb{1} + \sum_{j \neq i} \pi_j \otimes X$$

diagrammatically in **FdHilb**:

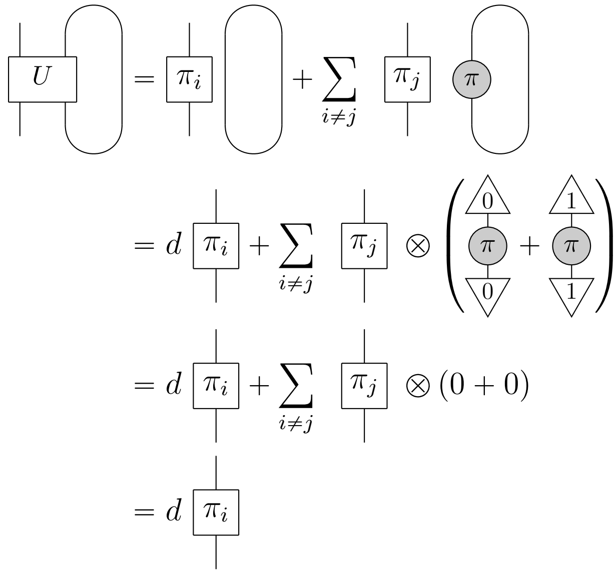


the map acts on $\mathcal{H}_{CR} \otimes \mathbb{C}^2$ where the first Hilbert space is associated with the CR system, the second space is associated to the ancillary CV qubit. We first notice that U is self adjoint as projectors are Hermitian operators. Straightforward algebraic manipulations imply that:



$$= \mathbb{1} \otimes \mathbb{1}$$

from which we conclude that $U \circ U^\dagger = U^\dagger \circ U = \mathbb{1}$ and U is unitary unitary operator. The completely positive map associated to the evolution $\rho \mapsto \mathrm{Tr}_{CV}[U]\rho\,\mathrm{Tr}_{CV}[U]^\dagger$ is given by:



which concludes the proof. □

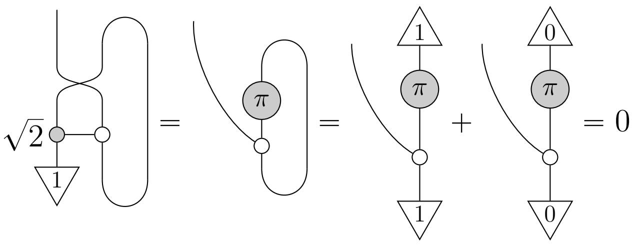
We can therefore force the outcome of any projective measurement. However it is possible to show (NC11) that unitary dynamics, projective measurements, and the ability to introduce ancillary systems, together allow the realisation of any general measurement.

This hints at the fact that the Lloyd theory may be particularly troublesome, with P-CTCs we are also automatically inducing the ability to perform arbitrary postselection.

5.1 Lloyd's Solutions to Paradoxes

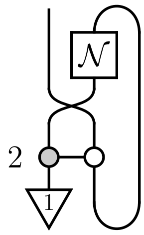
The graphical treatment allows us to see in a particularly clear way that the principle behind Lloyd's idea is to force the information entering the CTC to be in the same state as the information emerging from it. Renormalising the evolution we simply eliminate, get rid of all the inconsistent branches.

Returning to the grandfather paradox, let us calculate the evolution of the qubit $|1\rangle$:



(53)

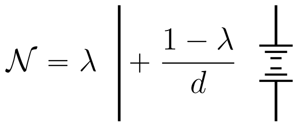
The P-CTCs do not provide an evolution for all the initial conditions, it merely tells us certain inputs lead to impossible evolutions. This might seem an incomplete description of reality, Lloyd's defense lies in the fact that all physical realisable gates come with an inherent amount of noise, for example we can choose to represent the noise as a depolarising gate acting on the CV qubit just before the postselection:



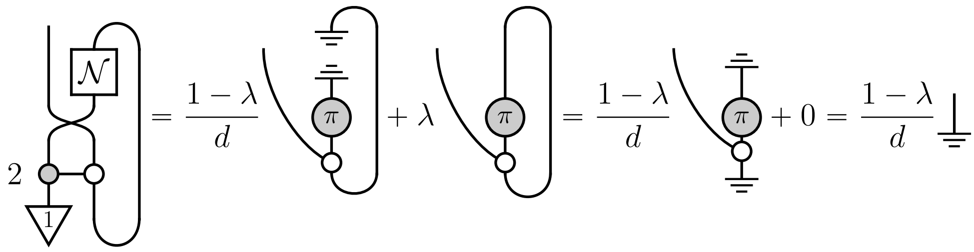
A depolarising gate $\mathcal{N}$ is a completely positive trace preserving map which act on a density matrix ρ as follows:

$$\mathcal{N}(\rho) = \lambda\rho + \frac{1-\lambda}{d}\mathbb{1}$$

graphically the gate can be represented as



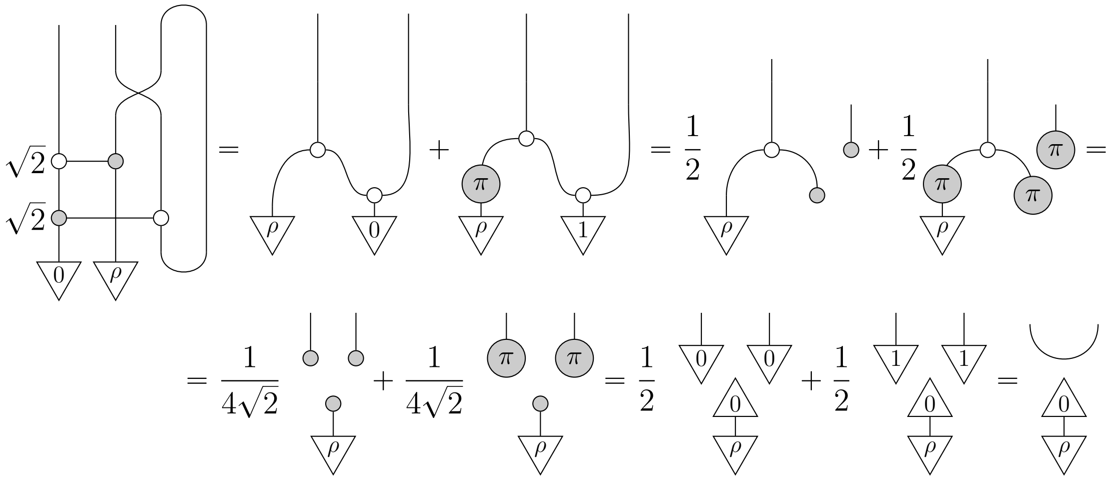
Repeating the calculations in Equation (53) with the additional gate we see that the non-normalised evolution is now given by:



(54)

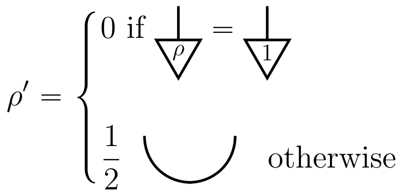
Renormalising the result we get an equal mixture of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, the maximally mixed state. We see that a randomly small noise guarantees that a solution always exists at the costs of amplifying the probability of a very improbable, even though still theoretically possible, outcome. This is independent on the value of λ, any arbitrarily small amount of noise would generate a consistent evolution.

To find out Lloyd's resolution of the informational paradox we apply the P-CTC formalism to Circuit (8) evaluating it using the thin-wire formalism:



(55)

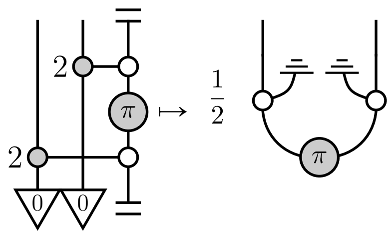
which after renormalisation equals to



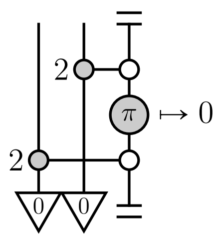
If ρ is equal to $|1\rangle\langle 1|$ then the evolution simply does not happen (in fact it would take us back to the grandfather paradox), in all other cases the solution of the evolution is the maximally entangled state:

$$\rho' = \frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle$$

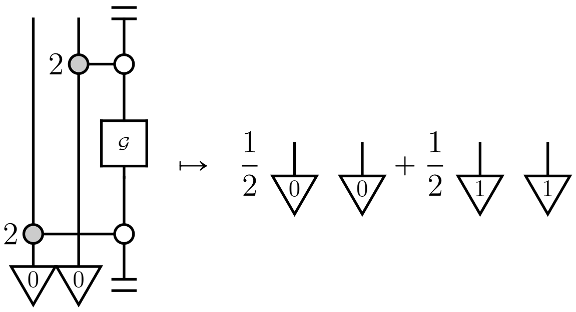
There are fundamental differences between D-CTCs and P-CTCs, according to the P-CTCs prescription certain evolutions are simply not allowed. The difference is not only in the existence of a class of "impossible" processes, do the two different models lead to observably different evolutions? We now consider Circuit (44). In order to present the multiverse interpretation provided by Deutsch we have calculated:



Under the P-CTC prescription it is easy to see that independently on the chronology-respecting region the circuit represents an "impossible" evolution:



If we add a depolarising channel after the NOT gate, independently on the degree of noisiness, we get the following behaviour:



We can interpret this result by saying that in D-CTCs the time traveller manages to kill its own grandfather, the two qubits are either measured to be $|0\rangle |1\rangle$ or $|1\rangle |0\rangle$. According to the P-CTCs, the NOT gate itself cannot be implemented as it always leads to an "impossible" evolution. If we assume the presence of an arbitrary small amount of noise, the random fluctuations get amplified and they prevent the time traveller to kill his own grandfather, we always measure the two qubits to have the same value, the grandfather either remains dead or remains alive after the action of the gate.

Lloyd provides this model in order to describe a local interaction, it doesn't talk about the possibility of extending the model compositionally, in the next subsection we will try to describe the categorical framework that describes the operational aspects of the P-CTC model.

5.2 The SMC of P-CTCs

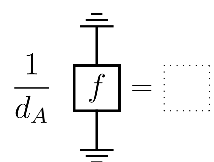
The operational framework that describes postselected quantum theory has been described by Oreshkov et al. in (OC16; OC15), they attempt to provide a more symmetric characterisation of the operational quantum theory that doesn't assume a predetermined direction of time. The formalism aims at removing the duality between the deterministic nature of states and the indeterminism implicit in the effects.

To realise this time-symmetric formulation it is therefore necessary to construct a theory were both pre- and post- selection are allowed. Clearly this formulation will contain the category modelling standard time-asymmetric quantum theory as a subcategory.

Definition 5.3 (Time Symmetric Quantum Theory)**.** We will call the denote the category by $\mathbf{Mix}_{sym}$. The objects of the category are as usual given by the finite dimension Hilbert spaces. The morphisms $f\colon A \to B$ are given by the completely positive maps satisfying the condition

$$\mathrm{Tr}\left[f\left(\frac{\mathbb{1}}{d_A}\right)\right] = 1 \tag{56}$$

which is equivalent to the diagrammatic formulation



(57)

now we define the composition, two morphisms $f\colon A \to B$ and $g\colon B \to C$ will compose according to the following rule

$$f \circ g = \frac{f \circ_{\mathbf{Mix}} g}{\mathrm{Tr}\left[(f \circ_{\mathbf{Mix}} g)\left(\frac{\mathbb{1}}{d_A}\right)\right]} \tag{58}$$

where $f \circ_{\mathbf{Mix}} g$ denotes the composition in the category of completely positive operators.

This definition is however incomplete as it is not closed under composition, it may happen in fact that

$$\mathrm{Tr}\left[(f \circ_{\mathbf{Mix}} g)\left(\frac{\mathbb{1}}{d_A}\right)\right] = 0$$

in this case the composition equals to the *zero process* between A and C, $0_{A,C}$ we therefore need to add such a process for every couple of objects. The zero process satisfies the following conditions:

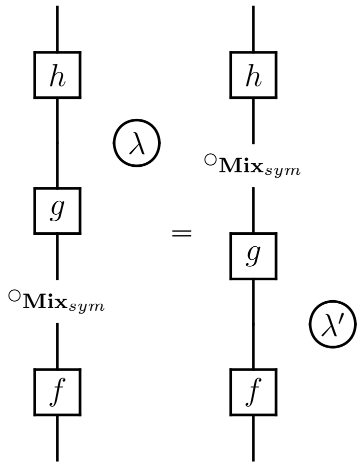
Definition 5.4 (Zero Process)**.** the *zero process* is a morphism for every pair of objects A, B such that

- $0_{A,B} \circ g = 0_{D,B}$ for all $g\colon D \to A$;
- $f \circ 0_{A,B} = 0_{A,C}$ for all $f\colon B \to C$;
- $0_{A,B} \otimes f = 0_{A\otimes C,B\otimes D}$ for all $f\colon C \to D$;
- $f \otimes 0_{A,B} = 0_{C\otimes A,D\otimes B}$ for all $f\colon C \to D$;

This shows that any collection of morphism that are connected by sequential or parallel composition to the zero object are equal to the zero object.

To show that the structure defined is a category we need to prove associativity of morphisms but this follows from straightforward algebraic manipulation:

$(h \circ g) \circ f =$

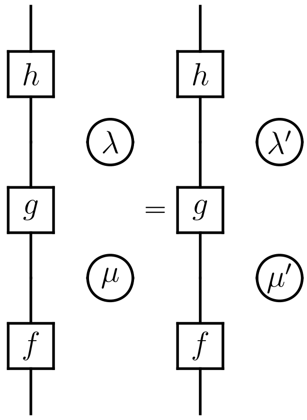


$= h \circ (g \circ f)$ (59)

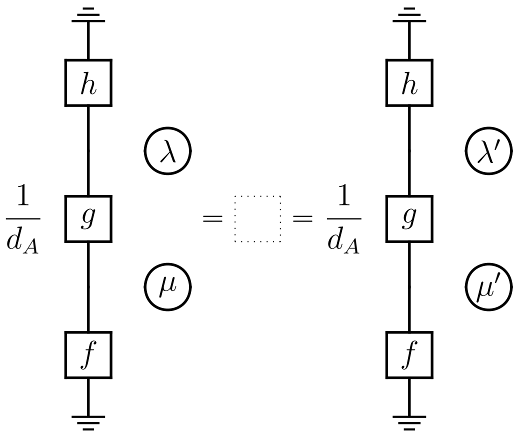
Where

$$\lambda = \frac{1}{\mathrm{Tr}\left[(h \circ_{\mathbf{Mix}} g)\left(\frac{\mathbb{1}}{d_A}\right)\right]} \quad \lambda' = \frac{1}{\mathrm{Tr}\left[(g \circ_{\mathbf{Mix}} f)\left(\frac{\mathbb{1}}{d_A}\right)\right]}$$

when we apply the second composition we have to "renormalise" it and add the scalars μ and μ'

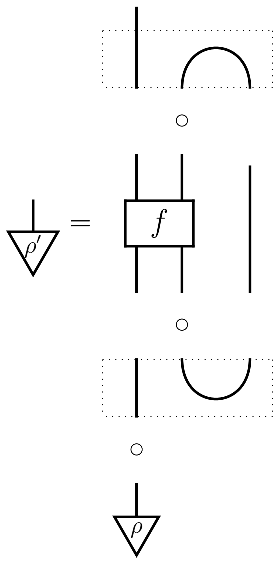


such that



but every scalar is invertible, from which we conclude that $\lambda\mu = 1 = \lambda'\mu'$.

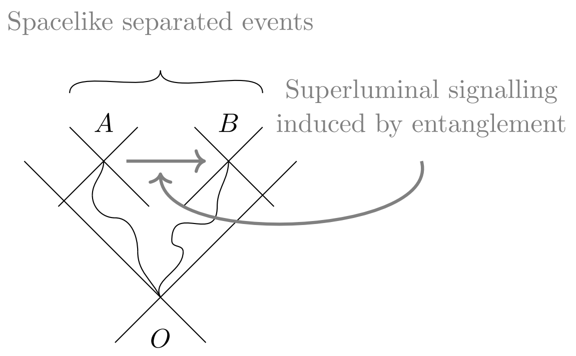
In fact the category modelling evolution of time asymmetrical quantum mechanics is consistent with Lloyd model, in fact given a completely positive map $f : A \otimes C \to B \otimes C$ the composition $\mathbf{Mix}_{sym}$ is equivalent to the prescription given by Lloyd:



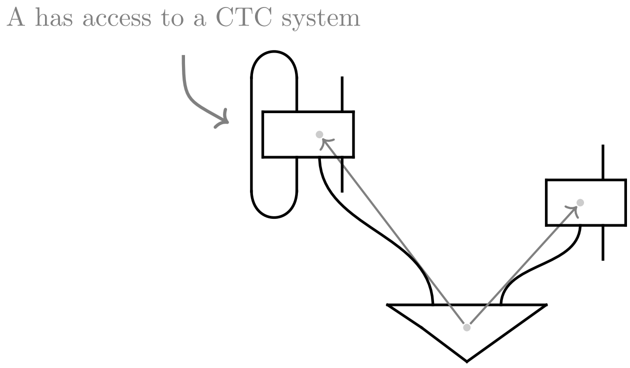
(60)

This is because the trace in **Mix** is equal to the evolution $\rho \mapsto \mathrm{Tr}_{CV}[U]\rho\,\mathrm{Tr}_{CV}[U]^\dagger$, the composition of the map with a normalised state ρ in $\mathbf{Mix}_{sym}$ introduces the normalising constant $1/\mathrm{Tr}[\mathrm{Tr}_{CV}[U]\rho\,\mathrm{Tr}_{CV}[U]^\dagger]$. In particular we see that the P-CTCs can be realised in a theory with postselection. Aaronson proved in (Aar05) that quantum mechanics with the ability of postselect is able to solve computational problems in the class PP, LLoyd in (LMG+11) claims that P-CTCs allows to solve problems in the same class.

