Axiomatic description of mixed states from Selinger's CPM-construction

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Quantum Mechanical Formalism

as it was created by von Neumann

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Quantum Mechanical Formalism

as it was denounced by von Neumann

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Quantum Mechanical Formalism

as it was made <u>cute</u> but <u>messy</u> by Dirac

von Neumann's axioms of quantum theory, . . .

Single system $\Longrightarrow$ Hilbert space

Evolving system $\Longrightarrow$ Unitary operator

Compound system $\Longrightarrow$ Tensor product

Measuring system $\Longrightarrow$ Self-adjoint operator

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von Neumann's axioms of quantum theory, . . .

System

Operations

Compoundness

Information retrieval

---

von Neumann's axioms of quantum theory, . . .

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von Neumann's analysis of quantum measurement, . . .

Quantum measurement:

$$H = \sum_i a_i \cdot \mathrm{P}_i \quad \text{with} \quad \psi \xmapsto{\langle \psi | \mathrm{P}_i(\psi) \rangle} \mathrm{P}_i(\psi)$$

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Fixed points = states which yield *answer with certainty* are **rays**

$$ray(\psi) = \{c \cdot \psi \mid c \in \mathbb{C}\}$$

and, more generally, **linear subspaces**.

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von Neumann's conclusions:

- The fully abstract space of states is the space of rays and the significant structure thereon is carried by the linear subspaces, which are the fixpoints of the corresponding projectors.
- State vectors ψ carry a redundancy: global scalar multiples

Birkhoff-von Neumann (re-)axiomatization attempt, . . .

Slogan:

The “non-distributive” lattice of linear subspaces $\mathbb{L}(\mathcal{H})$ captures the structural essence of quantum theory.

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Typically distributivity is weakened to “orthomodularity”:

$$\mathrm{P}_b(a) \leq c \iff a \leq b \rightsquigarrow c$$

in analogy to a Heyting algebra:

$$a \wedge b \leq c \iff a \leq b \Rightarrow c$$

But $\rightsquigarrow$ does not admit any useful logical mechanisms, causing so-called “quantum logic” to be “quantum non-logic”.

Defects of B-vN (re-)axiomatization, . . .

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$\Rightarrow$ again denounced by von Neumann

Dirac “notation” for projectors and probabilities, . . .

“ket”: $|\psi\rangle$

“bra”: $\langle\psi|$

“bra-ket”: $\langle\psi|\phi\rangle \in \mathbb{C}$

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"probability": $\langle\phi|\psi\rangle\langle\psi|\phi\rangle = \langle\psi|\mathrm{P}_i(\psi)\rangle$

"mixed states": $\sum_i w_i|\psi_i\rangle\langle\psi_i| \neq |\phi\rangle\langle\phi|$

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... as a consequence of this, it becomes a mess whenever $-\otimes-$ comes into play, since many expressions become superfluous

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... but it is extremely successful: almost everybody uses it!

von Neumann analysis of quantum theory

System

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HERE ⇒ Information retrieval ⇐ HERE

defect both of von Neumann's and Dirac's

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Information retrieval

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focus of our analysis of quantum theory

System

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our analysis of quantum compoundness, . . .

Mathematically we consider:

- **categorical algebra**

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Conceptually we introduce:

- **operational foundation**
- **types reflecting kinds**
- **compositionality**

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General result:

- **Quantum theory can be completely formulated in terms of an appropriate axiomatization of compoundness.**

- **Samson Abramsky** & **BC** ('04) *A categorical semantics for quantum protocols*; IEEE-LiCS; quant-ph/0402130

- **Peter Selinger** ('05) ... *mixed states & CPMs*; his www

- **BC** & **Dusko Pavlovic** ('06) *Quantum measurements* ...; soon

- **BC** & **Eric Paquette** ('06) *POVMs & Naimark's thm* ...; soon

- **BC** ('05) *Kindergarten Quantum Mechanics*; quant-ph/0510032

- **BC** ('05) *Introducing Categories to Practicing Physicists*; soon

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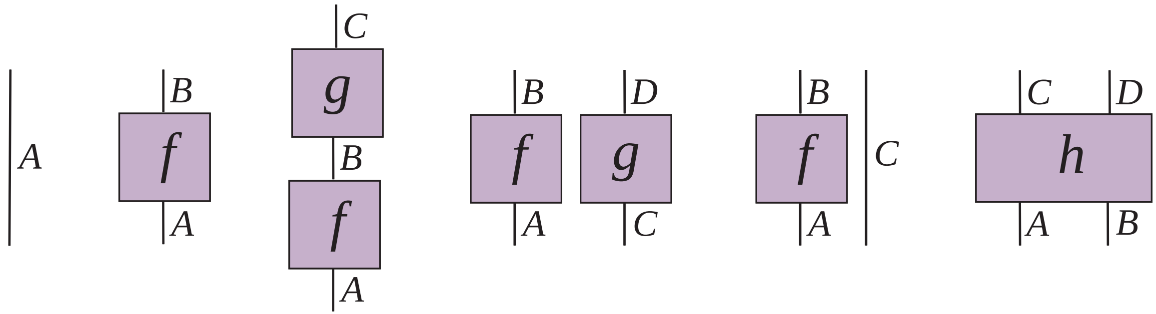
It both “extends” and “formalises” Dirac’s bra-kets in 2-D by distinguishing between sequential & parallel modes.

