Cooking with vegetables

1. Let A be a raw potato.

A admits many *states* e.g. dirty, clean, skinned, ...

2. We want to *process* A into cooked potato B.

B admits many *states* e.g. boiled, fried, deep fried, baked with skin, baked without skin, ... Let

$$A \xrightarrow{f} B \qquad A \xrightarrow{f'} B \qquad A \xrightarrow{f''} B$$

be boiling, frying, baking. *States* are *processes*

$$\mathrm{I} := \text{unspecified} \xrightarrow{\psi} A.$$

3. Let

$$A \xrightarrow{g \circ f} C$$

be the *composite process* of first boiling $A \xrightarrow{f} B$ and then salting $B \xrightarrow{g} C$. Let

$$X \xrightarrow{\mathbf{1}_X} X$$

be doing nothing. We have $\mathbf{1}_Y \circ \xi = \xi \circ \mathbf{1}_X = \xi$.

4. Let $A \otimes D$ be potato A and carrot D and let

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E$$

be boiling potato while frying carrot. Let

$$C \otimes F \xrightarrow{x} M$$

be mashing spice-cook-potato and spice-cook-carrot.

5. A recipe:

$$A \otimes D \xrightarrow{f \otimes h} B \otimes E \xrightarrow{g \otimes k} C \otimes F \xrightarrow{x} M$$

$$= A \otimes D \xrightarrow{x \circ (g \otimes k) \circ (f \otimes h)} M.$$

6. A law on recipes:

$$(\mathbf{1}_B \otimes g) \circ (f \otimes \mathbf{1}_C) = (f \otimes \mathbf{1}_D) \circ (\mathbf{1}_A \otimes g)$$

i.e.

boil potato then fry carrot

=

fry carrot then boil potato

7. A more general law on recipes:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

i.e.

boil pot then salt pot, while, fry car then pepper car

=

boil pot while fry car, then, salt pot while pepper car

Indeed,

$$(f \otimes 1_D) \circ (1_A \otimes g) = (f \circ 1_A) \otimes (1_D \circ g)$$

$$||$$

$$f \otimes g$$

$$||$$

$$(1_C \otimes g) \circ (f \otimes 1_B) = (1_B \circ f) \otimes (g \circ 1_C)$$

$\Rightarrow$ **BIFUNCTORIALITY** $\Leftarrow$

8. Logical laws:

$$A and (B or C) = (A and B) or (A and C)$$

vs.

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

⇒ reasoning about *propositions*

vs.

⇒ reasoning about ***interacting processes***

⇒ reasoning within a *Boolean lattice/algebra*

vs.

⇒ reasoning within a ***monoidal category***

⇒ reasoning in *monolithic mathematical object*

vs.

⇒ reasoning in ***many objects and processes thereon***

9. Applies to cooking, biological processes, chemical processes, social processes, physical processes, . . .

Very successful in **programming** and **proof theory**.

proof theory	**programming**	**physics**
Propositions	Data Types	Physical System
Proofs	Programs	Physical Operation

Goal is to identify the essential additional structure which is characteristic for a certain processes!

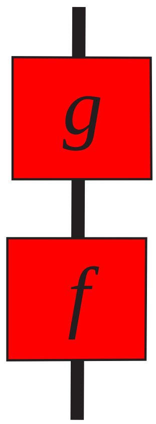
E.g. which additional structure allows for logical deduction, or, which structures witness the difference between quantum processes and classical processes.

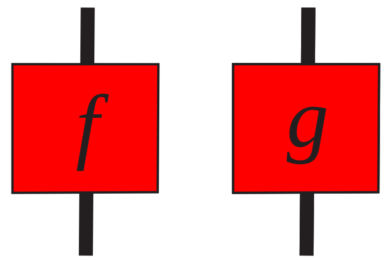
All this can be seen as **"structural phenomenology"**.

10. Main recent development in the area:

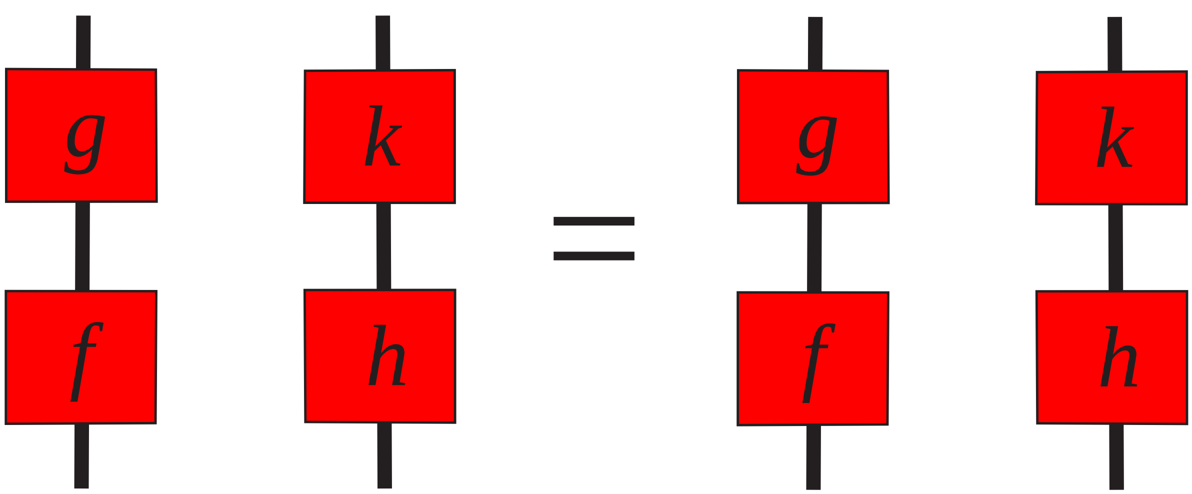
axiomatic reasoning ≡ diagrammatic reasoning

In particular, some axioms become trivial in diagrams!

