
Adaptive finite element approximation of the Francfort–Marigo model of brittle fracture

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Summary. The energy of the Francfort–Marigo model of brittle fracture can be approximated, in the sense of Γ -convergence, by the Ambrosio–Tortorelli functional. In this work we formulate and analyze an adaptive finite element algorithm, combining an inexact Newton method with residual-driven adaptive mesh refinement, for the computation its (local) minimizers. We prove that the sequence generated by this algorithm converges to a critical point.

Keywords: Adaptive finite element method, brittle fracture, free-discontinuity problems, Ambrosio–Tortorelli functional

1 Introduction

Beginning with the work of Francfort and Marigo [13], the mathematical theory of quasi-static brittle fracture mechanics has experienced a rapid and successful development. Upon recasting Griffith’s idea of balancing energy release rate with a fictitious surface energy [15] as an energy minimization problem, Francfort and Marigo were able to formulate a model that was free of the usual constraints of fracture mechanics such as a predefined and piecewise smooth crack path. With the help of the theory of free-discontinuity problems [2], this model was soon shown to be well-posed in a surprisingly general setting [9, 10, 12]. We briefly review the model in Section 1.1.

The model of Francfort and Marigo is posed in terms of the minimization of a highly irregular energy functional, which is also used in image segmentation where it is known as the Mumford–Shah functional [17]. Several methods have been proposed in the literature, which regularize this energy in order to render the problem accessible to numerical simulation [5]. Such methods typically use the theory of Γ -convergence to construct approximating functionals whose minimizers converge to those of the original functional.

In our experience, the Ambrosio–Tortorelli approximation [1, 5] is one of the most promising approaches. A particularly nice feature of the Ambrosio–Tortorelli functional is that its minimization can be reduced to the solution of elliptic boundary value problems that are straightforward to discretize, for example, by a finite element method. This approach has been used successfully by Bourdin *et al.* [3, 4] for the simulation of problems that are usually inaccessible to classical methods. A brief review of the Ambrosio–Tortorelli approximation is given in Section 1.2.

The Ambrosio–Tortorelli approximation can be understood as a phase field model for the crack set. To resolve the phase field variable, the mesh near the crack set has to be significantly finer than the mesh that would be required to resolve the elastic deformation away from the crack set. Since we do not know the crack set in advance, it is a natural idea to use an adaptive finite element method.

We shall formulate an optimization algorithm that is well-defined in the function space in which the minimization of the Ambrosio–Tortorelli functional is posed. Each step of the algorithm requires the solution of a linear self-adjoint second-order elliptic boundary value problem by an adaptive finite element algorithm. The adaptive algorithm can be controlled by adjusting the refinement tolerance, to yield a convergent adaptive optimization scheme with guaranteed convergence to a critical point of the Ambrosio–Tortorelli functional.

In order to lay out the main ideas, our analysis in the present work is restricted to linearized elasticity in anti-plane displacement, and to linear finite elements. We will extend our results to more general approximations and a wider range of models in future work.

1.1 The Francfort–Marigo Model of Brittle Fracture

In order to introduce the Francfort–Marigo model of brittle fracture, we briefly define the space of special functions of bounded variation [2]. Detailed knowledge of the properties of this function space is not necessary in order to follow the main ideas contained in the paper.

Let Ω be a domain in \mathbb{R}^N . For $p \in [1, \infty]$ we use $L^p(\Omega)$ to denote the standard L^p -spaces and $H^1(\Omega)$ to denote the standard Hilbertian Sobolev space of square-integrable functions on Ω whose distributional gradient is square-integrable on Ω . The N -dimensional Lebesgue and Hausdorff measures are denoted by \mathcal{L}^N and \mathcal{H}^N respectively.

We say that a function $f \in L^1(\Omega)$ is a special function of bounded variation (or $f \in \text{SBV}(\Omega)$) if its distributional gradient Du is a measure of bounded variation that has the form

$$Df = \nabla f \mathcal{L}^N + (f^+(x) - f^-(x)) \otimes \nu_f(x) \mathcal{H}^{N-1} \llcorner J(f).$$

Here, $\nabla f \in L^1(\Omega)^N$ is called the approximate gradient of f , $J(f)$ the jump sets, ν_f is the unit normal to $J(f)$, and f^\pm are the inner and outer traces of f on $J(u)$ with respect to ν_f .

The crack-free reference configuration of a linearly elastic body is denoted by a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$. For each $u \in \text{SBV}(\Omega)$, and for each Hausdorff measurable set Γ , the energy functional of the Francfort–Marigo model of brittle fracture [13] is defined by

$$E(u, \Gamma) := \begin{cases} \|\nabla u\|_{L^2(\Omega)}^2 + \mathcal{H}^{N-1}(\Gamma), & \text{if } \mathcal{H}^{N-1}(J(u) \setminus \Gamma) = 0, \\ +\infty, & \text{otherwise.} \end{cases} \quad (1)$$

The energy functional $E(u, \Gamma)$ reflects Griffith’s principle that to create a crack one has to *spend* an amount of elastic energy that is proportional to the area of the crack created [15] (here, the constant of proportionality is set to one). The crack set Γ and the jump set $J(u)$ are decoupled in the definition of the total energy in order to be able to impose irreversibility of the crack evolution.