1
Categorical Quantum Formalism

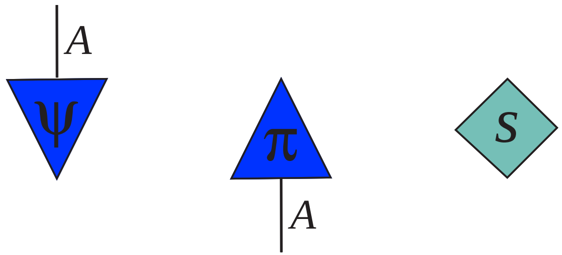
$\otimes$-structure of *symmetric monoidal category* comprising

$$A \otimes B \qquad A \otimes C \xrightarrow{f \otimes g} B \otimes D \qquad A \otimes C \xrightarrow{h} B \otimes D$$

is captured by the graphical representation:

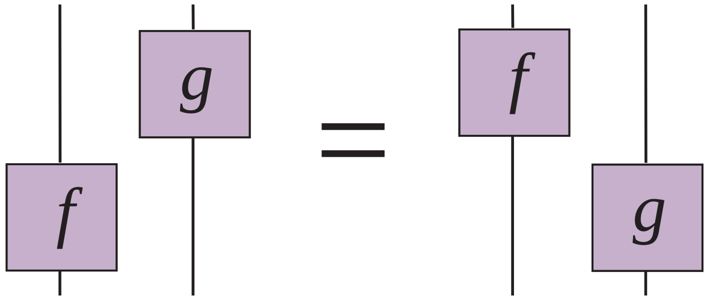


with has special cases kets, bras and scalars:

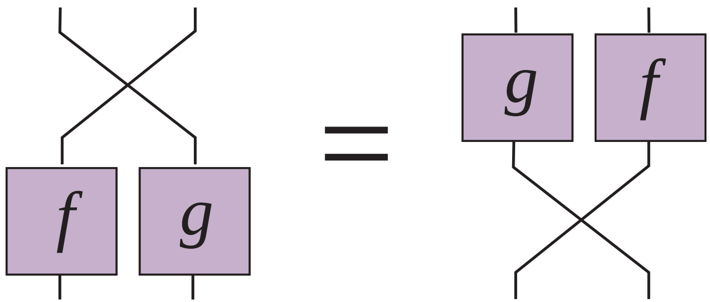


This graphical representation implicitly satisfies certain laws.

Examples are bifunctoriality

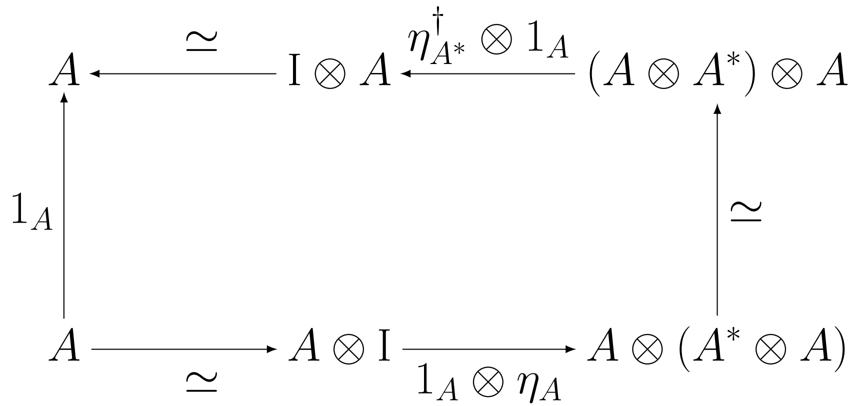


and naturality of symmetry:



Symmetric monoidal category with

- **contravariant $\otimes$-involution adjoint** $f_{A\to B} \mapsto f^{\dagger}_{B\to A}$ **;**
- **involution dual** $A \mapsto A^{*}$**;**
- **Units** $\eta_A : \mathrm{I} \to A^* \otimes A$ **with** $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$



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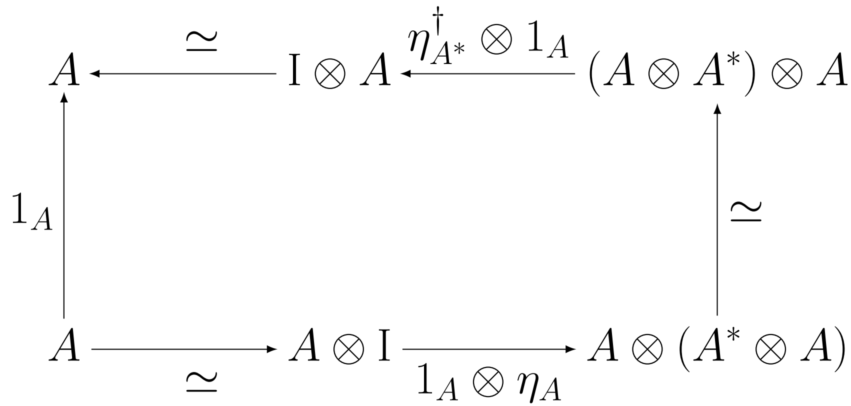

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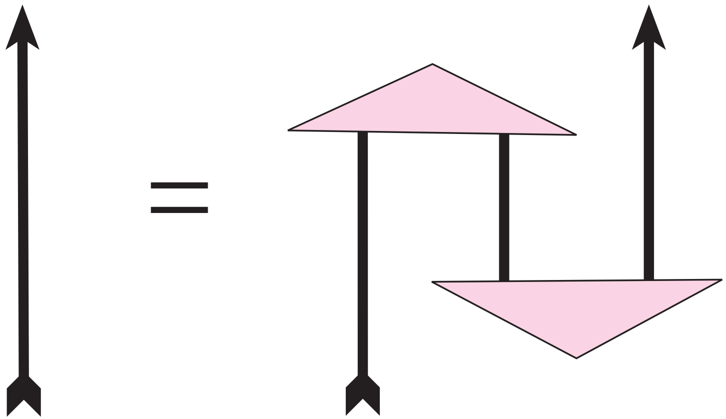

