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EXISTENCE OF GLOBAL WEAK SOLUTIONS TO FOKKER–PLANCK AND NAVIER–STOKES–FOKKER–PLANCK EQUATIONS IN KINETIC MODELS OF DILUTE POLYMERS

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Abstract. This survey paper reviews recent developments concerning the existence of global weak solutions to Fokker–Planck equations with unbounded drift terms, and coupled Navier–Stokes–Fokker–Planck systems of partial differential equations, that arise in finitely extensible nonlinear elastic (FENE) type kinetic models of incompressible dilute polymeric fluids in the case of general noncorotational flow.

1. **Introduction.** This paper is a survey of recent results concerning the question of existence of global weak solutions to coupled Navier–Stokes–Fokker–Planck systems of nonlinear partial differential equations that appear in kinetic models of dilute polymer solutions. We also explore the existence and uniqueness of global weak solutions to Fokker–Planck equations with unbounded drift terms that arise in such coupled Navier–Stokes–Fokker–Planck systems when, for a given divergence-free velocity field, the Fokker–Planck equation is decoupled from the Navier–Stokes system.

The solvent is assumed to be an incompressible, viscous, isothermal Newtonian fluid confined to a bounded open set $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega$. For the sake of simplicity of presentation, we shall suppose that Ω represents a closed container with solid boundary $\partial\Omega$; the velocity field $\underset{\sim}{u}$ will then satisfy the no-slip boundary condition $\underset{\sim}{u} = \underset{\sim}{0}$ on $\partial\Omega$. The polymer molecules, which are suspended in the solvent, are assumed not to interact with each other. The conservation of momentum and mass equations for the solvent then have the form of the incompressible Navier–Stokes equations in which the elastic *extra-stress* tensor $\underset{\approx}{\tau}$ (i.e., the polymeric part of the Cauchy stress tensor) appears as a source term:

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Given $T \in \mathbb{R}_{>0}$, find $\underset{\sim}{u} : (\underset{\sim}{x}, t) \in \overline{\Omega} \times [0, T] \mapsto \underset{\sim}{u}(\underset{\sim}{x}, t) \in \mathbb{R}^d$ and $p : (\underset{\sim}{x}, t) \in \Omega \times (0, T] \mapsto p(\underset{\sim}{x}, t) \in \mathbb{R}$ such that

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u} - \nu\,\Delta_x\,\underset{\sim}{u} + \underset{\sim}{\nabla}_x\, p = \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau} \quad \text{in } \Omega \times (0, T], \tag{1a}$$

$$\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0 \quad \text{in } \Omega \times (0, T], \tag{1b}$$

$$\underset{\sim}{u} = \underset{\sim}{0} \quad \text{on } \partial\Omega \times (0, T], \tag{1c}$$

$$\underset{\sim}{u}(\underset{\sim}{x}, 0) = \underset{\sim}{u}_0(\underset{\sim}{x}) \quad \forall \underset{\sim}{x} \in \Omega, \tag{1d}$$

where $\underset{\sim}{u}$ is the velocity field, p is the pressure of the fluid, $\nu \in \mathbb{R}_{>0}$ is the viscosity of the solvent, and $\underset{\sim}{f}$ is the density of body forces acting on the fluid.

In the kinetic models under consideration here the extra-stress tensor $\underset{\approx}{\tau}$ is defined as the weighted mean of ψ, the probability density function of the (random) conformation vector of the polymer molecules (cf. (6) below). The Kolmogorov equation satisfied by ψ is a Fokker–Planck-type second-order parabolic equation whose transport coefficients depend on the velocity field $\underset{\sim}{u}$.

Many of the interesting properties of dilute polymer solutions can be understood by modelling them as suspensions of simple coarse-grained objects (e.g. dumbbells) in a Newtonian fluid. This paper is devoted to the mathematical analysis of dumbbell models that are nonlinearly coupled Navier–Stokes–Fokker–Planck systems of partial differential equations: from the technical viewpoint these relatively simple models already exhibit many of the analytical difficulties encountered in the study of more complex models.

Suppose that the domain of admissible conformations $D \subset \mathbb{R}^d$ is a balanced convex open set in $\mathbb{R}^d$; the term *balanced* means that $\underset{\sim}{q} \in D$ if, and only if, $-\underset{\sim}{q} \in D$. Hence, in particular, $\underset{\sim}{0} \in D$. Typically, D is the whole of $\mathbb{R}^d$ or a bounded open d-dimensional ball centred at the origin $\underset{\sim}{0} \in \mathbb{R}^d$. Let $\mathcal{O} \subset [0, \infty)$ denote the image of D under the mapping $\underset{\sim}{q} \mapsto \frac{1}{2}|\underset{\sim}{q}|^2$, and consider the *spring-potential* $U \in \mathrm{C}^\infty(\mathcal{O}; \mathbb{R}_{\geq 0})$. Clearly, $0 \in \mathcal{O}$. We shall suppose that $U(0) = 0$ and that U is monotonic increasing and unbounded on $\mathcal{O}$. The elastic spring-force $\underset{\sim}{F} : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is then defined by

$$\underset{\sim}{F}(\underset{\sim}{q}) = U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\,\underset{\sim}{q}. \tag{2}$$

Example 1. *In the Hookean dumbbell model, the spring force is defined by* $\underset{\sim}{F}(\underset{\sim}{q}) = \underset{\sim}{q}$, *with* $\underset{\sim}{q} \in D = \mathbb{R}^d$, *corresponding to* $U(s) = s$, $s \in \mathcal{O} = [0, \infty)$. *Unfortunately, this simple model is physically unrealistic as it admits arbitrarily large extensions. We shall therefore assume in what follows that* D *is a bounded open ball in* $\mathbb{R}^d$, $d = 2, 3$, *centred at the origin* $\underset{\sim}{0} \in \mathbb{R}^d$. $\diamond$

We shall further suppose that there exist constants $c_i > 0$, $i = 1, 2, 3, 4$, and $\gamma > 1$ such that the (normalized) Maxwellian M, defined by

$$M(\underset{\sim}{q}) = \frac{\mathrm{e}^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)}}{\displaystyle\int_D \mathrm{e}^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)}\,\mathrm{d}\underset{\sim}{q}},$$

and the associated potential U satisfy

$$c_1\,[\mathrm{dist}(\underset{\sim}{q},\, \partial D)]^\gamma \leq M(\underset{\sim}{q}) \leq c_2\,[\mathrm{dist}(\underset{\sim}{q},\, \partial D)]^\gamma \quad \forall \underset{\sim}{q} \in D, \tag{3a}$$

$$c_3 \leq [\mathrm{dist}(\underset{\sim}{q},\, \partial D)]\, U'(\tfrac{1}{2}|\underset{\sim}{q}|^2) \leq c_4 \quad \forall \underset{\sim}{q} \in D. \tag{3b}$$

A simple application of the chain rule shows that

$$M(\underset{\sim}{q})\,\underset{\sim}{\nabla}_q\,[M(\underset{\sim}{q})]^{-1} = -[M(\underset{\sim}{q})]^{-1}\,\underset{\sim}{\nabla}_q\,M(\underset{\sim}{q}) = \underset{\sim}{\nabla}_q\,U(\tfrac{1}{2}|\underset{\sim}{q}|^2) = U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\,\underset{\sim}{q}. \quad (4)$$

Since $[U(\frac{1}{2}|\underset{\sim}{q}|^2)]^2 = (-\ln M(\underset{\sim}{q}) + \text{Const.})^2$, it follows from (3a,b) that (if $\gamma > 1$, as has been assumed here, then)

$$\int_D \left[1 + [U(\tfrac{1}{2}|\underset{\sim}{q}|^2)]^2 + [U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)]^2\right] M(\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q} < \infty. \quad (5)$$

Example 2. *In the FENE (finitely extensible nonlinear elastic) dumbbell model the spring force is given by*

$$\underset{\sim}{F}(\underset{\sim}{q}) = \frac{1}{1 - |\underset{\sim}{q}|^2/b}\,\underset{\sim}{q}, \qquad \underset{\sim}{q} \in D = B(\underset{\sim}{0}, b^{\frac{1}{2}}),$$

corresponding to $U(s) = -\frac{b}{2}\ln\left(1 - \frac{2s}{b}\right)$, $s \in \mathcal{O} = [0, \frac{b}{2})$. *Here* $B(\underset{\sim}{0}, b^{\frac{1}{2}})$ *is a bounded open ball in* $\mathbb{R}^d$, $d = 2, 3$, *centred at the origin* $\underset{\sim}{0} \in \mathbb{R}^d$ *and of fixed radius* $b^{\frac{1}{2}}$, *with* $b > 0$. *Direct calculations show that the Maxwellian* M *and the elastic potential* U *of the FENE model satisfy conditions (3a,b) with* $\gamma = \frac{b}{2}$ *provided that* $b > 2$*; thereby (5) also holds for* $b > 2$.

It is interesting to note that in the (equivalent) stochastic version of the FENE model a solution to the system of stochastic differential equations associated with the Fokker–Planck equation exists and has trajectorial uniqueness if, and only if, $b > 2$ *(cf. Jourdain, Lelièvre, and Le Bris [22] for details). Thus, the assumption* $\gamma > 1$ *can be seen as the weakest reasonable requirement on the decay-rate of* M *in (3a) as* dist$(\underset{\sim}{q}, \partial D) \to 0$. *See also the papers of Liu and Liu [34] and Masmoudi [37] for related discussion.* ⋄

Due to the flow-induced thermal agitation, polymer molecules are subjected to Brownian forces. Let $(\underset{\sim}{x}, \underset{\sim}{q}, t) \mapsto \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)$ denote the probability density function corresponding to the vector-valued stochastic process $(\underset{\sim}{X}(t), \underset{\sim}{Q}(t))$, where $\underset{\sim}{X}(t) \in \Omega$ is the position vector of the centre of mass of the dumbbell at time $t \geq 0$, and $\underset{\sim}{Q}(t) \in D$ is the conformation (or end-to-end) vector of the dumbbell at time $t \geq 0$. Roughly speaking, $\psi(\underset{\sim}{x}, \underset{\sim}{q}, t)$ represents the probability at time t of finding the centre of mass of a dumbbell at $\underset{\sim}{x}$ and having elongation vector $\underset{\sim}{q}$.

The governing equations of the coupled Navier–Stokes–Fokker–Planck model are (1a–d), where the extra-stress tensor $\underset{\approx}{\tau}$ is defined by Kramer's expression:

$$\underset{\approx}{\tau}(\underset{\sim}{x}, t) = k_B\,\mathcal{T}\left(\int_D \underset{\sim}{q}\,\underset{\sim}{q}^{\mathrm{T}}\,U'\left(\tfrac{1}{2}|\underset{\sim}{q}|^2\right)\psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,\mathrm{d}\underset{\sim}{q} - \rho(\underset{\sim}{x}, t)\,\underset{\approx}{I}\right), \quad (6)$$

with the density of polymer chains located at $\underset{\sim}{x}$ at time t given by

$$\rho(\underset{\sim}{x}, t) = \int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,\mathrm{d}\underset{\sim}{q}. \quad (7)$$

The probability density function ψ is a solution of the Fokker–Planck equation

$$\frac{\partial \psi}{\partial t} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x)\psi + \underset{\sim}{\nabla}_q\cdot(\underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}\,\psi) = \frac{1}{2\lambda}\,\underset{\sim}{\nabla}_q\cdot(\underset{\sim}{\nabla}_q\,\psi + U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\,\underset{\sim}{q}\,\psi) + \varepsilon\,\Delta_x\psi, \quad (8)$$

with $\underset{\approx}{\sigma}(\underset{\sim}{v}) \equiv \underset{\approx}{\nabla}_x\,\underset{\sim}{v}$, where $(\underset{\approx}{\nabla}_x\,\underset{\sim}{v})(\underset{\sim}{x}, t) \in \mathbb{R}^{d\times d}$ and $\{\underset{\approx}{\nabla}_x\,\underset{\sim}{v}\}_{ij} = \frac{\partial v_i}{\partial x_j}$ (cf. Barrett and Süli [5]). Here, $\varepsilon = \ell_0^2/(8\lambda)$ is the centre-of-mass diffusion coefficient of the dumbbells, $\ell_0 \ll \text{diam}(\Omega)$ is the characteristic microscopic length-scale (i.e. the characteristic dumbbell size) and $\lambda = \zeta/4H$. The parameter $\lambda \in \mathbb{R}_{>0}$ characterizes the elastic relaxation property of the fluid, $\zeta > 0$ is a friction coefficient, $H > 0$ is a spring-constant, $k_B > 0$ is the Boltzmann constant, $\mathcal{T} > 0$ is the absolute temperature, and $\underset{\approx}{I}$ is the $d \times d$ identity matrix.

A noteworthy feature of (11) compared to classical Fokker–Planck equations for bead-spring models in the literature is the presence of the $\underset{\sim}{x}$-dissipative centre-of-mass diffusion term $\varepsilon\,\Delta_x\psi \equiv (\ell_0^2/8\lambda)\,\Delta_x\psi$ on the right-hand side of the Fokker–Planck equation (8). We refer to Barrett and Süli [5] for the derivation of (8) and the mathematical justification of the presence of the centre-of-mass diffusion term $\varepsilon\,\Delta_x\psi$; see also the recent article by Schieber [41] concerning generalized dumbbell models with centre-of-mass diffusion. In standard derivations of bead-spring models the centre-of-mass diffusion term is routinely omitted, on the grounds that it is several orders of magnitude smaller than the other terms in the equation. Indeed, when $L \approx 1$ is a characteristic macroscopic length-scale (such as, for example, diam(Ω)), Bhave, Armstrong, and Brown [9] estimate the ratio ℓ_0^2/L^2 to be in the range of about 10^{-9} to 10^{-7}. On practical grounds at least, the omission of the centre-of-diffusion term $\varepsilon\,\Delta_x\psi$ from (8), in the case of a homogenous solvent velocity $\underset{\sim}{u} = \underset{\sim}{u}(t)$ exhibiting no spatial variation, is perhaps justified; however, in the case of a heterogeneous solvent velocity $\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{x}, t)$, this is a mathematically counterproductive model reduction. When $\varepsilon\,\Delta_x\psi$ is absent, (8) becomes a degenerate parabolic equation exhibiting hyperbolic behaviour with respect to $(\underset{\sim}{x}, t)$. Since the study of weak solutions to the coupled problem requires one to work with velocity fields $\underset{\sim}{u}$ that have very limited Sobolev regularity (typically $\underset{\sim}{u} \in \mathrm{L}^\infty(0, T; \underset{\sim}{\mathrm{L}}^2(\Omega)) \cap \mathrm{L}^2(0, T; \underset{\sim}{\mathrm{H}}_0^1(\Omega))$), one is then forced into the technically unpleasant territory of hyperbolically degenerate parabolic equations with rough transport coefficients (cf. Ambrosio [1] and DiPerna and Lions [15])[1]. The resulting difficulties are further exacerbated by the fact that, when D is bounded, a typical spring force $\underset{\sim}{F}(\underset{\sim}{q})$ for a finitely extensible model (such as FENE) explodes as $\underset{\sim}{q}$ approaches ∂D; see Example 2 above. For these reasons, here we shall retain the centre-of-mass diffusion term in (8). At the macroscopic level, centre-of-mass diffusion can be seen as stress diffusion: in the case of the Hookean model with centre-of-mass diffusion, the corresponding macroscopic model is Oldroyd-B with stress diffusion. For a careful numerical study of the Oldroyd-B model with stress diffusion, we refer to the paper of Sureshkumar and Beris [44]; see also the paper of Bhave, Armstrong and Brown [9].

We conclude this introduction with a brief survey of recent developments on the analysis of classical bead-spring models. We emphasize at the outset that, with the exception of the paper [5] mentioned above, our subsequent articles [6], [7] and [8], and a recent paper by Degond and Liu [13] surveyed below, all of the mathematical literature we are aware of, concerned with the existence, uniqueness or regularity of solutions to these equations, or the mathematical analysis of their numerical approximations, has so far focused on models that correspond to formally letting $\varepsilon \to 0_+$ in (8), i.e., omitting the centre-of-mass diffusion term.

An early contribution to the existence and uniqueness of local-in-time solutions to a family of bead-spring type polymeric flow models is due to Renardy [40]. While the class of potentials $\underset{\sim}{F}(\underset{\sim}{q})$ considered by Renardy [40] (cf. hypotheses (F) and (F′) on pp. 314–315) does include the case of Hookean dumbbells, it excludes the practically relevant case of the FENE model (see Example 2 above). More recently, E, Li, and Zhang [17] and Li, Zhang, and Zhang [28] have revisited the

[1] For a function-space Y, let $\underset{\sim}{Y} := [Y]^d$, $\underset{\approx}{Y} := [Y]^{d\times d}$, with the norm denoted by $\|\cdot\|_Y$ for each.

question of local existence of solutions for dumbbell models. A further development in this direction is the work of Zhang and Zhang [48], where the local existence of regular solutions to FENE-type models has been shown. All of these papers require high regularity of the initial data. More recently, Lin, Zhang and Zhang [31] have shown the global existence of smooth solutions to the two-dimensional *corotational* FENE dumbbell model, in which the velocity gradient $\underset{\approx}{\sigma}(\underset{\sim}{u}) = \underset{\approx}{\nabla}_x \underset{\sim}{u}$ in the drag term $\underset{\sim}{\nabla}_q \cdot (\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{q}\psi)$ in the Fokker–Planck equation is replaced by its skew-symmetric part, $\underset{\approx}{\sigma}_{\rm corot}(\underset{\sim}{u}) := \frac{1}{2}(\underset{\approx}{\nabla}_x \underset{\sim}{u} - (\underset{\approx}{\nabla}_x \underset{\sim}{u})^{\rm T})$. Replacement of $\underset{\approx}{\sigma}(\underset{\sim}{u})$ by $\underset{\approx}{\sigma}_{\rm corot}(\underset{\sim}{u})$ leads to cancelation of the drag term in the energy estimate for the Fokker–Planck equation when this is tested with the probability density function, which, in turn, simplifies the analysis. Here, instead, we shall be interested in a general class of noncorotational models, where by *noncorotational* we mean that $\underset{\approx}{\sigma}(\underset{\sim}{u}) = \underset{\approx}{\nabla}_x \underset{\sim}{u}$ in the drag term.

Constantin [10] has considered the Navier–Stokes equation coupled to a nonlinear Fokker–Planck equation describing the evolution of the probability distribution of the particles interacting with the fluid. He described, in the case when D is a Riemannian manifold, relations determining the coefficients of the stresses added in the fluid by the particles; these relations link the extra stresses to the kinematic effect of the fluid velocity on the particles and to the interparticle interaction potential. In equations (of Type 1, in the terminology of Constantin [10]) where the extra stresses depend linearly on the particle distribution density, as is the case in the present paper, the energy balance requires a response potential. In equations (of Type 2) where the added stresses depend quadratically on the particle distribution, it is shown that energy balance can be achieved without a dynamic response potential, and global existence of smooth solutions is shown if inertial effects are neglected. The necessary relationship (eq. (2.14) in Constantin [10]) for the existence of a Lyapunov function in the sense of Theorem 2.2 of Constantin [10] does not hold for the polymer models considered in the present paper. The regularity of solutions to coupled two-dimensional nonlinear Navier–Stokes–Fokker–Planck systems, with the velocity field in the Fokker–Planck equation replaced by a time-averaged velocity field, has been studied in the work of Constantin, Fefferman, Titi and Zarnescu [12]. See also Section 2 in the survey paper [11] by Constantin.

Otto and Tzavaras [39] have investigated the Doi model (which is similar to a Hookean model (cf. Example 1 above), except that $D = S^2$) for suspensions of rod-like molecules in the dilute regime. For certain parameter values, the velocity gradient vs. stress relation defined by the stationary and homogeneous flow is not rank-one monotone. They considered the evolution of possibly large perturbations of stationary flows and proved that, even in the absence of a microscopic cut-off, discontinuities in the velocity gradient cannot occur in finite time.

Jourdain, Lelièvre, and Le Bris [22] studied the existence of solutions to the FENE model in the case of a simple Couette flow. By using tools from the theory of stochastic differential equations, they established the existence of a unique local-in-time solution to the FENE model in two space dimensions ($d = 2$) when the velocity field $\underset{\sim}{u}$ is unidirectional and of the particular form $\underset{\sim}{u}(x_1, x_2) = (u_1(x_2), 0)^{\rm T}$. The notion of solution for which existence is proved in the paper of Jourdain, Lelièvre, and Le Bris [22] is mixed *deterministic-stochastic* in the sense that it is deterministic in the "macroscopic" variable $\underset{\sim}{x}$ but stochastic in the "microscopic" variable $\underset{\sim}{q}$. In contrast, our notion of solution (cf. Section 4 below) is deterministic both macroscopically and microscopically, since the microscales are modelled here by the probability density function $\psi(\underset{\sim}{x}, \underset{\sim}{q}, t)$. The choice between these different notions of solution has far-reaching consequences on computational simulation: mixed deterministic-stochastic notions of solution necessitate the use of Monte Carlo-type algorithms for the numerical approximation of polymer configurations, as proposed in the monograph of Öttinger [38] and, for example, in the paper of Jourdain, Lelièvre, and Le Bris [21]; whereas weak solutions in the sense considered herein can be approximated by entirely deterministic (e.g., Galerkin-type) schemes, as was done, for example, in Lozinski, Chauvière, Fang, and Owens [35] and Lozinski, Owens, and Fang [36]—at the cost of solving a Fokker–Planck equation in $2d$ spatial dimensions. See also the papers by Knezevic and Süli [24, 25] concerning the spectral approximation of FENE-type Fokker–Planck equations with unbounded drift, and the numerical analysis of heterogeneous alternating-direction methods for Fokker–Planck equations with unbounded drift on $\Omega \times D \subset \mathbb{R}^{2d}$, involving a finite element discretization over the flow domain $\Omega \subset \mathbb{R}^d$ and a spectral discretization over the configuration domain $D \subset \mathbb{R}^d$ of the Fokker–Planck equation; the paper [25] also includes numerical experiments for coupled Navier–Stokes–Fokker–Planck systems with FENE potentials where the Fokker–Planck equation is solved, using the proposed numerical algorithm, on $\Omega \times D \subset \mathbb{R}^6$. We note in passing that the convergence of a general family of Galerkin methods for the numerical approximation of weak solutions to corotational Navier–Stokes–Fokker–Planck systems with FENE type potentials has been studied in [7], while our subsequent paper [8] considers the convergence of a Galerkin finite element method to a weak solution of a general noncorotational Navier–Stokes–Fokker–Planck system with microscopic cut-off.

In the case of Hookean dumbbells, and assuming $\varepsilon = 0$, the coupled microscopic-macroscopic model described above yields, formally, taking the second moment of $\underset{\sim}{q} \mapsto \psi(\underset{\sim}{q}, \underset{\sim}{x}, t)$, the fully macroscopic, Oldroyd-B model of viscoelastic flow. Lions and Masmoudi [32] have shown the existence of global-in-time weak solutions to the Oldroyd-B model in a simplified corotational setting. The argument of Lions and Masmoudi [32] is based on exploiting the propagation in time of the compactness of the solution (i.e. the property that if one takes a sequence of weak solutions, which converges weakly and such that the corresponding sequence of initial data converges strongly, then the weak limit is also a solution) and the DiPerna–Lions [15] theory of renormalized solutions to linear hyperbolic equations with nonsmooth transport coefficients. It is not known if an identical global existence result for the Oldroyd-B model also holds in the absence of the crucial assumption that the drag term is corotational. We note in passing that, assuming $\varepsilon > 0$, the coupled microscopic-macroscopic model above yields, taking the appropriate moments in the case of Hookean dumbbells, a dissipative version of the Oldroyd-B model. In this sense, the Hookean dumbbell model has a macroscopic closure: it is the Oldroyd-B model when $\varepsilon = 0$, and a dissipative version of Oldroyd-B when $\varepsilon > 0$ (cf. Barrett and Süli [5]). In contrast, the FENE model is not known to have an exact closure at the macroscopic level, though Du, Liu and Yu [16] and Yu, Du, and Liu [47] have recently considered the analysis of approximate closures of the FENE model; see also the work of Hyon, Du and Liu [20] concerning an enhanced macroscopic closure approximation.

The subtle question of choice of boundary conditions for microscopic FENE models has been discussed in the works of Liu and Liu [34] and Masmoudi [37].

Lions and Masmoudi [33] proved the global existence of weak solutions for the corotational FENE dumbbell model, once again corresponding to the case of $\varepsilon = 0$, and the Doi model, also called the rod model. As in Lions and Masmoudi [32], the proof is based on propagation of compactness; see also the related paper of Masmoudi [37] already mentioned.

