Quantum operators: A classical perspective

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Abstract

We review current research in the field of categorical quantum mechanics. We place the category of relations within the same categorical framework and point out the similarities with the category of Hilbert spaces. Subsequently, we provide a categorical model for investigating the spectral theorem (as provided in [22]), within the context of an internally diagonalisable element and apply these ideas to both categories mentioned. We find out that while all normal operators can be internally diagonalisable in the category of Hilbert spaces – this is not the case for the category of Relations. We conclude by providing insights on how we can separate the internally diagonalisable elements within the set of normal operators .

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Chapter 1

Introduction

The birth of quantum mechanics was undoubtedly a cornerstone in the history of physics. Having being crafted by the great minds of the 20th century, quantum theory underwent intensive studying and testing after being established as one, if not the most, fundamental theory in physics.

With theoretical predictions agreeing to experimental data with up to 10^{-12} accuracy, it is no wonder that quantum electrodynamics, itself a child of quantum theory, is probably the most accurate theory physicists ever came up with. Indeed, even today, physicist all around the world use the mathematical formalism presented by Paul Dirac [11] and John Von Neumann [23] in seemingly diverse areas of physics: particle physics, solid state physics, nuclear physics etc - with great success. Because of its success even skeptics who criticized quantum theory over time now realize its importance. However, there are still questions about the simplicity of quantum theory. Consider the no-cloning theorem [24] and the quantum teleportation protocol [2]. Even though they now seem simple and trivial to prove, they were discovered considerably late, in 1982 and in 1993 respectively.

The new research trend of quantum informatics, partly initialized by the two theorems mentioned, put at the centre of attention what quantum theory can do, in the world of classical computation. It was not long after quantum information revealed some its enormous power: Shor's factoring algorithm [20] and Grover's searching algorithm [13] provide exponential speed-ups compared with their traditional classical counterparts. With all the experimentalists agreeing that a viable quantum computer could be made reality within the next decades, it is perhaps the right time to step back and re-visit the fundamentals of quantum theory, in order to establish a conceptually clear model.

Category theory provides an abstract mathematical environment in which mathematical structures can be expressed – with the concepts of *compositionality* and *type* having a central role. The benefit of having a single mathematical framework in which we can express theories on a more abstract level will prove to be very beneficial. Of course, this is not the only advantage that categories have. One of the most distinct features of monoidal categories is that they admit graphical calculations, in the sense that all mathematical expressions can be reduced to pictures.

Through their seminal paper [1], Abramsky and Coecke provided the categorical formulation for quantum mechanics, through the notion of a *dagger compact closed category*. In the category of Hilbert spaces - the category which admits the formulation of quantum mechanics – phenomena like the teleportation protocol, the no-cloning theorem can be formally axiomatized. The graphical counterparts of the mathematical expressions can be interpreted as the unveiling of 'quantum information flow'.

Quantum measurements play a significant role in quantum mechanics. In the classical world, when one measures something, then the measurement outcomes represents a true indication of the state of the system. However, in the quantum world, measurable quantities, i.e. observables, are represented by a self-adjoint operator – which admits *spectral decomposition*; that is, it can be decomposed into a sum of orthonormal projectors. These projectors exactly correspond to the change of state that takes place during the measurement while their coefficients represent the measurement outcome [7]. When measuring something in the quantum world, the equation that describes the quantum system, called the wavefunction, collapses and the new state of the system is affiliated with what you are measuring against.

In [1], quantum measurements where accounted by switching between the tensor and biproduct structure of Hilbert spaces. However, this kind of formulation does not provide us with a physical interpretation nor does it allow graphical calculations. To tackle this problem, Coecke and Pavlovic introduced in [9] the concept of a *classical structure* which is based on the so-called *Frobenius algebras*. This piece of structure exactly represents the possible outcomes of a quantum measurement.

Of course, category theory is not all about quantum mechanics. One can also define other categories; for example axiomatize relations over sets as a category. This categorical model of relations is of course purely classical, in the sense that we don't expect the 'weird' phenomena of quantum mechanics to be replicated. Surprisingly, as pointed out in [8], the category of relations turns out to be more quantum-like than classical, in the sense that we can replicate phenomena which are understood to be quantum. It should be evident that by studying quantum-like theories we can understand the fundamentals of quantum mechanics better by comparing the quantum-like theory with quantum mechanics.

The aim of this dissertation is to provide insights regarding the spectral theorem of quantum mechanics, but as applied to relations. By providing a categorical description of diagonalisation, we can 'move' to the abstract level of category theory in order to investigate diagonalisation in relations. In other words, we are going to study quantum mechanics, but within the context of relations. More explicitly, we are going to investigate relations (c.f. quantum operators) on the two and three element set; and examine whether or not they can be diagonalised.

The outline of this dissertation is:

– In Chapter 2 we present all the necessary category-theoretic background that we will be using throughout this dissertation, with the notion of a dagger compact category being the key definition.

– In Chapter 3 we axiomatize the category of Hilbert spaces and relations as a dagger compact category. We also present structural similarities between the two categories.

– In Chapter 4 we define quantum and classical structures, with the definition of a dagger Frobenius monoid being a critical element in exploring diagonalisation.

– In Chapter 5 we present the conventional spectral theorem along with its abstract categorical axiomatization.

- In Chapter 6 we investigate the diagonalisable elements and normal operators on the two and three element set.
- In Chapter 7 we conclude with a summary and provide directions for future work.

Chapter 2

Category Theory: Essentials

In this chapter we introduce the essential categorical framework that will be heavily used throughout this thesis. Category theory was proposed by Samuel Eilenberg and Saunders Mac Lane in 1945 as an attempt to study mathematical construction in their own right. Since then, category theory has developed to a modern language of mathematical structures.

Within this framework it is that we examine different categorical models – for example the category of Hilbert spaces or the category of relations. It is in this context that we see the true beauty of category theory. Even though it is traditionally considered an abstract language, the first thought would be to argue that it cannot teach us anything new. But it is exactly the opposite! Taking a step back and going slightly more abstract actually gives us more.

As we will see in Chapter 3 of this dissertation, category theory unifies all structures in the same language and it is there that we can examine how and why they are either different or similar. The two aforementioned categories, even though they express very different mathematical theories will turn out to be remarkably similar. The principal definition in this chapter is the notion of a *dagger compact (closed) category* – the category that has been proven to axiomatize a substantial portion of traditional quantum mechanics.

2.1 Categories and monoidal structure

Definition 2.1.1. A category $\mathcal{C}$ consists of:

1. A collection of *objects* $A, B, C...$, denoted by $|\mathcal{C}|$.
2. A collection of *morphisms* (or maps or arrows) between objects. These assign to each pair of objects $A, B \in |\mathcal{C}|$, a set of morphisms denoted by
$$\mathcal{C}(A, B),$$
the so-called hom-set. Any morphism $f \in \mathcal{C}(A, B)$, has domain A and codomain B and we write down the morphism as
$$f : A \longrightarrow B.$$
3. A composition operation $- \circ -$ such that for morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ we have that $g \circ f \in \mathcal{C}(A, C)$ and moreover, for $h : C \to D$ satisfies the associativity law:
$$h \circ (g \circ f) = (h \circ g) \circ f$$
4. For all objects $A, B \in |\mathcal{C}|$ there exist identities $id_A : A \to A$ and $id_B : B \to B$ satisfying the *identity law*:
$$id_B \circ f = f \qquad \text{and} \qquad f \circ id_A = f$$

The mathematical language that we will be using for categories are the so-called *commutative diagrams*. These are very handy since we can planely see the *type* of the objects when applying morphisms. As a simple example, consider having three objects A, B, C in some category $\mathcal{C}$ and two morphisms between them given by
$$f : A \to B, \quad g : B \to C$$
To begin with, we have an object which has type A, and we apply the morphism f, turn in into type B – then apply apply g finally become type C. But this is the same as applying $g \circ f : A \to C$ to our initial object A. To transform this to a commutative diagram, we have to 'split' the three different processes that we apply within our category. We say that "applying first

f to A and turning in into type B and secondly applying g to finally turn it into type C is the same as simultaneously applying $g \circ f$ to the system we started with". This is exactly the information that is neatly and explicitly stored in the following triangle:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Example A typical example of a category is $\mathcal{S}et$, the category with sets as objects and total functions between sets as morphisms. Restating the above definition to match the category-theoretic structure of the $\mathcal{S}et$ category we have:

1. Objects in $\mathcal{S}et$ are sets.

2. A morphism $f : A \to B$ in $\mathcal{S}et$ is a total function from the set A to another set B.

3. The composition of a total function $f : A \to B$ with another total function $g : B \to C$ is given by the total function $g \circ f : A \to C$, such that each element $a \in A$ is mapped to $g(f(a)) \in C$. Composition of total functions is associative and hence for $h : C \to D$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.

4. For each set in $\mathcal{S}et$, there exist an identity function which has the same domain and codomain. Hence, for sets A and B, we have that $id_A : A \to A$ and $id_B : B \to B$ which satisfy the identity law $id_B \circ f = f$ and similarly $f \circ id_A = f$.

Two other examples that we will be heavily working with in later parts of this dissertation are the $\mathcal{F}dHilb$ category, the category of finite dimensional Hilbert spaces with all Hilbert spaces as objects and linear maps as objects – and the $\mathcal{R}el$ category, with finite sets as objects and relations as morphisms but we postpone this discussion until Chapter 3.

A 'special' kind of morphism is an isomorphism and we now introduce its definition.

Definition 2.1.2. A morphism $f : A \to B$ is an isomorphism if there exist a inverse morphism $f^{-1} : B \to A$, such that $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$.

We say that objects A and B are *isomorphic*, or identical *up to isomorphism*, if there exist an isomorphism between them.

Another important notion, is that of a monoid.

Definition 2.1.3. A monoid $(M, \cdot, 1)$ is a set M equipped with a binary operation which takes pairs of elements of M into M, that is

$$- \cdot - : M \times M \to M.$$

Moreover, we have that for elements $x, y, z \in M$,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$

i.e. $- \cdot -$ is associative and there exists a distinguished element 1 such that

$$1 \cdot x = x \cdot 1 = x.$$

In category-theoretic terms, a monoid $(M, \cdot, 1)$ can be represented as a category with a single object. The elements of M are represented as morphisms from the object to itself, the identity element 1 is the identity morphism and the $- \cdot -$ operation represents morphism composition. Therefore, the hom-set $\mathcal{C}(A, A)$ defines a monoid.

We can also consider the category $\mathcal{M}on$, which has monoids as objects and the so-called monoid homomorphisms[1] as morphisms.

Definition 2.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* between these categories is a map

$$F : \mathcal{C} \to \mathcal{D},$$

such that:

- Objects $A \in |\mathcal{C}|$ are mapped onto $\mathcal{D}$ as

$$F : |\mathcal{C}| \to |\mathcal{D}| :: A \longmapsto F(A)$$

[1] A monoid homomorphism from $(M, \cdot, 1)$ to $(M', \cdot', 1')$ is a function $f : M \to M'$ such that $f(1) = 1'$ and $f(x \cdot y) = f(x) \cdot' f(y)$.

- For any objects $A, B \in |\mathcal{C}|$, a morphism $f \in \mathcal{C}(A, B)$ is mapped as

$$F : \mathcal{C}(A, B) \to \mathcal{D}\left(F(A), F(B)\right) :: f \longmapsto F(f)$$

A functor is said to be a 'structure-preserving map'. That is, it firstly preserves composition – so that for morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ we have

$$F(g \circ f) = F(g) \circ F(f).$$

and secondly preserves identities, such that for $A \in |\mathcal{C}|$,

$$F(id_A) = id_{F(A)}.$$

Example Consider the $\mathcal{S}et$ category. We can define an endo-functor

$$P : \mathcal{S}et \longrightarrow \mathcal{S}et$$

which maps a set (object) A to its usual powerset $P(A)$ and maps $f : A \to B$ to the function $P(f) : P(A) \to P(B)$ which takes $S \subset A$ and returns its image $f(S) \subset B$. We can trivially check that this defined a functor since we have $P(1_A) = 1_{PA}$ and $P(g \circ f) = P(g) \circ P(f)$.

Definition 2.1.5. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between categories $\mathcal{C}$ and $\mathcal{D}$. A *natural transformation*

$$\xi : F \Rightarrow G$$

between functors F and G is a family of morphisms

$$\{\xi_A \in \mathcal{D}(FA, GA) | A \in |\mathcal{C}|\}_A$$

such that, for $A, B \in |\mathcal{C}|$ and $f \in \mathcal{C}(A, B)$, we have the commutation of

$$\begin{tikzcd} F(A) \arrow[r, "\xi_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\xi_B"'] & GB \end{tikzcd}$$

If moreover each component ξ_A of ξ is also an isomorphism in $\mathcal{D}$ then we call ξ a natural isomorphism.

Example If we consider an identity natural transformation $id_F : F \to F$, where $F : \mathcal{C} \to \mathcal{D}$ is a functor, then for all objects $A \in |\mathcal{C}|$,

$$id_{F(A)} = id_A : F(A) \to F(A).$$

In fact, since $id_{F(A)}$ is an isomorphism in $\mathcal{D}$, it is a natural isomorphism.