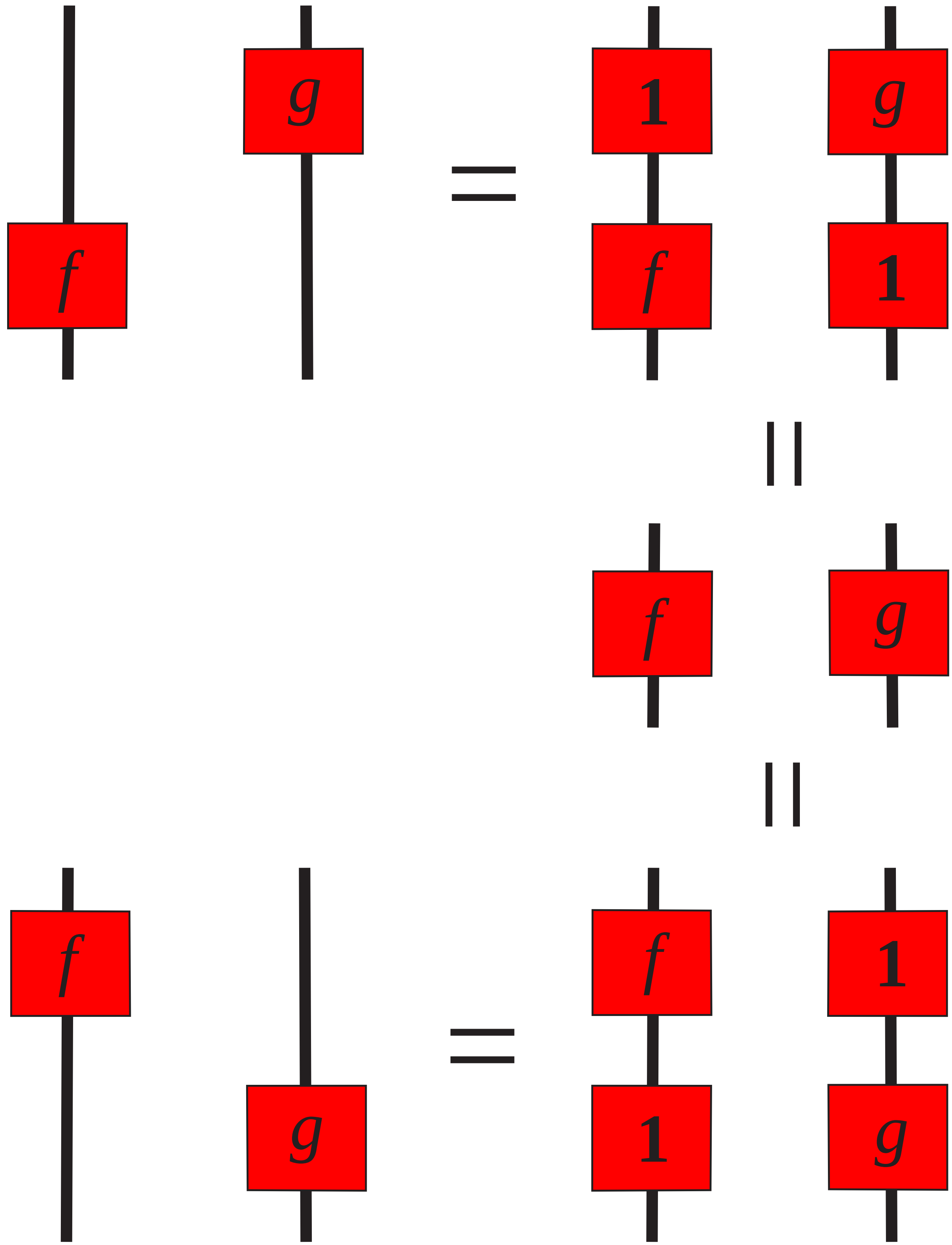
$g \circ f \equiv$ 

$f \otimes g \equiv$ 

$$(g \circ f) \otimes (k \circ h) \quad = \quad (g \otimes k) \circ (f \otimes h)$$



(wires = identities) ⇒ sliding boxes



NEW STRUCTURES FOR PHYSICS

A series of tutorials on category theory and its applications in physics to appear as a number of volumes in Springer Lecture Notes in Physics, including:

Introduction to categories and categorical logic
by Samson Abramsky and Nikos Tzevelekos.

Physics, topology, logic and computation: a Rosetta Stone
by John Baez and Mike Stay.

Categories for the practicing physicist
by Bob Coecke and Eric Oliver Paquette.

A survey of graphical languages for monoidal categories
by Peter Selinger.

For preliminary versions of these Google

`structures physics cafe`

which brings you to John Baez' n-category cafe.

Introduction to categories and categorical logic
by Samson Abramsky and Nikos Tzevelekos.

Physics, topology, logic and computation: a Rosetta Stone
by John Baez and Mike Stay.

Categories for the practicing physicist
by Bob Coecke and Eric Oliver Paquette.

A survey of graphical languages for monoidal categories
by Peter Selinger.

Linear logic for physicists
by Ross Duncan.

Proof nets as formal Feynman diagrams
by Rick Blute and Prakash Panangaden.

Geometry of Interaction and the dynamics of proof reduction
by Esfandiar Haghverdi and Philip Scott.

Dagger categories and formal distributions
by Rick Blute and Prakash Panangaden.