We wish to study how the body evolves in time under the action of a varying load $g(t)$, $t \in [0, T]$, with $T > 0$, which is applied on an open subset $\Omega_D \subset \Omega$ of positive N -dimensional Lebesgue measure. We assume that $g \in L^\infty(0, T; W^{1,\infty}(\Omega)) \cap W^{1,1}(0, T; H^1(\Omega))$, and we define

$$\mathcal{A}(t) := \{u \in \text{SBV}(\Omega) : u|_{\Omega_D} = g(t)|_{\Omega_D}\}, \quad t \in [0, T].$$

The fact that the Dirichlet boundary condition is imposed on a set of positive N -dimensional Lebesgue measure is mostly technical and ensures that the jump set on the Dirichlet boundary $\partial\Omega_D \cap \Omega$ is well-defined. We call $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$ the Neumann boundary.

The Francfort–Marigo model of irreversible brittle fracture is formulated as follows: Find a trajectory $(u(t), \Gamma(t))_{t \in [0, T]}$ such that the following conditions are satisfied:

1. Irreversibility:

$$\Gamma(s) \subset \Gamma(t) \quad \text{for all } s, t \in [0, T] \text{ such that } s \leq t;$$

2. Global stability:

$$E(u(t), \Gamma(t)) \leq E(v, J(v) \cup \Gamma(t)) \quad \text{for all } v \in \mathcal{A}(t);$$

3. Conservation of energy:

$$\frac{d}{dt} E(u(t), \Gamma(t)) = \int_{\Omega} \nabla \dot{g}(t) \cdot \nabla u(t) \, dx.$$

Existence of solutions to this model, in its full generality, was first established in the paper of Francfort & Larsen [12].

1.2 The Ambrosio–Tortorelli Approximation

The ability to predict complicated crack paths is the greatest strength of the Francfort–Marigo model. However, this generality makes the numerical approximation of the model particularly challenging. A promising approach is to work with the Ambrosio–Tortorelli regularization of $E(u, \Gamma)$, which represents the crack set by a phase field variable v , and which is easily discretized using standard numerical methods. The regularized functional is chosen to approximate $E(u, \Gamma)$ in the sense of Γ -convergence [1, 5]. Consequently, minimizers of the approximating functional converge to minimizers of $E(u, \Gamma)$ together with convergence of the minimized energy.

The Ambrosio–Tortorelli functional $I_\varepsilon : \mathbf{H}^1(\Omega; \mathbb{R}) \times \mathbf{H}^1(\Omega; [0, 1]) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined for $0 < \eta \ll \varepsilon \ll 1$ as follows:

$$I_\varepsilon(u, v) := \int_\Omega (v^2 + \eta)|\nabla u|^2 dx + \int_\Omega \left[\frac{1}{4\varepsilon}(1-v)^2 + \varepsilon|\nabla v|^2 \right] dx. \quad (2)$$

The Ambrosio–Tortorelli functional regularizes the Francfort–Marigo model in space. In addition we discretize the evolution in time. Let $0 = t_0 < t_1 < \dots < t_K = T$ be a discretization of the time interval $[0, T]$, with $\Delta t := \max_{k=1, \dots, K} (t_k - t_{k-1})$. At time $t = t_0$, a crack field $v(0) := v(\cdot, 0)$, with $0 \leq v(x, 0) \leq 1$ for all $x \in \Omega$, is prescribed. The corresponding elastic field $u(0) := u(\cdot, 0)$ is the (unique) minimizer of $I_\varepsilon(v, u)$ with $v = v(0)$ fixed, i.e., $u(0) = \operatorname{argmin}\{I_\varepsilon(\hat{u}, v(0)) : \hat{u} \in \mathbf{H}^1(\Omega), \hat{u}|_{\Omega_D} = g(0)\}$.

At subsequent times t_k , $k = 1, \dots, K$, we compute $(u_\varepsilon(t_k), v_\varepsilon(t_k))$ satisfying

$$\begin{aligned} (u_\varepsilon(t_k), v_\varepsilon(t_k)) \in \operatorname{argmin}\{I_\varepsilon(\hat{u}, \hat{v}) : \hat{u} \in \mathbf{H}^1(\Omega), \hat{u} = g(t_k) \text{ on } \Omega_D; \\ \hat{v} \in \mathbf{H}^1(\Omega), \hat{v} \leq v_\varepsilon(t_{k-1})\}. \end{aligned} \quad (3)$$

It was shown by Giacomini [14] that an evolution satisfying (3) converges in an appropriate sense, as $\Delta t, \varepsilon \rightarrow 0$, to a solution of the Francfort–Marigo model. We will therefore restrict our consideration to the problem of minimizing the Ambrosio–Tortorelli functional at a fixed moment in time t_k , and for fixed values of ε and η .