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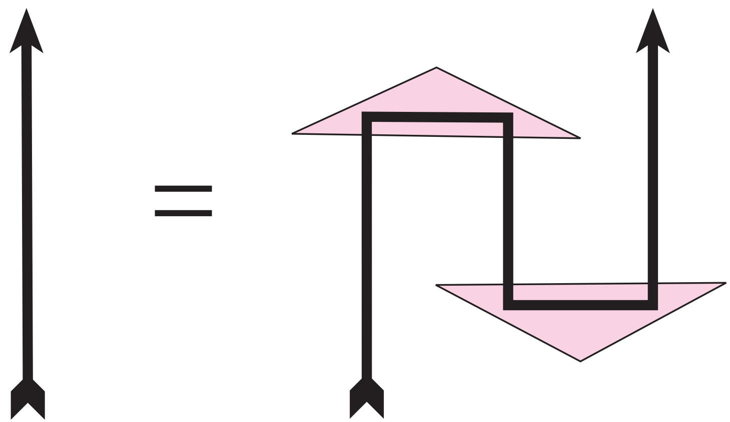

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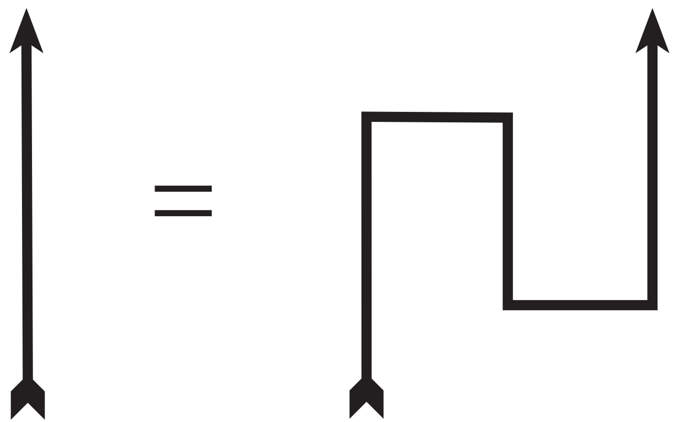

A trace structure:

$$f : C \otimes A \to C \otimes B \quad \mapsto \quad \mathrm{Tr}_C(f) : A \to B$$

arises as

$$\begin{CD}
B @<{\simeq}<< \mathrm{I} \otimes B @<{\eta_C^\dagger \otimes 1_B}<< C^* \otimes C \otimes B \\
@A{\mathrm{Tr}_C(f)}AA @. @AA{1_{C^*} \otimes f}A \\
A @>>{\simeq}> \mathrm{I} \otimes A @>>{\eta_C \otimes 1_A}> C^* \otimes C \otimes A
\end{CD}$$

Joyal, Street & Verity (1995) *Traced Monoidal Categories*.

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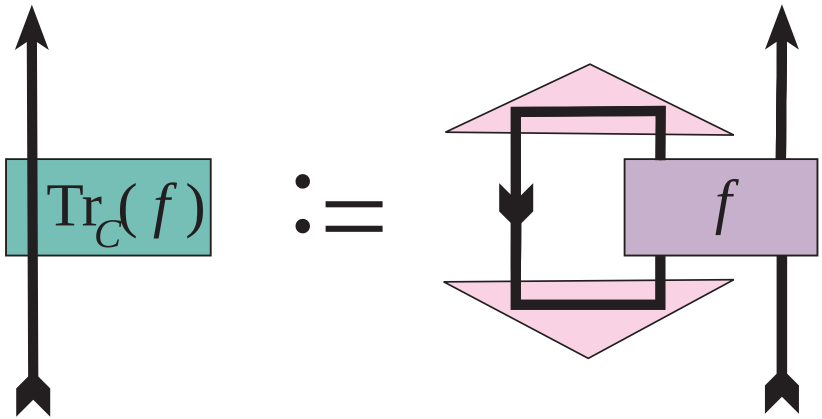

Joyal, Street & Verity (1995) ***Traced Monoidal Categories*****.**

A full trace:

$$f : A \to A \quad \mapsto \quad \mathrm{Tr}(f) : \mathrm{I} \to \mathrm{I}$$

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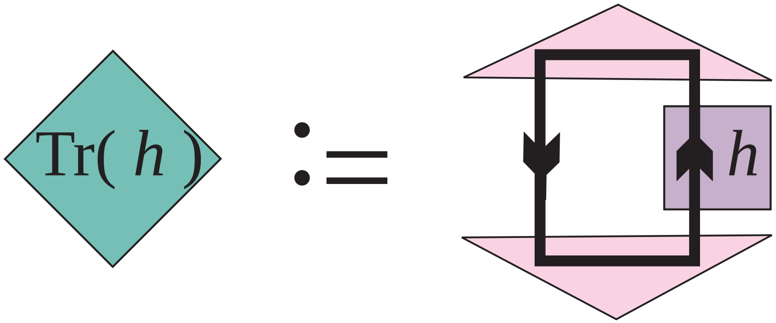

Joyal, Street & Verity (1995) ***Traced Monoidal Categories*****.**

The “contravariant” involution

$$f : A \to B \quad \mapsto \quad f^{*} : B^{*} \to A^{*}$$

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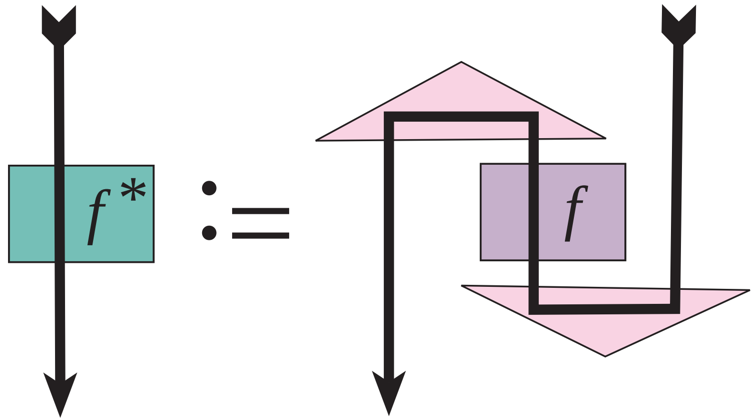
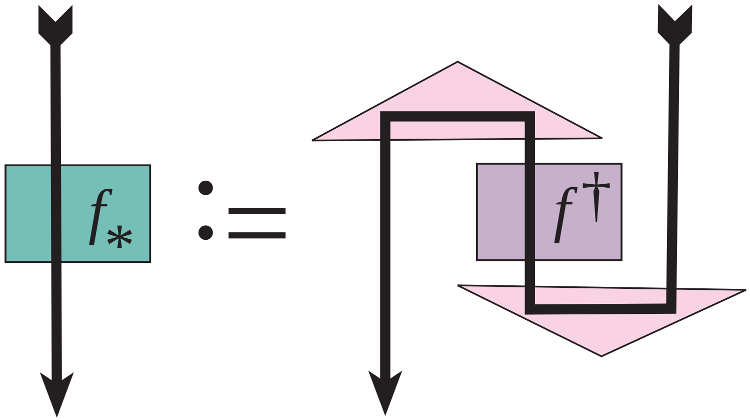