Compact monoidal categories from Linguistics to Physics
by Jim Lambek.

'What is a Thing?': Topos theory in the foundations of physics
by Andreas Döring and Chris Isham.

Domain theory and measurement
by Keye Martin.

Domain theory and general relativity
by Keye Martin and Prakash Panangaden.

Process, distinction, groupoids and Clifford algebras
by Basil Hiley.

Can a quantum computer run the von Neumann architecture?
by Peter Hines.

A categorical presentation of quantum computation with anyons
by Prakash Panangaden and Eric Oliver Paquette.

A **category** C consists of:

1. A family $|\mathbf{C}|$ of **objects** ;

2. For any $A, B \in |\mathbf{C}|$ a set $\mathbf{C}(A, B)$ of **morphisms** ;
   a shorthand for $f \in \mathbf{C}(A, B)$ is $A \xrightarrow{f} B$;

3. For all objects $A, B, C \in |\mathbf{C}|$ there is an associative **composition operation**

$$- \circ - : \mathbf{C}(A, B) \times \mathbf{C}(B, C) \to \mathbf{C}(A, C)$$

   with 2-sided units $\mathbf{1}_A \in \mathbf{C}(A, A)$, the **identities**.

Associativity justifies

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

and for identities we have

$$A \xrightarrow{\mathbf{1}_A} A \xrightarrow{f} B = A \xrightarrow{f} B = A \xrightarrow{f} B \xrightarrow{\mathbf{1}_B} B$$

Examples:

- $\mathbf{FdHilb}$: fin. dim. Hilbert spaces and linear maps.
- $\mathbf{CPM}$: fin. dim. Hilbert spaces and CP maps.
- $\mathbf{FdHilb}'$: $\mathbb{N}$ and linear maps.

Remark: The "lost" structure of the objects is "reincarnated" within the morphisms as "elements":

$$\mathcal{H} \quad \simeq \quad \left\{ f_\psi : \mathbb{C} \to \mathcal{H} :: 1 \mapsto \psi \mid \psi \in \mathcal{H} \right\}$$

The extra structure on these sets of morphisms can be captured in purely category theoretic terms.

- $\mathbf{Grp}$: groups and group homomorphisms.
- $\mathbf{Poset}$: posets and order-preserving maps.
- a group G: $\{\star\}$ and group elements.
- a poset P: **P** and $[- \leq -]$.

Remark: The extra structure of a group or a poset can be captured purely in category theoretic terms.

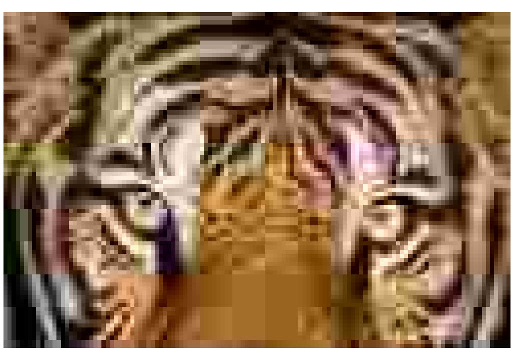
$\Longrightarrow$ group/order theory is instance of category theory.

‘Real world’ examples:

- **PhysProc**: physical systems, processes, after.
- **CProc**: classical systems, processes, after.
- **QProc**: quantum systems, processes, after.
- **QOpp**: quantum systems, operations, performed after, and with preparation procedures as elements.
- **ClQOpp**: closed quantum systems/operations.

Remark: Types keep us on worldlines for ‘after’.

Why does a tiger have stripes and a lion doesn’t?



$$\text{prey} \otimes \text{predator} \otimes \text{environment}$$

hunt ↓

$$\text{dead prey} \otimes \text{eating predator}$$

A **strict monoidal category** is a category with:

1. a **monoid structure** on $(|\mathbf{C}|, \otimes, \mathrm{I})$, that is,

$$A\otimes(B\otimes C) = (A\otimes B)\otimes C \qquad \mathrm{I}\otimes A = A = A\otimes\mathrm{I}\,,$$

2. for all $A, B, C, D \in |\mathbf{C}|$ an **operation**

$$-\otimes- : \mathbf{C}(A, B)\times\mathbf{C}(C, D) \to \mathbf{C}(A\otimes C, B\otimes D)$$

which is associative and has $\mathbf{1}_{\mathrm{I}}$ as its unit, and for all morphisms for which "types match" we have

$$(g\circ f)\otimes(k\circ h) = (g\otimes k)\circ(f\otimes h)$$

and for all $A, B \in |\mathbf{C}|$ we have

$$\mathbf{1}_A \otimes \mathbf{1}_A = \mathbf{1}_{A\otimes B}\,.$$

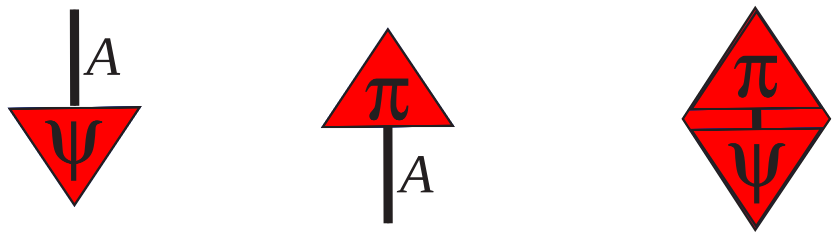
Examples:

- **CProc**: classical systems, processes, after, composite (classical) systems and processes.
- **QProc**: quantum systems, processes, after, composite (quantum) systems and processes.

