Mathematical Models and Methods in Applied Sciences

EXISTENCE OF SOLUTIONS TO A REGULARIZED MODEL OF DYNAMIC FRACTURE

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Existence and convergence results are proved for a regularized model of dynamic brittle fracture based on the Ambrosio–Tortorelli approximation. We show that the sequence of solutions to the time-discrete elastodynamics, proposed by Bourdin, Larsen & Richardson as a semidiscrete numerical model for dynamic fracture, converges, as the time-step approaches zero, to a solution of the natural time-continuous elastodynamics model, and that this solution satisfies an energy balance. We emphasize that these models do not specify crack-paths a priori, but predict them, including such complicated behavior as kinking, crack branching, and so forth, in any spatial dimension.

Keywords: dynamic fracture mechanics; phase-field approximation; crack path; existence of solutions; convergence; energy balance.

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1. Introduction

The starting point for models predicting fracture is Griffith's criterion[14], originally formulated in the quasi-static setting. It supposes that, as a crack grows, the displacement field is instantly in a new equilibrium (new, since the displacement may be discontinuous across the crack increment). The resulting decrease in stored elastic energy can then be balanced with the work required to create the crack increment, postulated to be proportional to the newly created area. The constant of proportionality is usually labelled *fracture toughness*. In other words, the rate of elastic energy decrease per unit area, the (quasi-static) *energy release rate*, is proportional to the fracture toughness. Griffith's criterion stipulates that the crack grows only if the energy release rate equals the fracture toughness. The crack is *stable*, if the energy release rate does not exceed the fracture toughness, and it is labelled *unstable* if it exceeds the fracture toughness[17] .

Traditionally, these ideas could be formalized only for relatively simple crack topologies, often only for a pre-defined crack path. Only recently was the theory of brittle fracture freed from this restriction[1,12]. Ambrosio & Braides[1] propose minimizing the sum of stored elastic energy and surface energy of discontinuity sets, to obtain displacements that are stable in the sense of Griffith. That is, for displacements $u \in \mathrm{SBV}(\Omega)$, the space of *special functions of bounded variation*, with Ω representing the reference configuration of a body (u taking real values, modeling antiplane displacement), they consider energy functionals of the form

$$E(u) := \frac{\mu}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x + G_c \mathcal{H}^{N-1}(S(u)). \tag{1.1}$$

We usually refer to $\frac{\mu}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x$ as the *elastic energy* and to $G_c\mathcal{H}^{N-1}(S(u))$ as the *surface energy*. Here, and throughout, μ denotes the stiffness and G_c the fracture toughness, $S(u)$ denotes the discontinuity set of u, $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure, and the minimization is preformed subject to a Dirichlet condition. (For the time being, we ignore the problem of a crack forming along $\partial\Omega$, releasing u from the Dirichlet data there; we will address this issue in Section 2.1). The idea is that, if u is a minimizer of E, then adding any increment to its crack set $S(u)$ cannot reduce the elastic energy by more than the cost of the increment in surface energy. Therefore, the 'crack' $S(u)$ is stable in the sense of Griffith.

The first well-posed (by which we mean, throughout the paper, that existence can be shown) mathematical models of quasi-static fracture can be found in Dal Maso, Francfort & Toader[9], Francfort & Larsen[11], and Francfort & Marigo[12]. In these references, the Dirichlet data u_D is varying in time and an evolution u is sought such that, at each time t, $u(t)$ minimizes E subject to the Dirichlet boundary condition, and subject to an irreversibility constraint on the crack set. More precisely, it is required that

$$\frac{\mu}{2}\int_\Omega |\nabla u(t)|^2 \,\mathrm{d}x \le \frac{\mu}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + G_c\mathcal{H}^{N-1}\big(S(w) \setminus C(t)\big) \qquad \forall w \in \mathrm{SBV}(\Omega) \text{ s.t. } w|_{\partial\Omega} = u_D(t)|_{\partial\Omega}, \tag{1.2}$$

where $C(t)$ denotes the *crack set* at time t, which is essentially the union of discontinuity sets $S(u(\tau))$, $\tau \le t$. Additionally, an energy balance formula is stipulated so that a suitably defined energy functional, including the work done by the boundary condition, is constant in time.

The strategy for proving existence of solutions to this model, proposed in the paper of Francfort & Marigo[12], is based on a time discretization. At step $t_i^n = i/n$, the solution $u_n(t_i^n)$ is a minimizer of

$$w \mapsto \frac{\mu}{2} \int_\Omega |\nabla w|^2 \, dx + G_c \mathcal{H}^{N-1}\big(S(w) \setminus \cup_{j<i} S(u_n(t_j^n))\big),$$

subject to $w = u_D(t_i^n)$ on $\partial\Omega$, $i = 1, \dots, n$, $n \geq 1$. It was hoped that limits of these discrete trajectories, as $n \to \infty$, would satisfy, among other things, the unilateral minimality condition (1.2) and the correct energy balance.

Proving the unilateral minimality was not straightforward (see Dal Maso & Toader[9]), but a method was introduced and shown to work in the anti-plane case in Francfort & Larsen[11], and then generalized in Dal Maso, Francfort & Toader[8] to the case of nonlinear elasticity. We emphasize that the main achievement of Ambrosio & Braides[1], Dal Maso, Francfort & Toader[8], Dal Maso & Toader[9], Francfort & Larsen[11], and Francfort & Marigo[12] was to formulate and establish well-posedness of a model able to *predict crack paths*. In particular, crack kinking, crack branching, or indeed the far more complex three-dimensional situation do not require additional modeling, but are naturally included in the formulation. This observation is also true for the dynamic model, which we propose in the following.

The difficulties in formulating models for dynamic fracture consistent with Griffith's criterion are readily apparent; indeed, we know of no well-posed models prior to this work. The main issue seems to be to find a precise mathematical principle corresponding to Griffith's criterion, which replaces unilateral minimality in the quasi-static setting. In our view, a dynamic model of fracture should obey the following three principles:

- *Elastodynamics:* Away from the crack set, the governing principle is elastodynamics, for example, for anti-plane displacements,

$$\rho\ddot{u} - \mu\Delta u = f \qquad \text{in } \Omega \setminus C,$$

  with traction-free boundary conditions on either side of the crack, or

$$\rho\ddot{u} - \Delta(\mu u + k\dot{u}) = f \qquad \text{in } \Omega \setminus C,$$

  where the term $-k\Delta\dot{u}$ models elastic dissipation.
- *Energy Balance:* The evolution should satisfy an energy balance formula, akin to that found in the quasi-static setting, but now including kinetic energy.
- *Maximal Dissipation:* If the crack can propagate while balancing energy, then it should propagate.

The first principle requires no further comment. The principle of energy balance in dynamic fracture is known as *Mott's extension* of Griffith's energy concept[17] . Finally, the *maximal dissipation* principle follows a recent formulation of Larsen[16]. It further narrows down the set of admissible trajectories, which could still be very large if only energy balance is imposed (for instance, an elastodynamic solution for a stationary crack always conserves energy), and replaces unilateral minimality in the quasi-static fracture model (indeed, in the quasi-static setting, the maximal dissipation principle implies unilateral minimality).

In Bourdin, Larsen & Richardson[4], a discrete-time candidate for such a model is proposed, based on the Ambrosio–Tortorelli approximation,

$$E_\varepsilon(u,v) := \frac{\mu}{2}\int_\Omega (v^2+\eta_\varepsilon)|\nabla u|^2\,\mathrm{d}x + G_c\int_\Omega \big[(4\varepsilon)^{-1}(1-v)^2+\varepsilon|\nabla v|^2\big]\,\mathrm{d}x,$$

which Γ-converges, as $0<\eta_\varepsilon \ll \varepsilon \to 0$, to the Griffith energy E; see Ambrosio & Tortorelli[2]. In fact, the regularized elastic and surface energies converge independently to their sharp-interface versions; cf. (1.1). An analysis of this approximation in the quasi-static setting is provided by Giacomini[13]. The Ambrosio–Tortorelli approximation is particularly convenient for numerical implementation and was proposed in Bourdin, Francfort & Marigo[5] and Bourdin[3] for the simulation of the quasi-static model. The observation which allows for an extension to dynamic fracture is that, for this approximation, there can be an instant decrease in the elastic energy when v decreases (i.e., the crack grows), even if u is held fixed. Hence, we will consider a model in which u follows elastodynamics (with stiffness $a(t) := v^2(t)+\eta_\varepsilon$) and v behaves identically as in the quasi-static setting, i.e., at every time t, $v(t)$ minimizes $v \mapsto E_\varepsilon(u(t),v)$ subject to an appropriate irreversibility constraint.

Bourdin, Larsen & Richardson[4] formulate this idea as a *numerical model*: given $T_f>0$ and a positive integer N_f, at each discrete time $t_i = ih$, $i=1,\dots,N_f$, with $h=T_f/N_f$, $u(t_i)$ is computed using a time-discrete wave equation (cf. Section 3.1) with stiffness $(v_h(t_{i-1})^2+\eta_\varepsilon)$, followed by the computation of $v_h(t_i)$ achieved by minimizing $v\mapsto E_\varepsilon(u_h(t_i),v)$ subject to $v \le v_h(t_{i-1})$. This approach was motivated by Bourdin[3] and Bourdin, Francfort & Marigo[5] where an alternate minimization procedure in the u and v variables was used for the simulation of the quasi-static problem. Note that, for a given discrete wave equation and time step size h, this algorithm uniquely determines the discrete trajectory $(u_h(t),v_h(t))_{t\in[0,T_f]}$, (or, briefly, (u_h,v_h)) obtained by continuous piecewise affine interpolation of the sequence of values $(u_h(t_i),v_h(t_i))_{i=0}^{N_f}$. We also remark that several steps in our convergence proof in Section 3 were inspired by the convergence analysis of the alternate minimization algorithm in Burke, Ortner & Süli[6].

If this numerical model is reasonable, then the pairs (u_h,v_h) it produces should balance energy (up to numerical dissipation) and converge to the solution of a corresponding time-continuous model. In the present paper, we will prove that this is indeed the case: any accumulation point (u,v) of the family $\{(u_h,v_h) : h>0\}$ of discrete trajectories is a solution to the time-continuous crack propagation problem: u solves the continuous-time wave equation, v is minimal, and the trajectory (u,v) balances energy. We were unable to prove our third postulate (maximal dissipation), and therefore believe that the formulation of our model might be underconstrained. We will return to this point in the conclusion of the article.