Mike Barr (1979) $*$-*Autonomous Categories*.

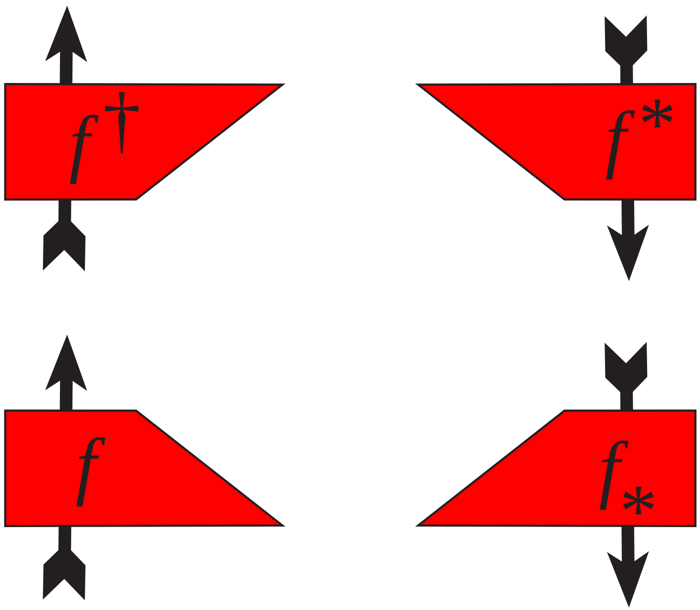
The "covariant" involution

$$f : A \to B \quad \mapsto \quad f_{*} : A^{*} \to B^{*}$$

arises as

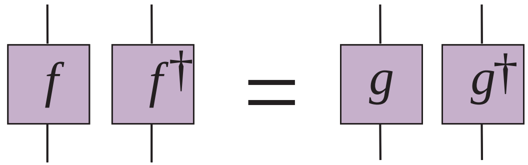



f*
=
f
f*
=
f†


f†
f*
f
f*

2
Projective Categorical Quantix

Theorem. If

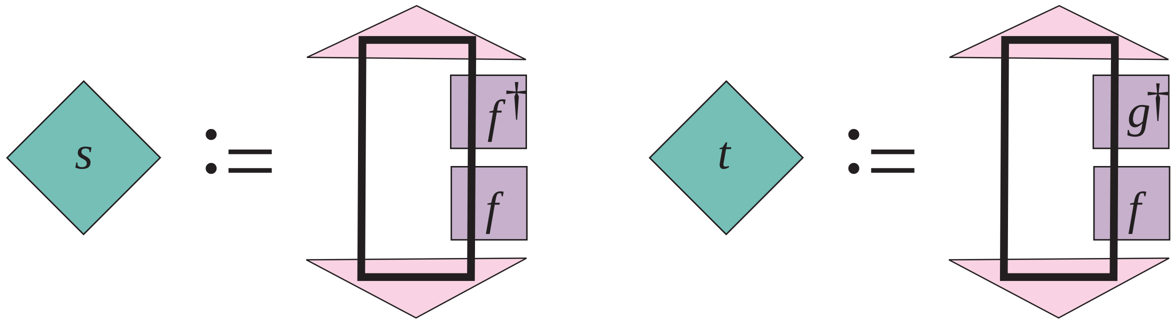


then there exist diamonds s, t such that:

Formally:

$$f \otimes f^{\dagger} = g \otimes g^{\dagger} \implies \exists s, t : \; s \bullet f = t \bullet g \; , \; s \circ s^{\dagger} = t \circ t^{\dagger}$$

Proof.



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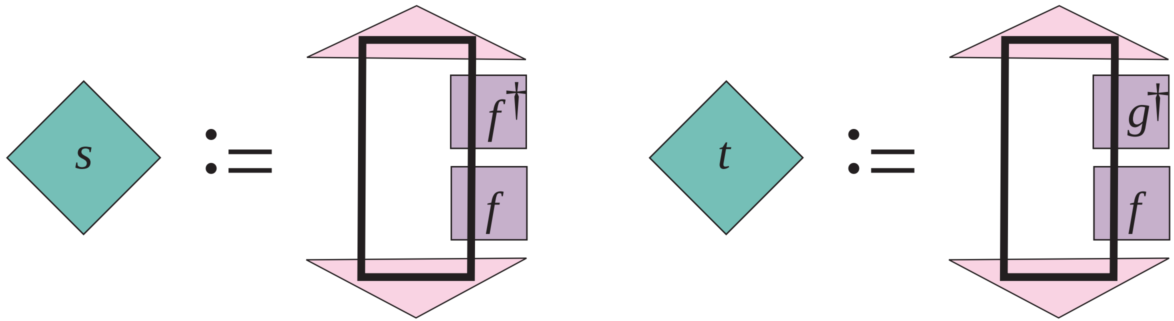

