A topos presentation of C*-algebra based physical systems

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Motivation

Motivation: Isham et.al.'s work using topos theory for quantum theory
We simplify and extend this by using the internal logic

Some problems in quantum theory:

- Kochen-Specker: no hidden variables in quantum mechanics.
  Quantum mechanics does not reduce to classical mechanics.
- External observer does not exist in quantum gravity.
  The universe is the only closed system.

Ideas (Isham): apply coarse graining (Kripke model)
Quantum theory in a topos should be the base for quantum gravity

Commutative C*-algebras

For $X \in \mathbf{CptHd}$, consider $C(X, \mathbb{C})$.

It is a complex vector space: $(f+g)(x) := f(x) + g(x)$,
$(z \cdot f)(x) := z \cdot f(x)$.

It is a complex associative algebra: $(f \cdot g)(x) := f(x) \cdot g(x)$.

It is a Banach algebra: $\|f\| := \sup\{|f(x)| \ : \ x \in X\}$.

It has an involution: $f^*(x) := \overline{f(x)}$.

It is a C*-algebra: $\|f^* \cdot f\| = \|f\|^2$.

It is a commutative C*-algebra: $f \cdot g = g \cdot f$.

In fact, X can be reconstructed from $C(X)$:
one can trade topological structure for algebraic structure.

Gelfand duality

More precisely, there is a categorical equivalence (Gelfand duality):

$$\mathbf{CommC^*} \underset{C(-,\mathbb{C})}{\overset{\sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CptHd}^{\mathrm{op}}$$

The structure space $\sigma(\mathcal{A})$ is called the (Gelfand) spectrum of $\mathcal{A}$.

C-algebras*

Now drop commutativity: a C*-algebra is a complex Banach algebra with involution $(-)^*$ satisfying $\|a^* \cdot a\| = \|a\|^2$.

Hence theory of C*-algebras 'is non-commutative topology'.

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Prime example: $B(H) = \{f : H \to H \mid a$ bounded linear$\}$, for $H \in \mathbf{Hilb}$

is a complex vector space:	$(f+g)(h) := f(h) + g(h)$,
	$(z \cdot f)(h) := z \cdot f(h)$,
is an associative algebra:	$f \cdot g := f \circ g$,
is a Banach algebra:	$\|f\| := \sup\{\|f(h)\| \; : \; \|h\| = 1\}$,
has an involution:	$\mathrm{mat}(f^*)_{ij} := \overline{\mathrm{mat}(f)_{ji}}$,
satisfies:	$\|f^* \cdot f\| = \|f\|^2$,
but not necessarily:	$f \cdot g = g \cdot f$.

C*-algebras

Theory of C*-algebras ‘is non-commutative topology’.
This is one instance of non-commutative geometry.
(Another: Von Neumann-algebras ‘is non-commutative measure theory’.
Prime example of a commutative Von Neumann-algebra: $L^\infty(X)$.)
Used in mathematical physics.

State space in classical mechanics is $X \in \mathbf{CptHd}$.

A state of a C*-algebra $\mathcal{A}$ is a functional $\rho : \mathcal{A} \to \mathbb{C}$ that is

linear:	$\rho(a+b) = \rho(a) + \rho(b)$,
	$\rho(z \cdot a) = z \cdot \rho(a)$,
positive:	$\rho(a^* \cdot a) \geq 0$ for all $a \in \mathcal{A}$,
unital:	$\rho(1) = 1$.

A (not necessarily commutative) C*-algebra models a quantum mechanical system. *Physical observables* are self-adjoint elements.

Toposes

A topos is a category resembling $\mathbf{Set}$. It has analogues of

- Subsets, characteristic functions, truth values $\Omega = \{0, 1\}$
- (Disjoint) union, empty set
- Products, singletons
- Power sets, element-of relation

The notion generalizes

- Set theory ($\mathbf{Set}$ is a topos)
- Topology ($\mathrm{Sh}(X)$ is a topos, for $X \in \mathbf{Top}$)
- Kripke models ($\mathrm{PSh}(K, \geq)$ is a topos, for $(K, \geq)$ a Kripke frame)
- Realisability ($\mathbf{Eff}$ is a topos)

Topos logic

Fix (first-order, many-sorted) language L.
Like L-structures are valued in $\mathbf{Set}$,
formulae of L can be interpreted consistently in any topos.
One can think of a topos as a ‘universe of discourse’.