We also note that, while there are other models for fracture based on crack regularization (see, e.g., Hakim & Karma[15]), they are typically based on phase-field models whose connection to the Griffith model is at best unclear. While we do not prove convergence of our model to a dynamic Griffith model, such a rigorous connection has been shown in the static and quasistatic settings by Giacomini[13]. We refer to section 4 of Bourdin, Larsen & Richardson[4] for a more complete discussion.

We conclude the introduction by noting that, in our analysis below, we add an elastic dissipation term, which helps in the analysis. Furthermore, we consider a more general, vector-valued case, instead of the anti-plane situation. Note also that, for simplicity of exposition, we take all physical constants (i.e., all constants except for ε and η_ε) to be 1.

2. Formulation of the Model

Suppose that Ω is a bounded open set in $\mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D, Γ_N are disjoint measurable sets and $\mathcal{H}^2(\Gamma_D) > 0$. We use the usual notation for Lebesgue and Sobolev spaces, omitting the domain Ω whenever it is obvious from the context what we mean. For example, we shall write H^1 instead of $\mathrm{H}^1(\Omega)$, and so forth. The space of displacements obeying the homogeneous Dirichlet boundary condition is denoted $\mathrm{H}^1_D(\Omega;\mathbb{R}^3) := \{u \in \mathrm{H}^1(\Omega;\mathbb{R}^3) : u|_{\Gamma_\mathrm{D}} = 0\}$ (or simply H^1_D). Spaces of trajectories are denoted, as usual, by $\mathrm{L}^p((0,T_f);X), \mathrm{W}^{k,p}((0,T_f);X)$, $\mathrm{C}^k([0,T_f];X)$, and so forth, where X is (a subset of) a Banach space. To simplify the notation, we shall usually write $\mathrm{L}^p(X), \mathrm{W}^{k,p}(X)$, $\mathrm{C}(X)$ instead. If, e.g., $u \in \mathrm{L}^p(\mathrm{H}^1)$, then we will usually write $u(t) := u(\cdot,t)$. Throughout, the symbol $\|\cdot\|$ denotes the L^2-norm on Ω.

We remark that the Arzelá–Ascoli Theorem for metric spaces (see Section IV.6.7 in Dunford & Schwartz[10]) implies that $\mathrm{H}^1(\mathrm{H}^1)$ is *compactly* embedded in $\mathrm{C}(\mathrm{L}^2)$. That is, if a sequence $(u_j)_{j=1}^\infty \subset \mathrm{H}^1(\mathrm{H}^1)$ is uniformly bounded in $\mathrm{H}^1(\mathrm{H}^1)$, then there exists a subsequence (not relabelled) and $u \in \mathrm{H}^1(\mathrm{H}^1)$ such that

$$u_j \to u \qquad \text{in } \mathrm{C}(\mathrm{L}^2). \tag{2.1}$$

Let $A \in \mathrm{L}^\infty(\Omega;\mathbb{R}^{3^4})$ be the elastic modulus tensor, with $A^{\alpha\beta}_{ij}(x) = A^{\beta\alpha}_{ji}(x)$ for a.e. $x \in \Omega$, satisfying the following ellipticity condition: there exists $c_0 > 0$ such that $A(x)\zeta : \zeta \geq c_0|\zeta|^2$ for all $\zeta \in \mathbb{R}^{3\times3}_{\mathrm{sym}} = \{\zeta \in \mathbb{R}^{3\times3} : \zeta = \zeta^\mathrm{T}\}$ and for a.e. $x \in \Omega$; equivalently,

$$\sum_{i,j=1}^{3}\sum_{\alpha,\beta=1}^{3} A^{\alpha,\beta}_{ij}(x)\zeta^i_\alpha\zeta^j_\beta \geq c_0 \sum_{i=1}^{3}\sum_{\alpha=1}^{3}|\zeta^i_\alpha|^2 \qquad \forall\zeta \in \mathbb{R}^{3\times3}_{\mathrm{sym}} \quad \text{for a.e } x \in \Omega.$$

For $\zeta \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$, and $x \in \Omega$, we define $|\zeta|^2_{A(x)} := A(x)\zeta : \zeta$. Further, for $u \in \mathrm{H}^1(\Omega;\mathbb{R}^3)$, let

$$e(u) := \tfrac{1}{2}(\nabla u + \nabla u^\mathrm{T}), \qquad \text{and} \quad \|e(u)\|^2_A := \textstyle\int_\Omega |e(u)|^2_A \,\mathrm{d}x.$$

For $\eta > 0$ and $\varepsilon > 0$, we define the elastic energy $\mathcal{E} : \mathrm{H}^1 \times \mathrm{H}^1 \to \mathbb{R} \cup \{+\infty\}$, and the (phase-field) surface energy $\mathcal{H} : \mathrm{H}^1 \to \mathbb{R}$, respectively, as

$$\mathcal{E}(u,v) := \frac{1}{2}\int_\Omega (v^2+\eta)|e(u)|^2_A \,\mathrm{d}x \quad \text{and} \quad \mathcal{H}(v) := \int_\Omega \left[(4\varepsilon)^{-1}(1-v)^2 + \varepsilon|\nabla v|^2\right] \mathrm{d}x.$$

The kinetic energy $\mathcal{K} : \mathrm{H}^1 \to \mathbb{R}$ is given by

$$\mathcal{K}(\dot u) := \frac{1}{2}\int_\Omega |\dot u|^2 \,\mathrm{d}x.$$

The external forces at time $t \in [0, T_f]$ are collected into a functional $\ell(t) \in \mathrm{H}^{-1}$, where $\mathrm{H}^{-1} = \mathrm{H}^{-1}(\Omega; \mathbb{R}^3)$ denotes the dual of $\mathrm{H}^1_D(\Omega; \mathbb{R}^3)$,

$$\langle \ell(t), \varphi\rangle := \int_\Omega f(t)\cdot\varphi \,\mathrm{d}x + \int_{\Gamma_N} g(t)\cdot\varphi\,\mathrm{d}s \qquad \forall \varphi \in \mathrm{H}^1_D,$$

where $f(t) \in \mathrm{L}^2(\Omega; \mathbb{R}^3)$ and $g(t) \in \mathrm{L}^2(\Gamma_N; \mathbb{R}^3)$. We assume that $\ell \in \mathrm{C}^1(\mathrm{H}^{-1})$; a sufficient condition for this would be $f \in \mathrm{C}^1(\mathrm{L}^2)$ and $g \in \mathrm{C}^1(\mathrm{L}^2)$. Finally, the total energy is given by

$$\mathcal{F}(t; u, \dot u, v) := \mathcal{K}(\dot u) + \mathcal{E}(u, v) - \langle \ell(t), u\rangle + \mathcal{H}(v).$$

In order to model a crack at the Dirichlet boundary, it is common to extend the domain, and to impose the 'Dirichlet condition' on a set of finite measure. In order to avoid distraction from the main issues (dynamics and energy balance), we chose to impose the boundary condition $v = 1$ on Γ_D. Intuitively, with this boundary condition, the Ambrosio–Tortorelli functional should still give a good aproximation to the Griffith functional, however, we stress that we do not know of a rigorous justification for this.

We seek a solution (u, v) of the system

$$\begin{aligned} \ddot u - \mathrm{div}\big(a(t)\, Ae(u + \dot u)\big) &= f(t) && \text{in } \Omega, \\ \nu^{\mathrm{T}} a(t)\, Ae(u + \dot u) &= g(t) && \text{on } \Gamma_{\mathrm{N}}, \\ (u, v) &= (0, 1) && \text{on } \Gamma_{\mathrm{D}}, \end{aligned} \tag{2.2}$$

for $t \in (0, T_f]$, where $a(t) = [v(t)]^2 + \eta$, with initial conditions $u(0) = u_0 \in \mathrm{H}^1_D$ and $\dot u(0) = u_1 \in \mathrm{H}^1_D$, and satisfying the crack stability condition

$$\mathcal{E}(u(t), v(t)) + \mathcal{H}(v(t)) = \inf_{\substack{v-1\in \mathrm{H}^1_D \\ v \le v(t)}} \mathcal{E}(u(t), v) + \mathcal{H}(v). \tag{2.3}$$

Note that we require (2.3) to hold for *every* $t \in [0, T_f]$. As initial condition for v we prescribe an arbitrary $v_0 \in 1 + \mathrm{H}^1_D$, $0 \le v_0 \le 1$ a.e. in Ω, that satisfies the unilateral minimality condition (2.3). Stated in this way the system is still severely under-constrained, hence we also impose the energy balance formula

$$\mathcal{F}(T; u(T), \dot u(T), v(T)) = \mathcal{F}(0; u_0, \dot u_0, v_0) - \int_0^T \|a^{1/2} e(\dot u)\|_A^2 \,\mathrm{d}t - \int_0^T \langle \dot\ell, u\rangle \,\mathrm{d}t \qquad \forall\, T \in [0, T_f]. \tag{2.4}$$

The main result of this paper is the following theorem, which is deduced as a direct consequence of Theorem 3.1 below.

Theorem 2.1. *Under the above conditions there exists at least one trajectory* $(u, v) \in (\mathrm{H}^2(\mathrm{L}^2) \cap \mathrm{W}^{1,\infty}(\mathrm{H}^1_D)) \times \mathrm{W}^{1,\infty}(1 + \mathrm{H}^1_D)$ *satisfying* (2.2) *in the weak sense, i.e.,*

$$(\ddot{u}, \varphi) + \big(a\, Ae(u + \dot{u}), e(\varphi)\big) = \langle \ell(t), \varphi \rangle \quad \forall \varphi \in \mathrm{H}^1_D(\Omega; \mathbb{R}^3) \quad \textit{for a.e. } t \in (0, T_f], \tag{2.5}$$

with $u(0) = u_0$*,* $\dot{u}(0) = u_1$*,* $v(0) = v_0$*. The unilateral minimality condition* (2.3) *and the energy balance condition* (2.4) *are satisfied for all times* $t \in (0, T_f]$*.*

Remark 2.1 (Boundary conditions). In order to avoid an overly cluttered notation, we restricted the generality of the boundary conditions in Theorem 2.1. As a matter of fact, our proof extends without major changes to the cases of (i) a time-dependent Dirichlet condition $u(t) = u_D(t)$ on Γ_D; and (ii) a pure traction problem (i.e., $\Gamma_D = \emptyset$).