Previously, El-Kareh and Leal [18] had proposed a macroscopic model, with added dissipation in the equation that governs the evolution of the conformation tensor $\underset{\approx}{A}(\underset{\sim}{x},t) := \int_D \underset{\sim}{q}\,\underset{\sim}{q}^{\mathrm{T}} U'(\frac{1}{2}|\underset{\sim}{q}|^2)\,\psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}$, in order to account for Brownian motion across streamlines; the model can be thought of as an approximate macroscopic closure of a FENE-type micro-macro model with centre-of-mass diffusion.

In a recent article, Degond and Liu [13] consider kinetic polymer models that arise from stochastic dynamics with inertial terms. The resulting Fokker–Planck equation is, for $d = 2, 3$, a degenerate parabolic equation in $4d$ space-like variables: the position vector of the dumbell, the conformation vector, the translational velocity and the orientational velocity. They show (cf. Theorem 1.1 in [13]) that in the limit of the mass-sizes of the beads in the dumbbell tending to zero, the integral average of the corresponding probability density function over all translational velocities and orientational velocities satisfies a Fokker–Planck equation with centre-of-mass diffusion.

Barrett, Schwab, and Süli [4] established the existence of, global in time, weak solutions to the coupled microscopic-macroscopic model (1a–d) and (8) with $\varepsilon = 0$, an $\underset{\sim}{x}$-mollified velocity gradient in the Fokker–Planck equation and an $\underset{\sim}{x}$-mollified probability density function ψ in the Kramers expression—admitting a large class of potentials U (including the Hookean dumbbell model as well as general FENE-type models); in addition to these mollifications, $\underset{\sim}{u}$ in the $\underset{\sim}{x}$-convective term $(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x)\psi$ in the Fokker–Planck equation was also mollified. Unlike Lions and Masmoudi [32], the arguments in Barrett, Schwab, and Süli [4] did not require the assumption that the drag term was corotational in the FENE case. The mollification S_α of the velocity field $\underset{\sim}{u}$ that was considered in Barrett, Schwab and Süli [4] was stimulated by the Leray-α model of the incompressible Navier–Stokes equations (the viscous Camassa–Holm equations), proposed by Foias, Holm, and Titi [19], with the mollified velocity field $S_\alpha\underset{\sim}{u}$ defined as the solution of a Helmholtz–Stokes problem, thus ensuring that, like $\underset{\sim}{u}$, $S_\alpha\underset{\sim}{u}$ is still divergence-free and satisfies the same boundary condition as $\underset{\sim}{u}$.

In [5], we derived the coupled Navier–Stokes–Fokker–Planck model with centre-of-mass diffusion stated above. The anisotropic Friedrichs mollifiers, which naturally arise in the derivation of the model in the Kramers expression for the extra stress tensor and in the drag term in the Fokker–Planck equation, were replaced by isotropic Friedrichs mollifiers. We established the existence of global-in-time weak solutions to the model for a general class of spring-force-potentials including in particular the FENE potential. We justified also, through a rigorous limiting process, certain classical reductions of this model appearing in the literature that exclude the centre-of-mass diffusion term from the Fokker–Planck equation on the grounds that the diffusion coefficient is small relative to other coefficients featuring in the equation. In the case of a corotational drag term we performed a rigorous passage to the limit as the Friedrichs mollifiers in the Kramers expression and the drag term converge to identity operators.

In the present paper neither the probability density function ψ in the Kramers expression (6) nor the velocity field $\underset{\sim}{u}$ in the drag term

$$\underset{\sim}{\nabla}_q \cdot (\underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}\,\psi) = \underset{\sim}{\nabla}_q \cdot \left[\underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}\,M\left(\frac{\psi}{M}\right)\right] \tag{9}$$

appearing in (8) will be mollified. Instead, motivated by recent papers of Jourdain, Lelièvre, Le Bris, and Otto [23] and Lin, Liu, and Zhang [29] (see also Arnold, Markowich, Toscani, and Unterreiter [3], Desvillettes and Villani [14], and Lin and Zhang [30]) concerning the convergence of the probability density function ψ to its equilibrium value $\psi_\infty(\underset{\sim}{x},\underset{\sim}{q}) := M(\underset{\sim}{q})$ (corresponding to the equilibrium value $\underset{\sim}{u}_\infty(\underset{\sim}{x}) := \underset{\sim}{0}$ of the velocity field) in the absence of body forces $\underset{\sim}{f}$, we observe that if ψ/M is bounded above then, for $L \in \mathbb{R}_{>0}$ sufficiently large, the drag term (9) is equal to

$$\underset{\sim}{\nabla}_q \cdot \left[\underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}\,M\beta^L\left(\frac{\psi}{M}\right)\right],$$

where $\beta^L \in \mathrm{C}(\mathbb{R})$ is a cut-off function defined as

$$\beta^L(s) := \begin{cases} s & \text{for } s \leq L, \\ L & \text{for } L \leq s. \end{cases} \tag{10}$$

It follows that, for $L \gg 1$, any solution ψ of (8), such that ψ/M is bounded above, also satisfies

$$\frac{\partial \psi}{\partial t} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x)\psi + \underset{\sim}{\nabla}_q \cdot \left[\underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}\,M\,\beta^L\left(\frac{\psi}{M}\right)\right]$$
$$= \frac{1}{2\,\lambda}\underset{\sim}{\nabla}_q \cdot \left(M\,\underset{\sim}{\nabla}_q\left(\frac{\psi}{M}\right)\right) + \varepsilon\,\Delta_x\,\psi \qquad \text{in } \Omega \times D \times (0,T]. \tag{11}$$

We impose the following boundary and initial conditions:

$$M\left[\frac{1}{2\,\lambda}\underset{\sim}{\nabla}_q\left(\frac{\psi}{M}\right) - \underset{\approx}{\sigma}(\underset{\sim}{u})\,\underset{\sim}{q}\,\beta^L\left(\frac{\psi}{M}\right)\right]\cdot \underset{\sim}{n}_{\partial D} = 0 \qquad \text{on } \Omega \times \partial D \times (0,T], \tag{12a}$$

$$\varepsilon\,(\underset{\sim}{\nabla}_x\,\psi)\cdot\underset{\sim}{n}_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times D \times (0,T], \tag{12b}$$

$$\psi(\underset{\sim}{x},\underset{\sim}{q},0) = \psi_0(\underset{\sim}{x},\underset{\sim}{q}) \geq 0 \qquad \forall (\underset{\sim}{x},\underset{\sim}{q}) \in \Omega \times D; \tag{12c}$$

where $\underset{\sim}{q}$ is normal to ∂D, as D is a bounded ball centred at the origin, so $\underset{\sim}{n}_{\partial D} := \underset{\sim}{q}/|\underset{\sim}{q}|$ is the unit outward normal vector to ∂D, and $\underset{\sim}{n}_{\partial\Omega}$ is the unit outward normal to $\partial\Omega$. Here $\int_D \psi_0(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$.

The coupled problem (1a–d), (6), (7), (11), (12a–c) will be referred to as a *dumbbell model with microscopic cut-off*. In order to highlight the dependence on ε and L, in subsequent sections the solution to (11), (12a–c) will be labelled $\psi_{\varepsilon,L}$. Due to the coupling of (11) to (1a) through (6), the velocity and the pressure will also depend on ε and L and we shall therefore denote them in subsequent sections by $\underset{\sim}{u}_{\varepsilon,L}$ and $p_{\varepsilon,L}$.

A detailed argument for introducing cut-off, albeit of a very different nature, was put forward in El-Kareh and Leal [18] (cf. (3.10a,b)); the authors used a nonnegative function $\underset{\sim}{q} \in D \mapsto g(|\underset{\sim}{q}|)$ that is compactly supported in D, in both the right-hand side of the momentum equation and in the macroscopic counterpart of the Fokker–Planck equation, in order to truncate the unbounded function $\underset{\sim}{q} \in D \mapsto U'(\frac{1}{2}|\underset{\sim}{q}|^2) = 1/(1-|\underset{\sim}{q}|^2/b)$, $|\underset{\sim}{q}|^2 < b$, to a bounded compactly supported function $\underset{\sim}{q} \in D \mapsto g(|\underset{\sim}{q}|)\,U'(\frac{1}{2}|\underset{\sim}{q}|^2)$.

The cut-off β^L proposed here has several attractive properties. We observe that the couple $\{\underset{\sim}{u}_\infty, \psi_\infty\}$, defined by $\underset{\sim}{u}_\infty(\underset{\sim}{x}) := \underset{\sim}{0}$ and $\psi_\infty(\underset{\sim}{x},\underset{\sim}{q}) := M(\underset{\sim}{q})$, is still an equilibrium solution of (1a–d) with $\underset{\sim}{f} = \underset{\sim}{0}$, (6), (7), (11), (12a–c) for all $L > 0$. Thus, unlike the truncation of the (unbounded) potential proposed in El-Kareh and Leal [18], the introduction of the cut-off function β^L into the Fokker–Planck equation (8) does not alter the equilibrium solution $(\underset{\sim}{u}_\infty, \psi_\infty)$ of the original Navier–Stokes–Fokker–Planck system. In addition, the boundary conditions for ψ on $\partial\Omega\times D\times(0,T]$ and $\Omega \times \partial D \times (0,T]$ ensure that

$$\frac{1}{|\Omega|}\int_{\Omega\times D} \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \frac{1}{|\Omega|}\int_{\Omega\times D} \psi_0(\underset{\sim}{x}, \underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 1 \qquad \forall t \in \mathbb{R}_{\geq 0}.$$

The paper is structured as follows. We begin, in Section 2, by stating the weak formulation of the coupled Navier–Stokes–Fokker–Planck system with centre-of-mass diffusion and microscopic cut-off, for the general class of potentials U under consideration. In particular, the FENE model fits into the general setting. In Section 3 we step back from the coupled Navier–Stokes–Fokker–Planck system, and consider the Fokker–Planck equation in isolation, with a given velocity field $\underset{\sim}{u} \in \mathrm{C}([0,T]; \underset{\sim}{\mathrm{W}}^{1,\infty}_0(\Omega))$. We introduce a family of weighted Sobolev spaces that provide the natural functional-analytic framework for the problem: the weight of the space is the Maxwellian induced by the potential U appearing in the Fokker–Planck equation. We then prove the existence of a unique weak solution to the Fokker–Planck equation; microscopic cut-off on the drag term will not be required in this case. We also show that the weak solution to the Fokker–Planck equation is a probability density function in the sense that it is nonnegative over $\Omega \times D$ for all $t \geq 0$ and $\int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,\mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$ and all $t \in [0,T]$, whenever $\psi_0 \geq 0$ on $\Omega \times D$ and $\int_D \psi_0(\underset{\sim}{x}, \underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$. In Section 4 we return to the coupled Navier–Stokes–Fokker–Planck system with microcopic cut-off (1a–d), (6), (7), (11), (12a–c), and reproduce from our paper [6] the main steps of the proof of existence of global weak solutions to this system. Our arguments require a special compact embedding result in Maxwellian-weighted Sobolev spaces, which is proved in the Appendix to paper [6] by combining compact embedding theorems by Antoci [2] and Shakhmurov [42]. A key ingredient, resulting in sufficiently strong a-priori bounds, is a special testing procedure based on the convex entropy function

$$s \in \mathbb{R}_{\geq 0} \mapsto \mathcal{F}(s) := s\,(\ln s - 1) + 1 \in \mathbb{R}_{\geq 0}$$

in the weak formulation of the Fokker–Planck equation. This leads to a fortuitous cancellation of the extra stress term on the right-hand side of the Navier–Stokes equation with the drag term in the Fokker–Planck equation and results in an $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$ bound on the Kullback–Leibler relative entropy $\mathcal{E}_M(\psi)$ of ψ with respect to the equilibrium solution $\psi_\infty = M$, where

$$\mathcal{E}_M(\psi) := \int_D \mathcal{F}\left(\frac{\psi}{M}\right) M(\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q}.$$

Our choice of the entropy function $\mathcal{F}$ has been motivated by the papers of Arnold, Markowich, Toscani, and Unterreiter [3], Desvillettes and Villani [14], Jourdain, Lelièvre, Le Bris, and Otto [23] and Lin, Liu, and Zhang [29] cited above. It is important to note that the cut-off function β^L and the entropy function $\mathcal{F}$ are closely related, viz.

$$\beta^L(s) = \min(1/\mathcal{F}''(s), L),$$

and this connection will play a crucial role in our argument. Due to the fact that $\mathcal{F}''(s)$ is unbounded at $s = 0$, in Section 4 the strictly convex entropy function $\mathcal{F}$ will be replaced by a strictly convex regularization $\mathcal{F}^L_\delta$ whose second derivative is bounded above by $1/\delta$ and bounded below by $1/L$, $\delta \in (0,1)$, $L > 1$; at the same time the cut-off function β^L will be replaced by a strictly positive cut-off function β^L_δ defined by

$$\beta^L_\delta(s) = 1/[\mathcal{F}^L_\delta]''(s).$$

The existence of global weak solutions to the regularized cut-off problem is shown in Section 4.2. In Section 4.4 we then pass to the limit $\delta \to 0_+$ with the regularization parameter δ, to deduce the existence of a global weak solution to the coupled Navier–Stokes–Fokker–Planck system (1a–d), (6), (7), (11), (12a–c) with microscopic cut-off. Ideally, one would like to replace $\beta^L(s) = \min(s, L)$ by $\beta(s) = s$ in the Fokker–Planck equation. However, our current proof of existence in the general noncorotational case requires the presence of the microscopic cut-off function β^L on the drag term. Nevertheless, in the case of a corotational drag term at least passage to the limit $L \to \infty$ recovers the Fokker–Planck equation (8), without cut-off (see Remark 3).

2. **The polymer model.** We term polymer models, under consideration here, microscopic–macroscopic-type models, since the continuum mechanical *macroscopic* equations of incompressible fluid flow are coupled to a *microscopic* model: the Fokker–Planck equation describing the statistical properties of particles in the continuum. We first present these equations and collect assumptions on the parameters in the model.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz-continuous boundary $\partial\Omega$, and suppose that the set D of admissible elongation vectors $\underset{\sim}{q}$ in (8) is a bounded open ball in $\mathbb{R}^d$, $d = 2$ or 3, centred at the origin.

Gathering (1a–d), (6) and (8) together, we then consider the following initial-boundary-value problem, dependent on the parameters $\varepsilon \ll 1$ and $L \gg 1$:

(P$_{\varepsilon,L}$) Find $\underset{\sim}{u}_{\varepsilon,L} : (\underset{\sim}{x}, t) \in \mathbb{R}^{d+1} \mapsto \underset{\sim}{u}_{\varepsilon,L}(\underset{\sim}{x}, t) \in \mathbb{R}^d$ and $p_{\varepsilon,L} : (\underset{\sim}{x}, t) \in \mathbb{R}^{d+1} \mapsto p_{\varepsilon,L}(\underset{\sim}{x}, t) \in \mathbb{R}$ such that

$$\frac{\partial \underset{\sim}{u}_{\varepsilon,L}}{\partial t} + (\underset{\sim}{u}_{\varepsilon,L} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u}_{\varepsilon,L} - \nu\, \Delta_x\, \underset{\sim}{u}_{\varepsilon,L} + \underset{\sim}{\nabla}_x\, p_{\varepsilon,L} = \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau}(\psi_{\varepsilon,L}) \quad \text{in } \Omega \times (0, T], \tag{13a}$$

$$\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_{\varepsilon,L} = 0 \quad \text{in } \Omega \times (0, T], \tag{13b}$$

$$\underset{\sim}{u}_{\varepsilon,L} = \underset{\sim}{0} \quad \text{on } \partial\Omega \times (0, T], \tag{13c}$$

$$\underset{\sim}{u}_{\varepsilon,L}(\cdot, 0) = \underset{\sim}{u}_0 \quad \text{on } \Omega, \tag{13d}$$

where $\underset{\approx}{\tau}(\psi_{\varepsilon,L}) : (\underset{\sim}{x}, t) \in \mathbb{R}^{d+1} \mapsto \underset{\approx}{\tau}(\psi_{\varepsilon,L})(\underset{\sim}{x}, t) \in \mathbb{R}^{d\times d}$ is the symmetric extra-stress tensor, dependent on a probability density function $\psi_{\varepsilon,L} : (\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}^{2d+1} \mapsto \psi_{\varepsilon,L}(\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}$, defined as

$$\underset{\approx}{\tau}(\psi_{\varepsilon,L}) = k_B\, \mathcal{T}\, (\underset{\approx}{C}(\psi_{\varepsilon,L}) - \rho(\psi_{\varepsilon,L})\, \underset{\approx}{I}), \tag{14}$$

with

$$\underset{\approx}{C}(\psi_{\varepsilon,L})(\underset{\sim}{x}, t) = \int_D \psi_{\varepsilon,L}(\underset{\sim}{x}, \underset{\sim}{q}, t)\, U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\, \underset{\sim}{q}\, \underset{\sim}{q}^{\mathrm{T}}\, \mathrm{d}\underset{\sim}{q} \tag{15a}$$

and

$$\rho(\psi_{\varepsilon,L})(\underset{\sim}{x}, t) = \int_D \psi_{\varepsilon,L}(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q}. \tag{15b}$$

The Fokker–Planck equation with microscopic cut-off satisfied by $\psi_{\varepsilon,L}$ is:

$$\frac{\partial \psi_{\varepsilon,L}}{\partial t} + (\underset{\sim}{u}_{\varepsilon,L} \cdot \underset{\sim}{\nabla}_x)\psi_{\varepsilon,L} + \underset{\sim}{\nabla}_q \cdot \left[\underset{\approx}{\sigma}(\underset{\sim}{u}_{\varepsilon,L})\, \underset{\sim}{q}\, M\, \beta^L\left(\frac{\psi_{\varepsilon,L}}{M}\right)\right]$$

$$= \frac{1}{2\,\lambda}\, \underset{\sim}{\nabla}_q \cdot \left(M\, \underset{\sim}{\nabla}_q \left(\frac{\psi_{\varepsilon,L}}{M}\right)\right) + \varepsilon\, \Delta_x\, \psi_{\varepsilon,L} \qquad \text{in } \Omega \times D \times (0,T]. \quad (16)$$

Here, $\underset{\approx}{\sigma}(\underset{\sim}{v}) \equiv \underset{\approx}{\nabla}_x\, \underset{\sim}{v}$ and, for a given $L \gg 1$, $\beta^L \in \mathrm{C}(\mathbb{R})$ is defined by (10).

We impose the boundary and initial conditions (12a–c), where now we write $\psi_{\varepsilon,L}$ instead of ψ in order to emphasize the dependence on ε and L:

$$M\left[\frac{1}{2\,\lambda}\underset{\sim}{\nabla}_q\left(\frac{\psi_{\varepsilon,L}}{M}\right) - \underset{\approx}{\sigma}(\underset{\sim}{u}_{\varepsilon,L})\, \underset{\sim}{q}\, \beta^L\left(\frac{\psi_{\varepsilon,L}}{M}\right)\right]\cdot \underset{\sim}{n}_{\partial D} = 0 \quad \text{on } \Omega \times \partial D \times (0,T], \quad (17a)$$

$$\varepsilon\, (\underset{\sim}{\nabla}_x\, \psi_{\varepsilon,L}) \cdot \underset{\sim}{n}_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times D \times (0,T], \quad (17b)$$

$$\psi_{\varepsilon,L}(\underset{\sim}{x}, \underset{\sim}{q}, 0) = \psi_0(\underset{\sim}{x}, \underset{\sim}{q}) \geq 0 \quad \forall (\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega \times D. \quad (17c)$$

The boundary conditions for $\psi_{\varepsilon,L}$ on $\partial\Omega \times D \times (0,T]$ and $\Omega \times \partial D \times (0,T]$ have been chosen so as to ensure that $\int_{\Omega\times D} \psi_{\varepsilon,L}(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} = \int_{\Omega\times D} \psi_0(\underset{\sim}{x}, \underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} = |\Omega|$ for all $t \geq 0$.

3. **Existence of weak solutions to the Fokker–Planck equation.** We begin by considering, for a given velocity field $\underset{\sim}{u} \in \mathrm{C}([0,T]; \mathrm{W}^{1,\infty}_0(\Omega))$, $\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0$, the Fokker–Planck equation

$$\frac{\partial \psi}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x \psi + \underset{\sim}{\nabla}_q \cdot \left((\underset{\approx}{\nabla}_x\, \underset{\sim}{u})\, \underset{\sim}{q}\, \psi\right) = \frac{1}{2\,\lambda}\, \underset{\sim}{\nabla}_q \cdot \left(\underset{\sim}{\nabla}_q \psi + U'\left(\tfrac{1}{2}|\underset{\sim}{q}|^2\right)\underset{\sim}{q}\, \psi\right) + \varepsilon\, \Delta_x \psi.$$

In this section we shall not explicitly indicate the dependence of ψ on the centre-of-mass diffusion coefficient ε, and there will be no need to introduce cut-off into the drag term. Following Kolmogorov [26], the equation can be recast as follows:

$$\frac{\partial \psi}{\partial t} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x \psi + \underset{\sim}{\nabla}_q \cdot \left((\underset{\approx}{\kappa}\, \underset{\sim}{q})\, \psi\right)$$

$$= \frac{1}{2\,\lambda}\, \underset{\sim}{\nabla}_q \cdot \left(M\, \underset{\sim}{\nabla}_q \left(\frac{\psi}{M}\right)\right) + \varepsilon\, \Delta_x \psi, \quad (18)$$

where $\underset{\approx}{\kappa} := \underset{\approx}{\sigma}(\underset{\sim}{u}) = \underset{\approx}{\nabla}_x\, \underset{\sim}{u}$. Equation (18) is supplemented with the following initial and boundary conditions, where $\underset{\sim}{n}_{\partial\Omega}$ is the unit outward normal vector to $\partial\Omega$ and $\underset{\sim}{n}_{\partial D} = \underset{\sim}{q}/|\underset{\sim}{q}|$ is the unit outward normal vector to D:

$$\psi(\underset{\sim}{x}, \underset{\sim}{q}, 0) = \psi_0(\underset{\sim}{x}, \underset{\sim}{q}), \qquad \text{for all } (\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega \times D, \quad (19a)$$

$$\varepsilon\, (\underset{\sim}{\nabla}_x\, \psi) \cdot \underset{\sim}{n}_{\partial\Omega} = 0, \qquad \text{on } \partial\Omega \times D \times (0,T] \quad (19b)$$

and

$$M\left[\frac{1}{2\,\lambda}\underset{\sim}{\nabla}_q\left(\frac{\psi}{M}\right) - \underset{\approx}{\kappa}\, \underset{\sim}{q}\left(\frac{\psi}{M}\right)\right]\cdot \underset{\sim}{n}_{\partial D} = 0 \qquad \text{on } \Omega \times \partial D \times (0,T]. \quad (19c)$$

Here, the initial datum ψ_0 is such that $\psi_0 \geq 0$ and $\int_D \psi_0(\underset{\sim}{x}, \underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$. The imposition of the boundary condition (19c) will ensure that this property is propagated in time, i.e. that $\int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$ and $t \in (0,T]$.

Let us define $\widehat{\psi} := \psi/M$; with this notation, since $\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0$, (18) becomes:

$$M\frac{\partial \widehat{\psi}}{\partial t} + \underset{\sim}{\nabla}_x \cdot \left(M\, \underset{\sim}{u}(\underset{\sim}{x},t)\widehat{\psi}\right) + \underset{\sim}{\nabla}_q \cdot \left(M(\underset{\approx}{\kappa}(\underset{\sim}{x},t)\,\underset{\sim}{q})\,\widehat{\psi}\right)$$
$$= \frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q \cdot \left(M\,\underset{\sim}{\nabla}_q \widehat{\psi}\right) + \varepsilon\, M\, \Delta_x \widehat{\psi}. \tag{20}$$

Equation (20) is considered subject to the following initial and boundary conditions:

$$\widehat{\psi}(\underset{\sim}{x},\underset{\sim}{q},0) = \widehat{\psi}_0(\underset{\sim}{x},\underset{\sim}{q}), \qquad \text{for all } (\underset{\sim}{x},\underset{\sim}{q}) \in \Omega \times D, \tag{21a}$$
$$\varepsilon\, M\,(\underset{\sim}{\nabla}_x\, \widehat{\psi}) \cdot \underset{\sim}{n}_{\partial\Omega} = 0, \qquad \text{on } \partial\Omega \times D \times (0,T] \tag{21b}$$

and

$$M\left[\frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q\, \widehat{\psi} - (\underset{\approx}{\kappa}\,\underset{\sim}{q})\,\widehat{\psi}\right] \cdot \underset{\sim}{n}_{\partial D} = 0 \qquad \text{on } \Omega \times \partial D \times (0,T], \tag{21c}$$

where $\widehat{\psi}_0 := \psi_0/M$. When restated in terms of $\widehat{\psi}$, the problem naturally lends itself to a weak formulation in a Maxwellian-weighted Sobolev space, which we shall next introduce.

