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FINITE ELEMENT APPROXIMATION OF KINETIC DILUTE POLYMER MODELS WITH MICROSCOPIC CUT-OFF

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Abstract. We construct a Galerkin finite method for the numerical approximation of weak solutions to a coupled microscopic-macroscopic bead-spring model that arises from the kinetic theory of dilute solutions of polymeric liquids with noninteracting polymer chains. The model consists of the unsteady incompressible Navier–Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, for the velocity and the pressure of the fluid, with an elastic extra-stress tensor as right-hand side in the momentum equation. The extra-stress tensor stems from the random movement of the polymer chains and is defined through the associated probability density function that satisfies a Fokker–Planck type parabolic equation, crucial features of which are the presence of a centre-of-mass diffusion term and a cut-off function $\beta^L(\cdot) := \min(\cdot, L)$ in the drag and convective terms, where $L \gg 1$. We focus on finitely-extensible nonlinear elastic, FENE-type, dumbbell models. We perform a rigorous passage to the limit as the spatial and temporal discretization parameters tend to zero, and show that a (sub)sequence of these finite element approximations converges to a weak solution of this coupled Navier–Stokes–Fokker–Planck system. The passage to the limit is performed under minimal regularity assumptions on the data. Our arguments therefore also provide a new proof of global existence of weak solutions to Fokker–Planck–Navier–Stokes systems with centre-of-mass diffusion and microscopic cut-off. The convergence proof rests on several auxiliary technical results, including the stability, in the Maxwellian-weighted H^1 norm, of the orthogonal projector, in the Maxwellian-weighted L^2 inner product, onto finite element spaces consisting of continuous piecewise linear functions. We establish optimal-order quasi-interpolation error bounds in the Maxwellian-weighted L^2 and H^1 norms, and prove a new elliptic regularity result in the Maxwellian-weighted H^2 norm.

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1. Introduction

This paper is concerned with the construction and convergence analysis of a Galerkin finite element approximation to weak solutions of a system of nonlinear partial differential equations that arises from the kinetic theory of dilute polymer solutions. The solvent is an incompressible, viscous, isothermal Newtonian fluid confined to an open set $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega$. For the sake of simplicity of presentation we shall suppose that Ω has solid boundary $\partial\Omega$; the velocity field $\underset{\sim}{u}$ will then satisfy the no-slip boundary condition $\underset{\sim}{u} = \underset{\sim}{0}$ on $\partial\Omega$. The polymer chains, which are suspended in the solvent, are assumed not to interact with each other.

Keywords and phrases: Finite element method, polymeric flow models, convergence analysis, existence of weak solutions, Navier–Stokes equations, Fokker–Planck equations, FENE.

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The conservation of momentum and mass equations for the solvent then have the form of the incompressible Navier–Stokes equations in which the elastic *extra-stress* tensor $\underset{\approx}{\tau}$ (*i.e.*, the polymeric part of the Cauchy stress tensor,) appears as a source term:

Find $\underset{\sim}{u} : (\underset{\sim}{x}, t) \in \overline{\Omega} \times [0, T] \mapsto \underset{\sim}{u}(\underset{\sim}{x}, t) \in \mathbb{R}^d$ and $p : (\underset{\sim}{x}, t) \in \Omega \times (0, T] \mapsto p(\underset{\sim}{x}, t) \in \mathbb{R}$ such that

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u} - \nu\, \Delta_x \underset{\sim}{u} + \underset{\sim}{\nabla}_x\, p = \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau} \qquad \text{in } \Omega \times (0, T], \tag{1.1a}$$

$$\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0 \qquad \text{in } \Omega \times (0, T], \tag{1.1b}$$

$$\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{on } \partial\Omega \times (0, T], \tag{1.1c}$$

$$\underset{\sim}{u}(\underset{\sim}{x}, 0) = \underset{\sim}{u}^0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega\,; \tag{1.1d}$$

where $\underset{\sim}{u}$ is the velocity field, p is the pressure, $\nu \in \mathbb{R}_{>0}$ is the viscosity of the solvent, and $\underset{\sim}{f}$ is the density of body forces acting on the fluid.

The extra stress tensor $\underset{\approx}{\tau}$ is defined via a weighted average of ψ, the probability density function of the (random) conformation vector of the polymer molecules (cf. (1.3) below); the progressive Kolmogorov equation satisfied by ψ is a Fokker–Planck type second-order parabolic equation whose transport coefficients depend on the velocity field $\underset{\sim}{u}$.

Kinetic theories of polymeric fluids ignore quantum mechanical and atomistic effects, and focus on 'coarse-grained' models of the polymeric *conformations*, *i.e.*, the orientation and the degree of stretching experienced by polymer molecules. The coarsest in the hierarchy of kinetic models of dilute polymers is the *dumbbell model*, which describes the polymer molecule by two beads connected by a massless elastic spring [7]; the elastic force $\underset{\sim}{F} : D \subseteq \mathbb{R}^d \to \mathbb{R}^d$ of the spring connecting the two beads is defined by a (sufficiently smooth) *spring potential* $U : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ through

$$\underset{\sim}{F}(\underset{\sim}{q}) = H\, U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\, \underset{\sim}{q}, \qquad \underset{\sim}{q} \in D, \tag{1.2}$$

where $H \in \mathbb{R}_{>0}$ is a spring constant. The elongation (or conformation) vector $\underset{\sim}{q}$, whose direction and length define the direction and length of the polymer chain represented by the dumbbell, is assumed to be confined to a balanced convex open set $D \subset \mathbb{R}^d$; the term *balanced* means that $\underset{\sim}{0} \in D$, and $-\underset{\sim}{q} \in D$ whenever $\underset{\sim}{q} \in D$. Typically, D is an open d-dimensional ball of fixed radius $r_D > 0$, or an ellipse with fixed half-axes, or the whole of $\mathbb{R}^d$. Our analytical results in this paper are concerned with the physically realistic case when D is bounded, although we shall also comment on the idealized situation when $D = \mathbb{R}^d$.

The governing equations of the dumbbell model considered here are (1.1a–d), where the *elastic extra-stress tensor* $\underset{\approx}{\tau}$ is defined by the *Kramers expression*:

$$\underset{\approx}{\tau}(\underset{\sim}{x}, t) = k\,\mu \left(\int_D \underset{\sim}{q}\, \underset{\sim}{q}^T\, U'\left(\tfrac{1}{2}|\underset{\sim}{q}|^2\right) \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q} - \rho(\underset{\sim}{x}, t)\, \underset{\approx}{I} \right); \tag{1.3}$$

here k is the *Boltzmann constant* and μ is the *absolute temperature*. Further,

$$\rho(\underset{\sim}{x}, t) = \int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q} \tag{1.4}$$

signifies *density*, and the *probability density function* $\psi(\underset{\sim}{x}, \underset{\sim}{q}, t)$ is a solution to the Fokker–Planck equation

$$\frac{\partial \psi}{\partial t} + (\underset{\sim}{u} \cdot \nabla_x)\psi + \underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x\, \underset{\sim}{u})\, \underset{\sim}{q}\, \psi) = \varepsilon\, \Delta_x \psi + \frac{1}{2\lambda}\, \underset{\sim}{\nabla}_q \cdot (\underset{\sim}{\nabla}_q\, \psi + U'\, \underset{\sim}{q}\, \psi). \tag{1.5}$$

Here $\lambda \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$ are fixed positive real numbers, called the *relaxation time* and the *centre-of-mass diffusion coefficient*, respectively. We refer to [4] for the derivation of the model; see also the recent paper of

Schieber [37] for a justification of the presence of the $\underset{\sim}{x}$-dissipative centre-of-mass diffusion term $\varepsilon\,\Delta_x\psi$ on the right-hand side of (1.5).

When D is $B(\underset{\sim}{0}, b^{\frac{1}{2}})$, a ball of radius $b^{\frac{1}{2}}$ in $\mathbb{R}^d$ centred at the origin, a typical spring force $\underset{\sim}{F}(\underset{\sim}{q})$ for a finitely-extensible model, such as the FENE (finitely-extensible nonlinear elastic) model for example in which

$$U(s) = -\frac{b}{2}\ln\left(1 - \frac{2s}{b}\right), \qquad s \in [0, \tfrac{b}{2}),$$

explodes as $\underset{\sim}{q}$ approaches ∂D; see Section 2.2 below. Parabolic PDEs with unbounded coefficients are studied, for example, in the monographs of Cerrai [11] and Lorenzi and Bertoldi [31]; see also the article of Da Prato and Lunardi [14] and references therein. We note in passing that, on letting $b \to +\infty$, the FENE potential converges to the (linear) Hookean spring potential $U(s) = s$ while D then becomes the whole of $\mathbb{R}^d$, — corresponding to a mathematically simple(r) albeit physically unrealistic scenario in which a polymer chain can have arbitrarily large elongation.

We note in passing that in contrast with the case of Hookean dumbbells, the FENE model does not have an exact closure at the macroscopic level, though Du, Yu, and Liu [15] and Yu, Du, and Liu [42] have recently considered the analysis of approximate closures of the FENE model. Previously, El-Kareh and Leal [17] had proposed a macroscopic model, with added dissipation in the equation which governs the evolution of the conformation tensor $\underset{\approx}{A}(\underset{\sim}{x}, t) := \int_D \underset{\sim}{q}\,\underset{\sim}{q}^T \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\,\mathrm{d}\underset{\sim}{q}$ in order to account for Brownian motion across streamlines; the model can be thought of as an approximate macroscopic closure of a FENE-type microscopic-macroscopic model with centre-of-mass diffusion.

An early effort to show the existence and uniqueness of local-in-time solutions to a family of bead-spring type polymeric flow models is due to Renardy [36]. While the class of potentials $\underset{\sim}{F}(\underset{\sim}{q})$ considered by Renardy [36] (cf. hypotheses (F) and (F$'$) on pp. 314–315) does include the case of Hookean dumbbells, it excludes the practically relevant case of the FENE model (see Section 2.2 below). More recently, E, Li, and Zhang [16] and Li, Zhang, and Zhang [28] have revisited the question of local existence of solutions for dumbbell models.

The existence of global weak solutions to the coupled Navier–Stokes–Fokker–Planck systems of the form (1.1a)–(1.5) with FENE type potentials, and related systems of partial differential equations, have been studied by Barrett, Schwab, and Süli [3], Constantin [13], Lions and Masmoudi [30], Barrett and Süli [4], [5], Otto and Tzavaras [35], and Masmoudi [34]. We refer to [5] for a detailed survey of the relevant literature.

For a survey of numerical algorithms for the approximation of kinetic models of dilute polymers see, for example, Section 4 of the survey article of Li and Zhang [29]; for recent progress on deterministic algorithms for the approximation of Fokker–Planck and coupled Navier–Stokes–Fokker–Planck systems, see, for example, Lozinski et al. [32, 33], and Knezevic and Süli [23, 24].

The present paper is a continuation of our recent work [6]; there, under very general assumptions on the finite-dimensional spaces used for the purpose of spatial discretization, including, in particular, classical conforming finite element spaces and spectral Galerkin subspaces, we showed the convergence of a (sub)sequence of numerical approximations to a weak solution of the coupled Navier–Stokes–Fokker–Planck system (1.1a)–(1.5), for a large class of unbounded spring potentials, including the FENE potential, in the case of the corotational model, where $\underset{\approx}{\nabla}_x\,\underset{\sim}{u}$ in the Fokker–Planck equation is replaced by its skew-symmetric part $\frac{1}{2}(\underset{\approx}{\nabla}_x\,\underset{\sim}{u} - (\underset{\approx}{\nabla}_x\,\underset{\sim}{u})^{\mathrm{T}})$.

Here, we shall be concerned with the general noncorotational model (1.1a)–(1.5), but where a cut-off function $\beta^L(\cdot) := \min(\cdot, L)$, with $L \gg 1$, is introduced into the drag and convective terms of (1.5). The paper is organized as follows. Section 2 is devoted to the statement of the problem, including our structural assumptions on the admissible class of nonlinear spring potentials. In addition, we review the energy law satisfied by the system. In Section 3, we introduce the appropriate function spaces for the problem. Finally, in Section 4 we introduce our Galerkin finite element method for this coupled Navier–Stokes–Fokker–Planck system with microscopic cut-off, which involves an additional regularization parameter $\delta > 0$. We show the existence of this numerical approximation, and that it satisfies a discrete analogue of the energy law for the continuous system. We then pass to the limit as the spatial discretization parameter h and the time step parameter Δt, as well as the regularization parameter δ, tend to zero; using a weak-compactness argument in Maxwellian-weighted Sobolev spaces we show that a subsequence of the sequence $\{\{\underset{\sim}{u}^{\Delta t}_{\delta,h}, \widehat{\psi}^{\Delta t}_{\delta,h}\}\}_{\delta>0,h>0,\Delta t>0}$ of numerical approximations to the velocity field $\underset{\sim}{u}$ and the scaled probability density function $\widehat{\psi} = \psi/M$, where M is the normalized Maxwellian

$$M(\underset{\sim}{q}) = Z^{-1}\exp(-U(\tfrac{1}{2}|\underset{\sim}{q}|^2)), \qquad \text{where} \qquad Z := \int_D \exp(-U(\tfrac{1}{2}|\underset{\sim}{q}|^2))\,\mathrm{d}\underset{\sim}{q}, \tag{1.6}$$

converges to a weak solution $\{\underset{\sim}{u}, \widehat{\psi}\}$ of the coupled Navier–Stokes–Fokker–Planck system with microscopic cut-off. We close the paper with an Appendix, where we use the Brascamp–Lieb inequality to construct a quasi-interpolation operator in Maxwellian-weighted Sobolev spaces. By applying an extension of the Bramble–Hilbert lemma due to Tartar, we prove sharp approximation error bounds; we also establish an, apparently new, elliptic regularity result in the Maxwellian-weighted H^2 norm on D; we then use these results to show that the orthogonal projection operator in the Maxwellian-weighted L^2 inner product is stable in the Maxwellian-weighted H^1 norm, — a result that plays a crucial role in our convergence proof of the numerical method.

The passage to the limit in the paper is performed under minimal regularity assumptions on the data. Our arguments therefore also provide a new proof of global existence of weak solutions to the general noncorotational Fokker–Planck–Navier–Stokes system with centre-of-mass diffusion and microscopic cut-off. The definition of the sequence of approximating solutions is completely constructive in the sense that it is based on a fully-discrete and practically implementable Galerkin finite element method. To the best of our knowledge this is the first rigorous result concerning the convergence of a sequence of numerical approximations to a global weak solution of the coupled Navier–Stokes–Fokker–Planck model in the case of a general, noncorotational, drag term.

2. Polymer models

We term polymer models under consideration here microscopic-macroscopic type models, since the continuum mechanical *macroscopic* equations of incompressible fluid flow are coupled to a *microscopic* model: the Fokker–Planck equation describing the statistical properties of particles in the continuum. We first present these equations and collect the assumptions on the parameters in the model.

2.1. Microscopic-macroscopic polymer models

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz-continuous boundary $\partial\Omega$, and suppose that the set $D \subseteq \mathbb{R}^d$, $d = 2$ or 3, of admissible elongation vectors $\underset{\sim}{q}$ in (1.5) is a balanced convex open set. For the sake of simplicity of presentation, we shall suppose that D is a bounded open ball in $\mathbb{R}^d$. Gathering (1.1a–d), (1.3) and (1.5), we then consider the following initial-boundary-value problem:

(P) Find $\underset{\sim}{u} : (\underset{\sim}{x}, t) \in \overline{\Omega} \times [0, T] \mapsto \underset{\sim}{u}(\underset{\sim}{x}, t) \in \mathbb{R}^d$ and $p : (\underset{\sim}{x}, t) \in \Omega \times (0, T) \mapsto p(\underset{\sim}{x}, t) \in \mathbb{R}$ such that

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u} - \nu\,\Delta_x\,\underset{\sim}{u} + \underset{\sim}{\nabla}_x\,p = \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau}(\psi) \qquad \text{in } \Omega \times (0, T], \tag{2.1a}$$

$$\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0 \qquad \text{in } \Omega \times (0, T], \tag{2.1b}$$

$$\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{on } \partial\Omega \times (0, T], \tag{2.1c}$$

$$\underset{\sim}{u}(\underset{\sim}{x}, 0) = \underset{\sim}{u}^0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega; \tag{2.1d}$$

where $\nu \in \mathbb{R}_{>0}$ is the given viscosity, $\underset{\sim}{f}$ is the given density of the body forces acting on the fluid, and $\underset{\approx}{\tau}(\psi) : (\underset{\sim}{x}, t) \in \Omega \times (0, T) \mapsto \underset{\approx}{\tau}(\psi)(\underset{\sim}{x}, t) \in \mathbb{R}^{d\times d}$ is the symmetric extra-stress tensor, dependent on a probability density function $\psi : (\underset{\sim}{x}, \underset{\sim}{q}, t) \in \Omega \times D \times (0, T) \mapsto \psi(\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}$, defined as

$$\underset{\approx}{\tau}(\psi) = k\,\mu\,(\underset{\approx}{C}(\psi) - \rho(\psi)\,\underset{\approx}{I}). \tag{2.2}$$

Here k, $\mu \in \mathbb{R}_{>0}$ are, respectively, the Boltzmann constant and the absolute temperature, $\underset{\approx}{I}$ is the unit $d \times d$ tensor,

$$\underset{\approx}{C}(\psi)(\underset{\sim}{x},t) = \int_D \psi(\underset{\sim}{x},\underset{\sim}{q},t)\, U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\, \underset{\sim}{q}\,\underset{\sim}{q}^T \,\mathrm{d}\underset{\sim}{q} \qquad \text{and} \qquad \rho(\psi)(\underset{\sim}{x},t) = \int_D \psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}. \tag{2.3}$$

In addition, the real-valued, continuous, nonnegative and strictly monotonic increasing function U, defined on a relatively open subset of $[0,\infty)$, is an elastic potential which gives the elastic force $\underset{\sim}{F} : D \to \mathbb{R}^d$ on the springs *via* (1.2).

The probability density $\psi(\underset{\sim}{x},\underset{\sim}{q},t)$ represents the probability at time t of finding the centre of mass of a dumbbell in the volume element $\underset{\sim}{x} + \mathrm{d}\underset{\sim}{x}$ and having the endpoint of its elongation vector within the volume element $\underset{\sim}{q} + \mathrm{d}\underset{\sim}{q}$. Hence $\rho(\psi)(\underset{\sim}{x},t)$ is the density of the polymer chains located at $\underset{\sim}{x}$ at time t. The function ψ satisfies the following Fokker–Planck equation, together with suitable boundary and initial conditions:

$$\frac{\partial \psi}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\psi + \underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x\, \underset{\sim}{u})\, \underset{\sim}{q}\, \psi) = \frac{1}{2\,\lambda}\, \underset{\sim}{\nabla}_q \cdot (\underset{\sim}{\nabla}_q\, \psi + U'\, \underset{\sim}{q}\, \psi) + \varepsilon\, \Delta_x\, \psi \qquad \text{in } \Omega \times D \times (0,T], \tag{2.4a}$$

$$\left[\frac{1}{2\,\lambda}\, (\underset{\sim}{\nabla}_q\, \psi + U'\, \underset{\sim}{q}\, \psi) - (\underset{\approx}{\nabla}_x\, \underset{\sim}{u})\, \underset{\sim}{q}\, \psi\right] \cdot \underset{\sim}{n}_{\partial D} = 0 \qquad \text{on } \Omega \times \partial D \times (0,T], \tag{2.4b}$$

$$\varepsilon\, \underset{\sim}{\nabla}_x\, \psi \cdot \underset{\sim}{n}_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times D \times (0,T], \tag{2.4c}$$

$$\psi(\underset{\sim}{x},\underset{\sim}{q},0) = \psi^0(\underset{\sim}{x},\underset{\sim}{q}) \geq 0 \qquad \forall (\underset{\sim}{x},\underset{\sim}{q}) \in \Omega \times D; \tag{2.4d}$$

where $\underset{\sim}{n}_{\partial D}$ and $\underset{\sim}{n}_{\partial\Omega}$ are the outward unit normal vectors to ∂D and $\partial\Omega$, respectively, and $U' := U'(\frac{1}{2}|\underset{\sim}{q}|^2)$. Here $\int_D \psi^0(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$. The boundary conditions for ψ on $\Omega \times \partial D \times (0,T]$ and $\partial\Omega \times D \times (0,T]$ have been chosen so as to ensure that $\rho(\psi)(\underset{\sim}{x},t) = \int_D \psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q} = \int_D \psi^0(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q} = 1$ for a.e. $(\underset{\sim}{x},t) \in \Omega_T$. In (2.4a–c) the parameters ε, $\lambda \in \mathbb{R}_{>0}$, with λ characterizing the elastic relaxation property of the fluid, and $(\underset{\approx}{\nabla}_x\, \underset{\sim}{u})(\underset{\sim}{x},t) \in \mathbb{R}^{d\times d}$ with $\{\underset{\approx}{\nabla}_x\, \underset{\sim}{u}\}_{ij} = \dfrac{\partial u_i}{\partial x_j}$.

