CATEGORIES

– for the 21st Century Working Physicist –

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(or Google ‘‘Bob Coecke’’)

Outline

1. **Concrete & abstract categories**
   · Operational and mathematical meaning; examples;
2. **Symmetric monoidal categories**
   · Operational, mathematical and graphical meaning; braided monoidal categories
3. **Scalars in categories**
4. **Some key categorical concepts**
   · Functor, naturality, duality, (co)products, adjoints;
5. **Enriched categories and n-categories**
6. **Categorical structure of matrix calculi**
   · Biproducts; compact closed categories; knots;

Why categories in physics?

It captures the core of physical practice!

or, slightly more daring, ...

A category is the exact mathematical structure of practicing physics!

Kinds of systems

$$A\,,\ B\,,\ C\,,\ \ldots$$

- e.g. one qubit, n qubits, an electron, ...

Operations on systems

$$A \xrightarrow{f} A\,,\ A \xrightarrow{g} B\,,\ B \xrightarrow{h} C\,,\ \ldots$$

- e.g. preparation, unitary operation, measurement, ...

Composition of operations

$$A \xrightarrow{g\circ f} C \quad := \quad A \xrightarrow{f} B \xrightarrow{g} C$$

'Doing nothing'-operations

$$A \xrightarrow{1_A} A\,,\ B \xrightarrow{1_B} B\,,\ C \xrightarrow{1_C} C\,,\ \ldots$$

Definition. A **Category** $\mathbf{C}$ consits of

- Objects $A, B, C, \ldots$
- Morphisms $f, g, h, \ldots \in \mathbf{C}(A, B)$ for each pair A, B
- Associative composition of morphisms i.e.

$$f \in \mathbf{C}(A, B)\ ,\ g \in \mathbf{C}(B, C) \ \Rightarrow\ g \circ f \in \mathbf{C}(A, C)$$

$$\boxed{(h \circ g) \circ f = h \circ (g \circ f)}$$

- An identity morphism $1_A \in \mathbf{C}(A, A)$ for each A i.e.

$$\boxed{f \circ 1_A = 1_B \circ f = f}$$

Parallel disciplines

- **Physics**:

  Physical System — Physical Operation

- **Computer Science**:

  Data-type — Program

- **Proof Theory and Logic**:

  Proposition — Proof

Key features of a category:

- **Structure lives on morphisms/operations**
- **Compositionality**
- **Types**

Important 'families' of examples:

- A monoid , a group, a quantale, ...

- A preorder, a poset, a lattice, a Heyting algebra, ...

- Mathematical objects and structure preserving maps
  e.g. **Set**, **Mon**, **Group**, **Top**, $\mathbf{Vect}_{\mathbb{K}}$, **Hilb**, **Rel**, **Pos**, **Lat**, **Sup**, **BAlg**, **Heyt**,... $\underline{\mathbf{Cat}}$, n-$\underline{\mathbf{Cat}}$, ...

$\Rightarrow$ Concrete vs. abstract categories

Abstract categories with one object

Monoid $(M, \circ, 1)$:= one-object category

Group $(G, \circ, 1, (-)^{-1})$:= a monoid with only isos

Definition. A morphism $A \xrightarrow{f} B$ is an **iso** if it has an inverse i.e. there exists $B \xrightarrow{f^{-1}} A$ such that

$$f^{-1} \circ f = 1_A \qquad \text{and} \qquad f \circ f^{-1} = 1_B$$

Quantale $(Q, \circ, 1, \bigvee)$:= a **Sup**-enriched monoid

Definition. A **C-enriched category** is . . .
(be patient for another 60 min)

Abstract categories with few morphisms

Preorder $(P, \preceq) := \mathbf{C}(A, B)$ at most singleton

Poset $(P, \leq) := \mathbf{C}(A, B) \dot{\cup} \mathbf{C}(B, A)$ at most singleton

Bounded poset $(P, \leq, 0, 1) :=$ terminal & initial objects

Definition. An object $\top$ is **terminal** if $\mathbf{C}(A, \top)$ is a singleton for all A. An object $\bot$ is **initial** if $\mathbf{C}(\bot, A)$ is a singleton for all A.

Lattice $(L, \vee, \wedge) :=$ products & coproducts

Definition. . . . **(co)product** . . . `(later)`

Heyting algebra $(L, \vee, \wedge)$:= products, coproducts & closedness (i.e. $a \wedge -$ has a right adjoint $a \Rightarrow -$)

Definition. . . . **closedness** . . . **left/right adjoint** . . . (again later)

In particular, a Heyting algebra is the 'few morphisms' version of a so-called **Cartesian Closed Category**, which is part of the structure of a **Topos**.

Categories of structure preserving maps

Set := sets and functions

Top := topological spaces and continuous functions

Group := groups and group homomorphisms

$\mathbf{Vect}_{\mathbb{K}}$:= $\mathbb{K}$-vector spaces and linear maps

Hilb := Hilbert spaces and linear maps

UHilb := Hilbert spaces and unitary maps

Cat := categories and 'functors' `(see further)`

Functions vs. relations

$\mathbf{Set}$:= sets and functions

$\mathbf{Rel}$:= sets and relations

$\mathbf{Vect}_{\mathbb{K}}$:= $\mathbb{K}$-vector spaces and linear maps

CLAIM: $\mathbf{Rel}$ and $\mathbf{Vect}_{\mathbb{K}}$ have the same categorical structure while $\mathbf{Rel}$ and $\mathbf{Set}$ are unrelated.

This common structure of $\mathbf{Rel}$ and $\mathbf{Vect}_{\mathbb{K}}$ includes "being a matrix calculus", <u>which is a categorical notion</u>.

But the difference is already exposed through 'elements'.

Elements in categories

In **Set** we have $X \times \{*\} \simeq X$ and X's elements are

$$\{*\} \xrightarrow{f} X :: * \mapsto x$$

In **Hilb** we have $\mathcal{H} \otimes \mathbb{C} \simeq \mathcal{H}$ and $\mathcal{H}$'s elements are

$$\mathbb{C} \xrightarrow{f} \mathcal{H} :: 1 \mapsto |\psi\rangle = \sum_{i \in I} |\psi_i\rangle$$

In **Rel** we have $X \times \{*\} \simeq X$ and X's elements are

$$\{*\} \xrightarrow{R} X :: * \mapsto Y = \bigcup_{i \in I} \{y_i\}$$

$\Rightarrow$ **Superposition**

Abstract categories

Symmetric Monoidal Category

Symmetric Monoidal Closed Category

... can be refined to ...

Cartesian Closed Category

Topos

... but in a very different direction we also have ...