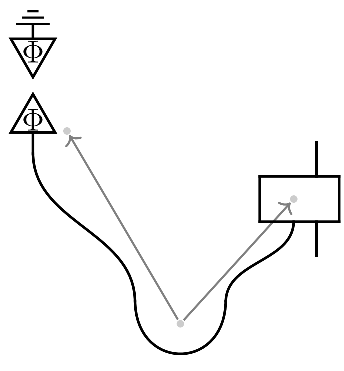
Clearly the ability to postselect makes Lloyd's theory technically extremely powerful but it also implies some fundamental conceptual weaknesses. For example, consider that there are two space-like separated regions of the spacetime A, B which are both in the lightcone of the region O.



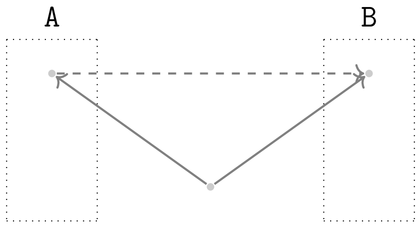
Furthermore we assume that in the region A there is a local interaction with a P-CTCs. This can allow an observer in that region of spacetime to send an instantaneous message to the space-like separated observer at B.



Assume in fact that at the event O we prepare an entangled system ρ_{AB} which subsystem $\mathrm{Tr}_B[\rho_{AB}]$ is accessible by A and subsystem $\mathrm{Tr}_A[\rho_{AB}]$ is accessible by B, we have shown in 5.2 that P-CTCs allows to deterministically create any projector, therefore:



We see that the theory implies the emergence of other causal correlations and that P-CTCs allow superluminal signalling.



We have managed to embed the P-CTCs into a compositional framework. It turns out to be more difficult and less clear how can this be done with the interactions described by Deutsch. In particular it is certainly not guaranteed that Deutsch's peculiar construction can be categorised at all. We devote the next section of the dissertation to the categorification of Deutsch's model.

6 The SMC of DMix

6.1 Introduction

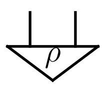
We have shown the working mechanisms of the Deutsch's model, the next step is to join the two aspects and from the local construction proposed by Deutsch create a theory that might be used to interpret causal structures described by cyclic directed graphs. We therefore want to be able to provide a coherent compositional theory of morphisms which can help us to model situations where we are in presence of multiple closed time-like curves.

The maps constructed by Deutsch can have a very hectic behaviour, first they are strongly nonlinear, secondly and even more worryingly, they can even produce discontinuities. However, we may hope that we can still describe at a more fundamental level the way such morphisms interact and the structure of this interaction. In fact in this chapter we show that it is quite surprisingly possible to extend the Deutsch model to a category of processes which is furthermore symmetric monoidal. The D-CTC model can therefore be extended to describe the physical evolutions of *mixed and pure states* – described categorically as morphisms $\mathbf{Mix}(\mathbb{C}, \mathcal{H}_A)$ – in a compositional way where there are notions of parallel composition, space-like composition and sequential, time-like composition. Before defining the category we have to comment on a interesting property of D-CTC maps:

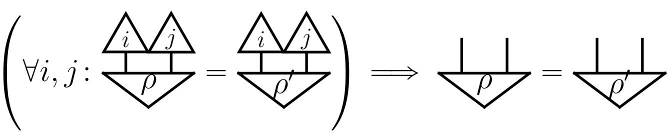
6.1.1 Failure of Process Tomography

In standard quantum mechanics, a quantum channel can be totally described by performing local measurement. This limits the degree of holism of the standard theory, if it is true that the total is more than the "sum" of its parts, we also have that any process can be reconstructed by the statistics of local measurements on its subsystems. Is this a fundamental requirement of a physical theory? As it is argued by Hardy and Wooters (HW12) the local tomography captures the idea that a reductionist science can meet up with a holistic theory. It is therefore a fundamental principle if we want to retain the notion of a reductionist science, the standard scientific method in which one studies smaller, simpler and more fundamental parts in order to infer the behaviour of more complex systems. There are fundamental issues mining a physical theory that ceases to be locally tomographic.

We will see that the entanglement breaking property of the Deutsch's maps poses certain problems regarding quantum tomography, however we will design the compositional properties of the maps to avoid to endanger local tomography. To do so, we need to distinguish between two different concepts: *local tomography* and *process tomography* (CK17). Local tomography is the property that guarantees that we can identify a bipartite state



by collecting statistics for each system individually, in fact we have that according to the **Mix** formalism:



process tomography on the other hand refers to the property of univocally identifying a quantum map $A \to B$ by describing it effect on all the states $\rho : \mathbb{C} \to A$.

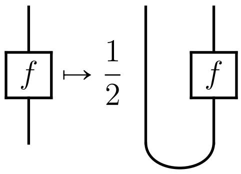
In $\mathbf{Mix}_{\top}$ however those two distinct tomographic notions concide. And this equivalence is established by one of the most interesting features of the formalism for quantum processes, the Choi-Jamilowkowski isomorphism (JLF13):

Definition 6.1 (CJ-isomorphism)**.** Let $\mathcal{M}\colon \mathcal{H}_a \to \mathcal{H}_b$ be a completely positive trace preserving map, therefore we can find a bijective correspondence between states and channels by defining:

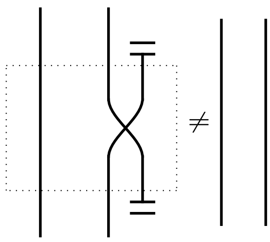
$$\mathcal{M} \mapsto \rho_{\mathcal{M}} := (\mathbb{1} \otimes \mathcal{M})(|\phi\rangle\langle\phi|) \tag{61}$$

where $|\phi\rangle$ is the canonical maximally entangled state in $\mathcal{H}_a \otimes \mathcal{H}_a$

Graphically, we map a causal morphism $f\colon A \to B$ into a normalised state $I \to A \otimes B$ in the following way:

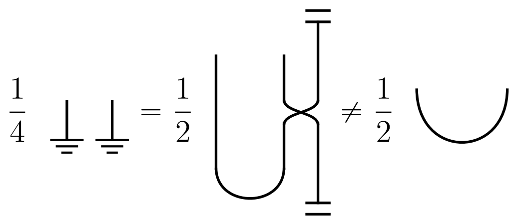


Interestingly this is not the case when we introduce the maps described by Deutsch. In the first chapter we have explained that the model has the ability to break entanglement, as such we expect the category **DMix** describing Deutsch's processes ceases to be process tomographic, in particular we have that the following two morphisms are not equal



(62)

The two maps above have exactly the same behaviour for all product bipartite states, however:



(63)

When constructing our process theory **D**, which models the interaction between chronology-respecting and chronology-violating regions, we will require it to satisfy certain properties. First we require the standard quantum mechanics formalism to be included in the theory, that the process theory describing any CR evolution directly embeds in **D**. Moreover we also impose **D** to have the same states as the original chronology-respecting **C**, i.e for every $A \in \mathrm{ob}(\mathbf{D})$,

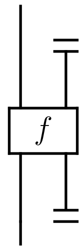
$$\mathbf{D}(I, A) = \mathbf{C}(I, A)$$

In our case we will take $\mathbf{C} = \mathbf{Mix}_{\top}$, and construct a bigger category $\mathcal{D} = \mathbf{DMix}$. Clearly the most important requirement is the full compatibility with the D-CTC model.

6.2 Constructing a New Category

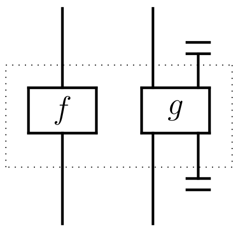
In order to define the category **DMix** we need to be particularly careful in defining the morphisms, this will be done in several steps. To start we formalise the notion of an *elementary box* (EB), the building block of our morphisms:

Definition 6.2 (Elementary Box). An *elementary box* is a morphism of the form:



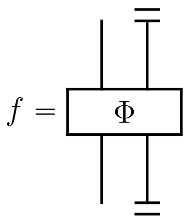
it therefore represents a choice of a completely positive map and an object (f, C) such that $f\colon A \otimes C \to B \otimes C$ and the Hilbert space C is assigned to denote the degrees of freedom of the chronology-violating region. We will also denote the elementary boxes symbolically by using the notation $\mathcal{D}(f, C)$.

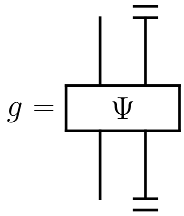
The elementary boxes by themselves don't form a category, they are not closed under composition. It is therefore strictly speaking meaningless to define a tensor product structure on the elementary boxes. However, with an abuse of notation which will be later justified, we can say that "tensoring" an EB with an arbitrary CPTP we obtain another EB in the following way, let f and g be CPTP maps, then we can obtain an elementary box from their parallel composition by saying that $f \otimes \mathcal{D}(g, C) = \mathcal{D}(f \otimes g, C)$



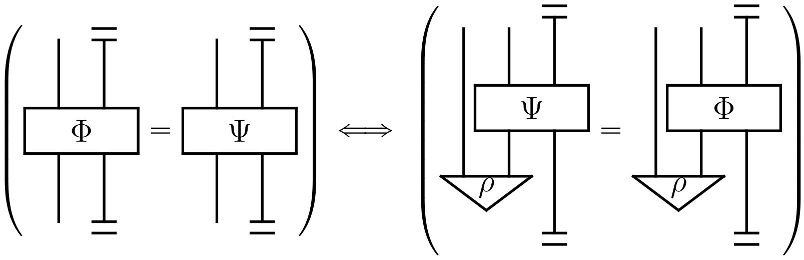
(64)

This has to be thought of as a straightforward extensions of the rules provided by Deutsch, we know how an object of the form $\mathcal{D}(f, C)$ where $f\colon A \otimes C \to B \otimes C$, acts on states of type A according to Deutsch's consistency rule. We can also define a notion of equivalence between two elementary boxes, we say that two EBs are equal if they have the same effect on all multipartite states when tensored by identities representing the environment, let $f\colon A \to B$ and $g\colon A \to B$ be given as:



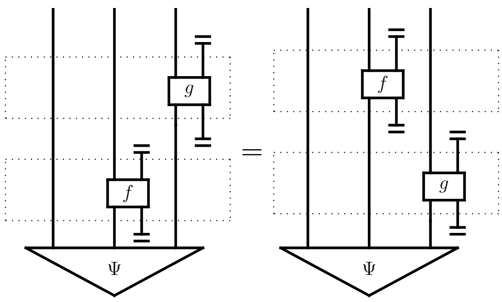


we have that


(65)

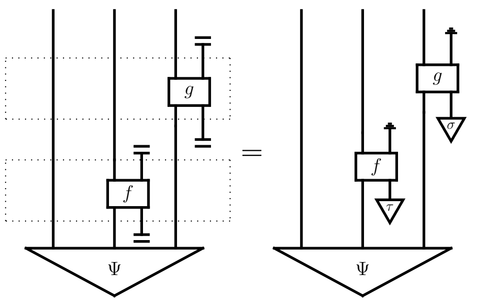
for all $\rho : \mathbb{C} \to E \otimes A$ where E is the space representing the environment. This way of defining the equivalence is a consequence of the discussion developed in the previous section. Two D-CTC maps can act analogously on all the state of their domain but have different actions on the correlations with the environment. Now we prove a lemma which will be used to guarantee that we can find a well defined notion of the parallel composition for all EBs and all the morphisms of the category that we will soon define.

Lemma 6.3 (Switching property)**.** Let $f\colon A \to C$ and $g\colon B \to D$ be two arbitrary TPCP maps, therefore for all $\psi : \mathbb{C} \to E \otimes A \otimes B$ we have that:

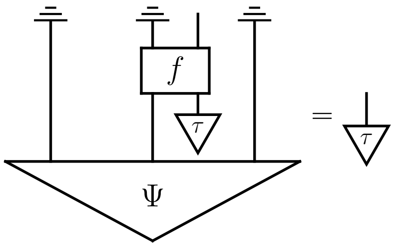

(66)

where the composition of the maps acting on the states is simply given by composition of functions.

Proof. Starting with the left hand side diagram, by definition the evolution of Ψ can be described by:

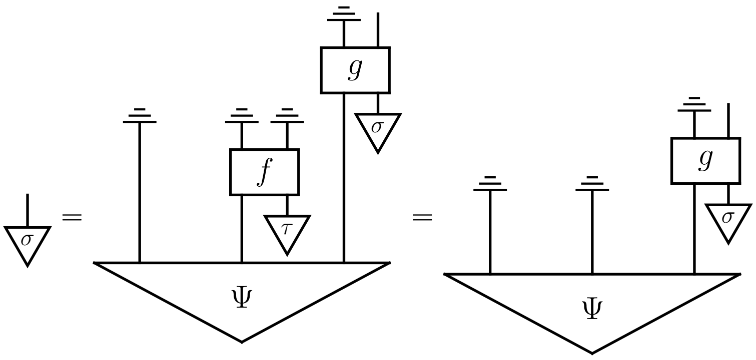

(67)

where the state τ satisfies the following equation:



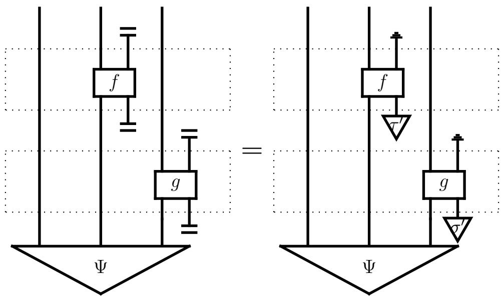
(68)

and the state σ satisfies:



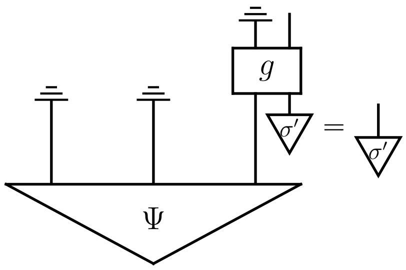
(69)

where the second equation is a consequence of causality of CPTP map. For the right hand side:



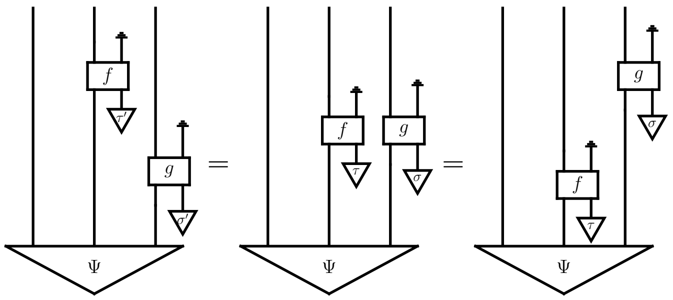
(70)

Now we have that τ' and σ' must be solutions of the same equation, for example for σ':



(71)

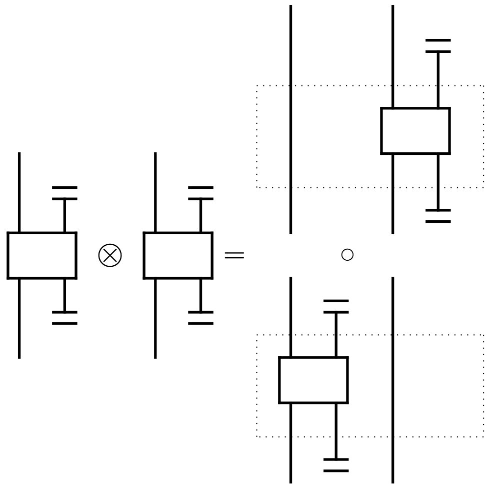
we therefore necessarily have that σ and σ' represent precisely the same fixed point. For τ the diagrammatic equation is analogous, we therefore conclude using the monoidal structure of CPTP maps that the sliding property is satisfied:



(72)

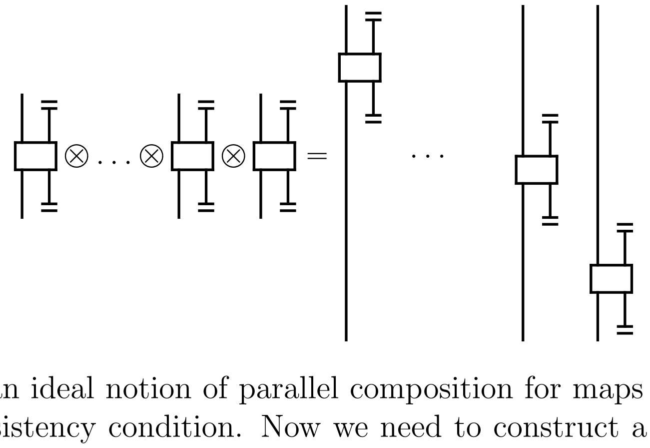
□

The theorem allows us to define what we mean by the tensor of two D-CTC maps, so what it means to apply the Deutsch's condition in parallel:



(73)

Moreover a straightforward inductive extension tells us that it is possible to define a space-like composition for multiple EBs, i.e to define what it means to compose in parallel more than two EBs in a way that is independent on the order of the bracketing.