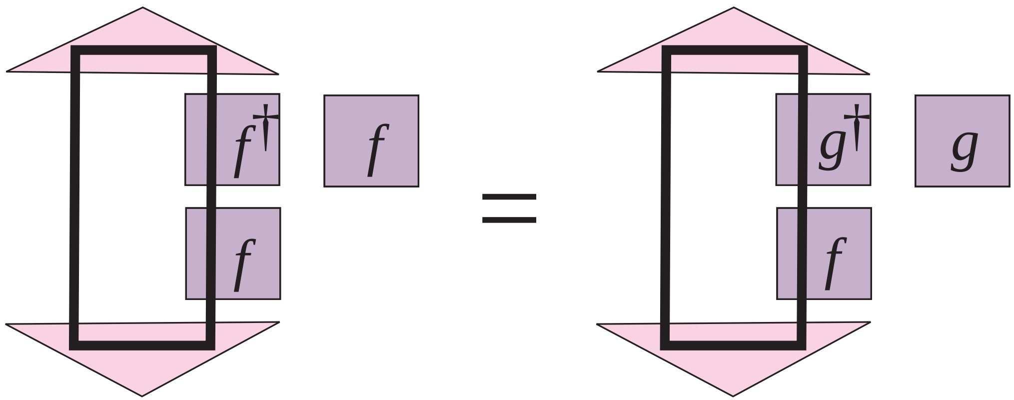

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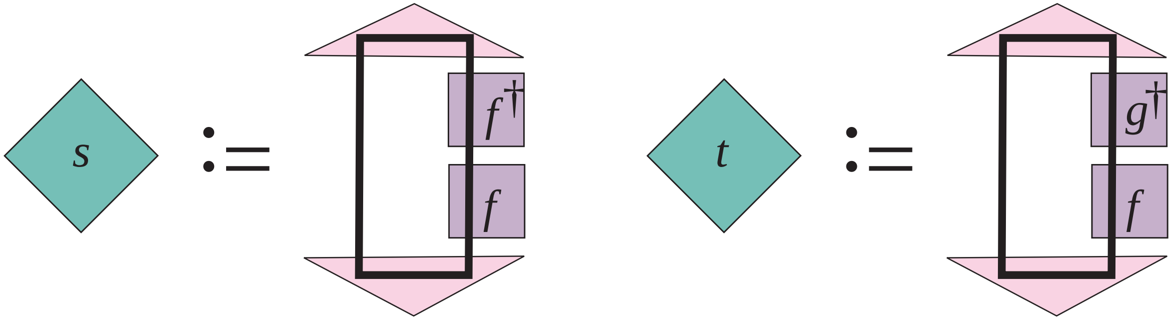

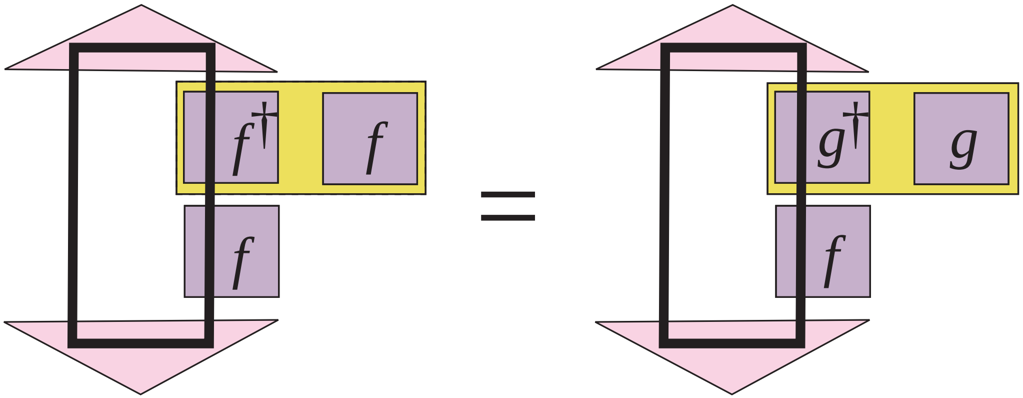

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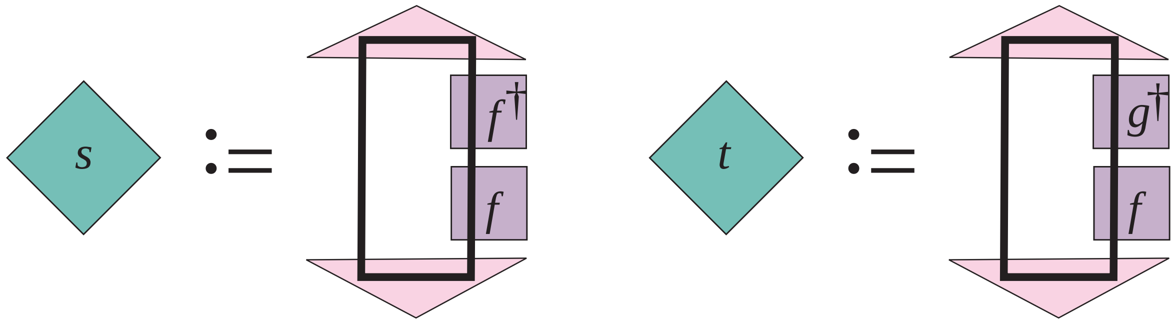