***-Autonomous Category**

Compact Closed Category

Strong Compact Closed Category

Biproduct Category

Compound Systems

We want a structure which captures compoundness i.e. given A & B and $A \xrightarrow{f} C$ & $B \xrightarrow{g} D$ we want things like

$$A \otimes B \qquad\qquad A \otimes B \xrightarrow{f \otimes g} C \otimes D$$

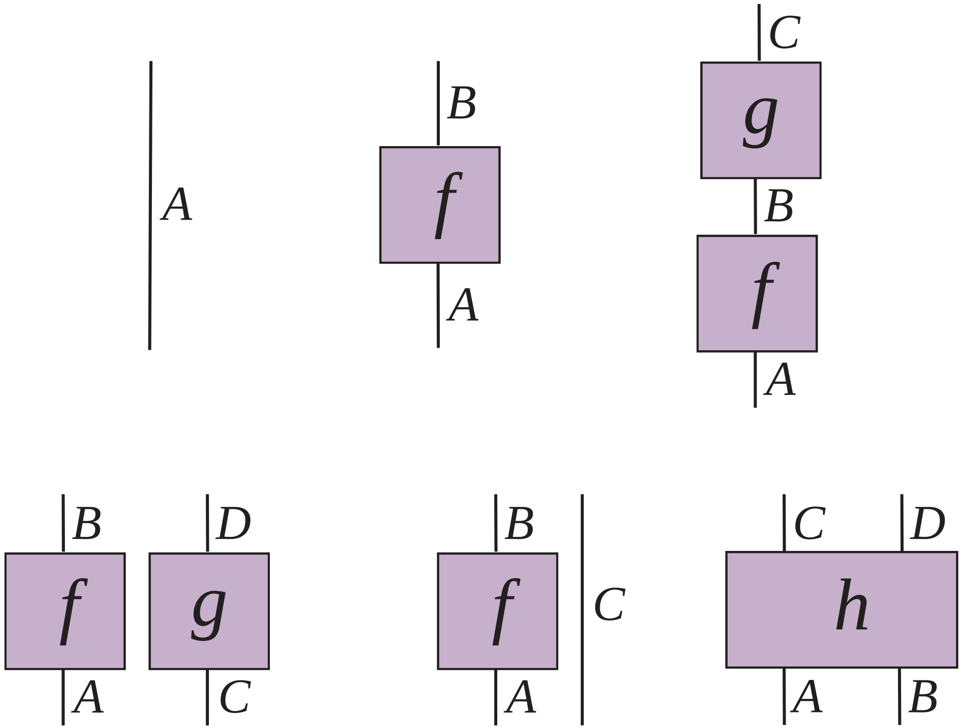
Definition. A **symmetric monoidal tensor**

$$- \otimes - : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

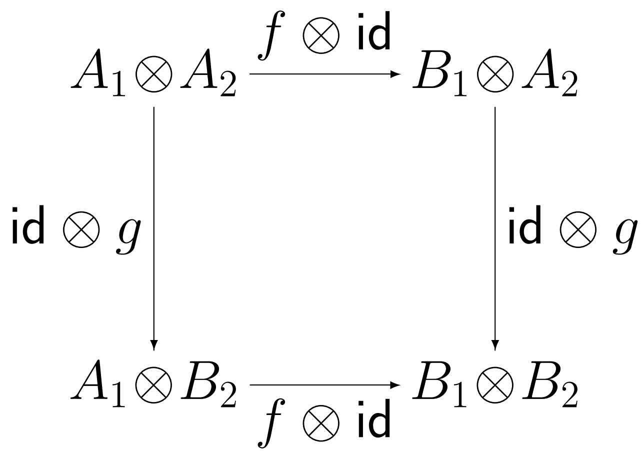
is a bifunctor together with left & right unit, symmetry and associativity natural iso(morphism)s.

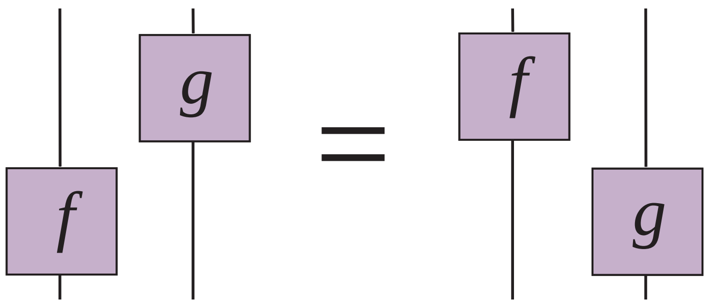
Definition. A **symmetric monoidal category** is a category with symmetric monoidal tensor.

Graphical Calculus

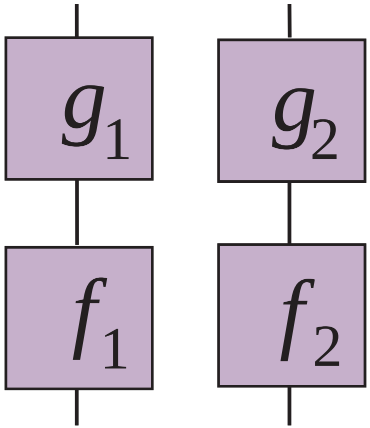


Bifunctoriality





$$(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2)$$



$$(1 \otimes g) \circ (f \otimes 1) = (1 \circ f) \otimes (g \circ 1)$$

$$=$$

$$(f \circ 1) \otimes (1 \circ g) = (f \otimes 1) \circ (1 \otimes g)$$

Natural isomorphisms

"Explicitly witnessed canonical isomorphisms"

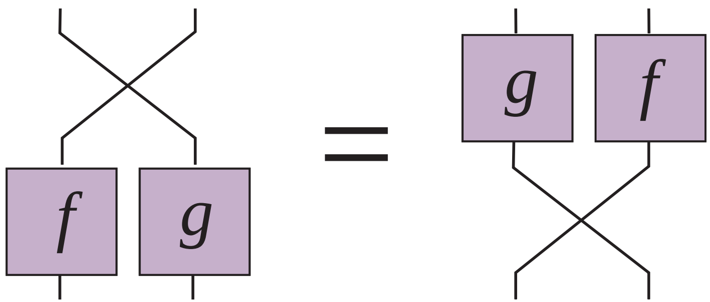
$$A \otimes B \xrightarrow{\simeq} B \otimes A \qquad \mathrm{I} \otimes A \xleftarrow{\simeq} A \xrightarrow{\simeq} A \otimes \mathrm{I}$$

$$(A \otimes B) \otimes C \xrightarrow{\simeq} A \otimes (B \otimes C)$$

$$\begin{array}{ccc}
\Lambda(A_1, A_2 \ldots) & \xrightarrow{\simeq} & \Xi(A_1, A_2 \ldots) \\
\Big\downarrow {\scriptstyle \Lambda(f, g, \ldots)} & & \Big\downarrow {\scriptstyle \Xi(f, g, \ldots)} \\
\Lambda(B_1, B_2, \ldots) & \xrightarrow[\simeq]{} & \Xi(B_1, B_2, \ldots)
\end{array}$$

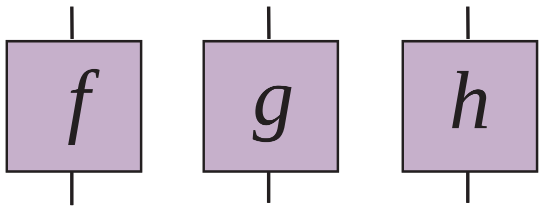
The symmetry natural isomorphism σ

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f \otimes g} & B_1 \otimes B_2 \\ \Big\downarrow \sigma_{A_1,A_2} & & \Big\downarrow \sigma_{B_1,B_2} \\ A_2 \otimes A_1 & \xrightarrow[g \otimes f]{} & B_2 \otimes B_1 \end{array}$$

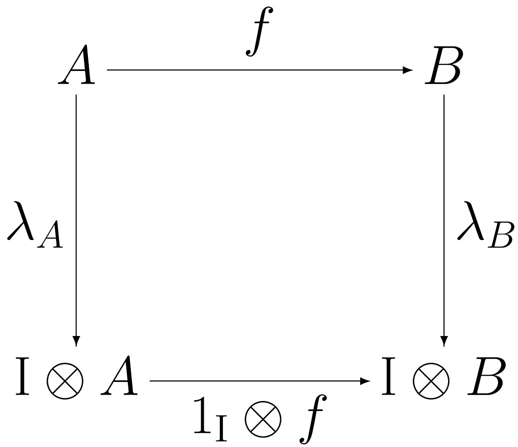


The associativity natural isomorphism α

$$
\begin{array}{ccc}
(A_1 \otimes A_2) \otimes A_3 & \xrightarrow{(f \otimes g) \otimes h} & (B_1 \otimes B_2) \otimes B_3 \\
\Big\downarrow{\scriptstyle \alpha_{A_1,A_2,A_3}} & & \Big\downarrow{\scriptstyle \alpha_{B_1,B_2,B_3}} \\
A_1 \otimes (A_2 \otimes A_3) & \xrightarrow[f \otimes (g \otimes h)]{} & B_1 \otimes (B_2 \otimes B_3)
\end{array}
$$