On introducing the (normalized) Maxwellian (1.6), we have that

$$M\, \underset{\sim}{\nabla}_q\, M^{-1} = -M^{-1}\, \underset{\sim}{\nabla}_q\, M = \underset{\sim}{\nabla}_q\, U = U'\, \underset{\sim}{q}. \tag{2.5}$$

Thus, the Fokker–Planck system (2.4a–d) can be rewritten in terms of the scaled probability density function $\widehat{\psi} = \psi/M$ as

$$M\left[\frac{\partial \widehat{\psi}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\widehat{\psi}\right] + \underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x\, \underset{\sim}{u})\, \underset{\sim}{q}\, M\widehat{\psi}) = \frac{1}{2\,\lambda}\, \underset{\sim}{\nabla}_q \cdot (M\, \underset{\sim}{\nabla}_q\, \widehat{\psi}) + \varepsilon\, M\, \Delta_x\, \widehat{\psi} \qquad \text{in } \Omega \times D \times (0,T], \tag{2.6a}$$

$$M\, \left[\frac{1}{2\,\lambda}\, (\underset{\sim}{\nabla}_q\, \widehat{\psi} - (\underset{\approx}{\nabla}_x\, \underset{\sim}{u})\, \underset{\sim}{q}\, \widehat{\psi}\right] \cdot \underset{\sim}{n}_{\partial D} = 0 \qquad \text{on } \Omega \times \partial D \times (0,T], \tag{2.6b}$$

$$\varepsilon\, M\, \underset{\sim}{\nabla}_x\, \widehat{\psi} \cdot \underset{\sim}{n}_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times D \times (0,T], \tag{2.6c}$$

$$M\, \widehat{\psi}(\underset{\sim}{x},\underset{\sim}{q},0) = M\widehat{\psi}^0(\underset{\sim}{x},\underset{\sim}{q}) = \psi^0(\underset{\sim}{x},\underset{\sim}{q}) \geq 0 \qquad \forall (\underset{\sim}{x},\underset{\sim}{q}) \in \Omega \times D. \tag{2.6d}$$

2.2. **FENE model**

We present an example of a spring potential: the FENE potential, where D is a bounded open ball in $\mathbb{R}^d$. In this widely used model

$$D = B(\underset{\sim}{0}, b^{\frac{1}{2}}) \quad \text{and} \quad U(s) = -\frac{b}{2}\ln\left(1 - \frac{2\,s}{b}\right), \quad \text{and hence} \quad \mathrm{e}^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)} = \left(1 - \frac{|\underset{\sim}{q}|^2}{b}\right)^{\frac{b}{2}}. \tag{2.7}$$

Here $B(\underset{\sim}{0}, s)$ is the bounded open ball of radius $s > 0$ in $\mathbb{R}^d$ centred at the origin, and $b > 0$ is an input parameter. Hence the length $|\underset{\sim}{q}|$ of the elongation vector $\underset{\sim}{q}$ cannot exceed $b^{\frac{1}{2}}$.

Letting $b \to \infty$ in (2.7) leads to the so-called Hookean dumbbell model where

$$D = \mathbb{R}^d \qquad \text{and} \qquad U(s) = s, \qquad \text{and therefore} \qquad \mathrm{e}^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)} = \mathrm{e}^{-\frac{1}{2}|\underset{\sim}{q}|^2}. \tag{2.8}$$

This particular kinetic model, with $\varepsilon \in \mathbb{R}_{>0}$, corresponds formally to a dissipative Oldroyd-B type model; see [4] for details.

2.3. General structural assumptions on the potential

As has been noted above, the choice of $D = \mathbb{R}^d$ (corresponding to the Hookean model) is physically unrealistic; thus, we shall henceforth suppose for simplicity that $D = B(\underset{\sim}{0}, r_D)$ is a bounded open ball in $\mathbb{R}^d$ of radius $r_D \in \mathbb{R}_{>0}$ centred at the origin. We assume that $\underset{\sim}{q} \mapsto U(\frac{1}{2}|\underset{\sim}{q}|^2) \in C^\infty(D)$; that $\underset{\sim}{q} \mapsto U(\frac{1}{2}|\underset{\sim}{q}|^2)$ is nonnegative, convex and has a positive definite Hessian at each $\underset{\sim}{q} \in D$; that $\underset{\sim}{q} \mapsto U'(\frac{1}{2}|\underset{\sim}{q}|^2)$ is positive on D; and that there exist constants $c_i > 0$, $i = 1 \to 5$, such that the Maxwellian M and the associated elastic potential U satisfy

$$c_1\,[\mathrm{dist}(\underset{\sim}{q},\,\partial D)]^\zeta \leq M(\underset{\sim}{q}) \;\leq c_2\,[\mathrm{dist}(\underset{\sim}{q},\,\partial D)]^\zeta \qquad \forall \underset{\sim}{q} \in D, \tag{2.9a}$$

$$c_3 \leq [\mathrm{dist}(\underset{\sim}{q},\,\partial D)]\,U'(\tfrac{1}{2}|\underset{\sim}{q}|^2) \leq c_4, \qquad [U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)]^2 \leq c_5\,U''(\tfrac{1}{2}|\underset{\sim}{q}|^2) \qquad \forall \underset{\sim}{q} \in D. \tag{2.9b}$$

It is an easy matter to show that the Maxwellian M and the elastic potential U of the FENE dumbbell model satisfy conditions (2.9a,b) with $D = B(\underset{\sim}{0}, b^{\frac{1}{2}})$ and $\zeta = \frac{b}{2}$. Since $[U(\underset{\sim}{q})]^2 = (-\ln M(\underset{\sim}{q}) + \mathrm{Const.})^2$, it follows from (2.9a,b) that if $\zeta > 1$, then

$$\int_D M\left[1 + U^2 + |U'|^2\right]\,\mathrm{d}\underset{\sim}{q} < \infty. \tag{2.10}$$

We shall therefore suppose that $\zeta > 1$. For the FENE model (2.7), $\zeta = \frac{b}{2}$, and so the condition $\zeta > 1$ translates into the requirement that $b > 2$. It is interesting to note that in the, equivalent, stochastic version of the FENE model, a solution to the system of stochastic differential equations associated with the Fokker–Planck equation exists and has trajectorial uniqueness if, and only if, $b > 2$ (cf. [22] for details). Thus, the assumption $\zeta > 1$ can be seen as the weakest reasonable requirement on the decay-rate of M as $\mathrm{dist}(\underset{\sim}{q}, \partial D) \to 0_+$.

2.4. Formal estimates

We end this section by identifying formally the energy structure for (P). Multiplying (2.1a) by $\underset{\sim}{u}$, integrating over Ω, and noting (2.1b,c) yields that

$$\begin{aligned}\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega |\underset{\sim}{u}|^2\,\mathrm{d}\underset{\sim}{x}\right] + \nu\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{u}|^2\,\mathrm{d}\underset{\sim}{x} - \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{u}\,\mathrm{d}\underset{\sim}{x} &= -\int_\Omega \underset{\approx}{\tau}(M\,\widehat{\psi}) : \underset{\approx}{\nabla}_x\,\underset{\sim}{u}\,\mathrm{d}\underset{\sim}{x}\\ &= -k\,\mu\int_\Omega \underset{\approx}{C}(M\,\widehat{\psi}) : \underset{\approx}{\nabla}_x\,\underset{\sim}{u}\,\mathrm{d}\underset{\sim}{x}.\end{aligned} \tag{2.11}$$

Let $\mathcal{F}(s) := (\ln s - 1)\,s + 1$ for $s > 0$, with $\mathcal{F}(0) := 1$. Multiplying the Fokker–Planck equation (2.6a) with $\mathcal{F}'(\widehat{\psi}) \equiv \ln\widehat{\psi}$, on assuming that $\widehat{\psi} > 0$, integrating over $\Omega \times D$ yields that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\Omega\times D} M\,\mathcal{F}(\widehat{\psi})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \int_{\Omega\times D} M\,\left[\frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_q\,[\mathcal{F}'(\widehat{\psi})] + \varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\,[\mathcal{F}'(\widehat{\psi})]\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}&\\ = \int_{\Omega\times D} M\,\widehat{\psi}\,[(\underset{\approx}{\nabla}_x\,\underset{\sim}{u})\,\underset{\sim}{q}]\cdot\underset{\sim}{\nabla}_q\,[\mathcal{F}'(\widehat{\psi})]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}.&\end{aligned} \tag{2.12}$$

It follows, on noting that $\mathcal{F}''(s) = s^{-1} > 0$ for $s > 0$ and hence that $\widehat{\psi}\,\underset{\sim}{\nabla}_q\,[\mathcal{F}'(\widehat{\psi})] = \underset{\sim}{\nabla}_q\,\widehat{\psi}$, (2.5), (2.1b) and $M = 0$ on ∂D that

$$
\begin{aligned}
\int_{\Omega\times D} M\,\widehat{\psi}\,[(\underset{\approx}{\nabla}_x\,\underset{\sim}{u})\,\underset{\sim}{q}]\cdot\underset{\sim}{\nabla}_q\,[\mathcal{F}'(\widehat{\psi})]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} &= \int_{\Omega\times D} M\,[(\underset{\approx}{\nabla}_x\,\underset{\sim}{u})\,\underset{\sim}{q}]\cdot\underset{\sim}{\nabla}_q\,\widehat{\psi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&= \int_{\Omega\times D} M\,U'\,\underset{\sim}{q}\cdot[(\underset{\approx}{\nabla}_x\,\underset{\sim}{u})\,\underset{\sim}{q}]\,\widehat{\psi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&= \int_{\Omega} \underset{\approx}{C}(M\,\widehat{\psi}) : \underset{\approx}{\nabla}_x\,\underset{\sim}{u}\,\mathrm{d}\underset{\sim}{x}, \qquad (2.13)
\end{aligned}
$$

on recalling (2.3). Combining (2.11)–(2.13), we obtain the following energy law for (P):

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_{\Omega}|\underset{\sim}{u}|^2\,\mathrm{d}\underset{\sim}{x} + k\,\mu\int_{\Omega\times D} M\,\mathcal{F}(\widehat{\psi})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \nu\int_{\Omega}|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}|^2\,\mathrm{d}\underset{\sim}{x} \\
&\quad + k\,\mu\int_{\Omega\times D} M\left[\frac{1}{2\,\lambda}\underset{\sim}{\nabla}_q\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_q\,[\mathcal{F}'(\widehat{\psi})] + \varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}\cdot\underset{\sim}{\nabla}_x\,[\mathcal{F}'(\widehat{\psi})]\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \int_{\Omega}\underset{\sim}{f}\cdot\underset{\sim}{u}\,\mathrm{d}\underset{\sim}{x}. \qquad (2.14)
\end{aligned}
$$

To make the above rigorous, and for computational purposes, we replace the convex function $\mathcal{F} \in C(\mathbb{R}_{\geq 0}) \cap C^{\infty}(\mathbb{R}_{>0})$ by the the convex regularization $\mathcal{F}_\delta^L \in C^{2,1}(\mathbb{R})$ defined, for any $\delta \in (0, 1)$ and $L > 1$, by

$$
\mathcal{F}_\delta^L(s) := \begin{cases} \frac{s^2-\delta^2}{2\,\delta} + (\ln\delta - 1)\,s + 1 & s \leq \delta, \\ \mathcal{F}(s) \equiv (\ln s - 1)\,s + 1 & \delta \leq s \leq L, \\ \frac{s^2-L^2}{2\,L} + (\ln L - 1)\,s + 1 & L \leq s. \end{cases} \qquad (2.15)
$$

Hence, it follows that

$$
[\mathcal{F}_\delta^L]'(s) = \begin{cases} \frac{s}{\delta} + \ln\delta - 1 & s \leq \delta, \\ \ln s & \delta \leq s \leq L, \\ \frac{s}{L} + \ln L - 1 & L \leq s, \end{cases} \quad \text{and} \quad [\mathcal{F}_\delta^L]''(s) = \begin{cases} \delta^{-1} & s \leq \delta, \\ s^{-1} & \delta \leq s \leq L, \\ L^{-1} & L \leq s. \end{cases} \qquad (2.16)
$$

In addition, we introduce

$$
\beta_\delta^L(s) := [[\mathcal{F}_\delta^L]''(s)]^{-1} = \begin{cases} \delta & s \leq \delta, \\ s & \delta \leq s \leq L, \\ L & L \leq s. \end{cases} \qquad (2.17)
$$

It follows from (2.17) for any sufficiently smooth $\widehat{\varphi}$ that

$$
\beta_\delta^L(\widehat{\varphi})\,\underset{\sim}{\nabla}_x\,([\mathcal{F}_\delta^L]'(\widehat{\varphi})\,) = \underset{\sim}{\nabla}_x\,\widehat{\varphi} \qquad \text{and} \qquad \beta_\delta^L(\widehat{\varphi})\,\underset{\sim}{\nabla}_q\,([\mathcal{F}_\delta^L]'(\widehat{\varphi})\,) = \underset{\sim}{\nabla}_q\,\widehat{\varphi}. \qquad (2.18)
$$

Let $\{\underset{\sim}{u}_\delta^L, \widehat{\psi}_\delta^L\}$ solve problem (P_δ^L), which is a regularization of the problem (P) where the drag term $\underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x\,\underset{\sim}{u})\,\underset{\sim}{q}\,\widehat{\psi})$ in the Fokker–Planck equation (2.6a) is replaced by

$$
\underset{\sim}{\nabla}_q\,\cdot((\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_\delta^L)\,\underset{\sim}{q}\,\beta_\delta^L(\widehat{\psi}_\delta^L)). \qquad (2.19)
$$

Multiplying the Fokker–Planck equation in (P_δ^L) with $[\mathcal{F}_\delta^L]'(\widehat{\psi}_\delta^L)$, integrating over $\Omega \times D$, noting (2.18) yields, similarly to (2.12) and (2.13), that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\Omega\times D} M\,\mathcal{F}_\delta^L(\widehat{\psi}_\delta^L)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \frac{1}{2\,\lambda}\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_\delta^L \cdot \underset{\sim}{\nabla}_q\,\left[[\mathcal{F}_\delta^L]'(\widehat{\psi}_\delta^L)\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$+\,\varepsilon\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_\delta^L \cdot \underset{\sim}{\nabla}_x\,\left[[\mathcal{F}_\delta^L]'(\widehat{\psi}_\delta^L)\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \int_\Omega \underset{\approx}{C}(M\,\widehat{\psi}_\delta^L) : \underset{\approx}{\nabla}_x\,\underset{\sim}{u}_\delta^L\,\mathrm{d}\underset{\sim}{x}. \tag{2.20}$$

Combining (2.20) and the (P_δ^L) version of (2.11), we obtain the following energy law for (P_δ^L), the regularized analogue of (2.14),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_\Omega |\underset{\sim}{u}_\delta^L|^2\,\mathrm{d}\underset{\sim}{x} + k\,\mu\int_{\Omega\times D} M\,\mathcal{F}_\delta^L(\widehat{\psi}_\delta^L)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \nu\int_\Omega |\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_\delta^L|^2\,\mathrm{d}\underset{\sim}{x}$$
$$+\,k\,\mu\int_{\Omega\times D} M\left[\frac{1}{2\,\lambda}\underset{\sim}{\nabla}_q\,\widehat{\psi}_\delta^L\cdot\underset{\sim}{\nabla}_q\,\left[[\mathcal{F}_\delta^L]'(\widehat{\psi}_\delta^L)\right] + \varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_\delta^L\cdot\underset{\sim}{\nabla}_x\,\left[[\mathcal{F}_\delta^L]'(\widehat{\psi}_\delta^L)\right]\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{u}_\delta^L\,\mathrm{d}\underset{\sim}{x}. \tag{2.21}$$

On noting that $[\mathcal{F}_\delta^L]'' \geq L^{-1}$, and

$$\min\{\mathcal{F}_\delta^L(s), s\,[\mathcal{F}_\delta^L]'(s)\} \geq \begin{cases} \frac{s^2}{2\,\delta} & \text{if } s \leq 0, \\ \frac{s^2}{4\,L} - C(L) & \text{if } s \geq 0, \end{cases} \tag{2.22}$$

one can establish from (2.21), on assuming that $\widehat{\psi}^0 \leq L$, that

$$\sup_{t\in(0,T)}\left[\int_\Omega |\underset{\sim}{u}_\delta^L|^2\,\mathrm{d}\underset{\sim}{x}\right] + \nu\int_{\Omega_T}|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_\delta^L|^2\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t + \delta^{-1}\sup_{t\in(0,T)}\left[\int_{\Omega\times D} M\,|[\widehat{\psi}_\delta^L]_-|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] \leq C. \tag{2.23}$$

In addition, one can establish that

$$\sup_{t\in(0,T)}\left[\int_{\Omega\times D} M\,|\widehat{\psi}_\delta^L|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \frac{1}{\lambda}\int_0^T\int_{\Omega\times D} M\left|\underset{\sim}{\nabla}_q\,\widehat{\psi}_\delta^L\right|^2\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+\,\varepsilon\int_0^T\int_{\Omega\times D} M\left|\underset{\sim}{\nabla}_x\,\widehat{\psi}_\delta^L\right|^2\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t + \sup_{t\in(0,T)}\left[\int_\Omega |\underset{\approx}{C}(M\,\widehat{\psi}_\delta^L)|^2\,\mathrm{d}\underset{\sim}{x}\right] \leq C(L,T). \tag{2.24}$$

The above formal bounds have been made rigorous and the existence of a global-in-time weak solution $\{\underset{\sim}{u}_\delta^L, \widehat{\psi}_\delta^L\}$ to (P_δ^L) has been established in [5], see also the next section. Moreover, one can take the limit $\delta \to 0_+$ in problem (P_δ^L) to establish the existence of a global-in-time weak solution $\{\underset{\sim}{u}^L, \widehat{\psi}^L\}$ to problem (P^L), which is a regularization of the problem (P) where the drag term $\underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x\,\underset{\sim}{u})\,\underset{\sim}{q}\,\widehat{\psi})$ in the Fokker–Planck equation (2.6a) is replaced by

$$\underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x\,\underset{\sim}{u}^L)\,\underset{\sim}{q}\,\beta^L(\widehat{\psi}_\delta^L)) \qquad \text{with} \qquad \beta^L(s) := \begin{cases} s & s \leq L, \\ L & L \leq s. \end{cases} \tag{2.25}$$

Once again, see [5] and the next section.

The aim of this paper is to construct a finite element approximation of problem (P_δ^L), which mimics the energy law (2.21) at a discrete level. Moreover, show that this approximation converges to a weak solution of (P^L), as the spatial discretization parameter h and the time step parameter Δt, as well as the regularization parameter δ, tend to zero.

3. Function spaces

Assuming that $\partial\Omega \in C^{0,1}$, let

$$\underset{\sim}{H} := \{\underset{\sim}{w} \in \underset{\sim}{L}^2(\Omega) : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} = 0\} \quad \text{and} \quad \underset{\sim}{V} := \{\underset{\sim}{w} \in \underset{\sim}{H}^1_0(\Omega) : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} = 0\}, \tag{3.1}$$

where the divergence operator $\underset{\sim}{\nabla}_x \cdot$ is to be understood in the sense of vector-valued distributions on Ω. Here, and throughout, we adopt, for example, the notation $\underset{\sim}{L}^2(\Omega) \equiv [L^2(\Omega)]^d$ and $\underset{\sim}{H}^1_0(\Omega) \equiv [H^1_0(\Omega)]^d$. Let $\underset{\sim}{V}'$ be the dual of $\underset{\sim}{V}$. Let $\underset{\sim}{S} : \underset{\sim}{V}' \to \underset{\sim}{V}$ be such that $\underset{\sim}{S}\,\underset{\sim}{v}$ is the unique solution to the Helmholtz–Stokes problem

$$\int_\Omega \underset{\sim}{S}\,\underset{\sim}{v} \cdot \underset{\sim}{w} \,\mathrm{d}\underset{\sim}{x} + \int_\Omega \underset{\approx}{\nabla}_x\, \underset{\sim}{S}\,\underset{\sim}{v} : \underset{\approx}{\nabla}_x\, \underset{\sim}{w} \,\mathrm{d}\underset{\sim}{x} = \langle \underset{\sim}{v}, \underset{\sim}{w}\rangle_V \qquad \forall \underset{\sim}{w} \in \underset{\sim}{V}, \tag{3.2}$$

where $\langle\cdot,\cdot\rangle_V$ denotes the duality pairing between $\underset{\sim}{V}'$ and $\underset{\sim}{V}$. We note that

$$\langle \underset{\sim}{v}, \underset{\sim}{S}\,\underset{\sim}{v}\rangle_V = \|\underset{\sim}{S}\,\underset{\sim}{v}\|^2_{H^1(\Omega)} \qquad \forall \underset{\sim}{v} \in \underset{\sim}{V}' \supset (\underset{\sim}{H}^1_0(\Omega))' \equiv \underset{\sim}{H}^{-1}(\Omega), \tag{3.3}$$

and $\|\underset{\sim}{S}\cdot\|_{H^1(\Omega)}$ is a norm on $\underset{\sim}{V}'$. Here, and throughout, we adopt, for example, the notation $\|\cdot\|_{H^1(\Omega)}$ for the norm, and $|\cdot|_{H^1(\Omega)}$ for the semi-norm, on $H^1(\Omega)$ or $\underset{\sim}{H}^1(\Omega)$. We require also the duality pairing $\langle\cdot,\cdot\rangle_{H^1_0(\Omega)}$ between $\underset{\sim}{H}^{-1}(\Omega)$ and $\underset{\sim}{H}^1_0(\Omega)$.