$\Rightarrow$ axioms for compoundness i.e. ***systems' interaction***

Theorem. An equational statement is derivable from the axioms of strict monoidal categories $\Longleftrightarrow$ it is derivable in their graphical language.

$$s : \mathrm{I} \to \mathrm{I} \qquad \psi : \mathrm{I} \to A \qquad \pi : A \to \mathrm{I} \qquad \pi \circ \psi : \mathrm{I} \to \mathrm{I}$$



Definition. A **strict monoidal †-category** is a strict monoidal one with for all $A, B \in |\mathbf{C}|$ an **involution**

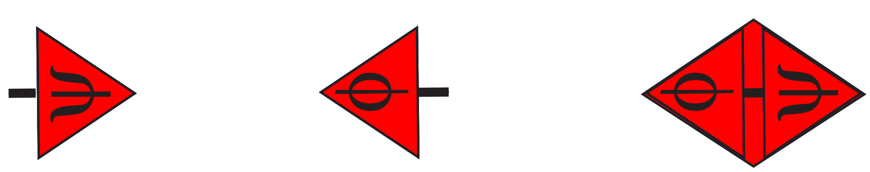
$$(-)^\dagger : \mathbf{C}(A, B) \to \mathbf{C}(B, A)$$

which is such that

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger \qquad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger \qquad \mathbf{1}_{\mathrm{I}}^\dagger = \mathbf{1}_{\mathrm{I}}$$

Theorem. Graphical language of strict monoidal †-categories is a 2-dim. extension of Dirac notation.

$$|\,\psi > \qquad < \phi\,| \qquad < \phi\,|\,\psi >$$



The set-theoretic verdict on strictness

For sets X, Y, Z we have

$$(x,(y,z)) \neq ((x,y),z) \qquad\qquad (\star,x) \neq x \neq (x,\star)$$

so neither

$$X \times (Y \times Z) = (X \times Y) \times Z \quad \text{nor} \quad \{\star\} \times X = X = X \times \{\star\}$$

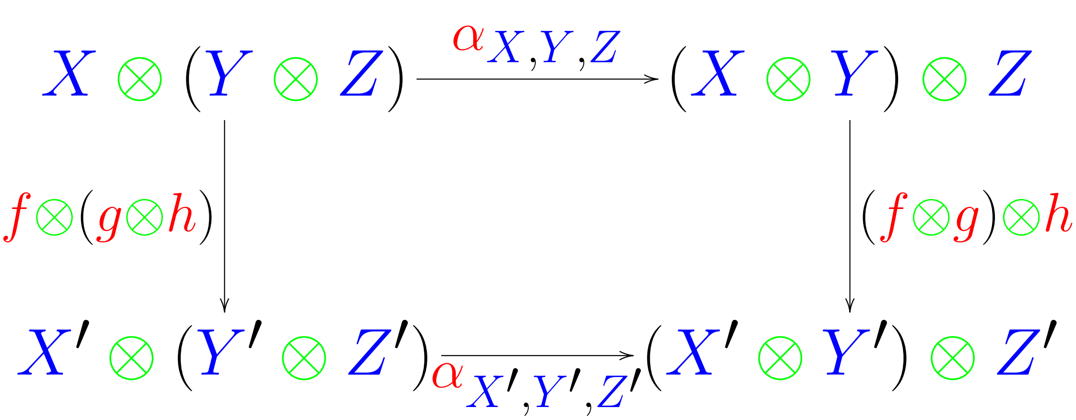
What we do have is

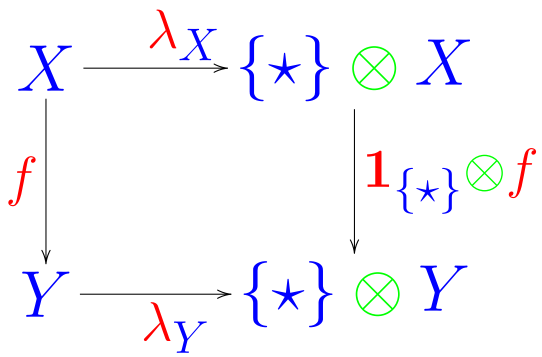
$$X \times (Y \times Z) \simeq (X \times Y) \times Z \qquad\qquad \{\star\} \times X \simeq X \simeq X \times \{\star\}$$

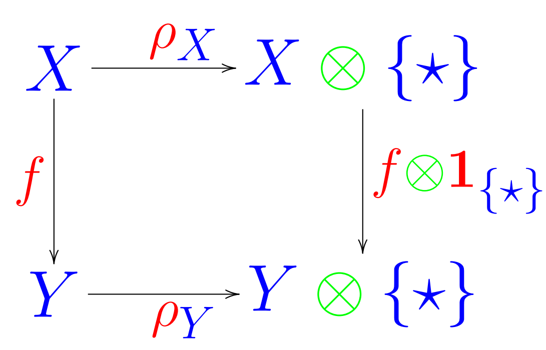
but this is a statement on objects only, hence not really of use.