To see this, note that case (i) can be reduced to our problem, provided $u_D \in \mathrm{C}^2(\mathrm{H}^1) \cap \mathrm{C}^3(\mathrm{L}^2)$. In case (ii), we face the potential difficulty that the Korn inequality

$$(Ae(w), e(w)) \geq c_0 \|\nabla w\|^2 \qquad \forall w \in \mathrm{H}^1(\Omega)^3$$

(where $c_0 > 0$) fails. However, the slightly weaker Gårding inequality,

$$(Ae(w), e(w)) \geq c_0 \|\nabla w\|^2 - c_1 \|w\|^2 \qquad \forall w \in \mathrm{H}^1(\Omega)^3,$$

still holds. Since the terms involving time-derivatives can be used to control the negative contribution, this is sufficient to extend our proofs.

Remark 2.2 (More general models). Furthermore, we note that our proofs apply verbatim to more general wave equations, including in particular the case of anti-plane strain, in-plane strain, and in-plane stress, as well as more general coefficients. For example, the wave equation

$$\rho \ddot{u} - \mathrm{div}\Big(aA\big(e(u) + ke(\dot{u})\big)\Big) = f,$$

where $\rho, k \in \mathrm{L}^\infty(\Omega)$ are uniformly positive can also be treated by the same analysis. We also point out that k, the dissipation, can be taken arbitrarily small.

The dissipation term is not only crucial for our analysis, but also opens up interesting modelling questions. For example, it may allow us to investigate whether time-rescaled limits of dynamic solutions converge to a quasi-static solution.

2.1. *A formal argument for energy balance*

In this section we review an entirely formal proof of the energy balance formula (2.4), which was the original motivation for pursuing the analysis in the present paper.

Suppose that (u, v) is a solution to the model introduced above. Let us assume, furthermore, that $u \in \mathrm{C}^2(\mathrm{H}^1)$, that $v \in \mathrm{C}^1(\mathrm{H}^1)$, and, for simplicity, that $\ell \equiv 0$.

Setting $a := v^2 + \eta$, and omitting for ease of writing the t-dependence from our notation on the right-hand side in the chain of equalities below, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t; u(t), \dot{u}(t), v(t)) &= \tfrac{1}{2}\big(\dot{a}Ae(u), e(u)\big) + \big(aAe(u), e(\dot{u})\big) + (\ddot{u}, \dot{u}) + \mathcal{H}'(v)[\dot{v}] \\ &= \Big\{(\ddot{u}, \dot{u}) + \big(aA(e(u + \dot{u})), e(\dot{u})\big)\Big\} - \big(aAe(\dot{u}), e(\dot{u})\big) \\ &\quad + \Big[\big(v|e(u)|_A^2, \dot{v}\big) + (2\varepsilon)^{-1}\big((v-1), \dot{v}\big) + 2\varepsilon(\nabla v, \nabla \dot{v})\Big].\end{aligned}$$

Since $\dot{u}(t) \in \mathrm{H}^1_D$, $t \in (0, T_f)$, the group of terms enclosed in curly brackets vanishes. Suppose, furthermore, that at $t \in (0, T_f)$ $v(t)$ is a *global minimizer* of $\mathcal{E}(u(t), \cdot) + \mathcal{H}(\cdot)$ (ignoring the inequality constraint); then, the group in square brackets represents the first-order criticality condition for this minimization problem (tested with $\dot{v}(t)$), and thus vanishes as well. Hence, we would obtain the desired energy balance formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t; u(t), \dot{u}(t), v(t)) = -\big(aAe(\dot{u}), e(\dot{u})\big).$$

This formal argument is made rigorous in Section 3.7 below.

3. Proof of the Existence Theorem

3.1. *Time discretization*

We set $v_h^0 = v_0$, $u_h^0 = u_0$, $u_h^0 - u_h^{-1} = hu_1$ and, for $n = 1, 2, \dots, N_f$, $N_f \geq 2$, $h = T_f/N_f$, solve

$$\big(\delta^2 u_h^n, \varphi\big) + \big(a_h^{n-1}\, Ae(u_h^n + \delta u_h^n), e(\varphi)\big) = \langle \ell(t_n), \varphi\rangle \qquad \forall \varphi \in \mathrm{H}^1_D(\Omega; \mathbb{R}^3), \tag{3.1}$$

$$v_h^n := \operatorname*{argmin}_{v-1 \in \mathrm{H}^1_D, v \leq v_h^{n-1}} \Big\{v \mapsto \mathcal{E}(u_h^n, v) + \mathcal{H}(v)\Big\}, \tag{3.2}$$

where

$$\delta^2 u_h^n := \frac{\delta u_h^n - \delta u_h^{n-1}}{h}, \quad n \geq 1, \qquad \delta u_h^n := \frac{u_h^n - u_h^{n-1}}{h}, \quad n \geq 0.$$

Due to the positivity of $a_h^{n-1} := [v_h^{n-1}]^2 + \eta$ and the uniform convexity of $\mathcal{E}(u, \cdot) + \mathcal{H}(\cdot)$ it is obvious that (3.1) and (3.2) are well-defined, i.e., there exists a *unique* family $(u_h^n)_{n=1}^{N_f}$ that solves the time-discrete problem.

For given $u_h^n \in \mathrm{H}^1_{\mathrm{D}}(\Omega; \mathbb{R}^3)$, the function v_h^n is also characterized as the solution in $1 + \mathrm{H}^1_D$ of the variational inequality

$$\partial_v \mathcal{E}(u_h^n, v_h^n)[\psi - v_h^n] + \mathcal{H}'(v_h^n)[\psi - v_h^n] \geq 0 \qquad \forall \psi \leq v_h^{n-1}, \quad \psi - 1 \in \mathrm{H}^1_D(\Omega; \mathbb{R}). \tag{3.3}$$

We remark furthermore that all v_h^n satisfy the maximum principle

$$0 \leq v_h^n \leq v_h^{n-1} \qquad \text{a.e. in } \Omega \qquad \forall n = 1, \dots, N_f. \tag{3.4}$$

The upper bound in (3.4) holds by definition, while the lower bound follows by testing (3.3) with $\psi = \max(0, v_h^n)$ (which is an admissible test function). Unless

$\psi = v_h^n$ a.e. in Ω, it would follow, in contradiction to (3.2), that $\mathcal{E}(u_h^n, v_h^n) + \mathcal{H}(v_h^n) > \mathcal{E}(u_h^n, \psi) + \mathcal{H}(\psi)$; hence $v_h^n \geq 0$. Moreover, testing (3.3) with $\psi = v_h^{n-1}$ and with $\psi = 2v_h^n - v_h^{n-1}$, we even obtain the equality,

$$\partial_v \mathcal{E}(u_h^n, v_h^n)[\delta v_h^n] + \mathcal{H}'(v_h^n)[\delta v_h^n] = 0. \tag{3.5}$$

Written out in full, this reads

$$\big(|e(u_h^n)|_A^2 v_h^n, \delta v_h^n\big) + (2\varepsilon)^{-1}\big(v_h^n - 1, \delta v_h^n\big) + 2\varepsilon\big(\nabla v_h^n, \nabla \delta v_h^n\big) = 0. \tag{3.6}$$

In the remainder of this section we shall prove that, upon defining suitable interpolants and extracting a subsequence, the family $(u_h^n, v_h^n)_{n=0}^{N_f}$ converges to a solution of (2.5), (2.3) and (2.4), as $h \to 0$.

Theorem 3.1. *For $N_f \in \mathbb{N}$ let $(u_h^n, v_h^n)_{n=0}^{N_f}$ be the solution of the time-discretization defined by* (3.1) *and* (3.2)*. Then, there exists a subsequence $h_k \searrow 0$ ($N_f^k \nearrow \infty$, with $T_f = N_f^k h_k$ fixed) and a trajectory $(u, v) \in (\mathrm{H}^2(\mathrm{L}^2) \cap \mathrm{W}^{1,\infty}(\mathrm{H}^1_D)) \times \mathrm{W}^{1,\infty}(1 + \mathrm{H}^1_D)$ such that*

$$(u_{h_k}, v_{h_k}) \to (u, v) \qquad \textit{strongly in } \mathrm{H}^1(\mathrm{H}^1 \times \mathrm{H}^1),$$

where u_{h_k}, v_{h_k} denote the piecewise affine interpolants as defined in (3.21) *below. Moreover, the trajectory (u, v) is a solution of* (2.3)*–*(2.5)*.*

3.2. *A priori estimates*

Testing (3.1) with $\varphi = h\,\delta u_h^n$, we obtain

$$\begin{aligned}\big(\delta u_h^n - \delta u_h^{n-1}, \delta u_h^n\big) &+ h\big(a_h^{n-1}\, Ae(\delta u_h^n), e(\delta u_h^n)\big) \\ &+\big(a_h^{n-1}\, Ae(u_h^n), e(u_h^n) - e(u_h^{n-1})\big) = \langle \ell(t_n), u_h^n - u_h^{n-1}\rangle.\end{aligned} \tag{3.7}$$

The first term on the left-hand side is rewritten as follows:

$$\begin{aligned}\big(\delta u_h^n - \delta u_h^{n-1}, \delta u_h^n\big) &= \tfrac{1}{2}\|\delta u_h^n\|^2 + \tfrac{1}{2}\|\delta u_h^n\|^2 - \big(\delta u_h^n, \delta u_h^{n-1}\big) \\ &\quad + \tfrac{1}{2}\|\delta u_h^{n-1}\|^2 - \tfrac{1}{2}\|\delta u_h^{n-1}\|^2 \\ &= \mathcal{K}(\delta u_h^n) - \mathcal{K}(\delta u_h^{n-1}) + \tfrac{1}{2}h^2\|\delta^2 u_h^n\|^2.\end{aligned} \tag{3.8}$$

A similar computation yields

$$\begin{aligned}\big(a_h^{n-1}\, Ae(u_h^n), e(u_h^n) - e(u_h^{n-1})\big) &= \mathcal{E}(u_h^n, v_h^n) - \mathcal{E}(u_h^{n-1}, v_h^{n-1}) \\ + \tfrac{1}{2}h^2\big\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\big\|^2 &- \tfrac{1}{2}\int_\Omega \big(a_h^n - a_h^{n-1}\big)|e(u_h^n)|_A^2 \,\mathrm{d}x.\end{aligned} \tag{3.9}$$