Let $\mathrm{L}^2_M(\Omega \times D)$ signify the Maxwellian-weighted L^2 space over $\Omega \times D$ with norm

$$\|\widehat{\varphi}\|_{\mathrm{L}^2_M(\Omega\times D)} := \left\{\int_{\Omega\times D} M\, |\widehat{\varphi}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right\}^{\frac{1}{2}}.$$

Similarly, we introduce $\mathrm{L}^2_M(D)$, the Maxwellian-weighted L^2 space over D.

On introducing

$$\|\widehat{\varphi}\|_{\mathrm{H}^1_M(\Omega\times D)} := \left\{\int_{\Omega\times D} M\, \left[|\widehat{\varphi}|^2 + |\underset{\sim}{\nabla}_x\, \widehat{\varphi}|^2 + |\underset{\sim}{\nabla}_q\, \widehat{\varphi}|^2\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right\}^{\frac{1}{2}}, \tag{22}$$

we then set

$$\widehat{\mathrm{X}} \equiv \mathrm{H}^1_M(\Omega \times D) := \left\{\widehat{\varphi} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega \times D) : \|\widehat{\varphi}\|_{\mathrm{H}^1_M(\Omega\times D)} < \infty\right\}. \tag{23}$$

The dual of $\widehat{\mathrm{X}}$ with respect to the pivot space $\mathrm{L}^2_M(\Omega \times D)$ will be denoted $\widehat{\mathrm{X}}'$.

It follows that

$$\mathrm{C}^\infty(\overline{\Omega \times D}) \text{ is dense in } \widehat{\mathrm{X}}.$$

This can be shown, for example, by a simple adaptation of Lemma 3.1 in Barrett, Schwab, and Süli [4], which appeals to fundamental results on weighted Sobolev spaces in Triebel [46] and Kufner [27]. We have from Sobolev embedding that

$$\mathrm{H}^1(\Omega; \mathrm{L}^2_M(D)) \hookrightarrow \mathrm{L}^s(\Omega; \mathrm{L}^2_M(D)), \tag{24}$$

where $s \in [1,\infty)$ if $d = 2$ or $s \in [1,6]$ if $d = 3$.

We are now ready to state the weak formulation of the Fokker–Planck initial-boundary-value problem (20), (21a–c). We begin by listing our hypotheses on the data:

$$\partial\Omega \in \mathrm{C}^{0,1}, \quad \widehat{\psi}_0 \in \mathrm{L}^2_M(\Omega \times D), \quad \widehat{\psi}_0 \geq 0, \quad \int_D M\, \widehat{\psi}_0(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q} = 1 \quad \text{a.e. } \underset{\sim}{x} \in \Omega;$$
$$\underset{\sim}{u} \in \mathrm{C}([0,T]; \underset{\sim}{\mathrm{W}}^{1,\infty}_0(\Omega)), \qquad \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0 \quad \text{on } \Omega \times [0,T]; \tag{25}$$
$$\text{and} \quad \gamma > 1 \quad \text{in (3a,b).}$$

Under these assumptions, and writing $\underset{\approx}{\kappa} := \underset{\approx}{\nabla}_x\, \underset{\sim}{u}$, we consider the following problem:

(Q) Find $\widehat{\psi} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D))\cap\mathrm{L}^2(0,T;\widehat{\mathrm{X}})\cap\mathrm{H}^1(0,T;\widehat{\mathrm{X}}')$ such that $\widehat{\psi}(\cdot,\cdot,0) = \widehat{\psi}_0(\cdot,\cdot)$ and

$$
\begin{aligned}
&\int_0^T \left\langle M\,\frac{\partial\widehat{\psi}}{\partial t}, \widehat{\varphi}\right\rangle_{\widehat{\mathrm{X}}} \mathrm{d}t - \int_0^T\int_{\Omega\times D} M\,\underset{\sim}{u}(\underset{\sim}{x},t)\,\widehat{\psi}(\underset{\sim}{x},\underset{\sim}{q},t)\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},t)\;\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&-\int_0^T\int_{\Omega\times D} M((\underset{\approx}{\kappa}(\underset{\sim}{x},t)\,\underset{\sim}{q})\,\widehat{\psi}(\underset{\sim}{x},\underset{\sim}{q},t))\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},t)\;\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad (26)\\
&+\frac{1}{2\lambda}\int_0^T\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_q\widehat{\psi}\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\;\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t + \varepsilon\int_0^T\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_x\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0\\
&\hspace{20em}\forall\widehat{\varphi}\in\mathrm{L}^2(0,T;\widehat{\mathrm{X}}),
\end{aligned}
$$

where $\widehat{\mathrm{X}}'$ is the dual space of $\widehat{\mathrm{X}}$ with respect to the pivot space $\mathrm{L}^2_M(\Omega\times D)$, and $\langle M\cdot,\cdot\rangle_{\widehat{\mathrm{X}}}$ denotes the duality pairing between $\widehat{\mathrm{X}}'$ and $\widehat{\mathrm{X}}$.

Theorem 3.1. *Under hypotheses* (25)*, there exists*

$$\widehat{\psi}\in\mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D))\cap\mathrm{L}^2(0,T;\widehat{\mathrm{X}})\cap\mathrm{H}^1(0,T;\widehat{\mathrm{X}}')$$

satisfying (26)*; moreover,* $\widehat{\psi}\in\mathrm{C}([0,T];\mathrm{L}^2_M(\Omega\times D))$,

$$\widehat{\psi}(\cdot,\cdot,0) = \widehat{\psi}_0(\cdot,\cdot), \qquad (27)$$

and $\widehat{\psi}$ *is unique.*

The function $\widehat{\psi}$ *will be referred to as the solution of problem (Q), and* $\psi := M\widehat{\psi}$ *will be called the weak solution of the initial-boundary-value problem* (18)*,* (19a–c)*.*

Proof. The proof proceeds in several steps.

Step 1. For $T>0$ and a positive integer N, let $N\Delta t = T$ and $t^n := n\Delta t$, $n = 0\to N$. We begin by considering the following semidiscrete problem: given $\widehat{\psi}^0 := \widehat{\psi}_0 = \psi_0/M \in \mathrm{L}^2_M(\Omega\times D)$, find $\widehat{\psi}^n\in\widehat{\mathrm{X}}$ for $n = 1\to N$ such that

$$
\begin{aligned}
&\int_{\Omega\times D} M\,\frac{\widehat{\psi}^n-\widehat{\psi}^{n-1}}{\Delta t}\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} - \int_{\Omega\times D} M\,\underset{\sim}{u}^n\,\widehat{\psi}^n\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\
&\quad-\int_{\Omega\times D} M\left[(\underset{\approx}{\kappa}^n\underset{\sim}{q})\,\widehat{\psi}^n - \frac{1}{2\lambda}\underset{\sim}{\nabla}_q\widehat{\psi}^n\right]\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\
&\qquad+\varepsilon\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_x\widehat{\psi}^n\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 0 \qquad \forall\widehat{\varphi}\in\widehat{\mathrm{X}}, \qquad (28)
\end{aligned}
$$

where $\underset{\approx}{\kappa}^n = \underset{\approx}{\nabla}_x\,\underset{\sim}{u}^n$ and $\underset{\sim}{u}^n = \underset{\sim}{u}(\cdot,t^n)$. It is easily shown that this problem possesses a unique solution for each $n = 1\to N$; to this end, consider the bilinear functional $\mathfrak{B}^n(\cdot,\cdot)$ on $\widehat{\mathrm{X}}\times\widehat{\mathrm{X}}$ defined by

$$
\begin{aligned}
\mathfrak{B}^n(\widehat{\psi},\widehat{\varphi}) &:= \int_{\Omega\times D} M\,\widehat{\psi}\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} - \Delta t\int_{\Omega\times D} M\,\underset{\sim}{u}^n\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\
&\quad-\Delta t\int_{\Omega\times D} M\left[(\underset{\approx}{\kappa}^n\underset{\sim}{q})\,\widehat{\psi} - \frac{1}{2\lambda}\underset{\sim}{\nabla}_q\widehat{\psi}\right]\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\
&\qquad+\varepsilon\,\Delta t\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_x\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \qquad \forall\widehat{\psi},\,\widehat{\varphi}\in\widehat{\mathrm{X}}, \qquad (29)
\end{aligned}
$$

and, for $\widehat{\psi}^{n-1} \in \mathrm{L}^2_M(\Omega \times D)$ fixed, we define the linear functional $\ell(\widehat{\psi}^{n-1})(\cdot)$ on $\widehat{\mathrm{X}}$ by

$$\ell(\widehat{\psi}^{n-1})(\widehat{\varphi}) := \int_{\Omega \times D} M\, \widehat{\psi}^{n-1}\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \qquad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}.$$

Hence, (28) amounts to solving, for $n = 1 \rightarrow N$, the problem: find $\widehat{\psi}^n \in \widehat{\mathrm{X}}$ such that

$$\mathfrak{B}^n(\widehat{\psi}^n, \widehat{\varphi}) = \ell(\widehat{\psi}^{n-1})(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}. \tag{30}$$

The existence of a unique solution to (30) is easily shown using the Lax–Milgram theorem on noting that: $\widehat{\mathrm{X}}$ is a Hilbert space; $\ell(\widehat{\psi}^{n-1})(\cdot)$ is a bounded linear functional on $\widehat{\mathrm{X}}$ for each $\widehat{\psi}^{n-1} \in \mathrm{L}^2_M(\Omega \times D)$; $\mathfrak{B}^n(\cdot,\cdot)$ is a bounded bilinear functional on $\widehat{\mathrm{X}} \times \widehat{\mathrm{X}}$; and, for Δt sufficiently small, $\mathfrak{B}^n(\cdot,\cdot)$ is a coercive bilinear functional on $\widehat{\mathrm{X}} \times \widehat{\mathrm{X}}$. For example, the last of these properties follows by recalling that $\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}^n = 0$ and noting the inequality

$$\begin{aligned}\mathfrak{B}^n(\widehat{\varphi}, \widehat{\varphi}) \geq \left(1 - \Delta t\, \lambda\, b\, \|\underset{\approx}{\kappa}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))}\right) \int_{\Omega \times D} M\, |\widehat{\varphi}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ + \frac{\Delta t}{4\lambda} \int_{\Omega \times D} M\, |\underset{\sim}{\nabla}_q \widehat{\varphi}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} + \varepsilon\, \Delta t \int_{\Omega \times D} M\, |\underset{\sim}{\nabla}_x \widehat{\varphi}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \qquad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}.\end{aligned} \tag{31}$$

Hence, on assuming that $\Delta t\, \lambda\, b\, \|\underset{\approx}{\kappa}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} < 1$ and letting

$$c_{\Delta t} := 1 - \Delta t\, \lambda\, b\, \|\underset{\approx}{\kappa}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))},$$

we deduce the coercivity of $\mathfrak{B}^n(\cdot,\cdot)$:

$$\mathfrak{B}^n(\widehat{\varphi}, \widehat{\varphi}) \geq \min\left(c_{\Delta t}, \frac{\Delta t}{4\lambda}, \varepsilon\, \Delta t\right) \|\widehat{\varphi}\|^2_{\widehat{\mathrm{X}}} \qquad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}. \tag{32}$$

Having shown that, for any $\Delta t = T/N$ such that $\Delta t\, \lambda\, b\, \|\underset{\approx}{\kappa}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} < 1$ and any $\widehat{\psi}^{n-1} \in \mathrm{L}^2_M(\Omega \times D)$, the problem (30) has a unique solution $\widehat{\psi}^n \in \widehat{\mathrm{X}}$ for each $n = 1 \rightarrow N$, and noting that $\widehat{\psi}^0 \in \mathrm{L}^2_M(\Omega \times D)$, we deduce that the semidiscrete problem (28) has a unique solution $(\widehat{\psi}^1, \ldots, \widehat{\psi}^N) \in \underbrace{\widehat{X} \times \cdots \times \widehat{X}}_{N}$.

Step 2. Next, on taking $\widehat{\varphi} = \widehat{\psi}^n$ in (28), we deduce that, for $n = 1 \rightarrow N$,

$$\begin{aligned}(1 - c_0 \Delta t) \int_{\Omega \times D} M\, |\widehat{\psi}^n|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} + \int_{\Omega \times D} M\, |\widehat{\psi}^n - \widehat{\psi}^{n-1}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ + \frac{\Delta t}{2\lambda} \int_{\Omega \times D} M\, |\underset{\sim}{\nabla}_q \widehat{\psi}^n|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} + 2\, \Delta t\, \varepsilon \int_{\Omega \times D} M\, |\underset{\sim}{\nabla}_x \widehat{\psi}^n|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\ \leq \int_{\Omega \times D} M\, |\widehat{\psi}^{n-1}|^2\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x},\end{aligned} \tag{33}$$

where $c_0 := 2\, b\, \lambda \|\underset{\approx}{\kappa}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))}$. Let us assume that

$$0 < c_0 \Delta t \leq \tfrac{1}{2} \qquad (\text{whereby, } \textit{a fortiori}, \ \Delta t\, \lambda\, b\, \|\underset{\approx}{\kappa}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} < 1). \tag{34}$$

On dividing (33) by $(1 - c_0\Delta t)$ and using that $1 \leq 1/(1 - c_0\Delta t) \leq 1 + 2c_0\Delta t$, we have that

$$\begin{aligned}\int_{\Omega\times D} M\,|\widehat{\psi}^n|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + \int_{\Omega\times D} M\,|\widehat{\psi}^n - \widehat{\psi}^{n-1}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ + \frac{\Delta t}{2\lambda}\int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_q \widehat{\psi}^n|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + 2\,\Delta t\,\varepsilon \int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_x \widehat{\psi}^n|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ \leq (1 + 2c_0\Delta t)\int_{\Omega\times D} M\,|\widehat{\psi}^{n-1}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \end{aligned} \tag{35}$$

for $n = 1 \rightarrow N$. Summing these inequalities through $n = 1 \rightarrow m$ gives

$$\begin{aligned}\int_{\Omega\times D} M\,|\widehat{\psi}^m|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + \sum_{n=1}^{m}\int_{\Omega\times D} M\,|\widehat{\psi}^n - \widehat{\psi}^{n-1}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ + \frac{1}{2\lambda}\sum_{n=1}^{m}\Delta t\int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_q \widehat{\psi}^n|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + 2\,\varepsilon \sum_{n=1}^{m}\Delta t\int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_x \widehat{\psi}^n|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ \leq \int_{\Omega\times D} M\,|\widehat{\psi}^0|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + 2c_0\sum_{n=0}^{m-1}\Delta t\int_{\Omega\times D} M\,|\widehat{\psi}^n|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \end{aligned} \tag{36}$$

for $m = 1 \rightarrow N$. Thus, by induction (or by using a discrete Grönwall lemma), we obtain the following energy estimate for the solution of (28):

$$\begin{aligned}\int_{\Omega\times D} M\,|\widehat{\psi}^m|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + \sum_{n=1}^{m}\Delta t\int_{\Omega\times D} M\left|\frac{\widehat{\psi}^n - \widehat{\psi}^{n-1}}{\sqrt{\Delta t}}\right|^2 \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ + \frac{1}{2\lambda}\sum_{n=1}^{m}\Delta t\int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_q \widehat{\psi}^n|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + 2\,\varepsilon \sum_{n=1}^{m}\Delta t\int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_x \widehat{\psi}^n|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ \leq (1 + 2c_0\Delta t)^m\int_{\Omega\times D} M\,|\widehat{\psi}^0|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ \leq \mathrm{e}^{2c_0 m\Delta t}\int_{\Omega\times D} M\,|\widehat{\psi}^0|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ \leq \mathrm{e}^{2c_0 T}\int_{\Omega\times D} M\,|\widehat{\psi}^0|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \end{aligned} \tag{37}$$

for $m = 1 \rightarrow N$.

Step 3. Let us denote by

$$\widehat{\psi}^{\Delta t} \in \mathrm{C}([0,T];\mathrm{L}^2_M(\Omega\times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}})$$

the continuous piecewise linear interpolant, with respect to $t \in [0,T]$, of the semidiscrete solution $\{\widehat{\psi}^n : n = 0 \rightarrow N\}$ to (28), defined for $n \geq 1$ by

$$\widehat{\psi}^{\Delta t}(\cdot,\cdot,t) := \frac{t - t^{n-1}}{\Delta t}\,\widehat{\psi}^n(\cdot,\cdot) + \frac{t^n - t}{\Delta t}\,\widehat{\psi}^{n-1}(\cdot,\cdot), \quad t \in [t^{n-1}, t^n];$$

and by $\widehat{\psi}^{\Delta t,+}$ and $\widehat{\psi}^{\Delta t,-}$ the piecewise constant interpolants defined, for $n \geq 1$, by

$$\widehat{\psi}^{\Delta t,+}(\cdot,\cdot,t) := \widehat{\psi}^n(\cdot,\cdot), \quad t \in (t^{n-1}, t^n], \quad \widehat{\psi}^{\Delta t,-}(\cdot,\cdot,t) := \widehat{\psi}^{n-1}(\cdot,\cdot), \quad t \in [t^{n-1}, t^n).$$

We shall denote by $\widehat{\psi}^{\Delta t,(\pm)}$ any one of the functions $\widehat{\psi}^{\Delta t}$, $\widehat{\psi}^{\Delta t,+}$, $\widehat{\psi}^{\Delta t,-}$ defined above; $\{\widehat{\psi}^{\Delta t,(\pm)}\}_{\Delta t}$ will signify the sequence of functions $\widehat{\psi}^{\Delta t,(\pm)}$, indexed by $\Delta t = T/N \rightarrow 0_+$, for T fixed, as $N \rightarrow \infty$.

Using analogous notation for $\underset{\approx}{\kappa}$, equation (28), with $\widehat{\varphi} \in \widehat{\mathrm{X}}$ replaced by $\widehat{\varphi}(\cdot,\cdot,t) \in \widehat{\mathrm{X}}$ for $t \in (0,T]$ where $\widehat{\varphi} \in \mathrm{L}^2(0,T;\widehat{\mathrm{X}})$, integrated over $[t^{n-1},t^n]$ and summed over $n = 1 \to N$, yields

$$\begin{aligned}\int_0^T \int_{\Omega\times D} M\, \frac{\partial \widehat{\psi}^{\Delta t}}{\partial t}\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t - \int_0^T \int_{\Omega\times D} M\, \underset{\sim}{u}^{\Delta t,+}\, \widehat{\psi}^{\Delta t,+} \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\ - \int_0^T \int_{\Omega\times D} M\, ((\underset{\approx}{\kappa}^{\Delta t,+} \underset{\sim}{q})\, \widehat{\psi}^{\Delta t,+}) \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\ + \frac{1}{2\lambda} \int_0^T \int_{\Omega\times D} M\, \underset{\sim}{\nabla}_q \widehat{\psi}^{\Delta t,+} \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\ + \varepsilon \int_0^T \int_{\Omega\times D} M\, \underset{\sim}{\nabla}_x \widehat{\psi}^{\Delta t,+} \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t = 0 \qquad \forall \widehat{\varphi} \in \mathrm{L}^2(0,T;\widehat{\mathrm{X}}). \end{aligned} \tag{38}$$

Inequality (37) implies that

$$\{\widehat{\psi}^{\Delta t,(\pm)}\}_{\Delta t} \quad \text{is bounded in} \quad \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D)), \tag{39a}$$

$$\{\widehat{\psi}^{\Delta t,(\pm)}\}_{\Delta t} \quad \text{is bounded in} \quad \mathrm{L}^2(0,T;\widehat{\mathrm{X}}), \tag{39b}$$

$$\left\{\frac{\widehat{\psi}^{\Delta t,+} - \widehat{\psi}^{\Delta t,-}}{\sqrt{\Delta t}}\right\}_{\Delta t} \quad \text{is bounded in} \quad \mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega\times D)). \tag{39c}$$

Now, (38) and the Cauchy–Schwarz inequality yield that

$$\begin{aligned}\left| \int_0^T \int_{\Omega\times D} M\, \frac{\partial \widehat{\psi}^{\Delta t}}{\partial t}\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \right| &\leq c_1 \|\widehat{\psi}^{\Delta t,+}\|_{\mathrm{L}^2(0,T;\widehat{\mathrm{X}})} \\ \times \left(\|\underset{\sim}{\nabla}_x \widehat{\varphi}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega\times D))} + \|\underset{\sim}{\nabla}_q \widehat{\varphi}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega\times D))} \right)^{1/2} &\qquad \forall \widehat{\varphi} \in \mathrm{L}^2(0,T;\widehat{\mathrm{X}}),\end{aligned}$$

where

$$c_1 := \left(\|\underset{\sim}{u}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} + b\|\underset{\approx}{\kappa}\|^2_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega))} + \varepsilon^2 + 1/(4\lambda^2) \right)^{\frac{1}{2}}.$$

Hence, using (39b), we deduce that

$$\left\{\frac{\partial \widehat{\psi}^{\Delta t}}{\partial t}\right\}_{\Delta t} \quad \text{is bounded in} \quad \mathrm{L}^2(0,T;\widehat{\mathrm{X}}'). \tag{39d}$$

Identifying, by means of the Riesz representation theorem, $\mathrm{L}^2_M(\Omega\times D)$ with $\mathrm{L}^2_M(\Omega\times D)'$, we deduce that $\widehat{\mathrm{X}} \subset \mathrm{L}^2_M(\Omega\times D) = \mathrm{L}^2_M(\Omega\times D)' \subset \widehat{\mathrm{X}}'$, so that each space is dense in the next one in the chain, with continuous and injective embedding. It thus follows from (39b) that $\{\widehat{\psi}^{\Delta t}\}_{\Delta t}$ is also bounded in $\mathrm{L}^2(0,T;\widehat{\mathrm{X}}')$. Combining this with (39d) implies that

$$\left\{\widehat{\psi}^{\Delta t}\right\}_{\Delta t} \quad \text{is bounded in} \quad \mathrm{H}^1(0,T;\widehat{\mathrm{X}}'). \tag{39e}$$

Now, (39a), (39b) and (39e) imply that we can extract a subsequence from $\{\widehat{\psi}^{\Delta t,(\pm)}\}_{\Delta t}$, which, for the sake of notational simplicity, we still label $\{\widehat{\psi}^{\Delta t,(\pm)}\}_{\Delta t}$, such that, as $\Delta t \to 0_+$,

$$\{\widehat{\psi}^{\Delta t,(\pm)}\}_{\Delta t} \quad \text{weak-}* \text{ converges in} \quad \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D)), \tag{40a}$$

$$\{\widehat{\psi}^{\Delta t,(\pm)}\}_{\Delta t} \quad \text{weakly converges in} \quad \mathrm{L}^2(0,T;\widehat{\mathrm{X}}), \tag{40b}$$

$$\{\widehat{\psi}^{\Delta t}\}_{\Delta t} \quad \text{weakly converges in} \quad \mathrm{H}^1(0,T;\widehat{\mathrm{X}}'). \tag{40c}$$

Specifically, (40a) implies the existence of $\widehat{\psi} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega \times D))$ such that

$$\int_0^T \int_{\Omega\times D} M\,(\widehat{\psi}^{\Delta t}(\underset{\sim}{x},\underset{\sim}{q},t) - \widehat{\psi}(\underset{\sim}{x},\underset{\sim}{q},t))\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \to 0 \quad \text{as} \quad \Delta t \to 0_+ \tag{41}$$

for all $\widehat{\varphi} \in \mathrm{L}^1(0,T;\mathrm{L}^2_M(\Omega \times D))$. On the other hand (40b) implies the existence of $\widehat{\psi}^* \in \mathrm{L}^2(0,T;\widehat{\mathrm{X}})$ such that

$$\int_0^T \langle M\,\widehat{\varphi}(t), \widehat{\psi}^{\Delta t}(t) - \widehat{\psi}^*(t)\rangle_{\widehat{\mathrm{X}}}\,\mathrm{d}t \to 0 \quad \text{as} \quad \Delta t \to 0_+ \tag{42}$$

for all $\widehat{\varphi} \in \mathrm{L}^2(0,T;\widehat{\mathrm{X}}')$.