Note that the condition $\hat{v} \leq v_\varepsilon(t_{k-1})$ enforces the irreversibility of the crack. In practise, however, we choose to implement the irreversibility criterion through the following equality constraint introduced by Bourdin [3]. At each time t_k , $k = 1, \dots, K$, we define the set

$$\operatorname{CR}(t_k) := \{x \in \overline{\Omega} : v_\varepsilon(x, t_{k-1}) < \operatorname{CRTOL}\}$$

for some small specified tolerance $\operatorname{CRTOL} > 0$, and we fix

$$v_\varepsilon(x, t_k) = 0 \text{ for all } x \in \operatorname{CR}(t_k).$$

Thus, if at a particular time, $t = t_k$, $v_\varepsilon(x, t_k)$ is close enough to zero to indicate that the point x lies on the crack path, then v_ε is set to zero at that point for

all subsequent time steps. This considerably simplifies the minimization over \hat{v} by allowing the use of an unconstrained minimization algorithm. However, it has yet to be shown that this modification of the irreversibility condition $\hat{v} \leq v_\varepsilon(t_{k-1})$ is equivalent to irreversibility of the crack as $\Delta t, \varepsilon \rightarrow 0$. We will address the question of imposing irreversibility via this pointwise monotonicity condition in future work.

1.3 Critical Points

In the Ambrosio–Tortorelli model, Lipschitz regularity of the domain is not required. Since, in practise, it is more convenient to model a pre-existing crack by a slit domain than by the initial crack field $v(0)$, we will not assume that Ω is a Lipschitz domain. Instead, motivated by the fact that we will need to partition Ω for the purpose of defining a finite element approximation, we shall assume that Ω is a polyhedral domain. By this, we simply mean that Ω possesses a finite partition into non-degenerate open N -simplices: there exist open, pairwise disjoint, non-degenerate simplices $T_1, \dots, T_K \subset \Omega$ such that $\mathcal{L}^N(\Omega \setminus \cup_k T_k) = 0$ (see also Section 2). This assumption guarantees that the usual trace and embedding theorems for Sobolev spaces hold on the domain Ω , while admitting domains with slits.

Since we consider the minimization of the Ambrosio–Tortorelli functional at the fixed time $t = t_k$, $k \in \{1, \dots, K\}$, it is useful to define the function spaces

$$\mathbf{H}_D^1(\Omega) := \{\varphi \in \mathbf{H}^1(\Omega) : \varphi = 0 \text{ on } \Omega_D\}, \quad \text{and} \quad (4)$$

$$\mathbf{H}_{\text{CR}(t_k)}^1(\Omega) := \{\psi \in \mathbf{H}^1(\Omega) : \psi = 0 \text{ on } \text{CR}(t_k)\}. \quad (5)$$

Fixing ε and η throughout, we relabel the Ambrosio–Tortorelli functional $I : \mathbf{H}^1(\Omega)^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ where

$$I(u, v) := \int_{\Omega} [(v^2 + \eta)|\nabla u|^2 + \alpha(1 - v)^2 + \varepsilon|\nabla v|^2] dx \quad \text{and} \quad \alpha = 1/(4\varepsilon). \quad (6)$$

It can be seen, using a truncation argument, that any local minimizer (u, v) of I (in the $\mathbf{H}^1(\Omega)^2$ topology) satisfies $0 \leq v(x) \leq 1$ a.e. in Ω . Thus all relevant test functions for v lie in the space $L^\infty(\Omega)$. As such, in the following discussion of differentiability of I , we work with test functions for v from the space $\mathbf{H}^1(\Omega) \cap L^\infty(\Omega)$.

It is easy to see that I is Fréchet-differentiable in $\mathbf{H}^1(\Omega) \times (\mathbf{H}^1(\Omega) \cap L^\infty(\Omega))$, however, we note that $I(u, v)$ is not finite for all $(u, v) \in \mathbf{H}^1(\Omega)^2$, and thus I is *not* Gateaux-differentiable in $\mathbf{H}^1(\Omega)^2$. This motivates the following definition of a critical point.

Definition 1. *Let $k \in \{1, \dots, K\}$. We say that $(u, v) \in (g(t_k) + \mathbf{H}_D^1(\Omega)) \times (\mathbf{H}_{\text{CR}(t_k)}^1(\Omega) \cap L^\infty(\Omega))$ is a critical point of I if $I'(u, v; \varphi, \psi) = 0$ for all $\varphi \in \mathbf{H}_D^1(\Omega)$ and for all $\psi \in \mathbf{H}_{\text{CR}(t_k)}^1(\Omega)$.*

Proposition 1. *Let $k \in \{1, \dots, K\}$. If $(u, v) \in \mathbf{H}^1(\Omega) \times (\mathbf{H}_{\text{CR}(t_k)}^1(\Omega) \cap \mathbf{L}^\infty(\Omega))$ is a critical point of I , then $0 \leq v(x) \leq 1$ for a.e. $x \in \Omega$.*

2 Adaptive Finite Element Discretization

2.1 The Alternating Minimization Algorithm

The minimization of the functional I is a particularly challenging task since the term $(v^2 + \eta)|\nabla u|^2$ renders the functional nonconvex. A number of minimization schemes can be employed; we note however that none would in general be able to find global minimizers, at least not easily. Instead we must be satisfied with being able to locate *local* minimizers.