(74)

We have described an ideal notion of parallel composition for maps acting on the states by the Deutsch consistency condition. Now we need to construct a category which will encompass this notion of space like composition and in which we can make sense of arbitrary sequential composition of EBs.

Definition 6.4 (DMix). The category **DMix** has finite dimensional Hilbert spaces as its objects and the morphisms $f : \mathcal{H}_A \to \mathcal{H}_B$ are all the diagrams obtained by composing a *finite number of elementary boxes*, all the diagrams that can be subdivided into finite slices of the type $\mathcal{D}(f, C)$.

We can more formally therefore represent every morphism $\mathbf{DMix}(A, B)$ as an ordered sequence of elementary boxes $\{\mathcal{D}(f_i, C_i)\}_{i=0}^{n-1}$ where

$$f_0 \in \mathbf{Mix}_\top(A \otimes C_0, A_1 \otimes C_0)$$

$$f_i \in \mathbf{Mix}_\top(A_i \otimes C_i, A_{i+1} \otimes C_i)$$

and

$$f_{n-1} \in \mathbf{Mix}_\top(A_{n-1} \otimes C_{n-1}, B \otimes C_{n-1})$$

The composition of two morphisms is therefore given by the composition of the two diagrams, let $\chi = \{\mathcal{D}(f_i, C_i)\}_{i=0}^{n-1} \in \mathbf{DMix}(A, B)$ and $\xi = \{\mathcal{D}(f_j, C_j)\}_{j=0}^{m-1} \in \mathbf{DMix}(B, C)$ therefore:

$$\xi \circ \chi = \{\mathcal{D}(f_k, C_k)\}_{k=0}^{n+m-1}$$

where $\mathcal{D}(f_k, C_k) = \mathcal{D}(f_i, C_i)$ for all $k \leqslant n-1$ and $\mathcal{D}(f_k, C_k) = \mathcal{D}(f_{j-n}, C_{j-n})$ for all $k > n-1$. The composition is clearly as associative as the graphical composition of diagrams is associative.

To conclude the definition of the category, we need to impose an additional equivalence equation on morphisms: we denote the set of normalised states of a given Hilbert space $\mathcal{H}$ to be $S(\mathcal{H})$, the functions between the set of normalised states of $\mathcal{H}_a$ and $\mathcal{H}_b$ is therefore $\mathbf{Set}(S(\mathcal{H}_a), S(\mathcal{H}_b))$. We say that for $f \in \mathbf{Mix}_\top(A \otimes C, B \otimes C)$ an *interpretation* of an elementary box $[\![\mathcal{D}(f, C)]\!] \in \mathbf{Set}(S(A), S(B))$ is the function sending each normalised state of A to a normalised state in B according to the D-CTC's prescription. We therefore say that two elementary boxes are equivalent if:

$$\mathcal{D}(f, C_1) \sim \mathcal{D}(g, C_2) \iff \forall \mathcal{H} \in \mathrm{ob}(\mathbf{DMix}) \colon [\![\mathcal{D}(\mathbb{1}_\mathcal{H} \otimes f, C_1)]\!] = [\![\mathcal{D}(\mathbb{1}_\mathcal{H} \otimes g, C_2)]\!]$$

This equivalence relation can be extended to arbitrary morphisms $\xi, \chi \in \mathbf{DMix}(A, B)$, where $\xi = \{\mathcal{D}(f_j, C_j)\}_{j=0}^{n-1}$ and $\chi = \{\mathcal{D}(g_i, D_i)\}_{i=0}^{m-1}$ as follows: $\xi \sim \chi$ if and only if:

$$\forall \mathcal{H} \in \mathrm{ob}(\mathbf{DMix}) \colon [\![\mathbb{1}_\mathcal{H} \otimes \xi]\!] = [\![\mathbb{1}_\mathcal{H} \otimes \chi]\!]$$

where

$$[\![\mathbb{1}_\mathcal{H} \otimes \xi]\!] \equiv [\![\mathcal{D}(\mathbb{1}_\mathcal{H} \otimes f_{n-1}, C_{n-1})]\!] \circ \ldots \circ [\![\mathcal{D}(\mathbb{1}_\mathcal{H} \otimes f_0, C_0)]\!]$$

and

$$[\![\mathbb{1}_\mathcal{H} \otimes \chi]\!] \equiv [\![\mathcal{D}(\mathbb{1}_\mathcal{H} \otimes g_{m-1}, C_{m-1})]\!] \circ \ldots \circ [\![\mathcal{D}(\mathbb{1}_\mathcal{H} \otimes g_0, C_0)]\!]$$

We can therefore informally say say that two diagrams are equal if they act in the same way on any state when tensored by an arbitrary identity. For example we see that the definition of the equivalence relation incorporates the D-CTC's evolution as part of this equivalence relation: consider for example a state in $\mathbf{Mix}_\top$, $\rho \colon \mathbb{C} \to A$ then we can clearly think of it as the elementary box given by $\mathcal{D}(\rho, \mathbb{C})$. Let us now postcompose ρ with a morphism in $\Phi \in \mathbf{DMix}(A, B)$, and let ρ' be given by

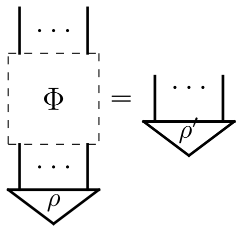
$$\rho' \equiv [\![\mathbb{1}_\mathbb{C} \otimes \Phi]\!](\rho) \in S(B)$$

therefore ρ' is a normalised state $\mathbb{C} \to B$ we can show that $\Phi \circ \mathcal{D}(\rho, \mathbb{C}) \sim \mathcal{D}(\rho', \mathbb{C})$. Fix an arbitrary identity $\mathbb{1}_{\mathcal{H}}$, we need to show the following equality between functions

$$[\![\mathbb{1}_{\mathcal{H}} \otimes \Phi]\!] \circ [\![\mathcal{D}(\mathbb{1}_{\mathcal{H}} \otimes \rho, \mathbb{C})]\!] = [\![\mathcal{D}(\mathbb{1}_{\mathcal{H}} \otimes \rho', \mathbb{C})]\!]$$

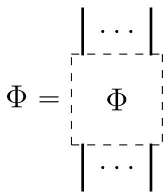
starting from the left hand side, the domain of definition is given by $\mathcal{H}$, any normalised state in $mathcalH \otimes \mathbb{C}$ is of the form $h \otimes 1 = h \in \mathcal{H}$ and gets mapped to $h \otimes \rho$, moreover by definition of ρ', applying $[\![\mathbb{1}_{\mathcal{H}} \otimes \Phi]\!]$ to $h \otimes \rho$ always returns $h \otimes \rho'$, which is therefore equal to the application of the function on the right hand side on an arbitrary normalised vector in $\mathcal{H}$.

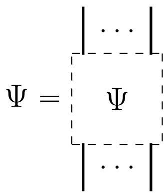
This shows that we can always substitute and replace diagrams obtained by applying the D-CTC condition on states:



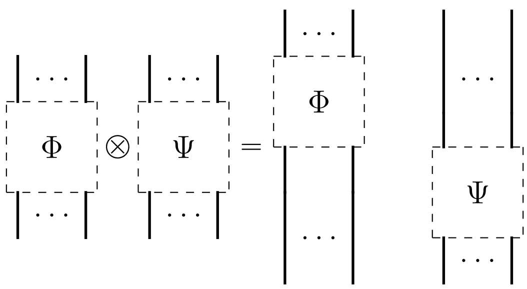
(75)

The category can be endowed with the symmetric monoidal structure $(\mathbf{DMix}, \otimes, \mathbb{C}, \sigma)$. On objects it inherits the monoidal structure given by the usual tensor product of finite dimensional Hilbert spaces, in particular the monoidal unit is given by the one-dimensional vector space $\mathbb{C}$, On morphisms we define the product as follows: take $\Phi \colon A \to B$ and $\Psi \colon C \to D$ be two morphisms, we will graphically represent general morphisms as follows:





compatibly to the notion of product between EBs we define the tensor product between morphisms to be equal to



(76)

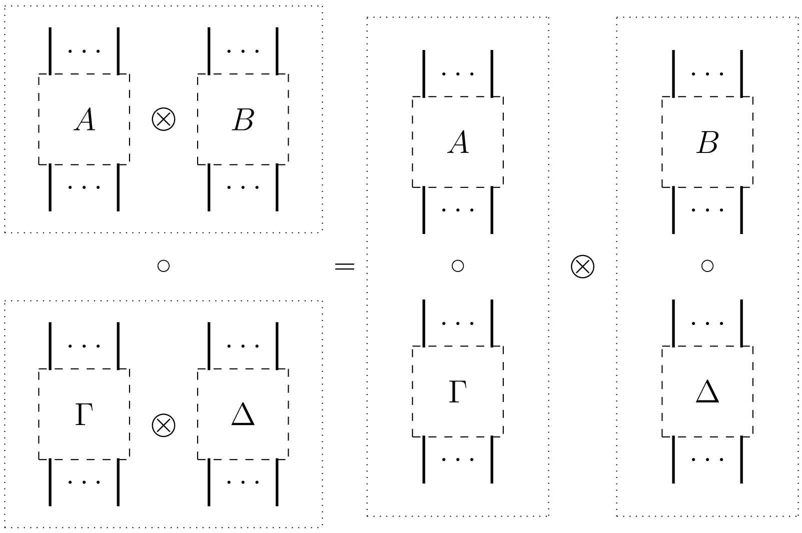
Let $\Phi \in \mathbf{DMix}(A, B)$ and $\Psi \in \mathbf{DMix}(C, D)$ with $\Phi = \{\mathcal{D}(f_j, P_j)\}_{j=0}^{n-1}$ and $\Psi = \{\mathcal{D}(g_i, Q_i)\}_{i=0}^{m-1}$.

Then we denote the tensor product $\Phi \otimes \Psi$ to be equal to

$$\Phi \otimes \Psi = \{\mathcal{D}(h_k, R_k)\}_{k=0}^{m+n-1}$$

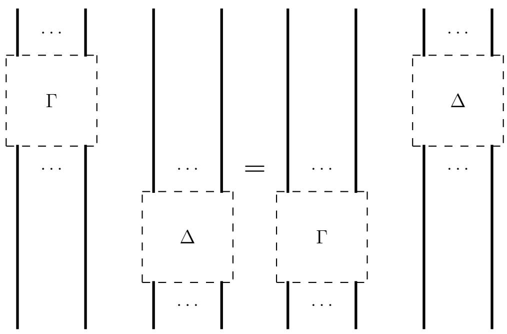
where $h_k = \mathbb{1}_A \otimes g_i$ and $R_k = Q_k$ for all $k \leqslant m-1$ and $h_k = f_{k-m} \otimes \mathbb{1}_D$ and $R_k = P_{k-m}$ for $k > m-1$.

We still need to check that the definition in the Equation (76) satisfies bifunctoriality, i.e that the the following two diagrams are equivalent:



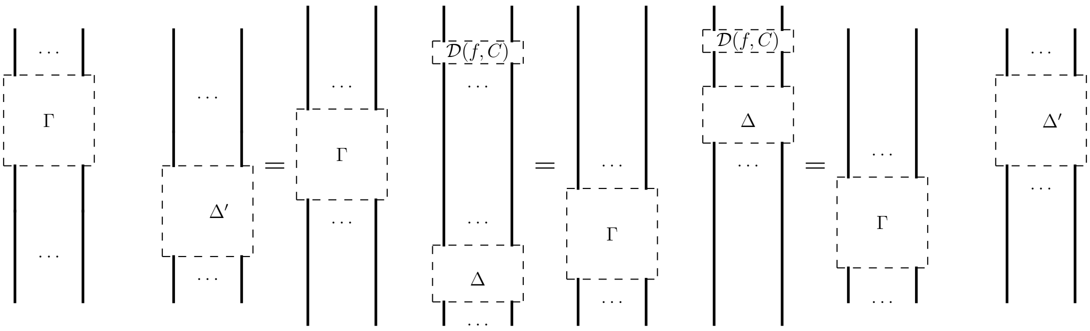
Bifunctoriality is thereforore in this case a direct consequence of the switching property for diagrams, which generalises the switching property for elementary boxes of Theorem (6.3). The generalisation can be proved by a straightforward inductive extension of Theorem (6.3).

Theorem 6.5.



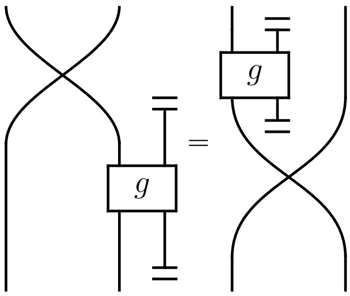
(77)

Proof. We can prove the theorem by induction on the sum of the finite number of elementary boxes composing the morphisms. The base case, one elementary box for each morphism, is proved in Theorem (6.5). Assume that the Equation (77) hold for all $\Delta = \{\mathcal{D}(f_j, P_j)\}_{j=0}^{n-1}$ and $\Gamma = \{\mathcal{D}(g_i, Q_i)\}_{i=0}^{m-1}$ such that $n + m \leqslant N$. Now consider an arbitrary $\Delta' = \mathcal{D}(f, C) \circ \Delta$ such that the sum of all elementary boxes now equals $N+1$. By inductive hypothesis we can slide Γ and $\mathcal{D}(f, C)$, moreover we see that applying the inductive hypothesis twice, we can also slide Δ and *Gamma* obtaining:



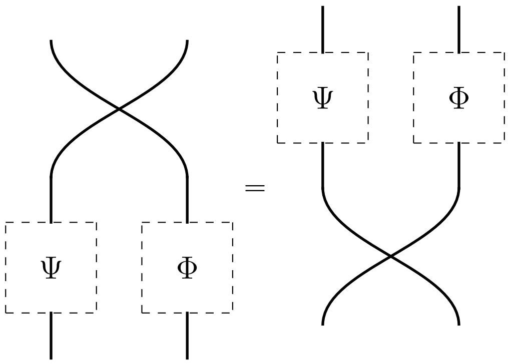
□

The category **DMix** is therefore a monoidal category, moreover it inherits the self-invertible braiding from $\mathbf{Mix}_{\top}$, the braiding automatically satisfies the two hexagonal equation defining braided monoidal categories (Sel11) and it is straightforward to show that the equation



(78)

induces by induction, similarly to the proof of Theorem (6.5), the naturality condition for all diagrams:



(79)

Now that we have defined our symmetric monoidal category we can interpret any acyclic diagram connecting morphisms as representing a unique and well defined morphism of the category (Sel11), moreover diagrams that are equivalent by exclusively applying the axioms of the monoidal category will be equal up to 4-dimensional framed isotopy (Sel11).

Now we present certain properties of the newly defined category, first we formalise the intuitively clear fact that $\mathbf{Mix}_{\top}$ embeds in **DMix**:

Definition 6.6. We say that a category **D** can be embedded in **C** if there exists a functor $F : \mathbf{D} \to \mathbf{C}$ which is faithful and injective on objects.

Next we show that it is possible to embed the category $\mathbf{Mix}_{\top}$ into the category **DMix**, in fact:

Theorem 6.7 (Standard Quantum mechanics can be embedded in **DMix**)**.** Consider the functor F defined to be the identity on objects and

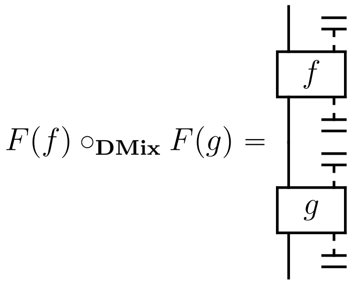
$$F(f) = \mathcal{D}(f, \mathbb{C})$$

so the functor that sends an arbitrary morphism in $\mathbf{Mix}_\top$ into an elementary boxes where the monoidal unit is the chronology-violating part, clearly there cannot be any meaningful interaction with the unit object.

