Mereotopology: a Survey

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- A topological space is a pair $\langle X, \mathcal{O}\rangle$ where $\mathcal{O}$ is a collection of subsets of X s.t. ....
- Let X be a topological space and p a subset of X. Then

$$-\left((p^{-0})^{-0}\right) \cup p^{-0} = X.$$

- Changing notation slightly, we obtain the modal logic formula

$$\neg\Box\Diamond\Box\Diamond p \vee \Box\Diamond p$$

  which happens to be an S4-theorem.

- More generally, McKinsey and Tarski, 1944, showed:

  Theorem: Let ϕ be a formula of modal logic. TFAE:

  1. ϕ is an S4-theorem;
  2. ϕ is valid in the class of topological spaces;
  3. (ϕ is valid in X, where X is any dense-in-itself, separable metric space).

- Consider the formal language $\mathcal{T}$:
  - *Terms*: $\tau ::\quad x \mid 0 \mid 1 \mid -\tau \mid \tau_1 \cup \tau_2 \mid \tau_1 \cap \tau_2 \mid \tau^- \mid \tau^0$
  - *Statements*: $\phi ::\quad \tau_1 = \tau_2 \mid \phi_1 \wedge \phi_2 \mid \neg\phi.$
- An *interpretation* for $\mathcal{T}$ is simply a topological space X (belonging to some class), with variables ranging over $\mathbb{P}(X)$.
- Using the obvious semantics for the above primitives, we obtain the notion of a $\mathcal{T}$-validity. For example:

$$\models -\left((p^{-0})^{-0}\right) \cup p^{-0} = 1$$

- The McKinsey-Tarski theorem tells us that the logic of $\mathcal{T}$, over the class of all topological spaces is (in effect) the logic S4. In particular, the corresponding satisfiability problems are PSPACE-complete.

- If X is a topological space, a subset $u \subseteq X$ is regular open if u is equal to the interior of its closure: $u = (u^-)^0$.
- the set of regular open subsets of X is denoted $\mathrm{RO}(X)$.
- $(\mathrm{RO}(X), \subseteq)$ is always a (complete) Boolean algebra under the interpretation:

$$
\begin{aligned}
1 &= X & \qquad x \cdot y &= x \cap y \\
0 &= \emptyset & \qquad x + y &= (x \cup y)^{-0} \\
& & -x &= (X \setminus x)^0.
\end{aligned}
$$

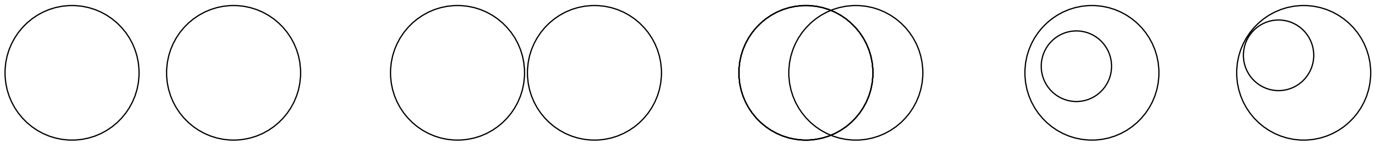
  It is called the regular open algebra of X.
- The valid formula in the previous example states (in effect) that the regular open sets are exactly those of the form p^{-0}.

- This leads to the following fragment of $\mathcal{T}$:
  - take variables to range only over regular open sets
  - take atomic formulas to be only those of the forms

    $\mathrm{DC}(x,y) \equiv x^{-} \cap y^{-} = 0$

    $\mathrm{EC}(x,y) \equiv x \cap y = 0 \wedge x^{-} \cap y^{-} \neq 0$

    etc.