This internal logic of a topos satisfies the intuitionistic axioms.

In fact, ‘usual’ mathematics (i.e. ZF-set theory with classical logic and AC) is just working in the *ambient topos* $\mathbf{Set}$.

Functor toposes

All functors

$$P : \mathbf{C} \to \mathbf{Set}$$

on $\mathbf{C}$ form a topos $\mathbf{Set}^{\mathbf{C}}$.

(Co)Limits are pointwise

Physical toposes

Theorem (Kochen-Specker): no hidden variables in quantum mechanics. (All observables having definite values at any given time contradicts that the values of those variables are intrinsic and independent of the device used to measure them.)

$\Rightarrow$ Collection of observables has no global sections.

$\Rightarrow$ Axiom of Choice does not hold in physical world ☺.

Physical toposes

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$\Rightarrow$ Collection of observables has no global sections.
$\Rightarrow$ Axiom of Choice does not hold in physical world ☺.

Isham-Döring: can still have neo-realistic interpretation.
Their (too?) general setting: a topos with a *state object* Σ and a *quantity object* R. Observables are then morphisms $\Sigma \to R$.

Use a presheaf topos to account for partiality.
'Coarse graining' then gives a neo-realistic interpretation:

"The physical world *is* intuitionistic" ... ☺

The Isham-topos

The Isham-topos is of a physical system with state space $H \in \mathbf{Hilb}$ is

$$\mathrm{PSh}(\mathcal{V}(H)) = \mathbf{Set}^{\mathcal{V}(H)^{\mathrm{op}}},$$

where

$$\mathcal{V}(H) = \{V \subseteq B(H) \mid V \text{ commutative unital von Neumann-algebra}\}$$

is a preorder category under inclusion.

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State object $\Sigma : \mathcal{V}(H)^{\mathrm{op}} \to \mathbf{Set}$ is spectral presheaf $\Sigma(V) = \sigma(V)$.
Quantity object R is ‘something like’ $\mathbb{R}$.
Observables are maps from Σ to R.

Coarse graining

Kochen-Specker:
It is impossible to assign a value to every observable (s.a. op):
$v : A_{sa} \to \mathbb{R}$ such that $v(f(A)) = f(v(A))$.

Coarse graining:
We take this as a fact of life (as Heisenberg uncertainty).
Do not force one global choice, but consider all choices.
Observation (Isham): This induces the poset $\mathcal{V}(H)$ above.

This provides a neo-realistic interpretation:
The question ‘Is this observable in this interval?’ is now assigned a precise (and intuitionistic) truth value.

C-toposes*

All the examples already fit in the C*-algebra framework.
Let's use this!
Let $\mathcal{A}$ be a C*-algebra. Put

$$\mathcal{V}(\mathcal{A}) := \{V \subseteq \mathcal{A} \mid V \text{ commutative C*-algebra}\}.$$

It is a preorder under inclusion. Elements V can be viewed as 'classical contexts', 'windows on the world'

The associated topos is

$$\mathcal{T}(\mathcal{A}) := \mathbf{Set}^{\mathcal{V}(\mathcal{A})}$$

The change in order is crucial.

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- State object $\underline{\Sigma}$ of $\overline{\mathcal{A}}$ (Gelfand)
- Quantity object $\mathbb{R}^{\leftrightarrow}$ in topos $\mathcal{T}(\mathcal{A})$ (interval domain)
- Observables as continuous functions $\underline{\Sigma} \to \mathbb{R}^{\leftrightarrow}$
- Valuation μ on $\underline{\Sigma}$
- Probability $\mu(a \in \Delta)$ in $\mathbb{R}^{\leftrightarrow}$

Internal C*-algebra

An internal C*-algebra in a topos $\mathbf{T}$ is an object A, equipped with maps

$$+\colon A\times A\to A,\quad \cdot\colon \mathbb{C}_{\mathbb{Q}}\times A\to A,\quad 0\colon 1\to A,\quad (-)^*\colon A\to A,$$
$$-\colon A\to A,\quad \cdot\colon A\times A\to A,\quad 1\colon 1\to A,\quad N\colon \mathbb{Q}^+\rightarrowtail \Omega^A,$$

satisfying

$$\mathbf{T}\models \forall_{a,b\in A}[(a+b)^*=a^*+b^*],$$
$$\mathbf{T}\models \forall_{a\in A}\forall_{q\in\mathbb{C}_{\mathbb{Q}}}[(qa)^*=\overline{q}a^*],$$
$$\mathbf{T}\models \forall_{a,b\in A}[(ab)^*=b^*a^*],$$
$$\mathbf{T}\models \ldots$$

If the norm is computable in the ambient topos, then also internally.