The last term on the right-hand side of (3.9) is further re-expressed, first by writing

$$a_h^n - a_h^{n-1} = (v_h^n)^2 - (v_h^{n-1})^2 = h(v_h^n + v_h^{n-1})\,\delta v_h^n = 2h\, v_h^n\,\delta v_h^n - h^2\big|\delta v_h^n\big|^2,$$

and then employing (3.6), by

$$\begin{aligned}
-\tfrac12 \int_\Omega & \big(a_h^n - a_h^{n-1}\big)|e(u_h^n)|_A^2 \,\mathrm{d}x \\
&= \Big\{(2\varepsilon)^{-1}\big(v_h^n - 1, v_h^n - v_h^{n-1}\big) + 2\varepsilon\big(\nabla v_h^n, \nabla v_h^n - \nabla v_h^{n-1}\big)\Big\} \\
&\quad + \tfrac12 h^2 \big\|(\delta v_h^n)\,|e(u_h^n)|_A\big\|^2.
\end{aligned}$$

Upon replacing $v_h^n - v_h^{n-1} = (v_h^n - 1) - (v_h^{n-1} - 1)$ in the first term on the right-hand side, the combined term in the curly brackets can be manipulated, with the same algebra as in (3.8), so that we arrive at

$$\begin{aligned}
-\tfrac12 \int_\Omega \big(a_h^n - a_h^{n-1}\big)|e(u_h^n)|_A^2 \,\mathrm{d}x &= \mathcal{H}(v_h^n) - \mathcal{H}(v_h^{n-1}) \\
&+ h^2\Big((4\varepsilon)^{-1}\|\delta v_h^n\|^2 + \varepsilon\|\nabla \delta v_h^n\|^2\Big) + \tfrac12 h^2 \|(\delta v_h^n)\,|e(u_h^n)|_A\|^2. \qquad (3.10)
\end{aligned}$$

Thus, summing (3.7) over n, and using (3.8)–(3.10) to replace the left-hand side, we obtain

$$\begin{aligned}
&\big[\mathcal{K}(\delta u_h^N) + \mathcal{E}(u_h^N, v_h^N) + \mathcal{H}(v_h^N)\big] - \big[\mathcal{K}(\delta u_0) + \mathcal{E}(u_0, v_0) + \mathcal{H}(v_0)\big] \\
&\quad + \sum_{n=1}^N h\big\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\big\|^2 + h\sum_{n=1}^N hD_h^n = \sum_{n=1}^N \langle \ell(t_n), u_h^n - u_h^{n-1}\rangle, \qquad (3.11)
\end{aligned}$$

where $N \in \{1, 2, \dots, N_f\}$, and

$$\begin{aligned}
D_h^n := {} & \tfrac12\|\delta^2 u_h^n\|^2 + \tfrac12\big\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\big\|^2 \\
& + \tfrac12\|(\delta v_h^n)\,|e(u_h^n)|_A\|^2 + (4\varepsilon)^{-1}\|\delta v_h^n\|^2 + \varepsilon\|\nabla\delta v_h^n\|^2.
\end{aligned}$$

We estimate the right-hand side in (3.11), using Korn's inequality, as follows:

$$\begin{aligned}
\sum_{n=1}^N h\langle \ell(t_n), \delta u_h^n\rangle &\le \Big(\sum_{n=1}^N h\|\ell(t_n)\|_{\mathrm{H}^{-1}}^2\Big)^{1/2}\Big(\sum_{n=1}^N h\|\nabla\delta u_h^n\|^2\Big)^{1/2} \\
&\le C\Big(\sum_{n=1}^N h\|\ell(t_n)\|_{\mathrm{H}^{-1}}^2\Big)^{1/2}\Big(\sum_{n=1}^N h\|e(\delta u_h^n)\|_A^2\Big)^{1/2}, \qquad (3.12)
\end{aligned}$$

and we use a Cauchy inequality to hide the second term on the right-hand side in the penultimate term on the left-hand side of (3.11) on noting that $a_h^{n-1} \ge \eta$. Using the coercivity of the different energies, we obtain the first a priori bound,

$$\max_{1\le n\le N_f}\Big\{\|\delta u_h^n\|^2 + \|e(u_h^n)\|_A^2 + \|v_h^n\|^2 + \|\nabla v_h^n\|^2\Big\} + \sum_{n=1}^{N_f} h\|e(\delta u_h^n)\|_A^2 + h\sum_{n=1}^{N_f} hD_h^n \le C_1, \qquad (3.13)$$

where C_1 depends on $\varepsilon > 0$, $\eta > 0$, $|\Omega|$ and on the initial and boundary data, but is independent of h.

Next, we estimate δv_h. Since $v_h^n \le v_h^{n-1}$, we obtain from (3.3), with indices shifted by 1, that

$$\big(|e(u_h^{n-1})|_A^2 v_h^{n-1}, \delta v_h^n\big) + (2\varepsilon)^{-1}\big(v_h^{n-1} - 1, \delta v_h^n\big) + (2\varepsilon)\big(\nabla v_h^{n-1}, \nabla\delta v_h^n\big) \ge 0. \qquad (3.14)$$

Subtracting (3.14) from (3.6) gives

$$\|(\delta v_h^n)\,|e(u_h^n)|_A\|^2 + (2\varepsilon)^{-1}\|\delta v_h^n\|^2 + (2\varepsilon)\|\nabla \delta v_h^n\|^2 \\ \le \frac{1}{h}\int_\Omega \Big(|e(u_h^{n-1})|_A^2 - |e(u_h^n)|_A^2 \Big) v_h^{n-1}\,\delta v_h^n \,\mathrm{d}x. \tag{3.15}$$

The fact that $0 \le v_h^n \le v_h^{n-1} \le 1$ (cf. (3.4)), and thereby $|v_h^n - v_h^{n-1}| \le 1$, gives $|\delta v_h^n| \le 1/h$. We can therefore rewrite and estimate the right-hand side of (3.15), using also the fact that $\delta v_h^n \le 0$, by

$$\begin{aligned}
&\frac{1}{h}\int_\Omega A\big(e(u_h^{n-1}) - e(u_h^n)\big) : \big(e(u_h^{n-1}) + e(u_h^n)\big) v_h^{n-1} \delta v_h^n \,\mathrm{d}x \\
&\quad = \frac{1}{h}\int_\Omega \big|\big(e(u_h^{n-1}) - e(u_h^n)\big)\big|_A^2 v_h^{n-1}\delta v_h^n \,\mathrm{d}x \\
&\qquad\qquad + \frac{2}{h}\int_\Omega A\big(e(u_h^{n-1}) - e(u_h^n)\big) : e(u_h^n) v_h^{n-1}\delta v_h^n \,\mathrm{d}x \\
&\quad \le \frac{2}{h}\int_\Omega \big(|e(u_h^n)|_A\,|\delta v_h^n|\big)\,\big(|v_h^{n-1}|\,|e(u_h^n) - e(u_h^{n-1})|_A\big)\,\mathrm{d}x \\
&\quad \le 2\big\|(\delta v_h^n)|e(u_h^n)|_A\big\|\,\big\|(v_h^{n-1})|e(\delta u_h^n)|_A\big\|.
\end{aligned}$$

Through an application of Cauchy's inequality we obtain from (3.15) that

$$(2\varepsilon)^{-1}\|\delta v_h^n\|^2 + 2\varepsilon\|\nabla \delta v_h^n\|^2 \le \|(v_h^{n-1})e(\delta u_h^n)\|_A^2, \qquad n = 1, \dots, N_f, \tag{3.16}$$

and in particular, by (3.13), that

$$\sum_{n=1}^{N_f} h\|\delta v_h^n\|_{\mathrm{H}^1}^2 \le C_2, \tag{3.17}$$

where C_2 depends on C_1 and on ε.

The bound (3.13) is, in some sense, the natural a priori bound for (2.2). Through the special structure of the coefficient a, we have additional regularity available, which we derive next. Testing (3.1) with $\varphi = h\,\delta w_h^n$, where $w_h^n = u_h^n + \delta u_h^n$, gives

$$h\|\delta^2 u_h^n\|^2 + (\delta u_h^n - \delta u_h^{n-1}, \delta u_h^n) + \big(a_h^{n-1} A e(w_h^n), e(w_h^n) - e(w_h^{n-1})\big) \\ = \langle \ell(t_n), w_h^n - w_h^{n-1}\rangle. \tag{3.18}$$

Using the same computations as above, the third term on the left-hand side can be estimated by

$$\big(a_h^{n-1} A e(w_h^n), e(w_h^n) - e(w_h^{n-1})\big) \ge \tfrac12\big\|(a_h^n)^{1/2}\,|e(w_h^n)|_A\big\|^2 - \tfrac12\big\|(a_h^{n-1})^{1/2}\,|e(w_h^{n-1})|_A\big\|^2 \\ + \tfrac12\big\|(a_h^{n-1})^{1/2}\,|e(w_h^n - w_h^{n-1})|_A\big\|^2 + \tfrac12\int_\Omega (a_h^{n-1} - a_h^n)|e(w_h^n)|_A^2 \,\mathrm{d}x.$$

Since $v_h^n \le v_h^{n-1}$, we have $a_h^{n-1} - a_h^n \ge 0$, and therefore

$$\big(a_h^{n-1}\, A e(w_h^n), e(w_h^n) - e(w_h^{n-1})\big) \ge \tfrac12\big\|(a_h^n)^{1/2}\,|e(w_h^n)|_A\big\|^2 - \tfrac12\big\|(a_h^{n-1})^{1/2}\,|e(w_h^{n-1})|_A\big\|^2 \\ + \tfrac12\eta\big\|e(w_h^n - w_h^{n-1})\big\|_A^2 + \tfrac12 h\int_\Omega |\delta a_h^n||e(w_h^n)|_A^2 \,\mathrm{d}x.$$

Summing (3.18) over n, using that $u_h^0 = u_0$ and $\delta u_h^0 = u_1$, and neglecting several terms, gives, for any $N \in \{1, \dots, N_f\}$,

$$\sum_{n=1}^{N} h\|\delta^2 u_h^n\|^2 + \tfrac{1}{2}\|\delta u_h^N\|^2 + \tfrac{1}{2}\big\|(a_h^n)^{1/2} e(w_h^N)\big\|_A^2 + \tfrac{1}{2}\sum_{n=1}^{N} h \int_\Omega |\delta a_h^n| |e(w_h^n)|_A^2 \,\mathrm{d}t$$
$$\le \tfrac{1}{2}\|u_1\|^2 + \tfrac{1}{2}\big\|(a_h^0)^{1/2} e(u_0+u_1)\big\|_A^2 + \sum_{n=1}^{N}\langle \ell(t_n), w_h^n - w_h^{n-1}\rangle.$$