DC EC O NTPP TPP

- Call this language $\mathcal{RCC}8$ (Egenhofer and Franzosa, Bennett $\ldots$).
- For example,

$$\models \mathrm{EC}(x,y) \wedge \mathrm{NTPP}(y,z) \rightarrow (\mathrm{O}(x,z) \vee \mathrm{TPP}(x,z) \vee \mathrm{NTPP}(x,z))$$

- We can extend $\mathcal{R}CC8$ by adding function symbols $+$, $\cdot$ and $-$ denoting the obvious operations in $\mathrm{RO}(X)$.
- Call this language $\mathcal{B}RCC8$ (Wolter and Zakharyaschev, 2000)
- For example, we have the validity

$$\models \mathrm{EC}(x, y+z) \rightarrow (\mathrm{EC}(x,y) \vee \mathrm{EC}(x,z)).$$

Theorem: Sat-$\mathcal{R}CC8$ is NP-hard, even for conjunctions of atomic formulas. Sat-$\mathcal{B}RCC8$ is in NP.

- Returning to the language $\mathcal{T}$, one can extend to obtain a language $\mathcal{TC}$ by adding an additional unary predicate c:
  - *Terms*: ...
  - *Statements*: $\phi :: \quad \ldots \mid c(\tau)$

  with the interpretation: $X \models c[s]$ iff $s \subseteq X$ is connected.

- Let X be a topological space and r, s subsets of X:
  - if r is connected and $r \subseteq s \subseteq r^-$, then s is connected;
  - if r and s are connected and $r \cap s \neq \emptyset$, then $r \cup s$ is connected.

- We can express these (textbook) results as the $\mathcal{TC}$-validities:
  - $\models c(x) \wedge -x \cup y = 1 \wedge -y \cup x^- = 1 \rightarrow c(y)$
  - $\models c(x) \wedge c(y) \wedge x \cap y \neq 0 \rightarrow c(x \cup y)$.

- The problem Sat-$\mathcal{TC}$ is in NEXPTIME.

- Obvious next step: add quantifiers.
- Consider the language $\mathcal{C}A$ defined as follows:
  - *Terms*: $\tau ::\quad x \mid 0 \mid 1 \mid -\tau \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$
  - *Statements* $\phi ::\quad \tau_1 = \tau_2 \mid C(\tau_1, \tau_2) \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \exists x \phi$,

  with variables ranging over certain collections (details to follow) of regular open of subsets of topological spaces belonging to some class, and and the predicate C is interpreted as:

  $$X \models C[r, s] \text{ iff } r^- \cap s^- \neq \emptyset.$$

- The predicate C is the contact predicate (and the relation it expresses, the contact relation).
- Whitehead (1919) introduced this relation, calling it "connection".
- Whitehead's motivation was metaphysical/epistemological, rather than computational.

- We now explain the 'certain collections' of regular open sets ...

  Definition: Let X be a topological space. A mereotopology over X is a Boolean sub-algebra M of $\mathrm{RO}(X)$ such that every neighbourhood in X contains a neighbourhood in M:

  if $q \in o \subseteq X$ with o open, there exists $r \in M$ such that $q \in r \subseteq o$.

- Where M is clear from context, we refer its elements as regions.
- Important: not every regular open subset of the space in question need count as a region.
- We shall always interpret the language $\mathcal{C}A$ over mereotopologies.

- A word on etymology:
  - Mereology (Leśniewski): the logic of the part-whole relationship ($\leq$).
  - Mereotopology is simply the study of topological spaces with regions functioning as the primary objects.
  - I am not sure where the term first appeared in print.
- It is easy to see that, for most interesting classes of mereotopologies, deciding satisfiability of $\mathcal{C}A$-formulas is undecidable. But there is plenty else we can ask about these logics ...

Definition: A contact algebra is a structure interpreting the signature $(C, \leq, +, \cdot, -, 0, 1)$ satisfying the usual axioms of Boolean algebra together with

(C0) $\forall x \neg C(x, 0)$

(C1) $\forall x(x > 0 \rightarrow C(x, x))$

(C2) $\forall x \forall y(C(x, y) \rightarrow C(y, x))$

(C3) $\forall x \forall y(C(x, y) \wedge y \leq z \rightarrow C(x, z))$

(C4) $\forall x \forall y(C(x, y + z) \rightarrow C(x, y) \vee C(x, z))$

- We consider also the following additional axioms:

(Ext) $\forall x \forall y(\forall z(C(x, z) \rightarrow C(y, z)) \rightarrow x \leq y)$

(Int) $\forall x \forall y(\neg C(x, y) \rightarrow \exists z(\neg C(x, -z) \wedge \neg C(y, z)))$

(Con) $\forall x \forall y(x + y = 1 \wedge x > 0 \wedge y > 0 \rightarrow C(x, y))$.