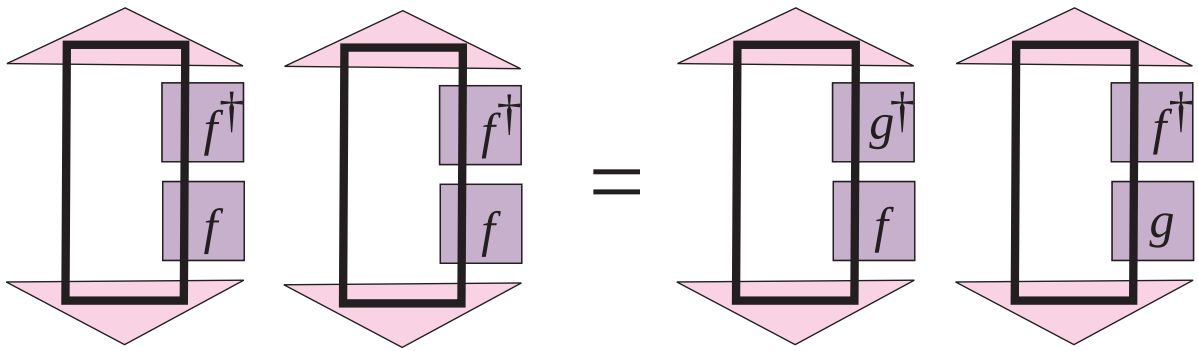

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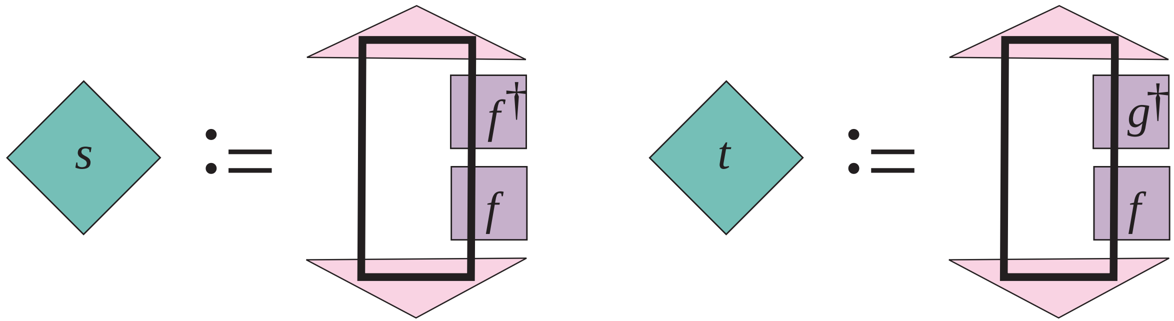

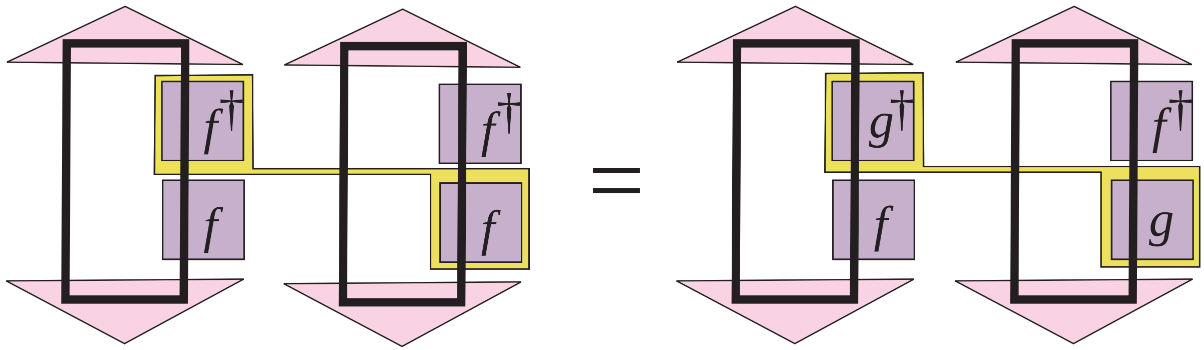

$$\mathbf{Proj} \circ \mathbf{Proj} \circ \mathbf{Proj} \circ \ldots \; ?$$

Preparation-state agreement axiom is, equivalently:

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$$\mathrm{P}_{f} = \mathrm{P}_{g} \implies f = g$$

$$\mathrm{P}_{\psi} = \mathrm{P}_{\phi} \implies \psi = \phi$$

3
Mixed State Categorical Quantix Mark I

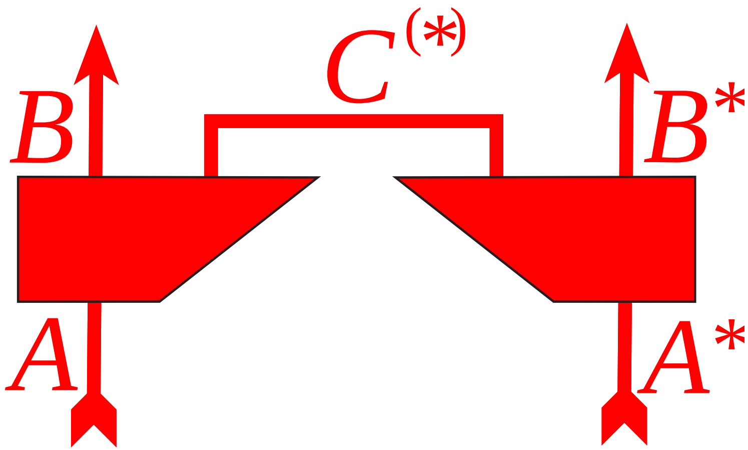
The †-compact functor

$$\mathsf{Pure} : \mathbf{C} \longrightarrow \mathbf{CPM}(\mathbf{C}) :: f \longrightarrow f \otimes f_* \, .$$

where

$$\mathbf{CPM}(\mathbf{C})(A, B) := \{(1_B \otimes \eta^{\dagger}_{C^*} \otimes 1_{B^*}) \circ (f \otimes f_*) \mid f : A \longrightarrow B \otimes C\}$$

embeds pure states within CPMs and mixed states i.e.