Definition 2.1.6. A category $\mathcal{C}$ is said to be *monoidal* if it is a structured triple $(\mathcal{C}, \otimes, I)$ such that

1. $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor [2] called the *monoidal product* or *monoidal tensor* or simply *tensor*[3]. For objects $A, B, C, D \in |\mathcal{C}|$ the tensor behaves as

   $$- \otimes - : |\mathcal{C}| \times |\mathcal{C}| \to |\mathcal{C}| :: (A, B) \longmapsto A \otimes B$$

   and for morphisms $f \in \mathcal{C}(A, B),\ g \in \mathcal{C}(C, D)$,

   $$- \otimes - : \mathcal{C}(A, B) \times \mathcal{C}(C, D) \to \mathcal{C}(A \otimes C, B \otimes D) :: (f, g) \longmapsto f \otimes g.$$

2. I is a special object called the *unit* or *identity object*.

3. There exist a *left unit*, a *right unit* and *associativity* natural isomorphisms, which are respectively given by

   $$\lambda : I \otimes A \to A, \quad \rho : A \otimes I \to A, \quad \alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C).$$

   For any map $f : A \to B$, the left and right unit obey the squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\lambda_A^{-1}"'] & B \arrow[d, "\lambda_B^{-1}"] \\ I \otimes A \arrow[r, "id_I \otimes f"'] & I \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\rho_A^{-1}"'] & B \arrow[d, "\rho_B^{-1}"] \\ A \otimes I \arrow[r, "f \otimes id_I"'] & B \otimes I \end{tikzcd}$$

[2]A bifunctor admits two arguments and maps pairs of objects of a category – which we denote by $\mathcal{C} \times \mathcal{C}$ onto the category.

[3]This should not be confused with the Kronecker tensor product – as we will see later it the tensor can also be the Cartesian product or the direct sum, depending in which category we are working.

For objects $A, B, C, D \in |\mathcal{C}|$, the associativity natural isomorphism obeys the so-called ‘pentagon axiom’,

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes id_D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A,B,C\otimes id_D}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

and also the ‘triangle axiom’,

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \oplus id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A \oplus \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

4. Importantly, in a monoidal category we have *bifunctoriality.* That is, for linear maps f, g, h, k of correct type, the equations

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h) \tag{2.1}$$
$$id_A \otimes id_B = id_{A\otimes B}$$

both hold.

Additionally, the tensor is associative and has id_I as its unit, i.e.

$$f \otimes (g \otimes h) = (f \otimes g) \otimes h \quad \text{and} \quad f \otimes id_I = id_I \otimes f = f.$$

In any category, monoidal or not, if we consider the associativity morphism then following diagram

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "f \otimes (g \otimes h)"'] & (A \otimes B) \otimes C \arrow[d, "(f \otimes g) \otimes h"] \\
A' \otimes (B' \otimes C') \arrow[r, "\alpha_{A',B',C'}"'] & (A' \otimes B') \otimes C'
\end{tikzcd}$$

commutes. This condition is the so-called *naturality condition.*

Consider a functor $F : \mathcal{C} \to \mathcal{D}$ acting upon monoidal categories. We say that the functor is *strong* if the natural transformations are isomorphisms and *strict* if they are identities.

In a monoidal category we also have that the left and right unit of the identity object are equal. That is,

$$\lambda_I = \rho_I$$

which makes the following triangles both commute.

$$\begin{tikzcd}
(A \otimes B) \otimes I \arrow[rr, "\alpha_{A,B,I}"] \arrow[dr, "\rho_{A \otimes B}"'] & & A \otimes (B \otimes I) \arrow[dl, "id_A \oplus \rho_B"] \\
& A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
(I \otimes A) \otimes B \arrow[rr, "\alpha_{I,A,B}"] \arrow[dr, "\lambda_A \oplus id_B"'] & & I \otimes (A \otimes B) \arrow[dl, "\lambda_{A \otimes B}"] \\
& A \otimes B &
\end{tikzcd}$$

Natural isomorphisms are very powerful tools, due to the following 'coherence theorem' by Mac Lane.

Theorem 2.1.7. All diagrams that are built up only with natural isomorphisms must commute.

Example The category $\mathcal{S}et$ comes is equipped with monoidal structure $(\mathcal{S}et, \times, *)$, with the Cartesian product $\times$ as the tensor and the singleton set $\{*\}$ as the monoidal unit object. Consider total functions

$$f : X \to Y, \quad f' : X' \to Y',$$

then the composition operation is the usual functional composition and the tensor is given by

$$f \times f' : X \times X \to Y \times Y' :: (x, x') \longmapsto (f(x), f'(x'))$$

Natural isomorphisms are given by

$$\alpha_{X,Y,Z} : X \times (Y \times Z) \to (X \times Y) \times Z :: (x, (y, z)) \longmapsto ((x, y), z)$$
$$\lambda_X : \{*\} \times X \to X :: (*, x) \mapsto x \qquad \rho_X : X \times \{*\} \to X :: (x, *) \mapsto x$$

2.2 Strictness and symmetry

Definition 2.2.1. A *strict monoidal category* is one for which the natural isomorphisms α, λ, ρ are all identities.
Importantly, in any strict monoidal category the following diagram commutes

$$\begin{tikzcd}
A \otimes B \arrow[r, "id_A \otimes g"] \arrow[d, "f \otimes id_B"'] & A \otimes D \arrow[d, "f \otimes id_D"] \\
C \otimes B \arrow[r, "id_C \otimes g"'] & C \otimes D
\end{tikzcd}$$

Reading from the upper-and-right side of the diagram we have:

$$(f \otimes id_D) \circ (id_A \otimes g) \overset{bifunct.}{=} (f \circ id_A) \otimes (id_D \circ g)$$

$$\overset{id.law}{=} f \otimes g \overset{id.law}{=} (id_C \circ f) \otimes (g \circ id_B) \overset{bifunct.}{=} (id_C \otimes g) \circ (f \otimes id_B) \tag{2.2}$$

which is the left-and-down side of the diagram[4].

It is now that we can understand what bifunctoriality gets. Conceptually we can understand this as having a space and time composition for the $- \otimes -$ and $- \circ -$ bifunctors respectively. Objects and morphisms 'separated' by the tensor can in no way affect two different systems, a feature that can be interpreted as expressing a sense of locality. Composed morphisms are applied one after the other, hence express a notion 'time flow'.

Strictness makes life much easier, due to the following theorem by Mac Lane.

Theorem 2.2.2. Every monoidal category is $\mathcal{C}$ 'equivalent' to strict monoidal category $\mathcal{C}'$.

This essentially means that no matter what is the category that we are working in, we can assume that there exist an equivalent one in which all natural isomorphisms are identities. Of course, this would have big simplifications when we will start dealing with more complicated morphisms and commutative diagrams.

[4]The pictorial proof of this is trivial as we will see in in the next section.

Definition 2.2.3. A *braided monoidal category* is a monoidal category equipped with a natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ such that the following two diagrams commute.

$$\begin{tikzcd}
 & A \otimes (B \otimes C) \arrow[r, "\sigma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\sigma_{A,B} \oplus id_C"'] & & & B \otimes (C \otimes A) \\
 & (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[ur, "id_B \oplus \sigma_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[r, "\sigma_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}_{C,A,B}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}_{A,B,C}"] \arrow[dr, "id_A \oplus \sigma_{B,C}"'] & & & (C \otimes A) \otimes B \\
 & A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[ur, "\sigma_{A,C} \oplus id_B"'] &
\end{tikzcd}$$

As an extension of braiding, we now proceed to introduce symmetric monoidal categories.

Definition 2.2.4. A *symmetric monoidal category* is a braided one, which moreover satisfies $\sigma_{A,B} \circ \sigma_{B,A} = id_{A\otimes B}$ or equivalently $\sigma_{A,B} = \sigma^{-1}_{B,A}$. Therefore the diagrams

$$\begin{tikzcd}
A \otimes B \arrow[r, "\sigma_{A,B}"] \arrow[dr, "id_{A\oplus B}"'] & B \otimes A \arrow[d, "\sigma_{B,A}"] \\
& A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\lambda^{-1}_A"] \arrow[dr, "\rho^{-1}_A"'] & I \otimes A \arrow[d, "\sigma_{I,A}"] \\
& A \otimes I
\end{tikzcd}$$

and finally of

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha^{-1}_{A,B,C}"] \arrow[d, "id_A \otimes \sigma_{B,C}"'] & (A \otimes B) \otimes C \arrow[r, "\sigma_{C,A\otimes B}"] & C \otimes (A \otimes B) \arrow[d, "\alpha^{-1}_{C,A,B}"] \\
A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[r, "\sigma_{A,C} \otimes id_B"'] & (C \otimes A) \otimes B
\end{tikzcd}$$

all commute.

Remark All the commutative diagrams introduced so far constitute the so-called coherence conditions.

If we consider a strict symmetric monoidal category we have that the diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "\sigma_{A,B}"] \arrow[d, "f \otimes g"'] & B \otimes A \arrow[d, "g \otimes f"] \\
C \otimes D \arrow[r, "\sigma_{C,D}"] & D \otimes C
\end{tikzcd}$$

commutes.

2.3 Dagger structure and compact closure

Definition 2.3.1. A *compact closed category*, introduced in [15], is a monoidal category which treats dual objects. In this category, every object has a dual[5] isomorphic object. Consider a pair of morphisms;
the unit

$$\eta : I \to B \otimes A$$

and counit

$$\epsilon : A \otimes B \to I.$$

We say that B is *dual* to A and denote it by A^* when we have

$$\lambda_A \circ (\epsilon_A \otimes id_A) \circ \alpha^{-1}_{A,A^*,A} \circ (id_A \otimes \eta_A) \circ \rho_A^{-1} = id_A \tag{2.3}$$

$$\rho^*_A \circ (id^*_A \otimes \epsilon_A) \circ \alpha_{A,A^*,A} \circ (\eta_A \otimes id^*_A) \circ \lambda^{-1}_{A^*} = id^*_A \tag{2.4}$$

These translate in a commutative diagram as follows:

$$\begin{tikzcd}
A \arrow[r, "\rho_A^{-1}"] \arrow[d, "id_A"'] & A \otimes I \arrow[r, "id_A \otimes \eta_A"] & A \otimes (A^* \otimes A) \arrow[d, "\alpha^{-1}_{A,A^*,A}"] \\
A & I \otimes A \arrow[l, "\lambda_A"] & (A \otimes A^*) \otimes A \arrow[l, "\epsilon_A \otimes id_A"]
\end{tikzcd}$$

$$\begin{tikzcd}
A^* \arrow[r, "\lambda^{-1}_{A^*}"] \arrow[d, "id_{A^*}"'] & I \otimes A^* \arrow[r, "\eta_A \otimes id_{A^*}"] & (A^* \otimes A) \otimes A^* \arrow[d, "\alpha_{A^*,A,A^*}"] \\
A^* & A^* \otimes I \arrow[l, "\rho_{A^*}"] & A^* \otimes (A \otimes A^*) \arrow[l, "id_{A^*} \otimes \epsilon_A"]
\end{tikzcd}$$

[5] In fact it has two duals, a left and a right one – but in a braided or symmetric category these are the same. Since we will be dealing primarily with symmetric categories there is no need to differentiate between the two.

Assuming strictness due Mac Lane's equivalence theorem, then the above two equations boil down to

$$(\epsilon_A \otimes id_A) \circ (id_A \otimes \eta_A) = id_A \quad \text{and} \quad (id_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes id_{A^*}) = id_{A^*}$$

Diagrammatically, that is:

$$\begin{tikzcd} A \arrow[d, "id_A \otimes \eta_A"'] \arrow[dr, "id_A"] & \\ A \otimes A^* \otimes A \arrow[r, "\epsilon_A \otimes id_A"'] & A \end{tikzcd} \qquad \begin{tikzcd} A^* \arrow[d, "\eta_A \otimes id_{A^*}"'] \arrow[dr, "id_{A^*}"] & \\ A^* \otimes A \otimes A^* \arrow[r, "id_{A^*} \otimes \epsilon_A"'] & A^* \end{tikzcd}$$

We call a compact structure on an object A a quadruple (A, A^*, η, ϵ).

Moreover, if every object $A \in \mathcal{C}$ in the compact closed category has a chosen dual A^*, one can define a contravariant functor[6]

$$(-)^* : \mathcal{C}^{op} \to C$$

such that objects are mapped as

$$A \longmapsto A^*$$

and morphisms as

$$f : A \to B \longmapsto f^* : B^* \to A^*.$$

We call the morphism f^* the *transpose* of f. Diagrammatically, the $(-)^*$ functor is defined by the commutation of

$$\begin{tikzcd} B^* \arrow[r, "\lambda^{-1}_{B^*}"] \arrow[d, "f^*"'] & I \otimes B^* \arrow[r, "\eta_A \otimes id_{B^*}"] & A^* \otimes A \otimes B^* \arrow[d, "id^*_A \otimes f \otimes id_{B^*}"] \\ A^* & A^* \otimes I \arrow[l, "\rho_{A^*}"] & A^* \otimes B \otimes B^* \arrow[l, "id_{A^*} \otimes \epsilon_B"] \end{tikzcd}$$

[6]A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ reverses the direction of composition, i.e. $F(g \circ f) = Ff \circ Fg$. We can also represent a contravariant functor as functors of type $F : \mathcal{C}^{op} \to \mathcal{D}$, where $\mathcal{C}^{op}$ is the opposite category of $\mathcal{C}$ in which for $f \in \mathcal{C}(A, B)$ we have $f^{op} \in \mathcal{C}(B, A)$ and $(g \circ f)^{op} = f^{op} \circ g^{op}$. Otherwise the opposite category is the same as the normal one.

Equivalently, in equational form is given by

$$f^* = (id_{A^*} \otimes \epsilon_B) \circ (id_{A^*} \otimes f \otimes id_{B^*}) \circ (\eta_A \otimes id_{B^*}),$$

where we have again assumed strictness and hence all natural isomorphisms are neglected.

We will now proceed to define a dagger category. Through the dagger structure we will consider a dagger compact category, initially called 'strongly compact closed category' in [1] – the category which provides us with all the necessary tools to express quantum mechanics.