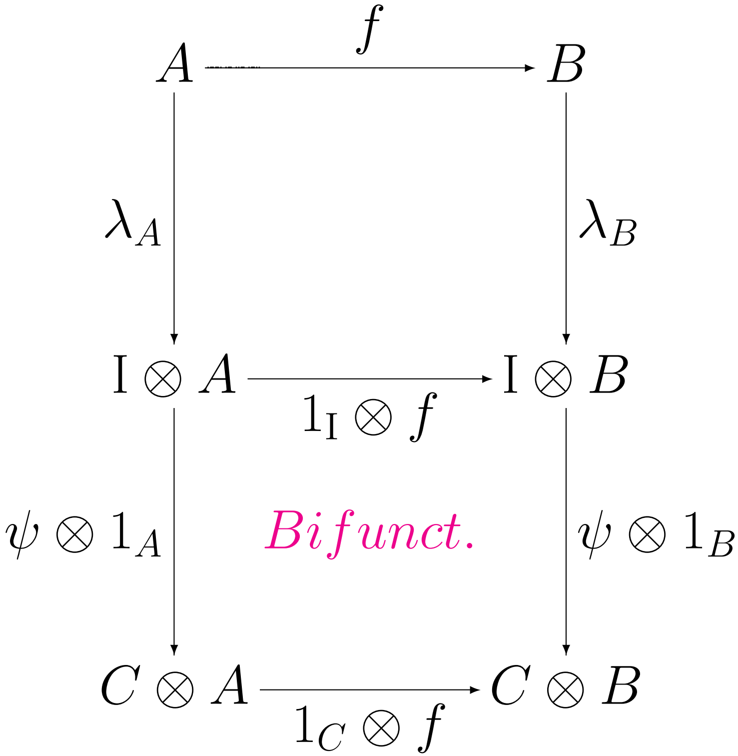


Unit natural isomorphisms λ, ρ

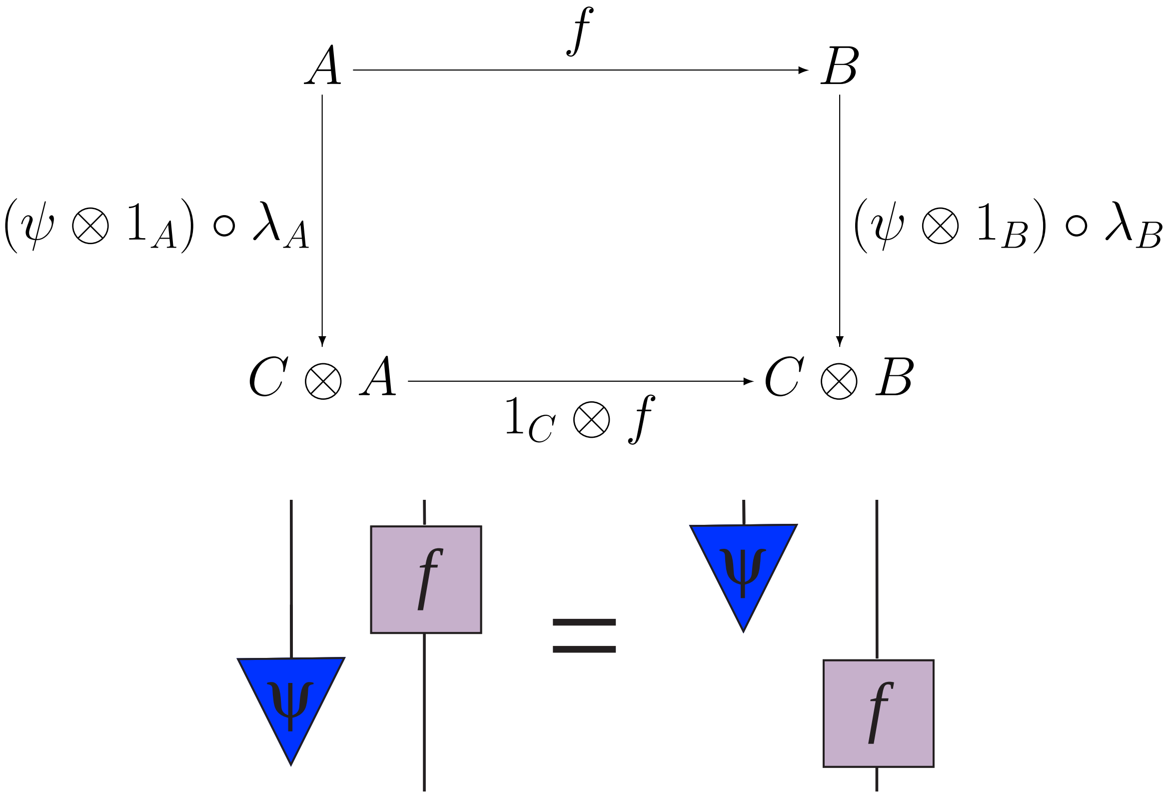


??? PICTURE ???

"it extends bifunctoriality to wireless entities"

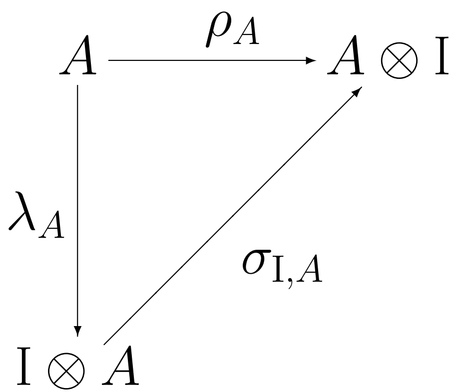


Unit natural isomorphisms λ, ρ



Coherence

"Tying up the loose ends behind the scene"



P.S. This is a very non-trivial part of 'building category theory' (as opposed to merely using category theory).

In $(\mathbf{Set}, \times)$ the unit is $\{*\}$

In $(\mathbf{Rel}, +)$ the unit is $\emptyset$

In $(\mathbf{Rel}, \times)$ the unit is $\{*\}$

In $(\mathbf{Vect}_{\mathbb{K}}, \oplus)$ the unit is $\{\underline{o}\}$

In $(\mathbf{Vect}_{\mathbb{K}}, \otimes)$ the unit is $\mathbb{K}$

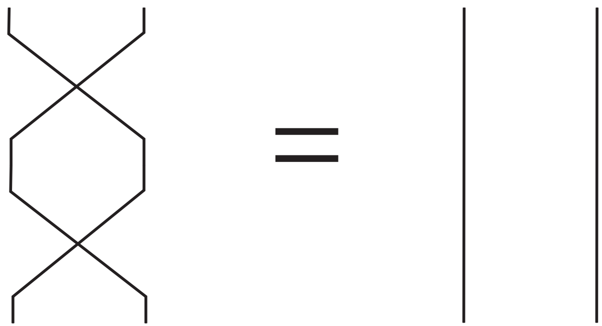
Braided Categories

Symmetric monoidal categories also assume

$$\sigma_{A,B}^{-1} = \sigma_{B,A}$$

i.e.

$$\sigma_{B,A} \circ \sigma_{A,B} = 1_{A \otimes B}$$

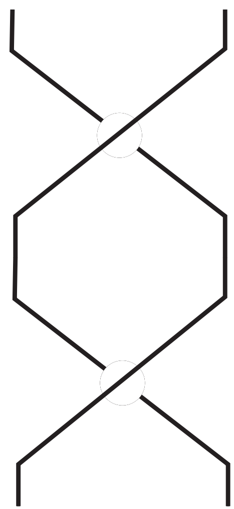


In **braided monoidal categories**

$$\sigma_{A,B}^{-1} \neq \sigma_{B,A}$$

i.e.