For later purposes, we recall the following well-known Gagliardo–Nirenberg inequality. Let $r \in [2,\infty)$ if $d = 2$, and $r \in [2,6]$ if $d = 3$ and $\theta = d\left(\frac{1}{2} - \frac{1}{r}\right)$. Then, there is a constant C, depending only on Ω, r and d, such that the following inequality holds for all $\eta \in H^1(\Omega)$:

$$\|\eta\|_{L^r(\Omega)} \leq C\, \|\eta\|^{1-\theta}_{L^2(\Omega)}\, \|\eta\|^{\theta}_{H^1(\Omega)}. \tag{3.4}$$

We make the following assumptions on the given initial data and the cut-off parameter L occurring in (2.15):

$$\underset{\sim}{u}^0 \in \underset{\sim}{H} \qquad \text{and} \qquad \widehat{\psi}^0 := M^{-1}\psi^0 \in L^\infty(\Omega \times D) \quad \text{with} \quad 0 \leq \widehat{\psi}^0 \leq L \ \text{ a.e. in } \Omega \times D; \tag{3.5a}$$

and the body force density

$$\underset{\sim}{f} \in L^2(0,T;\underset{\sim}{H}^{-1}(\Omega)). \tag{3.5b}$$

Let $L^2_M(\Omega\times D)$ be the Maxwellian-weighted L^2 space over $\Omega \times D$ with norm

$$\|\widehat{\varphi}\|_{L^2_M(\Omega\times D)} := \left\{\int_{\Omega\times D} M\,|\widehat{\varphi}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right\}^{\frac{1}{2}}.$$

Similarly, we consider $L^2_M(D)$, the Maxwellian-weighted L^2 space over D. On introducing

$$\|\widehat{\varphi}\|_{H^1_M(\Omega\times D)} := \left\{\int_{\Omega\times D} M\,\left[|\widehat{\varphi}|^2 + |\underset{\sim}{\nabla}_x\,\widehat{\varphi}|^2 + |\underset{\sim}{\nabla}_q\,\widehat{\varphi}|^2\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right\}^{\frac{1}{2}}, \tag{3.6}$$

we then set

$$\widehat{X} \equiv H^1_M(\Omega\times D) := \left\{\widehat{\varphi} \in L^1_{\mathrm{loc}}(\Omega\times D) : \|\widehat{\varphi}\|_{H^1_M(\Omega\times D)} < \infty\right\}. \tag{3.7}$$

It follows that

$$C^\infty(\overline{\Omega\times D}) \text{ is dense in } \widehat{X}. \tag{3.8}$$

This can be shown, for example, by a simple adaptation of Lemma 3.1 in Barrett, Schwab, and Süli [3], which appeals to fundamental results on weighted Sobolev spaces in Triebel [41] and Kufner [27]. We have from Sobolev embedding that

$$L^s(\Omega; L^2_M(D)) \hookrightarrow H^1(\Omega; L^2_M(D)), \tag{3.9}$$

where $s \in [1,\infty)$ if $d = 2$ or $s \in [1,6]$ if $d = 3$. Similarly to (3.4) we have, with r and θ as defined there, that there exists a constant C, depending only on Ω, r and d, such that

$$\|\widehat{\varphi}\|_{L^r(\Omega;L^2_M(D))} \leq C\,\|\widehat{\varphi}\|^{1-\theta}_{L^2(\Omega;L^2_M(D))}\,\|\widehat{\varphi}\|^{\theta}_{H^1(\Omega;L^2_M(D))} \qquad \forall \widehat{\varphi} \in H^1(\Omega; L^2_M(D)). \tag{3.10}$$

In addition, we note that the embeddings

$$L^2_M(D) \hookrightarrow H^1_M(D), \tag{3.11a}$$
$$L^2_M(\Omega\times D) \equiv L^2(\Omega; L^2_M(D)) \hookrightarrow H^1_M(\Omega\times D) \equiv L^2(\Omega; H^1_M(D)) \cap H^1(\Omega; L^2_M(D)) \tag{3.11b}$$

are compact if $\zeta \geq 1$ in (2.9a); see the Appendix in [5].

Let $\widehat{X}'$ be the dual space of $\widehat{X}$ with $L^2_M(\Omega\times D)$ being the pivot space. Then, similarly to (3.2), let $\mathcal{G} : \widehat{X}' \to \widehat{X}$ be such that $\mathcal{G}\,\widehat{\eta}$ is the unique solution of

$$\int_{\Omega\times D} M \left[(\mathcal{G}\,\widehat{\eta})\,\widehat{\varphi} + \underset{\sim}{\nabla}_q\ (\mathcal{G}\,\widehat{\eta}) \cdot \underset{\sim}{\nabla}_q\ \widehat{\varphi} + \underset{\sim}{\nabla}_x\ (\mathcal{G}\,\widehat{\eta}) \cdot \underset{\sim}{\nabla}_x\ \widehat{\varphi}\right] \mathrm{d}\underset{\sim}{q}\ \mathrm{d}\underset{\sim}{x} = \langle M\,\widehat{\eta}, \widehat{\varphi}\rangle_{\widehat{X}} \qquad \forall \widehat{\varphi} \in \widehat{X}, \tag{3.12}$$

where $\langle M\,\cdot,\cdot\rangle_{\widehat{X}}$ denotes the duality pairing between $\widehat{X}'$ and $\widehat{X}$. Then, similarly to (3.3), we have that

$$\langle M\,\widehat{\eta}, \mathcal{G}\,\widehat{\eta}\rangle_{\widehat{X}} = \|\mathcal{G}\,\widehat{\eta}\|^2_{\widehat{X}} \qquad \forall \widehat{\eta} \in \widehat{X}', \tag{3.13}$$

and $\|\mathcal{G}\cdot\|_{\widehat{X}}$ is a norm on $\widehat{X}'$.

We recall the following compactness result, see, e.g., Temam [40] and Simon [39]. Let $\mathcal{Y}_0$, $\mathcal{Y}$ and $\mathcal{Y}_1$ be Banach spaces, $\mathcal{Y}_i$, $i = 0,1$, reflexive, with a compact embedding $\mathcal{Y}_0 \hookrightarrow \mathcal{Y}$ and a continuous embedding $\mathcal{Y} \hookrightarrow \mathcal{Y}_1$. Then, for $\alpha_i > 1$, $i = 0,1$, the embedding

$$\{\,\eta \in L^{\alpha_0}(0,T;\mathcal{Y}_0) : \tfrac{\partial\eta}{\partial t} \in L^{\alpha_1}(0,T;\mathcal{Y}_1)\,\} \hookrightarrow L^{\alpha_0}(0,T;\mathcal{Y}) \tag{3.14}$$

is compact.

We note for future reference that (2.3) and (2.10) yield that, for $\widehat{\varphi} \in L^2_M(\Omega\times D)$,

$$\begin{aligned}\int_\Omega |\underset{\approx}{C}(M\,\widehat{\varphi})|^2\ \mathrm{d}\underset{\sim}{x} &= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \left(\int_D M\,\widehat{\varphi}\,U'\,q_i\,q_j\ \mathrm{d}\underset{\sim}{q}\right)^2 \mathrm{d}\underset{\sim}{x} \\ &\leq \left(\int_D M\,|U'|^2\,|\underset{\sim}{q}|^4\ \mathrm{d}\underset{\sim}{q}\right)\left(\int_{\Omega\times D} M\,|\widehat{\varphi}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right) \leq C\left(\int_{\Omega\times D} M\,|\widehat{\varphi}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right).\end{aligned} \tag{3.15}$$

In [5], for any $\varepsilon > 0$, $L > 1$ and $T > 0$ existence of a solution to the following weak formulation was established:

(P^L) Find $\underset{\sim}{u}^L \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}')$ and $\widehat{\psi}^L \in L^\infty(0,T;L^2_M(\Omega\times D)) \cap L^2(0,T;\widehat{X}) \cap W^{1,\frac{4}{d}}(0,T;\widehat{X}')$ with $\underset{\approx}{C}(M\,\widehat{\psi}^L) \in L^\infty(0,T;\underset{\approx}{L}^2(\Omega))$, such that $\underset{\sim}{u}^L(\cdot,0) = \underset{\sim}{u}^0(\cdot)$, $\widehat{\psi}^L(\cdot,\cdot,0) = \widehat{\psi}^0(\cdot,\cdot)$ and

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}^L}{\partial t}, \underset{\sim}{w} \right\rangle_V \mathrm{d}t + \int_{\Omega_T} \left[\left[(\underset{\sim}{u}^L \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u}^L \right] \cdot \underset{\sim}{w} + \nu \underset{\approx}{\nabla}_x \underset{\sim}{u}^L : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$= \int_0^T \langle \underset{\sim}{f}, \underset{\sim}{w} \rangle_{H^1_0(\Omega)}\,\mathrm{d}t - k\,\mu \int_{\Omega_T} \underset{\approx}{C}(M\,\widehat{\psi}^L) : \underset{\approx}{\nabla}_x \underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{4}{4-d}}(0,T;\underset{\sim}{V}); \tag{3.16a}$$

$$\int_0^T \left\langle \frac{\partial \widehat{\psi}^L}{\partial t}, \widehat{\varphi} \right\rangle_{\widehat{X}} \mathrm{d}t + \int_0^T \int_{\Omega\times D} M \left[\varepsilon \underset{\sim}{\nabla}_x \widehat{\psi}^L - \underset{\sim}{u}^L \widehat{\psi}^L \right] \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$+ \int_0^T \int_{\Omega\times D} \left[\frac{M}{2\,\lambda} \underset{\sim}{\nabla}_q \widehat{\psi}^L - (\underset{\approx}{\nabla}_x \underset{\sim}{u}^L)\,\underset{\sim}{q}\,\beta^L(\widehat{\psi}^L) \right] \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t = 0 \qquad \forall \widehat{\varphi} \in L^{\frac{4}{4-d}}(0,T;\widehat{X}). \tag{3.16b}$$

Remark 3.1. If $d=2$, then $\underset{\sim}{u}^L \in C([0,T];\underset{\sim}{H})$ (cf. Lemma 1.2 on p. 176 of Temam [40]), whereas if $d=3$, then $\underset{\sim}{u}^L$ is weakly continuous only as a mapping from $[0,T]$ into $\underset{\sim}{H}$ (similarly as in Theorem 3.1 on p. 191 in Temam [40]). It is in the latter, weaker sense that the imposition of the initial condition to the $\underset{\sim}{u}^L$-equation will be understood for $d=2,3$: that is, $\lim_{t\to 0_+} \int_\Omega (\underset{\sim}{u}^L(\underset{\sim}{x},t) - \underset{\sim}{u}_0(\underset{\sim}{x})) \cdot \underset{\sim}{v}(\underset{\sim}{x})\,\mathrm{d}\underset{\sim}{x} = 0$ for all $\underset{\sim}{v} \in \underset{\sim}{H}$. Similarly, for the initial conditions of the $\widehat{\psi}^L$-equation for $d=2,3$: $\lim_{t\to 0_+} \int_{\Omega\times D} M\,(\widehat{\psi}^L(\underset{\sim}{x},\underset{\sim}{q},t) - \widehat{\psi}_0(\underset{\sim}{x},\underset{\sim}{q}))\,\widehat{\varphi}(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 0$ for all $\widehat{\varphi} \in L^2_M(\Omega\times D)$.

Remark 3.2. Since the test functions in $\underset{\sim}{V}$ are divergence-free, the pressure has been eliminated in (3.16a,b); it can be recovered in a very weak sense following the same procedure as for the incompressible Navier–Stokes equations discussed on p.208 in Temam [40]; *i.e.*, one obtains that $\int_0^t p^L(\cdot,s)\,\mathrm{d}s \in C([0,T];L^2(\Omega))$.

4. Finite element approximation

Let us denote the measure of a bounded open region $\omega \subset \mathbb{R}^d$ by $\underline{m}(\omega)$. We make the following assumption on Ω and the partitions of Ω and D.

(A1) For ease of exposition, we shall assume that Ω is a convex polytope. Let $\{\mathcal{T}^x_h\}_{h>0}$ be a quasiuniform family of partitions of Ω into disjoint open nonobtuse simplices κ_x, so that

$$\overline{\Omega} \equiv \bigcup_{\kappa_x \in \mathcal{T}^x_h} \overline{\kappa_x} \quad \text{with} \quad h_{\kappa_x} := \mathrm{diam}(\kappa_x), \quad h_x := \max_{\kappa_x \in \mathcal{T}^x_h} h_{\kappa_x} \le \mathrm{diam}(\Omega)\,h \quad \text{and} \quad \underline{m}(\kappa_x) \ge C\,h^d.$$

Let $\{\mathcal{T}^q_h\}_{h>0}$ be a quasiuniform family of partitions of $D \equiv B(\underset{\sim}{0}, r_D)$, $r_D \in \mathbb{R}_{>0}$, into disjoint open nonobtuse simplices κ_q, with possibly one curved edge, $d=2$, or face, $d=3$, on ∂D; so that

$$\overline{D} \equiv \bigcup_{\kappa_q \in \mathcal{T}^q_h} \overline{\kappa_q} \quad \text{with} \quad h_{\kappa_q} := \mathrm{diam}(\kappa_q), \quad h_q := \max_{\kappa_q \in \mathcal{T}^q_h} h_{\kappa_q} \le \mathrm{diam}(D)\,h \quad \text{and} \quad \underline{m}(\kappa_q) \ge C\,h^d.$$

A "simplex" κ_q with a curved edge/face is nonobtuse if it is convex and the enclosed simplex with the same vertices is nonobtuse, in the sense that all of its dihedral angles are $\le \pi/2$. It follows from the above that

$$\frac{h_x}{h_q} + \frac{h_q}{h_x} \le C \qquad \text{as} \qquad h \to 0_+. \tag{4.1}$$

We note that such nonobtuse simplicial partitions of Ω and D are easily constructed in the case $d = 2$. For the construction of nonobtuse three-dimensional simplicial partitions we refer to the papers of Korotov and Křížek, [25] and [26], for example; the reader should note, however, that in [25] the authors use the term *acute* when they mean *nonobtuse*. Elsewhere in the computational geometry literature the term *acute* is reserved for a simplicial partition where all dihedral angles of any simplex in the partition are $< \pi/2$, which is a more restrictive requirement (especially in the case of $d = 3$) than what we assume here; see, for example, the articles of Brandts, Korotov, Křížek and Šolc [9], Eppstein, Sullivan and Üngör [18], and Itoh and Zamfirescu [21], and references therein. Nonobtuse simplicial partitions are sometimes also called *weakly acute* (cf. [38], p. 363).

We adopt the standard notation for L^2 inner products:

$$(\eta_1, \eta_2)_\Omega := \int_\Omega \eta_1\, \eta_2 \,\mathrm{d}\underset{\sim}{x} \quad \forall \eta_i \in L^2(\Omega) \qquad \text{and} \qquad (\eta_1, \eta_2)_{\Omega\times D} := \int_{\Omega\times D} \eta_1\, \eta_2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \quad \forall \eta_i \in L^2(\Omega \times D), \tag{4.2}$$

which are naturally extended to vector/matrix functions.

Let $\mathbb{P}^x_k$ and $\mathbb{P}^q_k$ denote polynomials of degree less than or equal to k in $\underset{\sim}{x}$ and $\underset{\sim}{q}$, respectively. We approximate the pressure and velocity with the lowest order Taylor–Hood element; that is,

$$R_h := \{\eta_h \in C(\overline{\Omega}) : \eta_h\,|_{\kappa_x} \in \mathbb{P}^x_1 \quad \forall \kappa_x \in \mathcal{T}^x_h\}, \tag{4.3a}$$

$$R^0_h := \{\eta_h \in R_h : (\eta_h, 1)_\Omega = 0\}, \tag{4.3b}$$

$$\underset{\sim}{W}_h := \{\underset{\sim}{w}_h \in [C(\overline{\Omega})]^d : \underset{\sim}{w}_h\,|_{\kappa_x} \in [\mathbb{P}^x_2]^d \quad \forall \kappa_x \in \mathcal{T}^x_h \text{ and } \underset{\sim}{w}_h = 0 \text{ on } \partial\Omega\} \subset [H^1_0(\Omega)]^d, \tag{4.3c}$$

$$\underset{\sim}{V}_h := \{\underset{\sim}{v}_h \in \underset{\sim}{W}_h : (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{v}_h, \eta_h)_\Omega = 0 \quad \forall \eta_h \in R_h\}. \tag{4.3d}$$

It is well-known that R^0_h and $\underset{\sim}{W}_h$ satisfy the inf-sup condition

$$\sup_{\underset{\sim}{w}_h \in \underset{\sim}{W}_h} \frac{(\underset{\sim}{\nabla}_x \cdot \underset{\sim}{w}_h, r_h)_\Omega}{\|\underset{\sim}{w}_h\|_{H^1(\Omega)}} \geq C_0\, \|r_h\|_{L^2(\Omega)} \qquad \forall r_h \in R^0_h, \tag{4.4}$$

see e.g. [10, §VI.6]. Hence for all $\underset{\sim}{v} \in \underset{\sim}{V}$, there exists a sequence $\{\underset{\sim}{v}_h\}_{h>0}$, with $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$, such that

$$\lim_{h\to 0_+} \|\underset{\sim}{v} - \underset{\sim}{v}_h\|_{H^1(\Omega)} = 0. \tag{4.5}$$

We require the L^2 projector $\underset{\sim}{Q}_h : \underset{\sim}{V} \to \underset{\sim}{V}_h$ defined by

$$(\underset{\sim}{v} - \underset{\sim}{Q}_h\,\underset{\sim}{v}, \underset{\sim}{w}_h)_\Omega = 0 \qquad \forall \underset{\sim}{w}_h \in \underset{\sim}{V}_h. \tag{4.6}$$

We note that the quasiuniformity of $\{\mathcal{T}^x_h\}_{h>0}$ implies that $\underset{\sim}{Q}_h$ is uniformly $H^1(\Omega)$ stable; that is,

$$\|\underset{\sim}{Q}_h\,\underset{\sim}{v}\|_{H^1(\Omega)} \leq C\, \|\underset{\sim}{v}\|_{H^1(\Omega)} \qquad \forall \underset{\sim}{v} \in \underset{\sim}{V}, \tag{4.7}$$

see [20].

For the approximation of the advection term in the Navier–Stokes equation we note that, for all $\underset{\sim}{v} \in \underset{\sim}{V}$ and $\underset{\sim}{w}, \underset{\sim}{z} \in \underset{\sim}{H}^1(\Omega)$, we have that

$$(\,(\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w}, \underset{\sim}{z})_\Omega \equiv \tfrac{1}{2}\left[(\,(\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w}, \underset{\sim}{z})_\Omega - (\,(\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{z}, \underset{\sim}{w})_\Omega\right]. \tag{4.8}$$

In addition, the choice $\underset{\sim}{w} = \underset{\sim}{z}$ leads to both sides of (4.8) vanishing. Obviously as $\underset{\sim}{V}_h \not\subset \underset{\sim}{V}$, the discrete analogue of the above does not hold; that is, it is *not* generally true that, for all $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$, $\underset{\sim}{w}_h, \underset{\sim}{z}_h \in \underset{\sim}{W}_h$,

$$((\underset{\sim}{v}_h \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w}_h, \underset{\sim}{z}_h)_\Omega \equiv \tfrac{1}{2}\left[((\underset{\sim}{v}_h \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w}_h, \underset{\sim}{z}_h)_\Omega - ((\underset{\sim}{v}_h \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{z}_h, \underset{\sim}{w}_h)_\Omega\right]. \tag{4.9}$$

We note that the right-hand side of (4.9) vanishes if $\underset{\sim}{w}_h = \underset{\sim}{z}_h$, which is not necessarily true for the left-hand side. Hence, we use the right-hand side form of (4.9) for the approximation of the advection term in the Navier–Stokes equation.

To approximate $\widehat{X}$, we first introduce

$$\widehat{X}_h^x := \{\widehat{\varphi}_h^x \in C(\overline{\Omega}) : \widehat{\varphi}_h^x\,|_{\kappa_x} \in \mathbb{P}_1^x \quad \forall \kappa_x \in \mathcal{T}_h^x\} \subset W^{1,\infty}(\Omega), \tag{4.10a}$$

$$\widehat{X}_h^q := \{\widehat{\varphi}_h^q \in C(\overline{D}) : \widehat{\varphi}_h^q\,|_{\kappa_q} \in \mathbb{P}_1^q \quad \forall \kappa_q \in \mathcal{T}_h^q\} \subset W^{1,\infty}(D). \tag{4.10b}$$

We then set

$$\widehat{X}_h := \widehat{X}_h^x \otimes \widehat{X}_h^q \subset \widehat{X}. \tag{4.11}$$

We note from (4.3a,d), (4.10a) and (4.11) that, for any $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ and any $\underset{\sim}{q} \in \overline{D}$,

$$(\underset{\sim}{\nabla}_x \cdot \underset{\sim}{v}_h, \widehat{\varphi}_h(\cdot, \underset{\sim}{q}))_\Omega = 0 \qquad \forall \widehat{\varphi}_h \in \widehat{X}_h. \tag{4.12}$$

We note that for (4.12) to hold in general, we require that $\widehat{X}_h^x \subseteq R_h$.