Again, since $\widehat{\mathrm{X}} \subset \mathrm{L}^2_M(\Omega \times D) = \mathrm{L}^2_M(\Omega\times D)' \subset \widehat{\mathrm{X}}'$, so that each space is dense in the next one in the chain, with continuous and injective embedding, it follows that $\langle M\,\widehat{\varphi}, \widehat{\psi}\rangle_{\widehat{\mathrm{X}}} = \int_{\Omega\times D} M\,\widehat{\varphi}\,\widehat{\psi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \int_{\Omega\times D} M\,\widehat{\psi}\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$ for all $\widehat{\psi} \in \widehat{\mathrm{X}}$ and all $\widehat{\varphi} \in \mathrm{L}^2_M(\Omega \times D)$. Returning to (42), we then deduce that

$$\begin{aligned}\int_0^T \int_{\Omega\times D} M\,(\widehat{\psi}^{\Delta t}(\underset{\sim}{x},\underset{\sim}{q},t) - \widehat{\psi}^*(\underset{\sim}{x},\underset{\sim}{q},t))\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\ = \int_0^T \langle M\,\widehat{\varphi}(t), \widehat{\psi}^{\Delta t}(t) - \widehat{\psi}^*(t)\rangle_{\widehat{\mathrm{X}}}\,\mathrm{d}t \to 0\end{aligned}$$

as $\Delta t \to 0_+$ for all $\widehat{\varphi} \in \mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega\times D))$ $(\subset \mathrm{L}^2(0,T;\widehat{\mathrm{X}}'))$; this and (41) yield

$$\widehat{\psi} = \widehat{\psi}^* \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}}).$$

It remains to show that the weak–$*$ limits $\widehat{\psi}^\pm$ of the sequences $\{\widehat{\psi}^{\Delta t,\pm}\}_{\Delta t}$ in $\mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega \times D))$ are also equal to $\widehat{\psi}$. We shall show below that $\widehat{\psi}^+ = \widehat{\psi}^-$. Once we have done so, recalling from the definitions of $\widehat{\psi}^{\Delta t}$ and $\widehat{\psi}^{\Delta t,\pm}$ that

$$\begin{aligned}&\widehat{\psi}^{\Delta t}(\cdot,\cdot,t) - \widehat{\psi}^\pm(\cdot,\cdot,t) \\ &\quad = \frac{t-t^{n-1}}{\Delta t}(\widehat{\psi}^{\Delta t,+}(\cdot,\cdot,t) - \widehat{\psi}^+(\cdot,\cdot,t)) + \frac{t^n - t}{\Delta t}(\widehat{\psi}^{\Delta t,-}(\cdot,\cdot,t) - \widehat{\psi}^-(\cdot,\cdot,t))\end{aligned}$$

for all $t \in [t^{n-1}, t^n]$ and $n = 1 \to N$, and passing to the weak–$*$ limit in the space $\mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D))$ as $\Delta t \to 0_+$, will imply that $\widehat{\psi} = \widehat{\psi}^\pm$.

To show that $\widehat{\psi}^+ = \widehat{\psi}^-$, observe that

$$\begin{aligned}&\left|\int_0^T \int_{\Omega\times D} M\,(\widehat{\psi}^{\Delta t,+}(\underset{\sim}{x},\underset{\sim}{q},t) - \widehat{\psi}^{\Delta t,-}(\underset{\sim}{x},\underset{\sim}{q},t))\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\right| \\ &\qquad \le \left\|\frac{\widehat{\psi}^{\Delta t,+} - \widehat{\psi}^{\Delta t,-}}{\sqrt{\Delta t}}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega\times D))} \sqrt{\Delta t}\,\|\widehat{\varphi}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega\times D))},\end{aligned}$$

for any $\widehat{\varphi} \in \mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega\times D)) \subset \mathrm{L}^1(0,T;\mathrm{L}^2_M(\Omega\times D))$. Since by (39c) the first factor on the right-hand side is bounded, independent of Δt, on passing to the limit $\Delta t \to 0_+$, it follows that

$$\lim_{\Delta t\to 0_+} \int_0^T \int_{\Omega\times D} M\,(\widehat{\psi}^{\Delta t,+}(\underset{\sim}{x},\underset{\sim}{q},t) - \widehat{\psi}^{\Delta t,-}(\underset{\sim}{x},\underset{\sim}{q},t))\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0$$

for all $\widehat{\varphi} \in \mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega\times D)) \subset \mathrm{L}^1(0,T;\mathrm{L}^2_M(\Omega\times D))$. Therefore,

$$\int_0^T \int_{\Omega\times D} M\,(\widehat{\psi}^{+}(\underset{\sim}{x},\underset{\sim}{q},t) - \widehat{\psi}^{-}(\underset{\sim}{x},\underset{\sim}{q},t))\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0$$

for all $\widehat{\varphi} \in \mathrm{L}^2(0,T;\mathrm{L}^2_M(\Omega \times D))$. This, in turn, implies that $\widehat{\psi}^+ = \widehat{\psi}^-$. Thereby, as has been argued above, $\widehat{\psi} = \widehat{\psi}^+ = \widehat{\psi}^- \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}})$. Finally, (40c) and an argument identical to the one that we used to show that $\widehat{\psi} = \widehat{\psi}^*$ implies that, also, $\{\widehat{\psi}^{\Delta t}\}_{\Delta t}$ converges weakly to $\widehat{\psi}$ in $\mathrm{H}^1(0,T;\widehat{\mathrm{X}}')$, and therefore

$$\widehat{\psi} = \widehat{\psi}^+ = \widehat{\psi}^- \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega \times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}}) \cap \mathrm{H}^1(0,T;\widehat{\mathrm{X}}').$$

As the sequences $\{\underset{\sim}{u}^{\Delta t,+}\}_{\Delta t}$ and $\{\underset{\approx}{\kappa}^{\Delta t,+}\}_{\Delta t}$ converge (strongly) in $\mathrm{L}^\infty(0,T;\underset{\sim}{\mathrm{L}}^\infty(\Omega))$ and $\mathrm{L}^\infty(0,T;\underset{\approx}{\mathrm{L}}^\infty(\Omega))$ to $\underset{\sim}{u}$ and $\underset{\approx}{\kappa}$, respectively, passing to the limit $\Delta t \to 0_+$ in (38) and using (40a–c) implies that $\widehat{\psi}$ satisfies (26).

We note that since $\widehat{\psi} \in \mathrm{L}^2(0,T;\widehat{\mathrm{X}})$ and $\frac{\partial \widehat{\psi}}{\partial t} \in \mathrm{L}^2(0,T;\widehat{\mathrm{X}}')$, Lemma 1.2 in Ch. 3, Sec. 1.4 of Temam [45] (with $V = \widehat{\mathrm{X}}$, $H = \mathrm{L}^2_M(\Omega\times D)$, $V' = \widehat{\mathrm{X}}'$ and $\langle\cdot,\cdot\rangle = \langle M\cdot,\cdot\rangle_{\widehat{\mathrm{X}}}$) implies that $\widehat{\psi}$ is a.e. equal to a continuous function from $[0,T]$ into $\mathrm{L}^2_M(\Omega\times D)$, and therefore $\widehat{\psi} \in \mathrm{C}([0,T];\mathrm{L}^2_M(\Omega \times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}}) \cap \mathrm{H}^1(0,T;\widehat{\mathrm{X}}')$; and the following identity holds in the sense of distributions on $(0,T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{\psi}\|^2_{\mathrm{L}^2_M(\Omega\times D)} = 2\left\langle M\,\frac{\partial\widehat{\psi}}{\partial t}, \widehat{\psi}\right\rangle_{\widehat{\mathrm{X}}}. \tag{43}$$

In fact, for $\widehat{\psi} \in \mathrm{L}^2(0,T;\widehat{\mathrm{X}}) \cap \mathrm{H}^1(0,T;\widehat{\mathrm{X}}')$ the right-hand side of (43) belongs $\mathrm{L}^1(0,T)$, so the same is true of its left-hand side; hence, (43) holds as an identity in $\mathrm{L}^1(0,T)$.

Step 4. Next we show that $\widehat{\psi}$ satisfies the initial condition $\widehat{\psi}(\cdot,\cdot,0) = \widehat{\psi}_0(\cdot,\cdot)$. Integrating by parts in the first term on the left-hand side of (38), with $\widehat{\varphi} \in \mathrm{H}^1(0,T;\widehat{\mathrm{X}}) \hookrightarrow \mathrm{C}([0,T];\widehat{\mathrm{X}}) \hookrightarrow \mathrm{L}^2(0,T;\widehat{\mathrm{X}})$, we deduce that

$$\begin{aligned}
&\int_{\Omega\times D} M\,\widehat{\psi}^{\Delta t}(\underset{\sim}{x},\underset{\sim}{q},T)\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},T)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} - \int_{\Omega\times D} M\,\widehat{\psi}^{\Delta t}(\underset{\sim}{x},\underset{\sim}{q},0)\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},0)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\
&\quad - \int_0^T\int_{\Omega\times D} M\,\widehat{\psi}^{\Delta t}\,\frac{\partial\widehat{\varphi}}{\partial t}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t - \int_0^T\int_{\Omega\times D} M\,\underset{\sim}{u}^{\Delta t,+}\,\widehat{\psi}^{\Delta t,+}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&\quad - \int_0^T\int_{\Omega\times D} M\,((\underset{\approx}{\kappa}^{\Delta t,+}\underset{\sim}{q})\,\widehat{\psi}^{\Delta t,+})\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&\quad + \frac{1}{2\lambda}\int_0^T\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_q\widehat{\psi}^{\Delta t,+}\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&\quad + \varepsilon\int_0^T\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_x\widehat{\psi}^{\Delta t,+}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \qquad \forall\widehat{\varphi} \in \mathrm{H}^1(0,T;\widehat{\mathrm{X}}).
\end{aligned} \tag{44}$$

As $\widehat{\psi}^{\Delta t}(\cdot,\cdot,0) := \widehat{\psi}_0(\cdot,\cdot)$, passing to the limit $\Delta t \to 0_+$ in (44) we deduce that

$$\begin{aligned}
&-\int_{\Omega\times D} M\,\widehat{\psi}_0(\underset{\sim}{x},\underset{\sim}{q})\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},0)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} - \int_0^T\int_{\Omega\times D} M\,\widehat{\psi}\,\frac{\partial\widehat{\varphi}}{\partial t}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&-\int_0^T\int_{\Omega\times D} M\,\underset{\sim}{u}\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t - \int_0^T\int_{\Omega\times D} M((\underset{\approx}{\kappa}\,\underset{\sim}{q})\,\widehat{\psi})\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&+\frac{1}{2\lambda}\int_0^T\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_q\widehat{\psi}\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t + \varepsilon\int_0^T\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_x\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0\\
&\qquad\qquad\qquad\qquad \forall\widehat{\varphi} \in \mathrm{H}^1(0,T;\widehat{\mathrm{X}}),\quad \widehat{\varphi}(\cdot,\cdot,T) = 0.
\end{aligned} \tag{45}$$

In particular, upon choosing $\widehat{\varphi} = \widehat{\zeta}\,\widehat{w} \in \mathrm{C}_0^\infty(0,T) \otimes \widehat{\mathrm{X}}$ in (45), where $\widehat{\zeta} \in \mathrm{C}_0^\infty(0,T)$ and $\widehat{w} \in \widehat{\mathrm{X}}$ are arbitrary, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widehat{\psi},\widehat{w}) - (\underset{\sim}{u},\widehat{\psi}\,\underset{\sim}{\nabla}_x \widehat{w}) - ((\underset{\approx}{\kappa}\,\underset{\sim}{q})\,\widehat{\psi},\underset{\sim}{\nabla}_q \widehat{w}) + \frac{1}{2\lambda}(\underset{\sim}{\nabla}_q \widehat{\psi},\underset{\sim}{\nabla}_q \widehat{w}) + \varepsilon(\underset{\sim}{\nabla}_x \widehat{\psi},\underset{\sim}{\nabla}_x \widehat{w}) = 0 \quad (46)$$

for all $\widehat{w} \in \widehat{\mathrm{X}}$, in the sense of distributions on $(0,T)$, where $(\cdot,\cdot)$ denotes the inner product in $\mathrm{L}^2_M(\Omega \times D)$.

Since, for any $\widehat{w} \in \widehat{\mathrm{X}}$, the second, third, fourth and fifth term on the left-hand side of (46) belong to $\mathrm{L}^1(0,T)$, the same is true of the first term; therefore the function $t \in [0,T] \mapsto (\widehat{\psi}(\cdot,\cdot,t),\widehat{w})$ is absolutely continuous on $[0,T]$ for any $\widehat{w} \in \widehat{\mathrm{X}}$. Consequently, it makes sense to multiply (46) by $\widehat{\zeta} \in \mathrm{H}^1(0,T)$ $(\subset \mathrm{C}[0,T])$, such that $\widehat{\zeta}(T) = 0$, integrate over $[0,T]$ and integrate by parts with respect to t in the first term to deduce, on writing $\widehat{\varphi} = \widehat{\zeta}\,\widehat{w}$, that

$$\begin{aligned}
-(\widehat{\psi},\widehat{\varphi})|_{t=0} &- \int_0^T \int_{\Omega\times D} M\,\widehat{\psi}\,\frac{\partial \widehat{\varphi}}{\partial t}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t - \int_0^T \int_{\Omega\times D} M\,\underset{\sim}{u}\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&- \int_0^T \int_{\Omega\times D} M\,((\underset{\approx}{\kappa}\,\underset{\sim}{q})\,\widehat{\psi})\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t + \frac{1}{2\lambda}\int_0^T \int_{\Omega\times D} M\,\underset{\sim}{\nabla}_q\widehat{\psi}\cdot\underset{\sim}{\nabla}_q\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&+ \varepsilon\int_0^T \int_{\Omega\times D} M\,\underset{\sim}{\nabla}_x\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \qquad (47)
\end{aligned}$$

for all $\widehat{\varphi} \in \mathrm{H}^1(0,T) \otimes \widehat{\mathrm{X}}$, $\widehat{\varphi}(\cdot,\cdot,T) = 0$.

Applying (45) with $\widehat{\varphi} \in \mathrm{H}^1(0,T) \otimes \widehat{\mathrm{X}} \subset \mathrm{H}^1(0,T;\widehat{\mathrm{X}})$ and comparing with (47), it follows that $(\widehat{\psi},\widehat{\varphi})|_{t=0} = (\widehat{\psi}_0,\widehat{\varphi})|_{t=0}$ for all $\widehat{\varphi} \in \mathrm{H}^1(0,T) \otimes \widehat{\mathrm{X}}$, $\widehat{\varphi}(\cdot,\cdot,T) = 0$, and therefore, since $\widehat{\mathrm{X}}$ is dense in $\mathrm{L}^2_M(\Omega \times D)$, it follows that $(\widehat{\psi} - \widehat{\psi}_0,\widehat{w})|_{t=0} = 0$ for all $\widehat{w} \in \mathrm{L}^2_M(\Omega \times D)$. Recalling that $\widehat{\psi} \in \mathrm{C}([0,T];\mathrm{L}^2_M(\Omega \times D))$, it follows that $(\widehat{\psi}|_{t=0} - \widehat{\psi}_0,\widehat{w}) = 0$ for all $\widehat{w} \in \mathrm{L}^2_M(\Omega \times D)$, and thus $\widehat{\psi}$ satisfies the initial condition $\widehat{\psi}|_{t=0} = \widehat{\psi}_0$, i.e. $\widehat{\psi}(\cdot,\cdot,0) = \widehat{\psi}_0(\cdot,\cdot)$.

Step 5. We shall show that problem (Q) has a unique solution. Suppose that $\widehat{\psi} \in \mathrm{C}([0,T];\mathrm{L}^2_M(\Omega \times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}}) \cap \mathrm{H}^1(0,T;\widehat{\mathrm{X}}')$ is a solution of the initial-boundary-value problem (Q). Then, by (43), for any $s \in (0,T]$,

$$\begin{aligned}
\int_0^s \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{\psi}\|^2_{\mathrm{L}^2_M(\Omega\times D)}\,\mathrm{d}t &= \int_0^s \left\langle M\,\frac{\partial\widehat{\psi}}{\partial t},\widehat{\psi}\right\rangle_{\widehat{\mathrm{X}}}\mathrm{d}t \\
&= \int_0^s \left\{(\underset{\sim}{u}\,\widehat{\psi},\underset{\sim}{\nabla}_x\widehat{\psi}) + ((\underset{\approx}{\kappa}\,\underset{\sim}{q})\widehat{\psi},\underset{\sim}{\nabla}_q\widehat{\psi}) - \frac{1}{2\lambda}(\underset{\sim}{\nabla}_q\widehat{\psi},\underset{\sim}{\nabla}_q\widehat{\psi}) - \varepsilon(\underset{\sim}{\nabla}_x\widehat{\psi},\underset{\sim}{\nabla}_x\widehat{\psi})\right\}\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\left(\|\widehat{\psi}(s)\|^2_{\mathrm{L}^2_M(\Omega\times D)} - \|\widehat{\psi}_0\|^2_{\mathrm{L}^2_M(\Omega\times D)}\right) \\
&\quad + \varepsilon\|\underset{\sim}{\nabla}_x\widehat{\psi}\|^2_{\mathrm{L}^2(0,s;\mathrm{L}^2_M(\Omega\times D))} + \frac{1}{2\lambda}\|\underset{\sim}{\nabla}_q\widehat{\psi}\|^2_{\mathrm{L}^2(0,s;\mathrm{L}^2_M(\Omega\times D))} \\
&\quad = \int_0^s \left((\underset{\sim}{u}\widehat{\psi},\underset{\sim}{\nabla}_x\widehat{\varphi}) + ((\underset{\approx}{\kappa}\,\underset{\sim}{q})\widehat{\psi},\underset{\sim}{\nabla}_q\widehat{\psi})\right)\mathrm{d}t \\
&\quad \le \left(\|\underset{\sim}{u}\|^2_{\mathrm{L}^\infty(0,s;\mathrm{L}^\infty(\Omega))} + b\|\underset{\approx}{\kappa}\|^2_{\mathrm{L}^\infty(0,s;\mathrm{L}^\infty(\Omega))}\right)^{1/2}\|\widehat{\psi}\|_{\mathrm{L}^2(0,s;\mathrm{L}^2_M(\Omega\times D))} \\
&\qquad \times \left(\|\underset{\sim}{\nabla}_x\widehat{\psi}\|^2_{\mathrm{L}^2(0,s;\mathrm{L}^2_M(\Omega\times D))} + \|\underset{\sim}{\nabla}_q\widehat{\psi}\|^2_{\mathrm{L}^2(0,s;\mathrm{L}^2_M(\Omega\times D))}\right)^{1/2}
\end{aligned}$$

for all $s \in (0,T]$. This implies that

$$\|\widehat{\psi}(s)\|^2_{L^2_M(\Omega\times D)} + \min\left(\varepsilon, \frac{1}{2\lambda}\right)\left(\|\underset{\sim}{\nabla}_x\widehat{\psi}\|^2_{L^2(0,s;L^2_M(\Omega\times D))} + \|\underset{\sim}{\nabla}_q\widehat{\psi}\|^2_{L^2(0,s;L^2_M(\Omega\times D))}\right)$$
$$\leq \|\widehat{\psi}_0\|^2_{L^2_M(\Omega\times D)} + \max\left(\frac{1}{\varepsilon}, 2\lambda\right)\left(\|\underset{\sim}{u}\|^2_{L^\infty(0,s;L^\infty(\Omega))} + b\|\underset{\approx}{\kappa}\|^2_{L^\infty(0,s;L^\infty(\Omega))}\right)$$
$$\times\|\widehat{\psi}\|^2_{L^2(0,s;L^2_M(\Omega\times D))} \qquad \text{for any } s \in (0,T].$$

Thus, by Grönwall's lemma, any solution $\widehat{\psi}$ of (Q) satisfies the following energy inequality

$$\|\widehat{\psi}(s)\|^2_{L^2_M(\Omega\times D)} + \min\left(\varepsilon, \frac{1}{2\lambda}\right)\left(\|\underset{\sim}{\nabla}_x\widehat{\psi}\|^2_{L^2(0,s;L^2_M(\Omega\times D))} + \|\underset{\sim}{\nabla}_q\widehat{\psi}\|^2_{L^2(0,s;L^2_M(\Omega\times D))}\right)$$
$$\leq \|\widehat{\psi}_0\|^2_{L^2_M(\Omega\times D)} \exp\left[s\max\left(\frac{1}{\varepsilon}, 2\lambda\right)\left(\|\underset{\sim}{u}\|^2_{L^\infty(0,T;L^\infty(\Omega))} + b\|\underset{\approx}{\kappa}\|^2_{L^\infty(0,T;L^\infty(\Omega))}\right)\right]$$

for all $s \in (0,T]$. Note, in particular, that if $\widehat{\psi}_0 = 0$, then $\widehat{\psi}(\cdot,\cdot,s) = 0$ in $L^2_M(\Omega\times D)$ for all $s \in (0,T]$, which in turn implies that the solution to (Q), whose existence we have shown, is unique. □

Next we shall show that $\psi = M\widehat{\psi}$ has the usual properties of a probability density function: if ψ_0 is nonnegative and has unit integral over D, then the same is true of $\psi(\cdot,\cdot,t)$ for all $t \in [0,T]$.

Lemma 3.2. *Let $\widehat{\psi}_0 := \psi_0/M \in L^2_M(\Omega \times D)$ and suppose that $\int_D \psi_0(\underset{\sim}{x}, \underset{\sim}{q})\,d\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$. Suppose further that $\psi = M\,\widehat{\psi}$, where*

$$\widehat{\psi} \in C([0,T]; L^2_M(\Omega \times D)) \cap L^2(0,T;\widehat{X}) \cap H^1(0,T;\widehat{X}')$$

is the solution to problem (Q) subject to the initial datum $\widehat{\psi}_0$. Then,

$$\int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,d\underset{\sim}{q} = 1 \qquad \text{for a.e. } \underset{\sim}{x} \in \Omega \text{ and all } t \in [0,T].$$

Furthermore, if $\psi_0 \geq 0$ a.e. on $\Omega \times D$, then $\psi(\cdot,\cdot,t) \geq 0$ a.e. on $\Omega \times D$ for all $t \in [0,T]$.

Proof. We begin by noting that since $\widehat{\psi}_0 := \psi_0/M \in L^2_M(\Omega \times D)$, we have that $\|\psi_0\|_{L^1(\Omega\times D)} \leq \|\widehat{\psi}_0\|_{L^2_M(\Omega\times D)}$, and therefore $\psi_0 \in L^1(\Omega \times D)$. By the Fubini–Tonelli theorem $\psi_0(\underset{\sim}{x},\cdot) = M(\cdot)\,\widehat{\psi}_0(\underset{\sim}{x},\cdot) \in L^1(D)$ for a.e. $\underset{\sim}{x} \in \Omega$. Let $\widehat{\psi}$ denote the (unique) solution to problem (Q), and define $\zeta(\underset{\sim}{x}, t) := \int_D M(\underset{\sim}{q})\,\widehat{\psi}(\underset{\sim}{x}, \underset{\sim}{q}, t)\,d\underset{\sim}{q}$ for $(\underset{\sim}{x}, t) \in \Omega \times [0,T]$ and $\zeta_0(\underset{\sim}{x}) := \zeta(\underset{\sim}{x}, 0) = \int_D M(\underset{\sim}{q})\,\widehat{\psi}_0(\underset{\sim}{x}, \underset{\sim}{q})\,d\underset{\sim}{q}$ for $\underset{\sim}{x} \in \Omega$.