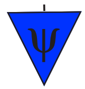
$$\sigma_{B,A} \circ \sigma_{A,B} \neq 1_{A \otimes B}$$



Numbers from compoundness

In a symmetric monoidal category with I the tensor unit:

- **State** :=



$$: \mathrm{I} \longrightarrow A$$

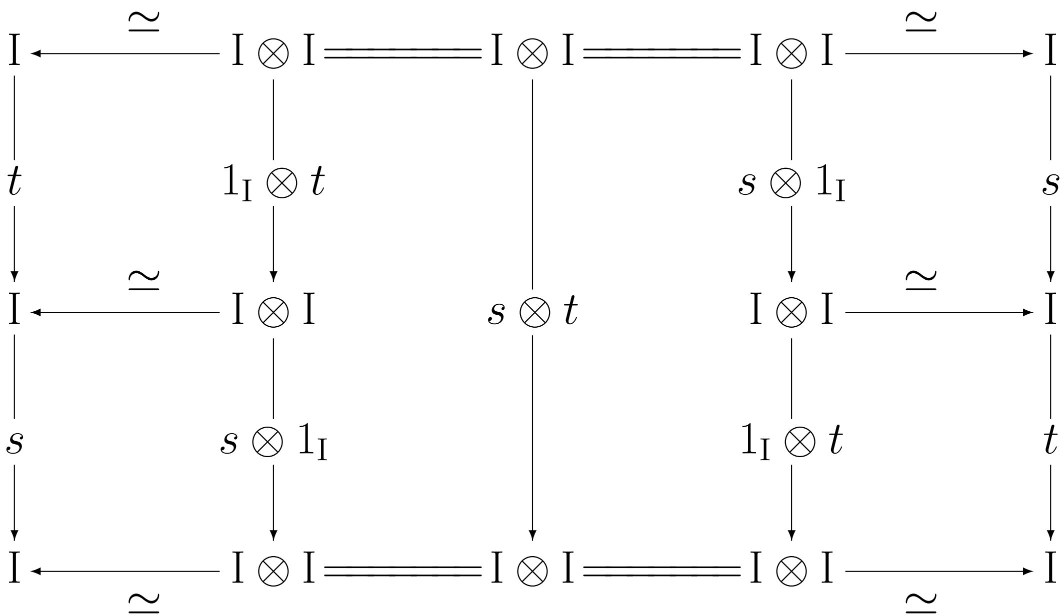
- **Number** :=

$$: \mathrm{I} \longrightarrow \mathrm{I}$$

- **State space of system A** $:= \mathbf{C}(\mathrm{I}, A)$

- **Scalar monoid**$:= \mathbf{C}(\mathrm{I}, \mathrm{I})$

Commutativity of the scalar monoid



— Kelly & Laplaza (1980) —

Scalars satisfy

$$s \circ t = \mathrm{I} \xrightarrow{\simeq} \mathrm{I} \otimes \mathrm{I} \xrightarrow{s \otimes t} \mathrm{I} \otimes \mathrm{I} \xrightarrow{\simeq} \mathrm{I}$$

and we define **scalar multiplication** as

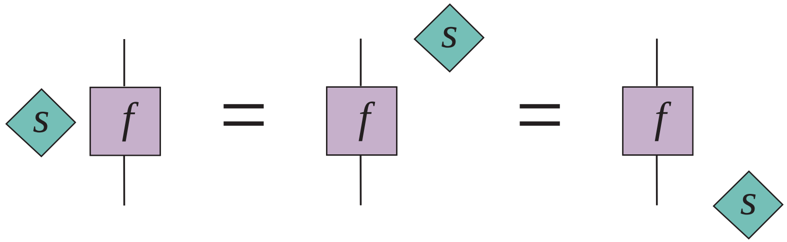
$$s \bullet f := A \xrightarrow{\simeq} A \otimes \mathrm{I} \xrightarrow{f \otimes s} B \otimes \mathrm{I} \xrightarrow{\simeq} B$$

for which we can then prove

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$$

$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$

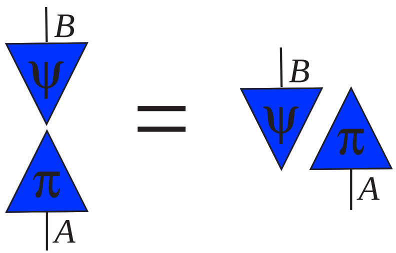
i.e. diamonds can move around freely in 'time' and 'space'



Similarly

$$\psi \circ \pi = A \xrightarrow{\simeq} \mathrm{I} \otimes A \xrightarrow{\psi \otimes \pi} B \otimes \mathrm{I} \xrightarrow{\simeq} B$$

i.e.



In $(\mathbf{Set}, \times)$ there is only one scalar

In $(\mathbf{Rel}, +)$ there is only one scalar

In $(\mathbf{Rel}, \times)$ there are two scalars

In $(\mathbf{Vect}_{\mathbb{K}}, \oplus)$ there is only one scalar

In $(\mathbf{Vect}_{\mathbb{K}}, \otimes)$ the scalers are isomorphic to $\mathbb{K}$

key categorical concepts

Definition. A **functor** $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a 'structure preserving map of categories' i.e.

$$A \mapsto FA \qquad\qquad A \xrightarrow{f} B \mapsto FA \xrightarrow{Ff} FB$$

$$\boxed{F(g \circ f) = Fg \circ Ff} \qquad \boxed{F(1_A) = 1_{FA}}$$

A tensor is a functor $\underline{\mathbf{C} \times \mathbf{C}} \longrightarrow \mathbf{C}$ hence

$$F(g_1 \circ f_1, g_2 \circ f_2) = F(g_1, g_2) \circ (f_1 \otimes f_2)$$

so for $F(-,-) := - \otimes -$

$$(g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2)$$

A group homomorphism is a functor of groups:

$$
\begin{aligned}
& a^{-1} \cdot a = e = a \cdot a^{-1} \\
\Rightarrow\ & F(a^{-1} \cdot a) = Fe = F(a \cdot a^{-1}) \\
\Rightarrow\ & F(a^{-1}) \cdot Fa = e = Fa \cdot F(a^{-1}) \\
\Rightarrow\ & (Fa)^{-1} = F(a^{-1})
\end{aligned}
$$

Definition. Given two functors $F, G : \mathbf{C} \longrightarrow \mathbf{D}$ a **natural transformation** $\xi : F \Rightarrow G$ is a family

$$\{\xi_A : FA \longrightarrow GA\}_A$$

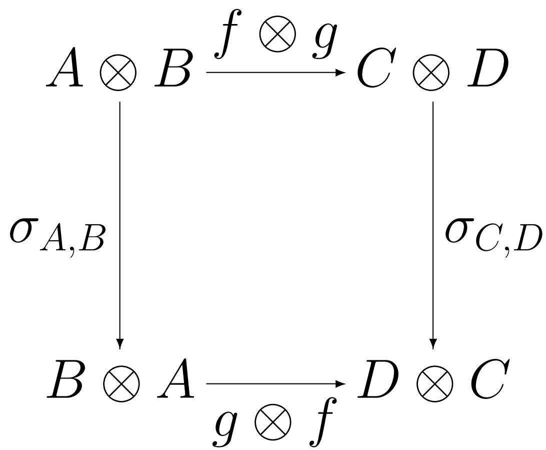
$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \xi_A}\downarrow & & \downarrow{\scriptstyle \xi_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