We introduce the interpolation operators $\pi_h^x : C(\overline{\Omega}) \to \widehat{X}_h^x$ and $\pi_h^q : C(\overline{D}) \to \widehat{X}_h^q$ such that

$$\pi_h^x\,\widehat{\varphi}^x(\underset{\sim}{P}_i^x) = \widehat{\varphi}^x(\underset{\sim}{P}_i^x), \quad i = 1 \to I^x, \qquad \text{and} \qquad \pi_h^q\,\widehat{\varphi}^q(\underset{\sim}{P}_i^q) = \widehat{\varphi}^q(\underset{\sim}{P}_i^q), \quad i = 1 \to I^q, \tag{4.13}$$

where $\{\underset{\sim}{P}_i^x\}_{i=1}^{I^x}$ and $\{\underset{\sim}{P}_i^q\}_{i=1}^{I^q}$ are the nodes (vertices) of $\mathcal{T}_h^x$ and $\mathcal{T}_h^q$, respectively. The associated basis functions are

$$\chi_i^x \in \widehat{X}_h^x \quad \text{such that} \quad \chi_i^x(P_j^x) = \delta_{ij} \quad \text{for } i,\, j = 1 \to I^x, \tag{4.14a}$$

$$\text{and} \qquad \chi_i^q \in \widehat{X}_h^q \quad \text{such that} \quad \chi_i^q(P_j^q) = \delta_{ij} \quad \text{for } i,\, j = 1 \to I^q. \tag{4.14b}$$

We introduce also $\pi_h : C(\overline{\Omega \times D}) \to \widehat{X}_h$ such that

$$(\pi_h\,\widehat{\varphi})(\underset{\sim}{P}_i^x, \underset{\sim}{P}_j^q) = \widehat{\varphi}(\underset{\sim}{P}_i^x, \underset{\sim}{P}_j^q) \quad \text{for } i = 1 \to I^x, \quad j = 1 \to I^q. \tag{4.15}$$

Of course, we have that $\pi_h \equiv \pi_h^x\,\pi_h^q \equiv \pi_h^q\,\pi_h^x$. The vector versions of the above interpolation operators are

$$\underset{\sim}{\pi}_h^x : [C(\overline{\Omega})]^d \to [\widehat{X}_h^x]^d, \qquad \underset{\sim}{\pi}_h^q : [C(\overline{D})]^d \to [\widehat{X}_h^q]^d \qquad \text{and} \qquad \underset{\sim}{\pi}_h : [C(\overline{\Omega \times D})]^d \to [\widehat{X}_h]^d. \tag{4.16}$$

We require also the local interpolation operators

$$\begin{aligned} &\pi_{h,\kappa_x}^x \equiv \pi_h^x\,|_{\kappa_x}, \qquad \pi_{h,\kappa_q}^q \equiv \pi_h^q\,|_{\kappa_q}, \qquad \pi_{h,\kappa_x \times \kappa_q} \equiv \pi_h\,|_{\kappa_x \times \kappa_q}, \qquad \underset{\sim}{\pi}_{h,\kappa_x}^x \equiv \underset{\sim}{\pi}_h^x\,|_{\kappa_x}, \\ &\underset{\sim}{\pi}_{h,\kappa_q}^q \equiv \underset{\sim}{\pi}_h^q\,|_{\kappa_q} \qquad \text{and} \qquad \underset{\sim}{\pi}_{h,\kappa_x \times \kappa_q} \equiv \underset{\sim}{\pi}_h\,|_{\kappa_x \times \kappa_q} \qquad\qquad \forall \kappa_x \in \mathcal{T}_h^x, \quad \forall \kappa_q \in \mathcal{T}_h^q. \end{aligned} \tag{4.17}$$

For any $\widehat{\varphi}_h \in \widehat{X}_h$, there exist $[\underset{\approx}{\Xi}^q_\delta(\widehat{\varphi}_h)](\underset{\sim}{x},\underset{\sim}{q})$, $[\underset{\approx}{\Xi}^x_\delta(\widehat{\varphi}_h)](\underset{\sim}{x},\underset{\sim}{q}) \in \mathbb{R}^{d\times d}$ for a.e. $(\underset{\sim}{x},\underset{\sim}{q}) \in \Omega \times D$ such that on $\kappa_x \times \kappa_q$, for all $\kappa_x \in \mathcal{T}^x_h$, $\kappa_q \in \mathcal{T}^q_h$,

$$\underset{\approx}{\Xi}^x_\delta(\widehat{\varphi}_h) \in [\mathbb{P}^q_1]^{d\times d} \qquad \text{and} \qquad \underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}\left[\underset{\approx}{\Xi}^x_\delta(\widehat{\varphi}_h)\,\underset{\sim}{\nabla}_x\left(\pi_h[\,[\mathcal{F}^L_\delta]'(\widehat{\varphi}_h)]\,\right)\right] = \underset{\sim}{\nabla}_x\,\widehat{\varphi}_h; \tag{4.18a}$$

$$\underset{\approx}{\Xi}^q_\delta(\widehat{\varphi}_h) \in [\mathbb{P}^x_1]^{d\times d} \qquad \text{and} \qquad \underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}\left[\underset{\approx}{\Xi}^q_\delta(\widehat{\varphi}_h)\,\underset{\sim}{\nabla}_q\left(\pi_h[\,[\mathcal{F}^L_\delta]'(\widehat{\varphi}_h)]\,\right)\right] = \underset{\sim}{\nabla}_q\,\widehat{\varphi}_h. \tag{4.18b}$$

Hence (4.18a,b) are discrete analogues of the relations (2.18). We now give the construction of $\underset{\approx}{\Xi}^x_\delta(\cdot)$ and $\underset{\approx}{\Xi}^q_\delta(\cdot)$. Let $\{\underset{\sim}{e}_i\}_{i=1}^d$ be the orthonormal vectors in $\mathbb{R}^d$, such that the j^{th} component of $\underset{\sim}{e}_i$ is δ_{ij}, $i,\,j = 1 \to d$. Let $\widetilde{\kappa}$ be the standard reference simplex in $\mathbb{R}^d$ with vertices $\{\underset{\sim}{\widetilde{P}}_i\}_{i=0}^d$, where $\underset{\sim}{\widetilde{P}}_0$ is the origin and $\underset{\sim}{\widetilde{P}}_i = \underset{\sim}{e}_i$, $i = 1 \to d$. Given $\widehat{\varphi}_h \in \widehat{X}_h$, $\kappa_x \in \mathcal{T}^x_h$ with vertices $\{\underset{\sim}{P}^x_{i_j}\}_{j=0}^d$ and $\kappa_q \in \mathcal{T}^q_h$ with vertices $\{\underset{\sim}{P}^q_{i_j}\}_{j=0}^d$, then for a fixed vertex $\underset{\sim}{P}^q_{i_k}$ of κ_q, let $\underset{\approx}{\Lambda}^x_\delta(\underset{\sim}{P}^q_{i_k}) \in \mathbb{R}^{d\times d}$ be diagonal with entries

$$[\underset{\approx}{\Lambda}^x_\delta(\underset{\sim}{P}^q_{i_k})]_{jj} = \begin{cases} \dfrac{\widehat{\varphi}_h(\underset{\sim}{P}^x_{i_j},\underset{\sim}{P}^q_{i_k}) - \widehat{\varphi}_h(\underset{\sim}{P}^x_{i_0},\underset{\sim}{P}^q_{i_k})}{[\mathcal{F}^L_\delta]'(\widehat{\varphi}_h(\underset{\sim}{P}^x_{i_j},\underset{\sim}{P}^q_{i_k})) - [\mathcal{F}^L_\delta]'(\widehat{\varphi}_h(\underset{\sim}{P}^x_{i_0},\underset{\sim}{P}^q_{i_k}))} & \text{if } \widehat{\varphi}_h(\underset{\sim}{P}^x_{i_j},\underset{\sim}{P}^q_{i_k}) \neq \widehat{\varphi}_h(\underset{\sim}{P}^x_{i_0},\underset{\sim}{P}^q_{i_k}), \\[2ex] \dfrac{1}{[\mathcal{F}^L_\delta]''(\widehat{\varphi}_h(\underset{\sim}{P}^x_{i_j},\underset{\sim}{P}^q_{i_k}))} = \beta^L_\delta(\widehat{\varphi}_h(\underset{\sim}{P}^x_{i_j},\underset{\sim}{P}^q_{i_k})) & \text{if } \widehat{\varphi}_h(\underset{\sim}{P}^x_{i_j},\underset{\sim}{P}^q_{i_k}) = \widehat{\varphi}_h(\underset{\sim}{P}^x_{i_0},\underset{\sim}{P}^q_{i_k}), \end{cases}$$
$$j = 1 \to d. \tag{4.19}$$

Let $B_{\kappa_x} \in \mathbb{R}^{d\times d}$ be such that the linear mapping $\mathcal{B}_{\kappa_x} : \underset{\sim}{y} \in \mathbb{R}^d \mapsto \underset{\sim}{P}^x_{i_0} + B_{\kappa_x}\,\underset{\sim}{y}$ maps the vertex $\underset{\sim}{\widetilde{P}}_j$ to $\underset{\sim}{P}^x_{i_j}$, $j = 0 \to d$, and hence $\widetilde{\kappa}$ to κ_x. For any $\widehat{\varphi}^x_h \in \widehat{X}^x_h$, let $\widehat{\varphi}^x_{h,y}(\underset{\sim}{x}) \equiv \widehat{\varphi}^x_h(\mathcal{B}_{\kappa_x}\underset{\sim}{y})$ for all $\underset{\sim}{y} \in \widetilde{\kappa}$. Hence it follows that

$$\underset{\sim}{\nabla}_x\,\widehat{\varphi}^x_h = [B^T_{\kappa_x}]^{-1}\,\underset{\sim}{\nabla}_y\,\widehat{\varphi}^x_{h,y}. \tag{4.20}$$

Therefore, for $k = 0 \to d$,

$$\underset{\approx}{\Xi}^x_\delta(\underset{\sim}{P}^q_{i_k}) = [B^T_{\kappa_x}]^{-1}\,\underset{\approx}{\Lambda}^x_\delta(\underset{\sim}{P}^q_{i_k})\,B^T_{\kappa_x} \tag{4.21}$$

is such that

$$\underset{\approx}{\Xi}^x_\delta(\underset{\sim}{P}^q_{i_k})\,\underset{\sim}{\nabla}_x\,\pi_h[[\mathcal{F}^L_\delta]'(\widehat{\varphi}_h)](\underset{\sim}{x},\underset{\sim}{P}^q_{i_k}) = \underset{\sim}{\nabla}_x\,\widehat{\varphi}_h(\underset{\sim}{x},\underset{\sim}{P}^q_{i_k}) \qquad \forall \underset{\sim}{x} \in \kappa_x. \tag{4.22}$$

Finally, on recalling (4.14b), we set

$$\underset{\approx}{\Xi}^x_\delta(\underset{\sim}{x},\underset{\sim}{q}) = \sum_{k=0}^d \underset{\approx}{\Xi}^x_\delta(\underset{\sim}{P}^q_{i_k})\,\chi^q_{i_k}(\underset{\sim}{q}) \qquad \forall \underset{\sim}{x} \in \kappa_x, \qquad \forall \underset{\sim}{q} \in \kappa_q. \tag{4.23}$$

Hence $\underset{\approx}{\Xi}^x_\delta$ satisfies (4.18a). A similar construction yields $\underset{\approx}{\Xi}^q_\delta$ satisfying (4.18b). The only difference is for those κ_q with a curved side or face, the corresponding linear mapping $\mathcal{B}_{\kappa_q}$ maps $\widetilde{\kappa}$ to the enclosed simplex with the same vertices as κ_q. As $\mathcal{T}^h_x$, $\mathcal{T}^h_q$ are quasiuniform partitions, we have from (4.23), (4.21) and (4.19), and their $\underset{\approx}{\Xi}^q_\delta$ counterparts that, for all $\widehat{\varphi}_h \in \widehat{X}_h$,

$$\|\underset{\approx}{\Xi}^x_\delta(\widehat{\varphi}_h)\|^2_{L^\infty(\Omega\times D)} + \|\underset{\approx}{\Xi}^q_\delta(\widehat{\varphi}_h)\|^2_{L^\infty(\Omega\times D)} \leq C\,L^2. \tag{4.24}$$

We note that the construction of $\underset{\approx}{\Xi}^x_\delta(\cdot)$ and $\underset{\approx}{\Xi}^q_\delta(\cdot)$ satisfying (4.18a,b) is an extension of ideas used in e.g. [19] and [2] for the finite element approximation of fourth-order degenerate nonlinear parabolic equations, such as the thin film equation.

As the partitions $\mathcal{T}_h^x$ and $\mathcal{T}_h^q$ are nonobtuse, we have that

$$\underset{\sim}{\nabla}_x\,\chi_i^x \cdot \underset{\sim}{\nabla}_x\,\chi_j^x \leq 0 \quad \text{on } \kappa_x \qquad i \neq j, \quad i,\, j = 1 \to I^x, \qquad \forall \kappa_x \in \mathcal{T}_h^x; \tag{4.25a}$$

$$\text{and} \qquad \underset{\sim}{\nabla}_q\,\chi_i^q \cdot \underset{\sim}{\nabla}_q\,\chi_j^q \leq 0 \quad \text{on } \kappa_q \qquad i \neq j, \quad i,\, j = 1 \to I^q, \qquad \forall \kappa_q \in \mathcal{T}_h^q. \tag{4.25b}$$

It follows from (4.25a,b) and the convexity of $\mathcal{F}_d^L$ that, for all $\kappa_x \in \mathcal{T}_h^x$, $\kappa_q \in \mathcal{T}_h^q$ and for all $\widehat{\varphi} \in \widehat{X}_h$,

$$\begin{aligned} &\delta \int_{\kappa_x\times\kappa_q} M\,\pi_{h,\kappa_x\times\kappa_q}\left[\left|\underset{\sim}{\nabla}_x\left(\pi_{h,\kappa_x\times\kappa_q}[\,[\mathcal{F}_\delta^L]'(\widehat{\varphi}_h)]\right)\right|^2\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ &\qquad\leq \int_{\kappa_x\times\kappa_q} M\,\pi_{h,\kappa_x\times\kappa_q}\left[\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h\cdot\underset{\sim}{\nabla}_x\left(\pi_{h,\kappa_x\times\kappa_q}[\,[\mathcal{F}_\delta^L]'(\widehat{\varphi}_h)]\right)\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}; \end{aligned} \tag{4.26a}$$

$$\begin{aligned} \text{and}\qquad &\delta \int_{\kappa_x\times\kappa_q} M\,\pi_{h,\kappa_x\times\kappa_q}\left[\left|\underset{\sim}{\nabla}_q\left(\pi_{h,\kappa_x\times\kappa_q}[\,[\mathcal{F}_\delta^L]'(\widehat{\varphi}_h)]\right)\right|^2\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ &\qquad\leq \int_{\kappa_x\times\kappa_q} M\,\pi_{h,\kappa_x\times\kappa_q}\left[\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h\cdot\underset{\sim}{\nabla}_q\left(\pi_{h,\kappa_x\times\kappa_q}[\,[\mathcal{F}_\delta^L]'(\widehat{\varphi}_h)]\right)\right]\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \end{aligned} \tag{4.26b}$$

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be a partition of the time interval $[0,T]$ into time steps $\Delta t_n = t_n - t_{n-1}$, $n = 1 \to N$. We set $\Delta t = \max_{n=1\to N} \Delta t_n$. We make the following assumptions on the time steps $\{\Delta t_n\}_{n=1}^N$ and the discrete initial data.

(A2) We assume that there exists $C \in \mathbb{R}_{>0}$ such that

$$\Delta t_n \leq C\,\Delta t_{n-1}, \qquad n = 2 \to N, \qquad \text{as} \quad \Delta t \to 0_+. \tag{4.27}$$

With Δt_1 and C as above, let $\Delta t_0 \in \mathbb{R}_{>0}$ be such that $\Delta t_1 \leq C\Delta t_0$. Given initial data satisfying (3.5a), we choose $\underset{\sim}{u}_h^0 \in \underset{\sim}{V}_h$ and $\widehat{\psi}_h^0 \in \widehat{X}_h$ such that

$$(\underset{\sim}{u}_h^0, \underset{\sim}{v}_h)_\Omega + \Delta t_0\,(\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_h^0, \underset{\approx}{\nabla}_x\,\underset{\sim}{v}_h)_\Omega = (\underset{\sim}{u}^0, \underset{\sim}{v}_h)_\Omega \qquad \forall \underset{\sim}{v}_h \in \underset{\sim}{V}_h, \tag{4.28a}$$

$$(M, \pi_h[\widehat{\psi}_h^0\,\widehat{\varphi}_h])_{\Omega\times D} = (M\,\widehat{\psi}^0, \widehat{\varphi}_h)_{\Omega\times D} \qquad \forall \widehat{\varphi}_h \in \widehat{X}_h. \tag{4.28b}$$

It follows from (4.28a,b) and (3.5a) that

$$\int_\Omega \left[|\underset{\sim}{u}_h^0|^2 + \Delta t_0\,|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_h^0|^2\right]\mathrm{d}\underset{\sim}{x} \leq C \qquad \text{and} \qquad 0 \leq \widehat{\psi}_h^0 \leq L. \tag{4.29}$$

We set

$$\underset{\sim}{f}^n(\cdot) := \frac{1}{\Delta t_n}\int_{t_{n-1}}^{t_n} \underset{\sim}{f}(\cdot,t)\,\mathrm{d}t \in \underset{\sim}{H}^{-1}(\Omega). \tag{4.30}$$

It is easily deduced from (3.5b) and (4.30) that

$$\sum_{n=1}^N \Delta t_n\,\|\underset{\sim}{f}^n\|_{H^{-1}(\Omega)}^r \leq \int_0^T \|\underset{\sim}{f}\|_{H^{-1}(\Omega)}^r\,\mathrm{d}t \leq C \qquad \text{for any } r \in [1,2], \tag{4.31a}$$

$$\text{and} \qquad \underset{\sim}{f}^n \to \underset{\sim}{f} \quad \text{strongly in } L^2(0,T;\underset{\sim}{H}^{-1}(\Omega)) \text{ as } \Delta t \to 0_+. \tag{4.31b}$$

Our numerical approximation of (P_δ^L) is then defined as follows.

$(\mathbf{P}_{\delta}^{h,\Delta t})$ For $n = 1 \to N$, given $\{\underset{\approx}{u}_{\delta,h}^{n-1}, \widehat{\psi}_{\delta,h}^{n-1}\} \in \underset{\sim}{V}_h \times \widehat{X}_h$, find $\{\underset{\approx}{u}_{\delta,h}^{n}, \widehat{\psi}_{\delta,h}^{n}\} \in \underset{\sim}{V}_h \times \widehat{X}_h$ such that

$$\left(\frac{\underset{\sim}{u}_{\delta,h}^{n} - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}, \underset{\sim}{w}_h\right)_{\Omega} + \nu\,(\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\delta,h}^{n}, \underset{\approx}{\nabla}_x\, \underset{\sim}{w}_h)_{\Omega} + \tfrac{1}{2}\left[((\underset{\sim}{u}_{\delta,h}^{n-1}\cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u}_{\delta,h}^{n}, \underset{\sim}{w}_h)_{\Omega} - ((\underset{\sim}{u}_{\delta,h}^{n-1}\cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w}_h, \underset{\sim}{u}_{\delta,h}^{n})_{\Omega}\right]$$

$$= \langle \underset{\sim}{f}^n, \underset{\sim}{w}_h\rangle_{H_0^1(\Omega)} - k\,\mu\,(\underset{\approx}{C}(M\,\widehat{\psi}_{\delta,h}^{n}), \underset{\approx}{\nabla}_x\, \underset{\sim}{w}_h)_{\Omega} \qquad \forall \underset{\sim}{w}_h \in \underset{\sim}{V}_h, \quad (4.32a)$$

$$\left(M, \pi_h\left[\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n}\,\widehat{\varphi}_h + \varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\delta,h}^{n}\cdot \underset{\sim}{\nabla}_x\,\widehat{\varphi}_h + \frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\delta,h}^{n}\cdot \underset{\sim}{\nabla}_q\,\widehat{\varphi}_h\right]\right)_{\Omega\times D}$$

$$= \left(M\,(\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\delta,h}^{n})\,\underset{\sim}{q}, \underset{\sim}{\pi}_h\left[\underset{\approx}{\Xi}_{\delta}^{q}(\widehat{\psi}_{\delta,h}^{n})\,\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h\right]\right)_{\Omega\times D} + \left(M\,\underset{\sim}{u}_{\delta,h}^{n}, \underset{\sim}{\pi}_h\left[\underset{\approx}{\Xi}_{\delta}^{x}(\widehat{\psi}_{\delta,h}^{n})\,\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h\right]\right)_{\Omega\times D} \quad \forall \widehat{\varphi}_h \in \widehat{X}_h; \ (4.32b)$$

where for ease of notation, we write π_h and $\underset{\sim}{\pi}_h$ in (4.32b) whereas it should really be $\pi_{h,\kappa_x\times\kappa_q}$ and $\underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}$, respectively, on each $\kappa_x\times\kappa_q$ of $\Omega\times D$. In addition, we have suppressed the dependence of the solution $\{\underset{\approx}{u}_{\delta,h}^{n}, \widehat{\psi}_{\delta,h}^{n}\}$ on L through the dependence of $\underset{\approx}{\Xi}_{\delta}^{x}$ and $\underset{\approx}{\Xi}_{\delta}^{q}$ on $\mathcal{F}_{\delta}^{L}$. This is because we will not be passing to the limit $L \to \infty$, but only to the limit $\delta \to 0_+$ in addition to letting the discretization parameters h, $\Delta t \to 0_+$.

We note that the approximations $\underset{\approx}{u}_{\delta,h}^{n}$ and $\widehat{\psi}_{\delta,h}^{n}$ at time level t_n to the velocity field and the scaled probability distribution satisfy a coupled nonlinear system, (4.32a,b). We will show existence of a solution to (4.32a,b) below, see Theorem 4.2, via a Brouwer fixed point theorem. First, assuming existence, we show that $(\mathrm{P}_{\delta}^{h,\Delta t})$ satisfies a discrete analogue of the energy equality (2.21). For all the following lemmas and theorems we assume the assumptions (A1) and (A2) hold.