The minimization is achieved using an alternate minimization algorithm proposed by Bourdin *et al.* [4]. We state the algorithm for the minimization of I over $(g(t_k) + \mathbf{H}_D^1(\Omega)) \times (\mathbf{H}_{\text{CR}(t_k)}^1(\Omega) \cap \mathbf{L}^\infty(\Omega))$ at time $t = t_k$, $k \in \{1, \dots, K\}$. The main observation is that, although the functional I is nonconvex with respect to the pair (u, v) , it is convex in each variable, taking the other variable fixed. Thus it is straightforward to minimize with respect to one variable at a time.

Algorithm 1. Alternating Minimization

1. Let $v^0 \in \mathbf{H}_{\text{CR}(t_k)}^1(\Omega) \cap \mathbf{L}^\infty(\Omega)$ be given
(normally the crack field $v(t_{k-1})$ from the previous, $(k-1)$ st, time step)
2. For $n = 1, 2, 3, \dots$ do:
 - 2.1. $u^n := \operatorname{argmin} \{I(z, v^{n-1}) : z \in g(t_k) + \mathbf{H}_D^1(\Omega)\}$
 - 2.2. $v^n := \operatorname{argmin} \{I(u^n, w) : w \in \mathbf{H}_{\text{CR}(t_k)}^1(\Omega)\}$
3. Set $u(t_k) = \lim_{n \rightarrow \infty} u^n$ and $v(t_k) = \lim_{n \rightarrow \infty} v^n$.

In practise, we terminate the iteration once $\|v^n - v^{n-1}\|_{\mathbf{L}^\infty(\Omega)}$ falls below a prescribed tolerance. Local convergence to isolated minimizers was established in [3, Theorem 2], while the first global convergence proof of alternating minimization was given in [6]. Since this proof is well-hidden inside more general results, we shall outline the main ideas.

Proposition 2. *Let $k \in \{1, \dots, K\}$. There exists a subsequence $((u^{n_\ell}, v^{n_\ell}))_{\ell=1}^\infty$ in $(g(t_k) + \mathbf{H}_D^1(\Omega)) \times (\mathbf{H}_{\text{CR}(t_k)}^1(\Omega) \cap \mathbf{L}^\infty(\Omega))$, and a critical point (u, v) of I in the same set, such that $(u^{n_\ell}, v^{n_\ell}) \rightarrow (u, v)$ strongly in $\mathbf{H}^1(\Omega)^2$ as $\ell \rightarrow \infty$.*

Proof. Using the fact that the energy I is nonincreasing along the sequence $((u^n, v^n))_{n=1}^\infty$, and that I is coercive in $\mathbf{H}^1(\Omega)^2$, we may extract a subsequence $n_\ell \nearrow \infty$ such that, as $\ell \rightarrow \infty$,

$$(u^{n_\ell}, v^{n_\ell}) \rightharpoonup (u, v) \quad \text{weakly in } \mathbf{H}^1(\Omega)^2 \quad \text{and} \quad v^{n_\ell-1} \rightharpoonup \bar{v} \quad \text{weakly in } \mathbf{H}^1(\Omega),$$

where $u \in g(t_k) + H_D^1(\Omega)$ and $v, \bar{v} \in H_{\text{CR}(t_k)}^1(\Omega) \cap L^\infty(\Omega)$. Since $0 \leq v^n \leq 1$ for all n (slightly strengthening Proposition 1) it follows that also $0 \leq v \leq 1$ a.e. in Ω . The task now is to verify that (u, v) is a critical point of I .

Due to the compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$ (we note that, although possibly non-Lipschitz, a domain Ω that is polyhedral in our sense satisfies the internal cone property, and therefore compactness of the embedding follows), $v^{n_\ell-1} \rightarrow \bar{v}$ strongly in $L^2(\Omega)$ as $\ell \rightarrow \infty$. From this, and the fact that $\partial_u I(u^{n_\ell}, v^{n_\ell-1}) = 0$ for all ℓ , it follows fairly easily that $\partial_u I(u, \bar{v}) = 0$. To prove that also $\partial_v I(u, v) = 0$, we first verify that $\nabla u^{n_\ell} \rightarrow \nabla u$ strongly in $L^2(\Omega)$. This can be deduced by noting that

$$\begin{aligned} \eta \|\nabla(u - u^{n_\ell})\|_{L^2(\Omega)}^2 &\leq \int_{\Omega} ([v^{n_\ell-1}]^2 + \eta) |\nabla(u - u^{n_\ell})|^2 dx \\ &= \int_{\Omega} (\bar{v}^2 + \eta) \nabla u \cdot \nabla(u - u^{n_\ell}) dx \\ &\quad - \int_{\Omega} ([v^{n_\ell-1}]^2 + \eta) \nabla u^{n_\ell} \cdot \nabla(u - u^{n_\ell}) dx \\ &\quad + \int_{\Omega} ([v^{n_\ell-1}]^2 - \bar{v}^2) \nabla u \cdot \nabla(u - u^{n_\ell}) dx. \end{aligned}$$

As $\partial_u I(u, \bar{v}) = 0$ and $\partial_u I(u^{n_\ell}, v^{n_\ell-1}) = 0$, the first and second term on the right vanish. The third term can be easily shown to converge to zero in the limit of $\ell \rightarrow \infty$ using Lebesgue's Dominated Convergence Theorem.