For natural isomorphisms all $\xi_A : FA \longrightarrow GA$ are isos.

$$F : \mathbf{C} \times \mathbf{C} \to \mathbf{C} :: \begin{cases} (A, B) & \mapsto & A \otimes B \\ (f, g) & \mapsto & f \otimes g \end{cases}$$

$$G : \mathbf{C} \times \mathbf{C} \to \mathbf{C} :: \begin{cases} (A, B) & \mapsto & B \otimes A \\ (f, g) & \mapsto & g \otimes f \end{cases}$$

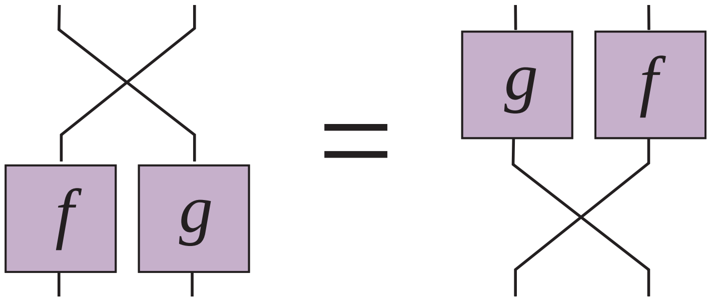
$$\{\sigma_{A,B} : A \otimes B \to B \otimes A\}_{A,B}$$



$$F : \mathbf{C} \times \mathbf{C} \to \mathbf{C} :: \begin{cases} (A, B) & \mapsto & A \otimes B \\ (f, g) & \mapsto & f \otimes g \end{cases}$$

$$G : \mathbf{C} \times \mathbf{C} \to \mathbf{C} :: \begin{cases} (A, B) & \mapsto & B \otimes A \\ (f, g) & \mapsto & g \otimes f \end{cases}$$

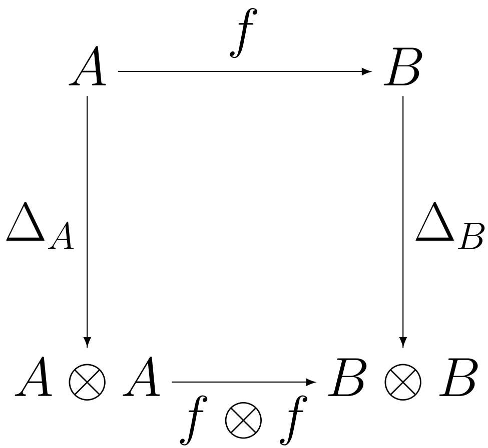
$$\{\sigma_{A,B} : A \otimes B \to B \otimes A\}_{A,B}$$



Diagonal

— the process of **copying** —

$$\{\Delta_A : A \longrightarrow A \otimes A\}_A$$



Duality

— Every concept in $\mathbf{C}$ yields a dual one in $\mathbf{C}^{op}$ —

$$\begin{cases} Objects(\mathbf{C}^{op}) & := & Objects(\mathbf{C}) \\ \mathbf{C}^{op}(A, B) & := & \mathbf{C}(B, A) \\ f \circ_{\mathbf{C}^{op}} g & := & g \circ_{\mathbf{C}} f \end{cases}$$

Terminal object $\overset{op}{\longleftrightarrow}$ Initial object

Diagonal $\overset{op}{\longleftrightarrow}$ Codiagonal

Product $\overset{op}{\longleftrightarrow}$ Coproduct

Limit $\overset{op}{\longleftrightarrow}$ Colimit

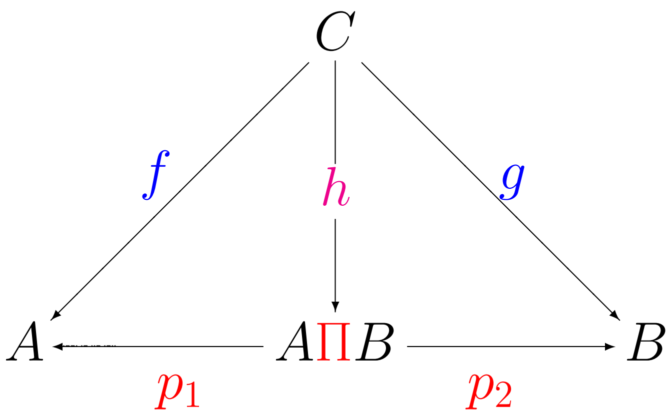
Products

— the dynamic concept of a **pair** —

Definition. The **product** of A, B is a triple

$$(A \Pi B \ , \ p_1 : A \Pi B \to A \ , \ p_2 : A \Pi B \to B)$$

such that $\forall f, g \ \exists! h$ within



Products

— pairing & copairing —

Definition. The **product** of A, B is a triple

$$(A \Pi B\ ,\ p_1 : A \Pi B \to A\ ,\ p_2 : A \Pi B \to B)$$

and **pairing** & **unpairing** operations

$$[-,-] : \mathbf{C}(C,A) \times \mathbf{C}(C,B) \to \mathbf{C}(C, A \Pi B)$$

$$(p_1 \circ -, p_2 \circ -) : \mathbf{C}(C, A \Pi B) \to \mathbf{C}(C,A) \times \mathbf{C}(C,B)$$

$$p_1 \circ [f,g] = f \quad \& \quad p_2 \circ [f,g] = g$$

$$[p_1 \circ h, p_1 \circ h] = h$$

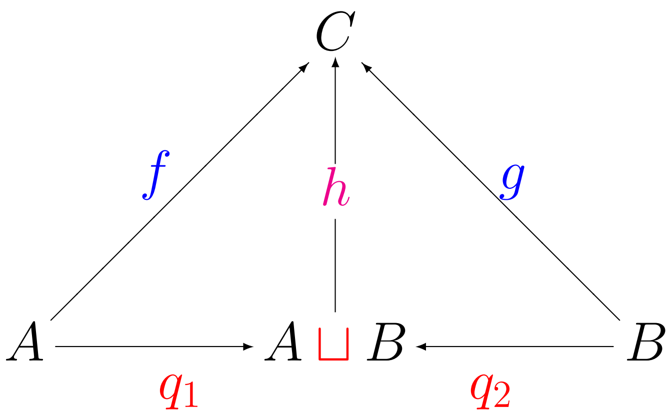
... and by duality: **Coproducts**

— the **dual** dynamic concept of a **pair** —

Definition. The **coproduct** of A, B is a triple

$$(A \sqcup B\ ,\ q_1 : A \to A \sqcup B\ ,\ q_2 : B \to A \sqcup B)$$

such that $\forall f, g\ \exists! h$ within



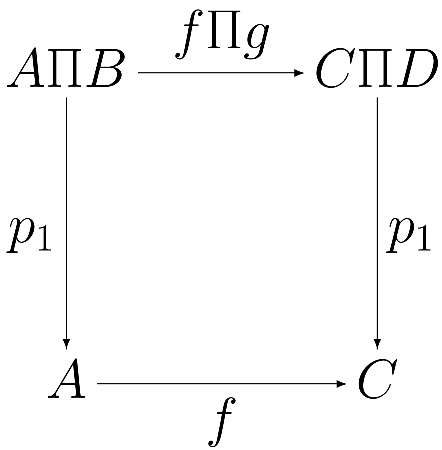
Proposition. Products yield a monoidal tensor

$$f \Pi g := [f \circ p_1, g \circ p_2] : A \Pi B \to C \Pi D$$

and a diagonal

$$\Delta_A := [1_A, 1_A] : A \to A \Pi A$$

and projections are natural



Cloning & deleting:

- $(\mathbf{Set}, \times)$
- $(\mathbf{Rel}, +)$
- $(\mathbf{Hilb}, \oplus)$

No-cloning & no-deleting:

- $(\mathbf{Rel}, \times)$
- $(\mathbf{Hilb}, \otimes)$

Internalizing morphisms

We have $\mathbf{Hilb}(\mathcal{H}_1, \mathcal{H}_2) \simeq \mathcal{H}_1^* \otimes \mathcal{H}_2$ as Hilbert spaces.