By taking $\widehat{\varphi}(\underset{\sim}{x}, \underset{\sim}{q}, t) = \varphi(\underset{\sim}{x}, t)$, independent of $\underset{\sim}{q}$, in (45), we have that

$$-\int_\Omega \zeta_0(\underset{\sim}{x})\,\varphi(\underset{\sim}{x},0)\,d\underset{\sim}{x} - \int_0^T\int_\Omega \zeta(\underset{\sim}{x},t)\,\frac{\partial\varphi(\underset{\sim}{x},t)}{\partial t}\,d\underset{\sim}{x}\,dt$$
$$-\int_0^T\int_\Omega \underset{\sim}{u}(\underset{\sim}{x},t)\,\zeta(\underset{\sim}{x},t)\cdot\underset{\sim}{\nabla}_x\varphi(\underset{\sim}{x},t)\,d\underset{\sim}{x}\,dt$$
$$+\,\varepsilon\int_0^T\int_\Omega \underset{\sim}{\nabla}_x\zeta\cdot\underset{\sim}{\nabla}_x\varphi\,d\underset{\sim}{x}\,dt = 0 \quad \forall\varphi \in H^1(0,T;H^1(\Omega)), \quad \varphi(\cdot,T) = 0. \tag{48}$$

Hence,

$$\zeta \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)')$$

is a weak solution to the parabolic equation

$$\zeta_t + \underset{\sim}{\nabla}_x \cdot (\underset{\sim}{u}(\underset{\sim}{x},t)\zeta) - \varepsilon \Delta_x \zeta = 0$$

subject to the initial condition $\zeta(\cdot,0) = \zeta_0 \in \mathrm{L}^2(\Omega)$ and the homogenous Neumann boundary condition $\varepsilon \underset{\sim}{\nabla}_x \zeta \cdot \underset{\sim}{n}_{\partial\Omega} = 0$ on $\partial\Omega$. This, however, has a unique weak solution. Since $\zeta_0(\underset{\sim}{x}) = 1$ for a.e. $\underset{\sim}{x} \in \Omega$, it follows that the corresponding unique weak solution must be the constant function $\zeta(\underset{\sim}{x},t) = 1$ for a.e. $\underset{\sim}{x} \in \Omega$ and all $t \in [0,T]$. Hence, $\int_D M\, \widehat{\psi}(\underset{\sim}{x},\underset{\sim}{q},t)\, \mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$ and all $t \in [0,T]$. Recalling that $\widehat{\psi} := \psi/M$, it follows that we have $\int_D \psi(\underset{\sim}{x},\underset{\sim}{q},t)\, \mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$ and all $t \in [0,T]$.

Now, suppose that $\psi_0 \geq 0$ on $\Omega \times D$; then, $\widehat{\psi}_0 := \psi_0/M \geq 0$ on $\Omega \times D$ and, by hypothesis, $\widehat{\psi}_0 \in \mathrm{L}^2_M(\Omega \times D)$. Selecting Δt small enough, as in (34), and defining $\widehat{\psi}^n$ as in the proof of the previous theorem, consider the sequence of functions $\{\widehat{\psi}^n\}_{n=0}^N \subset \widehat{\mathrm{X}}$. Then, by Lemma 3.3 below with $K = 0$, we have that $\{[\widehat{\psi}^n]_-\}_{n=0}^N \subset \widehat{\mathrm{X}}$. Hence,

$$\mathfrak{B}^n([\widehat{\psi}^n]_-\,,\,[\widehat{\psi}^n]_-) = \mathfrak{B}^n(\widehat{\psi}^n\,,\,[\widehat{\psi}^n]_-) = \ell(\widehat{\psi}^{n-1})([\widehat{\psi}^n]_-).$$

Suppose, for induction, that $\widehat{\psi}^{n-1} \geq 0$; this is certainly true for $n = 1$, since $\widehat{\psi}^0 = \widehat{\psi}_0 \geq 0$. Hence,

$$\ell(\widehat{\psi}^{n-1})([\widehat{\psi}^n]_-) = \int_{\Omega\times D} M\, \widehat{\psi}^{n-1}\, [\widehat{\psi}^n]_-\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \leq 0.$$

Therefore, $\mathfrak{B}^n([\widehat{\psi}^n]_-\,,\,[\widehat{\psi}^n]_-) \leq 0$; thus, (32) implies that $\|[\widehat{\psi}^n]_-\|_{\mathrm{H}^1_M(\Omega\times D)} \leq 0$, whereby $[\widehat{\psi}^n]_- = 0$ and hence $\widehat{\psi}^n \geq 0$. By induction, $\widehat{\psi}^n \geq 0$ for all $n = 0 \to N$. Therefore, each of the functions $\widehat{\psi}^{\Delta t}$, $\widehat{\psi}^+$ and $\widehat{\psi}^-$, defined in the proof of Theorem 3.1, is nonnegative on $\Omega \times D \times [0,T]$. Hence the common limiting function $\widehat{\psi}$ of the three sequences, as $\Delta t \to 0_+$, is also nonnegative on $\Omega \times D \times [0,T]$. $\square$

Lemma 3.3. *For $x \in \mathbb{R}$, let $[x]_\pm = (x \pm |x|)/2$; $[x]_+$ and $[x]_-$ are referred to as the positive and negative part of x, respectively. Suppose that $\widehat{\varphi} \in \widehat{\mathrm{X}}$ and $K \in \mathbb{R}_{\geq 0}$; then,*

$$\underset{\sim}{\nabla}_q [\,\widehat{\varphi} - K\,]_+ = \begin{cases} \underset{\sim}{\nabla}_q(\,\widehat{\varphi} - K\,) = \underset{\sim}{\nabla}_q \widehat{\varphi} & \text{if } \widehat{\varphi} > K, \\ \underset{\sim}{0} & \text{if } \widehat{\varphi} \leq K; \end{cases} \tag{49}$$

and

$$\underset{\sim}{\nabla}_q [\,\widehat{\varphi} - K\,]_- = \begin{cases} \underset{\sim}{\nabla}_q(\,\widehat{\varphi} - K\,) = \underset{\sim}{\nabla}_q \widehat{\varphi} & \text{if } \widehat{\varphi} < K, \\ \underset{\sim}{0} & \text{if } \widehat{\varphi} \geq K. \end{cases} \tag{50}$$

Analogous identities hold when the differential operator $\underset{\sim}{\nabla}_q$ is replaced by $\underset{\sim}{\nabla}_x$.

Furthermore, $[\,\widehat{\varphi} - K\,]_+$ and $[\,\widehat{\varphi} - K\,]_-$ belong to $\widehat{\mathrm{X}}$.

Proof. We shall prove (49); the proof of (50) is analogous, *mutatis mutandis*. We begin by noting that since $K \geq 0$ on $\Omega \times D$,

$$|[\,\widehat{\varphi} - K\,]_+| \leq |\widehat{\varphi}|. \tag{51}$$

Following [4], for any $\varepsilon > 0$, we define the following regularization of $[\,\cdot\,]_+$:

$$p_{+,\varepsilon}(s) := \begin{cases} (s^2 + \varepsilon^2)^{\frac{1}{2}} - \varepsilon & \text{if } s > 0, \\ 0 & \text{if } s \leq 0. \end{cases}$$

Clearly, $0 \leq p_{+,\varepsilon}(s) \leq [s]_+$ for all $s \in \mathbb{R}$. Let $\underset{\sim}{\eta} \in \underset{\sim}{\mathrm{C}}^\infty_0(\Omega \times D)$ be fixed. Thus,

$$\langle \underset{\sim}{\nabla}_q [\,\widehat{\varphi} - K\,]_+\,,\,\underset{\sim}{\eta} \rangle_{\mathrm{C}^\infty_0(\Omega\times D)} \quad = \quad -\langle [\,\widehat{\varphi} - K\,]_+\,,\,\underset{\sim}{\nabla}_q \cdot \underset{\sim}{\eta} \rangle_{\mathrm{C}^\infty_0(\Omega\times D)},$$

where $\langle\cdot,\cdot\rangle_{\mathrm{C}_0^\infty(\Omega\times D)}$ denotes the duality pairing between the topological vector space of distributions $\mathcal{D}'(\Omega\times D) = [\mathrm{C}_0^\infty(\Omega\times D)]'$ on $\Omega\times D$ and $\mathcal{D}(\Omega\times D) := \mathrm{C}_0^\infty(\Omega\times D)$ (on the right-hand side of this identity) and, with a minor abuse of notation, for d-fold vectorial versions of these spaces (on the left-hand side in this identity).

Let χ_S denote the characteristic function of a set $S\subset\Omega\times D$. Since $\underset{\sim}{\eta}$ has compact support in $\Omega\times D$, by Lebesgue's dominated convergence theorem we deduce that

$$\begin{aligned}\langle \underset{\sim}{\nabla}_q[\,\widehat{\varphi}-K\,]_+\,,\,\underset{\sim}{\eta}\rangle_{\mathrm{C}_0^\infty(\Omega\times D)} &= -\lim_{\varepsilon\to 0_+}\int_{\Omega\times D} p_{+,\varepsilon}(\widehat{\varphi}-K\,)\,(\underset{\sim}{\nabla}_q\cdot\underset{\sim}{\eta})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\ &= \lim_{\varepsilon\to 0_+}\int_{\Omega\times D} p'_{+,\varepsilon}(\widehat{\varphi}-K\,)\,\underset{\sim}{\nabla}_q(\widehat{\varphi}-K\,)\cdot\underset{\sim}{\eta}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\ &= \int_{\Omega\times D}\chi_{\widehat{\varphi}>K}(\underset{\sim}{q})\,\underset{\sim}{\nabla}_q(\widehat{\varphi}-K\,)\cdot\underset{\sim}{\eta}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \langle\chi_{\widehat{\varphi}>K}\,\underset{\sim}{\nabla}_q(\widehat{\varphi}-K\,)\,,\,\underset{\sim}{\eta}\rangle_{\mathrm{C}_0^\infty(\Omega\times D)}\end{aligned}$$

for all $\underset{\sim}{\eta}\in\underset{\sim}{\mathrm{C}}_0^\infty(\Omega\times D)$. Thus $\underset{\sim}{\nabla}_q[\,\widehat{\varphi}-K\,]_+ = \chi_{\widehat{\varphi}>K}(\underset{\sim}{q})\,\underset{\sim}{\nabla}_q(\widehat{\varphi}-K\,)$, and hence (49).

Now, since $\underset{\sim}{\nabla}_q[\,\widehat{\varphi}-K\,]_+ = \chi_{\widehat{\varphi}>K}\,\underset{\sim}{\nabla}_q\widehat{\varphi}$, and the right-hand side in this equality belongs to $\mathrm{L}^2_M(\Omega\times D)$ (recall that $\widehat{\varphi}\in\widehat{\mathrm{X}}$ by hypothesis), it follows that we have $\underset{\sim}{\nabla}_q[\,\widehat{\varphi}-K\,]_+\in\mathrm{L}^2_M(\Omega\times D)$; analogously, $\underset{\sim}{\nabla}_x[\,\widehat{\varphi}-K\,]_+\in\mathrm{L}^2_M(\Omega\times D)$. Hence, and by (51), $[\,\widehat{\varphi}-K\,]_+\in\widehat{\mathrm{X}}$, as required. $\square$

4. **Existence of global weak solutions to the coupled system.** Now we turn our attention to the coupled Navier–Stokes–Fokker–Planck system. Our aim is to establish the existence of global weak solutions, in the presence of microscopic cut-off on the drag term in the Fokker–Planck equation. The remaining sections represent an overview of the main results in our paper [6]. We begin with some preliminary considerations.

4.1. **Definitions and preliminary results.** Let

$$\underset{\sim}{\mathrm{H}} := \{\underset{\sim}{w}\in\underset{\sim}{\mathrm{L}}^2(\Omega) : \underset{\sim}{\nabla}_x\cdot\underset{\sim}{w}=0\}\quad\text{and}\quad\underset{\sim}{\mathrm{V}} := \{\underset{\sim}{w}\in\underset{\sim}{\mathrm{H}}^1_0(\Omega) : \underset{\sim}{\nabla}_x\cdot\underset{\sim}{w}=0\},\tag{52}$$

where the divergence operator $\underset{\sim}{\nabla}_x\cdot$ is to be understood in the sense of vector-valued distributions on Ω. Let $\underset{\sim}{\mathrm{V}}'$ be the dual of $\underset{\sim}{\mathrm{V}}$. Let $\underset{\approx}{S}:\underset{\sim}{\mathrm{V}}'\to\underset{\sim}{\mathrm{V}}$ be such that $\underset{\approx}{S}\,\underset{\sim}{v}$ is the unique solution to the Helmholtz–Stokes problem

$$\int_\Omega \underset{\approx}{S}\,\underset{\sim}{v}\cdot\underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x} + \int_\Omega \underset{\approx}{\nabla}_x\,(\underset{\approx}{S}\,\underset{\sim}{v}) : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x} = \langle\underset{\sim}{v},\underset{\sim}{w}\rangle_V\qquad\forall\underset{\sim}{w}\in\underset{\sim}{\mathrm{V}},\tag{53}$$

where $\langle\cdot,\cdot\rangle_V$ denotes the duality pairing between $\underset{\sim}{\mathrm{V}}'$ and $\underset{\sim}{\mathrm{V}}$. We note that

$$\langle\underset{\sim}{v},\underset{\approx}{S}\,\underset{\sim}{v}\rangle_V = \|\underset{\approx}{S}\,\underset{\sim}{v}\|^2_{\mathrm{H}^1(\Omega)}\qquad\forall\underset{\sim}{v}\in\underset{\sim}{\mathrm{V}}'\supset(\underset{\sim}{\mathrm{H}}^1_0(\Omega))',\tag{54}$$

and $\|\underset{\approx}{S}\cdot\|_{\mathrm{H}^1(\Omega)}$ is a norm on $\underset{\sim}{\mathrm{V}}'$.

For later purposes, we recall the following well-known Gagliardo–Nirenberg inequality. Let $r\in[2,\infty)$ if $d=2$, and $r\in[2,6]$ if $d=3$ and $\theta = d\left(\frac{1}{2}-\frac{1}{r}\right)$. Then, there is a constant C, depending only on Ω, r and d, such that the following inequality holds for all $\eta\in\mathrm{H}^1(\Omega)$:

$$\|\eta\|_{\mathrm{L}^r(\Omega)} \le C\,\|\eta\|^{1-\theta}_{\mathrm{L}^2(\Omega)}\,\|\eta\|^{\theta}_{\mathrm{H}^1(\Omega)}.\tag{55}$$

Our aim here is to prove existence of a (global-in-time) solution of a weak formulation of the problem $(\mathrm{P}_{\varepsilon,L})$ for any fixed parameters $\varepsilon\in(0,1]$ and $L>1$ under the following assumptions on the data:

$$\partial\Omega \in \mathrm{C}^{0,1}, \quad \underset{\sim}{u}_0 \in \underset{\sim}{\mathrm{H}}, \quad \widehat{\psi}_0 := M^{-1}\,\psi_0 \in \mathrm{L}^2_M(\Omega\times D) \text{ with } \widehat{\psi}_0 \geq 0 \text{ a.e. in } \Omega\times D, \tag{56}$$
$$\gamma > 1 \quad \text{in (3a,b)}, \qquad \text{and} \qquad \underset{\sim}{f} \in \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{V}}').$$

Similarly to (55) we have, with r and θ as defined there, that there exists a constant C, depending only on Ω, r and d, such that

$$\|\widehat{\varphi}\|_{\mathrm{L}^r(\Omega;\mathrm{L}^2_M(D))} \leq C\,\|\widehat{\varphi}\|^{1-\theta}_{\mathrm{L}^2(\Omega;\mathrm{L}^2_M(D))}\,\|\widehat{\varphi}\|^{\theta}_{\mathrm{H}^1(\Omega;\mathrm{L}^2_M(D))} \quad \forall \widehat{\varphi} \in \mathrm{H}^1(\Omega;\mathrm{L}^2_M(D)). \tag{57}$$

In addition, we note that the embeddings

$$\mathrm{H}^1_M(D) \hookrightarrow \mathrm{L}^2_M(D), \tag{58a}$$
$$\mathrm{H}^1_M(\Omega\times D) \equiv \mathrm{L}^2(\Omega;\mathrm{H}^1_M(D)) \cap \mathrm{H}^1(\Omega;\mathrm{L}^2_M(D)) \hookrightarrow \mathrm{L}^2_M(\Omega\times D) \equiv \mathrm{L}^2(\Omega;\mathrm{L}^2_M(D)) \tag{58b}$$

are compact if $\gamma \geq 1$ in (3a,b); see the Appendix in [8].

Let $\widehat{\mathrm{X}}'$ be the dual space of $\widehat{\mathrm{X}}$ with $\mathrm{L}^2_M(\Omega\times D)$ being the pivot space. Then, similarly to (53), let $\mathcal{G}:\widehat{\mathrm{X}}' \to \widehat{\mathrm{X}}$ be such that $\mathcal{G}\,\widehat{\eta}$ is the unique solution of

$$\int_{\Omega\times D} M\left[(\mathcal{G}\,\widehat{\eta})\,\widehat{\varphi} + \underset{\sim}{\nabla}_q\,(\mathcal{G}\,\widehat{\eta})\cdot\underset{\sim}{\nabla}_q\,\widehat{\varphi} + \underset{\sim}{\nabla}_x\,(\mathcal{G}\,\widehat{\eta})\cdot\underset{\sim}{\nabla}_x\,\widehat{\varphi}\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$= \langle M\,\widehat{\eta},\widehat{\varphi}\rangle_{\widehat{\mathrm{X}}} \qquad \forall \widehat{\varphi}\in\widehat{\mathrm{X}}, \tag{59}$$

where, as in the previous section, $\langle M\,\cdot,\cdot\rangle_{\widehat{\mathrm{X}}}$ denotes the duality pairing between $\widehat{\mathrm{X}}'$ and $\widehat{\mathrm{X}}$. Then, similarly to (54), we have that

$$\langle M\,\widehat{\eta},\mathcal{G}\,\widehat{\eta}\rangle_{\widehat{\mathrm{X}}} = \|\mathcal{G}\,\widehat{\eta}\|^2_{\widehat{\mathrm{X}}} \qquad \forall \widehat{\eta}\in\widehat{\mathrm{X}}', \tag{60}$$

and $\|\mathcal{G}\,\cdot\|_{\widehat{\mathrm{X}}}$ is a norm on $\widehat{\mathrm{X}}'$.

We recall the following compactness result, see, e.g., Temam [45] and Simon [43]. Let $\mathcal{B}_0$, $\mathcal{B}$ and $\mathcal{B}_1$ be Banach spaces, $\mathcal{B}_i$, $i=0,1$, reflexive, with a compact embedding $\mathcal{B}_0 \hookrightarrow \mathcal{B}$ and a continuous embedding $\mathcal{B}\hookrightarrow\mathcal{B}_1$. Then, for $\alpha_i > 1$, $i=0,1$, the embedding

$$\{\,\eta\in\mathrm{L}^{\alpha_0}(0,T;\mathcal{B}_0) : \tfrac{\partial\eta}{\partial t}\in\mathrm{L}^{\alpha_1}(0,T;\mathcal{B}_1)\,\} \hookrightarrow \mathrm{L}^{\alpha_0}(0,T;\mathcal{B}) \tag{61}$$

is compact.

We note that our proof, in the previous section, of Theorem 3.1, which concerns the existence and uniqueness of weak solutions to the Fokker–Planck equation for a given divergence-free velocity field $\underset{\sim}{u}\in\mathrm{C}^\infty([0,T];\underset{\sim}{\mathrm{W}}^{1,\infty}_0(\Omega))$, did not require the compact embedding of $\widehat{\mathrm{X}}$ into $\mathrm{L}^2_M(\Omega\times D)$ or of $\mathrm{H}^1_M(D)$ into $\mathrm{L}^2_M(D)$. In the case of the Navier–Stokes–Fokker–Planck system here the situation is radically different. Because of the (nonlinear) coupling between the Navier–Stokes and Fokker–Planck equations through the transport term $\underset{\sim}{u}\cdot\underset{\sim}{\nabla}_x\psi$ and the drag term $\underset{\sim}{\nabla}_q\cdot((\underset{\approx}{\nabla}_x\,\underset{\sim}{u})\,\underset{\sim}{q}\,\psi)$ in the Fokker–Planck equation, the regularity of $\underset{\sim}{u}$ cannot simply be hypothesized as before, but is dictated by the Navier–Stokes equations. The central difficulty is that weak solutions to the Navier–Stokes equations do not generally belong to $\mathrm{C}([0,T];\underset{\sim}{\mathrm{W}}^{1,\infty}_0(\Omega))$, so the energy estimate derived in the previous section no longer applies here. In [6] we therefore adopted a different strategy: the key element of our argument was an energy estimate based on the Kullback–Leibler relative entropy, which results in a fortuitous cancelation of the drag term in the Fokker–Planck equation with the extra-stress tensor appearing on the right-hand side of the Navier–Stokes equation. Applying this estimate to the time-semi-discrete Navier–Stokes–Fokker–Planck system gives rise to weakly convergent subsequences, as in the previous section; however now have to deal with products of weakly convergent sequences in the transport term $\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x \psi$ and the drag term $\underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x \underset{\sim}{u})\, \underset{\sim}{q}\, \psi)$ in the Fokker–Planck equation. To this end, we require the compact embedding of $\mathrm{H}^1_M(D)$ into $\mathrm{L}^2_M(D)$ and the compact embedding (61) with suitable choices of $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{B}_1$, α_0 and α_1, which were not needed before, when the Fokker–Planck equation was considered in isolation. We shall reproduce below the main steps of the argument from our paper [6] leading to the existence of weak solutions to coupled Navier–Stokes–Fokker–Planck systems with microscopic cut-off.

We will assume throughout that (56) hold, so that (5) and (58a,b) hold. We note for future reference that (15a) and (5) yield that, for $\widehat{\varphi} \in \mathrm{L}^2_M(\Omega \times D)$,

$$\begin{aligned} \int_\Omega |\underset{\approx}{C}(M\,\widehat{\varphi})|^2 \,\mathrm{d}\underset{\sim}{x} &= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \left(\int_D M\,\widehat{\varphi}\, U'\, q_i\, q_j \,\mathrm{d}\underset{\sim}{q} \right)^2 \mathrm{d}\underset{\sim}{x} \\ &\leq d \left(\int_D M\,(U')^2\, |\underset{\sim}{q}|^4 \,\mathrm{d}\underset{\sim}{q} \right) \left(\int_{\Omega\times D} M\,|\widehat{\varphi}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right) \\ &\leq C \left(\int_{\Omega\times D} M\,|\widehat{\varphi}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right), \end{aligned} \tag{62}$$

where $C = C(d)$ is a positive constant.

In order to prove existence of weak solutions to $(\mathrm{P}_{\varepsilon,L})$, we require a further regularization. Let $\mathcal{F} \in \mathrm{C}(\mathbb{R}_{>0})$ be defined by

$$\mathcal{F}(s) := s\,(\ln s - 1) + 1, \qquad s > 0. \tag{63}$$

As $\lim_{s\to 0_+} \mathcal{F}(s) = 1$, the function $\mathcal{F}$ can be considered to be defined and continuous on $[0,\infty)$, where it is a nonnegative, strictly convex function with $\mathcal{F}(1) = 0$.

We then introduce the following convex regularization $\mathcal{F}^L_\delta \in \mathrm{C}^{2,1}(\mathbb{R})$ of $\mathcal{F}$ defined, for any $\delta \in (0,1)$ and $L > 1$, by

$$\mathcal{F}^L_\delta(s) := \begin{cases} \frac{s^2-\delta^2}{2\,\delta} + s\,(\ln\delta - 1) + 1 & \text{for } s \leq \delta, \\ \mathcal{F}(s) \ \equiv s\,(\ln s - 1) + 1 & \text{for } \delta \leq s \leq L, \\ \frac{s^2-L^2}{2\,L} + s\,(\ln L - 1) + 1 & \text{for } L \leq s. \end{cases} \tag{64}$$

Hence,

$$[\mathcal{F}^L_\delta]'(s) = \begin{cases} \frac{s}{\delta} + \ln\delta - 1 & \text{for } s \leq \delta, \\ \ln s & \text{for } \delta \leq s \leq L, \\ \frac{s}{L} + \ln L - 1 & \text{for } L \leq s, \end{cases} \tag{65a}$$

$$[\mathcal{F}^L_\delta]''(s) = \begin{cases} \delta^{-1} & \text{for } s \leq \delta, \\ s^{-1} & \text{for } \delta \leq s \leq L, \\ L^{-1} & \text{for } L \leq s. \end{cases} \tag{65b}$$

We note that

$$\mathcal{F}^L_\delta(s) \geq \begin{cases} \frac{s^2}{2\,\delta} & \text{for } s \leq 0, \\ \frac{s^2}{4\,L} - C(L) & \text{for } s \geq 0; \end{cases} \tag{66}$$

and that $[\mathcal{F}^L_\delta]''(s)$ is bounded below by $1/L$ for all $s \in \mathbb{R}$. Finally, we set

$$\beta^L_\delta(s) := ([\mathcal{F}^L_\delta]'')^{-1}(s) = \max\{\beta^L(s), \delta\}\,, \tag{67}$$

and observe that $\beta^L_\delta(s)$ is bounded above by L for all $s \in \mathbb{R}$.

With the centre-of-mass diffusion coefficient $\varepsilon > 0$ and the microscopic cut-off parameter $L > 1$ fixed, our proof of existence of weak solutions to problem $(\mathrm{P}_{\varepsilon,L})$ will proceed in several steps, each of which is discussed in a separate section.