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\end{array}
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$$
\begin{aligned}
\mathbf{T} &\models \forall_{a,b \in A} [(a+b)^* = a^* + b^*], \\
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\mathbf{T} &\models \ldots
\end{aligned}
$$

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Internal C*-algebras in $\mathbf{Set}^{\mathbf{C}}$ are functors of the form $\mathbf{C} \to \mathbf{CStar}$. Since C*-algebras can (almost) be defined in geometric logic.

Canonical internal C*-algebra in a C*-topos

The C*-topos $\widehat{\mathcal{A}}$ has a canonical internal C*-algebra, namely

$$\begin{aligned} \overline{\mathcal{A}} : \mathcal{V}(\mathcal{A}) &\to \mathbf{Set}, \\ V &\mapsto V. \\ (i : U \rightarrowtail V) &\mapsto i. \end{aligned}$$

The internal C*-algebra $\overline{\mathcal{A}}$ is commutative!

Gelfand duality

Want to consider the states of this system, so we need constructive Gelfand duality (Banachewski, Mulvey):

$$\mathbf{ConstrCommC^*} \underset{C(-,\mathbb{C})}{\overset{\sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CptCmpRegLocale}^{\mathrm{op}}$$

Consider relation $N \subseteq \mathcal{A} \times \mathbb{Q}$ instead of norm $\| \cdot \| : \mathcal{A} \to \mathbb{R}$. (Intuitively, $(a, q) \in N$ iff $\|a\| < q$).

Recall:

$$\mathbf{CommC^*} \underset{C(-,\mathbb{C})}{\overset{\sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CptHd}^{\mathrm{op}}$$

Pointfree Topology

What is a locale? Pointfree topology.

In algebraic geometry there are often not enough points

Use topology without points (Grothendieck)

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What is a locale? Pointfree topology.
In algebraic geometry there are often not enough points
Use topology without points (Grothendieck)

Frame is a distributive lattice with finite meets and (arbitrary) joins.
Let $f : X \to Y$ be continuous, then $f^{-1} : O(Y) \to O(X)$ is a frame morphism.
Locales are the opposite of the category of frames.

Pointfree Topology

Choice is used to construct ideal points (real numbers, max. ideals). Avoiding points one can avoid choice and non-constructive reasoning (Mulvey?). Even explicit reasoning with lattices (Coquand?).
Point free approaches to topology:

- Pointfree topology (formal opens)
- commutative C*-algebras (formal continuous functions)

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Point free approaches to topology:

- Pointfree topology (formal opens)
- commutative C*-algebras (formal continuous functions)

These formal objects model basic observations
Logic is the theory of observations (Abramsky, Smyth and Vickers)
Topology via constructive logic (Vickers)

Pointfree topology

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distributive lattice of sets with finite intersection and arbitrary union
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pointfree topology = complete Heyting algebra
Recall a Heyting algebra is a model of propositional intuitionistic logic

$$\frac{\text{Classical logic}}{\text{Boolean algebra}} = \frac{\text{Intuitionistic logic}}{\text{Heyting algebra}}$$

State object in a C-topos*

Apply constructive Gelfand representation theorem.
The internal commutative C*-algebra has an (internal) spectrum.
Take its interpretation functor Σ to be the state object.

On objects, this is the same as that of Döring-Isham.
On morphisms they differ: continuous functions versus locale maps.

New: Σ is a compact, completely regular locale (within the C*-topos $\widehat{\mathcal{A}}$), without points.

Kochen-Specker= Σ has no (global) point.
Σ is a well-defined interesting compact regular locale
(Do we still have a no-go theorem?)

Geometric logic

The internal C*-algebra has no global points either.
Why does this work?
There are enough generic points.
Restricting to geometric logic we can reason ‘as if’ we have points.