- A topological space is semi-regular if it has a basis of regular open sets; a topological space is weakly regular if it is semi-regular and, for any non-empty open set u, there exists a non-empty open set v with $v^- \subseteq u$.

- X is regular $\Rightarrow$ X is weakly regular $\Rightarrow$ X is semi-regular.

Theorem: Let X be a topological space, and let M be a mereotopology over X, regarded as a structure interpreting the signature $(C, \leq, +, \cdot, -, 0, 1)$. Then $M \models$ (C0)–(C4). In addition:

1. If X is weakly regular, then $M \models$ (Ext).
2. If X is compact and Hausdorff, then $M \models$ (Int).
3. If X is connected, then $M \models$ (Con).

Proof: Routine.

Theorem: (Dimov and Vakarelov, 2006) Let $\mathfrak{A}$ be a structure interpreting $(C, \leq, +, \cdot, -, 0, 1)$, whose reduct to $(\leq, +, \cdot, -, 0, 1)$ is a Boolean algebra. If $\mathfrak{A} \models$ (C0)–(C4), then $\mathfrak{A}$ is isomorphic to a mereotopology over some topological space X. Moreover:

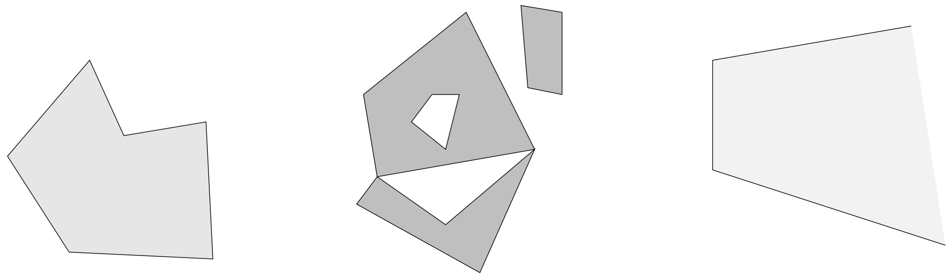
1. if $\mathfrak{A} \models$ (Ext), then X can be chosen to be weakly regular (Düntsch and Winter, 2004);
2. if $\mathfrak{A} \models$ (Int) and (Ext), then X can be chosen to be compact and Hausdorff (Roeper, 1997); and
3. if $M \models$ (Con), then X can be chosen to be connected.

Proof sketch: Define the points of X to be ultrafilter-like subsets of A; define a mapping $g : A \to \mathbb{P}(X)$ by

$$g(a) = \{x \in X \mid a \in X\};$$

use these sets as the basis of a topology.

- Examples of mereotopologies:
  - $\mathrm{RO}(X)$ for any semi-regular space X.
  - $\mathrm{ROS}(\mathbb{R}^n)$: the regular open semi-algebraic sets in $\mathbb{R}^n$;
  - $\mathrm{ROP}(\mathbb{R}^n)$: the regular open polyhedra in $\mathbb{R}^n$;
  - $\mathrm{ROQ}(\mathbb{R}^n)$: the regular open rational polyhedra in $\mathbb{R}^n$.



  - These have their closed-space analogues: $\mathrm{RO}(\mathbb{S}^n)$, $\mathrm{ROS}(\mathbb{S}^n)$, $\mathrm{ROP}(\mathbb{S}^n)$, $\mathrm{ROQ}(\mathbb{S}^n)$.