Definition 2.3.2. A †-*monoidal category* is one which is equipped with an involution on the morphisms, identity on objects, contravariant endofunctor

$$(-)^\dagger : \mathcal{C}^{op} \to \mathcal{C}.$$

This means that all morphisms, e.g. $f : A \to B$ and $g : B \to C$, have their corresponding *adjoint* such that:

$$\begin{aligned} (g \circ f)^\dagger &= f^\dagger \circ g^\dagger :: C \to A \\ f^{\dagger\dagger} &= f :: A \to B \end{aligned}$$

The dagger functor is identity-on-objects, i.e. we have the commutation of

$$\begin{tikzcd} A^\dagger \otimes B^\dagger \arrow[d, "id"'] \arrow[dr, "id"] & \\ (A \otimes B)^\dagger \arrow[r, "id"'] & A \otimes B \end{tikzcd}$$

and also preserves the identities, that is,

$$id_A^\dagger = id_A.$$

Finally, all natural natural isomorphisms of the monoidal structure, namely α, λ, ρ are all unitary[7]. In a †-*symmetric* monoidal category we also require the swapping map to be unitary.

[7]We say that a morphism χ is unitary when its adjoint and inverse coincide, that is, when $\chi^\dagger = \chi^{-1}$

As a result of unitarity, if we consider a compact closed category equipped with the dagger functor, that is, a *dagger compact (closed) category*, we have that the unit and counit isomorphisms

$$\eta_A : I \to A^* \otimes A \quad \text{and} \quad \epsilon_{\mathrm{A}} : \mathrm{A} \otimes \mathrm{A}^* \to \mathrm{I},$$

have corresponding adjoints given by

$$\eta_A^\dagger : A^* \otimes A \to I \quad \text{and} \quad \epsilon_{\mathrm{A}}^\dagger : \mathrm{I} \to \mathrm{A} \otimes \mathrm{A}^*.$$

Therefore, writing down $\eta_{A^*} : I \to A \otimes A^*$ we observe that

$$\eta_{A^*} = \epsilon_A^\dagger \qquad \text{and} \qquad \eta_{A^*}^\dagger = \epsilon_A$$

Moreover, writing down $\epsilon_{A^*}^\dagger : A^* \otimes A \to I$ we observe that

$$\eta_A = \epsilon_{A^*}^\dagger \qquad \text{and} \qquad \epsilon_{A^*} = \eta_A^\dagger$$

In a †-*symmetric compact closed monoidal category*, we also have that

$$\eta_{A^*} = \epsilon_A^\dagger = \sigma_{A^*,A} \circ \eta_A \text{ and } \eta_A = \epsilon_{A^*}^\dagger = \sigma_{A,A^*} \circ \epsilon_A^\dagger \tag{2.5}$$

In a commutative diagram, this is understood by the commutation of

$$\begin{tikzcd}
I \arrow[r, "\epsilon_A^\dagger"] \arrow[dr, "\eta_A"'] & A \otimes A^* \arrow[d, "\sigma_{A,A^*}"] \\
& A^* \otimes A
\end{tikzcd}$$

We can present the above commutative diagram as the definition of the *dagger symmetric compact closed category.*

Similarly to the definition of the transpose of a morphism f, we can now define – using the dagger – a similar covariant functor.

Definition 2.3.3. In a †-symmetric compact closed category, we define a covariant functor

$$(-)_* : \mathcal{C} \to \mathcal{C}$$

which maps objects A to A^* and morphisms $f : A \to B$ to their *conjugate* $f_* : A^* \to B^*$, such that we have the commutation of

$$\begin{tikzcd} A^* \arrow[r, "\lambda_{A^*}^{-1}"] \arrow[d, "f^*"'] & I \otimes B^* \arrow[r, "\eta_B \otimes id_{A^*}"] & B^* \otimes B \otimes A^* \arrow[d, "id_B^* \otimes f^\dagger \otimes id_{A^*}"] \\ B^* & B^* \otimes I \arrow[l, "\rho_{B^*}"] & B^* \otimes A \otimes A^* \arrow[l, "id_{B^*} \otimes \epsilon_A"] \end{tikzcd}$$

That is, we have that

$$f_* = (id_{B^*} \otimes \epsilon_A) \circ (id_{B^*} \otimes f^\dagger \otimes id_{A^*}) \circ (\eta_B \otimes id_{A^*}).$$

It can also be shown that each of the three functors $(-)_*, (-)^*, (-)^\dagger$ can be expressed using the other two, namely

$$f_* = (f^\dagger)^* = (f^*)^\dagger \qquad f^* = (f^\dagger)_* = (f_*)^\dagger \qquad f^\dagger = (f^*)_* = (f_*)^*$$

2.4 Scalars and traces

Definition 2.4.1. In a monoidal category $\mathcal{C}$, a scalar is an endomorphism in the set $\mathcal{C}(\mathcal{I}, \mathcal{I})$.

Since scalars are presented as morphisms $s : I \to I$, we can induce a scalar monoid by the composite

$$I \xrightarrow{\psi} A \xrightarrow{\pi} I$$

Interestingly, we have the following remarkable result for scalars.

Theorem 2.4.2. The scalars $\mathcal{C}(I, I)$ form a monoid which is always commutative. For $s, t \in \mathcal{C}(I, I)$, we have that

$$I \cong I \otimes I \xrightarrow{s \otimes t} I \otimes I \cong I$$

is equal to $s \circ t = t \circ s$.

Proof. We have the commutation of

$$\begin{tikzcd}
I \arrow[d, "t"] & & I \otimes I \arrow[ll, "\rho"'] \arrow[rr, "\rho"] \arrow[dd, "s \otimes t"] \arrow[dr, "s \oplus id_I"] & & I \arrow[d, "s"] \\
I \arrow[r, "\rho^{-1}"] \arrow[d, "s"] & I \otimes I \arrow[ur, "id_I \oplus t"] \arrow[dr, "id_I \oplus s"'] & & I \otimes I & I \arrow[l, "\rho^{-1}"'] \arrow[d, "t"] \\
I & & I \otimes I \arrow[ll, "\rho"] \arrow[rr, "\rho"'] \arrow[ur, "t \oplus id_I"'] & & I
\end{tikzcd}$$

□

We can now generalise this by introducing scalar multiplication.

Definition 2.4.3. Given a scalar $s : I \to I$ and a morphism $f : A \to B$, the scalar multiplication $- \bullet -$, is defined as

$$A \xrightarrow{\rho_A^{-1}} A \otimes I \xrightarrow{f \otimes s} B \otimes I \xrightarrow{\rho_B} B.$$

such that the diagram

$$\begin{tikzcd}
& A \otimes I \arrow[r] \arrow[dd, "u"] & B \otimes I \arrow[dd, "u"] \arrow[dr, "\rho_B"] & \\
A \arrow[ur, "\rho_A^{-1}"] \arrow[dr, "\lambda_A^{-1}"'] & & & B \\
& I \otimes A \arrow[r] & I \otimes B \arrow[ur, "\lambda_B"'] &
\end{tikzcd}$$

commutes[8].

[8] $u := \lambda^{-1} \circ \rho$ is a natural isomorphism.

It can also be proven [6] that for $s, t \in \mathcal{C}(I, I)$ and $f : A \to B, g : B \to C$

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g) \quad \text{and} \quad (\mathrm{s} \bullet \mathrm{f}) \otimes (\mathrm{t} \bullet \mathrm{g}) = (\mathrm{s} \circ \mathrm{t}) \bullet (\mathrm{f} \otimes \mathrm{g}). \tag{2.6}$$

So we proved that scalars do not respect the order of composition or tensor. In the same spirit but in slightly different context if we consider the composite

$$A \xrightarrow{\rho_A^{-1}} I \otimes A \xrightarrow{\psi \otimes \pi} B \otimes I \xrightarrow{\lambda_A} B \tag{2.7}$$

we see that $\psi \circ \pi = (\psi \otimes \pi)$.

Another important operation in quantum mechanics is the trace. Formal treatment was been given by Joyal, Street and Verity in [14].

Definition 2.4.4. In a compact closed monoidal category $\mathcal{C}$, given a morphism $f : A \otimes C \to B \otimes C$ a *trace* is defined as

$$tr^C_{A,B} := \rho_B^{-1} \circ (id_B \otimes \epsilon_C) \circ (f \otimes id_{C^*}) \circ (id_A \otimes \epsilon_C^\dagger) \circ \rho_A$$

Diagrammatically this is understood by the commutation of

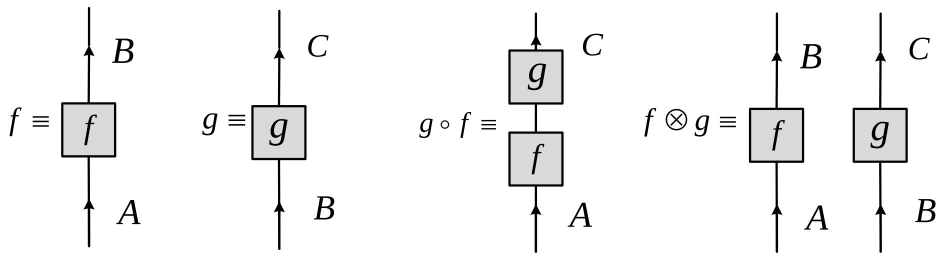
$$\begin{tikzcd}
A \arrow[r, "\rho_A"] \arrow[d, "tr^C_{A,B}"'] & A \otimes I \arrow[r, "id_A \otimes \epsilon_C^\dagger"] & A \otimes C \otimes C^* \arrow[d, "f \otimes id_{C^*}"] \\
B & B \otimes I \arrow[l, "\rho_B^{-1}"'] & B \otimes C \otimes C^* \arrow[l, "id_B \otimes \epsilon_C"']
\end{tikzcd}$$

2.5 Graphical calculi for monoidal categories

The most powerful feature for monoidal categories is that there exist a direct translation of the commutative diagrams involved, to pictures. For a complete description of graphical calculus for all different kinds of monoidal categories one may consult [19] by Selinger.

2.5.1 Morphisms, composition and tensor in pictures

We represent a morphism as a box, which has one input and one output line. The *type* of the morphism is shown in the input and output 'wire'. Composition of morphisms is performed by connecting the output wire of a morphism with the input one of another morphism, provided that the types match. The convention that is been used is having composition moving upwards – hence we read diagrams from bottom to top. The tensor depicts as having boxes side-by-side. For morphisms $f : A \to B$ and $g : B \to C$ we have:



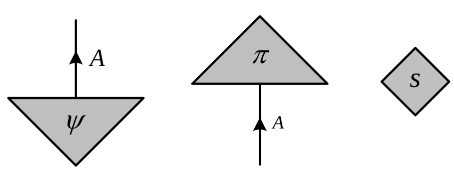
However, there are some other 'special' kind of morphisms, those that involve the domain or codomain of the morphism being the monoidal identity object I. Morphisms of the form

$$\psi : I \to A$$

are called *states* – these have no input and only one output and are represented by a triangle. Morphisms of the form

$$\pi : A \to I$$

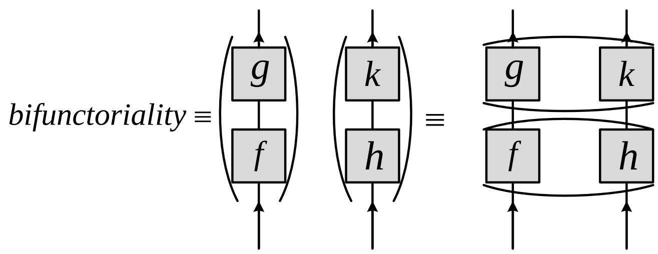
are called *co-states* or *preparation states* and have a single input, no output and are represented by an upside down triangle. Finally, morphisms of the form $s : I \to I$ are the so-called *scalars*; these have no input or output wires and are represented by diamonds. These are pictorially represented as



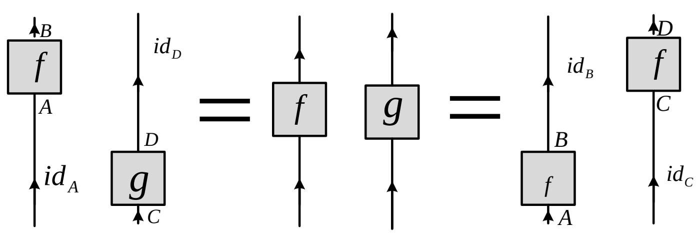
It is remarkable that this simple pictorial notation *directly* translated to maths. Consider the bifunctoriality equation 2.1;

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h).$$

It is trivial to see that both the left and right hand side of this equation yield the same pictorial result. For $f : A \to B$, $g : B \to C$, $k : D \to E$ and $h : E \to F$ the graphical equation is



We see that already the graphical calculus implicitly carries non-trivial graphical rules. As a second example, consider the 'strictness' equation 2.2.1 in which we already hinted at its simplicity. What we basically wrote down is:



That is, morphisms can freely 'slide' along lines.