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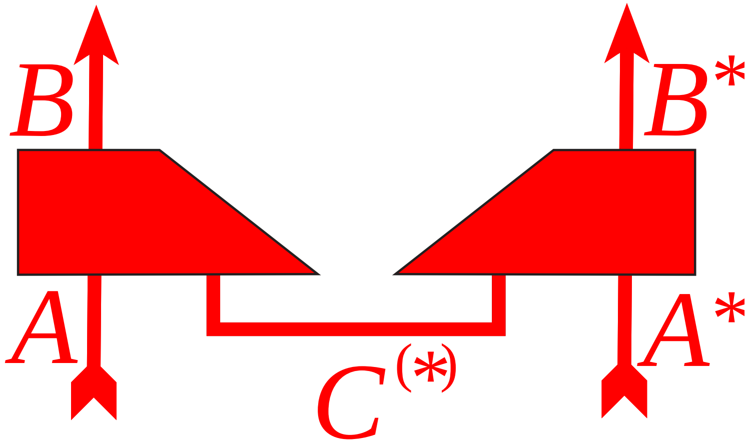

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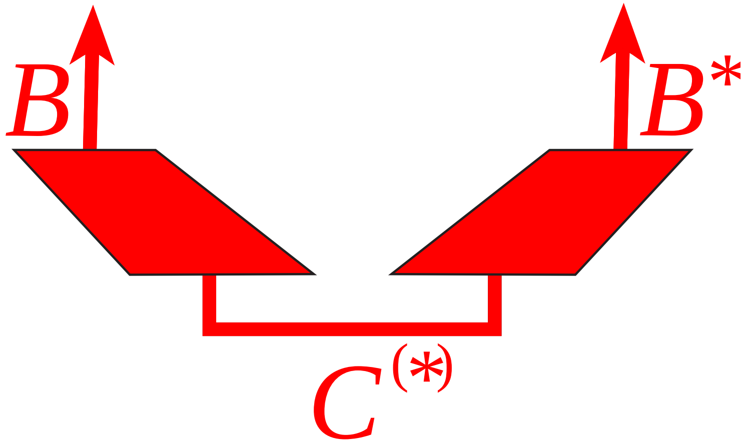

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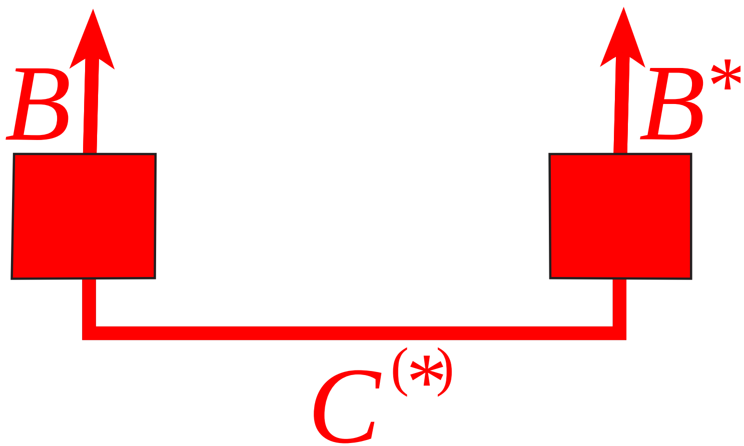

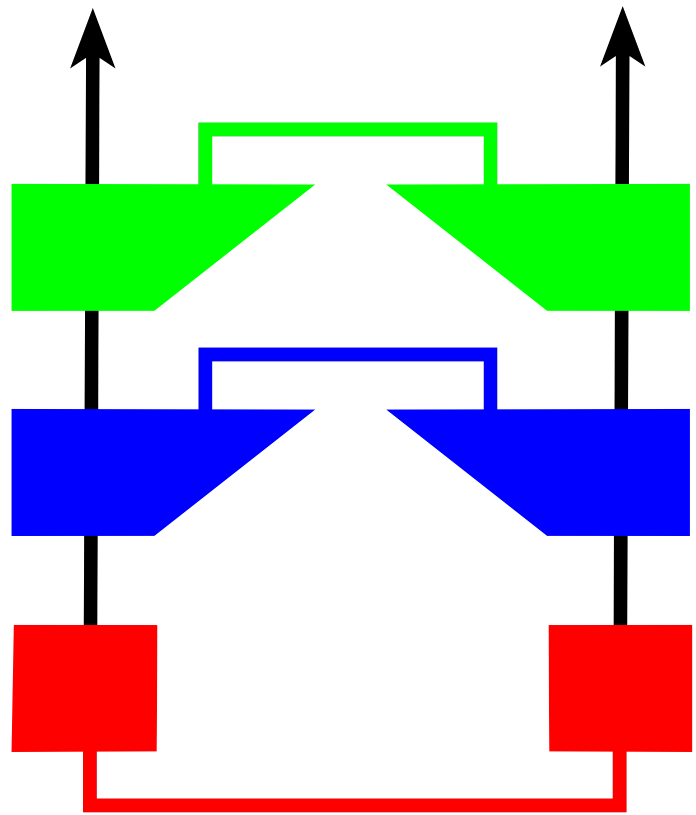

In Dirac notation:

$$|\,\psi\rangle\langle\psi\,| \mapsto f|\,\psi\rangle\langle\psi\,|f^\dagger$$

and for a CPM:

$$|\,\psi\rangle\langle\psi\,| \mapsto f(1_C \otimes |\,\psi\rangle\langle\psi\,|)f^\dagger$$

Compositionally we need to consider:

$$|\,\psi\rangle \otimes |\,\psi\rangle_*$$

yielding:

$$|\,\psi\rangle \otimes |\,\psi\rangle_* \mapsto (f \otimes f_*)(|\,\psi\rangle \otimes |\,\psi\rangle_*)$$

and for a CPM:

$$|\,\psi\rangle \otimes |\,\psi\rangle_* \mapsto (1_B \otimes \eta_C^\dagger \otimes 1_{B^*})(f \otimes f_*)(|\,\psi\rangle \otimes |\,\psi\rangle_*)$$

4
Mixed State Categorical Quantix Mark II

While

$$(\mathbf{Proj} \circ \mathbf{Proj})(\mathbf{FdHilb}) \neq \mathbf{Proj}(\mathbf{FdHilb})$$

we have

$$(\mathbf{CPM} \circ \mathbf{CPM})(\mathbf{FdHilb}) \neq \mathbf{CPM}(\mathbf{FdHilb})$$