Step 1. We shall begin by replacing the microscopic cut-off function β^L in the Fokker–Planck equation by the regularized cut-off function β^L_δ; the resulting problem will be labelled $(\mathrm{P}_{\varepsilon,L,\delta})$. Following our approach in the case of the Fokker–Planck equation, we shall then semi-discretize the resulting equations in time, with time step Δt. We will show using Schauder's fixed-point theorem that, at each time level, the resulting nonlinear elliptic system has a solution.

Step 2. Having established the existence of solutions to the semidiscretization of $(\mathrm{P}_{\varepsilon,L,\delta})$ we shall then derive the relevant energy estimates for the problem, which will allow us to pass to the limit $\Delta t \to 0_+$ with the time step. Thus we shall establish the existence of solutions to problem $(\mathrm{P}_{\varepsilon,L,\delta})$, for each $\delta > 0$.

Step 3. Finally, we shall pass to the limit $\delta \to 0_+$ with the regularization parameter, to deduce the existence of weak solutions to $(\mathrm{P}_{\varepsilon,L})$.

4.2. **Existence of solutions to problem $(\mathbf{P}_{\varepsilon,L,\delta})$.** Problem $(\mathrm{P}_{\varepsilon,L,\delta})$, with solution $\{\underset{\sim}{u}_{\varepsilon,L,\delta}, \psi_{\varepsilon,L,\delta}\}$, will denote problem $(\mathrm{P}_{\varepsilon,L})$, where $\beta^L(\cdot)$ in (16) and (17a) is replaced by $\beta^L_\delta(\cdot)$; recall (10) and (67). Here we shall prove existence of a solution to $(\mathrm{P}_{\varepsilon,L,\delta})$ for given parameters ε, $\delta \in (0,1]$ and $L > 1$ with $\widehat{\psi}_{\varepsilon,L,\delta} = \psi_{\varepsilon,L,\delta}/M$:

$(\mathrm{P}_{\varepsilon,L,\delta})$ Find $\underset{\sim}{u}_{\varepsilon,L,\delta} \in \mathrm{L}^\infty(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega)) \cap \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{V}}) \cap \mathrm{W}^{1,\frac{4}{d}}(0,T;\underset{\sim}{\mathrm{V}}')$ as well as $\widehat{\psi}_{\varepsilon,L,\delta} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}}) \cap \mathrm{W}^{1,\frac{4}{d}}(0,T;\widehat{\mathrm{X}}')$, with $\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L,\delta}) \in \mathrm{L}^\infty(0,T;\underset{\approx}{\mathrm{L}}^2(\Omega))$, such that $\underset{\sim}{u}_{\varepsilon,L,\delta}(\cdot,0) = \underset{\sim}{u}_0(\cdot)$, $\widehat{\psi}_{\varepsilon,L,\delta}(\cdot,\cdot,0) = \widehat{\psi}_0(\cdot,\cdot)$ and

$$\begin{aligned}&\int_0^T \left\langle \frac{\partial \underset{\sim}{u}_{\varepsilon,L,\delta}}{\partial t}, \underset{\sim}{w} \right\rangle_V \mathrm{d}t \\ &\qquad + \int_0^T\!\!\int_\Omega \left[\left[(\underset{\sim}{u}_{\varepsilon,L,\delta}\cdot\underset{\sim}{\nabla}_x)\,\underset{\sim}{u}_{\varepsilon,L,\delta}\right]\cdot\underset{\sim}{w} + \nu\,\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\varepsilon,L,\delta} : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\right]\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\ &\quad = \int_0^T \langle \underset{\sim}{f}, \underset{\sim}{w}\rangle_V\,\mathrm{d}t - k_B\,\mathcal{T}\int_0^T\!\!\int_\Omega \underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L,\delta}) : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad \forall \underset{\sim}{w} \in \mathrm{L}^{\frac{4}{4-d}}(0,T;\underset{\sim}{\mathrm{V}});\end{aligned} \tag{68a}$$

$$\begin{aligned}&\int_0^T \left\langle M\,\frac{\partial\widehat{\psi}_{\varepsilon,L,\delta}}{\partial t}, \widehat{\varphi}\right\rangle_{\widehat{\mathrm{X}}} \mathrm{d}t \\ &+ \int_0^T\!\!\int_{\Omega\times D} M\left[\frac{1}{2\lambda}\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\varepsilon,L,\delta} - [\,\underset{\approx}{\sigma}(\underset{\sim}{u}_{\varepsilon,L,\delta})\,\underset{\sim}{q}\,]\,\beta^L_\delta(\widehat{\psi}_{\varepsilon,L,\delta})\right]\cdot\underset{\sim}{\nabla}_q\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\ &+ \int_0^T\!\!\int_{\Omega\times D} M\left[\varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\varepsilon,L,\delta} - \underset{\sim}{u}_{\varepsilon,L,\delta}\,\widehat{\psi}_{\varepsilon,L,\delta}\right]\cdot\underset{\sim}{\nabla}_x\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \qquad \forall\widehat{\varphi} \in \mathrm{L}^{\frac{4}{4-d}}(0,T;\widehat{\mathrm{X}}).\end{aligned} \tag{68b}$$

Remark 1. If $d = 2$, then $\underset{\sim}{u}_{\varepsilon,L,\delta} \in \mathrm{C}([0,T];\underset{\sim}{\mathrm{H}})$ (cf. Lemma 1.2 on p. 176 of Temam [45]), whereas if $d = 3$, then $\underset{\sim}{u}_{\varepsilon,L,\delta}$ is weakly continuous only as a mapping from $[0,T]$ into $\underset{\sim}{\mathrm{H}}$ (similarly as in Theorem 3.1 on p. 191 in Temam [45]). It is in the latter, weaker sense that the imposition of the initial condition to the $\underset{\sim}{u}_{\varepsilon,L,\delta}$-equation will be understood for $d = 2,3$: that is, $\lim_{t\to 0_+}\int_\Omega(\underset{\sim}{u}_{\varepsilon,L,\delta}(\underset{\sim}{x},t) - \underset{\sim}{u}_0(\underset{\sim}{x}))\cdot\underset{\sim}{v}(\underset{\sim}{x})\,\mathrm{d}\underset{\sim}{x} = 0$ for all $\underset{\sim}{v} \in \underset{\sim}{\mathrm{H}}$. Similarly, for the initial conditions of the $\widehat{\psi}_{\varepsilon,L,\delta}$-equation for $d = 2,3$:

$\lim_{t\to 0_+}\int_{\Omega\times D} M\,(\widehat{\psi}_{\varepsilon,L,\delta}(\underset{\sim}{x},\underset{\sim}{q},t)-\widehat{\psi}_0(\underset{\sim}{x},\underset{\sim}{q}))\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}=0$ for all $\widehat{\varphi}\in \mathrm{L}^2_M(\Omega\times D)$. $\diamond$

In order to prove existence of a weak solution to $(\mathrm{P}_{\varepsilon,L,\delta})$, we proceed in the same manner as in the proof of Theorem 3.1: we discretize in time; and so, for any $T>0$, let $N\,\Delta t=T$ and $t^n=n\,\Delta t$, $n=0\to N$. To prove existence of weak solutions under minimal smoothness requirements on the initial data, recall (56), we introduce $\underset{\sim}{u}^0\in \underset{\sim}{\mathrm{V}}$ such that

$$\int_\Omega \left[\underset{\sim}{u}^0\cdot \underset{\sim}{v}+\Delta t\,\underset{\approx}{\nabla}_x\,\underset{\sim}{u}^0 : \underset{\approx}{\nabla}_x\,\underset{\sim}{v}\right]\mathrm{d}\underset{\sim}{x}=\int_\Omega \underset{\sim}{u}_0\cdot \underset{\sim}{v}\,\mathrm{d}\underset{\sim}{x}\qquad \forall \underset{\sim}{v}\in \underset{\sim}{\mathrm{V}}; \tag{69}$$

and so

$$\int_\Omega [\,|\underset{\sim}{u}^0|^2+\Delta t\,|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}^0|^2\,]\,\mathrm{d}\underset{\sim}{x}\le \int_\Omega |\underset{\sim}{u}_0|^2\,\mathrm{d}\underset{\sim}{x}\le C\,. \tag{70}$$

In addition, we have that $\underset{\sim}{u}^0$ converges to $\underset{\sim}{u}_0$ weakly in $\underset{\sim}{\mathrm{H}}$ in the limit of $\Delta t\to 0_+$.

Let $\underset{\sim}{u}^0_{\varepsilon,L,\delta}=\underset{\sim}{u}^0$ and $\widehat{\psi}^0_{\varepsilon,L,\delta}=\widehat{\psi}_0$. Then, for $n=1\to N$, given $\{\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta},\widehat{\psi}^{n-1}_{\varepsilon,L,\delta}\}\in \underset{\sim}{\mathrm{V}}\times \mathrm{L}^2_M(\Omega\times D)$, find $\{\underset{\sim}{u}^{n}_{\varepsilon,L,\delta},\widehat{\psi}^{n}_{\varepsilon,L,\delta}\}\in \underset{\sim}{\mathrm{V}}\times \widehat{\mathrm{X}}$ such that

$$\begin{aligned}\int_\Omega \left[\frac{\underset{\sim}{u}^n_{\varepsilon,L,\delta}-\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}}{\Delta t}+(\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}\cdot\underset{\sim}{\nabla}_x)\,\underset{\sim}{u}^n_{\varepsilon,L,\delta}\right]\cdot \underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}+\nu\int_\Omega \underset{\approx}{\nabla}_x\,\underset{\sim}{u}^n_{\varepsilon,L,\delta}:\underset{\approx}{\nabla}_x\,\underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\\ =\int_\Omega \underset{\sim}{f}^n\cdot \underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}-k_B\,T\int_\Omega \underset{\approx}{C}(M\,\widehat{\psi}^n_{\varepsilon,L,\delta}):\underset{\approx}{\nabla}_x\,\underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\qquad \forall \underset{\sim}{w}\in\underset{\sim}{\mathrm{V}},\end{aligned} \tag{71a}$$

$$\begin{aligned}&\int_{\Omega\times D} M\,\frac{\widehat{\psi}^n_{\varepsilon,L,\delta}-\widehat{\psi}^{n-1}_{\varepsilon,L,\delta}}{\Delta t}\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\ &\quad+\int_{\Omega\times D} M\left[\frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}^n_{\varepsilon,L,\delta}-[\,\underset{\approx}{\sigma}(\underset{\sim}{u}^n_{\varepsilon,L,\delta})\,\underset{\sim}{q}\,]\,\beta^L_\delta(\widehat{\psi}^n_{\varepsilon,L,\delta})\right]\cdot\underset{\sim}{\nabla}_q\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\\ &\quad+\int_{\Omega\times D} M\left[\varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}^n_{\varepsilon,L,\delta}-\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}\,\widehat{\psi}^n_{\varepsilon,L,\delta}\right]\cdot\underset{\sim}{\nabla}_x\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}=0\qquad\forall\widehat{\varphi}\in\widehat{\mathrm{X}};\end{aligned} \tag{71b}$$

where

$$\underset{\sim}{f}^n(\cdot):=\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\underset{\sim}{f}(\cdot,t)\,\mathrm{d}t\in\underset{\sim}{\mathrm{V}}'. \tag{72}$$

Now, letting $\underset{\sim}{f}^{\Delta t,+}(\cdot,t):=f^n(\cdot)$ for $t\in(t^{n-1},t^n]$, $n=1\to N$, (56) and (72) imply that

$$\underset{\sim}{f}^{\Delta t,+}\to\underset{\sim}{f}\quad\text{strongly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{V}}')\text{ as }\Delta t\to 0_+, \tag{73}$$

It is convenient to rewrite (71a) as

$$b(\underset{\sim}{u}^n_{\varepsilon,L,\delta},\underset{\sim}{w})=\ell_b(\widehat{\psi}^n_{\varepsilon,L,\delta})(\underset{\sim}{w})\qquad\forall\underset{\sim}{w}\in\underset{\sim}{\mathrm{V}}; \tag{74}$$

where for all $\underset{\sim}{w}_i\in\underset{\sim}{\mathrm{H}}^1_0(\Omega)$, $i=1,2$,

$$b(\underset{\sim}{w}_1,\underset{\sim}{w}_2):=\int_\Omega\left[\underset{\sim}{w}_1+\Delta t\,(\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}\cdot\underset{\sim}{\nabla}_x)\underset{\sim}{w}_1\right]\cdot\underset{\sim}{w}_2\,\mathrm{d}\underset{\sim}{x}+\Delta t\,\nu\int_\Omega\underset{\approx}{\nabla}_x\,\underset{\sim}{w}_1:\underset{\approx}{\nabla}_x\,\underset{\sim}{w}_2\,\mathrm{d}\underset{\sim}{x}, \tag{75a}$$

and, for all $\underset{\sim}{w} \in \underset{\sim}{\mathrm{H}}^1_0(\Omega)$ and $\widehat{\varphi} \in \mathrm{L}^2_M(\Omega \times D)$,

$$\ell_b(\widehat{\varphi})(\underset{\sim}{w}) := \Delta t \, \langle \underset{\sim}{f}^n, \underset{\sim}{w} \rangle_V + \int_\Omega \left[\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta} \cdot \underset{\sim}{w} - \Delta t \, k_B \, \mathcal{T} \, \underset{\approx}{C}(M \, \widehat{\varphi}) : \underset{\approx}{\nabla}_x \, \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x}. \tag{75b}$$

We note that

$$\int_\Omega [(\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{w}_1] \cdot \underset{\sim}{w}_2 \, \mathrm{d}\underset{\sim}{x} = - \int_\Omega [(\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{w}_2] \cdot \underset{\sim}{w}_1 \, \mathrm{d}\underset{\sim}{x} \qquad \forall \underset{\sim}{v} \in \underset{\sim}{\mathrm{V}}, \quad \forall \underset{\sim}{w}_1, \underset{\sim}{w}_2 \in \underset{\sim}{\mathrm{H}}^1_0(\Omega), \tag{76}$$

and hence $b(\cdot,\cdot)$ is a continuous nonsymmetric coercive bilinear functional on $\underset{\sim}{\mathrm{H}}^1_0(\Omega) \times \underset{\sim}{\mathrm{H}}^1_0(\Omega)$. In addition, $\ell_b(\widehat{\varphi})(\cdot)$ is a continuous linear functional on $\underset{\sim}{\mathrm{V}}$ for any $\varphi \in \mathrm{L}^2_M(\Omega \times D)$.

For $r > d$, let

$$\underset{\sim}{\mathrm{Y}}^r := \left\{ \underset{\sim}{v} \in \underset{\sim}{\mathrm{L}}^r(\Omega) : \int_\Omega \underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x \, w \, \mathrm{d}\underset{\sim}{x} = 0 \quad \forall \underset{\sim}{w} \in \underset{\sim}{\mathrm{W}}^{1,\frac{r}{r-1}}(\Omega) \right\}. \tag{77}$$

It is also convenient to rewrite (71b) as

$$a(\widehat{\psi}^n_{\varepsilon,L,\delta}, \widehat{\varphi}) = \ell_a(\underset{\sim}{u}^n_{\varepsilon,L,\delta}, \widehat{\psi}^n_{\varepsilon,L,\delta})(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}, \tag{78}$$

where, for all $\widehat{\varphi}_1$, $\widehat{\varphi}_2 \in \widehat{\mathrm{X}}$,

$$a(\widehat{\varphi}_1, \widehat{\varphi}_2) := \int_{\Omega \times D} M \left(\widehat{\varphi}_1 \, \widehat{\varphi}_2 + \Delta t \left[\varepsilon \, \underset{\sim}{\nabla}_x \, \widehat{\varphi}_1 - \underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta} \, \widehat{\varphi}_1 \right] \cdot \underset{\sim}{\nabla}_x \, \widehat{\varphi}_2 + \frac{\Delta t}{2\,\lambda} \underset{\sim}{\nabla}_q \, \widehat{\varphi}_1 \cdot \underset{\sim}{\nabla}_q \, \widehat{\varphi}_2 \right) \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}, \tag{79a}$$

and, for all $\underset{\sim}{v} \in \underset{\sim}{\mathrm{H}}^1(\Omega)$, $\widehat{\eta} \in \mathrm{L}^2_M(\Omega \times D)$ and $\widehat{\varphi} \in \widehat{\mathrm{X}}$,

$$\ell_a(\underset{\sim}{v}, \widehat{\eta})(\widehat{\varphi}) := \int_{\Omega \times D} M \left[\widehat{\psi}^{n-1}_{\varepsilon,L,\delta} \, \widehat{\varphi} + \Delta t \, [\, \underset{\approx}{\sigma}(\underset{\sim}{v}) \, \underset{\sim}{q} \,] \, \beta^L_\delta(\widehat{\eta}) \cdot \underset{\sim}{\nabla}_q \, \widehat{\varphi} \right] \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}, \tag{79b}$$

It follows from (77) and (24) that, for $r > d$,

$$\int_{\Omega \times D} M \, \underset{\sim}{v} \, \widehat{\varphi} \cdot \underset{\sim}{\nabla}_x \, \widehat{\varphi} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} = 0 \qquad \forall \underset{\sim}{v} \in \underset{\sim}{\mathrm{Y}}^r, \quad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}; \tag{80}$$

and hence that $a(\cdot,\cdot)$ is a continuous nonsymmetric coercive bilinear functional on $\widehat{\mathrm{X}} \times \widehat{\mathrm{X}}$. In addition, $\ell_a(\underset{\sim}{v}, \widehat{\eta})(\cdot)$ is a linear functional on $\widehat{\mathrm{X}}$ for all $\underset{\sim}{v} \in \underset{\sim}{\mathrm{H}}^1(\Omega)$ and $\widehat{\eta} \in \mathrm{L}^2_M(\Omega \times D)$.

In order to prove existence of a solution to (71a,b), we consider a fixed-point argument. Given $\widehat{\psi} \in \mathrm{L}^2_M(\Omega \times D)$ let $\{\underset{\sim}{u}^\star, \widehat{\psi}^\star\} \in \underset{\sim}{\mathrm{V}} \times \widehat{\mathrm{X}}$ be such that

$$b(\underset{\sim}{u}^\star, \underset{\sim}{w}) = \ell_b(\widehat{\psi})(\underset{\sim}{w}) \qquad \forall \underset{\sim}{w} \in \underset{\sim}{\mathrm{V}}, \tag{81a}$$

$$a(\widehat{\psi}^\star, \widehat{\varphi}) = \ell_a(\underset{\sim}{u}^\star, \widehat{\psi})(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}. \tag{81b}$$

The Lax–Milgram theorem yields the existence of a unique solution to (81a,b), and so the overall procedure (81a,b) is well defined.

The proof of the next lemma contains the key entropy estimate, and is therefore reproduced from [6] in its entirety.

Lemma 4.1. *Let $G : \mathrm{L}^2_M(\Omega \times D) \to \widehat{\mathrm{X}} \subset \mathrm{L}^2_M(\Omega \times D)$ denote the nonlinear map that takes $\widehat{\psi}$ to $\widehat{\psi}^\star = G(\widehat{\psi})$ via the procedure* (81a,b). *Then G has a fixed point. Hence there exists a solution $\{\underset{\sim}{u}^n_{\varepsilon,L,\delta}, \widehat{\psi}^n_{\varepsilon,L,\delta}\} \in \underset{\sim}{\mathrm{V}} \times \widehat{\mathrm{X}}$ to* (71a,b).

Proof. Clearly, a fixed point of G yields a solution of (71a,b). In order to show that G has a fixed point, we apply Schauder's fixed-point theorem; that is, we need to show that (i) $G : \mathrm{L}^2_M(\Omega\times D) \to \mathrm{L}^2_M(\Omega\times D)$ is continuous, that (ii) it is compact, and that (iii) there exists a $C_\star \in \mathbb{R}_{>0}$ such that

$$\|\widehat{\psi}\|_{\mathrm{L}^2_M(\Omega\times D)} \leq C_\star \tag{82}$$

for every $\widehat{\psi} \in \mathrm{L}^2_M(\Omega\times D)$ and $\omega \in (0,1]$ satisfying $\widehat{\psi} = \omega\, G(\widehat{\psi})$.

Let $\{\widehat{\psi}^{(i)}\}_{i\geq 0}$ be such that

$$\widehat{\psi}^{(i)} \to \widehat{\psi} \qquad \text{strongly in } \mathrm{L}^2_M(\Omega\times D) \qquad \text{as } i\to\infty. \tag{83}$$

It follows immediately from (67) and (62) that

$$M^{\frac{1}{2}}\,\beta^L_\delta(\widehat{\psi}^{(i)}) \to M^{\frac{1}{2}}\,\beta^L_\delta(\widehat{\psi}) \qquad \text{strongly in } \mathrm{L}^\infty(\Omega\times D) \qquad \text{as } i\to\infty, \tag{84a}$$

$$\underset{\approx}{C}(M\,\widehat{\psi}^{(i)}) \to \underset{\approx}{C}(M\,\widehat{\psi}) \qquad \text{strongly in } \mathrm{L}^2(\Omega) \qquad \text{as } i\to\infty. \tag{84b}$$

We need to show that

$$\widehat{\eta}^{(i)} := G(\widehat{\psi}^{(i)}) \to G(\widehat{\psi}) \quad \text{strongly in } \mathrm{L}^2_M(\Omega\times D) \quad \text{as } i\to\infty, \tag{85}$$

in order to prove (i) above. We have from the definition of G, see (81a,b), that, for all $i \geq 0$,

$$a(\widehat{\eta}^{(i)},\widehat{\varphi}) = \ell_a(\underset{\sim}{v}^{(i)},\widehat{\psi}^{(i)})(\widehat{\varphi}) \qquad \forall \widehat{\varphi}\in\widehat{\mathrm{X}}, \tag{86a}$$

where $\underset{\sim}{v}^{(i)} \in \underset{\sim}{\mathrm{V}}$ satisfies

$$b(\underset{\sim}{v}^{(i)},\underset{\sim}{w}) = \ell_b(\widehat{\psi}^{(i)})(\underset{\sim}{w}) \qquad \forall \underset{\sim}{w}\in\underset{\sim}{\mathrm{V}}. \tag{86b}$$

Choosing $\widehat{\varphi} = \widehat{\eta}^{(i)}$ in (86a) yields, on noting the simple identity

$$2\,(s_1 - s_2)\,s_1 = s_1^2 + (s_1-s_2)^2 - s_2^2 \qquad \forall s_1, s_2 \in \mathbb{R}, \tag{87}$$

(80) and (67) that, for all $i \geq 0$,

$$\int_{\Omega\times D} M\left[|\widehat{\eta}^{(i)}|^2 + |\widehat{\eta}^{(i)} - \widehat{\psi}^{n-1}_{\varepsilon,L,\delta}|^2 + \frac{\Delta t}{2\,\lambda}|\underset{\sim}{\nabla}_q\,\widehat{\eta}^{(i)}|^2 + 2\,\varepsilon\,\Delta t\,|\underset{\sim}{\nabla}_x\,\eta^{(i)}|^2\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$\leq \int_{\Omega\times D} M\,|\widehat{\psi}^{n-1}_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + C(L,\lambda)\,\Delta t\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{v}^{(i)}|^2\,\mathrm{d}\underset{\sim}{x}\,. \tag{88}$$

Choosing $\underset{\sim}{w} \equiv \underset{\sim}{v}^{(i)}$ in (86b), and noting (87), (76), (62), (53), a Poincaré inequality and (83) yields, for all $i \geq 0$, that

$$\int_\Omega \left[|\underset{\sim}{v}^{(i)}|^2 + |\underset{\sim}{v}^{(i)} - \underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}|^2\right]\mathrm{d}\underset{\sim}{x} + \Delta t\,\nu\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{v}^{(i)}|^2\,\mathrm{d}\underset{\sim}{x}$$
$$\leq \int_\Omega |\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{x} + C\,\Delta t\,\|\underset{\sim}{S}\,\underset{\sim}{f}^n\|^2_{\mathrm{H}^1(\Omega)} + C\,\Delta t\int_{\Omega\times D} M\,|\widehat{\psi}^{(i)}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq C\,. \tag{89}$$

Combining (88) and (89), we have for all $i \geq 0$ that

$$\|\widehat{\eta}^{(i)}\|_{\widehat{\mathrm{X}}} + \|\underset{\sim}{v}^{(i)}\|_{\mathrm{H}^1(\Omega)} \leq C(L,(\Delta t)^{-1})\,. \tag{90}$$

It follows from (90), (24) and the compactness of the embedding (58b) that there exists a subsequence $\{\widehat{\eta}^{(i_k)},\underset{\sim}{v}^{(i_k)}\}_{i_k\geq 0}$ and functions $\widehat{\eta}\in\widehat{\mathrm{X}}$ and $\underset{\sim}{v}\in\underset{\sim}{\mathrm{V}}$ such that, as

$i_k \to \infty$,

$$\widehat{\eta}^{(i_k)} \to \widehat{\eta} \qquad \text{weakly in } \mathrm{L}^s(\Omega; \mathrm{L}^2_M(D)), \tag{91a}$$

$$M^{\frac{1}{2}} \underset{\sim}{\nabla}_x \widehat{\eta}^{(i_k)} \to M^{\frac{1}{2}} \underset{\sim}{\nabla}_x \widehat{\eta} \qquad \text{weakly in } \underset{\sim}{\mathrm{L}}^2(\Omega \times D), \tag{91b}$$

$$M^{\frac{1}{2}} \underset{\sim}{\nabla}_q \widehat{\eta}^{(i_k)} \to M^{\frac{1}{2}} \underset{\sim}{\nabla}_q \widehat{\eta} \qquad \text{weakly in } \underset{\sim}{\mathrm{L}}^2(\Omega \times D), \tag{91c}$$

$$\widehat{\eta}^{(i_k)} \to \widehat{\eta} \qquad \text{strongly in } \mathrm{L}^2_M(\Omega \times D), \tag{91d}$$

$$\underset{\sim}{v}^{(i_k)} \to \underset{\sim}{v} \qquad \text{weakly in } \underset{\sim}{\mathrm{H}}^1(\Omega); \tag{91e}$$

where $s \in [1, \infty)$ if $d = 2$ or $s \in [1, 6]$ if $d = 3$. It follows from (86b), (75a,b), (91e) and (84b) that $\underset{\sim}{v} \in \underset{\sim}{\mathrm{V}}$ and $\widehat{\psi} \in \widehat{\mathrm{X}}$ satisfy

$$b(\underset{\sim}{v}, \underset{\sim}{w}) = \ell_b(\widehat{\psi})(\underset{\sim}{w}) \qquad \forall \underset{\sim}{w} \in \underset{\sim}{\mathrm{V}}. \tag{92}$$

It follows from (86a), (79a,b), (91a–e) and (84a) that $\widehat{\eta}$, $\widehat{\psi} \in \widehat{\mathrm{X}}$ and $\underset{\sim}{v} \in \underset{\sim}{\mathrm{V}}$, satisfy

$$a(\widehat{\eta}, \widehat{\varphi}) = \ell_a(\underset{\sim}{v}, \widehat{\psi})(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}. \tag{93}$$

Combining (93) and (92), we have that $\widehat{\eta} = G(\widehat{\psi}) \in \widehat{\mathrm{X}}$. Therefore the whole sequence $\widehat{\eta}^{(i)} \equiv G(\widehat{\psi}^{(i)}) \to G(\widehat{\psi})$ strongly in $\mathrm{L}^2_M(\Omega \times D)$ as $i \to \infty$, and so (i) holds.