2.5.2 Associator, symmetry and compact closure in graphical notation

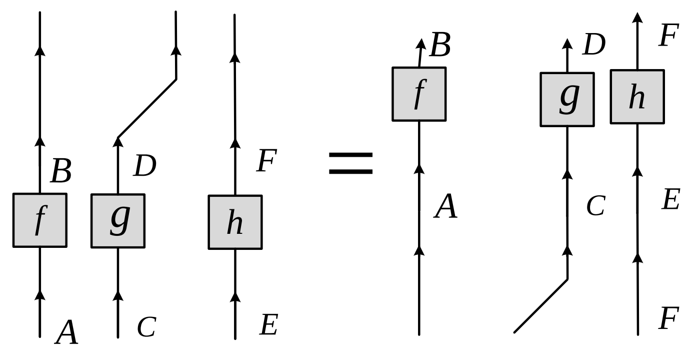
Consider now the associativity morphism

$$\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

applied to three morphisms $f : A \to B, g : C \to D$, and h : E $\to$ F. We depict the equivalence of

$$\alpha_{D,E,F} \circ (f \otimes (g \otimes h)) = ((f \otimes g) \otimes h) \circ \alpha_{A,B,C}$$

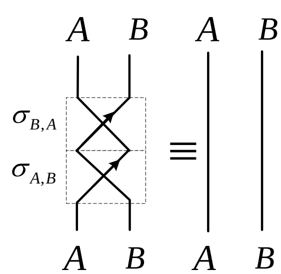
as



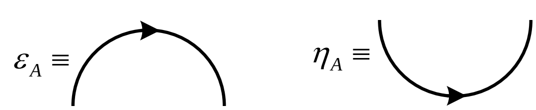
Now, since we will be mostly dealing with † compact symmetric categories, we moreover have symmetry. Recall that symmetry means we have a swap map;

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$

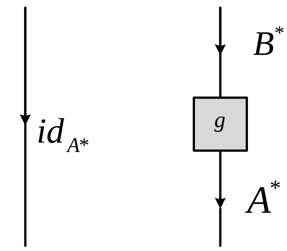
which is involutive. Hence, if we consider $\sigma_{B,A} \circ \sigma_{A,B}(A \otimes B) = id_A \otimes id_B$; i.e. in pictures we have:



The compact closure structure can be depicted by means of cups and caps. The maps $\eta : I \to A^* \otimes A$ and $\epsilon : A \otimes A^* \to I$ respectively depict as



We can see from these morphisms that the downward direction of arrows at the edges correspond to the 'dual part' of the morphism . In fact, identities $id_{A^*} : A^* \to A^*$ in the dual space depict as lines with arrows pointing downwards, and where we have morphisms that involve duals, for example

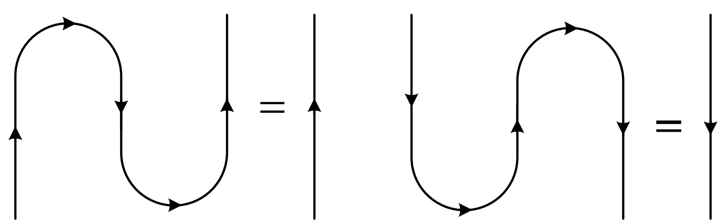


$g : B^* \to A^*$ we draw them with arrows pointing downwards. For example, $(id_{A^*} \otimes g)$ will depict as shown above.

To unveil some power of the graphical calculus, consider now the conditions for strict compact closed structure. Recall that these were given by

$$(\epsilon_A \otimes id_A) \circ (id_A \otimes \eta_A) = id_A \quad \text{and} \quad (id_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes id_{A^*}) = id_{A^*}.$$

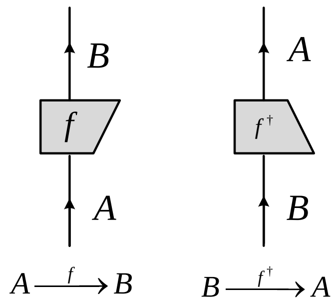
Rather surprisingly, these conditions translate to pictures as follows.



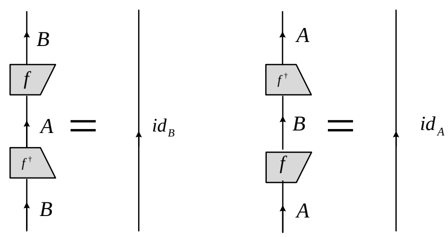
As we can see, the compact closed criterion basically boils down to the graphical rule of being able to straighten a bended line.

2.5.3 Adjoint, transpose and conjugate

In a similar sense we proceed to introduce graphical notations for the adjoint, the transpose and the conjugate. In † categories, we have the obvious $(-)^\dagger$ functor, such that for a morphism $f : A \to B$, $f^\dagger : B \to A$. In the graphical calculus we can account for this by introducing an assymetry in the square boxes that represent morphisms. A morphism f will henceforth have a negative gradient line on the right-hand-side, and its adjoint will be its reflection on a vertical axis, but keeping the arrows pointing in their original direction.



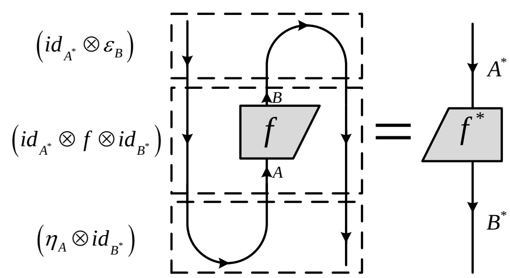
When $f : A \to B$ is unitary, we have that $f^\dagger \circ f = id_A$ and $f \circ f^\dagger = id_B$. So in pictures will correspond to the rule saying that when a unitary map meets its dagger counterpart, they will 'annihilate' and yield identity.



Moreover, we can take advantage of this assymetry to depict the already introduced transposed and conjugate functors. Recall the transpose:

$$f^* = (id_{A^*} \otimes \epsilon_B) \circ (id_{A^*} \otimes f \otimes id_{B^*}) \circ (\eta_A \otimes id_{B^*})$$

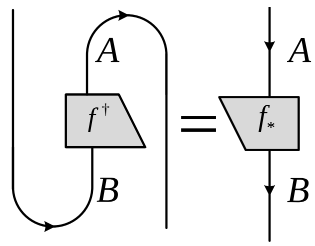
Therefore this will now depict as:



Notice that if we take the left picture and 'pull' the line so the bend in the middle straightens, we get the orientation of the box depicted on the right. Similarly for the conjugate functor,

$$f_* = (id_{B^*} \otimes \epsilon_A) \circ (id_{B^*} \otimes f^\dagger \otimes id_{A^*}) \circ (\eta_B \otimes id_{A^*})$$

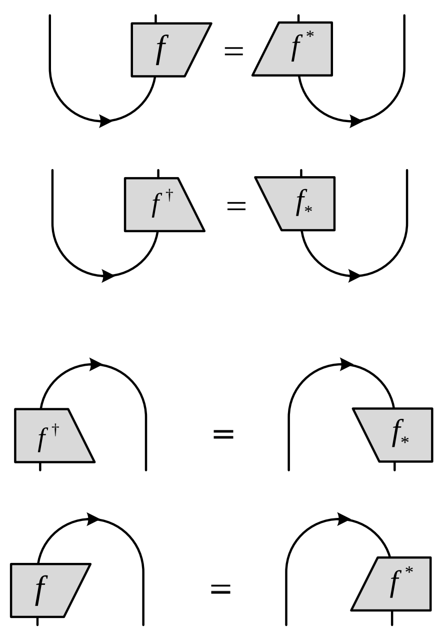
we have that the translation in picture is given by



Of course, the definition of the transpose and conjugate morphisms is not random. It is due to the following equivalences that we can establish the above equalities. Indeed, for $f : A \to B$ consider the following four equations [12]:

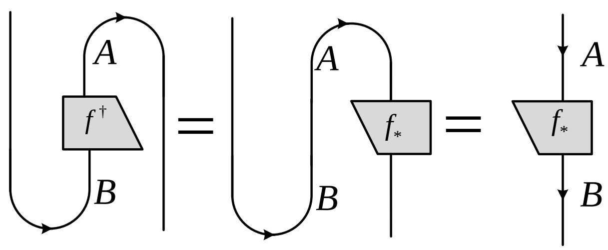
$$\begin{aligned}(id_{A^*} \otimes f) \circ \eta_A =& \quad (f^* \otimes id_B) \otimes \eta_B \\ (id_{B^*} \otimes f^\dagger) \otimes \eta_B =& \quad (f_* \otimes id_A) \circ \eta_A \\ \epsilon_B \circ (f \otimes id_{B^*}) =& \quad \epsilon_A \circ (id_A \otimes f^*) \\ \epsilon_A \circ (f^\dagger \otimes id_{A^*}) =& \quad \epsilon_B \circ (id_B \otimes f_*)\end{aligned}$$

In graphical calculus, these respectively correspond to the following pictorial equations.



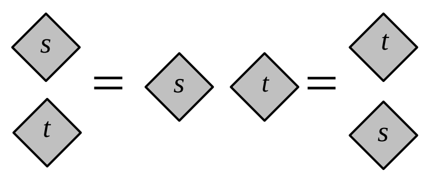
That is, morphisms can 'slide' along the unit and counit morphisms as well – and in the process, due to the assymetry, we can directly see how the morphisms evolve by the orientation of the box.

Hence, what we did above was slide the box through the counit morphism, and due to the compact structure the middle of the pictures below becomes identity.



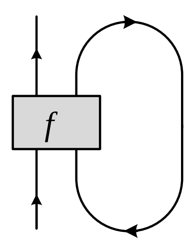
2.5.4 Scalars and trace pictorially

Recall Theorem 2.4.2, that scalars the scalar monoid is always commutative. This translate in the graphical calculus as:



This means that the scalars can freely move around the picture – which can be seen as a consequence of not having any input or output lines.

Traces as in Definition 2.4.4 depict as



While all these graphical calculi may seem trivial and not very formal, Selinger proved otherwise, as demonstrated in [19] and expressed the following theorem:

Theorem 2.5.1. A well-typed equational statement in the language of dagger compact categories holds if and only if it is derivable in the graphical calculus.

So it turns out that what we get is the exact opposite! The graphical calculus and the equational statements in monoidal categories are exactly equivalent and so we can choose whichever one we like depending on one's taste. If you are a pure mathematician and enjoy long and hard equations, then you can choose the equational way of expressing category theory; if you like things to be as simple and plain as they get then the pictorial calculus is a clear choice.

Chapter 3

Category of Hilbert spaces and Relations

In this dissertation we will be mainly working within two categories. The first one is category of finite dimensional Hilbert spaces with Hilbert spaces as objects and morphisms as linear maps between them. The second one is the category of relations; with finite sets as objects and relations as morphisms. We begin our investigations with the former.

3.1 Hilbert space formalism

3.1.1 ..as in quantum mechanics

At the very core of quantum mechanics lies the Hilbert space formalism, developed in the 1930s by John Von Neumann and Paul Dirac, with contributions from other great minds of the era. We will be firstly defining the notion of a Hilbert space, then stating the four postulates of quantum mechanics and subsequently discussing some of its important features.

Definition 3.1.1. A Hilbert space $\mathcal{H}$ is a vector space over the complex number plane $\mathbb{C}$, which is equipped with an inner product mapping pairs of elements of the 'state space' onto the complex number plane, i.e.

$$\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

This satisfies the usual inner-product axioms [11].

Postulate 3.1.1. (Superposition principle) Each physical system is represented by a finite dimensional Hilbert space – its state space – in which quantum states are vectors.

A typical example of a quantum state is the qubit, which 'lives' in a complex two-dimensional Hilbert space. Using Dirac notation we write vectors as a linear combination of a base vectors

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

It is a fundamental property of quantum mechanics that we allow superposition of states – a feature that is believed to be the source of the speed-up of quantum algorithms with reference their classical counterparts.

Postulate 3.1.2. (Born rule) To each physically measurable quantum there exist a corresponding operator O. This operator is self-adjoint and its eigenvectors form a complete orthonormal basis. The expectation value of O is given by $tr(\rho O)$ where ρ is an operator representing the state of the system.

For example the momentum operator is given by $p = -i\hbar\frac{\partial}{\partial x}$.

Postulate 3.1.3. (Projection postulate) Mixed systems are represented by the statistical operator ρ. After we perform the measurement P, the statistical operator is given by

$$\rho' = \frac{P_k \rho P_k}{tr(\rho P_k)}$$

Postulate 3.1.4. (Compound systems) Physical systems composed by more than one subsystems are described by the Hilbert space constructed by the tensor product $\otimes$ of their respective subsystem Hilbert spaces.

It is the tensor product that provides all the 'quantumness' in the Hilbert space formalism. States of the form

$$|\Psi_{Bell}\rangle = |00\rangle + |11\rangle,$$

are called *entangled*, since they cannot be decomposed to their constituent quantum systems and must be perceived as single system. While we can use Dirac notation to denote quantum states, we can also use the 'underlying' field $\mathbb{K}$ and denote states as matrices. For example, the above Bell state can be written as

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where the $\otimes$ is the usual Kronecker tensor product.

Another important feature of quantum theory is the so-called no-cloning theorem.

Theorem 3.1.2. There is no unitary operator that can copy an unknown quantum state.

Proof. Let Δ be a unitary copying operator, $|\psi\rangle$ and $|\phi\rangle$ some unknown quantum states that we wish to copy. The operation that we would perform are

$$\Delta|0\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\psi\rangle$$
$$\Delta|0\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\phi\rangle$$

where $0\rangle$ is a quantum state to be replaced by the unknown quantum state. Now, since we know that unitary operators preserve the inner product, we have that

$$\langle\phi|\psi\rangle = \langle 0\phi|0\psi\rangle = \langle 0\phi|\Delta^{\dagger}\Delta|0\psi\rangle = \langle\phi\phi|\psi\psi\rangle = \langle\phi|\psi\rangle\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2$$

So we have showed that $\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2$. This can only be true when $\langle\phi|\psi\rangle$ is either zero or one – and both of these cases describing elements of a known orthonormal basis. □

As an extension of the no-cloning theorem, we also have the following theorem.