Lemma 4.1. *For $n = 1 \to N$, a solution $\{\underset{\approx}{u}_{\delta,h}^{n}, \widehat{\psi}_{\delta,h}^{n}\} \in \underset{\sim}{V}_h \times \widehat{X}_h$ of (4.32a,b), if it exists, satisfies*

$$\begin{aligned}
&\tfrac{1}{2}\left[\|\underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 + \|\underset{\sim}{u}_{\delta,h}^{n} - \underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^2(\Omega)}^2\right] + k\,\mu\,(M, \pi_h[\mathcal{F}_{\delta}^{L}(\widehat{\psi}_{\delta,h}^{n})])_{\Omega\times D} + \Delta t_n\,\nu\,\|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 \\
&\quad + \Delta t_n\,k\,\mu\left(M, \pi_h\left[\varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\delta,h}^{n}\cdot \underset{\sim}{\nabla}_x\,(\pi_h[\,[\mathcal{F}_{\delta}^{L}]'(\widehat{\psi}_{\delta,h}^{n})]) + \frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\delta,h}^{n}\cdot \underset{\sim}{\nabla}_q\,(\pi_h[\,[\mathcal{F}_{\delta}^{L}]'(\widehat{\psi}_{\delta,h}^{n})])\right]\right)_{\Omega\times D} \\
&\qquad \le \tfrac{1}{2}\,\|\underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^2(\Omega)}^2 + k\,\mu\,(M, \pi_h[\mathcal{F}_{\delta}^{L}(\widehat{\psi}_{\delta,h}^{n-1})])_{\Omega\times D} + \Delta t_n\,\langle \underset{\sim}{f}^n, \underset{\sim}{u}_{\delta,h}^{n}\rangle_{H_0^1(\Omega)} \\
&\qquad \le \tfrac{1}{2}\,\|\underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^2(\Omega)}^2 + k\,\mu\,(M, \pi_h[\mathcal{F}_{\delta}^{L}(\widehat{\psi}_{\delta,h}^{n-1})])_{\Omega\times D} + \Delta t_n\left[\frac{\nu}{2}\,\|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 + C\,\|\underset{\sim}{f}^n\|_{H^{-1}(\Omega)}^2\right]. \qquad (4.33)
\end{aligned}$$

Proof. On choosing $\underset{\sim}{w}_h = \underset{\approx}{u}_{\delta,h}^{n}$ in (4.32a), it follows that

$$\tfrac{1}{2}\int_{\Omega}\left[|\underset{\approx}{u}_{\delta,h}^{n}|^2 + |\underset{\approx}{u}_{\delta,h}^{n} - \underset{\approx}{u}_{\delta,h}^{n-1}|^2 - |\underset{\approx}{u}_{\delta,h}^{n-1}|^2\right]\mathrm{d}\underset{\approx}{x} + \Delta t_n\,\nu\int_{\Omega}|\underset{\approx}{\nabla}_x\,\underset{\approx}{u}_{\delta,h}^{n}|^2\,\mathrm{d}\underset{\approx}{x}$$
$$= \Delta t_n\left[\langle \underset{\sim}{f}^n, \underset{\approx}{u}_{\delta,h}^{n}\rangle_{H_0^1(\Omega)} - k\,\mu\,(\underset{\approx}{C}(M\,\widehat{\psi}_{\delta,h}^{n}), \underset{\approx}{\nabla}_x\,\underset{\approx}{u}_{\delta,h}^{n})_{\Omega}\right], \qquad (4.34)$$

where we have noted the simple identity

$$2\,(s_1 - s_2)\,s_1 = s_1^2 + (s_1 - s_2)^2 - s_2^2 \qquad \forall s_1, s_2 \in \mathbb{R}. \qquad (4.35)$$

On choosing $\widehat{\varphi}_h = \pi_h[\,[\mathcal{F}^L_\delta]'(\widehat{\psi}^n_{\delta,h})]$ in (4.32b), and noting the convexity of $\mathcal{F}^L_\delta$, (4.18a,b), (2.5), (4.12) and (2.3), we have that

$$
\begin{aligned}
&(M, \pi_h[\mathcal{F}^L_\delta(\widehat{\psi}^n_{\delta,h}) - \mathcal{F}^L_\delta(\widehat{\psi}^{n-1}_{\delta,h})])_{\Omega\times D} \\
&\qquad + \Delta t_n \left(M, \pi_h \left[\varepsilon \,\underset{\sim}{\nabla}_x \widehat{\psi}^n_{\delta,h} \cdot \underset{\sim}{\nabla}_x (\pi_h[\,[\mathcal{F}^L_\delta]'(\widehat{\psi}^n_{\delta,h})]) + \frac{1}{2\,\lambda} \underset{\sim}{\nabla}_q \widehat{\psi}^n_{\delta,h} \cdot \underset{\sim}{\nabla}_q (\pi_h[\,[\mathcal{F}^L_\delta]'(\widehat{\psi}^n_{\delta,h})]) \right] \right)_{\Omega\times D} \\
&\qquad\qquad \le (M\,(\underset{\approx}{\nabla}_x \underset{\sim}{u}^n_{\delta,h})\, \underset{\sim}{q}, \underset{\sim}{\nabla}_q \widehat{\psi}^n_{\delta,h})_{\Omega\times D} + (M\,\underset{\sim}{u}^n_{\delta,h}, \underset{\sim}{\nabla}_x \widehat{\psi}^n_{\delta,h})_{\Omega\times D} \\
&\qquad\qquad = (M\,U'\,\underset{\sim}{q} \cdot [(\underset{\approx}{\nabla}_x \underset{\sim}{u}^n_{\delta,h})\, \underset{\sim}{q}\,], \widehat{\psi}^n_{\delta,h})_{\Omega\times D} - 2\,(M\,\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}^n_{\delta,h}, \widehat{\psi}^n_{\delta,h})_{\Omega\times D} \\
&\qquad\qquad = (\underset{\approx}{C}(M\,\widehat{\psi}^n_{\delta,h}), \underset{\approx}{\nabla}_x \underset{\sim}{u}^n_{\delta,h})_\Omega. \qquad (4.36)
\end{aligned}
$$

Combining (4.34) and (4.36) yields the first inequality (4.33). The second inequality follows from using a Young's inequality and a Poincaré inequality. $\square$

We now show using a Brouwer fixed point theorem that there exists a solution $\{\underset{\sim}{u}^n_{\delta,h}, \widehat{\psi}^n_{\delta,h}\}$ at time level t_n to (4.32a,b).

Theorem 4.2. *Given $\{\underset{\sim}{u}^{n-1}_{\delta,h}, \widehat{\psi}^{n-1}_{\delta,h}\} \in \underset{\sim}{V}_h \times \widehat{X}_h$ and for any time step $\Delta t_n > 0$, there exists at least one solution $\{\underset{\sim}{u}^n_{\delta,h}, \widehat{\psi}^n_{\delta,h}\} \in \underset{\sim}{V}_h \times \widehat{X}_h$ to (4.32a,b).*

Proof. We define the inner product, $((\cdot,\cdot))$, on the Hilbert space $\underset{\sim}{V}_h \times \widehat{X}_h$ as follows:

$$
((\{\underset{\sim}{u}_h, \widehat{\psi}_h\}, \{\underset{\sim}{w}_h, \widehat{\varphi}_h\})) := (\underset{\sim}{u}_h, \underset{\sim}{w}_h)_\Omega + (M, \pi_h[\widehat{\psi}_h\,\widehat{\varphi}_h])_{\Omega\times D} \qquad \forall \{\underset{\sim}{u}_h, \widehat{\psi}_h\},\ \{\underset{\sim}{w}_h, \widehat{\varphi}_h\} \in \underset{\sim}{V}_h \times \widehat{X}_h.
$$

Given $\{\underset{\sim}{u}^{n-1}_{\delta,h}, \widehat{\psi}^{n-1}_{\delta,h}\} \in \underset{\sim}{V}_h \times \widehat{X}_h$, let $\mathcal{H} : \underset{\sim}{V}_h \times \widehat{X}_h \mapsto \underset{\sim}{V}_h \times \widehat{X}_h$ be such that, for any $\{\underset{\sim}{u}_h, \widehat{\psi}_h\} \in \underset{\sim}{V}_h \times \widehat{X}_h$,

$$
\begin{aligned}
&((\mathcal{H}(\underset{\sim}{u}_h, \widehat{\psi}_h), \{\underset{\sim}{w}_h, \widehat{\varphi}_h\})) \\
&\quad := \left(\frac{\underset{\sim}{u}_h - \underset{\sim}{u}^{n-1}_{\delta,h}}{\Delta t_n}, \underset{\sim}{w}_h \right)_\Omega + \nu\,(\underset{\approx}{\nabla}_x \underset{\sim}{u}_h, \underset{\approx}{\nabla}_x \underset{\sim}{w}_h)_\Omega - \langle \underset{\sim}{f}^n, \underset{\sim}{w}_h \rangle_{H^1_0(\Omega)} + k\,\mu\,(\underset{\approx}{C}(M\,\widehat{\psi}_h), \underset{\approx}{\nabla}_x \underset{\sim}{w}_h)_\Omega \\
&\qquad + \tfrac{1}{2} \left[((\underset{\sim}{u}^{n-1}_{\delta,h} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u}_h, \underset{\sim}{w}_h)_\Omega - ((\underset{\sim}{u}^{n-1}_{\delta,h} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{w}_h, \underset{\sim}{u}_h)_\Omega \right] \\
&\qquad + \left(M, \pi_h \left[\frac{\widehat{\psi}_h - \widehat{\psi}^{n-1}_{\delta,h}}{\Delta t_n}\, \widehat{\varphi}_h + \varepsilon\, \underset{\sim}{\nabla}_x \widehat{\psi}_h \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}_h + \frac{1}{2\,\lambda} \underset{\sim}{\nabla}_q \widehat{\psi}_h \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}_h \right] \right)_{\Omega\times D} \\
&\qquad - \left(M\,(\underset{\approx}{\nabla}_x \underset{\sim}{u}_h)\,\underset{\sim}{q}, \pi_h \left[\underset{\approx}{\Xi}^q_\delta(\widehat{\psi}_h)\, \underset{\sim}{\nabla}_q \widehat{\varphi}_h \right] \right)_{\Omega\times D} - \left(M\,\underset{\sim}{u}_h, \pi_h \left[\underset{\approx}{\Xi}^x_\delta(\widehat{\psi}_h)\, \underset{\sim}{\nabla}_x \widehat{\varphi}_h \right] \right)_{\Omega\times D} \\
&\qquad\qquad \forall \{\underset{\sim}{w}_h, \widehat{\varphi}_h\} \in \underset{\sim}{V}_h \times \widehat{X}_h. \qquad (4.37)
\end{aligned}
$$

We note that a solution $\{\underset{\sim}{u}^n_{\delta,h}, \widehat{\psi}^n_{\delta,h}\}$ to (4.32a,b), if it exists, corresponds to a zero of $\mathcal{H}$; that is,

$$
((\mathcal{H}(\underset{\sim}{u}^n_{\delta,h}, \widehat{\psi}^n_{\delta,h}), \{\underset{\sim}{w}_h, \widehat{\varphi}_h\})) = 0 \qquad \forall \{\underset{\sim}{w}_h, \widehat{\varphi}_h\} \in \underset{\sim}{V}_h \times \widehat{X}_h 3. \qquad (4.38)
$$

On noting the construction of $\underset{\approx}{\Xi}^x_\delta$ and $\underset{\approx}{\Xi}^q_\delta$, (4.19)–(4.23), it is easily deduced that the mapping $\mathcal{H}$ is continuous.

For any $\{\underset{\sim}{u}_h, \widehat{\psi}_h\} \in \underset{\sim}{V}_h \times \widehat{X}_h$, on choosing $\{\underset{\sim}{w}_h, \widehat{\varphi}_h\} = \{\underset{\sim}{u}_h, \pi_h[[\mathcal{F}_\delta^L]'(\widehat{\psi}_h)]\}$, we obtain analogously to (4.33), on noting (4.26a,b) and neglecting some nonnegative terms, that

$$
\begin{aligned}
&((\mathcal{H}(\underset{\sim}{u}_h, \widehat{\psi}_h), \{\underset{\sim}{u}_h, \pi_h[[\mathcal{F}_\delta^L]'(\widehat{\psi}_h)]\})) \\
&\qquad \geq \frac{1}{\Delta t_n}\left[\tfrac{1}{2}\left(\|\underset{\sim}{u}_h\|^2_{L^2(\Omega)} - \|\underset{\sim}{u}^{n-1}_{\delta,h}\|^2_{L^2(\Omega)}\right) + k\,\mu\,(M, \pi_h[\mathcal{F}_\delta^L(\widehat{\psi}_h) - \mathcal{F}_\delta^L(\widehat{\psi}^{n-1}_{\delta,h})])_{\Omega\times D}\right] \\
&\qquad\qquad + \frac{\nu}{2}\,\|\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_h\|^2_{L^2(\Omega)} - C\,\|\underset{\sim}{f}^n\|^2_{H^{-1}(\Omega)}.
\end{aligned} \tag{4.39}
$$

Let us now assume that, for any $\gamma \in \mathbb{R}_{>0}$, the continuous mapping $\mathcal{H}$ has no zero $\{\underset{\sim}{u}^n_{\delta,h}, \widehat{\psi}^n_{\delta,h}\}$ satisfying (4.38), which lies in the ball

$$
\mathcal{Z}_\gamma := \{\{\underset{\sim}{w}_h, \widehat{\varphi}_h\} \in \underset{\sim}{V}_h \times \widehat{X}_h \;:\; |||\{\underset{\sim}{w}_h, \widehat{\varphi}_h\}||| \leq \gamma\}\,;
$$

where

$$
|||\{\underset{\sim}{w}_h, \widehat{\varphi}_h\}||| := [((\{\underset{\sim}{w}_h, \widehat{\varphi}_h\}, \{\underset{\sim}{w}_h, \widehat{\varphi}_h\}))]^{\frac{1}{2}} = \left[\|\underset{\sim}{w}_h\|^2_{L^2(\Omega)} + (M, \pi_h[(\widehat{\varphi}_h)^2])_{\Omega\times D}\right]^{\frac{1}{2}}.
$$

Then, for such γ, we can define the continuous mapping $\mathcal{E}_\gamma : \mathcal{Z}_\gamma \mapsto \mathcal{Z}_\gamma$ such that, for all $\{\underset{\sim}{w}_h, \widehat{\varphi}_h\} \in \mathcal{Z}_\gamma$,

$$
\mathcal{E}_\gamma(\underset{\sim}{w}_h, \widehat{\varphi}_h) := -\gamma \frac{\mathcal{H}(\underset{\sim}{w}_h, \widehat{\varphi}_h)}{|||\mathcal{H}(\underset{\sim}{w}_h, \widehat{\varphi}_h)|||}.
$$

By the Brouwer fixed point theorem, $\mathcal{E}_\gamma$ has at least one fixed point $\{\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h\}$ in $\mathcal{Z}_\gamma$; hence it satisfies

$$
|||\{\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h\}||| = |||\mathcal{E}_\gamma(\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h)||| = \gamma. \tag{4.40}
$$

It follows from (2.22) and (4.40) that

$$
\begin{aligned}
\frac{1}{2}\|\underset{\sim}{u}^\gamma_h\|^2_{L^2(\Omega)} + k\,\mu\,(M, \pi_h[\mathcal{F}_\delta^L(\widehat{\psi}^\gamma_h)])_{\Omega\times D} &\geq \frac{1}{2}\|\underset{\sim}{u}^\gamma_h\|^2_{L^2(\Omega)} + \frac{k\,\mu}{4L}\,(M, \pi_h[(\widehat{\psi}^\gamma_h)^2])_{\Omega\times D} - C(L) \\
&\geq \min\left\{\frac{1}{2}, \frac{k\,\mu}{4L}\right\} |||\{\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h\}|||^2 - C(L) \\
&= \min\left\{\frac{1}{2}, \frac{k\,\mu}{4L}\right\} \gamma^2 - C(L).
\end{aligned} \tag{4.41}
$$

Hence for all γ sufficiently large, it follows from (4.39) and (4.41) that

$$
((\mathcal{H}(\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h), \{\underset{\sim}{u}^\gamma_h, \pi_h[[\mathcal{F}_\delta^L]'(\widehat{\psi}^\gamma_h)]\})) > 0. \tag{4.42}
$$

On the other hand as $\{\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h\}$ is a fixed point of $\mathcal{E}_\gamma$, we have that

$$
((\mathcal{H}(\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h), \{\underset{\sim}{u}^\gamma_h, \pi_h[[\mathcal{F}_\delta^L]'(\widehat{\psi}^\gamma_h)]\})) = -\frac{|||\mathcal{H}(\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h)|||}{\gamma}\left[\|\underset{\sim}{u}^\gamma_h\|^2_{L^2(\Omega)} + (M, \pi_h[\widehat{\psi}^\gamma_h\,[\mathcal{F}_\delta^L]'(\widehat{\psi}^\gamma_h)])_{\Omega\times D}\right]. \tag{4.43}
$$

Similarly to (4.41), we have from (2.22) and (4.40) that

$$
\|\underset{\sim}{u}^\gamma_h\|^2_{L^2(\Omega)} + (M, \pi_h[\widehat{\psi}^\gamma_h\,[\mathcal{F}_\delta^L]'(\widehat{\psi}^\gamma_h)])_{\Omega\times D} \geq \frac{1}{4L}\,\gamma^2 - C(L). \tag{4.44}
$$

Therefore on combining (4.43) and (4.44), we have for all γ sufficiently large that

$$
((\mathcal{H}(\underset{\sim}{u}^\gamma_h, \widehat{\psi}^\gamma_h), \{\underset{\sim}{u}^\gamma_h, \pi_h[[\mathcal{F}_\delta^L]'(\widehat{\psi}^\gamma_h)]\})) < 0, \tag{4.45}
$$

which obviously contradicts (4.42). Hence the mapping $\mathcal{H}$ has a zero in $\mathcal{Z}_\gamma$ for γ sufficiently large. $\square$

In order to establish a stability result for our approximation $(\mathrm{P}_\delta^{h,\Delta t})$, we need first to prove a number of auxiliary results. We note that, for all $\kappa_x \in \mathcal{T}_x^h$ with vertices $\{\underset{\sim}{P}{}^x_{i_j}\}_{j=0}^d$,

$$|\,[\pi^x_{h,\kappa_x}\widehat{\varphi}^x](\underset{\sim}{x})|^2 = |\sum_{j=0}^d \widehat{\varphi}^x(\underset{\sim}{P}{}^x_{i_j})\,\chi^x_{i_j}(\underset{\sim}{x})|^2 \leq \sum_{j=0}^d [\widehat{\varphi}^x(\underset{\sim}{P}{}^x_{i_j})]^2\,\chi^x_{i_j}(\underset{\sim}{x}) \;= [\,\pi^x_{h,\kappa_x}[(\widehat{\varphi}^x)^2]\,](\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \kappa_x, \quad \forall \widehat{\varphi}^x \in C(\overline{\kappa_x}), \tag{4.46a}$$

where we have used (4.14a) and that $\chi^x_{i_j}$ are nonnegative, and $\sum_{j=0}^d \chi^x_{i_j}(\underset{\approx}{x}) = 1$ for all $\underset{\approx}{x} \in \kappa_x$. Similarly, we have for all $\kappa_x \in \mathcal{T}_x^h$, $\kappa_q \in \mathcal{T}_q^h$ that

$$|\,[\pi^q_{h,\kappa_q}\widehat{\varphi}^q](\underset{\sim}{q})|^2 \leq [\pi^q_{h,\kappa_q}[(\widehat{\varphi}^q)^2]\,](\underset{\sim}{q}) \qquad \forall \underset{\sim}{q} \in \kappa_q, \qquad \forall \widehat{\varphi}^q \in C(\overline{\kappa_q}), \tag{4.46b}$$

$$|\,[\pi_{h,\kappa_x\times\kappa_q}\widehat{\varphi}](\underset{\sim}{x},\underset{\sim}{q})|^2 \leq [\,\pi_{h,\kappa_x\times\kappa_q}[\widehat{\varphi}^2]\,](\underset{\sim}{x},\underset{\sim}{q}) \qquad \forall(\underset{\sim}{x},\underset{\sim}{q}) \in \kappa_x\times\kappa_q, \quad \forall \widehat{\varphi} \in C(\overline{\kappa_x\times\kappa_q}), \tag{4.46c}$$

$$|\,[\underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}\underset{\sim}{\widehat{\varphi}}](\underset{\sim}{x},\underset{\sim}{q})|^2 \leq [\pi_{h,\kappa_x\times\kappa_q}[\,|\underset{\sim}{\widehat{\varphi}}|^2]\,](\underset{\sim}{x},\underset{\sim}{q}) \qquad \forall(\underset{\sim}{x},\underset{\sim}{q}) \in \kappa_x\times\kappa_q, \quad \forall \underset{\sim}{\widehat{\varphi}} \in [C(\overline{\kappa_x\times\kappa_q})]^d. \tag{4.46d}$$

In addition, for all $\kappa_x \in \mathcal{T}_x^h$, $\kappa_q \in \mathcal{T}_q^h$ and for all $\widehat{\varphi}, \widehat{\psi} \in C(\overline{\kappa_x\times\kappa_q})$, $\underset{\sim}{\widehat{\varphi}}, \underset{\sim}{\widehat{\psi}} \in [C(\overline{\kappa_x\times\kappa_q})]^d$ the following inequalities are easily deduced for any $\eta \in \mathbb{R}_{>0}$

$$|\,[\pi_{h,\kappa_x\times\kappa_q}[\widehat{\varphi}\,\widehat{\psi}]\,](\underset{\sim}{x},\underset{\sim}{q})| \leq \tfrac{1}{2}\,[\pi_{h,\kappa_x\times\kappa_q}[\eta\,\widehat{\varphi}^2 + \eta^{-1}\,\widehat{\psi}^2]\,](\underset{\sim}{x},\underset{\sim}{q}) \qquad \forall(\underset{\sim}{x},\underset{\sim}{q}) \in \kappa_x\times\kappa_q, \tag{4.47a}$$

and

$$|\,[\pi_{h,\kappa_x\times\kappa_q}[\underset{\sim}{\widehat{\varphi}}\cdot\underset{\sim}{\widehat{\psi}}]\,](\underset{\sim}{x},\underset{\sim}{q})| \leq \tfrac{1}{2}\,[\pi_{h,\kappa_x\times\kappa_q}[\eta\,|\underset{\sim}{\widehat{\varphi}}|^2 + \eta^{-1}\,|\underset{\sim}{\widehat{\psi}}|^2]\,](\underset{\sim}{x},\underset{\sim}{q}) \qquad \forall(\underset{\sim}{x},\underset{\sim}{q}) \in \kappa_x\times\kappa_q. \tag{4.47b}$$

The following interpolation stability results are easily established for all $\kappa_x \in \mathcal{T}_h^x$, $\kappa_q \in \mathcal{T}_h^q$

$$\|\underset{\sim}{\nabla}_x\,\pi^x_{h,\kappa_x}\widehat{\varphi}^x\|_{L^\infty(\kappa_x)} \leq C\,\|\underset{\sim}{\nabla}_x\,\widehat{\varphi}^x\|_{L^\infty(\kappa_x)} \qquad \forall \widehat{\varphi}^x \in W^{1,\infty}(\kappa_x), \tag{4.48a}$$

$$\|\underset{\sim}{\nabla}_q\,\pi^q_{h,\kappa_q}\widehat{\varphi}^q\|_{L^\infty(\kappa_q)} \leq C\,\|\underset{\sim}{\nabla}_q\,\widehat{\varphi}^q\|_{L^\infty(\kappa_q)} \qquad \forall \widehat{\varphi}^q \in W^{1,\infty}(\kappa_q). \tag{4.48b}$$

It follows from (4.48a,b) that

$$\begin{aligned}\sum_{i=1}^d\sum_{j=1}^d \left\|\frac{\partial^2}{\partial x_i\,\partial q_j}\pi_{h,\kappa_x\times\kappa_q}\widehat{\varphi}\right\|_{L^\infty(\kappa_x\times\kappa_q)} &= \sum_{i=1}^d\sum_{j=1}^d \left\|\frac{\partial}{\partial x_i}\pi^x_{h,\kappa_x}\left[\frac{\partial}{\partial q_j}\pi^q_{h,\kappa_q}\widehat{\varphi}\right]\right\|_{L^\infty(\kappa_x\times\kappa_q)} \\ &\leq C\sum_{i=1}^d\sum_{j=1}^d \left\|\frac{\partial^2}{\partial x_i\,\partial q_j}\widehat{\varphi}\right\|_{L^\infty(\kappa_x\times\kappa_q)} \qquad \forall \widehat{\varphi} \in W^{2,\infty}(\kappa_x\times\kappa_q).\end{aligned} \tag{4.49}$$