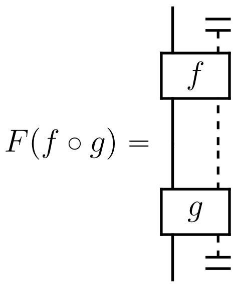
Proof. We check that F is a functor, i.e that

$$F(f) \circ_{\mathbf{DMix}} F(g) = F(f \circ g)$$

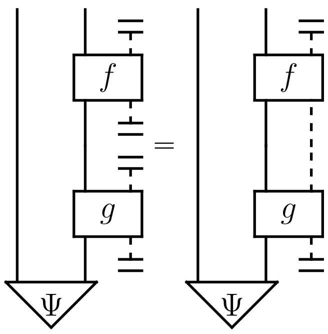
in fact $F(f) \circ_{\mathbf{DMix}} F(g)$ is just the graphical composition of the two elementary boxes, therefore



while for the right hand side



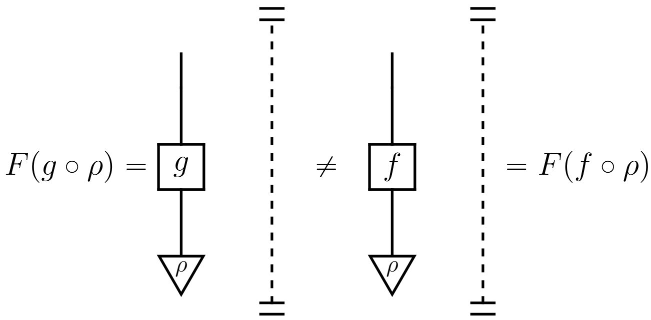
to show that the two morphisms are equal we need to show that they are both in the same equivalence class, i.e that for an arbitrary identity Id_E and state $\rho \colon \mathbb{C} \to E \otimes A$ we have that under the Deutsch prescription:



This is however clearly implied by the fact that the only normalised state in $\mathbf{Mix}(\mathbb{C}, \mathbb{C})$ is the identity $\mathbb{C} \to \mathbb{C}$, the identity scalar. All the fixed points are therefore equal to the scalar 1, which leaves the diagram unchanged.

To show that the morphism is faithful, suppose $f \neq g$, clearly if the domains or the codomains are different $F(f) \neq F(g)$, suppose therefore that $dom(f) = dom(g)$ and $cod(f) = cod(g)$, by the process tomography properties of $\mathbf{Mix}_\top$, there exists $\rho : I \to A$

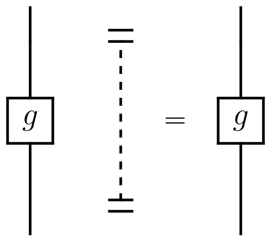
such that $f(\rho) \neq g(\rho)$ in $\mathbf{Mix}_\top$. In particular by definition of our category we have that



(80)

□

This embedding allows us to ignore the notationally trivial tracing out of the monoidal unit, we will therefore simple make use of the following graphical convention:



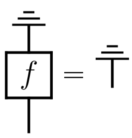
(81)

Therefore the morphisms and the states of $\mathbf{Mix}_\top$ can be treated as elementary boxes. The functor F is a bijection on objects but is clearly not full, however the fulness holds for morphisms of the type $\rho : I \to A$, i.e. if the states of the category **DMix** are precisely given by the image under the functor F of the states of the category $\mathbf{Mix}_\top$.

6.3 DMix is a Normalised Category

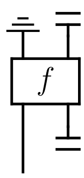
We note that the morphism $\top_A : A \to I$ is inherited in **DMix** since $\top_A \in \mathbf{Mix}_\top(A, I)$, we have $F(\top_A) \in$ **DMix**. We need to check that in the category **DMix** the unit object is terminal. In this section we will simply denote $F(\top_A)$ by $\top_A$.

Lemma 6.8. In the category **DMix** the elementary boxes satisfy the causality axiom. Ignoring the output of a process is equivalent to discarding the entire process, intuitively this tells us that a process doesn't have an influence on anything else which is not directly part of the input of that process.

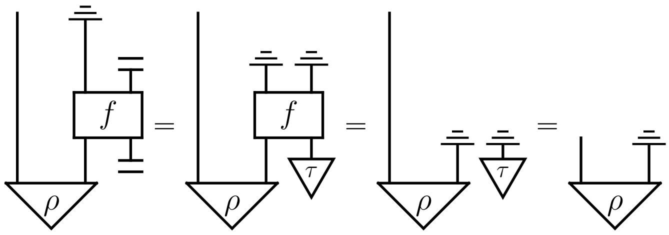


(82)

We start by showing that the elementary boxes $\mathcal{D}(f, C)$ satisfy causality, take $\mathcal{D}(f, C) : A \to B$ then $\top_B \circ \mathcal{D}(f, C)$ is given by



which is equal to $\top_A$:

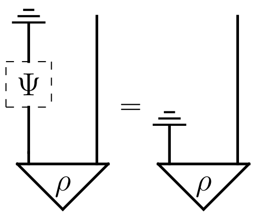


(83)

By the definition of the category any morphism can be obtained by sequential composition of finitely many elementary boxes, a simple induction on the number of elementary boxes therefore allows us to conclude that:

Theorem 6.9. The category **DMix** is terminal.

The terminality of the category guarantees that the non signalling principles are preserved, consider two space-like separated points A and B and assume that they share a bipartite entangled state. The correlation cannot be used to perform superluminal signalling, no matter what operation is performed at the location B, when its output is not accessible by A no information can be transferred by the entanglement. In fact if we ignore the output of B:



In this regard we need to confront our result with the proposal by Bub and Stairs that "one might conclude that D-CTCs are inconsistent" (BS13) because they would allow superluminal signalling and do not respect the relativistic covariance. What are the arguments that they provide and why it is so radically different from our approach?

We now summarise their argument show how the apparent incompatibility with our no-signalling argument derives from a misconception of the D-CTC model. In (BHW09) the authors propose a circuit (Figure (4)) based on the D-CTC model, which allows to perfectly distinguish between orthogonal states: they show that the input $|\Psi\rangle |0\rangle$ gets

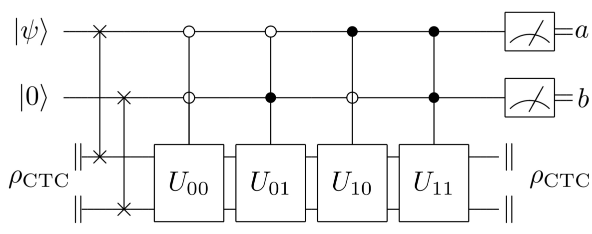
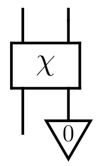


Figure 4: Figure taken from (BHW09) representing the circuit used to perfectly discriminate orthogonal states. The unitary gates U_{ij} is activated if the two qubits are in the state $|i\rangle |j\rangle$, $U_{00} \equiv SWAP, U_{01} \equiv X \otimes X, U_{10} \equiv (X \otimes I) \circ (H \otimes I), U_{11} \equiv (X \otimes H) \circ (SWAP)$

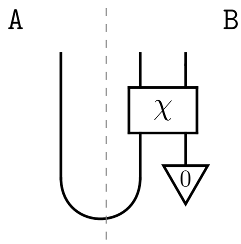
mapped to:

$$
\begin{aligned}
|0\rangle|0\rangle &\mapsto |0\rangle|0\rangle, \quad |+\rangle|0\rangle \mapsto |1\rangle|0\rangle \\
|1\rangle|0\rangle &\mapsto |0\rangle|1\rangle, \quad |-\rangle|0\rangle \mapsto |1\rangle|1\rangle
\end{aligned}
$$

Measuring both qubit we know exactly in which of the states $\{|0\rangle, |1\rangle, |-\rangle, |+\rangle\}$ the qubit $|\Psi\rangle$ has been prepared. We denote the map representing this circuit as a box χ:

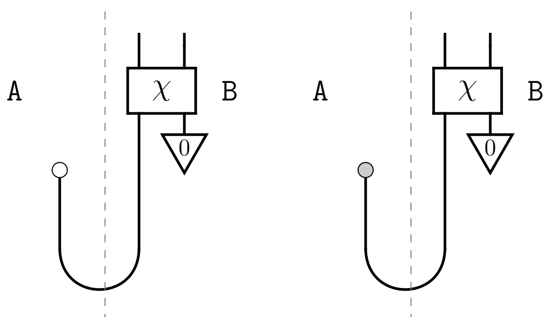


The argument by Bub et al. considers the following scenario:

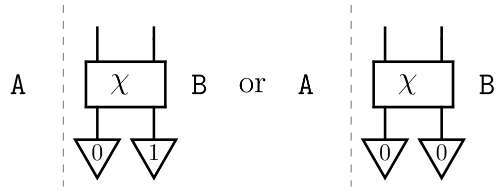


They argue that there are in fact two possibilities, one is the case in which A measures its qubit in either the Z (○) or X (●) basis before B measures its qubits, alternatively B can perform χ before the measurement at A has been carried out. If the regions A and B are space-like separated, then there exists a frame of reference where the measurement at A happens before the measurement at B and viceversa, in both cases the evolution that they describe must coincide. The monoidal structure of a category incorporates already incorporates this principle, it is an easy consequence of therminality that "local states do not depend on the choice of foliation of the diagram" (CL13).

Let us start assuming that from the frame of reference of the observer B the measurement at A happened first, graphically:

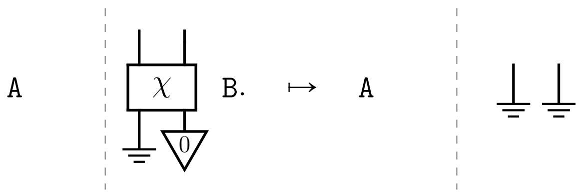


In this case the paper argues that if A takes a measurement in the Z basis then from the perspective of B one of the following scenarios happen with equal probability

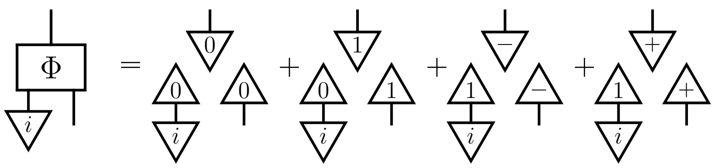


this implies that the observed state at B is with probability a half in $|0\rangle\,|1\rangle$ and half in $|0\rangle\,|0\rangle$ the state is therefore in an equiprobable mixture of the two. Alternatively if we perform a measurement in the X basis we obtain that the result at B is a state which is an equiprobable mixture of $|1\rangle\,|1\rangle$ or $|1\rangle\,|0\rangle$. The observer at B can therefore perfectly distinguish between the two cases measuring its the first qubit, if he obtains $|0\rangle$ he knows that A has chosen the Z basis, alternatively he can infer that the chosen basis is X. According to this way of thinking, the presence of CTC would allow for signalling, an observer at A would be able to transmit a bit of information by selecting one of the two basis.

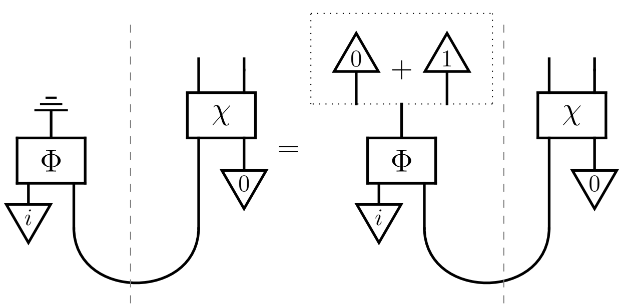
In (BS13) they also claim that this scenario is worrying for yet another reason. Assume that B measures the state before A's measurement, then B has only access to a maximally mixed state and ignoring the observer at A we obtain the following evolution:



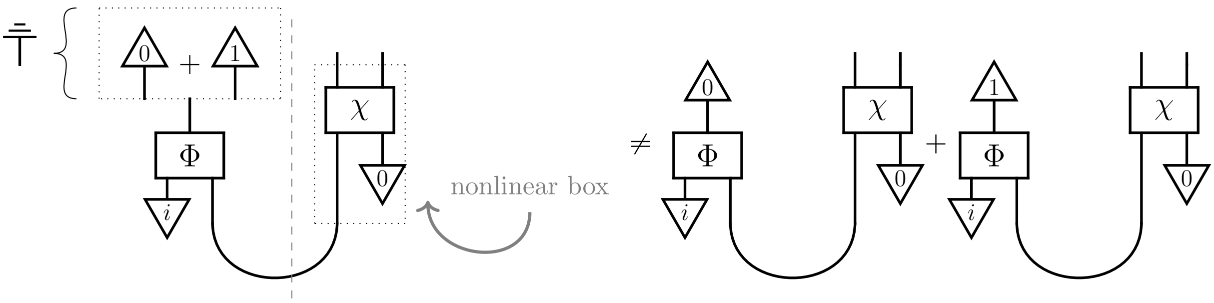
Those two ambiguities are a consequence of the fact that in the paper by Bub et al. they fell into what (BLSS09) calls a *linearity trap*. The essence of the problem can be easily seen diagrammatically, we can denote the controlled measurement as follows:



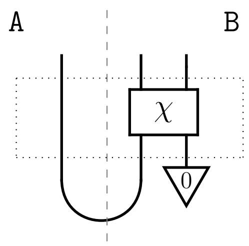
Depending on the choice of $|i\rangle\langle i|$, A choses the X or the Z basis (The bit of information that (BS13) claims to be transferrable is the value i of $|i\rangle\langle i|$).The diagram representing the situation is therefore given by



The idea that the sums distributes over the other parts of the diagram implies that we assume to be dealing with linear "boxes". The introduction of a non linear part of the diagram makes the apparently intuitive argument not justified:



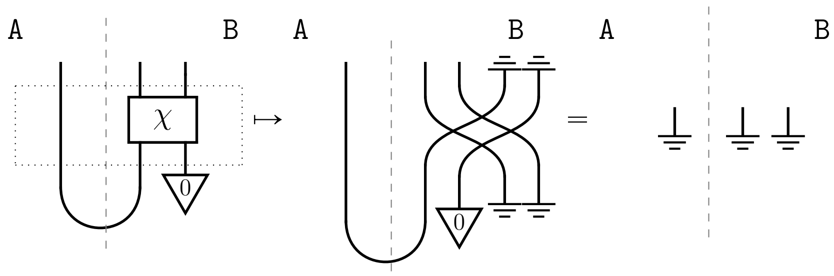
What happens to the dependence on the order of events? If we follow our categorical construction the covariance is preserved by the properties of the monoidal structure and the terminality of the category. In fact if we calculate the evolution of the maximally entangled state between A and B and we intepret it as morphisms in **DMix** we observe that no communication is possible, the regions A and B get even disentangled after the application of $(\mathbb{1} \otimes \chi)$, this can be shown by calculating to state which is equivalent to the following diagram in **DMix**, recall that χ is an elementary box:



the fixed point of the chronology-violating qubit is the same fixed point obtained when feeding the maximally mixed state as input, this is alreaddy calculated in (BS13) where they show that fixed point is in this case given by

$$\tau = \mathbb{1}/2 \otimes \mathbb{1}/2$$

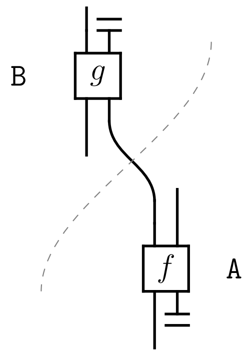
The maps U_{ij} (see Figure (4)) are all unitaries and the CTC qubits need to be discarded, if we unpack the definition of χ (see Figure (4)) we gat that the evolution is given by



Clearly this doesn't allow any type of communication, independently on the kind of measurement performed at A, the correlation gets destroyed and the information on the ancilla bit gets cancelled, in total accordance to the no-signalling theorem proved in this section.