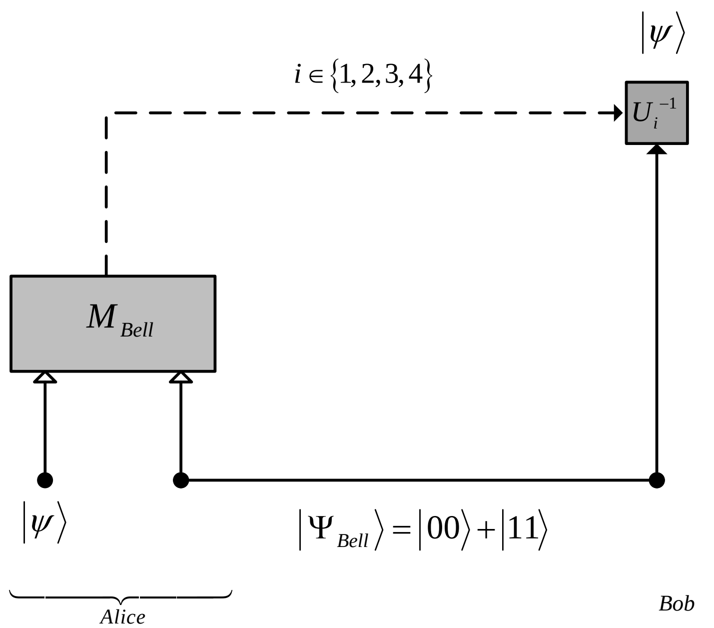
Theorem 3.1.3. There is no unitary operator that can uniformly delete a quantum state. That is, there is no $\Gamma : |\psi\rangle \mapsto 1$

Proof. See [16]. □

These two features of quantum theory are in total contrast with the classical world of information, where copying and deleting data is of course performed with a mouse click. This provides as with a distinct and explicit way to differentiate the quantum world from the classical world.

Another fundamental feature quantum computation and hence of quantum theory is the quantum teleportation protocol. It makes use of the non-local correlations of entanglement to teleport a single quantum state with the aid of entangled Bell-states, for example,

$$|\Psi_{Bell}\rangle = |00\rangle + |11\rangle.$$



To protocol is as follows. Alice has the input state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ that she wants to teleport to Bob. Both Alice and Bob have 1 qubit of the entangled Bell-state. Initially, the state of the whole quantum system is given by

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (|00\rangle + |11\rangle)$$

Subsequently Alice performs a Bell-base measurement on the unknown quantum state ψ and on the single qubit of the Bell-state that she possesses. That is, we perform

$$(|\Psi_i\rangle\langle\Psi_i| \otimes id) \otimes (|\psi\rangle \otimes |00\rangle + |11\rangle)$$

The measurement outcome has four possible outcomes; since the measurement $|\Psi_i\rangle\langle\Psi_i|$ has four possible values. Depending on what is observed we sent by means of classical communication two classical bits to Bob, in order to apply the inverse of that measurement outcome, thus recovering the initial state $|\phi\rangle$.

As already mentioned, since quantum theory does not allow copying, or broadcasting, teleportation is the only viable solution to send information. Today, quantum teleportation has been experimentally demonstrated [5] and is expected to be at the very core of the quantum computer; if one is to be ever built.

3.1.2 ..as a category

As already mentioned in the introduction of this dissertation, when we transfer from traditional Hilbert space formalism to category-theoretic quantum mechanics we come to the world of dagger categories and so we present the following theorem.

Theorem 3.1.4. The category of finite dimensional Hilbert spaces, $\mathcal{F}dHilb$, with Hilbert spaces as objects and linear maps[1] as morphisms is dagger symmetric compact closed.

Proof. [12] Strictness, symmetry and monoidal structure is trivial to show. The monoidal tensor is the usual tensor product of Hilbert spaces and the monoidal unit I is the 'underlying' structure of the Hilbert space, which is the complex plane $\mathbb{C}$.

Now consider the adjoint of a linear map $f : A \to B$. Let $\psi : I \to A$ and $\phi : I \to B$ be states, hence we have

$$\langle f \circ \psi | \phi \rangle = \langle \psi | f^\dagger \circ \phi \rangle.$$

since $\langle f \circ \psi | \phi \rangle = (f \circ \psi)^\dagger \circ \phi = \psi^\dagger \circ f^\dagger \circ \phi = \langle \psi | f^\dagger \circ \phi \rangle$.

To prove that $(-)^\dagger$ is a strict monoidal functor, consider a pair of morphisms f and g such that

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

Hence, for some states a and c we have

$$\langle (gf)^\dagger c | a \rangle = \langle c | gfa \rangle \qquad \Rightarrow (fg)^\dagger = g^\dagger f^\dagger$$

and also

$$\langle a | 1_A a \rangle = \langle 1_A a | a \rangle \qquad \Rightarrow 1_A^\dagger = 1_{A^\dagger} = 1_A.$$

Since we assume strictness, linear maps α, ρ, λ are all identities and hence all self-inverse

$$\alpha^{-1} = \alpha^\dagger \qquad \rho^{-1} = \rho^\dagger \qquad \lambda^{-1} = \lambda^\dagger.$$

[1] A linear operator between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is defined as the map $f : \mathcal{H}_1 \to \mathcal{H}_2$ such that $f(c_1 \cdot \psi_1 + c_2 \cdot \psi_2) = c_1 \cdot f(\psi_1) + c_2 \cdot f(\psi_2)$, where $c_1, c_2 \in \mathbb{C}$ and $\psi_1 \in \mathcal{H}_1, \psi_2 \in \mathcal{H}_2$.

To show that the swapping map satisfies $\sigma^{-1}_{A,B} = \sigma^{\dagger}_{A,B}$ is suffices to note that

$$\langle b \otimes a | \sigma_{A,B}\,(a \otimes b)\rangle = \langle \sigma^{-1}_{A,B}\,(b \otimes a)\,| a \otimes b\rangle.$$

Now consider

$$\begin{aligned}
\langle (f \otimes g)^{\dagger}\,(b \otimes c)\,| a \otimes b'\rangle &= \langle b \otimes c | (f \otimes g)(a \otimes b')\rangle \\
&= \langle b \otimes c | (fa \otimes gb')\rangle \\
&= \langle b | fa\rangle \langle c | gb'\rangle \\
&= \langle f^{\dagger} b | a\rangle \langle g^{\dagger} c | b'\rangle \\
&= \langle (f^{\dagger} \otimes g^{\dagger})(b \otimes c) | a \otimes b'\rangle
\end{aligned}$$

and hence $(-)^{\dagger} \otimes (-)^{\dagger} = (- \otimes -)^{\dagger}$, which completes the proof of the dagger being a strict monoidal functor.

Compact closure is guaranteed by defining the unit $\eta_{\mathcal{H}} : I \to \mathcal{H}^{*} \otimes \mathcal{H}$ as

$$\eta_{\mathcal{H}} : 1 \mapsto \sum_i |a_i\rangle \otimes |a_i\rangle$$

where $\{a_i\}_i$ forms a basis. For $\epsilon_{\mathcal{H}} : \mathcal{H} \otimes \mathcal{H}^{*} \to I$, suppose $\psi, \phi \in \mathcal{H}$; then

$$\begin{aligned}
\langle \phi \otimes \phi | \eta_A\rangle &= \langle \psi \otimes \phi | \textstyle\sum_i a_i \otimes a_i\rangle \\
&= \textstyle\sum_i \langle \psi | a_i\rangle \langle \phi | a_i\rangle \\
&= \langle \phi | \psi\rangle
\end{aligned}$$

and therefore the counit is the linear map given by

$$\epsilon_{\mathcal{H}} : \psi \otimes \phi \mapsto \langle \psi | \phi\rangle.$$

that is,

$$\epsilon_{\mathcal{H}} : \sum_i \langle a_i| \otimes \langle a_i| \mapsto 1$$

Hence, we have the commutation of

$$\begin{tikzcd}
\psi \arrow[r, "id_A \otimes \eta_A"] \arrow[dr, "id_A"'] & \psi \otimes \left(\sum_i a_i \otimes a_i\right) = \sum_i \psi \otimes a_i \otimes a_i \arrow[d, "\epsilon_A \otimes id_A"] \\
& \sum_i \langle \psi | a_i\rangle a_i = \psi
\end{tikzcd}$$

which suffices for compact closed structure.

3.2 Category of Relations

Consider now the category of relations, with relations as morphisms and finite sets as objects. But to begin with, let us first provide the definition of a relation.

Definition 3.2.1. Let X and Y be finite sets. A relation R from X to Y is a collection of ordered pairs of elements of their cartesian product $X \times Y$. In this sense, we say that a relation is just a subset of their cartesian product, i.e.

$$R \subseteq X \times Y.$$

Formally, we write a relation by its graph

$$R := \{(x, y) | x \in X, y \in Y\}$$

and we say that x is related to y and write that xRy to express exactly that.

However, instead of dealing with the graphs, we can see the relations as matrices, where the entries coincide with elements drawn from the two element Boolean semiring $\mathbb{B} = (\{0, 1\}, \wedge, \vee)$. This is the formal way of saying that having a relation $R : X \to Y$ where X has j elements and Y has i elements, then this relation would be written as a matrix with i rows and j columns. If some $x \in X$ is related to $y \in Y$ then their corresponding entry in matrix would equal 1 and if not, would be equal to 0. Multiplication and addition of entries is in accord with Boolean $\vee$ and $\wedge$ respectively.

Example Since we will primarily dealing with endo-relations, i.e. relations over the same set, consider the following relation for $X := \{a, b, c\}$

$$\begin{tikzcd} a \arrow[r] \arrow[dr] & a \\ b \arrow[r] & b \\ c & c \end{tikzcd}$$

This relation is just $R = \{(a, a), (a, b), (b, a)\} \subseteq X \times X$. In matrix form, we can present this as

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rows of the matrix express the domain and the columns the codomain of the relation.

We now proceed to introduce the category-theoretic axiomatization of relations.

Definition 3.2.2. The category of relations comes with a monoidal structure $(\mathcal{R}el, \times, \{*\})$ in which:

1. Objects $A, B, C..$ in $\mathcal{R}el$ are sets.

2. Morpshisms in $\mathcal{R}el$ are relations, $R : A \to B$.

3. A composition operation $- \circ -$, so that for $R : A \to B \subseteq A \times B$ and $R' : B \to C \subseteq B \times C$ the composite $R' \circ R \subseteq A \times C$ is given by

$$R' \circ R = \{(a, c)|\exists b \in B \quad \text{s.t} \quad \text{aRb} \quad \text{and} \quad \text{bR}'\text{c}\}$$

   If we use matrices, then the relational composition boils down to multiplying the corresponding matrices that express the relations involved. The identity relation is

$$id_A = \{(a, a)|a \in A\}$$

4. The monoidal unit is the singleton set $\{*\}$.

5. The monoidal tensor is simply the Cartesian product. Hence, for objects $A, B \in |\mathcal{R}el|$ their tensor would be pairs of elements from each system,

$$A \times B = \{(a, b)|a \in A, b \in B\}$$

   and for morphisms $R_1 : A_1 \to B_1$ and $R_2 : A_2 \to B_2$,

$$R_1 \times R_2 = \{((a, a'), (b, b'))|aR_1b \text{ and } \text{a}'\text{R}_2\text{b}'\} \subseteq (A_1 \times A_2) \to (B_1 \to B_2)$$

6. The left and right unit natural isomorphisms are given b y

$$\lambda_A = \{(a, (a, *))|a \in A\}\,, \quad \rho_A = \{(a, (*, a))|a \in A\}$$

   and also the associativity natural isomorphism

$$\alpha_{A,B,C} = \{((a, (b, c)), ((a, b), c))|a \in A, b \in B, c \in C\}$$

   which make all the necessary coherence conditions hold.

Similarly to the structure of the category of Hilbert spaces, $\mathcal{R}el$ is itself a dagger symmetric compact category. Firstly, to enable symmetry it suffices to define a swapping natural isomorphism

$$\sigma_{A,B} := \{((a,b),(b,a))|x \in X \ and y \in Y\}.$$

Secondly, $\mathcal{R}el$ also comes with compact closed structure, and the relations

$$\eta_A = \{(*,(a,a))|a \in A\}$$

and

$$\epsilon_A = \{((a,a),*)|x \in X\}$$

provide the unit and counit morphisms respectively. Objects $A \in |\mathcal{R}el|$ are self-dual.

Finally, $\mathcal{R}el$ is equipped with a dagger functor. If we consider a relation $R : A \to B$ then its *relational converse* is the relation $R^U : B \to A$ given by

$$\{(b,a)|aRb\}$$

To verify the dagger structure, we have that

$$(R \times R')^\dagger = \{((b,b'),(a,a'))|aRb \ and \ a'R'b'\} = R^\dagger \times R'^\dagger,$$

as required.

Unitarity of all natural isomorphisms follows trivially as the inverse of all these morphisms is the relational converse which establishes

$$a^{-1} = a^\dagger, \quad \rho^{-1} = \rho^\dagger, \quad \lambda^{-1} = \lambda^\dagger, \quad \sigma^{-1} = \sigma^\dagger.$$

3.3 Similarities of $\mathcal{F}dHilb$ and $\mathcal{R}el$

While the two categories express different theories, we encounter many similarities between them. Firstly, the obvious one is that they both admit matrix calculations; in $\mathcal{F}dHilb$ over the field $\mathbb{K}$ and in $\mathcal{R}el$ over a Boolean semiring. Both categories have isomorphisms over their respective underlying structure, namely

$$\mathcal{H} \otimes \mathbb{C} \cong \mathcal{H} \qquad A \times \{*\} \cong A.$$

Within the pictorial categorical framework this trivially follows from the fact that the identity object I is represented by nothing.