As the embedding $\widehat{\mathrm{X}} \hookrightarrow \mathrm{L}^2_M(\Omega \times D)$ is compact, it follows that (ii) holds.

As regards (iii), $\widehat{\psi} = \omega\, G(\widehat{\psi})$ implies that $\{\underset{\sim}{v}, \widehat{\psi}\} \in \underset{\sim}{\mathrm{V}} \times \widehat{\mathrm{X}}$ satisfies

$$b(\underset{\sim}{v}, \underset{\sim}{w}) = \ell_b(\widehat{\psi})(\underset{\sim}{w}) \qquad \forall \underset{\sim}{w} \in \underset{\sim}{\mathrm{V}}, \tag{94a}$$

$$a(\widehat{\psi}, \widehat{\varphi}) = \omega\, \ell_a(\underset{\sim}{v}, \widehat{\psi})(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in \widehat{\mathrm{X}}. \tag{94b}$$

The next two identities, and the resulting inequality (97), represent the key ingredients of our argument, highlighting the significance of using the Kullback–Leibler relative entropy in the testing procedure for the Fokker–Planck equation.

Choosing $\underset{\sim}{w} \equiv \underset{\sim}{v}$ in (94a) yields, similarly to (89), that

$$\frac{1}{2} \int_\Omega \left[|\underset{\sim}{v}|^2 + |\underset{\sim}{v} - \underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}|^2 - |\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}|^2 \right] \mathrm{d}\underset{\sim}{x} + \Delta t\, \nu \int_\Omega |\underset{\approx}{\nabla}_x \underset{\sim}{v}|^2 \,\mathrm{d}\underset{\sim}{x}$$
$$= \Delta t \left[\langle \underset{\sim}{f}^n, \underset{\sim}{v} \rangle_V - k_B\, \mathcal{T} \int_\Omega \underset{\approx}{C}(M\, \widehat{\psi}) : \underset{\approx}{\nabla}_x \underset{\sim}{v} \,\mathrm{d}\underset{\sim}{x} \right]. \tag{95}$$

Choosing $\widehat{\varphi} = [\mathcal{F}^L_\delta]'(\widehat{\psi})$ in (94b) and noting (65a), (67), (4), (15a) and that $\underset{\sim}{v}$ is divergence-free yield

$$\int_{\Omega \times D} M \left[\mathcal{F}^L_\delta(\widehat{\psi}) - \mathcal{F}^L_\delta(\omega\, \widehat{\psi}^{n-1}_{\varepsilon,L,\delta}) \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}$$
$$+ \Delta t \int_{\Omega \times D} M \left[\varepsilon\, \underset{\sim}{\nabla}_x \widehat{\psi} \cdot \underset{\sim}{\nabla}_x ([\mathcal{F}^L_\delta]'(\widehat{\psi})) + \frac{1}{2\,\lambda} \underset{\sim}{\nabla}_q \widehat{\psi} \cdot \underset{\sim}{\nabla}_q ([\mathcal{F}^L_\delta]'(\widehat{\psi})) \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}$$
$$\le \omega\, \Delta t \int_{\Omega \times D} M\, \underset{\approx}{\sigma}(\underset{\sim}{v})\, \underset{\sim}{q} \cdot \underset{\sim}{\nabla}_q \widehat{\psi} \,\mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}$$
$$= \omega\, \Delta t \int_\Omega \underset{\approx}{C}(M\, \widehat{\psi}) : \underset{\approx}{\sigma}(\underset{\sim}{v}) \,\mathrm{d}\underset{\sim}{x}. \tag{96}$$

Combining (95) and (96), and noting (53) and a Poincaré inequality yields that

$$\frac{\omega}{2}\int_{\Omega}\left[|\underset{\sim}{v}|^2+|\underset{\sim}{v}-\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}|^2\right]\mathrm{d}\underset{\sim}{x}+\omega\,\Delta t\,\nu\int_{\Omega}|\underset{\approx}{\nabla}_x\,\underset{\sim}{v}|^2\,\mathrm{d}\underset{\sim}{x}+k_B\,\mathcal{T}\int_{\Omega\times D}M\,\mathcal{F}^L_\delta(\widehat{\psi})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$+k_B\,\mathcal{T}\,\Delta t\int_{\Omega\times D}M\left[\varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\,([\mathcal{F}^L_\delta]'(\widehat{\psi}))+\frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_q\,([\mathcal{F}^L_\delta]'(\widehat{\psi}))\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$\leq\omega\,\Delta t\,\langle\underset{\sim}{f}^n,\underset{\sim}{v}\rangle_V+\frac{\omega}{2}\int_{\Omega}|\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{x}+k_B\,\mathcal{T}\int_{\Omega\times D}M\,\mathcal{F}^L_\delta(\omega\,\widehat{\psi}^{n-1}_{\varepsilon,L,\delta})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}.$$
$$\leq\frac{\omega}{2}\,\Delta t\,\nu\int_{\Omega}|\underset{\approx}{\nabla}_x\,\underset{\sim}{v}|^2\,\mathrm{d}\underset{\sim}{x}+\omega\,\Delta t\,C(\nu^{-1})\,\|\underset{\sim}{S}\,\underset{\sim}{f}^n\|^2_{\mathrm{H}^1(\Omega)}$$
$$+\frac{\omega}{2}\int_{\Omega}|\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{x}+k_B\,\mathcal{T}\int_{\Omega\times D}M\,\mathcal{F}^L_\delta(\omega\,\widehat{\psi}^{n-1}_{\varepsilon,L,\delta})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \tag{97}$$

It is easy to show that $\mathcal{F}^L_\delta(s)$ is nonnegative for all $s\in\mathbb{R}$, with $\mathcal{F}^L_\delta(1)=0$. Furthermore, for any $\omega\in(0,1]$,

$$\mathcal{F}^L_\delta(\omega\,s)\leq\mathcal{F}^L_\delta(s)\qquad\text{if } s<0 \text{ or } 1\leq\omega\,s,$$
$$\mathcal{F}^L_\delta(\omega\,s)\leq\mathcal{F}^L_\delta(0)\leq 1\qquad\text{if } 0\leq\omega\,s\leq 1.$$

Thus we deduce that

$$\mathcal{F}^L_\delta(\omega\,s)\leq\mathcal{F}^L_\delta(s)+1\qquad\forall s\in\mathbb{R},\quad\forall\omega\in(0,1]. \tag{98}$$

Hence, the bounds (97) and (98), on noting (66) and (65b), which implies that

$$[\mathcal{F}^L_\delta(s)]''\geq L^{-1}\qquad\text{for all } s\in\mathbb{R},$$

give rise to the desired bound (82) with C_* dependent only on L, k_B, $\mathcal{T}$ and $\widehat{\psi}^{n-1}_{\varepsilon,L,\delta}$. Therefore, (iii) holds, and so G has a fixed point. Thus we have proved existence of a solution to (71a,b). □

4.3. **Energy estimates for $(\mathrm{P}_{\varepsilon,L,\delta})$.** We refer the reader to [6] for details. Choosing $\underset{\sim}{w}\equiv\underset{\sim}{u}^n_{\varepsilon,L,\delta}$ in (74) and $\widehat{\varphi}\equiv[\mathcal{F}^L_\delta]'(\widehat{\psi}^n_{\varepsilon,L,\delta})$, and combining, then yields for $m=1\to N$, similarly to (97), but now with $\omega=1$,

$$\tfrac{1}{2}\left[\int_{\Omega}|\underset{\sim}{u}^m_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{x}+\sum_{n=1}^{m}\int_{\Omega}|\underset{\sim}{u}^n_{\varepsilon,L,\delta}-\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{x}\right]$$
$$+k_B\,\mathcal{T}\int_{\Omega\times D}M\,\mathcal{F}^L_\delta(\widehat{\psi}^m_{\varepsilon,L,\delta})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}+\sum_{n=1}^{m}\Delta t\left[\frac{\nu}{2}\int_{\Omega}|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}^n_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{x}\right.$$
$$+k_B\,\mathcal{T}\,\varepsilon\int_{\Omega\times D}M\,\underset{\sim}{\nabla}_x\,\widehat{\psi}^n_{\varepsilon,L,\delta}\cdot\underset{\sim}{\nabla}_x\,([\mathcal{F}^L_\delta]'(\widehat{\psi}^n_{\varepsilon,L,\delta}))\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$\left.+\frac{k_B\mathcal{T}}{2\,\lambda}\int_{\Omega\times D}M\,\underset{\sim}{\nabla}_q\,\widehat{\psi}^n_{\varepsilon,L,\delta}\cdot\underset{\sim}{\nabla}_q\,([\mathcal{F}^L_\delta]'(\widehat{\psi}^n_{\varepsilon,L,\delta}))\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right]\leq C; \tag{99}$$

where C is independent of δ, L and Δt, on assuming that L is chosen so that

$$0\leq\widehat{\psi}_0\leq L\qquad\text{a.e. in }\Omega\times D. \tag{100}$$

Choosing $\widehat{\varphi} = \widehat{\psi}^n_{\varepsilon,L,\delta}$ in (78) yields, for $m = 1 \to N$, using (99), that

$$\int_{\Omega\times D} M\,|\widehat{\psi}^m_{\varepsilon,L,\delta}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + \sum_{n=1}^{m}\int_{\Omega\times D} M\,|\widehat{\psi}^n_{\varepsilon,L,\delta} - \widehat{\psi}^{n-1}_{\varepsilon,L,\delta}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$+ \sum_{n=1}^{m}\Delta t \int_{\Omega\times D} M\left[2\,\varepsilon\left|\underset{\sim}{\nabla}_x\,\widehat{\psi}^n_{\varepsilon,L,\delta}\right|^2 + \frac{1}{2\,\lambda}\left|\underset{\sim}{\nabla}_q\,\widehat{\psi}^n_{\varepsilon,L,\delta}\right|^2\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \le C(L). \quad (101)$$

Choosing $\underset{\sim}{w} \equiv \underset{\sim}{S}\left(\frac{\underset{\sim}{u}^n_{\varepsilon,L,\delta}-\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}}{\Delta t}\right) \in \underset{\sim}{\mathrm{V}}$ in (74) yields, using (99) and (101), that

$$\sum_{n=1}^{N}\Delta t\left(\int_\Omega\left[\left|\underset{\approx}{\nabla}_x\left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}^n_{\varepsilon,L,\delta}-\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}}{\Delta t}\right)\right]\right|^2 + \left|\underset{\sim}{S}\left(\frac{\underset{\sim}{u}^n_{\varepsilon,L,\delta}-\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}}{\Delta t}\right)\right|^2\right]\mathrm{d}\underset{\sim}{x}\right)^{\frac{2}{d}}$$
$$\le C(L,T). \quad (102)$$

Choosing $\widehat{\varphi} \equiv \mathcal{G}\left(\frac{\widehat{\psi}^n_{\varepsilon,L,\delta}-\widehat{\psi}^{n-1}_{\varepsilon,L,\delta}}{\Delta t}\right) \in \widehat{\mathrm{X}}$ in (78) yields, using (99) and (101),

$$\sum_{n=1}^{N}\Delta t\left\|\mathcal{G}\left(\frac{\widehat{\psi}^n_{\varepsilon,L,\delta}-\widehat{\psi}^{n-1}_{\varepsilon,L,\delta}}{\Delta t}\right)\right\|^{\frac{4}{d}}_{\widehat{\mathrm{X}}} \le C(L,T)\,. \quad (103)$$

In line with (72) and (73), we now interpolate the time-semidiscrete sequences $\{\underset{\sim}{u}^n_{\varepsilon,L,\delta}\}_{n=0}^N$, $\{\widehat{\psi}^n_{\varepsilon,L,\delta}\}_{n=0}^N$ with respect to t in order to imbed them into the space-time function spaces in which the analytical solution is sought, and then we pass to the limit $\Delta t \to 0_+$. Analogously to the notation in Section 3, we introduce

$$\underset{\sim}{u}^{\Delta t}_{\varepsilon,L,\delta}(\cdot,t) := \frac{t-t^{n-1}}{\Delta t}\,\underset{\sim}{u}^n_{\varepsilon,L,\delta}(\cdot) + \frac{t^n-t}{\Delta t}\,\underset{\sim}{u}^{n-1}_{\varepsilon,L,\delta}(\cdot), \qquad t\in[t^{n-1},t^n], \quad n\ge 1, \quad (104a)$$

and

$$\underset{\sim}{u}^{\Delta t,+}_{\varepsilon,L,\delta}(\cdot,t) := \underset{\sim}{u}^n(\cdot), \quad t\in(t^{n-1},t^n], \quad \underset{\sim}{u}^{\Delta t,-}_{\varepsilon,L,\delta}(\cdot,t) := \underset{\sim}{u}^{n-1}(\cdot), \quad t\in[t^{n-1},t^n), \quad n\ge 1. \quad (104b)$$

We note for future reference that

$$\underset{\sim}{u}^{\Delta t}_{\varepsilon,L,\delta} - \underset{\sim}{u}^{\Delta t,\pm}_{\varepsilon,L,\delta} = (t-t^{n,\pm})\,\frac{\partial \underset{\sim}{u}^{\Delta t}_{\varepsilon,L,\delta}}{\partial t}, \qquad t\in(t^{n-1},t^n), \quad n\ge 1, \quad (105)$$

where $t^{n,+} := t^n$ and $t^{n,-} := t^{n-1}$. Using the above notation, and introducing analogous notation for $\{\widehat{\psi}^n_{\varepsilon,L,\delta}, \underset{\sim}{f}^n\}_{n=0}^N$, (74) summed for $n = 1 \to N$ can be restated as

$$\int_0^T\left\langle\frac{\partial \underset{\sim}{u}^{\Delta t}_{\varepsilon,L,\delta}}{\partial t}, \underset{\sim}{w}\right\rangle_V \mathrm{d}t$$
$$+ \int_0^T\int_\Omega\left[\left[(\underset{\sim}{u}^{\Delta t,-}_{\varepsilon,L,\delta}\cdot\underset{\sim}{\nabla}_x)\underset{\sim}{u}^{\Delta t,+}_{\varepsilon,L,\delta}\right]\cdot\underset{\sim}{w} + \nu\,\underset{\approx}{\nabla}_x\,\underset{\sim}{u}^{\Delta t,+}_{\varepsilon,L,\delta} : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\right]\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$= \int_0^T\left[\langle \underset{\sim}{f}^{\Delta t,+}, \underset{\sim}{w}\rangle_V - k_B\,\mathcal{T}\int_\Omega \underset{\approx}{C}(M\,\widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta}) : \underset{\approx}{\nabla}_x\,\underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\right]\mathrm{d}t \quad \forall \underset{\sim}{w}\in \mathrm{L}^{\frac{4}{4-d}}(0,T;\underset{\sim}{\mathrm{V}}). \quad (106)$$

Similarly, (78) summed for $n = 1 \to N$ can be restated as

$$\int_0^T \left\langle M \frac{\partial \widehat{\psi}^{\Delta t}_{\varepsilon,L,\delta}}{\partial t}, \widehat{\varphi} \right\rangle_{\widehat{X}} \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega \times D} M \left[\frac{1}{2\lambda} \underset{\sim}{\nabla}_q \widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta} - [\underset{\approx}{\sigma}(\underset{\sim}{u}^{\Delta t,+}_{\varepsilon,L,\delta})\, \underset{\sim}{q}]\, \beta^L_\delta(\widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta}) \right] \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega \times D} M \left[\varepsilon \underset{\sim}{\nabla}_x \widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta} - \underset{\sim}{u}^{\Delta t,-}_{\varepsilon,L,\delta} \widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta} \right] \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t = 0$$
$$\forall \widehat{\varphi} \in \mathrm{L}^{\frac{4}{4-d}}(0,T;\widehat{\mathrm{X}}). \tag{107}$$

We have from (99) and (104a,b), on noting (65b), that

$$\sup_{t\in(0,T)} \left[\int_\Omega |\underset{\sim}{u}^{\Delta t(,\pm)}_{\varepsilon,L,\delta}|^2 \,\mathrm{d}\underset{\sim}{x} \right] + \int_0^T \int_\Omega \frac{|\underset{\sim}{u}^{\Delta t,+}_{\varepsilon,L,\delta} - \underset{\sim}{u}^{\Delta t,-}_{\varepsilon,L,\delta}|^2}{\Delta t} \,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+ \nu \int_0^T \int_\Omega |\underset{\approx}{\nabla}_x \underset{\sim}{u}^{\Delta t(,\pm)}_{\varepsilon,L,\delta}|^2 \,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \leq C(T). \tag{108}$$

In the above, the notation $\underset{\sim}{u}^{\Delta t(,\pm)}_{\varepsilon,L,\delta}$ means $\underset{\sim}{u}^{\Delta t}_{\varepsilon,L,\delta}$ with or without the superscripts $\pm$. Similarly, we have from (101), (99), (66), (62), (102), (103) and (104a,b) that

$$\sup_{t\in(0,T)} \left[\int_{\Omega\times D} M\, |\widehat{\psi}^{\Delta t(,\pm)}_{\varepsilon,L,\delta}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right] + \frac{1}{\delta} \sup_{t\in(0,T)} \left[\int_{\Omega\times D} M\, [\widehat{\psi}^{\Delta t(,\pm)}_{\varepsilon,L,\delta}]_-^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right]$$
$$+ \frac{1}{\lambda} \int_0^T \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_q \widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta} \right|^2 \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t + \varepsilon \int_0^T \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_x \widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta} \right|^2 \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+ \sup_{t\in(0,T)} \left[\int_\Omega |\underset{\approx}{C}(M\, \widehat{\psi}^{\Delta t(,\pm)}_{\varepsilon,L,\delta})|^2 \,\mathrm{d}\underset{\sim}{x} \right] + \int_0^T \int_{\Omega\times D} M\, \frac{|\widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta} - \widehat{\psi}^{\Delta t,-}_{\varepsilon,L,\delta}|^2}{\Delta t} \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+ \int_0^T \int_{\Omega\times D} M \left[\left\| \underset{\sim}{S}\, \frac{\partial \underset{\sim}{u}^{\Delta t}_{\varepsilon,L,\delta}}{\partial t} \right\|^{\frac{4}{d}}_{\mathrm{H}^1(\Omega)} + \left\| \mathcal{G}\, \frac{\partial \widehat{\psi}^{\Delta t}_{\varepsilon,L,\delta}}{\partial t} \right\|^{\frac{4}{d}}_{\widehat{\mathrm{X}}} \right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \leq C(L,T)\,. \tag{109}$$

We are now in a position to prove the following convergence result.