- It is interesting to ask what first-order sentences (with various signatures of topological primitives) are true in mereotopologies over certain classes of spaces.
- Consider, for example, the sentence ψ_{con} given by

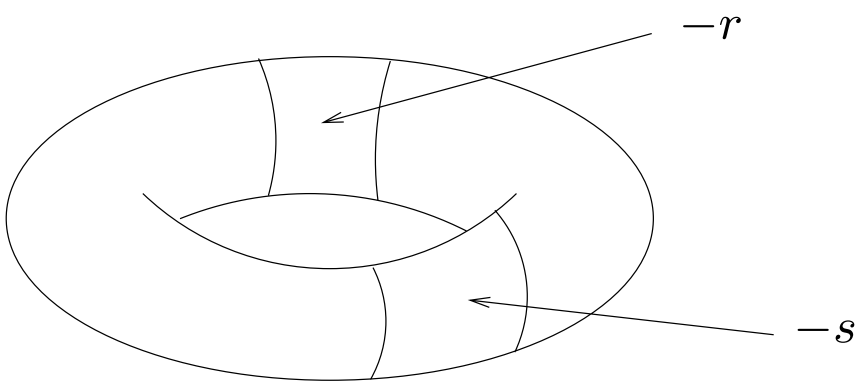
$$\forall x \forall y (c(x) \wedge c(y) \wedge x \cdot y > 0 \rightarrow c(x + y))$$

- If M is any mereotopology, then $M \models \psi_{\text{con}}$.

- Consider the sentence ψ_{Eucl} given by

$$\forall x \forall y (c(x) \wedge c(y) \rightarrow (c(x \cdot y) \vee C(-x, -y))).$$

- ψ_{Eucl} is not true in all mereotopologies:

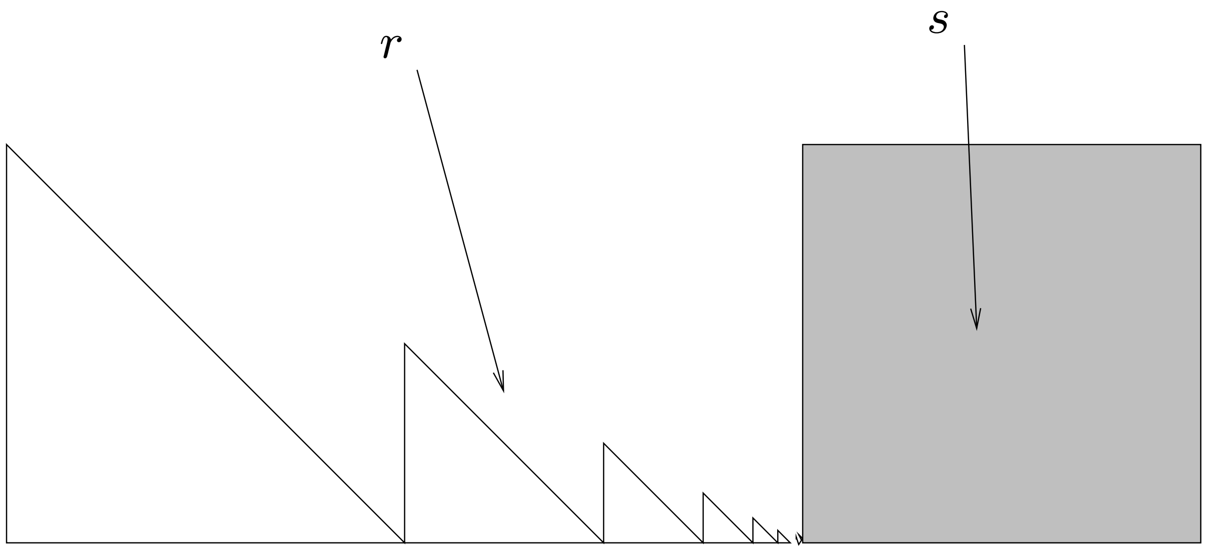


- However, if M is any mereotopology over $\mathbb{R}^n$ ($n \geq 1$), then $M \models \psi_{\text{Eucl}}$.