Moreover, both theories admit a 'map-state duality'. In $\mathcal{F}dHilb$ states, a state $|\psi\rangle$ can be also understood as the linear map

$$f_{|\psi\rangle} = \mathbb{C} \to \mathcal{H} :: 1 \mapsto |\psi\rangle$$

and similarly in $\mathcal{R}el$ we can express a state as the relation

$$R : \{*\} \to A$$

but since relations are multi-valued [6, 7] , what we actually get is the subsets $B \subseteq A$. Hence if we set

$$B_i := A \ \text{ if } \ \text{i} \in \text{B} \qquad \text{and} \qquad \text{B}_\text{i} := \{\ \} \ \text{ if } \ \text{i} \notin \text{B}$$

then one can see that a relation can be decomposed as

$$B = \bigcup_{i \in A} B_i \cap \{i\}$$

i.e. we can have state decomposition over some bases, in the same way we decompose a quantum state over the set of orthonormal bases:

$$|\psi\rangle = \sum_{i \in A} \psi_i |a_i\rangle$$

Now, let us consider the ability to copy data in both $\mathcal{F}dHilb$ and $\mathcal{R}el$. But first, what do we mean by copying in a category-theoretic sense? The notion of a *natural diagonal* in category theory provides us with a formal way to express copying. We define a copying operation as the family of natural transformation given by

$$\{\Delta : A \longrightarrow A \otimes A\}_A$$

which has to satisfy the obvious square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\ A \otimes A \arrow[r, "f \otimes f"'] & B \otimes B \end{tikzcd}$$

That is, copying and then performing an action f on the copies is the same as first applying the action and subsequently performing the copying operation.

In Hilbert space formalism, we would be tempted to define

$$\Delta_{|\psi\rangle} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |\psi\rangle \to |\psi\rangle \otimes |\psi\rangle$$

However, note that

$$\Delta_{|\psi\rangle} \sum_i c_i |a_i\rangle = \left(\sum_i c_i |a_i\rangle\right) \otimes \left(\sum_i c_i |a_i\rangle\right) = \sum_{ij} c_i c_j |a_i a_j\rangle \neq \sum_i c_i \Delta_{|\psi\rangle} a_i$$

In other words, the map $\Delta_{|\psi\rangle}$ is not linear. However, we can define a map that copies the base vectors, i.e.

$$\Delta_i : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |a_i\rangle \to |a_i\rangle \otimes |a_i\rangle$$

but this map is base-dependent and as a counterexample the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "1 \mapsto |0\rangle + |1\rangle"] \arrow[d, "1 \mapsto 1 \otimes 1"] & \mathbb{C} \oplus \mathbb{C} \arrow[d, "\begin{array}{l} |0\rangle \mapsto |0\rangle \otimes |0\rangle \\ |1\rangle \mapsto |1\rangle \otimes |1\rangle \end{array}"] \\
\mathbb{C} \otimes \mathbb{C} \arrow[r, "1 \otimes 1 \mapsto (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)"'] & (\mathbb{C} \oplus \mathbb{C}) \otimes (\mathbb{C} \oplus \mathbb{C})
\end{tikzcd}$$

fails to commute. We can think of this as the abstract counterpart of the no-cloning theorem.

Interestingly, we observe the same phenomenon in $\mathcal{R}el$. Here, the diagonal function is the relation

$$\{(x, (x, x)) |\ x \in X\}$$

which establishes $X \to X \times X :: x \longmapsto (x, x)$. However, again, the square

$$\begin{tikzcd}
\{*\} \arrow[r, "\{(*,0),(*,1)\}"] \arrow[d, "\{(*,(*,*)\}"'] & \{0,1\} \arrow[d, "\{(0,(0,0)),(1,(1,1))\}"] \\
\{(*,*)\} = \{*\} \times \{*\} \arrow[r, "\{(*,0),(*,1)\} \times \{(*,0),(*,1)\}"'] & \{0,1\} \times \{0,1\}
\end{tikzcd}$$

fails to commute.

Chapter 4

Quantum and classical structuralism in categories

In this chapter we will be strictly dealing with †-SMC since it is the category which 'hosts' finite dimensional Hilbert spaces as well as proving to be the structure of the category of Relations. Since this category is also compact closed we have the usual morphisms

$$\eta : I \to A^* \otimes A \ \text{ and } \ \epsilon : \mathrm{A} \otimes \mathrm{A}^* \to \mathrm{I}$$

which we will now exploit to first define quantum structures and prove that these naturally arise within the context of classical structures.

Perhaps the most distinct characteristic of quantum theory is that it does not allow for states to be copied. This is in full contrast with the classical theory of information and it is the this distinct difference that we will exploit to define quantum and classical structures.

We begin by introducing the definitions of of a quantum structure, subsequently introducing Frobenius algebras, showing their categorical axiomatization and finally providing the definition of a classical structure.

4.1 Quantum structures

A quantum structure [4] in a category $\mathcal{C}$ to be a pair $(X, \eta : I \to X \otimes X)$ Hence, the compact structure for this pair will just be $(X, X, \eta, \eta^\dagger)$, with X being self-dual.

4.2 Classical structures

4.2.1 Preliminaries: Frobenius algebras

An abstract representation of Frobenius algebras in a monoidal category $(\mathcal{C}, \otimes, I)$ –which we call a frobenius object – is a quintible $(A, m, e, \delta, \epsilon)$ where A is an object in $\mathcal{C}$ and

$$m_A : A \otimes A \to A, \quad e_A : I \to A, \quad \delta_A : A \to A \otimes A, \quad \epsilon_A : A \to I.$$

called the *multiplication*, *multiplicative unit*, *comultiplication* and *comultiplicative unit* respectively. A frobenius algebra satisfies:

1. Identity law:

$$m_A \circ (e_A \otimes id_A) = m_A \circ (id_A \otimes e_A) = id_A$$
$$(\epsilon_A \otimes id_A) \circ \delta_A = (id_A \otimes \epsilon_A) \circ \delta_A = id_A$$

2. Associativity and coassociativity law:

$$m_A \circ (m_A \otimes id_A) = m_A \circ (id_A \otimes m_A)$$
$$(\delta_A \otimes id_A) \circ \delta_A = (id_A \otimes \delta_A) \circ \delta_A$$

3. Commutativity and cocommutativity law:

$$m_A \circ \sigma_A = m$$
$$\sigma_A \circ \delta_A = \delta_A$$

4. Frobenius law:

$$(m_A \circ id_A) \circ (id_A \otimes \delta_A) = (id_A \otimes m_A) \circ (\delta_A \otimes id_A) = \delta_A \circ m_A$$

5. If it is special(or isometric), it satisfies

$$m \circ \delta_A = id_A$$

Formally, in category-theoretic terms, the triples (A, m, e) and (A, δ, ϵ) express the notions of an internal commutative monoid and internal co-commutative co-monoid.

Definition 4.2.1. The triple (A, m_A, e_A) defines a *internal commutative monoid*, such that the diagrams

$$\begin{tikzcd}
A & & A \otimes A \arrow[ll, "m_A"'] \\
A \otimes A \arrow[u, "m_A"] & & A \otimes A \otimes A \arrow[ll, "m_A \otimes id_A"] \arrow[u, "id_A \otimes m_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "\lambda_A^{-1}"'] \arrow[dr, "\rho_A^{-1}"] & \\
I \otimes A \arrow[r, "e_A \otimes id_A"'] & A \otimes A \arrow[u, "m_A"] & A \otimes I \arrow[l, "id_A \otimes e_A"]
\end{tikzcd}$$

both commute.

Dually, we define the internal co-commutative co-monoid.

Definition 4.2.2. The triple (A, δ, ϵ) forms an *internal co-commutative comonoid* such that the following diagrams commute.

$$\begin{tikzcd}
A \arrow[rr, "\delta_A"] \arrow[d, "\delta_A"'] & & A \otimes A \arrow[d, "id_A \otimes \delta_A"] \\
A \otimes A \arrow[rr, "\delta_A \otimes id_A"'] & & A \otimes A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[d, "\delta_A"] & \\
I \otimes A \arrow[ur, "\lambda_A"] & A \otimes A \arrow[l, "\epsilon_A \otimes id_A"] \arrow[r, "id_A \otimes \epsilon_A"'] & A \otimes I \arrow[ul, "\rho_A"']
\end{tikzcd}$$

We can see that through the above definition, we establish the commutation of

$$\begin{tikzcd}
A \otimes A \arrow[rr, "A \otimes \delta_A"] \arrow[d, "\delta_A \otimes A"'] \arrow[dr, "m_A"] & & A \otimes A \otimes A \arrow[d, "m_A \otimes A"] \\
 & A \arrow[dr, "\delta_A"] & \\
A \otimes A \otimes A \arrow[rr, "A \otimes m_A"'] & & A \otimes A
\end{tikzcd}$$

which is nothing else than the frobenius equation. We also observe that within the context of a classical structure we can induce a quantum structure by defining

$$\eta_A = \delta_A \circ e_A.$$

4.2.2 Graphical language for Frobenius algebras

What is more about the Frobenius algebras, is that their counterparts in monoidal categories also have translations in pictures. We depict the four morphisms that define a frobenius object as follows.

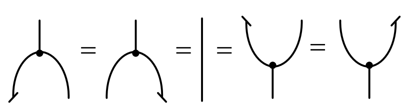
Since we are 'living' in †-symmetric compact closed monoidal categories[1] we have the $(-)^\dagger$ involution, and hence we can see that these four morphism can be express in just two. Hence we can directly write down that:

$$\delta = m^\dagger \qquad \epsilon = e^\dagger$$

or equivalently,

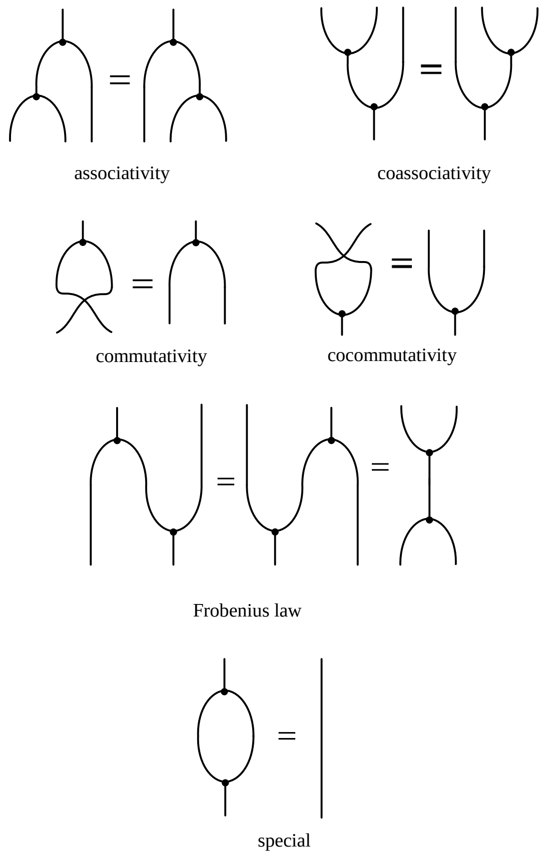
$$\delta^\dagger = m \qquad \epsilon^\dagger = e$$

Consider now the laws that a frobenius algebra must satisfy. These directly translate to pictures as follows.



identity laws

[1]Frobenius algebras can be defined in any monoidal category.



4.2.3 Definition of a classical structure

A classical structure in a †-SMC is defined as a triple (A, δ, ϵ) – an object A along with two morphisms[2];

$$\epsilon : A \rightarrow I$$

and

$$\delta : A \rightarrow A \otimes A.$$

This piece of structure exactly corresponds to a commutative dagger Frobenius algebra.

[2]Equivalently we can define the classical structure by the monoid structure of the Frobenius algebra, i.e. as a structured triple of (A, m, e)

In the category of finite dimensional Hilbert spaces, these correspond to the linear maps which copy and delete 'data'. In quantum mechanical formulation, these are:

$$\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |i\rangle \mapsto |ii\rangle \qquad \epsilon : \mathcal{H} \to \mathbb{C} :: |i\rangle \mapsto 1$$

But what does exactly these maps copy and delete?

Theorem 4.2.3. In a †-commutative Frobenius monoid, in the category of $\mathcal{F}dHilb$, the only vectors that can be copied by the comultiplication morphism $\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ are the basis vectors.

Proof. We have that $|\psi\rangle = \sum_N c_i|i\rangle$. The map $\delta : H \to H \otimes H$ applied to ψ yields

$$\delta|\psi\rangle = \sum_N c_i\delta|i\rangle = \sum_i^N c_i|ii\rangle$$

Assume δ clones the states, that is,

$$\delta_{clone}|\psi\rangle = |\psi\rangle \otimes |\psi\rangle = \sum_{i,j}^N c_ic_j|ij\rangle$$

Hence equating we have

$$\sum_i^N c_i|ii\rangle = \sum_{i,j}^N c_ic_j|ij\rangle$$

Multiplying both sides by $\langle kh|$ where k and h are some indices yields

$$\sum_i^N c_i\langle kh|ii\rangle = \sum_{i,j}^N c_ic_j\langle kh|ij\rangle$$

When we have $h = k$ we get that

$$c_k^2 = c_k$$

so c_k is 0 or 1. When $h \neq k$ we have that

$$c_kc_h = 0$$

and hence c_k or c_h is zero.

Then for every c_i we require it to be pairwise zero i.e. $c_ic_j = 0$ and hence only one c_i can be non-zero. As a result of this, the set of elements of $|\psi\rangle$ must be a singleton and hence $|\psi\rangle$ is a base vector. □

This is unsurprising since we already know that quantum states cannot be copied due to the no-cloning theorem. In fact, given the comultiplication morphism $\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, we can see that solving

$$\delta(|\psi\rangle) = |\psi\rangle \otimes |\psi\rangle$$

yields the orthonormal basis set. Therefore we can say that the triple the triple (H, δ, ϵ) faithfully encodes the orthonormal basis set $\{i\}_i$.

In fact, it turns out that we have the following theorem due to Coecke, Pavlovic and Vicary [10].

Theorem 4.2.4. In $\mathcal{F}dHilb$ there is a one-to-one correspondence between dagger special commutative Frobenius monoids and orthonormal bases.

So what we get is a new way to describe orthonormal bases, in a category-theoretic sense. We will be exploiting this theorem in the following chapter.