We recall the well-known approximation results for all $\kappa_x \in \mathcal{T}_h^x$ and $\kappa_q \in \mathcal{T}_h^q$

$$\|(I-\pi^x_{h,\kappa_x})\widehat{\varphi}^x\|_{L^\infty(\kappa_x)} \leq C\,h_x^2\,|\widehat{\varphi}^x|_{W^{2,\infty}(\kappa_x)} \qquad \forall \widehat{\varphi}^x \in W^{2,\infty}(\kappa_x), \tag{4.50a}$$

$$\|(I-\pi^q_{h,\kappa_q})\widehat{\varphi}^q\|_{L^\infty(\kappa_q)} \leq C\,h_q^2\,|\widehat{\varphi}^q|_{W^{2,\infty}(\kappa_q)} \qquad \forall \widehat{\varphi}^q \in W^{2,\infty}(\kappa_q). \tag{4.50b}$$

We require the following inverse bounds for all $\widehat{\varphi}_h^x \in \mathbb{P}_1^x$, $\widehat{\varphi}_h^q \in \mathbb{P}_1^q$ and for all $\kappa_x^\star \subset \kappa_x \in \mathcal{T}_h^x$, $\kappa_q^\star \subset \kappa_q \in \mathcal{T}_h^q$ with $\underline{m}(\kappa_x) \leq C\,\underline{m}(\kappa_x^\star)$, $\underline{m}(\kappa_q) \leq C\,\underline{m}(\kappa_q^\star)$:

$$\|\widehat{\varphi}_h^x\|_{L^\infty(\kappa_x)}^2 \leq C\,[\underline{m}(\kappa_x^\star)]^{-1} \int_{\kappa_x^\star} |\widehat{\varphi}_h^x|^2 \,\mathrm{d}\underset{\sim}{x}, \tag{4.51a}$$

$$\|\widehat{\varphi}_h^q\|_{L^\infty(\kappa_q)}^2 \leq C\,[\underline{m}(\kappa_q^\star)]^{-1} \int_{\kappa_q^\star} |\widehat{\varphi}_h^q|^2 \,\mathrm{d}\underset{\sim}{q}, \tag{4.51b}$$

$$\int_{\kappa_x^\star} |\underset{\sim}{\nabla}_x\, \widehat{\varphi}_h^x|^2 \,\mathrm{d}\underset{\sim}{x} \leq C\,h_x^{-2} \int_{\kappa_x^\star} |\widehat{\varphi}_h^x|^2 \,\mathrm{d}\underset{\sim}{x} \leq C\,h_x^{-2} \int_{\kappa_x^\star} \pi_{h,\kappa_x}^x\,[\,|\widehat{\varphi}_h^x|^2]\,\mathrm{d}\underset{\sim}{x}, \tag{4.51c}$$

$$\int_{\kappa_q^\star} |\underset{\sim}{\nabla}_q\, \widehat{\varphi}_h^q|^2 \,\mathrm{d}\underset{\sim}{q} \leq C\,h_q^{-2} \int_{\kappa_q^\star} |\widehat{\varphi}_h^q|^2 \,\mathrm{d}\underset{\sim}{q} \leq C\,h_q^{-2} \int_{\kappa_q^\star} \pi_{h,\kappa_q}^q\,[\,|\widehat{\varphi}_h^q|^2]\,\mathrm{d}\underset{\sim}{q}. \tag{4.51d}$$

The bounds (4.51a,b) are standard inverse bounds in the case $\kappa_x^\star \equiv \kappa_x$ and $\kappa_q^\star \equiv \kappa_q$. However, these results are easily generalized to $\kappa_x^\star \subset \kappa_x$ and $\kappa_q^\star \subset \kappa_q$ under the stated conditions. The first inequalities in (4.51c,d) then follow immediately from (4.51a,b), respectively; whereas the second inequalities in (4.51c,d) follow from (4.46a,b), respectively. The following bounds follow immediately from (4.51a,b) under the same stated conditions:

$$\int_{\kappa_x^\star} \pi_{h,\kappa_x}^x\,[\,|\widehat{\varphi}_h^x|^2]\,\mathrm{d}\underset{\sim}{x} \leq C \int_{\kappa_x^\star} |\widehat{\varphi}_h^x|^2 \,\mathrm{d}\underset{\sim}{x} \qquad \text{and} \qquad \int_{\kappa_q^\star} \pi_{h,\kappa_q}^q\,[\,|\widehat{\varphi}_h^q|^2]\,\mathrm{d}\underset{\sim}{q} \leq C \int_{\kappa_q^\star} |\widehat{\varphi}_h^q|^2 \,\mathrm{d}\underset{\sim}{q}. \tag{4.52}$$

In addition, we require the following weighted bounds.

Lemma 4.3. *For all $\kappa_q \in \mathcal{T}_q^h$ and for all $\widehat{\varphi}_h^q \in \mathbb{P}_1^q$ we have that*

$$\int_{\kappa_q} M\,|\underset{\sim}{\nabla}_q\, \widehat{\varphi}_h^q|^2 \,\mathrm{d}\underset{\sim}{q} \leq C\,h_q^{-2} \int_{\kappa_q} M\,|\widehat{\varphi}_h^q|^2 \,\mathrm{d}\underset{\sim}{q} \leq C\,h_q^{-2} \int_{\kappa_q} M\,\pi_{h,\kappa_q}^q\,[\,|\widehat{\varphi}_h^q|^2]\,\mathrm{d}\underset{\sim}{q}, \tag{4.53a}$$

$$\int_{\kappa_q} M\,\pi_{h,\kappa_q}^q\,[\,|\widehat{\varphi}_h^q|^2]\,\mathrm{d}\underset{\sim}{q} \leq \left(\int_{\kappa_q} M \,\mathrm{d}\underset{\sim}{q}\right) \|\widehat{\varphi}_h^q\|_{L^\infty(\kappa_q)}^2 \leq C \int_{\kappa_q} M\,|\widehat{\varphi}_h^q|^2 \,\mathrm{d}\underset{\sim}{q}. \tag{4.53b}$$

Proof. If κ_q has no vertices on ∂D, let $\underset{\sim}{q}_{\min}$ be the nearest point of κ_q to ∂D. It follows from the quasiuniformity of $\mathcal{T}_q^h$ that $\mathrm{dist}(\underset{\sim}{q}_{\min}, \partial D) \geq C\,h_q$, and hence, on noting (2.9a), it follows that

$$\frac{\max_{\underset{\sim}{q} \in \kappa_q} M(\underset{\sim}{q})}{\min_{\underset{\sim}{q} \in \kappa_q} M(\underset{\sim}{q})} \leq \frac{c_2\,[\mathrm{dist}(\underset{\sim}{q}_{\min}, \partial D) + h_q]^\zeta}{c_1\,[\mathrm{dist}(\underset{\sim}{q}_{\min}, \partial D)]^\zeta} \leq C. \tag{4.54}$$

The first inequality in (4.53a) then follows immediately from (4.51d) and (4.54). Similarly, (4.53b) follows immediately from (4.51b) and (4.54).

If κ_q has vertices on ∂D, we introduce, for appropriate $C_i \in \mathbb{R}_{>0}$,

$$\kappa_q^\star := \{\underset{\sim}{q} \in \kappa_q : \mathrm{dist}(\underset{\sim}{q}, \partial D) \geq C_1\,h_q\} \subset \kappa_q \qquad \textit{and} \qquad \underline{m}(\kappa_q) \leq C_2\,\underline{m}(\kappa_q^\star). \tag{4.55}$$

Similarly to (4.54), we have from (4.54) and (2.9a) that

$$\frac{\max_{\underset{\sim}{q} \in \kappa_q} M(\underset{\sim}{q})}{\min_{\underset{\sim}{q} \in \kappa_q^\star} M(\underset{\sim}{q})} \leq C. \tag{4.56}$$

It follows from (4.55), (2.9a), (4.56) and (4.51d) that

$$\int_{\kappa_q} M\,|\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h^q|^2\,\mathrm{d}\underset{\sim}{q} \leq C_2 \int_{\kappa_q^\star} M\,|\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h^q|^2\,\mathrm{d}\underset{\sim}{q} \leq C\,h_q^{-2} \int_{\kappa_q^\star} M\,|\widehat{\varphi}_h^q|^2\,\mathrm{d}\underset{\sim}{q} \leq C\,h_q^{-2} \int_{\kappa_q} M\,|\widehat{\varphi}_h^q|^2\,\mathrm{d}\underset{\sim}{q}, \tag{4.57}$$

and hence the first inequality in (4.53a). Similarly, the bound (4.53b) in this case follows immediately from (4.51b), (4.55) and (4.56).

Finally, the second inequality in (4.53a) follows in both cases from (4.46b). □

In addition, we require the following inverse inequalities.

Lemma 4.4. *For all $\widehat{\varphi}_h \in \mathbb{P}_1^x \otimes \mathbb{P}_1^q$ and for all $\kappa_x \in \mathcal{T}_h^x$, $\kappa_q \in \mathcal{T}_h^q$ we have that*

$$\int_{\kappa_x\times\kappa_q} M\,\pi_{h,\kappa_x\times\kappa_q}\,[\,|\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h|^2]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq \int_{\kappa_x\times\kappa_q} M\,|\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq C\,h_x^{-2}\int_{\kappa_x\times\kappa_q} M\,|\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}, \tag{4.58a}$$

$$\int_{\kappa_x\times\kappa_q} M\,\pi_{h,\kappa_x\times\kappa_q}\,[\,|\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h|^2]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq \int_{\kappa_x\times\kappa_q} M\,|\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \leq C\,h_q^{-2}\int_{\kappa_x\times\kappa_q} M\,|\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \tag{4.58b}$$

Proof. The first inequalities in (4.58a,b) follow immediately from (4.53b) and (4.52), respectively. The second inequalities in (4.58a,b) follow immediately from the first inequalities in (4.51c) and (4.53a), respectively. □

We require the following results.

Lemma 4.5. *For all $\kappa_x \in \mathcal{T}_h^x$, $\kappa_q \in \mathcal{T}_h^q$ and for all $\widehat{\psi}_h$, $\widehat{\varphi}_h \in \widehat{X}_h$ we have that*

$$\begin{aligned}
&\left|\int_{\kappa_x\times\kappa_q} M\,(I-\pi_{h,\kappa_x\times\kappa_q})\,[\underset{\sim}{\nabla}_q\,\widehat{\psi}_h\cdot\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right| \\
&\qquad\leq C\,h_x\left(\int_{\kappa_x\times\kappa_q} M\,|\underset{\sim}{\nabla}_q\,\widehat{\psi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}\sum_{j=1}^{d}\int_{\kappa_x\times\kappa_q} M\left|\frac{\partial^2\widehat{\varphi}_h}{\partial x_i\partial q_j}\right|^2\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}},
\end{aligned} \tag{4.59a}$$

$$\begin{aligned}
&\left|\int_{\kappa_x\times\kappa_q} M\,(I-\pi_{h,\kappa_x\times\kappa_q})\,[\underset{\sim}{\nabla}_x\,\widehat{\psi}_h\cdot\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right| \\
&\qquad\leq C\,h_q\left(\int_{\kappa_x\times\kappa_q} M\,|\underset{\sim}{\nabla}_x\,\widehat{\psi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}\sum_{j=1}^{d}\int_{\kappa_x\times\kappa_q} M\left|\frac{\partial^2\widehat{\varphi}_h}{\partial x_i\partial q_j}\right|^2\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}},
\end{aligned} \tag{4.59b}$$

and

$$\begin{aligned}
&\left|\int_{\kappa_x\times\kappa_q} M\,(I-\pi_{h,\kappa_x\times\kappa_q})\,[\widehat{\psi}_h\,\widehat{\varphi}_h]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right| \\
&\qquad\leq C\,h_x^2\left(\int_{\kappa_x\times\kappa_q} M\,|\underset{\sim}{\nabla}_x\,\widehat{\psi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}}\left(\int_{\kappa_x\times\kappa_q} M\,|\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}} \\
&\qquad\quad + C\,h_q^2\left(\int_{\kappa_x\times\kappa_q} M\,|\underset{\sim}{\nabla}_q\,\widehat{\psi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}}\left(\int_{\kappa_x\times\kappa_q} M\,|\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.59c}$$

Proof. As $\underset{\sim}{\nabla}_q \widehat{\psi}_h$, $\underset{\sim}{\nabla}_q \widehat{\varphi}_h \in [\mathbb{P}_1^x]^d$ on $\kappa_x \times \kappa_q$, it follows from (4.50a) that

$$
\begin{aligned}
&\left| \int_{\kappa_x \times \kappa_q} M \, (I - \pi_{h,\kappa_x \times \kappa_q}) \, [\underset{\sim}{\nabla}_q \widehat{\psi}_h \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}_h] \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right| \\
&\qquad \leq \left(\int_{\kappa_x \times \kappa_q} M \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right) \|(I - \pi_{h,\kappa_x}^x) \, [\underset{\sim}{\nabla}_q \widehat{\psi}_h \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}_h]\|_{L^\infty(\kappa_x)} \\
&\qquad \leq C \, h_x^2 \left(\int_{\kappa_x \times \kappa_q} M \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right) \|\underset{\sim}{\nabla}_q \widehat{\psi}_h \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}_h\|_{W^{2,\infty}(\kappa_x)} \\
&\qquad \leq C \, h_x^2 \left(\sum_{i=1}^d \sum_{j=1}^d \int_{\kappa_x \times \kappa_q} M \left| \frac{\partial^2 \widehat{\psi}_h}{\partial x_i \partial q_j} \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} \left(\sum_{i=1}^d \sum_{j=1}^d \int_{\kappa_x \times \kappa_q} M \left| \frac{\partial^2 \widehat{\varphi}_h}{\partial x_i \partial q_j} \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} . \qquad (4.60)
\end{aligned}
$$

The desired result (4.59a) then follows from (4.60) on applying (4.58a) to the first integral.

Similarly, as $\underset{\sim}{\nabla}_x \widehat{\psi}_h$, $\underset{\sim}{\nabla}_x \widehat{\varphi}_h \in [\mathbb{P}_1^q]^d$ on $\kappa_x \times \kappa_q$, it follows from (4.50b) that

$$
\begin{aligned}
&\left| \int_{\kappa_x \times \kappa_q} M \, (I - \pi_{h,\kappa_x \times \kappa_q}) \, [\underset{\sim}{\nabla}_x \widehat{\psi}_h \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}_h] \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right| \\
&\qquad \leq C \, h_q^2 \left(\sum_{i=1}^d \sum_{j=1}^d \int_{\kappa_x \times \kappa_q} M \left| \frac{\partial^2 \widehat{\psi}_h}{\partial x_i \partial q_j} \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} \left(\sum_{i=1}^d \sum_{j=1}^d \int_{\kappa_x \times \kappa_q} M \left| \frac{\partial^2 \widehat{\varphi}_h}{\partial x_i \partial q_j} \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} . \qquad (4.61)
\end{aligned}
$$

The desired result (4.59b) then follows from (4.61) on applying (4.58b) to the first integral.

To prove (4.59c), we first note that $I - \pi_{h,\kappa_x \times \kappa_q} \equiv (I - \pi_{h,\kappa_x}^x) + (I - \pi_{h,\kappa_q}^q) \, \pi_{h,\kappa_x}^x$. It follows from (4.50a) that

$$
\begin{aligned}
\int_{\kappa_x \times \kappa_q} M \, \|(I - \pi_{h,\kappa_x}^x) \, [\widehat{\psi}_h \, \widehat{\varphi}_h]\|_{L^\infty(\kappa_x)} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} &\leq C \, h_x^2 \int_{\kappa_x \times \kappa_q} M \, |\underset{\sim}{\nabla}_x \widehat{\psi}_h| \, |\underset{\sim}{\nabla}_x \widehat{\varphi}_h| \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \\
&\leq C \, h_x^2 \left(\int_{\kappa_x \times \kappa_q} M \, |\underset{\sim}{\nabla}_x \widehat{\psi}_h|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} \left(\int_{\kappa_x \times \kappa_q} M \, |\underset{\sim}{\nabla}_x \widehat{\varphi}_h|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} . \qquad (4.62)
\end{aligned}
$$

It follows from (4.50b) and (4.51a) that

$$
\begin{aligned}
&\left(\int_{\kappa_x \times \kappa_q} M \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right) \|(I - \pi_{h,\kappa_x}^q) \pi_{h,\kappa_x}^x \, [\widehat{\psi}_h \, \widehat{\varphi}_h]\|_{L^\infty(\kappa_x \times \kappa_q)} \\
&\qquad \leq C \, h_q^2 \left(\int_{\kappa_x \times \kappa_q} M \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right) \sum_{i=1}^d \sum_{j=1}^d \left\| \pi_{h,\kappa_x}^x \, [\frac{\partial \widehat{\psi}_h}{\partial q_i} \, \frac{\partial \widehat{\varphi}_h}{\partial q_j}] \right\|_{L^\infty(\kappa_x)} \\
&\qquad \leq C \, h_q^2 \left(\int_{\kappa_x \times \kappa_q} M \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right) \|\underset{\sim}{\nabla}_q \widehat{\psi}_h\|_{L^\infty(\kappa_x)} \, \|\underset{\sim}{\nabla}_q \widehat{\varphi}_h\|_{L^\infty(\kappa_x)} \\
&\qquad \leq C \, h_q^2 \left(\int_{\kappa_x \times \kappa_q} M \, |\underset{\sim}{\nabla}_q \widehat{\psi}_h|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} \left(\int_{\kappa_x \times \kappa_q} M \, |\underset{\sim}{\nabla}_q \widehat{\varphi}_h|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} . \qquad (4.63)
\end{aligned}
$$

Hence combining (4.62) and (4.63) yields the desired result (4.59c). $\square$

Lemma 4.6. *For all $\kappa_x \in \mathcal{T}_h^x$, $\kappa_q \in \mathcal{T}_h^q$ and for all $\widehat{\psi}_h, \widehat{\varphi}_h \in \widehat{X}_h$ we have that*

$$\int_{\kappa_x \times \kappa_q} M \left[\left| (\underset{\sim}{I} - \underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}) \, [\underset{\approx}{\Xi}{}_\delta^q(\widehat{\psi}_h) \, \underset{\sim}{\nabla}_q \, \widehat{\varphi}_h] \right|^2 + \left| (\underset{\sim}{I} - \underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}) \, [\underset{\approx}{\Xi}{}_\delta^x(\widehat{\psi}_h) \, \underset{\sim}{\nabla}_x \, \widehat{\varphi}_h] \right|^2 \right] \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$
$$\leq C(L)\,(h_x^2 + h_q^2) \left(\sum_{i=1}^d \sum_{j=1}^d \int_{\kappa_x\times\kappa_q} M \left| \frac{\partial^2 \widehat{\varphi}_h}{\partial x_i \partial q_j} \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right). \tag{4.64}$$

Proof. As $\underset{\approx}{\Xi}{}_\delta^q(\widehat{\psi}_h) \in [\mathbb{P}_1^x]^{d\times d}$ and $\underset{\sim}{\nabla}_q \, \widehat{\varphi}_h \in [\mathbb{P}_1^x]^d$ on $\kappa_x \times \kappa_q$, it follows from (4.50a), (4.51c) and (4.24) that

$$\int_{\kappa_x \times \kappa_q} M \left| (\underset{\sim}{I} - \underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}) \, [\underset{\approx}{\Xi}{}_\delta^q(\widehat{\psi}_h) \, \underset{\sim}{\nabla}_q \, \widehat{\varphi}_h] \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x}$$
$$\leq C\, h_x^4 \left(\int_{\kappa_x\times\kappa_q} M \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right) \left(\sum_{i=1}^d \sum_{j=1}^d \| \underset{\sim}{\nabla}_x \, [\underset{\approx}{\Xi}{}_\delta^q(\widehat{\psi}_h)]_{ij} \|_{L^\infty(\kappa_x)}^2 \right) \left(\sum_{i=1}^d \sum_{j=1}^d \| \frac{\partial^2 \widehat{\varphi}_h}{\partial x_i \partial q_j} \|_{L^\infty(\kappa_x)}^2 \right)$$
$$\leq C(L)\, h_x^2 \left(\sum_{i=1}^d \sum_{j=1}^d \int_{\kappa_x\times\kappa_q} M \left| \frac{\partial^2 \widehat{\varphi}_h}{\partial x_i \partial q_j} \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right). \tag{4.65}$$

Similarly, as $\underset{\approx}{\Xi}{}_\delta^x(\widehat{\psi}_h) \in [\mathbb{P}_1^q]^{d\times d}$ and $\underset{\sim}{\nabla}_x \, \widehat{\varphi}_h \in [\mathbb{P}_1^q]^d$ on $\kappa_x \times \kappa_q$, it follows from (4.50b), (4.53a) and (4.24) that

$$\int_{\kappa_x \times \kappa_q} M \left| (\underset{\sim}{I} - \underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}) \, [\underset{\approx}{\Xi}{}_\delta^x(\widehat{\psi}_h) \, \underset{\sim}{\nabla}_x \, \widehat{\varphi}_h] \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \leq C(L)\, h_q^2 \left(\sum_{i=1}^d \sum_{j=1}^d \int_{\kappa_x\times\kappa_q} M \left| \frac{\partial^2 \widehat{\varphi}_h}{\partial x_i \partial q_j} \right|^2 \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right). \tag{4.66}$$

Combining (4.65) and (4.66) yields the desired result (4.64). $\square$

In addition, we introduce $Q_h^M : \widehat{X} \mapsto \widehat{X}_h$ and $\widetilde{Q}_h^M : \widehat{X} \mapsto \widehat{X}_h$ such that

$$(M\, Q_h^M \widehat{\psi}, \widehat{\varphi}_h)_{\Omega\times D} = (M\widehat{\psi}, \widehat{\varphi}_h)_{\Omega\times D} \qquad \forall \widehat{\varphi}_h \in \widehat{X}_h, \tag{4.67a}$$
$$(M, \pi_h[(\widetilde{Q}_h^M \widehat{\psi})\, \widehat{\varphi}_h])_{\Omega\times D} = (M\widehat{\psi}, \widehat{\varphi}_h)_{\Omega\times D} \qquad \forall \widehat{\varphi}_h \in \widehat{X}_h. \tag{4.67b}$$

In the Appendix, it is shown that

$$\|Q_h^M \widehat{\psi}\|_{\widehat{X}}^2 \leq C\, \|\widehat{\psi}\|_{\widehat{X}}^2 \qquad \forall \widehat{\psi} \in \widehat{X}. \tag{4.68}$$

We require a related result for $\widetilde{Q}_h^M$.