Closedness:

$$\mathbf{C}(A \otimes B, C) \simeq \mathbf{C}(A, B \Rightarrow C)$$

$\Rightarrow$ Multiplicative Logic

$*$-Autonomy:

$$\mathbf{C}(A \otimes B, C) \simeq \mathbf{C}(A, (B \otimes C^*)^*)$$

$\Rightarrow$ Multiplicative Logic with Negation

Compact Closure:

$$\mathbf{C}(A \otimes B, C) \simeq \mathbf{C}(A, B^* \otimes C)$$

$\Rightarrow$ Weird Logic

By closedness

$$\mathbf{C}(\mathrm{I} \otimes B, C) \simeq \mathbf{C}(\mathrm{I}, B \Rightarrow C)$$

$$B \xrightarrow{f} C \quad \stackrel{\simeq}{\longleftrightarrow} \quad \mathrm{I} \xrightarrow{\Psi} B \Rightarrow C$$

$\Rightarrow$ $\mathbf{C}(B, C)$-morphisms internalize as $B \Rightarrow C$-elements

By compact closure

$$\mathbf{C}(\mathrm{I} \otimes B, C) \simeq \mathbf{C}(\mathrm{I}, B^* \otimes C)$$

$$\mathbf{C}(B \otimes C^*, \mathrm{I}) \simeq \mathbf{C}(B, C \otimes \mathrm{I})$$

$$B \otimes C^* \xrightarrow{\Phi^*} \mathrm{I} \quad \stackrel{\simeq}{\longleftrightarrow} \quad B \xrightarrow{f} C \quad \stackrel{\simeq}{\longleftrightarrow} \quad \mathrm{I} \xrightarrow{\Psi} B^* \otimes C$$

$\Rightarrow$ internalize as $B^* \otimes C$-elements & $B \otimes C^*$-coelements

If $\otimes := \Pi \Rightarrow$ **Cartesian Closed Category**

> **Proposition.** If a symmetric monoidal category is both a cartesian closed category and a $*$-autonomous category then it is a preorder.

... things get worse when passing to compact closure.

$\Rightarrow$ The Hilbert space tensor product is highly incompatible with cartesian closure (hence with topos structure)

Closedness is a special case an **adjunction** $F \dashv G$

$$\mathbf{C}(FA, B) \simeq \mathbf{C}(A, GB)$$

for the case of $F := (C \otimes -)$ and $G := (C \Rightarrow -)$

Few morphism examples are:

- Galois adjoints – weakest preconditions
- Heyting algebra – deduction & modus ponens cf. CCC

All mathematical constructions arise from adjunctions

From a mathematical perspective it is probably the most important concept of category theory as a whole.

Enriched Categories

We have $\mathbf{Hilb}(\mathcal{H}_1, \mathcal{H}_2) \simeq \mathcal{H}_1^* \otimes \mathcal{H}_2$ as Hilbert spaces.

> **Definition.** (**sketch**) A 'category' $\mathbf{C}$ is $(\mathbf{D}, \otimes)$-enriched if each $\mathbf{C}(A, B)$ is an object of $\mathbf{D}$.

Each category is $\mathbf{Set}$-enriched

$\mathbf{Vect}_{\mathbb{K}}$ is $\mathbf{ComMon}$-enriched and $\mathbf{Vect}_{\mathbb{K}}$-enriched

Each symmetric monoidal closed category is self-enriched

2-categories are $\mathbf{Cat}$-enriched e.g. $\mathbf{Mon}$-, $\mathbf{Pos}$-, $\mathbf{Sup}$-

3-categories are 2-$\mathbf{Cat}$-enriched

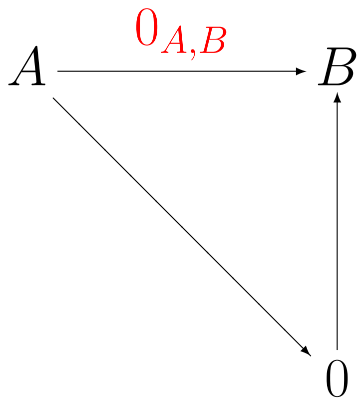
$\vdots$

n-categories are $(n-1)$-$\mathbf{Cat}$-enriched

Categorics of matrix calculi

0-maps

If an object 0 is terminal and initial it is a **0-object**.



$\Rightarrow$ **0-maps** are uniquely defined

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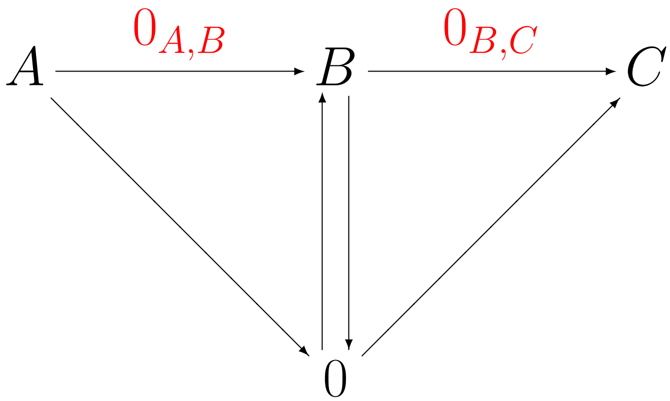


$\Rightarrow$ **0-maps** are closed under composition

Sums

Denote coinciding products and coproducts by $- \oplus -$.
For codiagonal $\nabla_A : A \oplus A \longrightarrow A$ and $f, g : A \longrightarrow B$

$$f + g := A \xrightarrow{\Delta} A \oplus A \xrightarrow{f \oplus g} A \oplus A \xrightarrow{\nabla} A$$

By naturality of Δ and ∇

$$(f_1 + f_2) \circ g = (f_1 \circ g) + (f_2 \circ g)$$
$$f \circ (g_1 + g_2) = (f \circ g_1) + (f \circ g_2)$$
$$f + g = g + f$$
$$0_{A,B} + f = f$$

$\Rightarrow$ **ComMon**-enrichment

Matrix

For projections p_1, p_2 and coprojections q_1, q_2

$$A \sqcup B \xrightarrow{f} C \Pi D$$

can be represented by a matrix

$$(f_{ij})_{ij} = \begin{pmatrix} p_1 \circ f \circ q_1 & p_1 \circ f \circ q_2 \\ p_2 \circ f \circ q_1 & p_2 \circ f \circ q_2 \end{pmatrix}$$

We can reconstruct f using pairing and copairing.

Definition. If $\mathbf{C}$ has a 0-object, products and coproducts and if all morphisms with matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are isos then $\mathbf{C}$ has **biproducts**.

Definition. If $\mathbf{C}$ is $\mathbf{ComMon}$-enriched and if there are morphisms

$$A \underset{q_1}{\overset{p_1}{\leftrightarrows}} A \oplus B \underset{q_2}{\overset{p_2}{\rightleftarrows}} B$$

with

$$p_i \circ q_j = \delta_{ij} \qquad \sum_i q_i \circ p_i = 1_{A\oplus B}$$

then $\mathbf{C}$ has **biproducts**.

Proposition. Each biproduct category admits an additive and multiplicative matrix calculus.

Proposition. Each category with numbers as objets and $n \times m$-matrices in a commutative semiring as morphisms yields a biproduct category.