To bound the final term on the right-hand side we reorder the sum as follows:

$$\sum_{n=1}^{N}\langle \ell(t_n), w_h^n - w_h^{n-1}\rangle = -\sum_{n=0}^{N-1}\langle \ell(t_{n+1}) - \ell(t_n), w_h^n\rangle + \langle \ell(t_N), w_h^N\rangle - \langle \ell(t_0), w_h^0\rangle$$
$$\le \sum_{n=0}^{N-1} h\big\|h^{-1}(\ell(t_{n+1}) - \ell(t_n))\big\|_{\mathrm{H}^{-1}}\|\nabla w_h^n\| + \|\ell(t_N)\|_{\mathrm{H}^{-1}}\|\nabla w_h^N\| + \|\ell(t_0)\|_{\mathrm{H}^{-1}}\|\nabla w_h^0\|$$
$$\le C\sum_{n=0}^{N-1} h\big\|h^{-1}(\ell(t_{n+1}) - \ell(t_n))\big\|_{\mathrm{H}^{-1}}\|e(w_h^n)\|_A$$
$$+ C\|\ell(t_N)\|_{\mathrm{H}^{-1}}\|e(w_h^N)\|_A + C\|\ell(t_0)\|_{\mathrm{H}^{-1}}\|e(w_h^0)\|_A,$$

where, in the transition to the last line, we used Korn's inequality. Using the assumption that $\ell \in \mathrm{C}^1(\mathrm{H}^{-1})$ and $w_h^0 = u_0 + u_1$, we obtain

$$\sum_{n=1}^{N_f} h\|\delta^2 u_h^n\|^2 + \max_{n=1,\dots,N_f}\Big(\|e(\delta u_h^n)\|_A^2 + \|\nabla \delta v_h^n\|^2\Big) \le C_3, \tag{3.19}$$

where C_3 is a positive constant, independent of h.

3.3. *Discrete energy inequality*

Starting from (3.11), we deduce an energy inequality for the time-discretization (3.1), (3.2). Identity (3.11) gives, for $1 \le N \le N_f$,

$$\mathcal{F}(t_N; u_h^N, \delta u_h^N, v_h^N) - \mathcal{F}(0; u_0, \delta u_0, v_0)$$
$$= -\langle \ell(t_N), u_h^N\rangle + \langle \ell(0), u_0\rangle + \sum_{n=1}^{N}\langle \ell(t_n), u_h^n - u_h^{n-1}\rangle$$
$$- \sum_{n=1}^{N} h\big\|(a_h^{n-1})^{1/2}\,|e(\delta u_h^n)|_A\big\|^2 - h\sum_{n=1}^{N} hD_h^n.$$

We reorder the sum over the forcing terms as follows,

$$-\langle \ell(t_N), u_h^N\rangle + \langle \ell(0), u_0\rangle + \sum_{n=1}^{N}\langle \ell(t_n), u_h^n - u_h^{n-1}\rangle = -\sum_{n=1}^{N}\langle \ell(t_n) - \ell(t_{n-1}), u_h^{n-1}\rangle.$$

Hence, we obtain the discrete energy inequality

$$\mathcal{F}(t_N; u_h^N, \delta u_h^N, v_h^N) \le \mathcal{F}(0; u_0, \delta u_0, v_0) - \sum_{n=1}^{N} h\Big\{ \|(a_h^{n-1})^{1/2}|e(\delta u_h^n)|_A\|^2 + \langle \delta\ell_h^n, u_h^{n-1}\rangle \Big\}. \tag{3.20}$$

Note that (3.20) is in fact an equality up to the numerical dissipation $h\sum_{n=1}^N hD_h^n$, which we would expect to be of order $\mathcal{O}(h)$. However, we will not require this fact in our analysis.

3.4. *Passage to the limit*

Let u_h denote the piecewise affine interpolant of the sequence $(u_h^n)_{n=0}^{N_f}$, defined as

$$u_h(t) = u_h^n + (t - t_n)\delta u_h^n, \qquad t \in [t_{n-1}, t_n], \quad n = 1, \dots, N_f. \tag{3.21}$$

In the same way, we define v_h to be the piecewise affine interpolant of $(v_h^n)_{n=0}^{N_f}$ and u_h' that of $(\delta u_h^n)_{n=0}^{N_f}$. Let $a_h = v_h^2 + \eta$. Furthermore, we define the backward interpolant $u_h^+(\cdot,t)$ and the forward interpolant $a_h^-(\cdot,t)$:

$$\begin{aligned} u_h^+(\cdot,t) &= u_h^n, \qquad && t \in (t_{n-1}, t_n], \\ a_h^-(\cdot,t) &= a_h^{n-1}, \qquad && t \in [t_{n-1}, t_n), \end{aligned}$$

with analogous definitions of v_h^+, v_h^-, ℓ_h^+. Finally, we define u_h'' to be the backward interpolant of $(\delta^2 u_h^n)_{n=1}^{N_f}$. We emphasize that, while $u_h'' = \dot u_h'$, u_h' is *not* the derivative of u_h. Instead, $\dot u_h$ is the backward interpolant of $(\delta u_h^n)_{n=1}^{N_f}$.

With this notation, (3.1) reads

$$(u_h''(t), \varphi) + \big(a_h^-(t)\, Ae(u_h^+(t) + \dot u_h(t)), e(\varphi)\big) = \langle \ell_h^+(t), \varphi\rangle \qquad \forall \varphi \in \mathrm{H}^1_D \quad \forall t \in (0, T_f]. \tag{3.22}$$

We will pass to the limit in this formulation.

Since $\dot u_h$ and $\dot v_h$ are the backward interpolants of, respectively, $(\delta u_h^n)_{n=0}^{N_f}$ and $(\delta v_h^n)_{n=0}^{N_f}$, the a priori bound (3.19) and Korn's inequality imply that

$$\|u_h\|_{\mathrm{W}^{1,\infty}(\mathrm{H}^1)} + \|v_h\|_{\mathrm{W}^{1,\infty}(\mathrm{H}^1)} \le C.$$

(We remark that only the bounds on $\|\dot u_h\|_{\mathrm{L}^\infty(\mathrm{H}^1)}$ and on $\|\dot v_h\|_{\mathrm{L}^\infty(\mathrm{H}^1)}$ are required to deduce this.) Furthermore $0 \le v_h(\cdot,t) \le 1$ and $\dot v_h(\cdot,t) \le 0$ a.e. on Ω and a.e. $t \in (0, T_f]$. Hence, there exists a subsequence $h_j \searrow 0$ (but we just write h instead of h_j) and $u \in \mathrm{W}^{1,\infty}(\mathrm{H}^1_D)$ and $v \in \mathrm{W}^{1,\infty}(1 + \mathrm{H}^1_D)$, $0 \le v(\cdot,t) \le 1$, and $\dot v(\cdot,t) \le 0$ a.e. in Ω, and for a.e. $t \in (0, T_f]$, such that

$$u_h \overset{*}{\rightharpoonup} u \qquad \text{in } \mathrm{W}^{1,\infty}(\mathrm{H}^1), \quad \text{and} \tag{3.23}$$

$$v_h \overset{*}{\rightharpoonup} v \qquad \text{in } \mathrm{W}^{1,\infty}(\mathrm{H}^1). \tag{3.24}$$

In particular, we obtain from (3.23) and (3.24) that

$$u_h(t) \rightharpoonup u(t) \quad \text{and} \quad v_h(t) \rightharpoonup v(t) \qquad \text{in } \mathrm{H}^1 \quad \forall t \in [0, T_f]. \tag{3.25}$$

To see this, note that (2.1) implies that $u_h(t) \to u(t)$ in L^2, for every $t \in [0, T_f]$. Since the sequences $(u_h(t))_{h>0}$ and $(v_h(t))_{h>0}$ (for fixed t) are bounded in H^1, the convergence must also be weak in H^1 for every $t \in [0, T_f]$.

It follows immediately from the definition of u_h that $u_h(t) = u_h^+(t) + (t - t_n)\dot{u}_h$, which implies

$$\|u_h - u_h^+\|_{\mathrm{L}^\infty(\mathrm{H}^1)} \le h\|\dot{u}_h\|_{\mathrm{L}^\infty(\mathrm{H}^1)} \le hC. \tag{3.26}$$

Hence, we deduce from (3.25) and from (3.26) that

$$u_h^+ \overset{*}{\rightharpoonup} u \quad \text{in } \mathrm{L}^\infty(\mathrm{H}^1), \text{ and } u_h^+(t) \rightharpoonup u(t) \quad \text{in } \mathrm{H}^1 \quad \forall t \in (0, T_f]. \tag{3.27}$$

Next, we show that $u \in \mathrm{H}^2(\mathrm{L}^2)$. Similarly as in the above argument we obtain (note that u_h' is the piecewise affine interpolant and $\dot{u}_h$ is the backward interpolant of $(\delta u_h^n)_{n=1}^{N_f}$)

$$\|u_h'(t) - \dot{u}_h(t)\| \le h\|u_h''(t)\| \qquad \forall t \in (0, T_f].$$

Estimate (3.19) gives an a priori bound on $\|u_h''\|_{\mathrm{L}^2(\mathrm{L}^2)}$, which implies, using also (3.23) to deduce that $\dot{u}_h \rightharpoonup \dot{u}$ weakly in $\mathrm{L}^2(\mathrm{L}^2)$, that

$$u_h' \rightharpoonup \dot{u} \qquad \text{in } \mathrm{L}^2(\mathrm{L}^2).$$

Furthermore, it implies that $\|u_h'\|_{\mathrm{H}^1(\mathrm{L}^2)}$ is bounded, which shows that, in fact,

$$u_h' \rightharpoonup \dot{u} \qquad \text{in } \mathrm{H}^1(\mathrm{L}^2). \tag{3.28}$$

In particular, we deduce that $u \in \mathrm{H}^2(\mathrm{L}^2)$ and that

$$u_h'' \rightharpoonup \ddot{u} \qquad \text{in } \mathrm{L}^2(\mathrm{L}^2). \tag{3.29}$$

Since v_h is uniformly bounded in $\mathrm{W}^{1,\infty}(\mathrm{H}^1)$, (2.1) implies that $v_h \to v$ in $\mathrm{C}(\mathrm{L}^2)$. Since $0 \le v_h(t) \le 1$ a.e. in Ω, for all t, this convergence also implies that

$$a_h \to a \qquad \text{in } \mathrm{C}(\mathrm{L}^2) \quad \text{and} \quad a_h^\pm \to a \qquad \text{in } \mathrm{L}^\infty(\mathrm{L}^2), \tag{3.30}$$

were $a = v^2 + \eta$. The latter convergence follows from estimate (3.19) and an argument similar to the one given in (3.26).