Having shown that $\nabla u^{n_\ell} \rightarrow \nabla u$ strongly in $L^2(\Omega)$ it is now an easy exercise to prove that $\partial_v I(u, v) = 0$ and that $v^{n_\ell} \rightarrow v$ strongly in $H^1(\Omega)$ as $\ell \rightarrow \infty$. Due to the boundedness of v^{n_ℓ} in $L^\infty(\Omega)$ an application of Fatou's Lemma shows that $I(u^{n_\ell}, v^{n_\ell}) \rightarrow I(u, v)$.

To prove that (u, v) is a critical point of I , it suffices to show that $\bar{v} = v$. To see this, note that monotonicity of the energy along the computed sequence and, in particular, $I(u^{n_\ell}, v^{n_\ell-1}) \leq I(u^{n_{\ell-1}}, v^{n_{\ell-1}})$ implies

$$I(u, \bar{v}) \leq \liminf_{\ell \rightarrow \infty} I(u^{n_\ell}, v^{n_\ell-1}) \leq \liminf_{\ell \rightarrow \infty} I(u^{n_\ell}, v^{n_\ell}) = I(u, v).$$

Since $\partial_v I(u, v) = 0$ and I is strictly convex for fixed u , it follows that $\bar{v} = v$, and therefore that (u, v) is a critical point of I . ///

In what follows, we shall solve each step of the alternating minimization algorithm by an adaptive finite element method.

2.2 Finite Element Discretization

For simplicity, we shall now restrict our presentation to the physically relevant case of $N \in \{2, 3\}$. In particular, Ω is assumed to be a polygonal ($N = 2$) or polyhedral ($N = 3$) domain in \mathbb{R}^N , in the sense that it has a finite

partition into pairwise disjoint nondegenerate open N -simplices, the complement (relative to Ω) of whose union is of N -dimensional Lebesgue measure zero. We consider a sequence $\{\mathcal{T}_j\}_{j \in J}$ of regular partitions \mathcal{T}_j of Ω , where $J := \mathbb{N}_{\geq 0} \cup (\frac{1}{2} + \mathbb{N}_{\geq 0})$. (The index set describes the half-steps in the alternating minimization algorithm.) We also assume that Ω_D is partitioned exactly by \mathcal{T}_j for all $j \in J$. The associated P1 finite element space is defined as follows:

$$X_j := \{w \in C(\overline{\Omega}) : w \text{ is piecewise affine with respect to } \mathcal{T}_j\}.$$

For simplicity, we shall assume that $g(t_k) \in X_j$ for all $j \in J$ and $k \in \{1, \dots, K\}$. We also define the discrete test spaces for the variables u and v , respectively, as

$$X_{j,D} := X_j \cap H_D^1(\Omega) \quad \text{and} \quad X_{j,\text{CR}(t_k)} := X_j \cap H_{\text{CR}(t_k)}^1(\Omega).$$

Note that the discrete trial space for the u variable at $t = t_k$ is given by $g(t_k) + X_{j,D}$, while the discrete trial space for the v variable at $t = t_k$ is simply $X_{j,\text{CR}(t_k)}$.

In our subsequent analysis, we require the discrete formulation to satisfy the analogue of Proposition 1. In order to accomplish this we use a ‘mass-lumping’ approximation [19, Chapter 11] for I together with the following additional hypothesis on the mesh regularity.

Hypothesis A. We assume that all off-diagonal elements of the finite element stiffness matrix, associated with the finite element space X_j on the partition \mathcal{T}_j , $j \in J$, are non-positive.

This condition has been studied in detail in two dimensions [8], [18, p.78] as well as in three dimensions [16]. A sufficient condition for Hypothesis A to hold is that \mathcal{T}_j , $j \in J$, is a nonobtuse partition of Ω in the sense that, for $N = 2$, each internal angle of each $T \in \mathcal{T}_j$ is $\leq \pi/2$; and, for $N = 3$, each internal dihedral angle of each $T \in \mathcal{T}_j$ is $\leq \pi/2$. We note that Hypothesis A is only adopted for technical purposes, and in practise we do not expect it to be necessary for the conclusions of Proposition 3 below (in whose proof, in [6], Hypothesis A is crucially used) to hold.