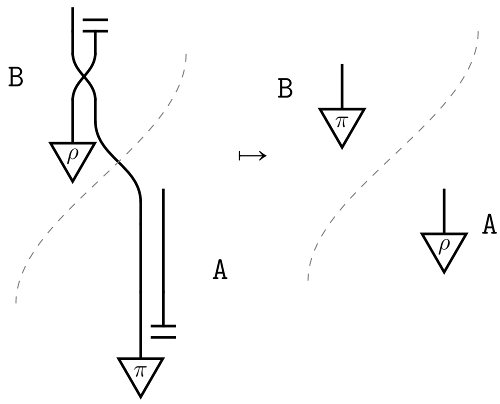
6.3.1 A Note on Composition And a Postcard From the Future

We may now ask whether a CTC could allow us to communicate to the past, consider now the example of two time-like separated events A and B which share a CTC in the following way:

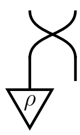


(84)

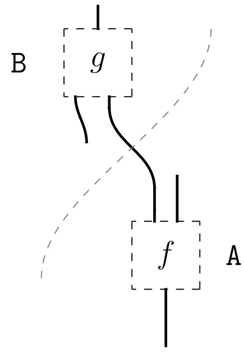
Where we assume that A and B are time-like separated events, setting $g := SWAP$, $f := \mathbb{1}$ and feeding as input at the location A an arbitrary state π and at B the state ρ, the evolution is:



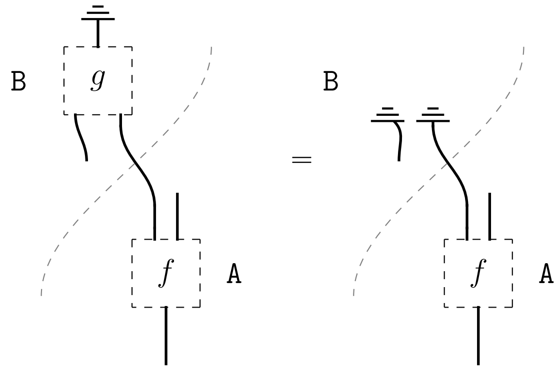
is this an actual communication to the past? At first sight it may appear to be such, it seems that Deutsch's consistency condition has forced ρ to be teleported to A. However the scenario described above while it is a well defined morphism in the category **DMix**, *cannot* be used two describe two time-like separated events, it is not factorable into a "time-like" composition of two morphisms. As such, it is a fallacy to assume that an experimenter at B is in the future of the experimenter at A and no one is free to perform a morphism



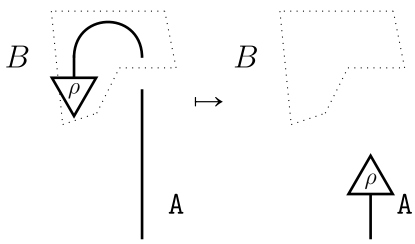
for an arbitrary state ρ chosen at its own discretion independently of what happens at A. Every morphism in the category assumes an entirely localised interaction with a CTC and "sharing" the same CTC is an indication that all those interaction occur locally. It is therefore not possible to send a message back in the future as Diagram (84) does not represent to spacelike separated events. In particular, similarly to the standard prescription of quantum mechanics this is prevented by the causality principle (KHC17). It is easy to show that if we assign to every different region of the spacetime a different morphism in **DMix**, as it is prescribed by the model, causality will prevent any possible communication to the past:



if we ignore the output at the location B:

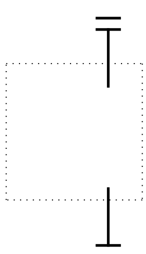


According to the observer at A no information can be extracted from the future. This is not the case for P-CTCs, if B has access to a P-CTC he can postselect an outcome and retrospectively communicate with the past:



6.4 CTCs as Superoperators

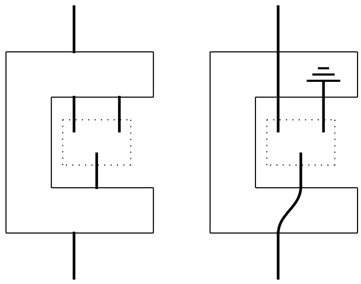
In the next parts of the dissertation we will abstract the Deutsch model in order to understand what are the properties that a compositional theory modelling chronology violations should satisfy. A first move would be to consider the following graphical ornament



as representing a superoperator taking the maps in $\mathbf{Mix}_\top$ to other morphisms, living in a suitably large category. The method that we will use will therefore be similar in spirit to the introduction of the notion of discarding into the framework of **FdHilb** and thus allowing for the definition of the interaction with the environment and the appearance of mixed states.

In fact, we can consider the transformation of CPTP map into Deutsch maps to be a particular way of discarding an input with an output of the same type.

This also underlines the fact that we do not have an effective and direct control over what happens in the CTC, no preparation nor direct test is possible in the CV region. We treat the CTC's as a resource to produce other types of transformation which are otherwise not allowed in the linear theory $\mathbf{Mix}_{\top}$ in the same way the environment is a resource that allows to discard parts of an unitary evolution otherwise representable in **FdHilb**. To make this conceptual point more clear we can refer to the language of second order processes (see for example (KS)) in fact, the classical discarding can be represented as a second order process using the comb notation (KS):



where the comb represents a second order process taking process to process. For example the comb describing the discarding map may take morphisms of the type

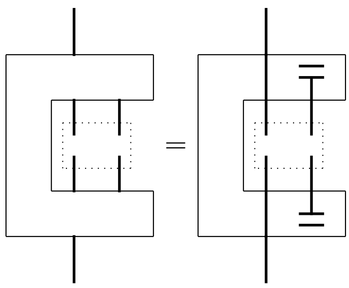
$$A \to B \otimes C$$

to morphisms of the type

$$A \to B$$

by sending $f : A \to B \otimes C$ to the morphism $(id_B \otimes \top_C) \circ f \circ id_A : A \to B$.

Similarly we may wish to consider the effect of the CTC as being a second order process taking morphisms of the type $A \otimes C \to B \otimes C$ to morphisms of the type $A \to B$. It is important to underline however, that the operator *"discarding by CTC"* is not defined on the entire category **DMix** but only on the subcategory $\mathbf{Mix}_{\top}$.

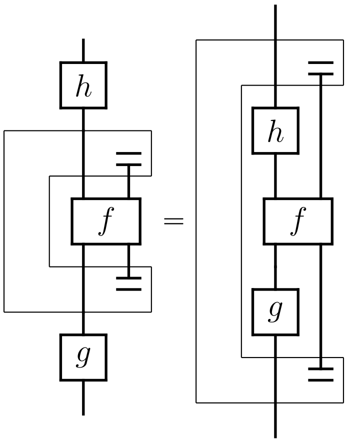


(85)

6.4.1 Properties of the D-CTC super-operator

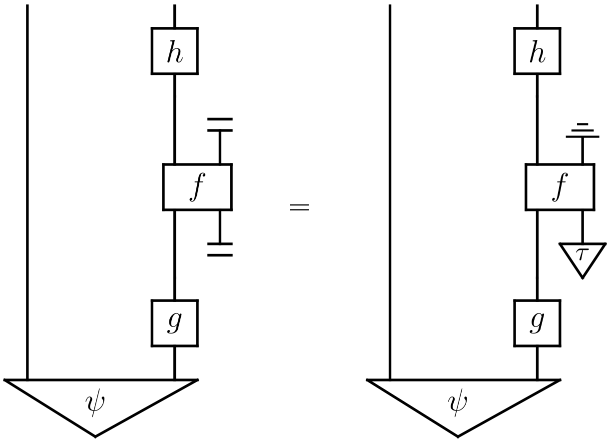
We now observe that the D-CTC operator satisfies certain properties which make its structure similar to a symmetric monoidal trace define on the subcategory $\mathbf{Mix}_{\top}$ for this reason we invite the reader to have a look at the definition of symmetric traced monoidal categories in Definition 2.10.

Theorem 6.10 (Naturality in the CR region)**.** Let $f\colon A\otimes C \to B\otimes C$ and $g\colon D \to A$ be CPTP maps, therefore we have that $\mathcal{D}((h\otimes \mathbb{1}_C)\circ f\circ(g\otimes \mathbb{1}_C), C) = (h\otimes \mathbb{1}_C)\circ \mathcal{D}(f, C)\circ (g\otimes \mathbb{1}_C)$, or graphically:

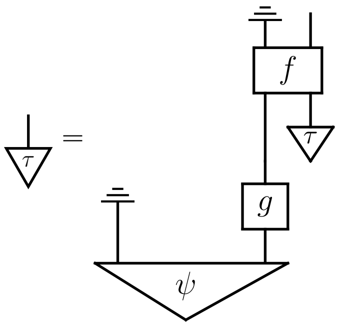


(86)

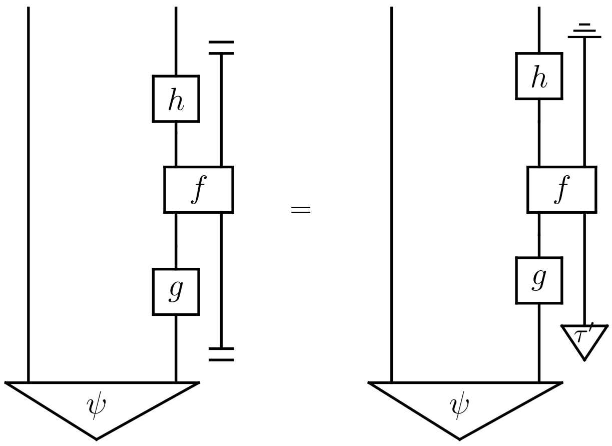
Proof. Starting from the left hand side:



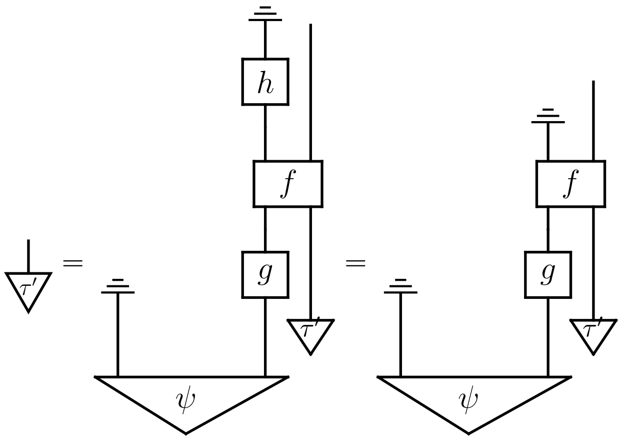
where τ satisfies:



For the right hand side:



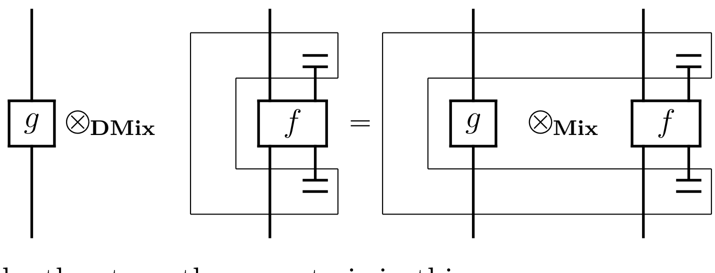
But τ' satisfies the same condition as τ:



Therefore $\tau = \tau'$ and the lead to the same evolution for all $\psi\colon \mathbb{C} \to E \otimes A$ □

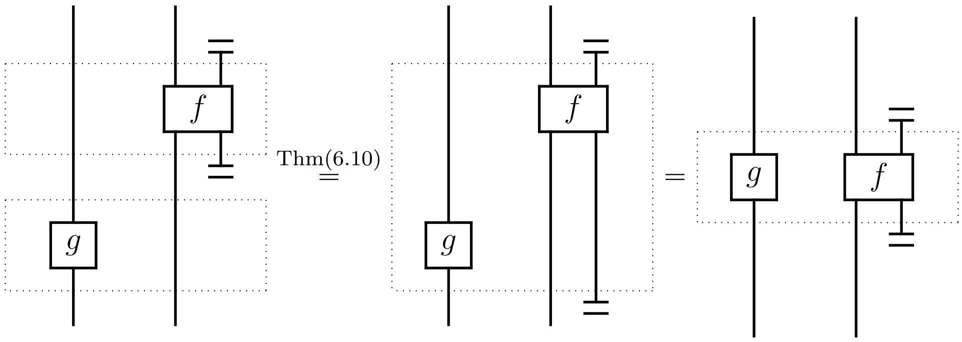
Theorem 6.11 (Strength)**.** Let g be any CPTP map therefore:

$$g \otimes \mathcal{D}(f, C) = \mathcal{D}(g \otimes f, C)$$



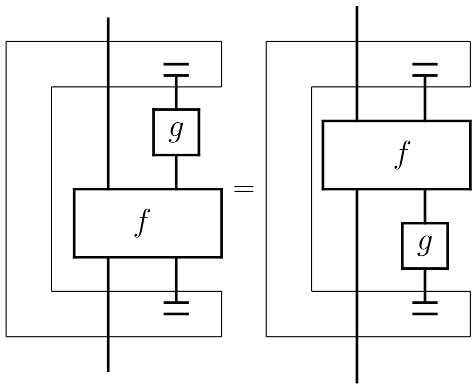
(87)

Proof. Interestingly, the strength property is in this case a consequence of the definition of the tensor product in **DMix** and of the Theorem (6.10):



□

Theorem 6.12 (Sliding)**.** Let $g\colon C \to D$ and $f\colon A \otimes D \to B \otimes C$ be completely positive trace preserving maps, we have that $\mathcal{D}((id_b \otimes g) \circ f, D) = \mathcal{D}(f \circ (id_A \otimes g), C)$, this is graphically represented by the sliding of the map g along the CTC:



(88)

Proof. First we introduce a metric which defines a concept of distance on density matrices:

Definition 6.13 (Trace Distance, (NC11))**.** The *trace distance* between two density matrices is

$$\mathcal{T}(\rho, \sigma) := \frac{1}{2} \operatorname{Tr}[\sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)}]$$

The density matrices are Hermitian and the trace distance can be therefore expressed as

$$\mathcal{T}(\rho, \sigma) = \frac{1}{2} \sum_i |\lambda_i|$$

where $\{\lambda_i\}_i$ are the eigenvalues of the difference $(\rho - \sigma)$.

It can also be shown that the trace distance gives a measure of how distinguishable are two states with an optimal measurement (NC11), for a proof we refer to the exhaustive discussion in (NC11). For example, the maximally mixed state for $d = 2$ is at the center of the Bloch sphere, it is the only point which is *equally* indistinguishable from any other pure state (recall that pure state are represented on the surface of the sphere). The following lemma shows that the trace distance is well behaved under the action of CPTP maps:

Lemma 6.14. Trace preserving completely positive maps are contractive. Let f be a CPTP map, therefore for arbitrary density matrices ρ, σ:

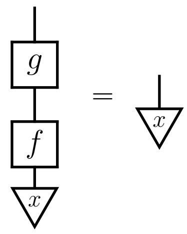
$$\mathcal{T}(f(\rho), f(\sigma)) \leqslant \mathcal{T}(\rho, \sigma)$$

Proof. We omit the proof, a simple and concise argument can be found in (NC11) pp.406-407. □

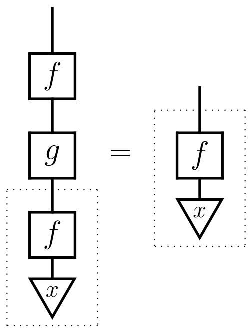
Let f and g be arbitrary CPTP maps. We now establish a strong similarity between the sets of fixed points of the compositions $f \circ g$ and $g \circ f$:

Lemma 6.15. Let $f : A \to B$ and $g : B \to A$ be completely positive maps, therefore the set of fixed density matrices P of $g \circ f$ is isometric to the set of fixed density matrices of $f \circ g$, denoted by Q.

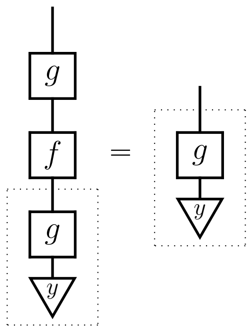
Proof. In Theorem 4.1 we have shown that all completely positive trace preserving have at least a fixed point. The restriction of the CPTP map f on the set P is entirely contained in Q: let $x \in P$ be a fixed point



Applying f to the both side of the equation we obtain:



similarly, g applied to the fixed points of $f \circ g$ gives:



Moreover the following two triangles necessarily commute by definition of P and Q:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ & \overset{id_P}{\searrow} & \downarrow g \\ & & P \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{g} & P \\ & \overset{id_P}{\searrow} & \downarrow f \\ & & Q \end{array}$$

The sets of fixed points are convex subsets of the real vector space of density matrices, if we define on them the trace metric we get the metric spaces $(P, \mathcal{L})$ and $(Q, \mathcal{L})$ and the maps between them are therefore isometric isomorphisms by virtue of Lemma (6.14). □

Lemma 6.16. Let χ be a completely positive trace preserving map and let K be the convex set of density matrices fixed by χ, let $G < U(d)/U(1)$ be the subgroup fixing K in the unitary action given by:

$$u(\rho) := u\rho u^\dagger$$

Let $\tau \in K$ such that for all $u \in G$:

$$u(\tau) = \tau$$

then τ is the density matrix of K with greatest entropy.

Proof. Let $\sigma \in K$ such that $\sigma \neq \tau$. If we take the orbital integral, the average over the orbit of σ we see that it is fixed by an arbitrary element $g \in G$:

$$g\left(\int_G h(\sigma)\ dh\right) = \int_G gh(\sigma)\ dh = \int_G h'(\sigma)\ dh'$$

therefore $\int_G h(\sigma)\ dh = \tau$, the unique point fixed by the action of the elements of G. The Von Neumann entropy S is convex (NC11):

$$S(\tau) = S\left(\int_G h(\sigma)\ dh\right) \geqslant \int_G S(h(\sigma))\ dh$$

and it is invariant under the action of the unitary elements of G:

$$\int_G S(h(\sigma))\ dh = \int_G S(\sigma)\ dh = S(\sigma)\int_G dh = S(\sigma)$$

We get that $S(\tau) \geqslant S(\sigma)$. Since σ was arbitrary, τ is the point of maximal entropy.