We are now in a position to take the limit $h \searrow 0$ in (3.22). For any fixed $t_1, t_2 \in [0, T_f]$, and $\varphi \in \mathrm{H}_D^1$, we make the following split:

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big[(u_h'', \varphi) + \big(a_h^- \, Ae(u_h^+ + \dot{u}_h), e(\varphi)\big) - \langle \ell_h^+, \varphi\rangle\Big]\,\mathrm{d}t \\
&\quad = \int_{t_1}^{t_2} \Big[(u_h'', \varphi) + \big(a\, Ae(u_h^+ + \dot{u}_h), e(\varphi)\big) - \langle \ell, \varphi\rangle\Big]\,\mathrm{d}t \\
&\qquad + \int_{t_1}^{t_2} \big((a_h^- - a)\, Ae(u_h^+ + \dot{u}_h), e(\varphi)\big)\,\mathrm{d}t + \int_{t_1}^{t_2} \langle \ell_h^+ - \ell, \varphi\rangle\,\mathrm{d}t.
\end{aligned} \tag{3.31}$$

We estimate the third term on the right-hand side of (3.31) by

$$\int_{t_1}^{t_2} \langle \ell_h^+ - \ell, \varphi\rangle\,\mathrm{d}t \le h(t_2 - t_1)\|\dot{\ell}\|_{\mathrm{C}(\mathrm{H}^{-1})}\|\varphi\|_{\mathrm{H}^1}.$$

For the second term on the right-hand side we use the Cauchy–Schwarz inequality to obtain

$$\int_{t_1}^{t_2} \big((a_h^- - a)\, Ae(w_h), e(\varphi)\big)\, \mathrm{d}t \le \|(a_h^- - a)|e(\varphi)|_A\|_{\mathrm{L}^2(\mathrm{L}^2)} \||e(w_h)|_A\|_{\mathrm{L}^2(\mathrm{L}^2)},$$

where $w_h = u_h^+ + \dot{u}_h$. Since $(a_h^- - a) \to 0$ a.e. in $\Omega \times (0, T_f)$, and $|a_h^- - a|^2 \le 4$ and $|\nabla\varphi|^2 \in \mathrm{L}^1(\Omega \times (0, T_f))$, Lebesgue's dominated convergence theorem implies

$$\lim_{h\searrow 0} \int_{t_1}^{t_2} \big((a_h^- - a)\, Ae(w_h), e(\varphi)\big)\, \mathrm{d}t = 0.$$

Finally, noting that $a \in \mathrm{L}^\infty(\mathrm{L}^\infty)$, we can simply take the (weak) limits as $h \searrow 0$ in each component of the first term on the right-hand side of (3.31), using (3.29), (3.27), and (3.23), to deduce that

$$\int_{t_1}^{t_2} \Big[(\ddot{u}, \varphi) + \big(a\, Ae(u + \dot{u}), e(\varphi)\big) - \langle \ell, \varphi\rangle\Big]\, \mathrm{d}t = 0 \qquad \forall t_1, t_2 \in [0, T_f].$$

It follows immediately from Lebesgue's differentiation theorem that

$$(\ddot{u}(t), \varphi) + \big(a(t)\, Ae(u(t) + \dot{u}(t)), e(\varphi)\big) = \langle \ell(t), \varphi\rangle \qquad \forall\varphi \in \mathrm{H}^1_D \quad \text{for a.e. } t \in (0, T_f]. \tag{3.32}$$

3.5. *Strong convergence*

To obtain strong convergence of u_h to u, we estimate the truncation error and then use a discrete stability estimate followed by an application of a discrete Gronwall inequality.

For $h = T_f/N_f$ and for $n = 1, 2, \ldots$, define $U_h^n = u(nh)$ and $U_h^0 = u_0$, $U_h^{-1} = u_0 - hu_1$. Furthermore, let U_h denote the piecewise affine interpolant and U_h^+ the backward piecewise constant interpolant of the samples $(U_h^n)_{n=0}^{N_f}$, and let U_h'' denote the backward piecewise constant interpolant of $(\delta^2 U_h^n)_{n=1}^{N_f}$. The same notation is used for interpolants of other discrete functions.

With this notation, and recalling from Theorem 3.1 that $u \in \mathrm{H}^2(\mathrm{L}^2) \cap \mathrm{W}^{1,\infty}(\mathrm{H}^1_D)$, we have the following result.

Lemma 3.1. *The following convergence results hold:*

$$U_h \to u \qquad \textit{in } \mathrm{H}^1(\mathrm{H}^1), \tag{3.33}$$

$$U_h^+ \to u \qquad \textit{in } \mathrm{L}^2(\mathrm{H}^1), \quad \textit{and} \tag{3.34}$$

$$U_h'' \to \ddot{u} \qquad \textit{in } \mathrm{L}^2(\mathrm{L}^2). \tag{3.35}$$

Proof. We begin by showing (3.33). Since $u \in \mathrm{H}^1(\mathrm{H}^1) \subset \mathrm{C}(\mathrm{H}^1)$, it follows that $I_h u := U_h$ is correctly defined as an element of $\mathrm{H}^1(\mathrm{H}^1)$. If we had $u \in \mathrm{C}^2(\mathrm{H}^1)$ then (3.33) would trivially follow. However, we can approximate u by a sequence $(u_\delta)_{\delta>0} \subset \mathrm{C}^2(\mathrm{H}^1)$ of such smooth functions and use the uniform boundedness of the linear operator $I_h - I : \mathrm{H}^1(\mathrm{H}^1) \to \mathrm{H}^1(\mathrm{H}^1)$, where I is the identity operator, to deduce (3.33).

The uniform boundedness of $I_h - I : \mathrm{H}^1(\mathrm{H}^1) \to \mathrm{H}^1(\mathrm{H}^1)$ is shown as follows. First, observe that

$$|I_h w|_{\mathrm{H}^1(\mathrm{H}^1)} \le |w|_{\mathrm{H}^1(\mathrm{H}^1)}, \quad \text{and thus} \quad |I_h w - w|_{\mathrm{H}^1(\mathrm{H}^1)} \le 2|w|_{\mathrm{H}^1(\mathrm{H}^1)} \quad \forall w \in \mathrm{H}^1(\mathrm{H}^1).$$

Furthermore,

$$\|I_h w - w\|_{\mathrm{L}^2(\mathrm{H}^1)} \le \sqrt{2}h|w|_{\mathrm{H}^1(\mathrm{H}^1)} \qquad \forall w \in \mathrm{H}^1(\mathrm{H}^1),$$

and therefore, for any $w \in \mathrm{H}^1(\mathrm{H}^1)$,

$$\|I_h w - w\|_{\mathrm{H}^1(\mathrm{H}^1)} \le (1 + 2h^2)^{1/2}|w|_{\mathrm{H}^1(\mathrm{H}^1)} \le (1 + 2T_f^2)^{1/2}\|w\|_{\mathrm{H}^1(\mathrm{H}^1)}. \tag{3.36}$$

Hence, we have that

$$\|U_h - u\|_{\mathrm{H}^1(\mathrm{H}^1)} = \|I_h u - u\|_{\mathrm{H}^1(\mathrm{H}^1)} \le \|I_h u_\delta - u_\delta\|_{\mathrm{H}^1(\mathrm{H}^1)} + \|(I_h - I)(u - u_\delta)\|_{\mathrm{H}^1(\mathrm{H}^1)}.$$

For $\delta > 0$ fixed, the first term on the right-hand side tends to zero as $h \searrow 0$, while the second term, by (3.36) with $w = u - u_\delta$, is bounded by a constant multiple of $\|u - u_\delta\|_{\mathrm{H}^1(\mathrm{H}^1)}$, which, in turn, can be made arbitrarily small by letting $\delta \searrow 0$; this implies (3.33). The convergence result (3.34) can be deduced exactly as in (3.26).

The same argument can be employed for proving (3.35). One proves, first, that the interpolation operator $u \mapsto U_h''$ is bounded from $\mathrm{H}^2(\mathrm{L}^2)$ to $\mathrm{L}^2(\mathrm{L}^2)$, and then repeats the regularization argument. □

To simplify the notation, we define $e_h^n := U_h^n - u_h^n$. Subtracting (3.1) from the same equation with U_h in place of u_h gives

$$(\delta^2 e_h^n, \varphi) + (a_h^{n-1} Ae(e_h^n + \delta e_h^n), e(\varphi)) = T_h^n(\varphi), \tag{3.37}$$

where

$$T_h^n(\varphi) = (\delta^2 U_h^n, \varphi) + (a_h^{n-1}\, Ae(U_h^n + \delta U_h^n), e(\varphi)) - \langle \ell(t_n), \varphi \rangle.$$

We test (3.37) with $\varphi = w_h^n = e_h^n + \delta e_h^n$, and sum over n, to obtain, for $1 \le N \le N_f$,

$$\sum_{n=1}^N h(\delta^2 e_h^n, w_h^n) + \sum_{n=1}^N h\big(a_h^{n-1}\, Ae(e_h^n + \delta e_h^n), e(w_h^n)\big) = \sum_{n=1}^N h T_h^n(w_h^n). \tag{3.38}$$

The first term on the left-hand side of (3.38) is estimated as

$$\begin{aligned}\sum_{n=1}^N h(\delta^2 e_h^n, e_h^n + \delta e_h^n) &= (\delta e_h^N, e_h^N + \delta e_h^N) - \sum_{n=1}^N h(\delta e_h^{n-1}, \delta e_h^n + \delta^2 e_h^n) \\ &\ge \|\delta e_h^N\|^2 + \frac{1}{2h}\Big(\|e_h^N\|^2 - \|e_h^{N-1}\|^2\Big) \\ &\quad - \tfrac{1}{2}\sum_{n=1}^N h\Big(\|\delta e_h^n\|^2 + \|\delta e_h^{n-1}\|^2\Big) - \sum_{n=1}^N h(\delta e_h^{n-1}, \delta^2 e_h^n).\end{aligned}$$