The ‘mass-lumping’ approximation of I is the functional $I_j : X_j \times X_j \rightarrow \mathbb{R}$, $j \in J$, defined by

$$I_j(u, v) := \int_{\Omega} [(P_j(v^2) + \eta)|\nabla u|^2 + \alpha P_j((v - 1)^2) + \varepsilon|\nabla v|^2] dx, \quad (7)$$

where $P_j : C(\overline{\Omega}) \mapsto X_j$ is the standard nodal interpolation operator on \mathcal{T}_j .

Definition 2. Let $k \in \{1, \dots, K\}$. We say that $(u, v) \in (g(t_k) + X_{j,D}) \times X_{j,\text{CR}(t_k)}$ is a critical point of I_j if $I'_j(u, v; \varphi, \psi) = 0$ for all $\varphi \in X_{j,D}$ and $\psi \in X_{j,\text{CR}(t_k)}$, where

$$\begin{aligned}
I'_j(u, v; \varphi, \psi) &= 2a_j(v; u, \varphi) + 2b_j(u_h; v, \psi), \\
a_j(v; u, \varphi) &:= \int_{\Omega} (P_j(v^2) + \eta) \nabla u \cdot \nabla \varphi \, dx, \quad \text{and} \\
b_j(u; v, \psi) &:= \int_{\Omega} [P_j(v\psi) |\nabla u|^2 + \alpha P_j((v-1)\psi)] + \varepsilon \nabla v \cdot \nabla \psi \, dx.
\end{aligned}$$

Proposition 3. *Let $k \in \{1, \dots, K\}$. Suppose that $(u, v) \in (g(t_k) + X_{j,D}) \times X_{j, \text{CR}(t_k)}$ satisfies $b_j(u; v, \psi) = 0$ for all $\psi \in X_{j, \text{CR}(t_k)}$; then, $0 \leq v(x) \leq 1$ for all $x \in \Omega$.*

2.3 Adaptive Alternating Minimization

Steps 2.1 and 2.2 in the alternating minimization algorithm require the solution of elliptic boundary-value problems that arise from the criticality conditions for the respective minimization problems.

We shall modify the algorithm, requiring that the criticality conditions in each step are only satisfied approximately. The theory of residual-based a-posteriori error estimation for finite element methods provides powerful tools for achieving approximate criticality, up to a prescribed tolerance. In particular, we have the following residual-based a-posteriori estimates (cf. [6]).

Proposition 4 (Residual Estimates). *Suppose that $k \in \{1, \dots, K\}$.*

(i) *Let $u \in X_j, v \in X_{j, \text{CR}(t_k)}$ such that $a_j(v; u, \varphi) = 0$ for all $\varphi \in X_{j,D}$; then,*

$$|\partial_u I(u, v; \varphi)| \leq C\mu_j(u, v) \|\nabla \varphi\|_{L^2(\Omega)} \quad \forall \varphi \in H_D^1(\Omega),$$

where

$$\begin{aligned}
[\mu_j(u, v)]^2 &:= \sum_{\substack{T \in \mathcal{T}_j \\ T \cap \Omega_D = \emptyset}} \left[h_T^4 \|\nabla v\|_{L^\infty(T)}^4 \|\nabla u\|_{L^2(T)}^2 + h_T^2 \|2v(\nabla v \cdot \nabla u)\|_{L^2(T)}^2 \right. \\
&\quad \left. + h_T \| (v^2 + \eta) |\nabla u|^2 \|_{L^2(\partial T \setminus \overline{\Omega}_D)}^2 \right] =: \sum_{\substack{T \in \mathcal{T}_j \\ T \cap \Omega_D = \emptyset}} [\mu_j(u, v; T)]^2.
\end{aligned}$$

(ii) *Let $u \in X_j, v \in X_{j, \text{CR}(t_k)}$ such that $b_j(u; v, \psi) = 0$ for all $\psi \in X_{j, \text{CR}(t_k)}$; then,*

$$|\partial_v I(u, v; \psi)| \leq C\nu_j(u, v) \|\nabla \psi\|_{L^2(\Omega)} \quad \forall \psi \in H_{\text{CR}(t_k)}^1(\Omega),$$

where

$$\begin{aligned}
[\nu_j(u, v)]^2 &:= \sum_{\substack{T \in \mathcal{T}_j \\ T \cap \text{CR}(t_k) = \emptyset}} \left[h_T^4 \|\nabla v\|_{L^\infty(T)}^2 \|\alpha + |\nabla u|^2\|_{L^2(T)}^2 + \varepsilon^2 h_T \|[\nabla v]\|_{L^2(\partial T \setminus \text{CR}(t_k))}^2 \right. \\
&\quad \left. + h_T^2 \|(\alpha + |\nabla u|^2)v - \alpha\|_{L^2(T)}^2 \right] =: \sum_{\substack{T \in \mathcal{T}_j \\ T \cap \text{CR}(t_k) = \emptyset}} [\nu_j(u, v; T)]^2.
\end{aligned}$$

We note that the constant C in the preceding proposition depends on the mesh quality, which must be controlled during mesh refinement.

The residual estimates motivate the following formulation of our adaptive algorithm, where TOL_n is a sequence of residual tolerances, which satisfy $\text{TOL}_n \searrow 0$ as $n \nearrow \infty$.