Lemma 4.2. *There exists a subsequence of* $\{\underset{\sim}{u}^{\Delta t}_{\varepsilon,L,\delta}, \widehat{\psi}^{\Delta t}_{\varepsilon,L,\delta}\}_{\Delta t>0}$*, and functions* $\underset{\sim}{u}_{\varepsilon,L,\delta} \in \mathrm{L}^\infty(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega)) \cap \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{V}}) \cap \mathrm{W}^{1,\frac{4}{d}}(0,T;\underset{\sim}{\mathrm{V}}')$ *and* $\widehat{\psi}_{\varepsilon,L,\delta} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}}) \cap \mathrm{W}^{1,\frac{4}{d}}(0,T;\widehat{\mathrm{X}}')$ *such that, as* $\Delta t \to 0_+$,

$$\underset{\sim}{u}^{\Delta t(,\pm)}_{\varepsilon,L,\delta} \to \underset{\sim}{u}_{\varepsilon,L,\delta} \qquad \textit{weak* in } \mathrm{L}^\infty(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega)), \tag{110a}$$

$$\underset{\sim}{u}^{\Delta t(,\pm)}_{\varepsilon,L,\delta} \to \underset{\sim}{u}_{\varepsilon,L,\delta} \qquad \textit{weakly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{V}}), \tag{110b}$$

$$\underset{\approx}{S}\, \frac{\partial \underset{\sim}{u}^{\Delta t}_{\varepsilon,L,\delta}}{\partial t} \to \underset{\approx}{S}\, \frac{\partial \underset{\sim}{u}_{\varepsilon,L,\delta}}{\partial t} \qquad \textit{weakly in } \mathrm{L}^{\frac{4}{d}}(0,T;\underset{\sim}{\mathrm{V}}), \tag{110c}$$

$$\underset{\sim}{u}^{\Delta t(,\pm)}_{\varepsilon,L,\delta} \to \underset{\sim}{u}_{\varepsilon,L,\delta} \qquad \textit{strongly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{L}}^r(\Omega)), \tag{110d}$$

where $r \in [1,\infty)$ *if* $d = 2$ *and* $r \in [1,6)$ *if* $d = 3$*; and*

$$M^{\frac{1}{2}}\,\widehat{\psi}^{\Delta t(,\pm)}_{\varepsilon,L,\delta} \to M^{\frac{1}{2}}\,\widehat{\psi}_{\varepsilon,L,\delta} \qquad \textit{weak* in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega\times D)), \tag{111a}$$

$$M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta} \to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\varepsilon,L,\delta} \qquad \textit{weakly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega\times D)), \tag{111b}$$

$$M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x\,\widehat{\psi}^{\Delta t,+}_{\varepsilon,L,\delta} \to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\varepsilon,L,\delta} \qquad \textit{weakly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega\times D)), \tag{111c}$$

$$\mathcal{G}\,\frac{\partial\widehat{\psi}^{\Delta t}_{\varepsilon,L,\delta}}{\partial t} \to \mathcal{G}\,\frac{\partial\widehat{\psi}_{\varepsilon,L,\delta}}{\partial t} \qquad \textit{weakly in } \mathrm{L}^{\frac{4}{d}}(0,T;\widehat{\mathrm{X}}), \tag{111d}$$

$$M^{\frac{1}{2}}\,\widehat{\psi}^{\Delta t(,\pm)}_{\varepsilon,L,\delta} \to M^{\frac{1}{2}}\,\widehat{\psi}_{\varepsilon,L,\delta} \qquad \textit{strongly in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega\times D)), \tag{111e}$$

$$M^{\frac{1}{2}}\,\beta^L_\delta(\widehat{\psi}^{\Delta t(,\pm)}_{\varepsilon,L,\delta}) \to M^{\frac{1}{2}}\,\beta^L_\delta(\widehat{\psi}_{\varepsilon,L,\delta}) \qquad \textit{strongly in } \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega\times D)), \tag{111f}$$

$$\underset{\approx}{C}(M\,\widehat{\psi}^{\Delta t(,\pm)}_{\varepsilon,L,\delta}) \to \underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L,\delta}) \qquad \textit{strongly in } \mathrm{L}^2(0,T;\underset{\approx}{\mathrm{L}}^2(\Omega)). \tag{111g}$$

Furthermore, the pair $\{\underset{\sim}{u}_{\varepsilon,L,\delta}, \widehat{\psi}_{\varepsilon,L,\delta}\}$ *is a global weak solution to* ($\mathrm{P}_{\varepsilon,L,\delta}$)*,* (68a,b) *satisfying the bounds:*

$$\sup_{t\in(0,T)}\left[\int_\Omega |\underset{\sim}{u}_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{x}\right] + \nu\int_0^T\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \le C(T), \tag{112a}$$

$$\begin{aligned}
&\sup_{t\in(0,T)}\left[\int_{\Omega\times D} M\,|\widehat{\psi}_{\varepsilon,L,\delta}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \frac{1}{\delta}\sup_{t\in(0,T)}\left[\int_{\Omega\times D} M\,[\widehat{\psi}_{\varepsilon,L,\delta}]_-^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right]\\
&\quad + \int_0^T\int_{\Omega\times D} M\left[\frac{1}{\lambda}\left|\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\varepsilon,L,\delta}\right|^2 + \varepsilon\left|\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\varepsilon,L,\delta}\right|^2\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\\
&\quad + \sup_{t\in(0,T)}\left[\int_\Omega |\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L,\delta})|^2\,\mathrm{d}\underset{\sim}{x}\right]\\
&\quad + \int_0^T\left[\left\|\underset{\sim}{S}\,\frac{\partial\underset{\sim}{u}_{\varepsilon,L,\delta}}{\partial t}\right\|^{\frac{4}{d}}_{\mathrm{H}^1(\Omega)} + \left\|\mathcal{G}\,\frac{\partial\widehat{\psi}_{\varepsilon,L,\delta}}{\partial t}\right\|^{\frac{4}{d}}_{\widehat{\mathrm{X}}}\right]\mathrm{d}t \le C(L,T)\,.
\end{aligned} \tag{112b}$$

Proof. We refer the reader to [6]. □

Remark 2. Since the test functions in $\underset{\sim}{\mathrm{V}}$ are divergence-free, the pressure has been eliminated in (68a,b); it can be recovered in a very weak sense following the same procedure as for the incompressible Navier–Stokes equations discussed on p. 208 in Temam [45]; i.e., one obtains that $t \mapsto \int_0^t p_{\varepsilon,L,\delta}(\cdot,s)\,\mathrm{d}s \in \mathrm{C}([0,T];\mathrm{L}^2(\Omega))$. ⋄

4.4. **Existence for ($\mathbf{P}_{\varepsilon,L}$).** As the bounds (112a,b) are independent of the parameter δ, it follows immediately, similarly to (110a–d), (111a–g), and (112a,b), that the following lemma holds.

Lemma 4.3. *There exists a subsequence of* $\{\underset{\sim}{u}_{\varepsilon,L,\delta}, \widehat{\psi}_{\varepsilon,L,\delta}\}_{\delta>0}$*, and functions*

$$\underset{\sim}{u}_{\varepsilon,L} \in \mathrm{L}^\infty(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega)) \cap \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{V}}) \cap \mathrm{W}^{1,\frac{4}{d}}(0,T;\underset{\sim}{\mathrm{V}}')$$

and

$$\widehat{\psi}_{\varepsilon,L} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}}) \cap \mathrm{W}^{1,\frac{4}{d}}(0,T;\widehat{\mathrm{X}}'),$$

with $\widehat{\psi}_{\varepsilon,L} \geq 0$ *a.e. in* $\Omega \times D \times (0,T)$*, such that, as* $\delta \to 0_+$*,*

$$\underset{\sim}{u}_{\varepsilon,L,\delta} \to \underset{\sim}{u}_{\varepsilon,L} \qquad \textit{weak* in } \mathrm{L}^\infty(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega)), \tag{113a}$$

$$\underset{\sim}{u}_{\varepsilon,L,\delta} \to \underset{\sim}{u}_{\varepsilon,L} \qquad \textit{weakly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{V}}), \tag{113b}$$

$$\underset{\sim}{S}\,\frac{\partial \underset{\sim}{u}_{\varepsilon,L,\delta}}{\partial t} \to \underset{\sim}{S}\,\frac{\partial \underset{\sim}{u}_{\varepsilon,L}}{\partial t} \qquad \textit{weakly in } \mathrm{L}^{\frac{4}{d}}(0,T;\underset{\sim}{\mathrm{V}}), \tag{113c}$$

$$\underset{\sim}{u}_{\varepsilon,L,\delta} \to \underset{\sim}{u}_{\varepsilon,L} \qquad \textit{strongly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{L}}^r(\Omega)), \tag{113d}$$

where $r \in [1,\infty)$ *if* $d = 2$ *and* $r \in [1,6)$ *if* $d = 3$*; and*

$$M^{\frac{1}{2}}\,\widehat{\psi}_{\varepsilon,L,\delta} \to M^{\frac{1}{2}}\,\widehat{\psi}_{\varepsilon,L} \qquad \textit{weak* in } \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega\times D)), \tag{114a}$$

$$M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\varepsilon,L,\delta} \to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\varepsilon,L} \qquad \textit{weakly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega\times D)), \tag{114b}$$

$$M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\varepsilon,L,\delta} \to M^{\frac{1}{2}}\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\varepsilon,L} \qquad \textit{weakly in } \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega\times D)), \tag{114c}$$

$$\mathcal{G}\,\frac{\partial \widehat{\psi}_{\varepsilon,L,\delta}}{\partial t} \to \mathcal{G}\,\frac{\partial \widehat{\psi}_{\varepsilon,L}}{\partial t} \qquad \textit{weakly in } \mathrm{L}^{\frac{4}{d}}(0,T;\widehat{\mathrm{X}}), \tag{114d}$$

$$M^{\frac{1}{2}}\,\widehat{\psi}_{\varepsilon,L,\delta} \to M^{\frac{1}{2}}\,\widehat{\psi}_{\varepsilon,L} \qquad \textit{strongly in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega\times D)), \tag{114e}$$

$$M^{\frac{1}{2}}\,\beta^L_\delta(\widehat{\psi}_{\varepsilon,L,\delta}) \to M^{\frac{1}{2}}\,\beta^L(\widehat{\psi}_{\varepsilon,L}) \qquad \textit{strongly in } \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega\times D)), \tag{114f}$$

$$\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L,\delta}) \to \underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L}) \qquad \textit{strongly in } \mathrm{L}^2(0,T;\underset{\approx}{\mathrm{L}}^2(\Omega)). \tag{114g}$$

In addition, we have that

$$\sup_{t\in(0,T)}\left[\int_\Omega |\underset{\sim}{u}_{\varepsilon,L}|^2\,\mathrm{d}\underset{\sim}{x}\right] + \nu\int_0^T\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\varepsilon,L}|^2\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \leq C(T), \tag{115a}$$

$$\begin{aligned}
&\sup_{t\in(0,T)}\left[\int_{\Omega\times D} M^{\frac{1}{2}}\,|\widehat{\psi}_{\varepsilon,L}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \sup_{t\in(0,T)}\left[\int_\Omega |\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L})|^2\,\mathrm{d}\underset{\sim}{x}\right] \\
&\quad + \int_0^T\int_{\Omega\times D} M\left[\frac{1}{\lambda}\left|\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\varepsilon,L}\right|^2 + \varepsilon\left|\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\varepsilon,L}\right|^2\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \\
&\quad + \int_0^T\left[\left\|\underset{\sim}{S}\,\frac{\partial \underset{\sim}{u}_{\varepsilon,L}}{\partial t}\right\|^{\frac{4}{d}}_{\mathrm{H}^1(\Omega)} + \left\|\mathcal{G}\,\frac{\partial \widehat{\psi}_{\varepsilon,L}}{\partial t}\right\|^{\frac{4}{d}}_{\widehat{\mathrm{X}}}\right]\mathrm{d}t \leq C(L,T).
\end{aligned} \tag{115b}$$

The nonnegativity of $\widehat{\psi}_{\varepsilon,L}$ in the above lemma follows from the second bound in (112b). Thus we can now pass to limit $\delta \to 0_+$ in $(\mathrm{P}_{\varepsilon,L,\delta})$ to obtain global existence of a weak solution to the following problem for given $\varepsilon \in (0,1]$ and $L > 1$:

$(\mathrm{P}_{\varepsilon,L})$ Find functions $\underset{\sim}{u}_{\varepsilon,L} \in \mathrm{L}^\infty(0,T;\underset{\sim}{\mathrm{L}}^2(\Omega)) \cap \mathrm{L}^2(0,T;\underset{\sim}{\mathrm{V}}) \cap \mathrm{W}^{1,\frac{4}{d}}(0,T;\underset{\sim}{\mathrm{V}}')$ and $\widehat{\psi}_{\varepsilon,L} \in \mathrm{L}^\infty(0,T;\mathrm{L}^2_M(\Omega\times D)) \cap \mathrm{L}^2(0,T;\widehat{\mathrm{X}}) \cap \mathrm{W}^{1,\frac{4}{d}}(0,T;\widehat{\mathrm{X}}')$, with $\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L}) \in$

$\mathrm{L}^\infty(0,T;\underset{\approx}{\mathrm{L}}^2(\Omega))$, such that $\underset{\sim}{u}_{\varepsilon,L}(\cdot,0)=\underset{\sim}{u}_0(\cdot)$, $\widehat{\psi}_{\varepsilon,L}(\cdot,0)=\widehat{\psi}_0(\cdot)$ and

$$\int_0^T\left\langle\frac{\partial\underset{\sim}{u}_{\varepsilon,L}}{\partial t},\underset{\sim}{w}\right\rangle_V \mathrm{d}t+\int_0^T\int_\Omega\left[\left[(\underset{\sim}{u}_{\varepsilon,L}\cdot\underset{\sim}{\nabla}_x)\underset{\sim}{u}_{\varepsilon,L}\right]\cdot\underset{\sim}{w}+\nu\,\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\varepsilon,L}:\underset{\approx}{\nabla}_x\,\underset{\sim}{w}\right]\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$=\int_0^T\langle\underset{\sim}{f},\underset{\sim}{w}\rangle_V\,\mathrm{d}t-k_B\,\mathcal{T}\int_0^T\int_\Omega\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon,L}):\underset{\approx}{\nabla}_x\,\underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t\qquad\forall\underset{\sim}{w}\in\mathrm{L}^{\frac{4}{4-d}}(0,T;\underset{\sim}{\mathrm{V}}),$$

$$\int_0^T\left\langle M\,\frac{\partial\widehat{\psi}_{\varepsilon,L}}{\partial t},\widehat{\varphi}\right\rangle_{\widehat{X}}\mathrm{d}t+\int_0^T\int_{\Omega\times D}M\left[\varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\varepsilon,L}-\underset{\sim}{u}_{\varepsilon,L}\,\widehat{\psi}_{\varepsilon,L}\right]\cdot\underset{\sim}{\nabla}_x\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+\int_0^T\int_{\Omega\times D}M\left[\frac{1}{2\,\lambda}\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\varepsilon,L}-[\,\underset{\approx}{\sigma}(\underset{\sim}{u}_{\varepsilon,L})\,\underset{\sim}{q}\,]\,\beta^L(\widehat{\psi}_{\varepsilon,L})\right]\cdot\underset{\sim}{\nabla}_q\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t=0$$
$$\forall\widehat{\varphi}\in\mathrm{L}^{\frac{4}{4-d}}(0,T;\widehat{\mathrm{X}}).$$

Remark 3. Although we have introduced x-diffusion and a cut-off above to $\widehat{\psi}=\psi/M$ in the drag term in the Fokker–Planck equation through the parameters $\varepsilon\in(0,1]$ and $L>1$ in the model $(\mathrm{P}_{\varepsilon,L})$ compared to the standard polymer model, (P), we wish to stress that the bounds on $\underset{\sim}{u}_{\varepsilon,L}$, the variable of real physical interest, in (115a) are *independent* of these parameters ε and L. In fact, in the case of a corotational model, the right-hand side in the estimate (101) becomes independent of L, as one can exploit additional cancellations due to the skew-symmetry of $\underset{\approx}{\sigma}_{\mathrm{corot}}(\underset{\sim}{v})$. Hence, (102) is then also independent of L. This raises the question whether in a corotational model one can pass to the limit $L\to\infty$ to recover the Fokker–Planck equation, *without* cut-off. The answer is positive; the necessary modifications to our arguments to show this are indicated below.

In our discussion above, because of the cut-off, we also control the time derivative of $\widehat{\psi}_{\varepsilon,L,\delta}$; without cut-off this does not appear to be possible. Also, one should avoid (103) as the right-hand side of this inequality remains L-dependent regardless of whether or not the drag term is corotational. It is possible to get around these technical difficulties by proceeding as in Barrett and Süli [5]: i.e., (i) the time derivative has to be transferred from $\widehat{\psi}_{\varepsilon,L,\delta}$ to the (time-dependent) test function in the weak formulation of the Fokker–Planck equation, similarly as in Section 3; (ii) as we will no longer have strong convergence of a subsequence of $\{\widehat{\psi}_{\varepsilon,L,\delta}\}_{\delta>0}$ to $\widehat{\psi}_{\varepsilon,L}$ as $\delta\to0_+$, and of $\{\widehat{\psi}_{\varepsilon,L}\}_{L>1}$ to $\widehat{\psi}_\varepsilon$ as $L\to\infty$, the drag term has to be rewritten using that for all $\underset{\sim}{v}\in\underset{\sim}{\mathrm{H}}^1_0(\Omega)$ and $\underset{\sim}{\widehat{\varphi}}\in\underset{\sim}{\mathrm{H}}^1(\Omega;\mathrm{L}^2_M(D))$

$$\int_{\Omega\times D}M\,[\underset{\approx}{\sigma}_{\mathrm{corot}}(\underset{\sim}{v})\,\underset{\sim}{q}]\cdot\underset{\sim}{\widehat{\varphi}}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}=\tfrac{1}{2}\int_{\Omega\times D}M\left[(\underset{\sim}{v}\cdot\underset{\sim}{q})\,(\underset{\sim}{\nabla}_x\cdot\underset{\sim}{\widehat{\varphi}})-[(\underset{\approx}{\nabla}_x\,\underset{\sim}{\widehat{\varphi}})\,\underset{\sim}{q}]\cdot\underset{\sim}{v}\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}.$$

One can then pass to the simultaneous limit $\delta\to0_+$ and $L\to\infty$ in a very similar manner as we did in the final section of Barrett and Süli [5]. ⋄

Remark 4. For any $s\in(0,T)$ and Δt sufficiently small such that $0<\Delta t<s$, we can choose $\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q},t)=\{[s-t]_+-[s-\Delta t-t]_+\}/\Delta t$ in $(\mathrm{P}_{\varepsilon,L})$; hence,

$$\frac{1}{\Delta t}\int_{s-\Delta t}^s\int_{\Omega\times D}M\,\widehat{\psi}_{\varepsilon,L}(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t=\int_{\Omega\times D}M\,\widehat{\psi}_0(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}.$$

Letting $\Delta t\to0_+$ yields $\int_{\Omega\times D}M\,\widehat{\psi}_{\varepsilon,L}(\underset{\sim}{x},\underset{\sim}{q},s)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}=\int_{\Omega\times D}M\,\widehat{\psi}_0(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$ for all $s\in(0,T)$. An identical statement can be made about $\widehat{\psi}_{\varepsilon,L,\delta}$ in $(\mathrm{P}_{\varepsilon,L,\delta})$. ⋄

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REFERENCES

[1] L. Ambrosio, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math. **158** (2004), 227–260.

[2] F. Antoci, *Some necessary and some sufficient conditions for the compactness of the embedding of weighted Sobolev spaces*, Ricerche Mat., **52** (2003), 55–71.

[3] A. Arnold, P. Markowich, G. Toscani and A. Unterreiter, *On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations*, Comm. Partial Differential Equations, **26** (2001), 43–100.

[4] J. W. Barrett, Ch. Schwab and E. Süli, *Existence of global weak solutions for some polymeric flow models*, Math. Models Methods Appl. Sci., **15** (2005), 939–983.

[5] J. W. Barrett and E. Süli, *Existence of global weak solutions to some regularized kinetic models for dilute polymers*, Multiscale Model. Simul., **6** (2007), 506–546 (electronic).

[6] J. W. Barrett and E. Süli, *Existence of global weak solutions to dumbbell models for dilute polymers with microscopic cut-off*, Math. Models Methods Appl. Sci., **18** (2008), 935–971.

[7] J. W. Barrett and E. Süli, *Numerical approximation of corotational dumbbell models for dilute polymers*, IMA J. Numer. Anal., **29** (2009), 937–959.

[8] J. W. Barrett and E. Süli, *Finite element approximation of kinetic dilute polymer models with microscopic cut-off*, M2AN Math. Model. Numer. Anal., submitted for publication.

[9] A. V. Bhave, R. C. Armstrong and R. A. Brown, *Kinetic theory and rheology of dilute, nonhomogeneous polymer solutions*, J. Chem. Phys., **95** (1991), 2988–3000.

[10] P. Constantin, *Nonlinear Fokker–Planck Navier–Stokes systems*, Commun. Math. Sci., **3** (2005), 531–544.

[11] P. Constantin, *Partial differential equation problems from simple to complex fluids*, Nonlinearity, **21** (2008), T239–T244.

[12] P. Constantin, C. Fefferman, E. S. Titi and A. Zarnescu, *Regularity of coupled two-dimensional nonlinear Fokker–Planck and Navier–Stokes systems*, Comm. Math. Phys., **270** (2007), 789–811, 2007.

[13] P. Degond and H. Liu, *Kinetic models for polymers with inertial effects*, Networks and Heterogeneous Media, **4** (2009), 625–647.

[14] L. Desvillettes and C. Villani, *On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation*, Invent. Math., **159** (2005), 245–316.

[15] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., **98** (1989), 511–547.

[16] Q. Du, C. Liu and P. Yu, *FENE dumbbell model and its several linear and nonlinear closure approximations*, Multiscale Model. Simul., **4** (2005), 709–731.

[17] W. E, T. Li and P. Zhang, *Well-posedness for the dumbbell model of polymeric fluids*, Comm. Math. Phys., **248** (2004), 409–427.

[18] A. E. El Kareh and L. G. Leal, *Existence of solutions for all Deborah numbers for a non-newtonian model modified to include diffusion*, J. Non-Newtonian Fluid Mech., **33** (1989), 257–287.

[19] C. Foias, D. D. Holm and E. S. Titi, *The Navier–Stokes-alpha model of fluid turbulence. Advances in nonlinear mathematics and science*, Phys. D, **152/153** (2001), 505–519.

[20] Y. Hyon, Q. Du and C. Liu, *An enhanced macroscopic closure approximation to the micro-macro FENE model for polymeric materials*, Multiscale Model. Simul., **7** (2008), 978–1002.

[21] B. Jourdain, T. Lelièvre and C. Le Bris, *Numerical analysis of micro-macro simulations of polymeric fluid flows: A simple case*, Math. Models Methods Appl. Sci., **12** (2002), 1205–1243.

[22] B. Jourdain, T. Lelièvre and C. Le Bris, *Existence of solution for a micro-macro model of polymeric fluid: the FENE model*, J. Funct. Anal., **209** (2004), 162–193.

[23] B. Jourdain, T. Lelievre, C. Le Bris, and F. Otto, *Long-time asymptotics of a multiscle model for polymeric fluid flows*, Arch. Rat. Mech. Anal., **181** (2006), 97–148.

[24] D. Knezevic and E. Süli, *Spectral Galerkin approximation of Fokker–Planck equations with unbounded drift*, M2AN: Mathematical Modeling and Numerical Analysis, **43** (2009), 445–485.

[25] D. Knezevic and E. Süli, *A heterogeneous alternating-direction method for a micro-macro dilute polymeric fluid model*, M2AN: Mathematical Modeling and Numerical Analysis, accepted for publication.

[26] A. N. Kolmogorov, *Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung*, Math. Ann., **104** (1931).

[27] A. Kufner, "Weighted Sobolev Spaces", Teubner-Texte zur Mathematik. Teubner, 1980.

[28] T. Li, H. Zhang and P. Zhang, *Local existence for the dumbbell model of polymeric fuids*, Comm. Partial Differential Equations, **29** (2004), 903–923.

[29] F.-H. Lin, C. Liu and P. Zhang, *On a micro-macro model for polymeric fluids near equilibrium*, Comm. Pure Appl. Math., **60** (2007), 838–866.

[30] F. H. Lin and P. Zhang, *The FENE dumbbell model near equilibrium*, Acta Math. Sin. (Engl. Ser.), **24** (2008), 529–538.

[31] F.-H. Lin, P. Zhang and Z. Zhang, *On the global existence of smooth solution to the 2-D FENE dumbbell model*, Comm. Math. Phys., **277** (2008), 531–553.

[32] P.-L. Lions and N. Masmoudi, *Global solutions for some Oldroyd models of non-Newtonian flows*, Chinese Ann. Math. Ser. B, **21**, (2000), 131–146.

[33] P.-L. Lions and N. Masmoudi, *Global existence of weak solutions to some micro-macro models*, C. R. Math. Acad. Sci. Paris, **345** (2007), 15–20.

[34] C. Liu and H. Liu, *Boundary conditions for the microscopic FENE models*, SIAM J. Appl. Math., **68** (2008), 1304–1315.

[35] A. Lozinski, C. Chauvière, J. Fang and R. G. Owens, *Fokker–Planck simulations of fast flows of melts and concentrated polymer solutions in complex geometries*, J. Rheology, **47** (2003), 535–561.

[36] A. Lozinski, R. G. Owens and J. Fang, *A Fokker–Planck-based numerical method for modelling non-homogeneous flows of dilute polymeric solutions*, J. Non-Newtonian Fluid Mech., **122** (2004), 273–286.

[37] N. Masmoudi, *Well-posedness for the FENE dumbbell model of polymeric flows*, Comm. Pure Appl. Math., **61** (2008), 1685–1714.

[38] H. C. Öttinger, "Stochastic Processes in Polymeric Fluids", Springer, 1996.

[39] F. Otto and A. E. Tzavaras, *Continuity of velocity gradients in suspensions of rod-like molecules*, Comm. Math. Phys., **277** (2008), 729–758.

[40] M. Renardy, *An existence theorem for model equations resulting from kinetic theories of polymer solutions*, SIAM J. Math. Anal., **22** (1991), 313–327.

[41] J. D. Schieber, *Generalized Brownian configuration field for Fokker–Planck equations including center-of-mass diffusion*, J. Non-Newtonian Fluid Mech., **135** (2006), 179–181.

[42] V. B. Shakhmurov, *Embedding and maximal regular differential operators in Sobolev–Lions spaces*, Acta Math. Sin. (Engl. Ser.), **22** (2006), 1493–1508.

[43] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4), **146** (1987), 65–96.

[44] R. Sureshkumar and A.N. Beris, *Effect of artificial stress diffusivity on the stability of numerical calculations and the flow dynamics of time-dependent viscoelastic flows*, J. Non-Newtonian Fluid Mech., **60** (1995), 53–80.

[45] R. Temam, "Navier–Stokes Equations: Theory and Numerical Analysis", North-Holland, Amsterdam, 3rd edition, 1984.

[46] H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators", second edition, Joh. Ambrosius Barth Publ., 1995.

[47] P. Yu, Q. Du and C. Liu, *From micro to macro dynamics via a new closure approximation to the FENE model of polymeric fluids*, Multiscale Model. Simul., **3** (2005), 895–917.

[48] H. Zhang and P. Zhang, *Local existence for the FENE-dumbbell model of polymeric fluids*, Arch. Ration. Mech. Anal., **181** (2006), 373–400.

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