$(\mathbf{Rel}, +)$ and $(\mathbf{Vect}_{\mathbb{K}}, \oplus)$ are biproduct categories.

The distributivity natural isomorphism τ

$$
\begin{array}{ccc}
(A_1 \oplus A_2) \otimes C & \xrightarrow{(f_1 \oplus f_2) \otimes g} & (B_1 \oplus B_2) \otimes D \\
\Big\downarrow \tau_{A_1,A_2,C} & & \Big\downarrow \tau_{B_1,B_2,D} \\
(A_1 \otimes C) \oplus (A_2 \otimes C) & \xrightarrow[(f_1 \otimes g) \oplus (f_2 \otimes g)]{} & (B_1 \otimes D) \oplus (B_2 \otimes D)
\end{array}
$$

by ‘type-matching’

$$\tau_{A_1,A_2,C} := \langle p_1 \otimes 1_C, p_2 \otimes 1_C \rangle \qquad \tau^{-1}_{A_1,A_2,C} := [q_1 \otimes 1_C, q_2 \otimes 1_C]$$

Operational significance of τ

1. Outcomes in measurements interpretation:

$$(\mathrm{I} \oplus \mathrm{I}) \otimes (\mathrm{I} \oplus \mathrm{I}) \simeq \mathrm{I} \oplus \mathrm{I} \oplus \mathrm{I} \oplus \mathrm{I}$$

2. Distribution of data:

$$(\mathrm{I} \oplus \mathrm{I}) \otimes Agent \simeq (\mathrm{I} \otimes Agent) \oplus (\mathrm{I} \otimes Agent)$$

Let $\mathbf{BP}$ be a biproduct category with an object I such that $\mathbf{BP}(\mathrm{I},\mathrm{I})$ is commutative. Define a subcategory $\mathbf{D}$.

- $\mathit{Objects} := \mathbb{N} \simeq \{\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n} \mid n \in \mathbb{N}\}$

- $\mathbf{D}(n,m) = n \times m$ matrices in $\mathbf{BP}(\mathrm{I},\mathrm{I})$

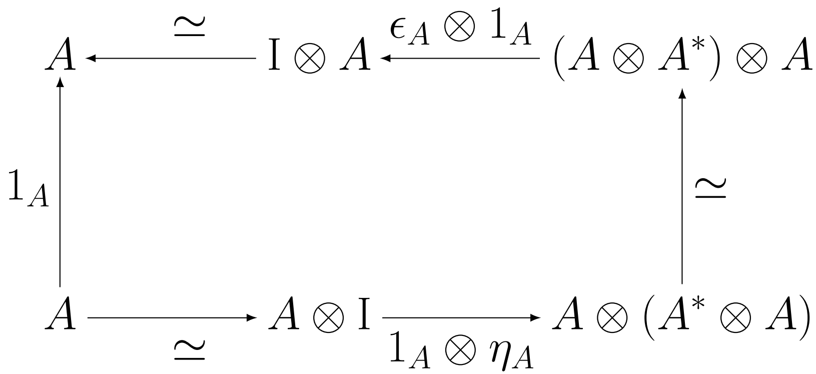
- $(\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n}) \otimes (\underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{m}) := \underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n \times m}$

- $n^* := n$ and $\eta_n = \epsilon_n^T := \Delta^{(n)} =: \mathrm{I} \to \underbrace{\mathrm{I} \oplus \ldots \oplus \mathrm{I}}_{n \times n}$

$\Rightarrow$ we obtain a compact closed category

Compact closure

Symmetric Monoidal Tensor with $\otimes$-involution $A \mapsto A^*$, **units** $\eta_A : \mathrm{I} \to A^* \otimes A$, **counits** $\epsilon_A : A \otimes A^* \to \mathrm{I}$



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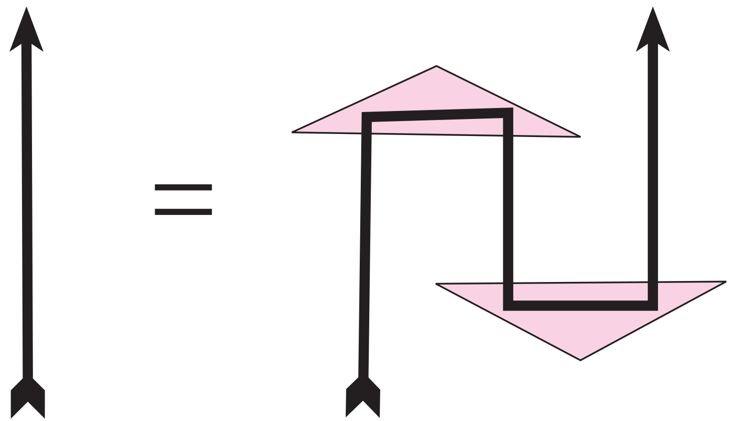
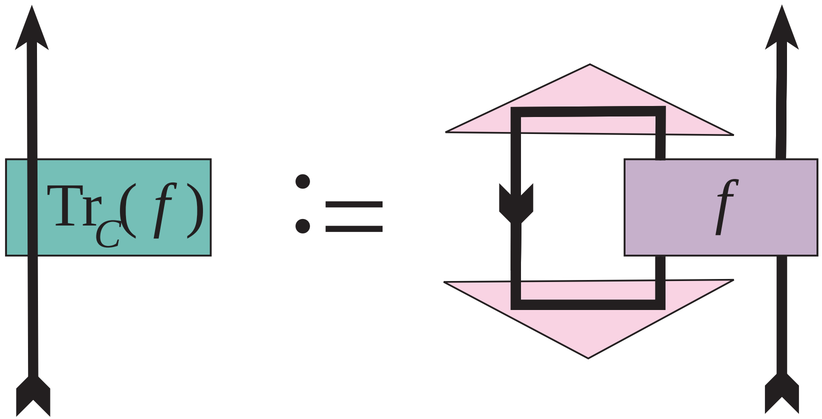


Trace Structure

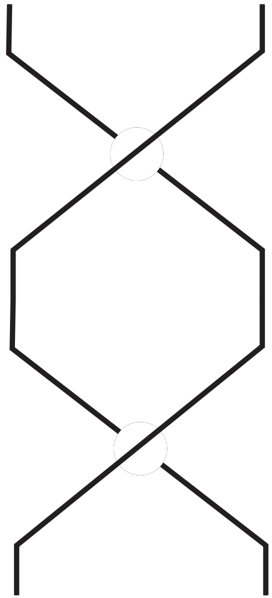
A **Joyal-Street-Verity** (**partial**) **trace**

$$f_{C\otimes A\to C\otimes B} \;\mapsto\; \mathrm{Tr}_C(f)_{A\to B}$$

arises as



categories of knots



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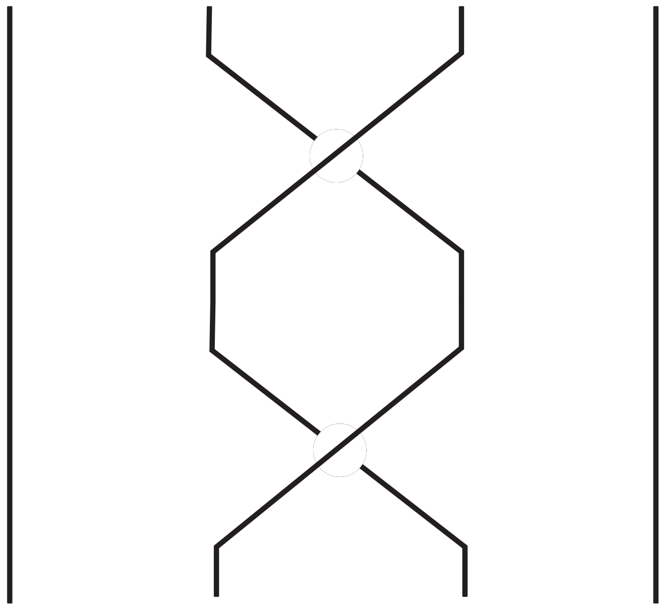