- Consider the sentence ψ_{inf} given by

$$\forall x \forall y (C(x,y) \rightarrow \exists z (c(z) \wedge z \leq x \wedge C(y,z)))$$

- ψ_{inf} is not true in $\mathrm{RO}(\mathbb{R}^2)$:

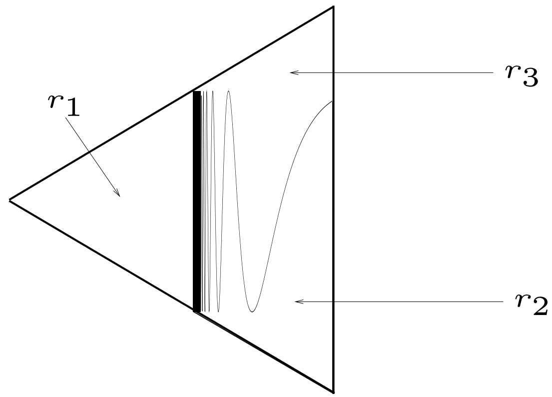


- However, ψ_{inf} is true in $\mathrm{ROQ}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$, $\mathrm{ROS}(\mathbb{R}^2)$.

- Consider the sentence ψ_{wiggly} given by

$$\forall x_1 \forall x_2 \forall x_3 (c(x_1) \wedge c(x_2) \wedge c(x_3) \wedge c(x_1 + x_2 + x_3) \rightarrow (c(x_1 + x_2) \vee c(x_1 + x_3))).$$

- ψ_{wiggly} is not true in $\mathrm{RO}(\mathbb{R}^2)$:



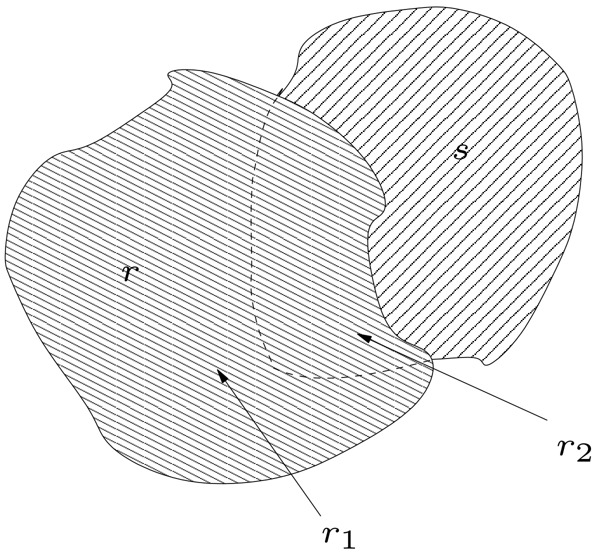
- However, ψ_{wiggly} is true in $\mathrm{ROQ}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$, $\mathrm{ROS}(\mathbb{R}^2)$.

- We can characterize mereotopologies over large classes of topological spaces abstractly; but what about familiar mereotopologies, such as $\mathrm{ROQ}(\mathbb{R}^n)$, $\mathrm{ROP}(\mathbb{R}^n)$ and $\mathrm{ROS}(\mathbb{R}^n)$?
- We proceed to give a partial answer to this question where $n = 2$.
- Here it turns out to be more convenient to employ the signature $(c, \leq, +, \cdot, -, 0, 1)$ (rather than $(C, \leq, +, \cdot, -, 0, 1)$).
- In fact, for this signature, we have $\mathrm{RO}(\mathbb{R}^n) \simeq \mathrm{RO}(\mathbb{S}^n)$, and similarly, $\mathrm{ROP}(\mathbb{R}^n) \simeq \mathrm{ROP}(\mathbb{S}^n)$ etc.