Using Cauchy's inequality, and the fact that (by definition) $\delta e_h^0 = 0$, we have

$$-\sum_{n=1}^{N} h(\delta e_h^{n-1}, \delta^2 e_h^n) \geq -\tfrac{1}{2}\sum_{n=1}^{N}\left(\|\delta u_h^n\|^2 - \|\delta u_h^{n-1}\|^2\right) = -\tfrac{1}{2}\|\delta e_h^N\|^2,$$

and so, we arrive at

$$\sum_{n=1}^{N} h(\delta^2 e_h^n, e_h^n + \delta e_h^n) \geq \tfrac{1}{2}\|\delta e_h^N\|^2 + \frac{1}{2h}\Big(\|e_h^N\|^2 - \|e_h^{N-1}\|^2\Big) - \sum_{n=1}^{N} h\|\delta e_h^n\|^2. \quad (3.39)$$

For the second term on the left-hand side of (3.38), we have

$$\sum_{n=1}^{N} h\big\|(a_h^{n-1})^{1/2}\,|e(e_h^n + \delta e_h^n)|_A\big\|^2 \geq \eta\sum_{n=1}^{N} h\big\|e(e_h^n + \delta e_h^n)\big\|_A^2$$

$$\geq \tfrac{1}{2}\eta\sum_{n=1}^{N} h\big\|e(e_h^n + \delta e_h^n)\big\|_A^2 + \tfrac{1}{2}\eta\sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 + \eta\sum_{n=1}^{N} h(Ae(e_h^n), e(\delta e_h^n)).$$

(Note that we have neglected several terms in this estimate.) Since

$$\sum_{n=1}^{N} h(Ae(e_h^n), e(\delta e_h^n)) \geq \sum_{n=1}^{N}\left(\tfrac{1}{2}\|e(e_h^n)\|_A^2 - \tfrac{1}{2}\|e(e_h^{n-1})\|_A^2\right) = \tfrac{1}{2}\|e(e_h^N)\|_A^2,$$

we therefore obtain

$$\sum_{n=1}^{N} h\big\|(a_h^{n-1})^{1/2}\,|e(e_h^n + \delta e_h^n)|_A\big\|^2 \geq \tfrac{1}{2}\eta\sum_{n=1}^{N} h\big\|e(e_h^n + \delta e_h^n)\big\|_A^2 \quad (3.40)$$

$$+ \tfrac{1}{2}\eta\sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 + \tfrac{1}{2}\eta\|e(e_h^N)\|_A^2.$$

The sum on the right-hand side of (3.38) can be rewritten as

$$\sum_{n=1}^{N} hT_h^n(w_h^n) = \int_0^T \Big\{(U_h'', w_h^+) + \big(a_h^-\, Ae(U_h^+ + \dot{U}_h), e(w_h^+)\big) - \langle \ell_h^+, w_h^+\rangle\Big\}\,\mathrm{d}t.$$

Testing (3.32) with $\varphi = w_h^+(t)$, and applying Korn's inequality, we obtain

$$\begin{aligned}
\sum_{n=1}^{N} hT_h^n(w_h^n) = &\int_0^T \Big\{(U_h'' - \ddot{u}, w_h^+) + (a_h^-\, Ae(U_h^+ - u + \dot{U}_h - \dot{u}), e(w_h^+)) \\
&+ \langle \ell - \ell_h^+, w_h^+\rangle + \big((a_h^- - a)\, Ae(u + \dot{u}), e(w_h^+)\big)\Big\}\,\mathrm{d}t \\
\leq\ & C\Big(\int_0^{T_f} \Big\{\|U_h'' - \ddot{u}\|_{\mathrm{L}^2}^2 + \|e(U_h^+) - e(u)\|_A^2 + \|e(\dot{U}_h) - e(\dot{u})\|_A^2 \\
&+ \|\ell - \ell_h^+\|_{-1}^2 + \big\|(a_h^- - a)\,|e(u + \dot{u})|_A\big\|^2\Big\}\,\mathrm{d}t\Big)^{1/2}\Big(\sum_{n=1}^{N} h\|e(w_h^n)\|_A^2\Big)^{1/2} \\
=:\ & \tilde{T}_h\Big(\sum_{n=1}^{N} h\|e(w_h^n)\|_A^2\Big)^{1/2}.
\end{aligned}$$

Using (3.33)–(3.35) and the assumption that $\ell \in \mathrm{C}^1(\mathrm{H}^{-1})$, it follows that $\tilde{T}_h \to 0$ as $h \to 0$. The only term in the definition of $\tilde{T}_h$ that is nontrivial to handle is $\|(a_h^- - a)\,|e(u + \dot{u})|_A\|$; it can be estimated by extracting a subsequence, which attains its upper limit and for which $a_h^- \to a$ pointwise a.e. in $\Omega \times (0, T_f)$. Lebesgue's dominated convergence theorem then implies that the term tends to zero as $h \searrow 0$.

Combining the last estimate with (3.39) and (3.40), we deduce that

$$\|\delta e_h^N\|^2 + \frac{1}{h}\Big(\|e_h^N\|^2 - \|e_h^{N-1}\|^2\Big) + \sum_{n=1}^{N} h\|e(\delta e_h^n)\|_A^2 + \|e(e_h^N)\|_A^2 \le C_\eta\Big\{\sum_{n=1}^{N} h\|\delta e_h^n\|^2 + \tilde{T}_h^2\Big\}. \tag{3.41}$$

Omitting the last two terms on the left-hand side and applying a discrete Gronwall argument we obtain, first,

$$\|e_h^N\|^2 + \sum_{n=1}^{N} h\|\delta e_h^n\|^2 \le C(\varepsilon, \eta, T_f)\,\tilde{T}_h^2 \to 0, \tag{3.42}$$

as $h \searrow 0$, and for all N. Moreover, upon summing (3.41) over N from 1 to N_f and using the positive definiteness of A and Korn's inequality, we also obtain

$$\int_0^{T_f} \|\nabla \dot{e}_h(t)\|^2\,\mathrm{d}t = \sum_{n=1}^{N_f} h\|\nabla \delta e_h^n\|^2 \to 0 \qquad \text{as } h \searrow 0,$$

where e_h denotes the piecewise affine interpolant of $(e_h^n)_{n=0}^{N_f}$. Together with (3.42), and the fact that $e_h^0 = 0$, this gives

$$e_h = U_h - u_h \to 0 \qquad \text{in } \mathrm{H}^1(\mathrm{H}^1) \qquad \text{as } h \searrow 0.$$

Using the interpolation error estimate (3.33), and the fact that $\|u_h^+(t) - u_h(t)\|_{\mathrm{H}^1} \le h\|\dot{u}_h(t)\|_{\mathrm{H}^1}$ for all $t \in [0, T_f]$, we deduce that

$$u_h \to u \qquad \text{in } \mathrm{H}^1(\mathrm{H}^1), \qquad \text{and} \tag{3.43}$$

$$u_h^+ \to u \qquad \text{in } \mathrm{L}^\infty(\mathrm{H}^1). \tag{3.44}$$

(Recall, however, that both (3.43) and (3.44) are understood in the sense of a subsequence which we had previously extracted.)

3.6. *Minimality of v*

We use the strong convergence result (3.44) to establish the unilateral minimality of v, i.e., that (2.3) holds. The associated variational inequality is

$$\partial_v \mathcal{E}(u, v)[\psi - v] + \mathcal{H}'(v)[\psi - v] \ge 0 \qquad \forall\, \psi \in 1 + \mathrm{H}_D^1, \quad \psi \le v, \tag{3.45}$$

or equivalently, upon substituting $\chi = \psi - v \le 0$,

$$\partial_v \mathcal{E}(u(t), v(t))[\chi] + \mathcal{H}'(v(t))[\chi] \ge 0 \qquad \forall\, \chi \in \mathrm{H}_D^1, \quad \chi \le 0, \quad t \in [0, T_f]. \tag{3.46}$$

Since $v_h^+(t)$ is minimal among $\psi \le v_h^-(t)$, it is also minimal among $\psi \le v_h^+(t)$ which again gives

$$\partial_v \mathcal{E}(u_h^+(t), v_h^+(t))[\chi] + \mathcal{H}'(v_h^+(t))[\chi] \ge 0 \qquad \forall \chi \in \mathrm{H}^1_D, \quad \chi \le 0, \quad t \in [0, T_f].$$

Since $\mathcal{H}'(v)[\chi]$ is linear in v, and since $v_h^+(t) \rightharpoonup v(t)$ weakly in H^1 for every t, it follows that

$$\mathcal{H}'(v_h^+(t))[\chi] \to \mathcal{H}'(v(t))[\chi],$$

where $\chi \in \mathrm{H}^1_D, \chi \le 0$ is held fixed. Furthermore, for every $t \in [0, T_f]$, $\nabla u_h^+(t) \to \nabla u(t)$ strongly in H^1, which implies that $|e(u_h^+(t))|_A^2 \to |e(u(t))|_A^2$ strongly in L^1. After extracting a subsequence $h' \subset h$ for which $v_{h'}^+(t) \to v(t)$ pointwise a.e. in Ω, for all $t \in [0, T_f]$, Lebesgue's dominated convergence theorem implies

$$\int_\Omega v_h^+ \chi |e(u_h^+)|_A^2 \,\mathrm{d}x \to \int_\Omega v\chi |e(u)|_A^2 \,\mathrm{d}x \qquad \forall \chi \in \mathrm{L}^\infty.$$

Thus, we have shown that (3.46) holds for all $\chi \in \mathrm{L}^\infty \cap \mathrm{H}^1$, $\chi \le 0$. Since the only reasonable competitors ψ for the energy satisfy $0 \le \psi \le v$, this is sufficient to deduce unilateral minimality of v and thus concludes the proof of (3.45), and equivalently of (2.3).