Chapter 5

The spectral theorem

In this chapter we examine the spectral theorem which is at the core of quantum measurements. We introduce the conventional definition through the concept a diagonalisable operator and subsequently provide the framework that captures diagonalisation in a categorical sense.

5.1 Conventional algebraic definition

A normal endo-operator $N : \mathcal{H} \to \mathcal{H}$ acting on a Hilbert space $\mathcal{H}$ is one such that it commutes with its adjoint, that is,

$$N \circ N^{\dagger} = N^{\dagger} \circ N.$$

The 'conventional' spectral theorem for normal operators in Hilbert spaces is traditionally expressed as follows.

Theorem 5.1.1. Every normal operator in a complex Hilbert space $\mathcal{H}$ admits a diagonal matrix form. More functionally put, there exists a basis such that the matrix corresponding to the normal operator N diagonalizes.

Example Let $N = \begin{pmatrix} i & 0 \\ 0 & 2 \end{pmatrix}$. Since N is diagonal according to the above theorem it must commute with its adjoint. We have that, $N^{\dagger} = \begin{pmatrix} -i & 0 \\ 0 & 2 \end{pmatrix}$ and indeed we verify that $N \circ N^{\dagger} = N^{\dagger} \circ N = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

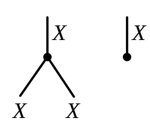
Now suggest we have a non-diagonal matrix.

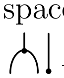
Example Let $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (in the computational base). We have that $Q \circ Q^\dagger = Q^\dagger \circ Q$; hence Q is a normal operator and can be diagonalized in some other basis. First we need to compute its eigenvalues which are found to be $\lambda_1 = 1$ and $\lambda_2 = 2$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Now, writing down the matrix P which has as columns the two eigenvectors, we have $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The diagonal form of the matrix Q can now be found by computing PQP^{-1} which yields $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the basis expressed by the eigenvectors.

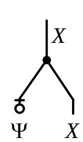
5.2 Category-theoretic axiomatization

In the category of finite-dimensional Hilbert spaces $\mathcal{F}dHilb$ as we saw in theorem 4.2.4, we know that there exist a bijective correspondence of orthonormal basis and commutative special dagger Frobenius monoids.

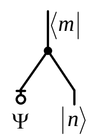
Suggest we choose an orthonormal basis for X, defined by a multiplication and a unit of a Frobenius monoid, shown below.



Let an arbitrary element of Hilbert space $\mathcal{H}$, $\psi = (\psi_1 \quad \psi_2 \quad \ldots \quad \psi_N)^T$ expressed in the basis of X defined by  – and where N is the dimension of $\mathcal{H}$. Now consider the *left action* of this element,



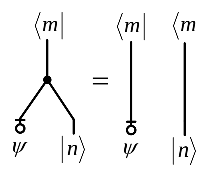
Let the chosen basis of X be $|1\rangle,\ |2\rangle,\ \ldots |N\rangle$, where $|i\rangle$ has i^{th} horizontal element equal to 1, then the elements of the matrix given by the above picture simplifies to



where $|n\rangle : \mathbb{C} \to X$ and $\langle m| : X \to \mathbb{C}$. Since we know that our chosen basis corresponds to a classical structure then we can show that the above picture corresponds to

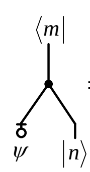
$$\psi_m \delta_{nm}$$

i.e. we have the pictorial equivalence of



$$= \psi_m \delta_{nm}$$

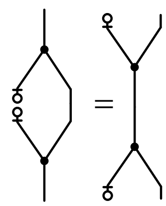
since $\psi\langle m| = \psi_m$ and $\langle m|n\rangle$ is just the inner product. In matrix form, this corresponds to



$$= \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ 0 & \psi_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \psi_N \end{pmatrix}$$

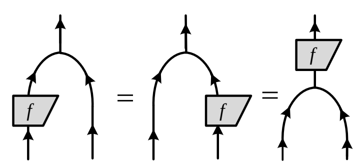
This matrix is in a diagonal form and therefore due to the spectral theorem a normal operator, which gives rise to the following proposition.

Proposition 5.2.1. In a †-SMC of Hilbert spaces, a left action of an element $\psi \in \mathcal{H}$ acting upon a specific choice of basis defining by fixing a dagger Frobenius algebra $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ defines a normal operator i.e. in a picture we have that:



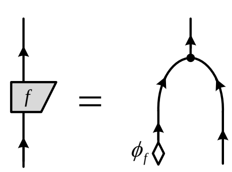
But all we have proved, is that structures like the above are normal operators – but only in the $\mathcal{F}dHilb$ category. What about in an arbitrary monoidal †-category? Is the above structure always normal in an arbitrary †-category? The answer is yes, and the proof follows. First we have to introduce some category-theoretic way to describe diagonalisation. Vicary in [22] introduced the notions of a *compatible monoid* and an *internally diagonalisable element* to abstractly express the spectral theorem.

Definition 5.2.2. In a monoidal category, an endomorphism $f : A \to A$ is *compatible* with a monoid (A, m, u) if the following following equations hold:



$$m \circ (f \otimes A) \;=\; m \circ (A \otimes f) \;=\; f \circ m$$

Definition 5.2.3. In a braided monoidal †-category, an endomorphism $f : A \to A$ is *internally diagonalisable* if it can be written as an action of an element of a commutative †-Frobenius algebra on A. That is, where

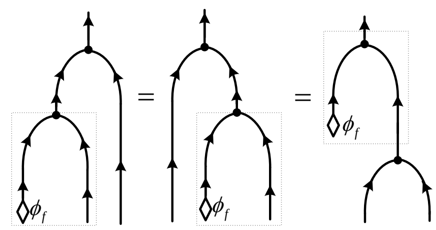


$$f \qquad\qquad m \circ \left(\phi_f \otimes A\right)$$

$m : A \otimes A \to A$ is the usual multiplication of a commutative †-Frobenius algebra and $\phi_f : I \to A$ is a state of A.

Theorem 5.2.4. An endomorphism $f : A \to A$ is internally diagonalisable if and only if it is compatible with a commutative †-Frobenius monoid.

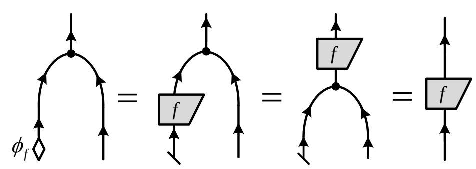
Proof. Substituting pictorially the definition of an internally diagonalisable endomorphism we have that:



But these pictures also correspond to the associativity and commutativity laws in a †-Frobenius monoid – and therefore explicitly express compatibility as well. Conversely, assuming f is compatible with a commutative †-Frobenius monoid (A, m, u) and defining $\phi_f = f \circ u$ we have

$$m \circ (\phi \otimes id_A) = m \circ ((f \circ u) \otimes id_A) = f \circ m \circ (u \otimes id_A)$$

so f is internally diagonalisable. The above equations depict as,



□

Proposition 5.2.5. If an endomorphism $f : A \to A$ is internally diagonalisable, then it is normal.

Proof. For a morphism $f : A \to A$ to be internally diagonalisable we said that is suffices to show that it can be written as the left action of a state a, i.e. $f = m \circ (\phi_f \otimes id_A)$. Since we are 'living' in a *commutative* †-Frobenius monoid, this is the same as a right action – which in turn is equal to

$$R_\alpha = m \circ (id_A \otimes \alpha) :: A \to A$$

where $\alpha \in A$. It can be proven [22] that R_α is involutive on the $\dagger$ functor and that $R^\dagger$ corresponds to a right action of some other element α'. Hence, we have that:

$$f \circ f^\dagger = R_\alpha \circ R_\alpha^\dagger = R_\alpha \circ R_{\alpha'} \overset{\text{commut.}}{=} R_{\alpha'} \circ R_\alpha = f^\dagger \circ f$$

as required. $\square$

So we have proved that morphism f which is internally diagonalisable –i.e. can be written as a left or right action of a classical structure – must be diagonal and hence normal. But is the opposite true? Is every normal endomorphism internally diagonalisable? The answer is yes, and we have the following theorem.

Theorem 5.2.6. In $\mathcal{F}dHilb$ every normal endomorphism $f : A \to A$ is internally diagonalisable.

Proof. Suggest we choose an orthonormal basis set

$$a_i : \mathbb{C} \to A$$

such that each vector each vector a_i is an eigenvector for f. Then the orthonormal property can be written as $a_i^\dagger \circ a_j = \delta_{ij} id$. This basis is unique if and only if f is a non-degenerate operator. We can use the morphisms defined by the orthonormal basis a_i to construct an internal commutative monoid (A, m, e) defining

$$\begin{aligned} m = & \sum_i a_i \circ (a_i^\dagger \otimes a_i^\dagger) \\ u = & \sum_i a_i \end{aligned}$$

$\square$

But what about the category of relations $\mathcal{R}el$? Do we have equivalent theorems that establish correlations between internally diagonalisable elements and normal operators? We will be examining this question in the next chapter.

Chapter 6

Spectral theorem in $\mathcal{R}el$

First of all, why should we care about the normal operators in $\mathcal{R}el$? Why should we care about categories other than the principal category for quantum mechanics, $\mathcal{F}dHilb$? The ability to 'simulate' some quantum features in categories other than $\mathcal{F}dHilb$ provides us with many insights. As we already pointed out in Chapter 3, the category of relations and the category of of Hilbert spaces surprisingly have a lot in common. As proved in [8], the $\mathcal{R}el$ category is rich enough to simulate the quantum teleportation and dense coding protocols, as well as a variation of classical communication and decoherence due to measurement.

Interestingly, if we consider the category $\mathcal{S}pek$ as a sub-category of $\mathcal{R}el$ we are closer than ever in describing quantum mechanics with a seemingly classical world. The $\mathcal{S}pek$ category provides a categorical framework for Spekkens' toy theory [21] – a category that reproduces inter alia a no-cloning theorem, a dense coding protocol, the non commutativity of measurements and mutually unbiased states.

The core difference of Spekkens' toy theory with traditional quantum mechanics is that the former is a completely local theory. However, this can be thought as an advantage is this case. We can formally express both theories in the same language and reveil their difference from a category-theoretic perspective – for example by finding the structural source non-locality [3], entanglement and other non-trivial phenomena.

In this chapter we will be investigating the normal operators and internally diagonalisable elements on the two and three element set.

6.1 The II-element set

6.1.1 Classical structures on the II-element set

We can define a two element set, by consulting the two element groupoid structure. If we consider the $\mathbb{Z}_2$ set, we obviously have a two element set defined by the set

$$\mathbb{Z}_2 := \{1, a\}$$

where we have

$$1.a = a.1 = a \qquad \text{and} \qquad \text{a.a} = 1.$$

However, we can also define a two element set as a cyclic group of two single element sets as

$$\mathbb{Z}_1 + \mathbb{Z}_1 := \{1\} + \{1'\}$$

where we would have

$$1.1 = 1 \qquad \text{and} \qquad 1'.1' = 1'.$$

Using this group-to-set correlation we would be able to define two classical structures, one for each two element group, in accordance with Pavlovic's theorem in [17].

Theorem 6.1.1. Every abelian group in the $\mathcal{S}et$ category induces a classical structure in $\mathcal{R}el$ and every classical structure is induced by an abelian group.

Let us first consider the $\mathbb{Z}_2$ group $\{1, a\}$. Taking the cartesian product with itself yields

$$II \times II = \{(1,1), (1,a), (a,1), (a,a)\}.$$

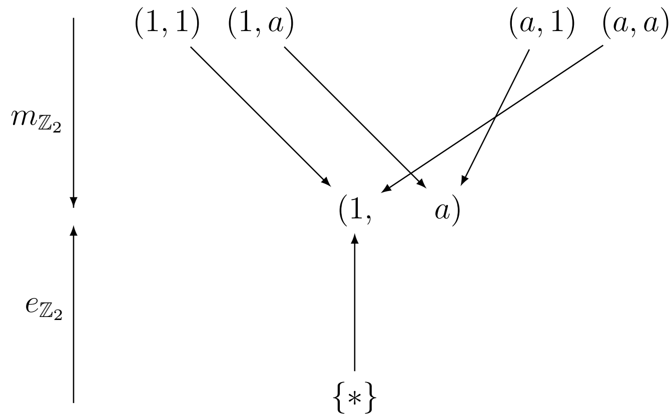
We define the multiplication

$$m_{\mathbb{Z}_2} : II \times II \to II$$

and the multiplicative unit

$$e_{\mathbb{Z}_2} : I \to II$$

operations of the classical structure, as



In matrix form this is written as

$$m_{\mathbb{Z}_2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{e}_{\mathbb{Z}_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

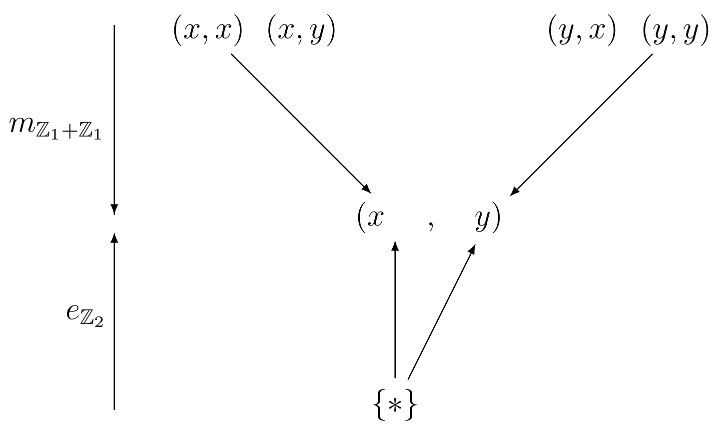
In a slightly more compact notation, we write

$$m_{\mathbb{Z}_2} : II \times II \to II :: \left\{ \begin{array}{l} \{(1,1),(a,a)\} \mapsto 1 \\ \{(1,a),(a,1)\} \mapsto a \end{array} \right. , \qquad e_{\mathbb{Z}_2} : I \to II :: \{\{*\} \mapsto 1$$

Now consider the II-element set obtained by the cyclic group $\mathbb{Z}_1 + \mathbb{Z}_1$. Consider the cartesian product $(\mathbb{Z}_1 + \mathbb{Z}_1) \times (\mathbb{Z}_1 + \mathbb{Z}_1)$. Analytically this would be given by

$$II \times II = \{(x,x),(x,y),(y,x),(y,y)\}$$

where x and y are the single elements from the first group and second group respectively. We define the multiplication and unit operations as



In matrix form this is given by

$$m_{\mathbb{Z}_1+\mathbb{Z}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathrm{e}_{\mathbb{Z}_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

or equivalently,

$$m_{\mathbb{Z}_1+\mathbb{Z}_1} : II \times II \to II :: \begin{cases} (1,1) \mapsto 1 \\ (a,a) \mapsto a \end{cases} \qquad e_{\mathbb{Z}_1+\mathbb{Z}_1} : I \to II :: \begin{cases} 1 \mapsto \{*\} \\ a \mapsto \{*\} \end{cases}$$

6.1.2 Diagonalisable and normal operators on II-element set

An endo-operator $N : II \to II$ is said to be normal when it commutes with its adjoint, that is

$$N \circ N^\dagger = N^\dagger \circ N \tag{6.1}$$

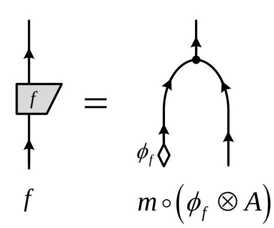
In the II-element set, there is a total of $2^4 = 16$ operators which can be constructed as a two-by-two matrix. We can verify that only nine of these are normal, namely

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

All of these are self-adjoint, except the last two which are each others adjoint.