Algorithm 2. Adaptive Alternating Minimization

1. Let \mathcal{T}_0 , v^0 , $g(t_k)$ and $\text{CR}(t_k)$ be given.
2. For $n = 1, 2, 3, \dots$ do
 - 2.1. Construct a sub-mesh $\mathcal{T}_{n-1/2}$ of \mathcal{T}_{n-1} such that the solution $u^n \in g(t_k) + X_{n-1/2,D}$ of

$$a_{n-1/2}(v^{n-1}; u^n, \varphi) = 0 \quad \forall \varphi \in X_{n-1/2,D},$$
 satisfies $\mu_{n-1/2}(u^n, v^{n-1}) \leq \text{TOL}_n$.
 - 2.2. Construct a sub-mesh \mathcal{T}_n of $\mathcal{T}_{n-1/2}$ such that the solution $v^n \in X_{n,\text{CR}(t_k)}$ of

$$b_n(u^n; v^n, \psi) = 0 \quad \forall \psi \in X_{n,\text{CR}(t_k)}$$
 satisfies $\nu_n(u^n, v^n) \leq \text{TOL}_n$.

In practice, we terminate the algorithm once $\|v^n - v^{n-1}\|_{L^\infty(\Omega)}$ is sufficiently small.

For details of the implementation of the adaptive mesh-refinement in steps 2.1 and 2.2 in Algorithm 2 we refer to the adaptive finite element literature, for example, [7, 11, 20], and to our extended paper [6]. We note that for Algorithm 2 to be well-posed one needs to prove convergence of the adaptive finite element algorithm that we use for solving the linear self-adjoint second-order elliptic boundary-value problems with ‘mass-lumping’ in steps 2.1 and 2.2. Convergence results of this type have been established in the literature, though, to the best of our knowledge, only in the absence of ‘mass-lumping’. Having said this, the extension of these convergence results to the case of a ‘mass-lumped’ approximation is not foreseen to be technically demanding.

Provided that each step of Algorithm 2 can be executed, we obtain the following convergence theorem. Its proof follows essentially along the lines of the proof of Proposition 2, requiring only minor modifications (see, [6]).

Theorem 1. *Let $((u^n, v^n))_{n=1}^\infty$ be a sequence generated by Algorithm 2. Then, assuming convergence of the adaptive finite element approximation of the linear self-adjoint second-order elliptic problems with ‘mass-lumping’ in steps 2.1 and 2.2, there exists a subsequence $((u^{n_\ell}, v^{n_\ell}))_{\ell=1}^\infty$ and a critical point (u, v) of the functional I such that $(u^{n_\ell}, v^{n_\ell}) \rightarrow (u, v)$ strongly in $\mathbf{H}^1(\Omega)^2$ as $\ell \rightarrow \infty$.*

3 A Computational Example

We now briefly illustrate the performance of the adaptive alternating minimization algorithm in practice with a numerical example. Consider the rectangular domain $(-0.5, 6.5) \times (0, 6)$ with two initial slits along $\{2\} \times [0, 2]$ and

$\{4\} \times [4, 6]$. The domain is shown in Figure 1, where the shaded region is Ω_D , together with the initial mesh. The time-dependent displacement imposed on Ω_D is given as follows:

$$g(x, t) := \begin{cases} -t, & \text{on } (-0.5, 0) \times (0, 6), \\ t, & \text{on } (6, 6.5) \times (0, 6), \end{cases}$$

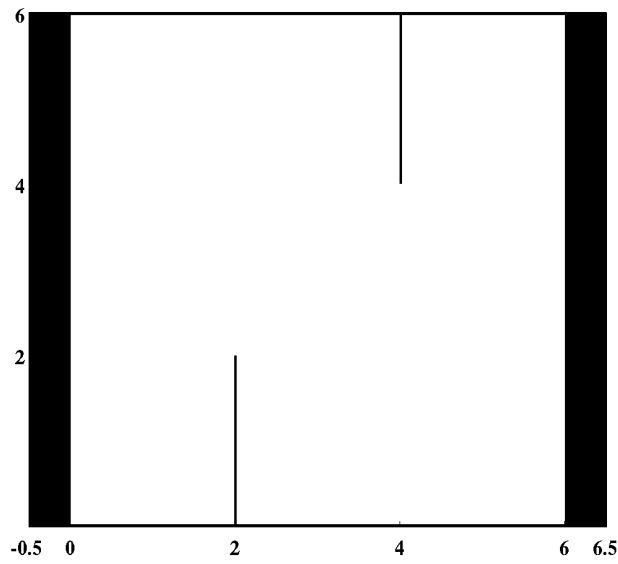
where $t = 0.01s$, for $s = 0, 1, \dots, 160$. We set the following values for the parameters in the computation: $\varepsilon = 0.05$, $\eta = 1 \times 10^{-5}$, $\text{CRTOL} = 1 \times 10^{-4}$ and $\text{TOL}_n = 0.3$. For each $s = 0, 1, \dots, 160$, we terminate the adaptive alternating minimization algorithm when $\|v^n - v^{n-1}\|_{L^\infty(\Omega)} < 1 \times 10^{-3}$.