Of use is the existence of indexed morphisms

$$\{\alpha_{X,Y,Z}\}_{X,Y,Z} \quad \{\lambda_X\}_X \quad \{\rho_X\}_X$$







A **monoidal category** consists of the following data:

1. a category $\mathbf{C}$;
2. an object $\mathrm{I} \in |\mathbf{C}|$;
3. a *bifunctor* $-\otimes-$, that is, an operation both on objects and on morphisms as in the definition of strict monoidal category, which moreover satisfies

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h) \quad \text{and} \quad 1_A \otimes 1_B = 1_{A \otimes B}$$

   for objects $A, B \in |\mathbf{C}|$ and morphisms f, g, h, k of appropriate type ;

4. three natural isomorphisms

$$\alpha = \{A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C \mid A, B, C \in |\mathbf{C}|\},$$

$$\lambda = \{A \xrightarrow{\lambda_A} \mathrm{I} \otimes A \mid A \in |\mathbf{C}|\} \quad \text{and} \quad \rho = \{A \xrightarrow{\rho_A} A \otimes \mathrm{I} \mid A \in |\mathbf{C}|\},$$

   satisfying the above commutative diagrams, and such that we also have

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha_-} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_-} & ((A \otimes B) \otimes C) \otimes D \\ \Big\downarrow{\scriptstyle 1_A \otimes \alpha_-} & & & & \Big\uparrow{\scriptstyle \alpha_- \otimes 1_D} \\ A \otimes ((B \otimes C) \otimes D) & & \xrightarrow[\alpha_-]{\hspace{6em}} & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

   for all $A, B, C, D \in |\mathbf{C}|$, and,

$$\begin{array}{ccc} A \otimes B & \xrightarrow{1_A \otimes \lambda_B} & A \otimes (\mathrm{I} \otimes B) \\ & {\scriptstyle \rho_A \otimes 1_B} \searrow & \Big\downarrow{\scriptstyle \alpha_{A,\mathrm{I},B}} \\ & & (A \otimes \mathrm{I}) \otimes B \end{array}$$

   for all $A, B \in |\mathbf{C}|$, and, finally,

$$\lambda_{\mathrm{I}} = \rho_{\mathrm{I}} .$$

A monoidal category is *symmetric* if there is a fourth natural isomorphism

$$\sigma = \{A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A \mid A, B \in |\mathbf{C}|\}$$

satisfying

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\sigma_{A,B}} & B \otimes A \\
{\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\
C \otimes D & \xrightarrow[\sigma_{C,D}]{} & D \otimes C
\end{array}$$

and such that we also have

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\sigma_{A,B}} & B \otimes A \\
& {\scriptstyle 1_{A\otimes B}}\searrow & \downarrow{\scriptstyle \sigma_{B,A}} \\
& & A \otimes B
\end{array}$$

for all $A, B \in |\mathbf{C}|$, and

$$\begin{array}{ccc}
A & \xrightarrow{\lambda_A} & \mathrm{I} \otimes A \\
& {\scriptstyle \rho_A}\searrow & \downarrow{\scriptstyle \sigma_{\mathrm{I},A}} \\
& & A \otimes \mathrm{I}
\end{array}$$

for all $A \in |\mathbf{C}|$, and

$$\begin{array}{ccccc}
A \otimes (B \otimes C) & \xrightarrow{\alpha_-} & (A \otimes B) \otimes C & \xrightarrow{\sigma_{(A\otimes B),C}} & C \otimes (A \otimes B) \\
{\scriptstyle 1_A\otimes\sigma_{B,C}}\downarrow & & & & \downarrow{\scriptstyle \alpha_-} \\
A \otimes (C \otimes B) & \xrightarrow[\alpha_-]{} & (A \otimes C) \otimes B & \xrightarrow[\sigma_{A,B}\otimes 1_C]{} & (C \otimes A) \otimes B
\end{array}$$

for all $A, B, C \in |\mathbf{C}|$.

The set-theoretic verdict on strictness is harsh!

Coherence: Any two ways to go from expression

$$\Lambda(A_1, \ldots, A_n, C_1, \ldots, C_m) \quad \text{to} \quad \Xi(A_1, \ldots, A_n, C_1, \ldots, C_m)$$

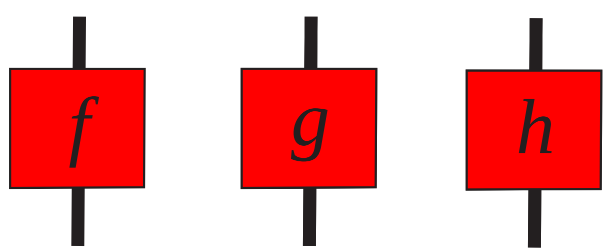
by composing natural isos and identities are equal.

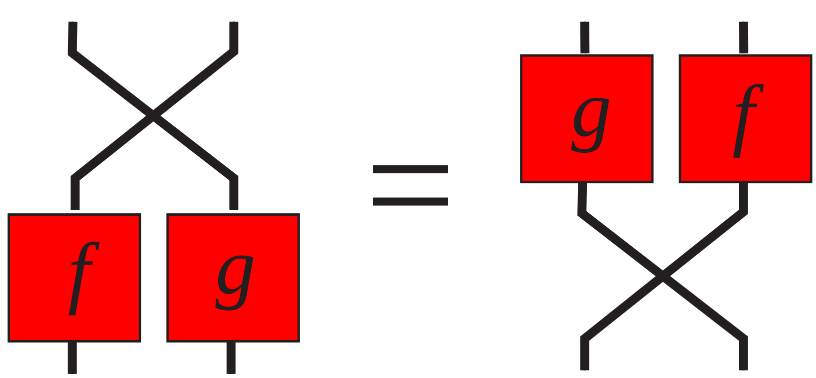
Coherence theorem - MacLane. The 3/7 conditions stated in the above definition suffice for this purpose.

Pffffffffffffffffffffff . . .

. . . sometimes miracles do happen:

Strictification theorem - MacLane. Any monoidal category **C** is strongly monoidally categorically equivalent to a strict monoidal category **D**.