Positive formula: $\wedge, \bigvee, \exists$.
Geometric formula: $\phi \rightarrow \psi \quad (\phi, \psi$ positive)

The construction of the spectrum is geometric!

Constructive interval domain

The interval domain (generalised reals) in a topos $\mathbf{T}$ is the locale $\mathbb{R}^{\leftrightarrow}$ with as points pairs (L, U) of a lowercut L and an uppercut U satisfying

$$\exists_{r\in\mathbb{Q}}[r \in L],$$
$$\exists_{r\in\mathbb{Q}}[r \in U],$$
$$q \in L \Leftrightarrow \exists_{r\in\mathbb{Q}}[q < r \wedge r \in L],$$
$$q \in U \Leftrightarrow \exists_{r\in\mathbb{Q}}[r < q \wedge r \in U],$$
$$q \in L \wedge r \in U \Rightarrow q < r.$$

It is canonical (geometric) object: it exists in any topos (with NNO) and it is preserved by geometric morphisms.

Essentially, it is the Dedekind real numbers object, except that the lower- and uppercuts don't come arbitrarily close.
Used widely in theoretical computer science.

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Daseinisation: observables in a C-topos*

We would like to make an observable $a \in \mathcal{A}_{\mathrm{sa}}$ into an observable in $\overline{\mathcal{A}_{\mathrm{sa}}}$.
Hence we'll have to approximate a from $V \in \mathcal{V}(\mathcal{A})$: daseinisation.

$$\begin{aligned} L_a(V) &= \{b \in V \mid \exists_{\varepsilon \in \mathbb{Q}^+}[b \leq a - \varepsilon]\}, \\ U_a(V) &= \{b \in V \mid \exists_{\varepsilon \in \mathbb{Q}^+}[a + \varepsilon \leq b]\}, \\ \delta(a)(V) &= (L_a(V), U_a(V)). \end{aligned}$$

Now $\delta(a)(V) \in C(\Sigma, \mathbb{R})^{\leftrightarrow} \subseteq \overline{\mathcal{A}_{\mathrm{sa}}}$, and δ is an injection.

It factors as follows: $\downarrow a = \{b \in A_{\mathrm{sa}} \mid b \leq a\}$, $\uparrow a = \{b \in A_{\mathrm{sa}} \mid a \leq b\}$.

$$\delta : A_{\mathrm{sa}} \xrightarrow{\delta_1} A_{\mathrm{sa}}^{\leftrightarrow} \xrightarrow{\delta_2} \underline{A}_{\mathrm{sa}}^{\leftrightarrow}$$

$$a \longmapsto (\downarrow a, \uparrow a) \longmapsto (\downarrow a \cap C, \uparrow a \cap C)$$

Quantity object in a C*-topos

Use $\mathbb{R}^{\leftrightarrow}$ to be the quantity object R.
Observables will be maps $\Sigma \to \mathbb{R}^{\leftrightarrow}$.

Quantity object in a C-topos*

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Thus within our C*-topos $\widehat{\mathcal{A}}$, consider $\mathrm{Sh}(\Sigma)$.

Then R in $\widehat{\mathcal{A}}$
  is $\mathbb{R}^{\leftrightarrow}$ in $\mathrm{Sh}(\Sigma)$,
  is $C(\Sigma, \mathbb{R}^{\leftrightarrow})$ in $\widehat{\mathcal{A}}$,
  is $C(\Sigma, \mathbb{R})^{\leftrightarrow}$ in $\widehat{\mathcal{A}}$.
Theorem: R and δ are the same as Isham-Döring.

New: "observables are generalised reals". Cf. random variables...

Interpretation in a C-topos*

We can now naturally interpret an observation.

Theorem: for $a \in \mathcal{A}_{\mathrm{sa}}$, and an open $\Delta \subseteq \mathbb{R}^{\leftrightarrow}_{\mathrm{Sh}(\Sigma)}$, the interpretation of "$\delta(a) \in \Delta$" in $\overline{\mathcal{A}}$ is $\{\rho \in \mathcal{C} \subset \mathcal{A} \mid \rho(a) \in \Delta\}$.