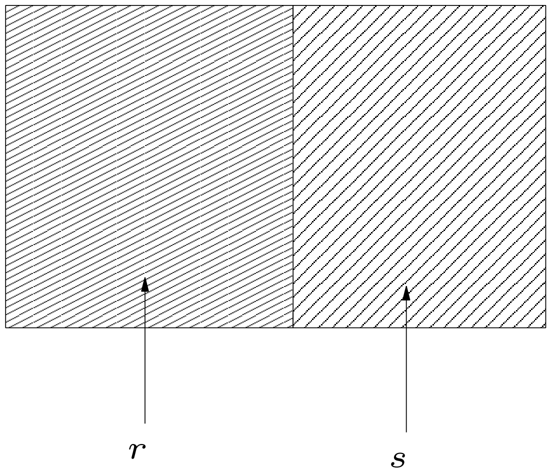
- Notice that ROQ($\mathbb{R}^n$), ROP($\mathbb{R}^n$) and ROS($\mathbb{R}^n$) are all *tame*, in the following sense:
  - They are all finitely decomposable: each region is the sum of finitely many connected regions (Cell Decomposition Theorem).
  - They exhibit curve-selection: if $r \in M$ and $q \in \mathcal{F}(r)$ there exists a Jordan arc have end q as one of its endpoints, lying in $r \cup \{q\}$ (Curve Selection Lemma).
- They are also all splittable: they make true the following *splitting axiom*:

$$\forall x \forall y \big(x,\, y \text{ and } -(x+y) \text{ are non-empty and connected} \rightarrow \exists u \exists v \big(u_1 \oplus u_2 = x \wedge c(u_1 + y) \wedge \neg c(u_1 + -(x+y)) \wedge c(u_2 + -(x+y)) \wedge \neg c(u_2 + y)\big).$$

• We can illustrate the splitting axiom diagrammatically:



a)



b)

- Consider the following axioms

  1. the usual axioms of Boolean algebra, and the axiom $0 \neq 1$;
  2. the axiom $\forall x \forall y (c(x) \wedge c(y) \wedge x \cdot y \neq 0 \rightarrow c(x+y))$.
  3. where $n > 2$, the axioms
  $$\forall x_1 \ldots \forall x_n \Big(c(x_1 + \cdots + x_n) \wedge \bigwedge_{1 \leq i \leq n} c(x_i) \rightarrow \bigvee_{2 \leq i \leq n} c(x_1 + x_i) \Big).$$
  4. two planarity axioms, e.g.
  $$\neg \exists x_1 \ldots \exists x_5 \Big(\bigwedge_{1 \leq i \leq 5} (c(x_i) \wedge x_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq 5} (c(x_i + x_j) \wedge x_i \cdot x_j = 0) \Big);$$
  5. the axioms $c(0)$ and $c(1)$;
  6. the splitting axiom;
  7. another dreadful axiom to do with splitting up regions.

- If $n \geq 1$, we let $\psi_c^n(x)$ stand for the formula

$$\exists z_1 \ldots \exists z_n \Big(\bigwedge_{1 \leq i \leq n} c(z_i) \wedge (x = z_1 + \cdots + z_n) \Big)$$

stating that x can be formed by adding together n connected regions.

- Thus, for any finitely decomposable mereotopology, the following infinitary rule of inference is valid:

$$\frac{\{\forall x(\psi_c^n(x) \rightarrow \phi(x)) \mid n \geq 1\}}{\forall x \phi(x)}.$$

- This rule simply says that, if a property holds of all n-component regions, for all n, then it holds of all regions.

- We have the following:

  Theorem: Let M be a finitely decomposable mereotopology over $\mathbb{R}^2$ having curve-selection, and satisfying the splitting axiom. Then M satisfies all the above axioms, and makes the infinitary rule of inference valid.

  Proof: Routine.

- More interestingly, we have a converse. Let $T_{c,\leq}$ denote the set of sentences which are consequences of the above axioms and the infinitary proof rule.

  Theorem: $T_{c,\leq}$ is the complete theory of any finitely decomposable mereotopology over $\mathbb{R}^2$ having curve-selection and satisfying the splitting axiom.

  Proof: Use the omitting types theorem to get a finitely decomposable model of $T_{c,\leq}$; embed it in $\mathrm{ROP}(\mathbb{R}^2)$, and show that the embedding is elementary.