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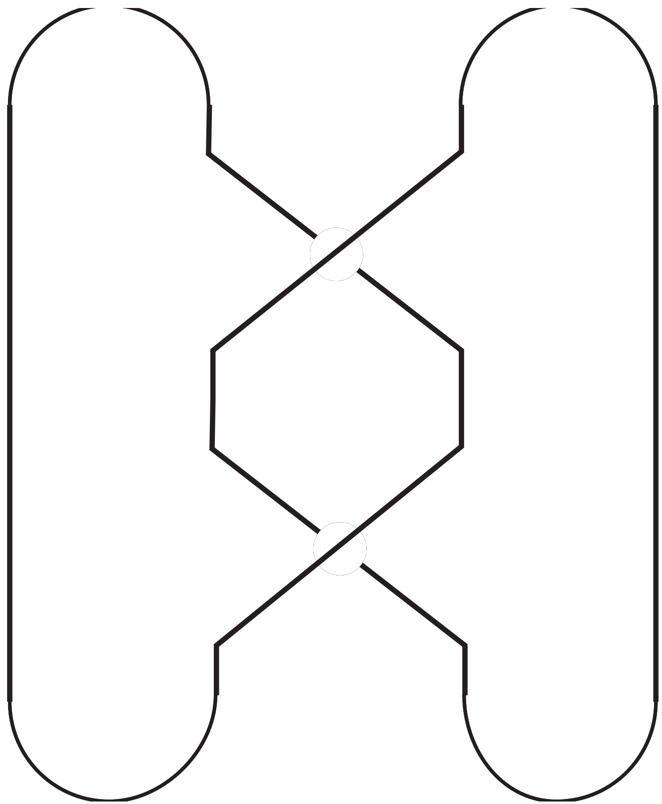

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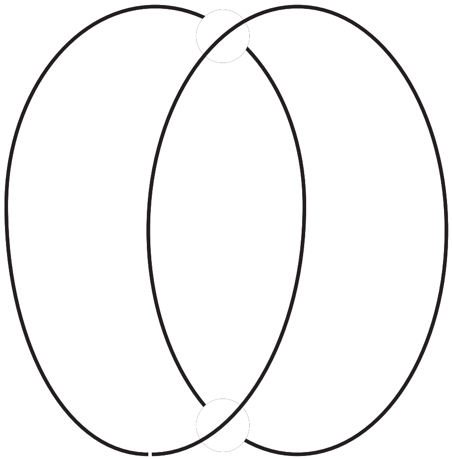

Strong Compact Closure

Symmetric Monoidal Tensor with

- $\otimes$-involution **dual** $A \mapsto A^*$;
- contravariant $\otimes$-involution **adjoint** $f_{A\to B} \mapsto f^{\dagger}_{B\to A}$;
- **Units** $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$;

$$\begin{array}{ccccc}
A & \xleftarrow{\ \simeq\ } & \mathrm{I} \otimes A & \xleftarrow{\ \eta^{\dagger}_{A^*} \otimes 1_A\ } & (A \otimes A^*) \otimes A \\
\uparrow{\scriptstyle 1_A} & & & & \uparrow{\scriptstyle \simeq} \\
A & \xrightarrow[\ \simeq\ ]{} & A \otimes \mathrm{I} & \xrightarrow[\ 1_A \otimes \eta_A\ ]{} & A \otimes (A^* \otimes A)
\end{array}$$

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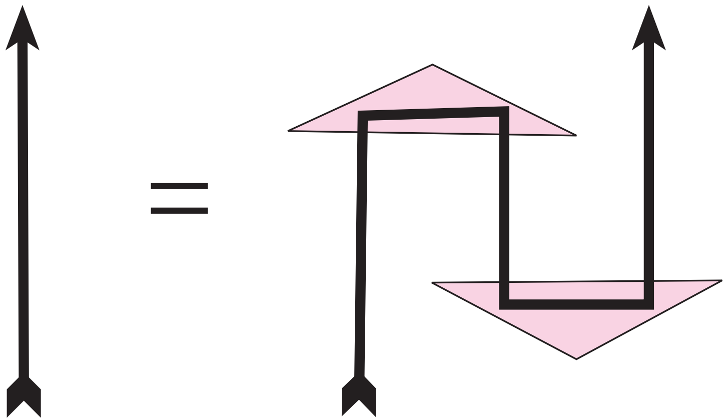

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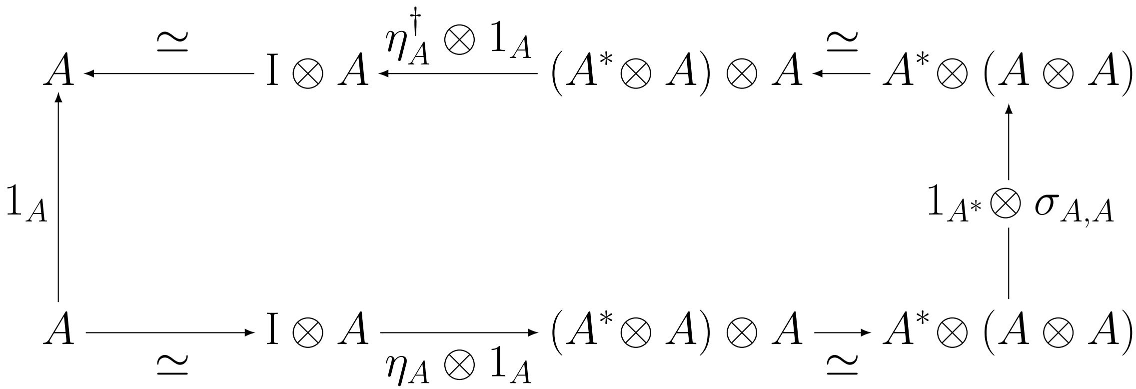

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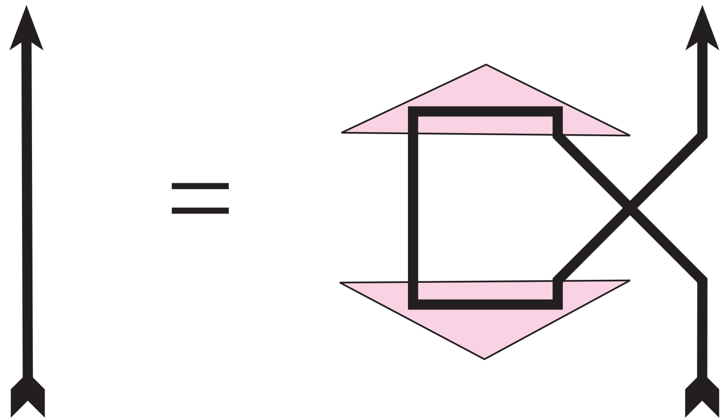

Structural content of strong compact closure:

- (Self-)adjointness, unitarity, projector, positivity and complete positivity, inner-product, Hilbert-Schmidt norm, Hilbert-Schmidt inner-product, ...

- **Quantum information-flow**

Related stuff at QICL

Samson Abramsky's introductory lecture addresses Linear Logic, Categorical Types and Categorical Logic.

Categorical Quantum Mechanics talks at workshop:

S. A.: "Information is physical, but physics is logical"

B. C.: "kindergaten quantum mechanics"

Also on Categorical Quantum Mechanics:

Ross Duncan's poster.

Eric Oliver Paquette's poster.