Examples of monoidal categories:

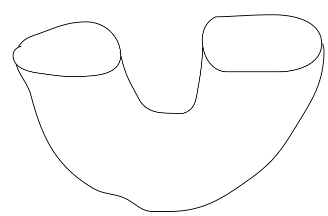
category	classical-like	quantum-like	other
Set	$\times$		$+$
Rel	$+$	$\times$	
FdHilb	$\oplus$	$\otimes$	
nCob		$+$	

Examples of quantum-like monoidal categories:

FdHilb :

$$\eta_{\mathcal{H}} : \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H} :: 1 \mapsto \sum_i |ii\rangle$$

n-**Cob** :



Rel :

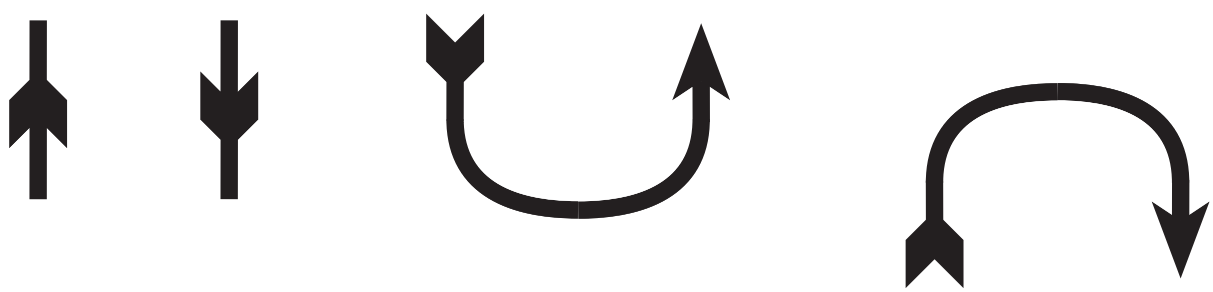
$$\eta_X := \{(*, (x, x)) | x \in X\}$$

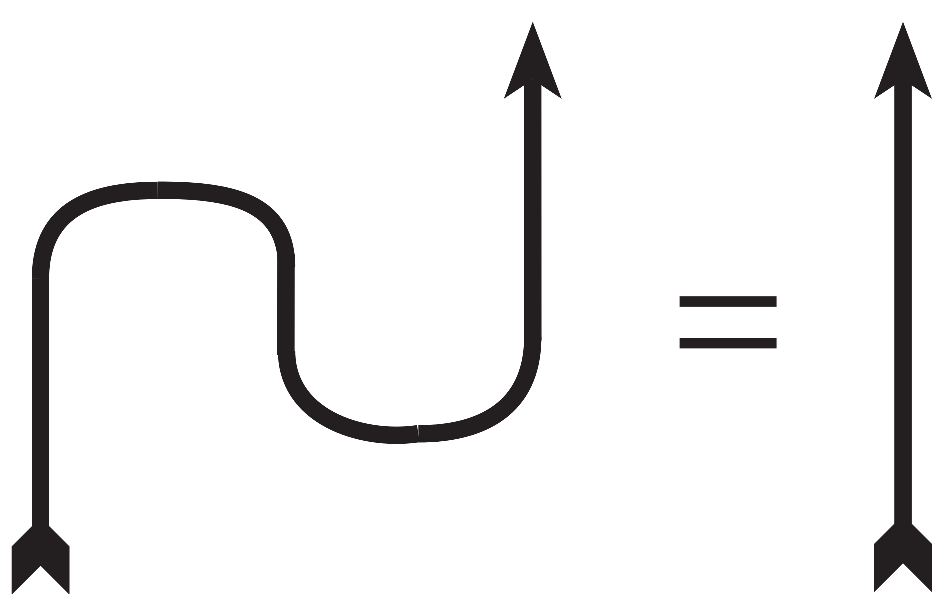
Completeness theorem - Selinger. An equational statement in the language of **†-compact categories** is provable from the axioms if and only if it holds for **finite dimensional Hilbert spaces**.

Quantum like = compact:

$$(A\,,A^*,\ \eta_A : \mathrm{I} \to A^* \otimes A\,,\ \epsilon_A : A \otimes A^* \to \mathrm{I})$$

$$\begin{array}{ccccc} A & \xleftarrow{\ \simeq\ } & \mathrm{I}\otimes A & \xleftarrow{\ \epsilon_A\otimes \mathbf{1}_A\ } & (A\otimes A^*)\otimes A \\ {\scriptstyle \mathbf{1}_A}\uparrow & & & & \uparrow{\scriptstyle\simeq} \\ A & \xrightarrow[\simeq]{} & A\otimes \mathrm{I} & \xrightarrow[\mathbf{1}_A\otimes\eta_A]{} & A\otimes(A^*\otimes A)\end{array}$$