Lemma 4.7. *The following bounds hold*

$$\|\widetilde{Q}_h^M \widehat{\psi}\|_{\widehat{X}}^2 \leq \Big(M, \pi_h \left[|\widetilde{Q}_h^M \widehat{\psi}|^2 + |\underset{\sim}{\nabla}_x \, (\widetilde{Q}_h^M \widehat{\psi})|^2 + |\underset{\sim}{\nabla}_q \, (\widetilde{Q}_h^M \widehat{\psi})|^2 \right] \Big)_{\Omega\times D} \leq C\, \|\widehat{\psi}\|_{\widehat{X}}^2 \qquad \forall \widehat{\psi} \in \widehat{X}. \tag{4.69}$$

Proof. Given $\widehat{\psi} \in \widehat{X}$, let $E = (Q_h^M - \widetilde{Q}_h^M)\widehat{\psi}$. It follows from (4.46c), (4.67a,b), (4.59c), (4.68), (4.58a,b) that

$$(M, E^2)_{\Omega\times D} \leq (M, \pi_h[E^2])_{\Omega\times D} = (M, (\pi_h - I)[(Q_h^M \widehat{\psi})\, E])_{\Omega\times D}$$
$$\leq C\, \|\widehat{\psi}\|_{\widehat{X}} \left[h_x^2 \left(\int_{\Omega\times D} M\, |\underset{\sim}{\nabla}_x \, E|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} + h_q^2 \left(\int_{\Omega\times D} M\, |\underset{\sim}{\nabla}_q \, E|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right)^{\frac{1}{2}} \right]$$
$$\leq C\,(h_x + h_q)\, \|\widehat{\psi}\|_{\widehat{X}}\, [\, (M, E^2)_{\Omega\times D}\,]^{\frac{1}{2}} \leq C\,(h_x + h_q)^2\, \|\widehat{\psi}\|_{\widehat{X}}^2. \tag{4.70}$$

It follows from (4.58a,b), (4.70) and (4.1) that

$$\|(Q_h^M - \widetilde{Q}_h^M)\widehat{\psi}\|_{\widehat{X}}^2 \leq C\,\|\widehat{\psi}\|_{\widehat{X}}^2. \tag{4.71}$$

The desired result (4.69) then follows from (4.71), (4.68), (4.58a,b), (4.59c) and (4.46c,d). □

We are now in a position to prove the following stability result for $(\mathrm{P}_\delta^{h,\Delta t})$.

Lemma 4.8. *A solution* $\{\underset{\sim}{u}_{\delta,h}^n, \widehat{\psi}_{\delta,h}^n\}_{n=1}^N$ *of* $(\mathrm{P}_\delta^{h,\Delta t})$ *satisfies the following stability bounds:*

$$\begin{aligned}
&\max_{n=1\to N} \|\underset{\sim}{u}_{\delta,h}^n\|_{L^2(\Omega)}^2 + \max_{n=1\to N} (M, \pi_h[\mathcal{F}_\delta^L(\widehat{\psi}_{\delta,h}^n)])_{\Omega\times D} + \sum_{n=1}^N \Delta t_n\,\|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\delta,h}^n\|_{L^2(\Omega)}^2 + \sum_{n=1}^N \|\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^2(\Omega)}^2 \\
&\quad + \delta \sum_{n=1}^N \Delta t_n \left[(M, \pi_h[\,|\underset{\sim}{\nabla}_x\,(\pi_h[\,[\mathcal{F}_\delta^L]'(\widehat{\psi}_{\delta,h}^n)])|^2])_{\Omega\times D} + (M, \pi_h[\,|\underset{\sim}{\nabla}_q\,(\pi_h[\,[\mathcal{F}_\delta^L]'(\widehat{\psi}_{\delta,h}^n)])|^2])_{\Omega\times D}\right] \\
&\qquad \leq C\,\left[\|\underset{\sim}{u}_{\delta,h}^0\|_{L^2(\Omega)}^2 + (M, \pi_h[\mathcal{F}_\delta^L(\widehat{\psi}_{\delta,h}^0)])_{\Omega\times D} + \sum_{n=1}^n \Delta t_n\,\|\underset{\sim}{f}^n\|_{H^{-1}(\Omega)}^2\right] \leq C,
\end{aligned} \tag{4.72a}$$

$$\begin{aligned}
&\max_{n=1\to N} (M, \pi_h[\,|\widehat{\psi}_{\delta,h}^n|^2]\,)_{\Omega\times D} + \sum_{n=1}^N \Delta t_n\,(M, \pi_h[\,|\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\delta,h}^n|^2 + |\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\delta,h}^n|^2\,]\,)_{\Omega\times D} + \sum_{n=1}^N (M, \pi_h[\,|\widehat{\psi}_{\delta,h}^n - \widehat{\psi}_{\delta,h}^{n-1}|^2\,]\,)_{\Omega\times D} \\
&\qquad \leq C(L) + C\,(M, \pi_h[\,|\widehat{\psi}_h^0|^2\,]\,)_{\Omega\times D} \leq C(L),
\end{aligned} \tag{4.72b}$$

and

$$\max_{n=1\to N} \left[\int_\Omega |\underset{\approx}{C}(M\,\widehat{\psi}_{\delta,h}^n)|^2\,\mathrm{d}\underset{\sim}{x}\right] + \sum_{n=1}^N \Delta t_n \left\|\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right\|_{H^1(\Omega)}^{\frac{4}{\vartheta}} + \sum_{n=1}^N \Delta t_n \left\|\mathcal{G}\left(\frac{\widehat{\psi}_{\delta,h}^n - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right\|_{\widehat{X}}^2 \leq C(L,T), \tag{4.72c}$$

where

$$\vartheta \in (2,4) \quad \textit{if } d = 2 \quad \textit{and} \quad \vartheta = 3 \quad \textit{if } d = 3. \tag{4.73}$$

Proof. Summing (4.33) from $n = 1 \to m$, for $m = 1 \to N$, yields the desired result (4.72a) on noting (4.26a,b), (4.29), (2.15) and (4.31a).

On choosing $\widehat{\varphi}_h = \widehat{\psi}_{\delta,h}^n$ in (4.32b) and noting (4.35), we obtain

$$\begin{aligned}
\mathrm{T}^n &:= \left(M, \pi_h\left[|\widehat{\psi}_{\delta,h}^n|^2 + |\widehat{\psi}_{\delta,h}^n - \widehat{\psi}_{\delta,h}^{n-1}|^2\right]\right)_{\Omega\times D} + \Delta t_n\,\left(M, \pi_h\left[2\,\varepsilon\,|\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\delta,h}^n|^2 + \frac{1}{\lambda}\,|\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\delta,h}^n|^2\right]\right)_{\Omega\times D} \\
&= \left(M, \pi_h\left[|\widehat{\psi}_{\delta,h}^{n-1}|^2\right]\right)_{\Omega\times D} + 2\,\Delta t_n\,\left(M\,(\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\delta,h}^n)\,\underset{\sim}{q}, \underset{\sim}{\pi}_h\left[\underset{\approx}{\Xi}_\delta^q(\widehat{\psi}_{\delta,h}^n)\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\delta,h}^n\right]\right)_{\Omega\times D} \\
&\qquad + 2\,\Delta t_n\,\left(M\,\underset{\sim}{u}_{\delta,h}^n, \underset{\sim}{\pi}_h\left[\underset{\approx}{\Xi}_\delta^x(\widehat{\psi}_{\delta,h}^n)\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\delta,h}^n\right]\right)_{\Omega\times D}.
\end{aligned}$$

Hence, recalling (4.46d) and (4.24), for any $\eta \in \mathbb{R}_{>0}$, we have that

$$
\begin{aligned}
\mathrm{T}^n \leq{}& \left(M, \pi_h\left[|\widehat{\psi}_{\delta,h}^{n-1}|^2\right]\right)_{\Omega\times D} + \Delta t_n\, C(\eta^{-1})\, \left[\|\underset{\sim}{u}_{\delta,h}^n\|_{L^2(\Omega)}^2 + \|\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\delta,h}^n\|_{L^2(\Omega)}^2\right] \\
&+ \Delta t_n\, \eta\, \left(M, \left|\underset{\sim}{\pi}_h\left[\underset{\approx}{\Xi}_\delta^x(\widehat{\psi}_{\delta,h}^n)\, \underset{\sim}{\nabla}_x\, \widehat{\psi}_{\delta,h}^n\right]\right|^2 + \left|\underset{\sim}{\pi}_h\left[\underset{\approx}{\Xi}_\delta^q(\widehat{\psi}_{\delta,h}^n)\, \underset{\sim}{\nabla}_q\, \widehat{\psi}_{\delta,h}^n\right]\right|^2\right)_{\Omega\times D} \\
\leq{}& \left(M, \pi_h\left[|\widehat{\psi}_{\delta,h}^{n-1}|^2\right]\right)_{\Omega\times D} + \Delta t_n\, C(\eta^{-1})\, \left[\|\underset{\sim}{u}_{\delta,h}^n\|_{L^2(\Omega)}^2 + \|\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\delta,h}^n\|_{L^2(\Omega)}^2\right] \\
&+ \Delta t_n\, C\, L^2\, \eta\, \left(M, \pi_h\left[|\underset{\sim}{\nabla}_x\, \widehat{\psi}_{\delta,h}^n|^2 + |\underset{\sim}{\nabla}_q\, \widehat{\psi}_{\delta,h}^n|^2\right]\right)_{\Omega\times D}.
\end{aligned}
\tag{4.74}
$$

On noting the definition of T^n, summing (4.74) from $n = 1 \to m$, for $m = 1 \to N$, with η chosen sufficiently small, and recalling inequalities (4.72a) and (4.29), yields the desired result (4.72b).

The first bound in (4.72c) follows immediately from the first bound in (4.72b), (3.15) and (4.46c).

On choosing $\underset{\sim}{w}_h = \underset{\sim}{Q}_h\left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right] \in \underset{\sim}{V}_h$ in (4.32a) yields, on noting (4.6), (3.3), (4.7) and Sobolev embedding, that

$$
\begin{aligned}
\left\|\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right\|_{H^1(\Omega)}^2 ={}& \left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}, \underset{\sim}{Q}_h\left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right]\right)_\Omega \\
={}& -\nu\, \left(\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\delta,h}^n, \underset{\approx}{\nabla}_x\, \left[\underset{\sim}{Q}_h\left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right]\right]\right)_\Omega \\
&- k\,\mu\, \left(\underset{\approx}{C}(M\,\widehat{\psi}_{\delta,h}^n), \underset{\approx}{\nabla}_x\, \left[\underset{\sim}{Q}_h\left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right]\right]\right)_\Omega \\
&- \frac{1}{2}\, \left((\underset{\sim}{u}_{\delta,h}^{n-1}\cdot \underset{\sim}{\nabla}_x\,)\underset{\sim}{u}_{\delta,h}^n, \underset{\sim}{Q}_h\left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right]\right)_\Omega \\
&+ \frac{1}{2}\, \left(\underset{\sim}{u}_{\delta,h}^n, (\underset{\sim}{u}_{\delta,h}^{n-1}\cdot \underset{\sim}{\nabla}_x\,)\left[\underset{\sim}{Q}_h\left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right]\right]\right)_\Omega \\
&+ \left\langle \underset{\sim}{f}^n, \underset{\sim}{Q}_h\left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}_{\delta,h}^n - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n}\right)\right]\right\rangle_{H_0^1(\Omega)} \\
\leq{}& C\,\big[\|\underset{\approx}{C}(M\,\widehat{\psi}_{\delta,h}^n)\|_{L^2(\Omega)}^2 + \|\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\delta,h}^n\|_{L^2(\Omega)}^2 + \|\,|\underset{\sim}{u}_{\delta,h}^{n-1}|\,|\underset{\sim}{u}_{\delta,h}^n|\,\|_{L^2(\Omega)}^2 \\
&+ \|\,|\underset{\sim}{u}_{\delta,h}^{n-1}|\,|\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\delta,h}^n|\,\|_{L^{1+\theta}(\Omega)}^2 + \|\underset{\sim}{f}^n\|_{H^{-1}(\Omega)}^2\big],
\end{aligned}
\tag{4.75}
$$

for any $\theta > 0$ if $d = 2$ and for $\theta = \frac{1}{5}$ if $d = 3$. Applying the Cauchy–Schwarz and the algebraic-geometric mean inequalities, in conjunction with (3.4) and a Poincaré inequality yields that

$$
\begin{aligned}
\|\,|\underset{\sim}{u}_{\delta,h}^{n-1}|\,|\underset{\sim}{u}_{\delta,h}^n|\,\|_{L^2(\Omega)}^2 \leq{}& \|\underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^4(\Omega)}^2\, \|\underset{\sim}{u}_{\delta,h}^n\|_{L^4(\Omega)}^2 \leq \tfrac{1}{2} \sum_{m=n-1}^{n} \|\underset{\sim}{u}_{\delta,h}^m\|_{L^4(\Omega)}^4 \\
\leq{}& C \sum_{m=n-1}^{n} \left[\|\underset{\sim}{u}_{\delta,h}^m\|_{L^2(\Omega)}^{4-d}\, \|\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_{\delta,h}^m\|_{L^2(\Omega)}^d\right].
\end{aligned}
\tag{4.76}
$$

Similarly, we have for any $\theta \in (0,1)$, if $d=2$, that

$$\| \, |\underset{\sim}{u}_{\delta,h}^{n-1}| \, |\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{n}| \, \|_{L^{1+\theta}(\Omega)}^2 \leq \|\underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^{\frac{2(1+\theta)}{1-\theta}}(\Omega)}^2 \, \|\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 \leq C \, \|\underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^2(\Omega)}^{\frac{2(1-\theta)}{1+\theta}} \sum_{m=n-1}^{n} \|\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{m}\|_{L^2(\Omega)}^{\frac{2(1+3\theta)}{1+\theta}} \; ; \tag{4.77a}$$

and if $d=3$, ($\theta = \frac{1}{5}$), that

$$\| \, |\underset{\sim}{u}_{\delta,h}^{n-1}| \, |\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{n}| \, \|_{L^{\frac{6}{5}}(\Omega)}^2 \leq \|\underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^3(\Omega)}^2 \, \|\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 \leq C \, \|\underset{\sim}{u}_{\delta,h}^{n-1}\|_{L^2(\Omega)} \sum_{m=n-1}^{n} \|\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{m}\|_{L^2(\Omega)}^3. \tag{4.77b}$$

On taking the $\frac{2}{\vartheta}$ power of both sides of (4.75), recall (4.73), multiplying by Δt_n, summing from $n=1 \to N$ and noting (4.76), (4.77a) with $\theta = (\vartheta-2)/(6-\vartheta)$, (4.77b), (4.27), (4.31a), (4.72a,b), (4.29) and the first bound in (4.72c) yields that

$$\begin{aligned}
&\sum_{n=1}^{N} \Delta t_n \left\| \underset{\sim}{S} \left(\frac{\underset{\sim}{u}_{\delta,h}^{n} - \underset{\sim}{u}_{\delta,h}^{n-1}}{\Delta t_n} \right) \right\|_{H^1(\Omega)}^{\frac{4}{\vartheta}} \\
&\quad \leq C \left[\sum_{n=1}^{N} \Delta t_n \, \|\underset{\approx}{C}(M\,\widehat{\psi}_{\delta,h}^{n})\|_{L^2(\Omega)}^{\frac{4}{\vartheta}} \right] + C(T) \left[\sum_{n=1}^{N} \Delta t_n \left[\|\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 + \|\underset{\sim}{f}^{n}\|_{H^{-1}(\Omega)}^2 \right] \right]^{\frac{2}{\vartheta}} \\
&\qquad + C \left[\max_{n=0\to N} \left(\|\underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 \right)^{\frac{4}{\vartheta}-1} \right] \left[\sum_{n=0}^{N} \Delta t_n \, \|\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 \right] \\
&\quad \leq C(L,T);
\end{aligned} \tag{4.78}$$

and hence the second bound in (4.72c).

On choosing $\widehat{\varphi}_h = \widetilde{Q}_h^M \left[\mathcal{G} \left(\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n} \right) \right] \in \widehat{X}_h$ in (4.32b) yields, on noting (4.67b), (3.13), (4.47b), (4.46d), (4.24) and (4.69), that

$$\begin{aligned}
\left\| \mathcal{G} \left(\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n} \right) \right\|_{\widehat{X}}^2 &= \left(M, \pi_h \left[\left(\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n} \right) \widetilde{Q}_h^M \left[\mathcal{G} \left(\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n} \right) \right] \right] \right)_{\Omega \times D} \\
&= -\frac{1}{2\,\lambda} \left(M, \pi_h \left[\underset{\sim}{\nabla}_q \, \widehat{\psi}_{\delta,h}^{n} \cdot \underset{\sim}{\nabla}_q \left[\widetilde{Q}_h^M \left[\mathcal{G} \left(\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n} \right) \right] \right] \right] \right)_{\Omega \times D} \\
&\qquad - \varepsilon \left(M, \pi_h \left[\underset{\sim}{\nabla}_x \, \widehat{\psi}_{\delta,h}^{n} \cdot \underset{\sim}{\nabla}_x \left[\widetilde{Q}_h^M \left[\mathcal{G} \left(\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n} \right) \right] \right] \right] \right)_{\Omega \times D} \\
&\qquad + \left(M \, (\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{n}) \, \underset{\sim}{q}, \underset{\sim}{\pi}_h \left[\underset{\approx}{\Xi}_{\delta}^{q}(\widehat{\psi}_{\delta,h}^{n}) \, \underset{\sim}{\nabla}_q \left[\widetilde{Q}_h^M \left[\mathcal{G} \left(\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n} \right) \right] \right] \right] \right)_{\Omega \times D} \\
&\qquad + \left(M \, \underset{\sim}{u}_{\delta,h}^{n}, \underset{\sim}{\pi}_h \left[\underset{\approx}{\Xi}_{\delta}^{x}(\widehat{\psi}_{\delta,h}^{n}) \, \underset{\sim}{\nabla}_x \left[\widetilde{Q}_h^M \left[\mathcal{G} \left(\frac{\widehat{\psi}_{\delta,h}^{n} - \widehat{\psi}_{\delta,h}^{n-1}}{\Delta t_n} \right) \right] \right] \right] \right)_{\Omega \times D} \\
&\leq C(L) \left[\|\underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 + \|\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_{\delta,h}^{n}\|_{L^2(\Omega)}^2 \right] + C \left(M, \pi_h \left[|\underset{\sim}{\nabla}_q \, \widehat{\psi}_{\delta,h}^{n}|^2 + |\underset{\sim}{\nabla}_x \, \widehat{\psi}_{\delta,h}^{n}|^2 \right] \right)_{\Omega \times D}.
\end{aligned} \tag{4.79}$$

Multiplying (4.79) by Δt_n, summing from $n = 1 \to N$ and noting (4.72a,b) yields the desired result (4.72c). $\square$

Now we introduce some definitions prior to passing to the limit δ, h, $\Delta t \to 0_+$. Let

$$\underset{\sim}{u}^{\Delta t}_{\delta,h}(\cdot,t) := \frac{t - t_{n-1}}{\Delta t_n}\, \underset{\sim}{u}^{n}_{\delta,h}(\cdot) + \frac{t_n - t}{\Delta t_n}\, \underset{\sim}{u}^{n-1}_{\delta,h}(\cdot), \qquad t \in [t_{n-1}, t_n], \quad n \geq 1, \tag{4.80a}$$

$$\underset{\sim}{u}^{\Delta t,+}_{\delta,h}(\cdot,t) := \underset{\sim}{u}^{n}_{\delta,h}(\cdot), \qquad \underset{\sim}{u}^{\Delta t,-}_{\delta,h}(\cdot,t) := \underset{\sim}{u}^{n-1}_{\delta,h}(\cdot), \qquad t \in (t_{n-1}, t_n], \quad n \geq 1, \tag{4.80b}$$

and
$$\Delta(t) := \Delta t_n, \qquad t \in (t_{n-1}, t_n], \quad n \geq 1. \tag{4.80c}$$