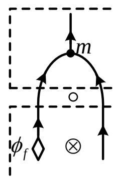
Recall that we proved that an internally diagonalisable operator $f : N \to N$ can be written as the left action of a commutative dagger Frobenius monoid, as shown below



In the II-element case, ϕ_f can be any of the following four morphisms which have type $\phi_f : I \to II$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Since it is never interesting to be working only with zeros, we will be disregarding the zero matrix. The above picture can seen in a somewhat more mathematical term if we explicitly write what the 'legs' of the pictures represent; i.e.



That is, in mathematical terms

$$m \circ (\phi_f \otimes id_{II}).$$

Hence in order to find the internally diagonalisable elements on the II-element set it suffices to compute the above expression. The results are tabulated below.

ϕ_f	$m_1 \circ (\phi_f \otimes id_{II})$	$m_2 \circ (\phi_f \otimes id_{II})$
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We observe that 5 normal operators, out of possible 9 operators are internally diagonalisable. Indeed, the four ones that are missing are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Interestingly, all the normal operators which quite 'dense', i.e those which have exactly three non-zero entries, are not included in the internally diagonalisable elements set. Therefore, we seek a condition to establish when a normal operator is internally diagonalisable and when it is not.

In order to explore this, we need to find more operators that cannot be diagonalised – and hence proceed to investigate the three element set.

6.2 The III-element set

6.2.1 Classical structures on the III-element set

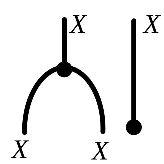
Similarly to the II-element set case above where we considered groups $\mathbb{Z}_2$ and $\mathbb{Z}_1 + \mathbb{Z}_1$, we can construct a three element set with many ways. The straight-forward case would be to define it as the three-element group $\mathbb{Z}_3$ consisting of elements $\{1, a, a^2\}$ with the group multiplication being the usual multiplication but satisfying

$$a^3 = 1 \quad \text{and} \quad \mathrm{a} \cdot 1 = 1 \cdot \mathrm{a} = 1$$

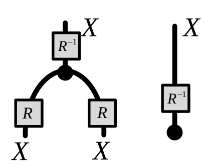
and similarly for a^2.

We can also define a three element cyclic group by 'adding' a two and the single element group, for example $\mathbb{Z}_2 + \mathbb{Z}_1$ and the final way would be by constructing a three single element set, by adding three $\mathbb{Z}_1$ groups.

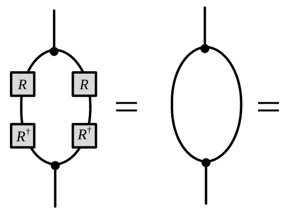
However, we must also consider permutations of the internally diagonalisable elements. If we consider that



defines a classical structure, then so does



where R is each of the five unitary permutation matrices[1]. It is trivial to show that this also satisfies all the frobenius laws. For example, consider the specialness axiom. Exploiting unitarity we have that $R^\dagger = R^{-1}$ and hence when the the R box 'meets' its dagger counterpart they will annihilate.

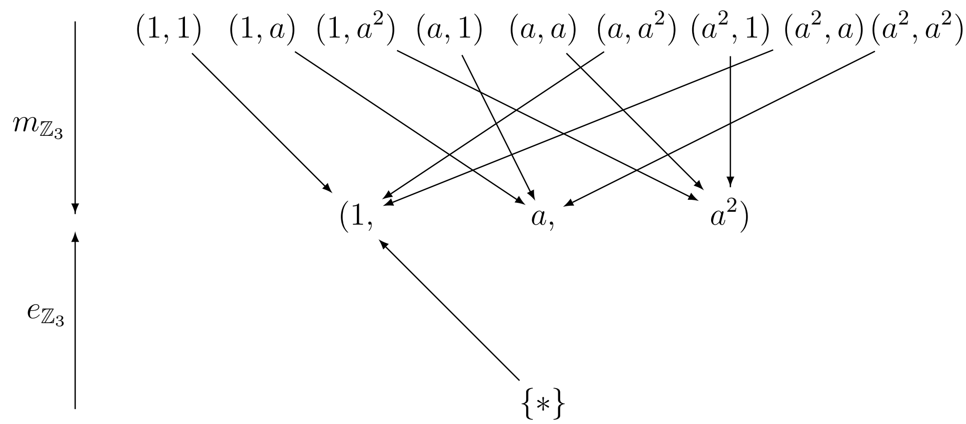


Now let us spell out the classical structures obtain by the III-element set. We would write explicitly the correlations for the first one and only write down the matrix form for the others. Permutations will not be explicitly considered.

1. *III-element set as a $\mathbb{Z}_3$ group.*
   We have that $III = (1, a, a^2)$, with $a^3 = 1$ and 1 being the unit element. If we consider the cartesian product of

   $$(1, a, a^2) \times (1, a, a^2)$$

   then we define the multiplication operation $m : III \times III \to III$ and the deleting operation as



[1] Excluding the identity matrix these are:
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Writing these correlations matrix-wise, we have

$$m_{\mathbb{Z}_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{e}_{\mathbb{Z}_3} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

2. *III-element set as a $\mathbb{Z}_2 + \mathbb{Z}_1$ group.*

$$m_{\mathbb{Z}_2+\mathbb{Z}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{e}_{\mathbb{Z}_2+\mathbb{Z}_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

3. *III-element set as a $\mathbb{Z}_1 + \mathbb{Z}_1 + \mathbb{Z}_1$ group.*

$$m_{\mathbb{Z}_1+\mathbb{Z}_1+\mathbb{Z}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{e}_{\mathbb{Z}_1+\mathbb{Z}_1+\mathbb{Z}_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

To compute the internally diagonalisable elements that arise from the permutation of these classical structures, it suffices to compute

$$R^{-1} \circ (m \circ ((R \circ \phi_f) \otimes R))$$

for each of the six permutation matrices, each of the three multiplication matrices and each of the seven states on the three element set.

6.2.2 Diagonalisable and normal operators on the III-element set

In the three element case there are a total of $2^9 = 512$ operators. Using Mathematica we were able to find all the operators that commute with their adjoint – 158. Exluding the internally diagonalisable elements that we will compute shortly, these are:

1. Three entries:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

2. Four entries:

$$\begin{pmatrix} 0&0&1\\0&0&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&1&0\\1&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1\\1&0&0\\1&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0\\0&0&1\\0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&0\\1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&1\\0&1&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0\\1&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&1&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0\\1&0&0\\0&0&1 \end{pmatrix}$$

3. Five entries:

$$\begin{pmatrix} 0&0&1\\0&0&1\\1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&1&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&1&0\\1&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&1\\0&1&1 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0\\1&1&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&1\\1&0&0\\1&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1\\1&1&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&1\\1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\1&0&0\\1&1&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0\\1&1&0\\0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\1&1&1\\1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&0&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&1&0\\0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\1&1&0\\0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&0\\0&1&0\\1&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0\\0&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0\\1&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 1&1&1\\0&0&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&1\\0&1&0\\0&1&0 \end{pmatrix},$$

$$\begin{pmatrix} 1&1&1\\1&0&0\\1&0&0 \end{pmatrix}$$

4. Six entries:

$$\begin{pmatrix} 0&0&1\\0&1&1\\1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&1\\1&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&1&1\\0&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&1&1\\1&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&1&1\\1&1&1 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0\\1&1&0\\1&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&1&1\\0&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&1&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&1\\0&0&1\\1&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&1\\0&1&0\\1&1&1 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&1\\1&1&0\\1&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1\\1&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1\\1&1&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\1&0&1\\1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\1&1&0\\1&1&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0\\1&1&1\\1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\1&1&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&0&1\\1&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&1&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1\\1&0&0\\1&1&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&1\\1&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\1&1&1\\1&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&0\\0&1&0\\1&1&1 \end{pmatrix}, \begin{pmatrix} 1&1&0\\1&0&0\\1&1&1 \end{pmatrix}, \begin{pmatrix} 1&1&0\\1&0&1\\0&1&1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Subsequently, using the multiplication matrices and the permutation matrices we were able to find all the internally diagonalisable elements. While we await $7 \cdot 6 \cdot 3 = 126$ three-by-three internally diagonalisable matrices, only 26 are unique. These are:

1.
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2.
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3.
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5.
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

6.
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

7.
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

6.3 Normal operators vs internally diagonalisable elements

In this section we will try to explore how we can separate the internally diagonalisable elements within the normal operator set.

From the internally diagonalisable elements on the three element set we can directly observe that:

i All normal operators that have a single or two entries are internally diagonalisable

ii The size[2] of the relation of all internally diagonalisable elements does not exceed six if we exclude the total matrix

iii The matrices with size equal to three are the permutation matrices and

iv The matrices with five entries resemble the three rotation matrices.

Interestingly, we can see that internally diagonalisable relations on the III-element satisfy some more properties. For instance:

→ the matrices with size equal to six can be constructed by combining two permutation matrices such that resulting relation is either reflexive or irreflexive[3] (Similarly we can subtract from the total matrix one of the permutations that yield reflexive or irreflexive relations).

→ the four-entries matrices can be constructed by subtracting the single-entry matrices from the five-entry matrices such that the resulting relations is transitive.

[2]We call the size of a relation the number of ones in its matrix representation.

[3]A relation is reflexive if for all $x \in X$ we have that xRx (i.e. the diagonal of the matrix is all ones). On the other hand a relation is irreflexive if for all $x \in X$ then x is not relation to x(the diagonal of the matrix is zeros). In a transitive relation we have that if xRy and yRz then xRz(A relation with matrix R is transitive if $R \circ R = R$.)

On the II-element set we observe a similar phenomenon; all normal operators which have a single entry are internally diagonalisable. The two permutation matrices are also internally diagonalisable and there is no relation with size bigger than two.

Therefore we can present the following conjecture:

Conjecture 6.3.1. Consider the relations (operators) which are normal on the two and three element set. A relation can be diagonalised if: (i) It is the zero or total matrix. (ii) It has a single or two entries. (iii) Represents a permutation or a rotation matrix. (iv) Can be constructed by adding two two-entry matrices and the resulting matrix is transitive (v) and finally can by constructed by adding two three-entry matrices and the results matrix if either reflexive or irreflexive. The size of the relation must not exceed $n-1$, where n is the dimension of the set.

Chapter 7

Discussion

7.1 Summary

We reviewed most categorical concepts that enable quantum mechanics to be expressed in a category-theoretic sense. Quantum measurements are accounted for through the definition of a classical structure which is in turn defined through a Frobenius algebra. Categorically, measurement outcomes – as exposed through the spectral theorem – exactly correspond to classical structures.

Subsequently, we axiomatized relations over sets in the same categorical framework and pointed out similarities with the category of Hilbert spaces. We investigated the concept of diagonalisation in both categories.

Finally, we concluded by providing insights on how to separate the normal operators and the internally diagonalisable elements in the category of relations. It turns out that there is a key difference between the normal operators in the category of Hilbert spaces and the ones in the category of relations. We show that in the category of relations, the number of internally diagonalisable elements does not equal the number of normal operators. This is in contrast to the category of Hilbert spaces, where all the normal operators can be diagonalised and visa versa. We then proceed by providing insights on how the we can distinguish the internally diagonalisabe elements within the set of normal operators of the two and three element case.

7.2 Future directions of work

An obvious extension of this work would be to investigate the four elements set and examine if the same conditions hold in the internally diagonalisable elements. It would also be very interesting to compare the four element set from *Rel* with the four element set of Spekkens' category $\mathcal{S}pek$ and investigate how the spectral theorem applies in that category. By doing do, we will also have more evidence on *why* some normal operators can be internally diagonalisable whilst other cannot.

Another direction would be to 'trace back' the piece of structure of the category relations that makes $\mathcal{R}el$ behave differently than $\mathcal{F}dHilb$. By doing so we would provide answers on exactly why the two categories behave differently.

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