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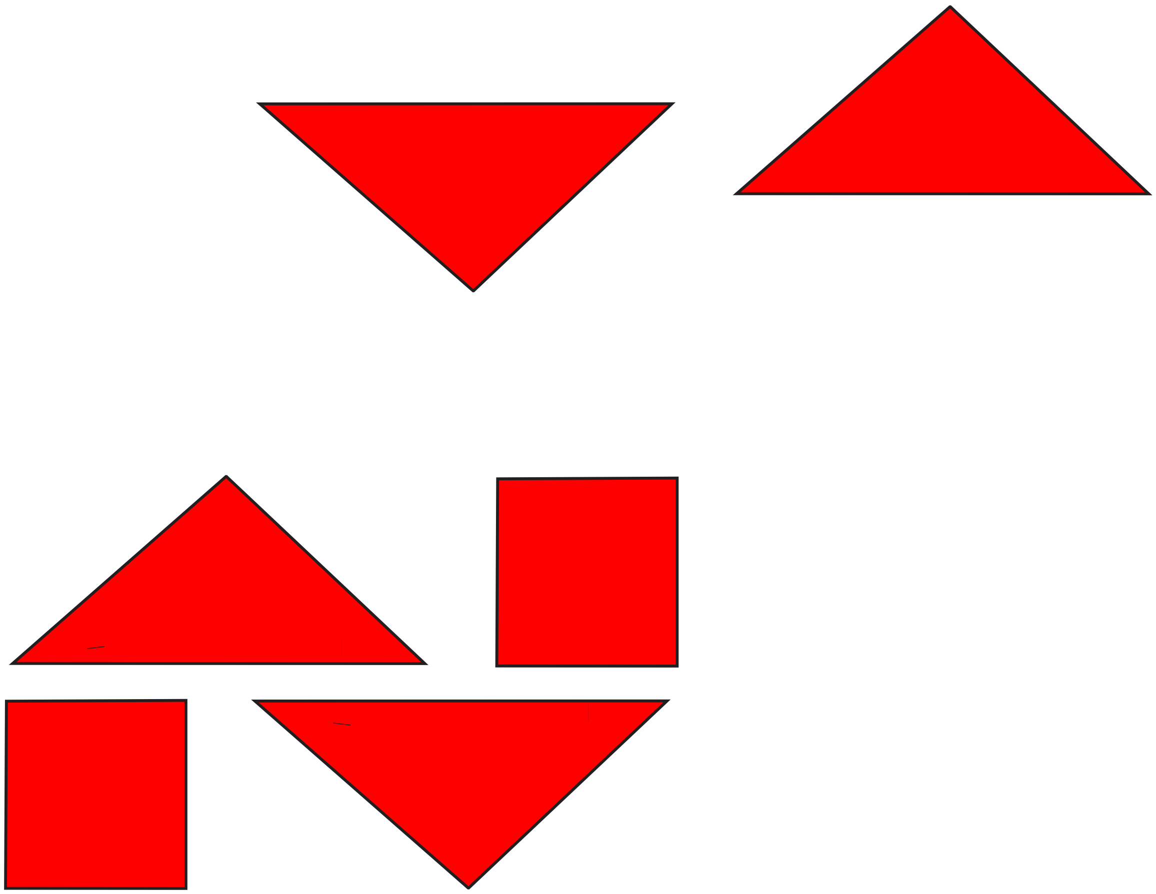