□

To continue the proof of the sliding theorem, we use the fact that every metric induces a measure in the space of mixed quantum quantum states (ZS01), let τ_P be:

$$\tau_P = \int_P \rho\ d\rho$$

and

$$\tau_Q = \int_Q \sigma\ d\sigma$$

by linearity and continuity of f:

$$f(\tau_P) = f\left(\int_P \rho \; d\rho\right) = \int_P f(\rho) \; d\rho$$

but f is an isometry therefore

$$\int_P f(\rho) \; d\rho = \int_Q \rho' \; d\rho' = \tau_Q$$

we therefore conclude that

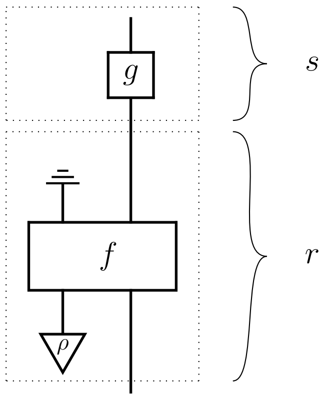
$$f(\tau_P) = \tau_Q$$

analogously we can show that:

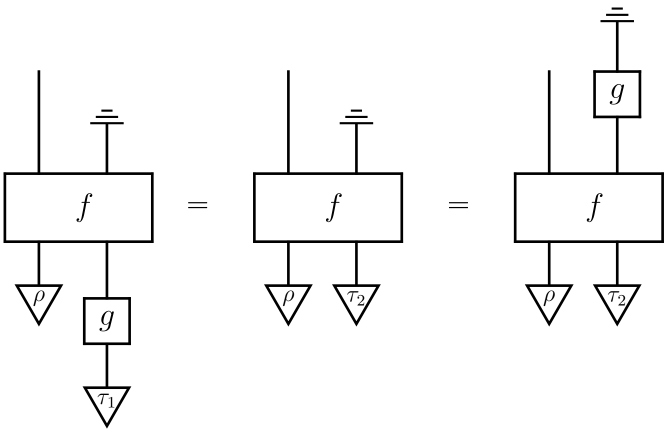
$$g(\tau_Q) = \tau_P$$

It can be shown (see (BKNPV08) for a characterisation of the set of fixed points of quantum channels) that for the sets of fixed density matrices, such as P and Q, the average obtained by taking the integral above is the only fixed point of the group formed by those unitaries sending the set of fixed points into itself. By this observation, as a consequence of Lemma 6.16, we can conclude that τ_P and τ_Q are indeed the fixed point of maximal entropy. And that therefore f maps the point of maximal entropy of P to the point of maximal entropy of Q.

Now can finalise the proof of the sliding theorem, consider the two completely positive trace preserving maps to be r and s given by:

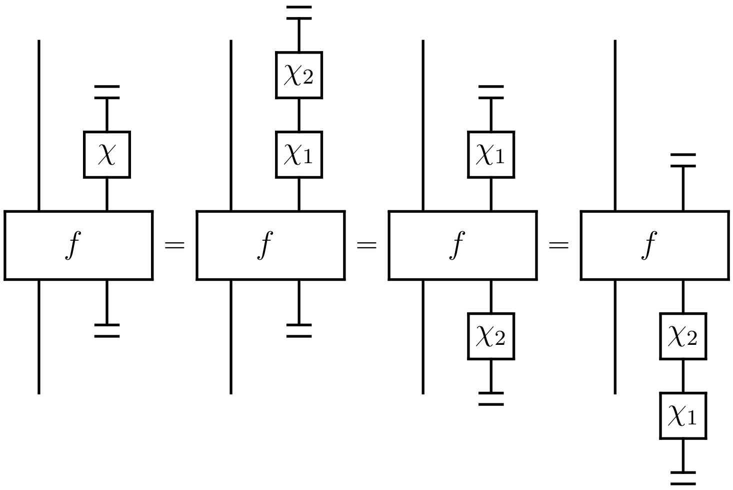


Let τ_1 be the fixed point with maximal entropy of $r \circ s$, therefore the fixed point with maximal entropy of $s \circ r$ is given by $\tau_2 = s \circ \tau_1 = g \circ \tau_1$ we conclude:



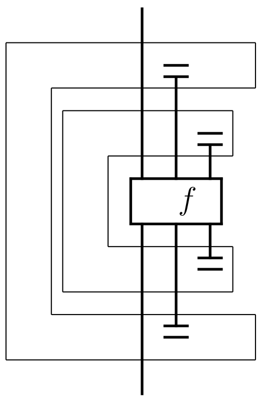
Which concludes the proof of the sliding theorem. □

The sliding theorem is a property with a particular physical relevance. It shows that it doesn't matter where we impose the beginning of the chronology-violating region, for example if a completely positive map χ is applied on the chronology-violating region and χ can be factorised into two maps χ_1 and χ_2, it doesn't matter what we consider to be the starting point of the chronology violation:

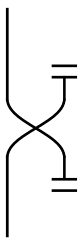


(89)

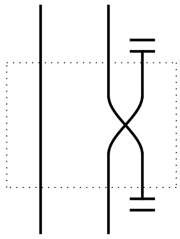
There are however two properties holding in symmetric traced monoidal categories which are not satisfied by our theory. First of all the Vanishing II axiom fails for a very simple reason, the action of the super-operator $\mathcal{D}$ is not defined on morphisms which are not in $\mathbf{Mix}_{\top}$, therefore the statement of the axiom is completely vacuous as the following diagram is not well defined:



moreover, another axiom that ceases to be true is the Unitality, or the Yanking axiom, Equation (5). We have shown in Chapter 4.2.2 that the following map



even though it acts as a unit on the states it is most definitely not equal to a unit as witnessed by the entanglement breaking map:



The fact that the theory satisfies those axioms is not simply fortuitous, we will discover in the next section that those properties – even the unsatisfiability of unitality– are related to the capability of a model to describe certain discrete graphs representing causal relationships. The beginning of such an analysis will be developed in the next section of this dissertation.

7 Causal Graphs, a Discrete Spacetime

In the category that we have crafted to contain the special maps defined by Deutsch we were able to preserve all the main elements that are needed to describe interaction of processes in spacetime. For example we have seen that the maps themselves satisfy the important constraint of causality, discarding the output of a process is equivalent to discarding the process itself. We also observed that Deutsch's consistency condition can be augmented with a well define notion of space-like composition, the tensor product. We stress that both results are in fact not at all foregone and they show that there is a undercovered compositional potential in the theory described by Deutsch.

Analysing the properties of the construction $\mathcal{D}\colon \mathbf{C} \to \mathbf{G}$ we have also seen that it satisfies certain properties derived from the definition of abstract traces in symmetric monoidal categories.

We can therefore go one step further and try to understand what are the properties that have to be satisfied by a general theory representing chronology violation. We will see that the presence of this trace-like structure can be used to describe and model networks that represent a discretised version of the spacetime, where we drop the usual assumption that such a model must be an *acyclic* graph. At the end of this chapter we will see why the **DMix** category is fully mature to describe spacetimes where the loops of information have a local interaction with the chronology-respecting part of the directed graph.

7.1 Framed Causal Sets

7.1.1 Causal Sets

Spacetime is the setting stage of all physical phenomena, several results have shown that it is possible for temporally oriented spacetimes to recover the metric from the class of time-like curves, see for example Malament's theorem (Mal77), moreover as described in (CL13), for an important class of spacetimes the Martin-Panangaden theorem allows to conclude that there is "no need to use information about smoothness and continuity of curves connecting points of the spacetime" (CL13; MP06). This discrete representation of the fabric of spacetime is also amenable to a categorical description, where the space-like and time-like interactions can be described between discrete events.

Each discrete location of the spacetime can be seen as a local laboratory where an observer physically realises a quantum process. The strings of the diagram tell us how are the events causally interlinked, the flow of information between those localised laboratories. The rules of standard quantum theory therefore describe in which way can information spread in this net of related spacetime events when there are no closed cycles.

Before we continue, a brief comment on the notation. For a directed graph Γ we will consider the set of vertices or nodes to be $V(\Gamma)$, the set of directed edges to be given by $E(\Gamma)$. For two nodes u and v we denote the directed edge connecting u to v by (u, v).

Definition 7.1 (Causalset)**.** A causal set $(C, \prec)$ is a set C endowed with a binary relation $\prec$ possessing the following properties:

- *transitivity*: For all $x, y, z \in C$, $x \prec y$, $y \prec z$ implies that $x \prec z$.
- *reflexivity*: For all $x \in C$, $x \prec x$.

- *local finiteness*: Between any two events in C, there are finitely many elements y such that $x \prec y \prec z$.
- *non-circularity*: if $x \prec y$ and $y \prec x$ then $x = y$;

The local finiteness condition means that causal sets can be equivalently represented by *non-transitive* acyclic directed graphs, i.e a graph satisfying the property that if (x, y) and (y, z) are directed edges in the digraph, then (x, z) is not a directed edge in the digraph.

The construction establishing the equivalence works by setting $V(\Gamma) := C$ and $(v_0, v_1) \in E(\Gamma)$ if and only if $v_0 \prec v_1$ and there is no $z \in C$ such that $v_0 \prec z \prec v_1$. We note that the converse is also true as every acyclic directed graph uniquely defines a cusalset: let Γ be an acyclic directed graph, then $C = V(\Gamma)$ and for $a, b \in C$ we say that $a \prec b$ if and only if there exists a directed path joining a to b.

As anticipated we do not want to focus exclusively in acyclic graphs, we will need to formalise a more flexible and compositional setting by introducing the category of *framed causal graphs*:

7.2 Framed Causal Graphs

We use the directed graph perspective to extend this framework to chronology-violating (CV) scenarios.

Definition 7.2 (Framed Causal Graphs)**.** We define a *framed causal graph* Γ to be a non-transitive digraph equipped with:

- a sub-set $in(\Gamma)$ of nodes of Γ—the *input nodes*—such that each $i \in in(\Gamma)$ zero incoming edges and a single outgoing edge;
- a sub-set $out(\Gamma)$ of nodes of Γ—the *output nodes*—such that each $o \in out(\Gamma)$ has zero outgoing edges and a single incoming edge;
- a *framing* for Γ, which consists of the following data:
  - a total order on $in(\Gamma)$;
  - a total order on $out(\Gamma)$;
  - for each node $x \in \Gamma$, a total order on the edges outgoing from x, compatible with the total order on $out(\Gamma)$, where relevant.;
  - for each node $x \in \Gamma$, a total order on the edges incoming to x,compatible with the total order on $in(\Gamma)$, where relevant.

The inputs and outputs should be thought of as "open ends" in the digraphs: because of the way they are defined, it is always true that $in(\Gamma)$ and $out(\Gamma)$ are disjoint subsets of the nodes in G.

Because we allow cycles, framed causal graphs can be used to described chronology-violating scenarios: we refer to framed causal graphs with cycles as *chronology-violating*, and to acyclic framed causal graphs as *chronology-respecting*.

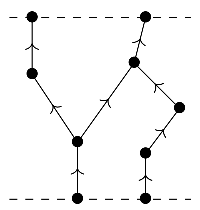


Figure 5: Example of a framed causal graph, we underline that the graphical notation implies a chosen order of the edges if we read the diagram of the graph from left to right.

7.2.1 The SMC of Framed Causal Graphs

Definition 7.3 (The SMC of Framed Causal Graphs)**.** The *category of framed causal graphs* **CausGraphs** has the natural numbers as its objects and morphisms $n \to m$ the framed causal graphs G with $\#in(G) = n$ and $\#out(G) = m$. Using the framing, we can canonically identify $in(G)$ with the total order $\{0, ..., n-1\}$, and $in(G)$ with the total order $\{0, ..., m-1\}$.

Composition $H \circ G$ of morphisms $G : n \to m$ and $H : m \to r$ in the category is given by gluing the intermediate "open ends" $out(G)$ and $in(H)$, i.e. the graph $H \circ G$ has set of nodes $(V(H)_{\setminus in(H)}) \sqcup (V(G)_{\setminus out(G)})$, input nodes $in(G)$, output nodes $out(H)$, and the following edges:

$$(x, y) \text{ in } H \circ G \text{ iff either } \begin{cases} (x, y) \text{ in } E(G) \text{ and } y \notin out(G) \\ (x, y) \text{ in } E(H) \text{ and } x \notin in(H) \\ \exists b \in \{0, ..., m-1\} \text{ s.t. } (x, b) \text{ in } E(G) \text{ and } (b, y) \text{ in } E(H) \end{cases}$$

where in the last case we have identified both $out(G)$ and $in(H)$ with the total order $\{0, ..., m-1\}$. The identity $id_A : A \to A$ on a total order A is given by the digraph with $A \times \{0\} \sqcup A \times \{1\}$ as set of nodes and $((a, 0), (a, 1))$ for all $a \in A$ as edges.

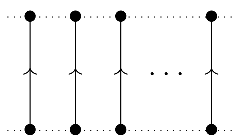


Figure 6: Identity on **CausGraphs**

The category **CausGraphs** can be endowed with the following symmetric monoidal structure $(\mathbf{CausGraphs}, \oplus, \varnothing, \sigma)$:

- on objects, $A \oplus B$ is the sum total order $A + B$, where all elements of A are taken to come before all elements of B;
- on morphisms, $G \oplus H$ is the disjoint union $G \sqcup H$ of digraphs G and H, with $in(G \oplus H) := in(G) \sqcup in(H)$ and $out(G \oplus H) := out(G) \sqcup out(H)$;
- the tensor unit $\varnothing$ is the empty digraph, with $in(\varnothing) = \varnothing = out(\varnothing)$;
- the symmetry isomorphism $\sigma_{A,B} : A \oplus B \to B \oplus A$ is the digraph with $(A + B) \times \{0\} \sqcup (B + A) \times \{1\}$ as its set of nodes, and edges $((a, 0), (a, 1))$ and $((b, 0), (b, 1))$ for all $a \in A$ and all $b \in B$.

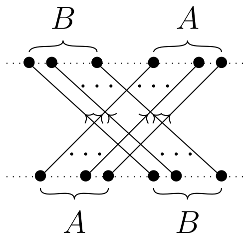


Figure 7: The symmetry isomorphism $\sigma_{A,B}$

Because of the way composition is defined in **CausGraphs**, framed causal graphs cannot be used to describe scenarios in which inputs/outputs live in a chronology-violating region: no new cycles can ever be created by sequential or parallel composition.

From a physical perspective, this means that framed causal graphs can be used to describe chronology-violating regions of space-time with CTCs, but with boundaries constrained to live in the chronology-respecting sector. We refer to acyclic framed causal graphs as *chronology-respecting*, and we write $\mathbf{CausGraphs}^{\mathrm{CR}}$ for the sub-SMC of **CausGraphs** which they span.

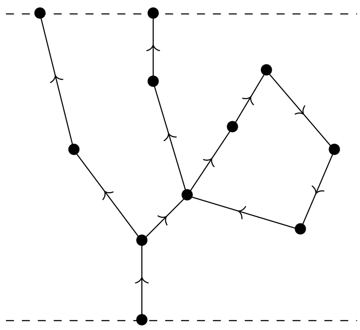
7.2.2 CV-local Framed Causal Graphs

For now we will be interested in a sub-class of framed causal graphs, which we will refer to as "CV-local", we want the CTC to appear in our graphs, but we want the interaction with the CR part to remain "local", in a suitable sense which we describe below.

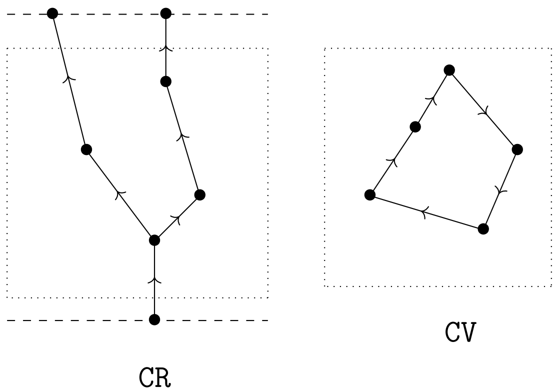
Definition 7.4 (CV-local Framed Causal Graph)**.** A framed causal graph G is said to be *CV-local* if the following conditions holds: every simple cycle, by which we mean a cycle which is both edge-disjoint and vertex-disjoint, has at most one node of degree higher than 2. Simple cycles are then identified to be CTCs, and to the node of degree higher than 2 as the *interaction node* for the CTC.

CV-local framed causal graphs are closed under the composition and tensor product of the SMC **CausGraphs** defined above: they form a sub-SMC, which we refer to as $\mathbf{CausGraphs}^{\text{CV-Loc}}$. In particular, $\mathbf{CausGraphs}^{\text{CV-Loc}}$ contains $\mathbf{CausGraphs}^{\mathrm{CR}}$ as a sub-SMC.