A *$\perp$-structure* on a $(\otimes, \dagger)$-category $\mathbf{C}$ comprises

1. a *maximally mixed state* $\perp_A : \mathrm{I} \to A$ for each object with

$$\perp_{\mathrm{I}} = 1_{\mathrm{I}} \qquad\qquad \perp_{A\otimes B} = (\perp_A \otimes \perp_B) \circ \lambda_{\mathrm{I}}$$

2. a sub-$(\otimes, \dagger)$-category $\mathbf{C}_\Sigma$ of *pure states* with $\ulcorner 1 \urcorner$-structure,

which are such that for all $f, g \in \mathbf{C}_\Sigma$ we have

$$f \circ f^\dagger = g \circ g^\dagger \iff f \circ \perp_{\mathsf{dom}(f)} = g \circ \perp_{\mathsf{dom}(g)}$$

From

$$f \circ f^\dagger = g \circ g^\dagger \iff f \circ \bot_{\mathsf{dom}(f)} = g \circ \bot_{\mathsf{dom}(g)}$$

setting $\mathsf{dom}(f) = \mathsf{dom}(g) := \mathrm{I}$ **follows**

$$\psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi$$

and setting $g := 1_{\mathsf{codom}(f)}$ **follows**

$$f \circ \bot_{\mathsf{dom}(f)} = \bot_{\mathsf{codom}(f)} \iff f \circ f^\dagger = 1_{\mathsf{codom}(f)}$$

Theorem. **If C carries a $\perp$-structure then**

$$\mathrm{CPM}(\mathrm{C}_\Sigma) \simeq \mathrm{C}^{\mathsf{purif}}$$

where $\mathrm{C}^{\mathsf{purif}}$ are all purifiable states/operations of C.

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Theorem. **If $\mathbf{C}$ is a $\dagger$-compact category and if $\mathrm{CPM}(\mathbf{C})$ satisfies preparation-state agreement then**

$$\bot_A := \eta_A \qquad \text{and} \qquad \mathbf{CPM}(\mathbf{C})_\Sigma := F_{\mathbf{CPM}}[\mathbf{C}]$$

define a $\bot$-structure on $\mathrm{CPM}(\mathbf{C})$.

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define a $\perp$-structure on $\mathrm{CPM}(\mathbf{C})$.

$\perp$**-structure** $\equiv$ **CPM-construction + preparation-state**

A $\perp$-structure defines an *internal trace*-structure:

$$\mathsf{tr}_{A,B}^{C} : \mathbf{C}(A, C \otimes B) \to \mathbf{C}(A, B) :: f \mapsto \lambda_B^\dagger \circ (\perp_C^\dagger \otimes 1_B) \circ f$$

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A *purification* of an operation $f : A \to B$ to be a pure operation $g : A \to C \otimes B$ (i.e. in $\mathbf{C}_\Sigma$) which is such that $f = \mathsf{tr}^C_{A,B}(g)$.

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Proposition. The $\perp$-structure axiom is equivalent to

$$\mathsf{Tr}(\Psi) : \mathbf{C}(A \otimes E, A \otimes E') \to \mathbf{C}(E, E') ::$$
$$f \mapsto \lambda^\dagger_{E'} \circ (\Psi \otimes 1_{E'})^\dagger \circ (1_C \otimes f) \circ (\Psi \otimes 1_E) \circ \lambda_E$$

not depending on the choice of purification of $\Psi : \mathrm{I} \to C \otimes A$.

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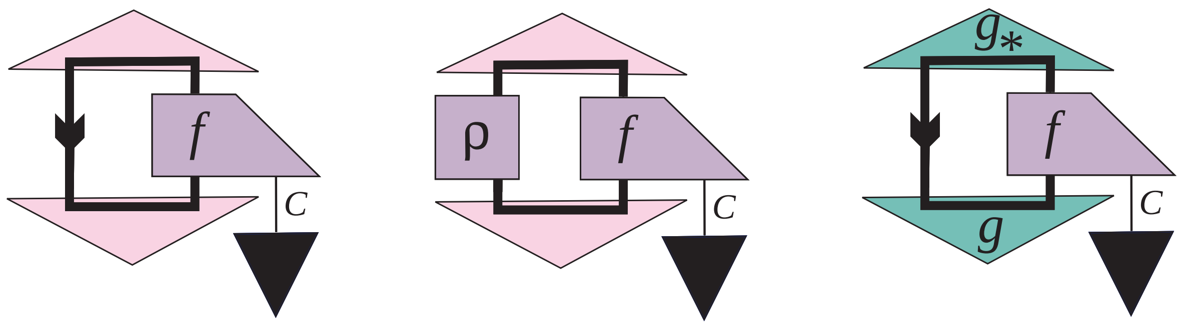
Example. **Schumacher's entanglement fidelity.**

Pictures of quantities: Reimpell and Werner's *channel fidelity*, Schumacher's *entanglement fidelity* and Devetak's *entanglement generating capacity*:



Theorem. Setting

$$\mathsf{Tr}_g : \mathbf{C}(A \otimes E, A \otimes E') \to \mathbf{C}(B \otimes E, B \otimes E') ::$$
$$f \mapsto (h \otimes 1_{E'})^\dagger \circ (1_C \otimes f) \circ (h \otimes 1_E)$$

for any purification $h : B \to C \otimes A$ **of not necessarily pure** $g : B \to A$**, the** $\perp$**-structure axiom becomes**

$$\mathsf{Tr}_f = \mathsf{Tr}_g \iff f \circ \perp_{\mathsf{dom}(f)} = g \circ \perp_{\mathsf{dom}(g)}$$

Outlook:

- **Compositional theory on Q-informatic resources**

- **Thick counterparts to domains as informatics**