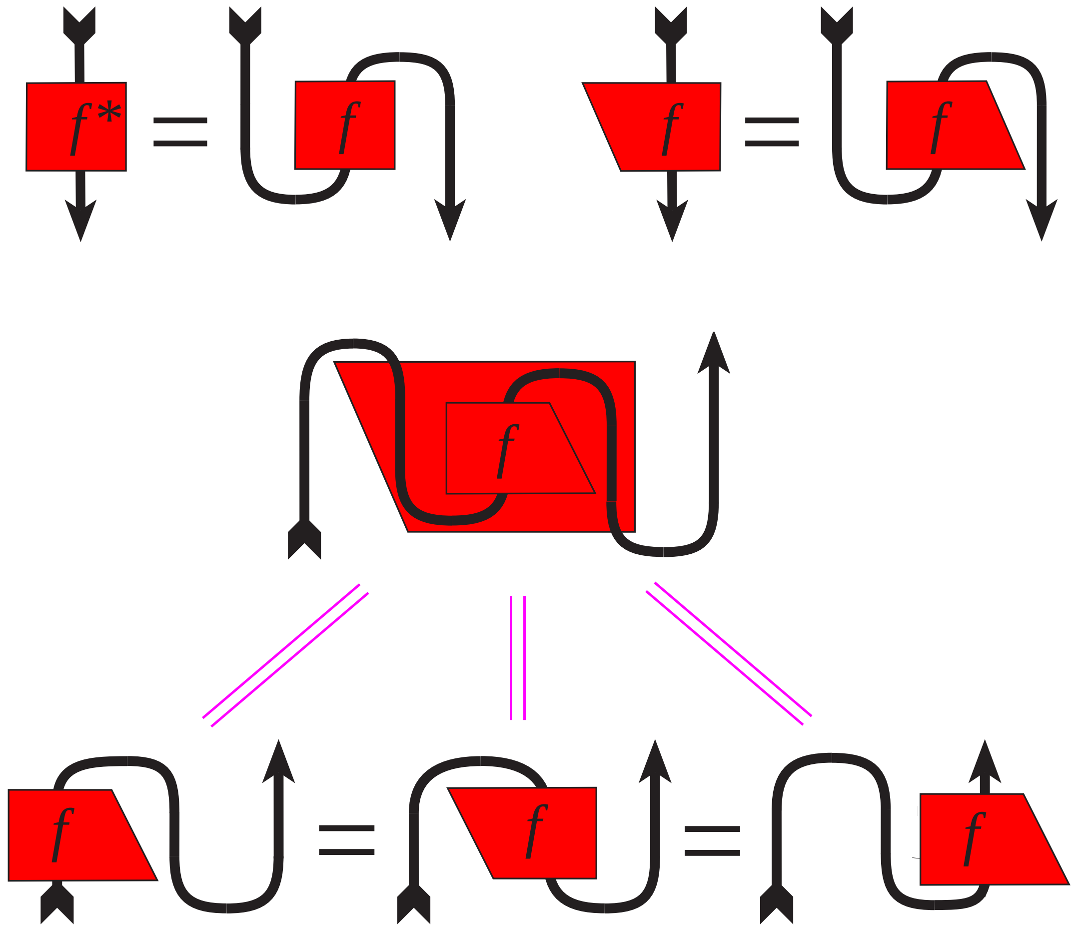

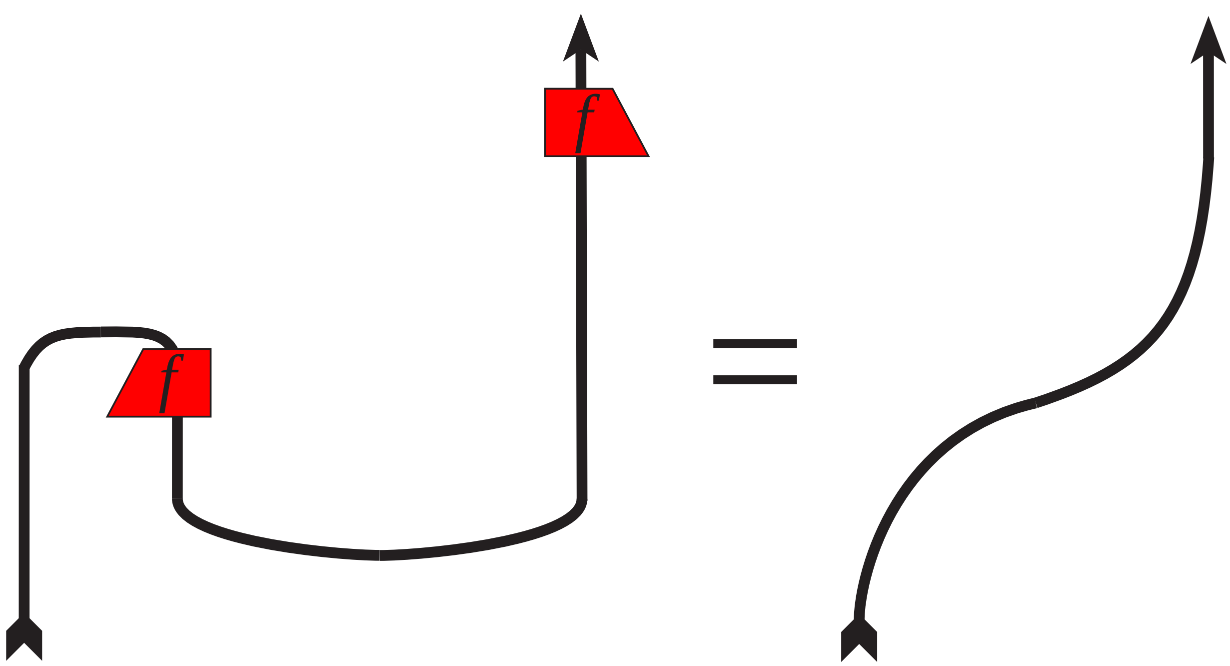

S. Abramsky & B. C. A categorical semantics of quantum protocols. quant-ph/0402130

B. C. Kindergarten quantum mechanics. quant-ph/0510032

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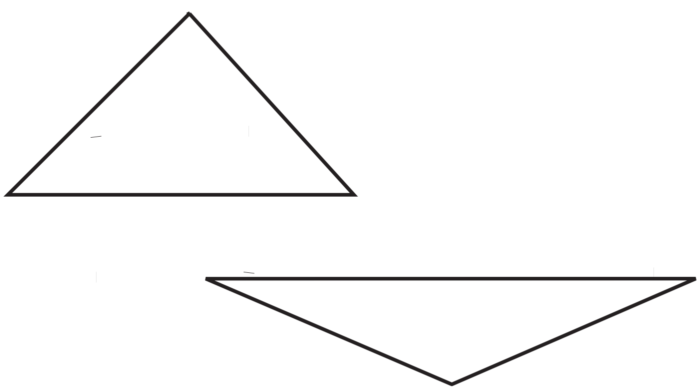

ALICE

BOB

ALICE

BOB

Classical like = Cartesian = product structure:

A **product** of objects A_1 and A_2 in a category $\mathbf{C}$ is a triple consisting of another object $A_1 \times A_2 \in |\mathbf{C}|$ and

$$\pi_1 : A_1 \times A_2 \to A_1 \quad \text{and} \quad \pi_2 : A_1 \times A_2 \to A_2$$

which are such that

$$(\pi_1 \circ -, \pi_2 \circ -) : \mathbf{C}(C, A_1 \times A_2) \to \mathbf{C}(C, A_1) \times \mathbf{C}(C, A_2)$$

admits an inverse $\langle -, - \rangle_{C,A_1,A_2}$ for all $C \in |\mathbf{C}|$. A category $\mathbf{C}$ is **Cartesian** if any pair of objects $A, B \in |\mathbf{C}|$ admits a (not necessarily unique) product.

category	classical-like	quantum-like	other
Set	$\times$		$+$
Rel	$+$	$\times$	
FdHilb	$\oplus$	$\otimes$	
nCob		$+$	

No-cloning theorem - Abramsky. Quantum-like categories that are also <u>semi-classical</u>-like are <u>trivial</u>.

Hierarchy of categorical concepts:

((strict) mon.) category

$$\mathbf{C}$$

e.g. group

((strict) mon.) functor := morph. of categories

$$F:\mathbf{C}\longrightarrow\mathbf{D}$$

e.g. group representation

((strict) mon.) natural transf. := morph. of functors

$$\alpha:F\longrightarrow G$$

e.g. change of group representation

Topological QFT := symmetric monoidal functor

$$F:n\mathbf{Cob}\longrightarrow\mathbf{FdHilb}$$

OTHER TOPICS

- **internal structures e.g. (unbiassed) bases**
- **categorical matrix calculus**
- **monads and algebras**
- **comonads and coalgebras e.g. measurement**
- **adjoint functors**
- **categorical logic**
- **enriched categories**
- **higher-dimensional categories**