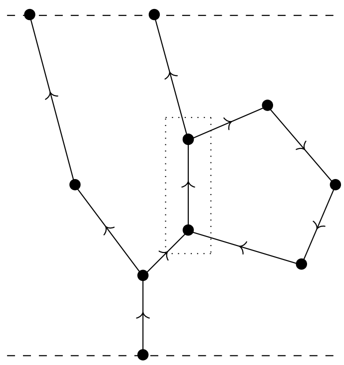
We refer to the sub-graph of G obtained by removing all edges in all CTCs and all nodes in all CTCs except for the interaction nodes as the *CR region*: by definition, the CR region is a CR framed causal graph with the same input and output nodes as G. We refer to the collection of all edges and nodes on all CTCs of G as the *CV region*. The CV separability requirement can then be re-stated to say that the intersection between the CR region and the CV region is a discrete sub-graph of G, which we take to capture the intuition that CTCs interact with the CR region in a fully localised way. The following is a CV-local graph:



which decomposes in the following chronology-violating and chronology-respecting regions:



while in the following diagram the intersection between the CR and the CV region is not a discrete subgraph and the graph is not CV-local.



We can assign morphisms to *acyclic* framed causal graphs G to obtain a morphism of the type $\bigotimes in(G) \to \bigotimes out(G)$, we will formalise this assignment introducing the following definition:

Definition 7.5 (Diagrams over Causal Graphs)**.** Let G be a causal graph and $\mathbf{C}$ a symmetric monoidal category. We say that that a diagram of $\mathbf{C}$ over the graph G is a pair of functions (α, β) with $\alpha\colon V(G)\backslash_{in(G) \sqcup out(G)} \to \hom(\mathbf{C})$, $\beta\colon E(G) \to \mathrm{ob}(\mathbf{C})$ such that for every node v in G, $\alpha(v)$ is a morphism of the type

$$\bigotimes_i \beta(e_i) \to \bigotimes_j \beta(f_j)$$

where $(e_1, \ldots, e_n)$ are the ordered incoming edges and $(f_1, \ldots, f_m)$ the ordered outgoing edges of v.

An acyclic framed causal graph uniquely defines a morphism, any acyclic diagram

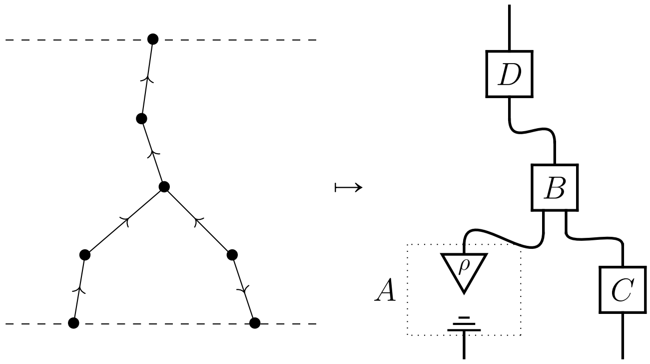
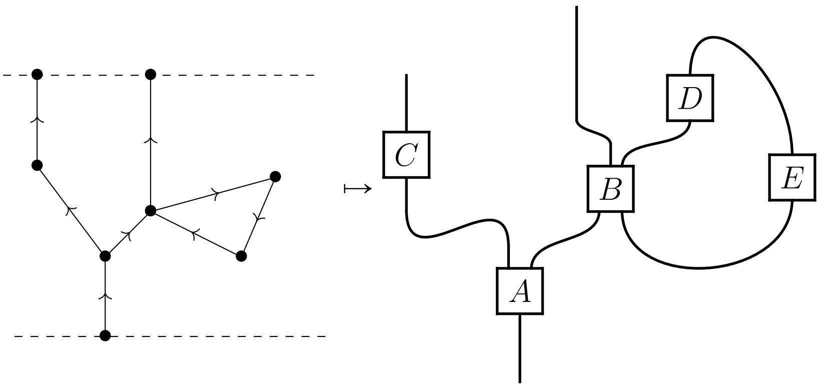


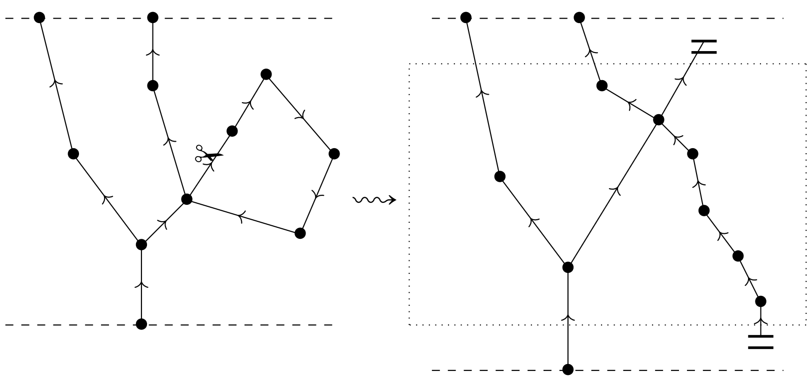
Figure 8: Diagram over an acyclic causal graph

of morphisms can itself be interpreted as a well defined morphism by the virtue of the graphical calculus for symmetric monoidal categories (Sel11). However, when we are in presence of loops, the definition ceases to be meaningful and the following diagram is not a well defined morphism in a symmetric monoidal category:



(90)

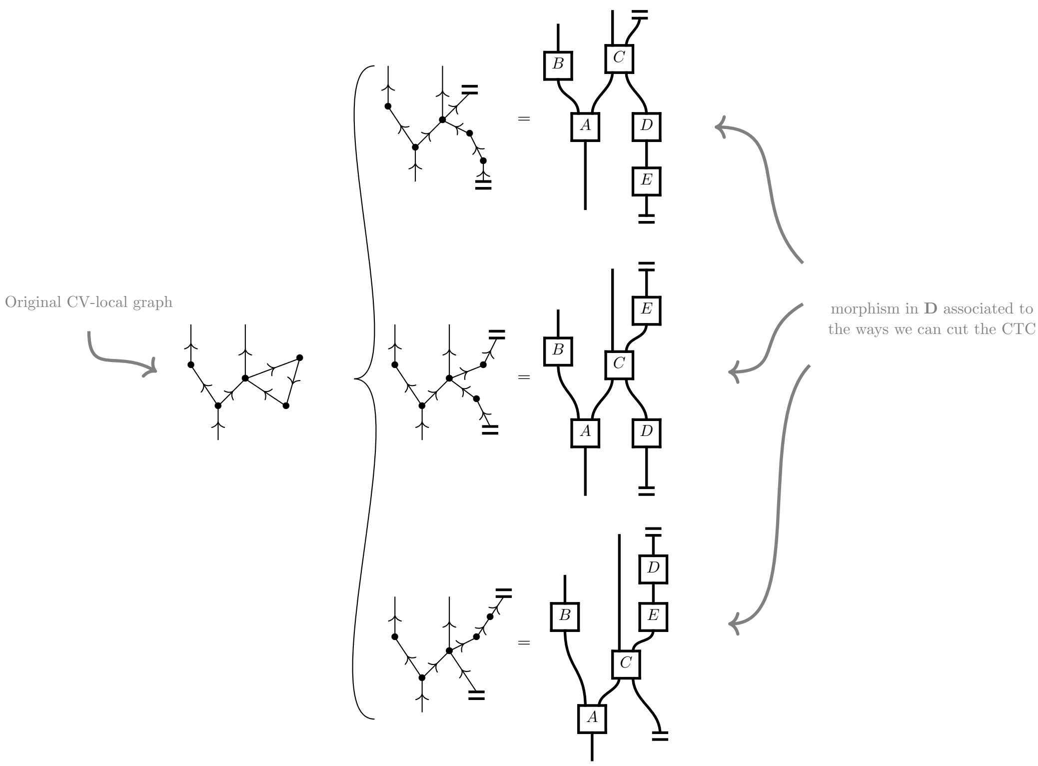
How can we interpret diagram with loops? If we cut the edges of the loops, so if cut edges in the chronology-violating part of the diagram we can transform a cyclic graph into an acyclic one:



(91)

Note that while performing the cut we want to preserve the ordering relations imposed by framing, in particular the total order defined for edges incoming and outgoing for each node. We will call such a cut a *sequenceable interpretation* of a CV-local graph, we underline that the relative position of the double lines representing the application of the operator is irrelevant the only thing that matters is the combinatorial structure of the graph, including the total orders of the edges. If we now impose on the sequenceable interpretation a choice of morphisms and objects (α, β) and the additional constraint that the same object must get mapped to the two cuts of the original edge, we obtain a well defined morphism in **C**. After that, applying the superoperator on the object associated to the cut embeds the morphism defined on the acyclic graph of a sequenceable interpretation into the bigger category **D**.

However we see that there is a fundamental ambiguity in constructing the sequenceable interpretations of CV-local graph: when "cutting" the chronology-violating diagrams it should not matter where we open the CTC, each different sequenceable interpretation must lead to the same global morphisms provided that we stick to the same choice of (α, β). Returning to the framed causal graph in Diagram (90), we can open up the CTC in three different ways and we would like to impose all the resultant morphisms to be equivalent:



In particular the two different cuts of the causal graph in Figure (9) imply that the superoperator must satisfy the sliding property for all the morphisms living in the subcategory **C**. To finish the analysis we also have to consider that we allow multiple CTC interacting

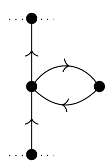
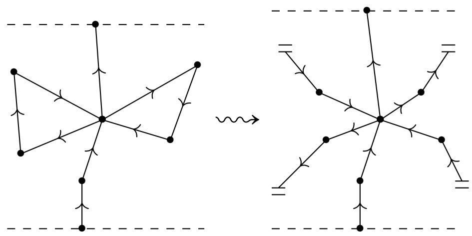


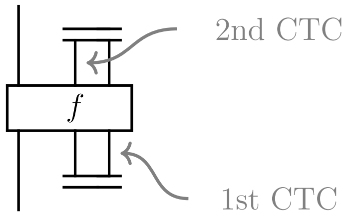
Figure 9: CV-local graph proving the necessity of the sliding property

with the CV region, for example in the following causal graph, there are two closed single cycles meeting at the same point:



(92)

However we will see that this consistency can be recovered from the sliding property if we define the interaction of two chronology-violating curves to be simply given by:



(93)

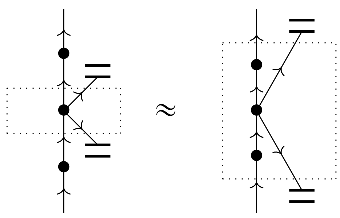
The notation defines a physical situation describing a local interaction with two CTCs intersecting at the same chronology-respecting region of the spacetime. The discussion that we have introduced above can be synthesised in the following theorem:

Theorem 7.6 (Interpretations of CV-local graphs)**.** Let **C** be a symmetric monoidal category describing the chronology-respecting evolution. Let Γ be a CV-local connected graph, let $(e_i)_i$ be the ordered edges connecting the nodes in $in(\Gamma)$ to the rest of the graph and $(f_j)_j$ the ordered edges connected to a node in $out(\Gamma)$. Every sequenceable interpretation of Γ, gives rise to the same morphism:

$$\bigotimes_{i \in in(\Gamma)} \beta(e_i) \to \bigotimes_{j \in out(\Gamma)} \beta(f_j)$$

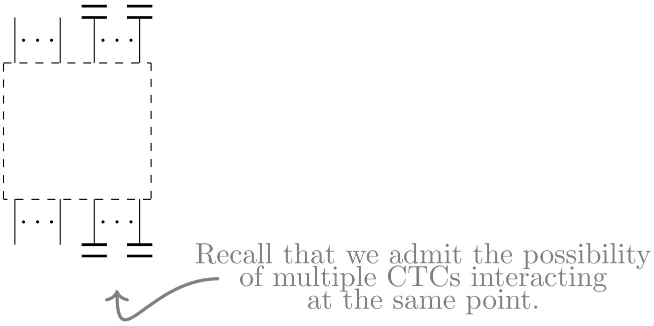
for any choice of (α, β), if and only if there exists a superoperator $\mathcal{D}\colon \mathbf{C} \to \mathbf{G}$ satisfying the sliding property and the naturality in the CR region. The category **G** can then be used to univocally describe CV-local causal graphs.

Proof. We have already shown that the sliding property is implied by the equivalence between different chronology respecting interpretations (using the CV-local graph in Figure 9). Similarly, the naturality in the CR respecting region is a consequence of the fact that it doesn't matter, once we open the cycles, where we define the starting and the ending point of the CTC. The sequenceable interpretations are combinatorial notions even though their topological representations is a powerful visualising aid. The following two graphs must lead, once we assign morphisms to the vertices, to the same evolution:

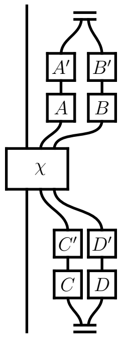


from which we can easily read off the naturality property with respect to the CR region.

Now we show that the two properties are also sufficient to establish the equivalence for every chronology respecting interpretation of a CV-Local graph. We will proceed by induction on the number of interaction points, for the base case we consider CV-local graphs with a unique interaction point. Every chronology respecting interpretation of such a CV-Local graph is of the following form

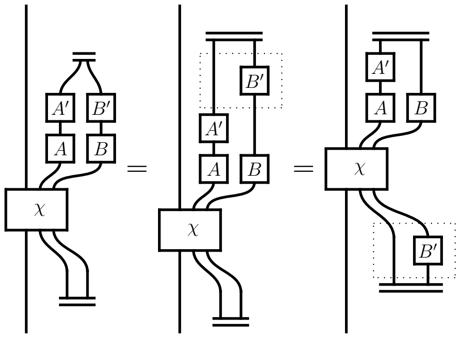


Where the dashed area contains an acyclic framed causal graph. When assigning fixed morphisms to the vertices, the general morphism associated to that acylic framed causal graph can clearly depend on the choice of the cut. However, every diagram over the cut CV-local graph must be of the following normal form

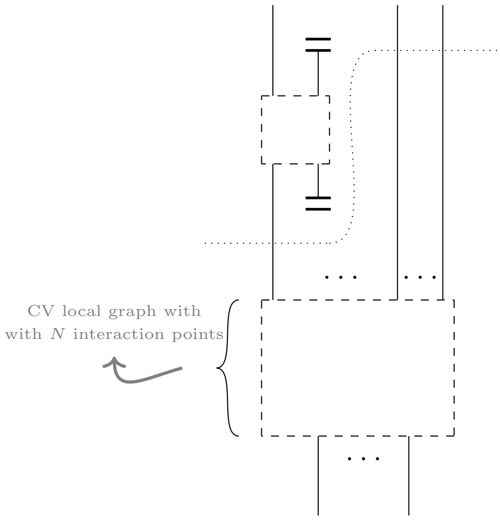


where the map χ is fixed by every cut cut and includes all the morphism living in the chronology-respecting region: when assigning morphisms to such a graph we can by the assumption of naturality, gather all the maps with inputs and outputs in the the CR part of the diagram, including the morphism $\alpha(v)$ (v is the point where the CTCs interact with the chronology-respecting region) under the same label. The map χ will be therefore be invariant under any cut performed on the CV region as it is exclusively defined on the CR region.

To prove that any cut is equivalent we need to show that it is possible to independently slide the boxes, this is an easy consequence of the monoidal structure of the category **C** and the sliding property. Assume without loss of generality that $(C' \otimes D') \circ (C \otimes D) = \mathrm{Id}$, therefore:



It is clear that distinct cuts can only differ by a cyclic permutation of the boxes lying in the CV region, and all such combinations will lead to the same morphism in **D** after we apply $\mathcal{D}$. Consider now the case where we have $N+1$ interaction nodes, we can always isolate a subgraph with a single interaction point:



by the inductive hypothesis this is just a composition of well defined morphism independent on the cut choices. □

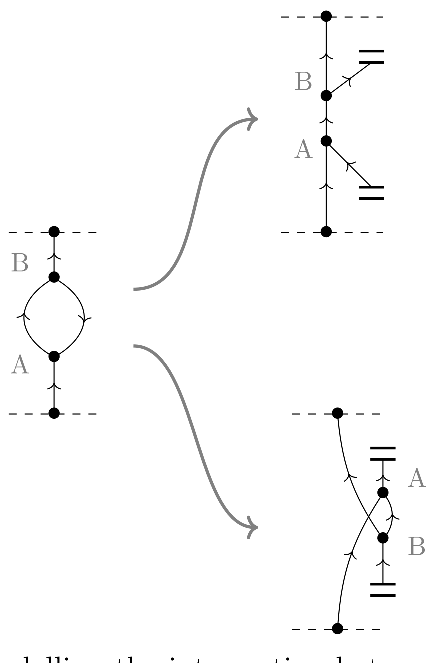
7.3 Extending the Interaction with the CV region

In the previous discussion we have explicitly not allowed framed causal graphs where the intersection between the CR and the CV region was not a discrete subgraph. We have seen that the trace-like properties identified in Deutsch's method are sufficient and necessary conditions to model causal graphs which are CV-local, what happens if we relax this requirement? Is the Deutsch model able to simulate situations where at least an edge is both part of the chronology violating and chronology respecting region? It is not entirely straightforward to provide a physical interpretation of such causal graph but it would also be a pity to surrender to the dryness of the formalism without fighting such an uphill battle.