- The above theorem entails that all finitely decomposable, splittable mereotopologies over $\mathbb{R}^2$ having curve-selection, considered as $\{c, \leq\}$-structures, are elementarily equivalent.
- Actually, over the closed plane, more is true:

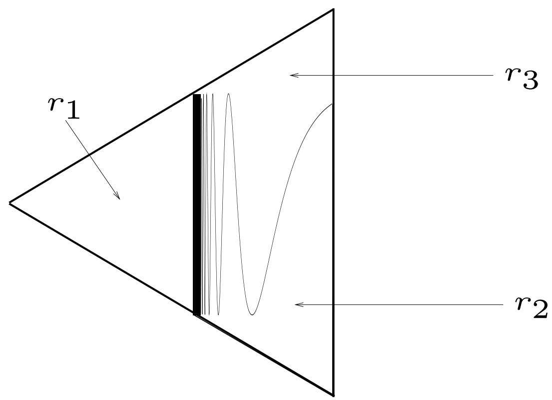
  Theorem: All splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ with curve-selection have the same L_Σ-theory for any topological signature Σ.

- The theory $T_{c,\leq}$ is well-behaved. It is atomic, with $\mathrm{ROQ}(\mathbb{R}^2)$ a prime model.
- In addition, we have:

Theorem: All countable finitely decomposable models of the theory $T_{c,\leq}$ are isomorphic.

- Thus, we can get reasonably close to characterizing the tame-region-based topology of the Euclidean plane axiomatically.

- We conclude with an open problem regarding contact algebras.
- All of the results so far concern *mereotopologies* over various topological spaces X—that is, subalgebras of $\mathrm{RO}(X)$ which form a basis of X.
- But what about the contact structure of the whole algebra $\mathrm{RO}(X)$? Can we characterize that?
- The example ψ_{wiggly}, which is true in $\mathrm{ROS}(\mathbb{R}^2)$, but not true in $\mathrm{RO}(\mathbb{R}^2)$, suggests that this problem may not be so simple:



Theorem Suppose $\mathfrak{A} \models \Phi_{\mathrm{CA}} \cup \{\phi_{\mathrm{int}}, \phi_{\mathrm{ext}}\}$, and $\mathfrak{A}$ is a (non-trivial) complete Boolean algebra. Then

$$\mathfrak{A} \models \exists x \exists y (C(x,y) \wedge \forall z (z \leq x \wedge$$

$$\forall z_1 \forall z_2 (z_1 > 0 \wedge z_2 > 0 \wedge z = z_1 + z_2 \rightarrow C(z_1, z_2)) \rightarrow \neg C(z,y)))$$

- But this sentence is false in any *finitely decomposable* mereotopology over a topological space.
- Let $\mathcal{X}$ be the class of all topological spaces, and set

$$\begin{aligned} \mathcal{R} &= \{\mathrm{RO}(X) \mid X \in \mathcal{X}\} \\ \mathcal{M} &= \{M \mid M \text{ a mereotopology over } X \text{ for some } X \in \mathcal{X}\}. \end{aligned}$$

Then $\mathrm{Th}(\mathcal{M}) \neq \mathrm{Th}(\mathcal{R})$.

Open problem: What is the elementary theory (over a suitable signature) of classes $\{\mathrm{RO}(X) \mid X \in \mathcal{X}\}$, where $\mathcal{X}$ is some salient class of topological spaces?

- Summary
  - Two important ideas:
    - formal language interpreted over classes of geometrical structures (spatial logic),
    - study of topology from a region-based viewpoint (Whitehead's vision).
  - These ideas led us to the notion of a mereotopology.
  - We can prove representation theorems for the first-order theories of various classes of mereotopologies.
  - We can prove an almost-first order representation theorem for the rational polygonal mereotopology over the Euclidean plane.
- See Aiello, Pratt-Hartmann and van Benthem: *Handbook of Spatial Logic* (Springer, 2007) for details ...