3.7. *Energy balance*

Testing (3.32) with $\varphi = \dot{u}$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{u}\|^2 + \left\|a^{1/2}|e(\dot{u})|_A\right\|^2 + (a\,Ae(u), e(\dot{u})) = \langle \ell, \dot{u} \rangle. \tag{3.47}$$

Using the fact that $\ell \in \mathrm{C}^1(\mathrm{H}^{-1})$, the left-hand side can be rewritten as

$$\langle \ell, \dot{u} \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \ell, u \rangle - \langle \dot{\ell}, u \rangle. \tag{3.48}$$

In what follows, we will find a way to bypass the technically subtle product rule formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|a^{1/2}|e(u)|_A\|^2 = \tfrac{1}{2}(\dot{a}\,Ae(u), e(u)) + (a\,Ae(u), e(\dot{u})),$$

which, formally, would quickly lead to the energy balance condition (2.4).

First, we use the discrete energy inequality (3.20) to deduce a corresponding result for the limit. Using only the weak convergence of v_h and u_h and the strong convergence of a_h^- in $\mathrm{L}^2(\mathrm{L}^2)$, as well as a standard lower-semicontinuity property of convex integrands[7] , we obtain

$$\begin{aligned}
&\mathcal{F}(T,u(T),\dot u(T),v(T)) + \int_0^T \big\|a^{1/2}|e(\dot u)|_A\big\|^2\,\mathrm{d}t + \int_0^T \langle \dot\ell(t),u\rangle\,\mathrm{d}t \qquad (3.49)\\
&\le \liminf_{h\searrow 0}\Big\{\mathcal{F}(T,u_h(T),\dot u_h(T),v_h(T)) + \int_0^T \big\|(a_h^-)^{1/2}|e(\dot u_h)|_A\big\|^2\,\mathrm{d}t + \int_0^T \langle \dot\ell_h(t),u_h\rangle\,\mathrm{d}t\Big\}\\
&\le \limsup_{h\searrow 0}\Big\{\mathcal{F}(T,u_h(T),\dot u_h(T),v_h(T)) + \int_0^T \big\|(a_h^-)^{1/2}|e(\dot u_h)|_A\big\|^2\,\mathrm{d}t + \int_0^T \langle \dot\ell_h(t),u_h\rangle\,\mathrm{d}t\Big\}\\
&\le \mathcal{F}(0,u_0,\dot u_0,v_0),
\end{aligned}$$

i.e.,

$$\mathcal{F}(T,u(T),\dot u(T),v(T)) + \int_0^T \big\|a^{1/2}|e(\dot u)|_A\big\|^2\,\mathrm{d}t + \int_0^T \langle \dot\ell(t),u\rangle\,\mathrm{d}t \le \mathcal{F}(0,u(0),\dot u(0),v(0)). \qquad (3.50)$$

It remains to prove the reverse inequality,

$$\mathcal{F}(T,u(T),\dot u(T),v(T)) + \int_0^T \big\|a^{1/2}|e(\dot u)|_A\big\|^2\,\mathrm{d}t + \int_0^T \langle \dot\ell(t),u\rangle\,\mathrm{d}t \ge \mathcal{F}(0,u(0),\dot u(0),v(0)). \qquad (3.51)$$

By integrating (3.47) and using (3.48), we find that (3.51) is equivalent to

$$\mathcal{E}(u(T),v(T)) + \mathcal{H}(v(T)) - \int_0^T (a\,Ae(u),e(\dot u))\,\mathrm{d}t \ge \mathcal{E}(u(0),v(0)) + \mathcal{H}(v(0)),$$

which can be rearranged as

$$\mathcal{E}(u(T),v(T)) - \mathcal{E}(u(0),v(0)) \ge -\mathcal{H}(v(T)) + \mathcal{H}(v(0)) + \int_0^T (a\,Ae(u),e(\dot u))\,\mathrm{d}t. \qquad (3.52)$$

We prove (3.52) by a time-discretization. In order to avoid confusion with the earlier time-step discretization, let $M \in \mathbb{N}$, let $\tau = \tau_M = T/M$, and set $s_i = i\tau$, $i = 0,\dots,M$. For each i we write

$$\begin{aligned}
&\mathcal{E}(u(s_i),v(s_i)) - \mathcal{E}(u(s_{i-1}),v(s_{i-1}))\\
&= \frac12\int_\Omega \Big(a(s_i)|e(u(s_i))|_A^2 - a(s_{i-1})|e(u(s_{i-1}))|_A^2\Big)\,\mathrm{d}x\\
&= \frac12\int_\Omega \big(a(s_i)-a(s_{i-1})\big)|e(u(s_{i-1}))|_A^2\,\mathrm{d}x\\
&\quad + \frac12\int_\Omega a(s_i)\big(|e(u(s_i))|_A^2 - |e(u(s_{i-1}))|_A^2\big)\,\mathrm{d}x\\
&=: \mathbf{A}_i + \mathbf{B}_i. \qquad (3.53)
\end{aligned}$$

We estimate the terms $\mathbf{A}_i$ and $\mathbf{B}_i$ separately.

Thanks to the unilateral minimality of $v(s_{i-1})$, we have

$$\frac12\int_\Omega a(s_i)|e(u(s_{i-1}))|_A^2\,\mathrm{d}x + \mathcal{H}(v(s_i)) \ge \frac12\int_\Omega a(s_{i-1})|e(u(s_{i-1}))|_A^2\,\mathrm{d}x + \mathcal{H}(v(s_{i-1})),$$

and hence

$$\mathbf{A}_i = \frac{1}{2}\int_\Omega \big(a(s_i) - a(s_{i-1})\big)|e(u(s_{i-1}))|_A^2 \,\mathrm{d}x \geq \mathcal{H}(v(s_{i-1})) - \mathcal{H}(v(s_i)). \tag{3.54}$$

To estimate the term $\mathbf{B}_i$ we note first that

$$\begin{aligned}\mathbf{B}_i =\ & \frac{1}{2}\int_\Omega a(s_i)\, Ae(u(s_i) + u(s_{i-1})) : e(u(s_i) - u(s_{i-1}))\,\mathrm{d}x \\ =\ & \int_{s_{i-1}}^{s_i}\int_\Omega a_\tau^-(s)\, Ae(\bar{u}_\tau(s)) : e(\dot{u}(s))\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{3.55}$$

where

$$a_\tau^-(s) = a(s_{i-1}) \ \text{ and } \ \bar{u}_\tau(s) = \tfrac{1}{2}(u(s_{i-1}) + u(s_i)) \quad \text{for } s \in (s_{i-1}, s_i), \quad i = 1, \ldots, M.$$

Due to the regularity of u and v, it follows that

$$\begin{aligned} a_\tau^- &\to a \qquad \text{strongly in } \mathrm{L}^\infty(\mathrm{L}^2), \text{ and} \\ \bar{u}_\tau &\to u \qquad \text{strongly in } \mathrm{L}^\infty(\mathrm{H}^1).\end{aligned}$$

Summing (3.55) over $i = 1, \ldots, M$ gives

$$\begin{aligned}\sum_{i=1}^M \mathbf{B}_i = \int_0^T \big(a_\tau^-\, Ae(\bar{u}_\tau), e(\dot{u})\big)\,\mathrm{d}s = \int_0^T \big(a\, Ae(u), e(\dot{u})\big)\,\mathrm{d}s \\ + \int_0^T \big(a_\tau^-\, Ae(\bar{u}_\tau - u), e(\dot{u})\big)\,\mathrm{d}s + \int_0^T \big((a_\tau^- - a)\, Ae(u), e(\dot{u})\big)\,\mathrm{d}s.\end{aligned}$$

The second term on the right-hand side clearly converges to zero as $\tau \to 0$. For the third term on the right-hand side, we use again Lebesgue's dominated convergence theorem to prove that, after extracting a suitable subsequence, so that $a_\tau^- \to a$ pointwise, the third term tends to zero as well. Summing (3.53) and (3.54) over $i = 1, \ldots, M$ as well, we have shown that

$$\begin{aligned}\mathcal{E}(u(T), v(T)) - \mathcal{E}(u(0), v(0)) =\ & \limsup_{\tau\searrow 0} \sum_{i=1}^M \big(\mathbf{A}_i + \mathbf{B}_i\big) \\ \geq\ & \mathcal{H}(v(0)) - \mathcal{H}(v(T)) + \int_0^T \big(a\, Ae(u), e(\dot{u})\big)\,\mathrm{d}t,\end{aligned}$$

which, as we have argued above, implies (3.51). Together with (3.50), we have shown that the limit indeed satisfies the energy conservation condition (2.4).

A forteriori we also obtain strong convergence of v_h to v. Notice that, by (2.4), we have in fact equality in all inequalities in the chain of estimates in (3.49), which implies immediately that

$$\mathcal{H}(v_h(t)) \to \mathcal{H}(v(t)) \qquad \forall t \in [0, T_f].$$

Since weak convergence together with convergence of the norm implies strong convergence, this gives

$$v_h(t) \to v(t) \qquad \text{strongly in } \mathrm{H}^1(\Omega;\mathbb{R}) \qquad \forall t \in [0, T_f]. \tag{3.56}$$

This concludes the proof of the convergence Theorem 3.1, which immediately implies Theorem 2.1 as well.

4. Conclusions

In this paper, we have developed the first steps of a theory for a regularized model of dynamic crack propagation based on the time-discrete model put forward in Bourdin, Larsen & Richardson[4]. By proving convergence of a time-discretization, as the time-step tends to zero, we have established existence of solutions to a time-continuous formulation as well as balance of total energy of the system. We stress, once again, that this model and our theory do not require any a priori assumptions on the crack topology and is in particular dimension-independent.

Of course, a number of questions remain open: For example, we were unable to perform our analysis without the damping term. We believe that it should be possible to establish existence of solutions in that case, however, we could not see a possibility to guarantee energy balance.

Second, taking the limit as $\varepsilon \searrow 0$ poses a formidable challenge. Note, for example, that the unilateral minimality of the v variable has no obvious counterpart in a sharp-interface model. While some possibilities are proposed in Larsen[16], proof that any of these is the limiting model is open.

Third, we were unable to establish a sufficiently strong notion of maximal dissipation. Intuitively, it seems reasonable to us that unilateral minimality of v and energy balance could provide such a condition.

Finally, a fascinating question is to rescale time, and to investigate the quasi-static limit of our dynamic model. Here, the damping is crucial. It would be interesting to see whether one can, at all, recover the model of Francfort & Marigo[12], as well as discover situations in which the model in Francfort & Marigo[12] does not give the limiting model.

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