If the CV-CR interaction is a single point we are basically only producing local maps, local gates that are broadly speaking aware of the value of its output given any input. We have described in great length the fact that such maps are able to simulate many well known paradoxes but can it be considered to be real time travel? In the various interviews and explanations of the model given by Deutsch he seems to argue that this is indeed the case, that the microscopic quantum reality that he is describing could be potentially translated into the macroscopic realm.

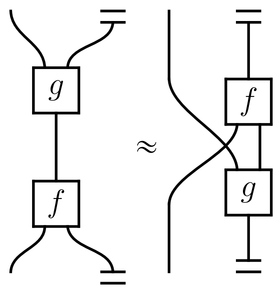
If we allow the interaction to be more complex and we lift the assumption of CV-locality, the picture changes quite radically and the conceptual difference between chronology respecting and chronology violating timelines gets blurred. Let us what happens when we try to cut such a cyclic graph, we will see that in this case the D-CTC description doesn't respect this covariance; and there is also a fascinating physical idea that suggests that the Deutsch model has to imply a distinction between the chronology respecting and the chronology violating region of spacetime. We will start by analysing the simplest example of a non CV-local framed causal graph:

Example 7.7.

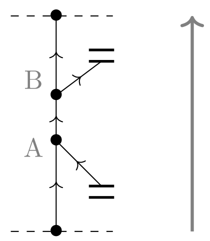


(94)

In this case the subgraph modelling the intersection between the CR and the CV region is an entire edge and there are two ways to transform the cyclic graph into an acyclic graph, two possible cuts. To preserve the equivalence of those physical scenarios we need the following equivalence to hold for any f, g associated to the vertices:

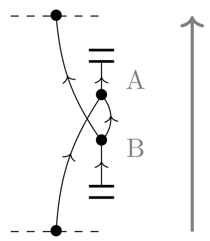


The first thing we notice is that in Example (7.7), the order of morphisms in what we might have considered the chronology-respecting region has been upset by a cut. If we consider the direction of time to flow from the bottom to the top of the diagram, one cut keeps fixed the global ordering of events of the original graph: A before the event B.



(95)

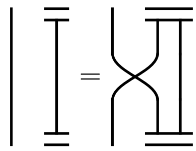
The other one inevitably reverses this order:



(96)

Suppose that event at A represents the birth of the father of an hypothetical time traveller and the event B the birth of the traveller itself. We can consider the scenarios obtained by considering in accordance to the traveller's and the father's internal clocks. If we assume that the clock of the father is aligned to the asymptotic time flow, we get the ordinary description, its clock starts at A and he witnesses all the strange phenomena related to the emergence of a visitor from another time. If we take a different perspective and describe the scneario from the perspective of the time traveller, he then just enters in a loophole at a certain point of his existence and finds itself in another time, it is the rest of the universe that somehow "travelled in time", the birth of his father appears in its future, while he is just following his timeline aligned to the asymptotic direction. In Diagram (96), the birth of the time traveller and its influence to the past event is not considered part of a CTC but it is the standard chronology of events, the order of the chapters that an hypothetical time-traveller would choose while writing his autobiography.

Clearly this narration is of a mythological kind, we ask the reader to read in it the fragile and thin skeleton of an idea. Returning to causal graphs, it is straightforward to show that those different scenarios lead to different evolutions in the case of D-CTCs, if we assign $f = g = \mathrm{Id}_{A\otimes A}$ we get the following equation between morphisms



(97)

which as we have shown before is not a valid equation in **DMix**, the morphism on the right hand side of Equation (97) breaks the correlation with the environment while the morphism on the left is simply the identity.

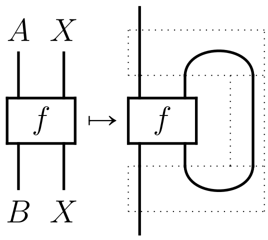
The failure of this equivalence is however clearly in line with the philosophy of the model. Deutsch distinguishes *two* different ways of "travelling" trough spacetime, a wordline which is confined to its universe and rapresents the standard chronology of events and a "time-travelling" wordline which defines a movement trough the different branches of the multiverse. The quantum evolution in the timeline of the time traveller is allowed to be different from the global picture, the evolution obtained by evaluating the diagram in **DMix** which takes in consideration the ensembles of states spread across the multiverses. If we recall the discussion about breaking entanglement and violating the principle of the "Church of the Larger Hilbert Space" the entanglement can be thought to be preserved across the multiverses, however it is broken it is global description.

What about Lloyd's model? We reiterate that Lloyd is unknowingly introducing a trace in the time symmetric formalism introduced by Oreshkov (OC16).

Lemma 7.8. Let $\mathcal{L}^X_{A,B}\colon \mathbf{Mix}_{sym} \to \mathbf{Mix}_{sym}$ be the family of functions

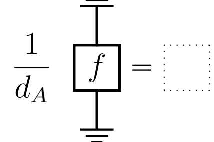
$$\mathbf{Mix}_{sym}(A\otimes X, B\otimes X) \to \mathbf{Mix}_{sym}(A,B)$$

given by:



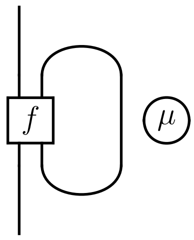
Then $\mathcal{L}$ defines a trace in the category $\mathbf{Mix}_{sym}$.

Proof. Recall the properties of the trace (Definition (2.10)), in our case the proofs are all straightforward, almost trivial when approached diagrammatically. To be concise we only prove *strength*, *unitality* and *sliding*. We start by proving strength. Recall that in the category $\mathbf{Mix}_{sym}$ we always need to renormalise after a composition requiring that every morphism satisfies

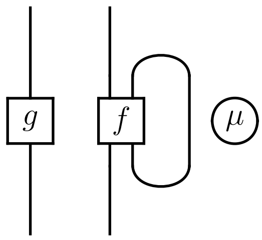


(98)

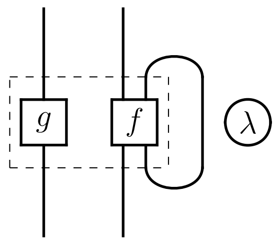
Let $g\colon A' \to B'$ and $f\colon A \otimes C \to B \otimes C$. Consider $\mathcal{L}^C_{A,B}(f)$, we get:

$\mathcal{L}^C_{A,B}(f) =$ 

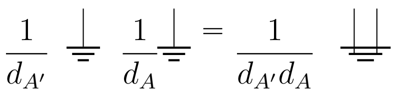
where μ is a normalising factor which makes the composition satisfy Equation (98). Tensoring with g gives:

$g \otimes \mathcal{L}^C_{A,B}(f) =$ 

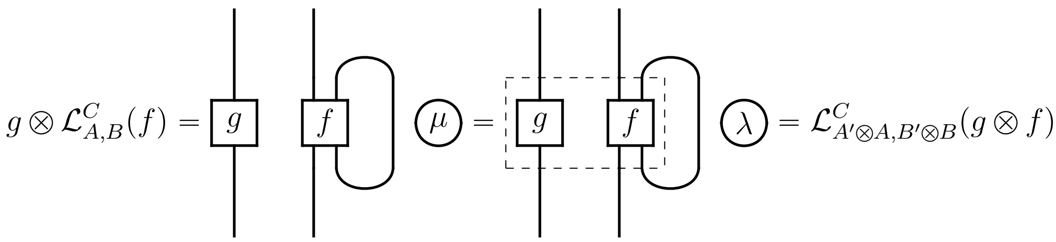
since g is already in $\mathbf{Mix}_{sym}$ no renormalisation is needed. Now we start with $g \otimes f$ and apply $\mathcal{L}$ to the tensor product:

$\mathcal{L}^C_{A'\otimes A, B'\otimes B}(g \otimes f) =$ 

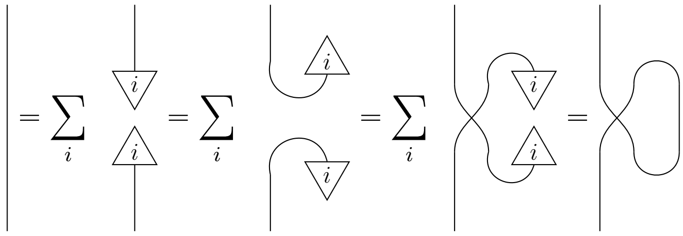
A possibly different factor λ appears, however the following property of the maximally mixed state



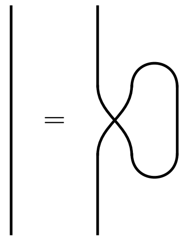
implies that as an straightforward consequence of testing the requirement given by Equation (eqn:normalisation) that $\lambda = \mu$ and



An analogous argument applies to the proof of *vanishing II* while *vanishing I* is entirely trivial. To show unitality, we first recall that $\mathrm{Id}_A \in \mathbf{Mix}_{sym}$ as every completely positive trace preserving map satisfies Equation (98). We can show that in **FdHilb** the identity can be written as:



therefore in **Mix**:

$\mathrm{Id}_A =$  $= \mathcal{L}^A_{A,A}(\sigma_A)$

All the equations valid in **Mix** are therefore valid in $\mathbf{Mix}_{sym}$ and unitality is satisfied. The sliding property is an immediate consequence of the fact that the standard partial trace of linear maps is invariant under cyclic permutation, for $U\colon B \to C$ and f:

$$\mathrm{Tr}_C[(\mathrm{Id}_A \otimes U) \circ f)] = \mathrm{Tr}_C[f \circ (\mathrm{Id}_A \otimes U)]$$

therefore the map $\rho \mapsto E\rho E^\dagger$ of Equation (52) is unchanged by sliding and so id the renormalisation constant. □

We have established that precomposing and postcomposing with a cup and a cap does indeed describe a trace and $\mathbf{Mix}_{sym}$ and it is therefore a symmetric traced category. To explain what is the relationship between traces and non CV-local causal graph we quote verbatim the following theorem from (Sel11):

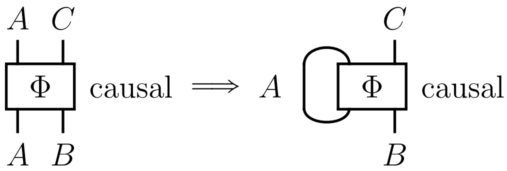
Theorem 7.9 (Coherence for symmetric traced categories, (Sel11))**.** A well formed equation betweeen morphisms in the language of symmetric traced categories follows from the axioms of symmetric traced categories if and only if it holds in the graphical language up to isomorphism of diagrams.

Two diagrams are meant to be isomorphic if there is a bijective correspondence between boxes and wires preserving the connections (Sel11). If we consider the superoperator $\mathcal{D}$ to be the restriction of $\mathcal{L}$ acting on the subcategory **Mix** then clearly the different cuts produce isomorphic diagrams and the equivalence is always preserved independently on the type of CV-CR interaction.

In this context it is interesting to compare our observation with Kissinger's No Timetavel theorem (KS). Kissinger proves that for precausal categories, i.e *compact closed categories* with a discarding map and satisfying certain other reasonable physical assumptions, the superoperator described above, obtained using the *cups* and *caps* of the compact closed structure, cannot send all causal processes to causal processes.

Theorem 7.10 (No Time-travel, (KS))**.** No non-trivial system A in a precausal category **C** admits *time travel*. That is, if there exists systems B and C such that for all processes

Φ we have:



then $A \simeq I$.

The category $\mathbf{Mix}_{\top}$ is a causal subcategory of $\mathbf{Mix}_{sym}$, Theorem (5.2) is a witness of the type of failure described by the No Time-Travel theorem since we have shown that the Lloyd model can send a causal process into a non-causal projector. Kissinger's theorem shows that a trace operator satisfying causality cannot be defined using the canonical trace arising from a compact closed structure.

We have analysed two different models. The D-CTC model entails a structure which is similar but not entirely equivalent to a trace, there is therefore an ambiguity in interpreting certain non CV-local causal graph. The second model, provided by Lloyd allows more flexibility in interpreting causal graphs but it deeply upsets the causal relationships of events by producing non causal morphisms, leading to signalling and retrocausality.

8 Conclusions and Future Works

Hopefully the reader is now convinced of the importance of category theoretical considerations in theory building. In particular, if we want to consider time asymmetric quantum theory as a restriction of a bigger physical theory without a predetermined time-flowing asymmetry, then the categorical approach may be particularly fruitful in defining the non negotiable and fundamental structural properties that such a theory must satisfy.

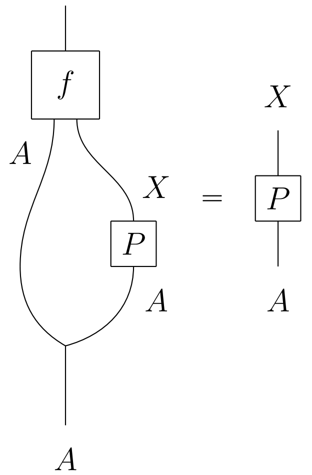
We have also seen how categorisation may be convenient in assessing the peculiarities of existing models and be able to study their differences. Often the approaches on quantum time-travel lack the necessary maturity for a very simple reason, there is a fundamental impossibility of testing hypothesis empirically. It is precisely here that the structuralist approach provided by category theory manifests its biggest potential. In requiring that a class of physical transformations must interact with each other in line with the framework of process-theories, the use of category theory is particularly useful in order to assess what could be the structural consequences of certain physical assumptions and thus tailor the model accordingly. Bringing into play category theory allows to access a standardised vocabulary and an abstract intuitive framework with the potential to connect and unify the realm of physical theories.

To provide an example which valorises our claims, we recall that in Theorem 6.12 we have shown that the choice of the fixed point with maximal entropy is not only in accordance to the principle invoked by Deutsch: knowledge comes into existence only by evolutionary processes, but it is also connected to the sliding property and therefore to fundamental aspects of the interpretive power of the model. An attempt to derive the maximal entropy rule has been made by the development of the "equivalent circuit model" (RM10) but it has been shown to be incorrect in (All14). This has recently led to question this rule as "there is no physical principle for making such a choice and the maximum entropy rule may not be an essential component of the D-CTCs" (DCZ17); on the contrary, we have shown that there reasons to believe that the maximal entropy rule is a fundamental requirement of the model.

Deutsch's model can only unambiguously describe a particular class of causal graphs which do not entail what is commonly considered to be time travel. P-CTCs offer more flexibility but at the expense of a violation of the causality principle. We have made some additional progress in the direction of characterising with greater precision the sufficient and necessary conditions that a theory should satisfy in order to be able to uniquely interpret all the morphisms of **CausGraphs**, however the work was not mature enough to be included in this dissertation and we had to postpone its presentation. By performing a categorisation of the D-CTCs we have identified certain common core properties that must be shared by all models describing time travel at a quantum mechanical level. The natural progression of the work would be to use those considerations to create a theory which satisfies causality and at the same time describes a broad class of causal graph or –perhaps even more interestingly– show that such a model is impossible by constructing a No-go theorem in the spirit of Theorem (7.10).

This dissertation may be seen as the beginning of a journey, as there are many questions to be answered. While one can construct a model such us **DMix**, the mathematical nature of Deutsch's map has not been explored in this work, where we only showed the non-linear and discontinuous behaviour. We have shown that the model allows more flexibility regarding the clonability of states. However, we conjecture that it is not powerful enough to produce an exact CTC assisted cloner. Applying the No-cloning theorem

(Abr09) to the context of **DMix** is however not possible as the theory cannot be embedded in a compact structure such us **FdHilb**, we cannot recover an isomorphisms between channels and states a la Choi-Jamiłowkowski. Another important direction would be to try to characterise the categories for which a trace-like operator exists. In this area there might be an interesting analogy to be uncovered. It has been independently observed by Hyland and Hasegawa (PS14; HHP08) that in a *cartesian* closed category giving the notion of a trace is equivalent to assigning to each morphism $A \times X \to X$ a fixed point $A \to X$ such that



Interestingly D-CTCs are based on the selection of a fixed point while the P-CTC model, acts on a different principle, requiring the probabilistic amplification of the projections which are consistent with the chronology violating scenario. In (BWW13) the authors comments on the possibility of cloning a quantum state with arbitrary fidelity as a witness of the fact that "Deutsch's model turns quantum theory into a classical theory, in the sense that each density operator becomes a distinct, distinguishable point in a classical phase space". Quantum theory is in its fundamental constituent aspects a non-cartesian theory, it would be interesting to understand, also in the light of the Hasegawa analogy between fixed points and traces in cartesian categories, to what extent is the claim in (BWW13) justified.

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