"the interpretation of a proposition is the collection of all states (of a classical observer) making it true"

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Quantum ‘logic’

The lattice of projections as ‘quantum logic’.
We see this differently:
von Neumann algebras are non-commutative probability spaces
commutative von Neumann algebra ‘is’ L^∞
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Has been used as a logic:
Scott's model of sheaves over BA of measurable sets of $[0,1]$.
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(Random topos, contradicts Continuum Hypothesis!)

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Similar situation: sheaves over measurable sets (Σ).
How does this connect with measure theory?

States in a C*-topos

An integral is a pos lin functional I on a commutative C*-algebra, with $I(1) = 1$.
A state is a pos lin functional ρ on a C*-algebra, with $\rho(1) = 1$.

Theorem: There is a one-to-one correspondence between (quasi)-states (linear on commutative parts) of $\mathcal{A}$ and integrals on $C(\Sigma)$ in $\overline{\mathcal{A}}$.

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Proof: (in-out) If $a \in V \in \mathcal{V}(\mathcal{A})$, then also $a^* \in V$, and hence a is normal ($aa^* = a^*a$). Hence $I(a) = \rho(a)$ is defined by its value in the commutative C*-algebra generated by a. For a general operator $a + ib$ we define $I(a + ib) = \rho(a) + i\rho(b)$. If $a + ib$ is normal, then $ab = ba$, and hence a, b are in a common V; thus $I(a + ib) = \rho(a) + i\rho(b)$ by linearity, and I is well-defined. $\square$

Constructive integration

Quasi-states = states most of the time (Mackey, Gleason, ...)

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Segal-Kunze developed integration theory in this way, with intended interpretation: an expectation defined on an algebra of observables. We need a variant of this in geometric logic, since we have no points! (See my talk on Thursday.)

Constructive integration

Integral on commutative C*-algebras of functions (Daniell,Segal/Kunze)

An integral is a positive linear functional on a space of continuous functions on a topological space

Prime example: Lebesgue integral $\int$

Linear: $\int af + bg = a \int f + b \int g$

Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$

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Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$

Other example: Dirac measure $\delta_t(f) := f(t)$.

Integration theory

A number of results generalize to states on C*-algebras

1. integrable, measurable functions, L_p-spaces
2. Riemann-Stieltjes
3. Dominated convergence
4. Radon-Nikodym
5. Giry monad (Pure states $\to$ states)
6. Valuations
7. ...

We are now able to apply this also to the non-commutative context (provided they are geometric)

Valuations

There is a geometrically defined homeomorphism between integrals on Riesz spaces and valuations on there spectrum. (Coquand/S)
A valuation is a map $\mu : O(X) \to \mathbb{R}^{\leftarrow}$, (lower semicontinuous).
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This is the internal form of a measure on projections!

For a C*-algebra there are not enough projections (effect algebras)
But we can still define the valuation, with a possibly imprecise outcome.
Thus an open $\delta(a) \in \Delta$ can be assigned a probability (in the interval domain)!

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Algebraic Quantum Field Theory

An AQFT is a functor $(O(M), \subseteq) \to \mathbf{C}^*$ (satisfying certain properties), where M is a manifold (*e.g.* Minkowski space-time).
It unifies relativity theory and quantum mechanics.

Theorem: An AQFT is an internal C*-algebra in the functor topos $\mathbf{Set}^{O(M)}$.

Can be extended to general relativity.

Now have stack of 3 nested topos constructions.

More ...

Symmetries Physicists like to ‘divide out’ symmetries, like the unitary group U of the C*-algebra, or the Poincaré-group P of an AQFT. Can simply add another layer to stack of nested topos constructions: P-$\mathbf{Set}$, or U-$\mathbf{Set}$.

Davis: A Relativity Principle in Quantum Mechanics

Computability All this can be done in a computable setting: Use an effective ambient topos.

Conclusions

Bohr's doctrine suggests a functor topos making a C*-algebra commutative

- Interpretation of coarse graining using internal logic
- State space via internal Gelfand duality
- Observables are partially defined reals (domains)
- Quasi-states as internal integrals
- Geometric logic, since we have no points
- AQFT??

References

- A topos for C*-algebra based quantum theory (with Heunen)
- Constructive algebraic integration theory without choice
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- Located and overt locales (with Coquand)
- Integrals and valuations (with Coquand)