Figure 2 shows the final mesh and crack path, together with the change in bulk, surface and total energies over time. As the incremental displacement is applied to Ω_D the bulk energy increases. There is an initial period in which this increase is not sufficient to cause the initial slits in the domain to extend. At time $t \sim 0.7$ the body starts to experience a period of steady crack growth in which both slits grow simultaneously. This continues until time $t = 1.43$ when the two cracks grow rapidly, meeting at the centre of the domain, thus causing the body to fracture into two pieces.

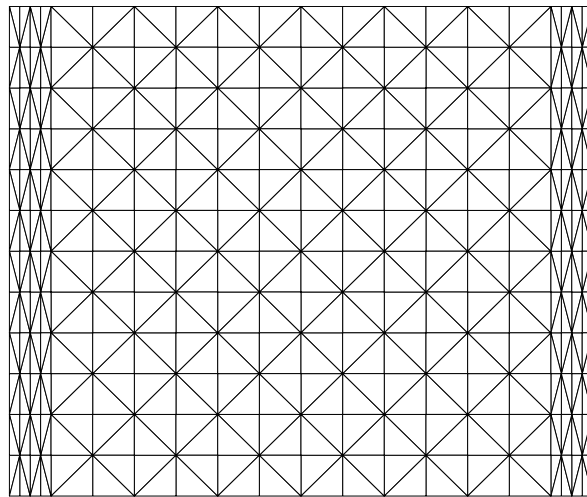
It can be seen that the adaptive algorithm refines the mesh around the growing cracks whilst retaining coarser elements elsewhere. The number of elements in the final mesh is 76937. The smallest and largest elements have diameters of size 1×10^{-3} and 0.25 respectively. In particular, a (quasi-)uniform mesh with elements of diameter 1×10^{-3} would require roughly 8×10^7 elements.

Conclusion

We have presented an adaptive algorithm for computing finite element approximations of local minimizers of the Ambrosio–Tortorelli functional. We have primarily focused on convergence results for the algorithm. We have been able to show that, provided the associated residuals converge to zero, the algorithm generates a sequence of numerical solutions that converges to a critical point of the energy functional I (as the sequence of termination tolerances tends to zero). Our preliminary computational results demonstrate the potential of using this method. In particular, we believe that the algorithm enables us to accurately and reliably compute the evolution of the crack path using considerably fewer elements compared to simulations with uniform meshes. We believe that the results presented are easily extendable to the cases of planar and three-dimensional elasticity. We are currently working on extending the theory and implementation to these models.



(a) Domain



(b) Initial mesh

Fig. 1. Computational domain and initial mesh on the domain.

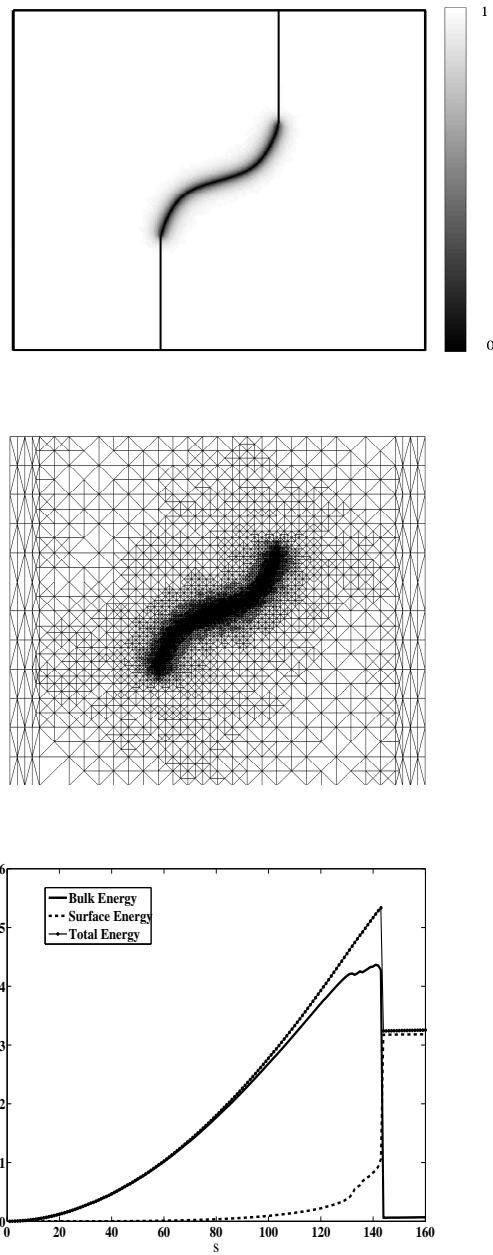


Fig. 2. Evolution of the body: The figure at the top shows v , indicating the crack-path at the final time, $T = 1.6$. The middle figure depicts the final mesh, with 76937 elements. The figure at the bottom shows the evolution of bulk, surface and total energies, for $t \in [0, 1.6]$.

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