We note for future reference that

$$\underset{\approx}{u}^{\Delta t}_{\delta,h} - \underset{\approx}{u}^{\Delta t,\pm}_{\delta,h} = (t - t_n^{\pm})\, \frac{\partial \underset{\sim}{u}^{\Delta t}_{\delta,h}}{\partial t}, \qquad t \in (t_{n-1}, t_n), \quad n \geq 1, \tag{4.81}$$

where $t_n^+ := t_n$ and $t_n^- := t_{n-1}$. Using the above notation, and introducing analogous notation for $\{\widehat{\psi}^n_{\delta,h}\}_{n=0}^N$ and $\{\underset{\sim}{f}^n\}_{n=1}^N$, (4.32a) multiplied by Δt_n and summed for $n = 1 \to N$ can be restated as:

$$\begin{aligned}
&\int_0^T \int_\Omega \left[\frac{\partial \underset{\sim}{u}^{\Delta t}_{\delta,h}}{\partial t} \cdot \underset{\sim}{w}_h + \nu \underset{\approx}{\nabla}_x \underset{\sim}{u}^{\Delta t,+}_{\delta,h} : \underset{\approx}{\nabla}_x \underset{\sim}{w}_h \right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\
&\quad + \frac{1}{2} \int_0^T \int_\Omega \left[\left[(\underset{\sim}{u}^{\Delta t,-}_{\delta,h} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u}^{\Delta t,+}_{\delta,h} \right] \cdot \underset{\sim}{w}_h - \left[(\underset{\sim}{u}^{\Delta t,-}_{\delta,h} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{w}_h \right] \cdot \underset{\sim}{u}^{\Delta t,+}_{\delta,h} \right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\
&\qquad = \int_0^T \left[\langle \underset{\sim}{f}^+, \underset{\sim}{w}_h \rangle_{H^1_0(\Omega)} - k\, \mu \int_\Omega \underset{\approx}{C}(M\, \widehat{\psi}^{\Delta t,+}_{\delta,h}) : \underset{\approx}{\nabla}_x \underset{\sim}{w}_h \, \mathrm{d}\underset{\sim}{x} \right] \mathrm{d}t \qquad \forall \underset{\sim}{w}_h \in L^{\frac{4}{4-\vartheta}}(0,T; \underset{\sim}{V}_h),
\end{aligned} \tag{4.82}$$

where ϑ is as defined in (4.73). Similarly, (4.32b) multiplied by Δt_n and summed for $n = 1 \to N$ can be restated as:

$$\begin{aligned}
&\int_0^T \int_{\Omega \times D} M\, \pi_h \left[\frac{\partial \widehat{\psi}^{\Delta t}_{\delta,h}}{\partial t}\, \widehat{\varphi}_h \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t - \int_0^T \int_{\Omega \times D} M\, \underset{\sim}{u}^{\Delta t,+}_{\delta,h} \cdot \underset{\sim}{\pi}_h \left[\underset{\approx}{\Xi}^x_\delta(\widehat{\psi}^{\Delta t,+}_{\delta,h})\, \underset{\sim}{\nabla}_x \widehat{\varphi}_h \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega \times D} M\, \pi_h \left[\frac{1}{2\,\lambda} \underset{\sim}{\nabla}_q \widehat{\psi}^{\Delta t,+}_{\delta,h} \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}_h + \varepsilon\, \underset{\sim}{\nabla}_x \widehat{\psi}^{\Delta t,+}_{\delta,h} \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}_h \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \\
&\quad - \int_0^T \int_{\Omega \times D} M\, (\underset{\approx}{\nabla}_x \underset{\sim}{u}^{\Delta t,+}_{\delta,h} \underset{\sim}{q}) \cdot \underset{\sim}{\pi}_h \left[\underset{\approx}{\Xi}^q_\delta(\widehat{\psi}^{\Delta t,+}_{\delta,h})\, \underset{\sim}{\nabla}_q \widehat{\varphi}_h \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t = 0 \qquad \forall \widehat{\varphi}_h \in L^2(0,T; \widehat{X}_h).
\end{aligned} \tag{4.83}$$

It follows from (4.72a–c), (4.80a–c), (2.22) and (4.46c,d) that

$$\begin{aligned}
&\sup_{t \in (0,T)} \left[\|\underset{\sim}{u}^{\Delta t(,\pm)}_{\delta,h}\|^2_{L^2(\Omega)} \right] + \frac{1}{\delta} \sup_{t \in (0,T)} \left[(M, \pi_h[\, [\widehat{\psi}^{\Delta t(,\pm)}_{\delta,h}]^2_- \,])_{\Omega \times D} \right] + \int_0^T \|\underset{\approx}{\nabla}_x \underset{\sim}{u}^{\Delta t(,\pm)}_{\delta,h}\|^2_{L^2(\Omega)} \, \mathrm{d}t \\
&\quad + \delta \int_0^T \left[(M, |\underset{\sim}{\nabla}_x (\pi_h[\, [\mathcal{F}^L_\delta]'(\widehat{\psi}^n_{\delta,h})])|^2 + |\underset{\sim}{\nabla}_q (\pi_h[\, [\mathcal{F}^L_\delta]'(\widehat{\psi}^n_{\delta,h})])|^2)_{\Omega \times D} \right] \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \frac{|\underset{\sim}{u}^{\Delta t,+}_{\delta,h} - \underset{\sim}{u}^{\Delta t,-}_{\delta,h}|^2}{\Delta(t)} \, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \leq C
\end{aligned} \tag{4.84a}$$

and

$$\sup_{t\in(0,T)}\left[(M,|\widehat{\psi}_{\delta,h}^{\Delta t(,\pm)}|^2)_{\Omega\times D}\right]+\int_0^T(M,|\underset{\sim}{\nabla}_q\,\widehat{\psi}_{\delta,h}^{\Delta t,+}|^2+|\underset{\sim}{\nabla}_x\,\widehat{\psi}_{\delta,h}^{\Delta t,+}|^2)_{\Omega\times D}\;\mathrm{d}t$$
$$+\int_0^T\left[\int_{\Omega\times D}M\,\frac{|\widehat{\psi}_{\delta,h}^{\Delta t,+}-\widehat{\psi}_{\delta,h}^{\Delta t,-}|^2}{\Delta(t)}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right]\mathrm{d}t+\sup_{t\in(0,T)}\left[\|\underset{\approx}{C}(\widehat{\psi}_{\delta,h}^{\Delta t(,\pm)})\|^2_{L^2(\Omega)}\right]$$
$$+\int_0^T\left\|\underset{\sim}{S}\,\frac{\partial \underset{\sim}{u}_{\delta,h}^{\Delta t}}{\partial t}\right\|^{\frac{4}{\vartheta}}_{H^1(\Omega)}\mathrm{d}t+\int_0^T\left\|\mathcal{G}\,\frac{\partial\widehat{\psi}_{\delta,h}^{\Delta t}}{\partial t}\right\|^2_{\widehat{X}}\mathrm{d}t\le C(L,T), \tag{4.84b}$$

where ϑ is as defined in (4.73). In the above and throughout, the notation $\underset{\sim}{u}_{\delta,h}^{\Delta t(,\pm)}$ means $\underset{\sim}{u}_{\delta,h}^{\Delta t}$ with or without the superscripts $\pm$, and similarly $\widehat{\psi}_{\delta,h}^{\Delta t(,\pm)}$.

Before proving a convergence result for $(\mathrm{P}_\delta^{h,\Delta t})$, we need the following result.

Lemma 4.9. *For all $\kappa_x\in\mathcal{T}_h^x$, $\kappa_q\in\mathcal{T}_h^q$ and for all $\widehat{\varphi}_h\in\widehat{X}_h$ we have that*

$$\int_{\kappa_x\times\kappa_q}M\,|\underset{\approx}{\Xi}_\delta^x(\widehat{\varphi}_h)-\beta^L(\widehat{\varphi}_h)\,\underset{\approx}{I}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$\le C\left(\delta^2+h_x^2\int_{\kappa_x\times\kappa_q}M\,|\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}+\int_{\kappa_x\times\kappa_q}M\,\pi_{h,\kappa_x\times\kappa_q}\left[[\widehat{\varphi}_h]_-^2\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right), \tag{4.85a}$$

$$\int_{\kappa_x\times\kappa_q}M\,|\underset{\approx}{\Xi}_\delta^q(\widehat{\varphi}_h)-\beta^L(\widehat{\varphi}_h)\,\underset{\approx}{I}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$\le C\left(\delta^2+h_q^2\int_{\kappa_x\times\kappa_q}M\,|\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}+\int_{\kappa_x\times\kappa_q}M\,\pi_{h,\kappa_x\times\kappa_q}\left[[\widehat{\varphi}_h]_-^2\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right). \tag{4.85b}$$

Proof. Firstly, we have from (4.21), (4.19), (2.17) and (4.53b) that

$$\int_{\kappa_x\times\kappa_q}M\,|\underset{\approx}{\Xi}_\delta^x(\widehat{\varphi}_h)-\beta_\delta^L(\widehat{\varphi}_h)\,\underset{\approx}{I}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\le\left(\int_{\kappa_x\times\kappa_q}M\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)\|\underset{\approx}{\Lambda}_\delta^x(\widehat{\varphi}_h)-\beta_\delta^L(\widehat{\varphi}_h)\,\underset{\approx}{I}\|^2_{L^\infty(\kappa_x\times\kappa_q)}$$
$$\le C\,h_x^2\left(\int_{\kappa_x\times\kappa_q}M\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right)\|\underset{\sim}{\nabla}_x\,[\beta_\delta^L(\widehat{\varphi}_h)]\|^2_{L^\infty(\kappa_x\times\kappa_q)}$$
$$\le C\,h_x^2\int_{\kappa_x\times\kappa_q}M\,|\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \tag{4.86}$$

Similarly, we have from (4.21), (4.19), (2.17) and (4.51a) that

$$\int_{\kappa_x\times\kappa_q}M\,|\underset{\approx}{\Xi}_\delta^q(\widehat{\varphi}_h)-\beta_\delta^L(\widehat{\varphi}_h)\,\underset{\approx}{I}|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\le C\,h_q^2\int_{\kappa_x\times\kappa_q}M\,|\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h|^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \tag{4.87}$$

Next we note from (2.17) and (2.25) that, for all $s\in\mathbb{R}$,

$$|\beta_\delta^L(s)-\beta^L(s)|\le\delta-[s]_-. \tag{4.88}$$

In addition, we note that

$$[\widehat{\varphi}_h]_-(\underset{\sim}{x},\underset{\sim}{q})\ge\pi_{h,\kappa_x\times\kappa_q}\,\left[[\widehat{\varphi}_h]_-\right](\underset{\sim}{x},\underset{\sim}{q})\qquad\forall(\underset{\sim}{x},\underset{\sim}{q})\in\kappa_x\times\kappa_q. \tag{4.89}$$

Hence (4.88), (4.89) and (4.46c) yield that

$$\begin{aligned}\int_{\kappa_x \times \kappa_q} M\, |\beta^L_\delta(\widehat{\varphi}_h) - \beta^L(\widehat{\varphi}_h)|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} &\leq \int_{\kappa_x \times \kappa_q} M\, |\delta - [\widehat{\varphi}_h]_-|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ &\leq \int_{\kappa_x \times \kappa_q} M\, |\delta - \pi_{h,\kappa_x\times\kappa_q}\, [\widehat{\varphi}_h]_-|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\ &\leq C \left[\delta^2 + \int_{\kappa_x \times \kappa_q} M\, |\pi_{h,\kappa_x\times\kappa_q}\, [\widehat{\varphi}_h]_-|^2 \right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}. \end{aligned} \tag{4.90}$$

Combining (4.86), (4.87) and (4.90) yields the desired results (4.85a,b). □

We are now in a position to prove the following convergence result for $(\mathrm{P}^{h,\Delta t}_\delta)$.

Theorem 4.10. *There exists a subsequence of* $\{\{\underset{\sim}{u}^{\Delta t}_{\delta,h}, \widehat{\psi}^{\Delta t}_{\delta,h}\}\}_{\delta>0,h>0,\Delta t>0}$*, and functions* $\underset{\sim}{u} \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{\vartheta}}(0,T;\underset{\sim}{V}')$ *and* $\widehat{\psi} \in L^\infty(0,T;L^2_M(\Omega\times D)) \cap L^2(0,T;\widehat{X}) \cap H^1(0,T;\widehat{X}')$ *with* $\widehat{\psi} \geq 0$ *a.e. in* $\Omega \times D \times (0,T)$ *such that, as* δ, h, $\Delta t \to 0_+$,

$$\underset{\sim}{u}^{\Delta t(,\pm)}_{\delta,h} \to \underset{\sim}{u} \qquad \text{weak* in } L^\infty(0,T;\underset{\sim}{L}^2(\Omega)), \tag{4.91a}$$
$$\underset{\sim}{u}^{\Delta t(,\pm)}_{\delta,h} \to \underset{\sim}{u} \qquad \text{weakly in } L^2(0,T;\underset{\sim}{H}^1_0(\Omega)), \tag{4.91b}$$
$$\underset{\sim}{S}\frac{\partial \underset{\sim}{u}_{\delta,h}}{\partial t} \to \underset{\sim}{S}\frac{\partial \underset{\sim}{u}}{\partial t} \qquad \text{weakly in } L^{\frac{4}{\vartheta}}(0,T;\underset{\sim}{V}), \tag{4.91c}$$
$$\underset{\sim}{u}^{\Delta t(,\pm)}_{\delta,h} \to \underset{\sim}{u} \qquad \text{strongly in } L^2(0,T;\underset{\sim}{L}^r(\Omega)), \tag{4.91d}$$

and

$$M^{\frac12}\, \widehat{\psi}^{\Delta t(,\pm)}_{\delta,h} \to M^{\frac12}\, \widehat{\psi} \qquad \text{weak* in } L^\infty(0,T;L^2(\Omega\times D)), \tag{4.92a}$$
$$M^{\frac12}\, \underset{\sim}{\nabla}_q\, \widehat{\psi}^{\Delta t,+}_{\delta,h} \to M^{\frac12}\, \underset{\sim}{\nabla}_q\, \widehat{\psi} \qquad \text{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), \tag{4.92b}$$
$$M^{\frac12}\, \underset{\sim}{\nabla}_x\, \widehat{\psi}^{\Delta t,+}_{\delta,h} \to M^{\frac12}\, \underset{\sim}{\nabla}_x\, \widehat{\psi} \qquad \text{weakly in } L^2(0,T;\underset{\sim}{L}^2(\Omega\times D)), \tag{4.92c}$$
$$\mathcal{G}\frac{\partial \widehat{\psi}^{\Delta t}_{\delta,h}}{\partial t} \to \mathcal{G}\frac{\partial \widehat{\psi}}{\partial t} \qquad \text{weakly in } L^2(0,T;\widehat{X}), \tag{4.92d}$$
$$M^{\frac12}\, \widehat{\psi}^{\Delta t(,\pm)}_{\delta,h} \to M^{\frac12}\, \widehat{\psi} \qquad \text{strongly in } L^2(0,T;L^2(\Omega\times D)), \tag{4.92e}$$
$$M^{\frac12}\, \underset{\approx}{\Xi}^x_\delta(\widehat{\psi}^{\Delta t(,\pm)}_{\delta,h}) \to M^{\frac12}\, \beta^L(\widehat{\psi})\, \underset{\approx}{I} \qquad \text{strongly in } L^2(0,T;\underset{\approx}{L}^2(\Omega\times D)), \tag{4.92f}$$
$$M^{\frac12}\, \underset{\approx}{\Xi}^q_\delta(\widehat{\psi}^{\Delta t(,\pm)}_{\delta,h}) \to M^{\frac12}\, \beta^L(\widehat{\psi})\, \underset{\approx}{I} \qquad \text{strongly in } L^2(0,T;\underset{\approx}{L}^2(\Omega\times D)), \tag{4.92g}$$
$$\underset{\approx}{C}(M\, \widehat{\psi}^{\Delta t(,\pm)}_{\delta,h}) \to \underset{\approx}{C}(M\, \widehat{\psi}) \qquad \text{strongly in } L^2(0,T;\underset{\approx}{L}^2(\Omega)); \tag{4.92h}$$

where ϑ *is defined by (4.73) and* $r \in [1,\infty)$ *if* $d = 2$ *and* $r \in [1,6)$ *if* $d = 3$.

Furthermore, $\{\underset{\sim}{u}, \widehat{\psi}\}$ *solves the following problem:*

(**P**) *Find* $\underset{\sim}{u} \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{\vartheta}}(0,T;\underset{\sim}{V}')$ *and* $\widehat{\psi} \in L^\infty(0,T;L^2_M(\Omega\times D)) \cap L^2(0,T;\widehat{X}) \cap H^1(0,T;\widehat{X}')$ *with* $\widehat{\psi} \geq 0$ *a.e. in* $\Omega \times D \times (0,T)$ *and* $\underset{\approx}{C}(M\,\widehat{\psi}) \in L^\infty(0,T;\underset{\approx}{L}^2(\Omega))$*, such that* $\underset{\sim}{u}(\cdot,0) = \underset{\sim}{u}^0(\cdot)$,

$\widehat{\psi}(\cdot,\cdot,0) = \widehat{\psi}^0(\cdot,\cdot)$ *and*

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}}{\partial t}, \underset{\sim}{w} \right\rangle_V \mathrm{d}t + \int_{\Omega_T} \left[\left[(\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u} \right] \cdot \underset{\sim}{w} + \nu \underset{\approx}{\nabla}_x \underset{\sim}{u} : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$= \int_0^T \langle \underset{\sim}{f}, \underset{\sim}{w} \rangle_{H_0^1(\Omega)} \,\mathrm{d}t - k\,\mu \int_{\Omega_T} \underset{\approx}{C}(M\,\widehat{\psi}) : \underset{\approx}{\nabla}_x \underset{\sim}{w}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{4}{4-\vartheta}}(0,T;\underset{\sim}{V}); \tag{4.93a}$$

$$\int_0^T \left\langle \frac{\partial \widehat{\psi}}{\partial t}, \widehat{\varphi} \right\rangle_{\widehat{X}} \mathrm{d}t + \int_0^T \int_{\Omega \times D} M \left[\varepsilon \underset{\sim}{\nabla}_x \widehat{\psi} - \underset{\sim}{u}\, \beta^L(\widehat{\psi}) \right] \cdot \underset{\sim}{\nabla}_x \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$+ \int_0^T \int_{\Omega \times D} \left[\frac{M}{2\,\lambda} \underset{\sim}{\nabla}_q \widehat{\psi} - (\underset{\approx}{\nabla}_x \underset{\sim}{u})\, \underset{\sim}{q}\, \beta^L(\widehat{\psi}) \right] \cdot \underset{\sim}{\nabla}_q \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t = 0 \qquad \forall \widehat{\varphi} \in L^2(0,T;\widehat{X}). \tag{4.93b}$$

Proof. The results (4.91a–c) follow immediately from the bounds (4.84a,b) on noting the notation (4.80a–c). The denseness of $\bigcup_{h>0} R_h$ in $L^2(\Omega)$ and (4.3d) yield that $\underset{\sim}{u} \in L^2(0,T;\underset{\sim}{V})$. The strong convergence result (4.91d) for $\underset{\sim}{u}^{\Delta t}_{\delta,h}$ follows immediately from (4.91a–c), (3.3) and (3.14), on noting that $\underset{\sim}{V} \subset \underset{\sim}{H}^1_0(\Omega)$ is compactly embedded in $\underset{\sim}{L}^r(\Omega)$ for the stated values of r. We now prove (4.91d) for $\underset{\sim}{u}^{\Delta t,\pm}_{\delta,h}$. First we obtain from the bound on the last term on the left-hand side of (4.84a) and (4.81) that

$$\|\underset{\sim}{u}^{\Delta t}_{\delta,h} - \underset{\sim}{u}^{\Delta t,\pm}_{\delta,h}\|^2_{L^2(0,T,L^2(\Omega))} \leq C\,\Delta t. \tag{4.94}$$

Second, we note from Sobolev embedding that, for all $\eta \in L^2(0,T;H^1(\Omega))$,

$$\|\eta\|_{L^2(0,T;L^r(\Omega))} \leq \|\eta\|^{\theta}_{L^2(0,T;L^2(\Omega))} \|\eta\|^{1-\theta}_{L^2(0,T;L^s(\Omega))} \leq C\,\|\eta\|^{\zeta}_{L^2(0,T;L^2(\Omega))} \|\eta\|^{1-\theta}_{L^2(0,T;H^1(\Omega))} \tag{4.95}$$

for all $r \in [2,s)$, with any $s \in (2,\infty)$ if $d=2$ or any $s \in (2,6]$ if $d=3$, and $\theta = [2\,(s-r)]/[r\,(s-2)] \in (0,1]$. Hence, combining (4.94), (4.95) and (4.91d) for $\underset{\sim}{u}^{\Delta t}_{\delta,h}$ yields (4.91d) for $\underset{\sim}{u}^{\Delta t,\pm}_{\delta,h}$.

The result (4.92a) follows immediately from the bounds on the first and third terms on the left-hand side of (4.84b). It follows immediately from the bound on the second term on the left-hand side of (4.84b) that (4.92b) holds for some limit $\underset{\sim}{g} \in L^2(0,T;\underset{\sim}{L}^2(\Omega \times D))$, which we need to identify. However for any $\underset{\sim}{\eta} \in L^2(0,T;\underset{\sim}{C}^\infty_0(\Omega \times D))$, it follows from (2.5) and the compact support of $\underset{\sim}{\eta}$ on D that $[\underset{\sim}{\nabla}_q \cdot (M^{\frac{1}{2}}\,\underset{\sim}{\eta})]/M^{\frac{1}{2}} \in L^2(0,T;L^2(\Omega \times D))$ and hence the above convergence implies, on noting (4.92a), that

$$\int_0^T \int_{\Omega \times D} \underset{\sim}{g} \cdot \underset{\sim}{\eta}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \quad \leftarrow \quad -\int_0^T \int_{\Omega \times D} M^{\frac{1}{2}}\, \widehat{\psi}^{\Delta t,+}_{\delta,h}\, \frac{\underset{\sim}{\nabla}_q \cdot (M^{\frac{1}{2}}\,\underset{\sim}{\eta})}{M^{\frac{1}{2}}}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$
$$\rightarrow \quad -\int_0^T \int_{\Omega \times D} M^{\frac{1}{2}}\, \widehat{\psi}\, \frac{\underset{\sim}{\nabla}_q \cdot (M^{\frac{1}{2}}\,\underset{\sim}{\eta})}{M^{\frac{1}{2}}}\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t \tag{4.96}$$

as δ, h, $\Delta t \to 0_+$. Hence the desired result (4.92b) follows from (4.96) on noting the denseness of $C^\infty_0(\Omega \times D)$ in $L^2(\Omega \times D)$. Similar arguments also prove (4.92c,d) on noting (4.92a) and the second and sixth bounds in (4.84b). The strong convergence result (4.92e) for $\widehat{\psi}^{\Delta t}_{\delta,h}$ follows immediately from (4.92a–c), (3.13), (3.14) and (3.11b). Similarly to (4.94), the third bound in (4.84b) then yields that (4.92e) holds for $\widehat{\psi}^{\Delta t(,\pm)}_{\delta,h}$. The desired results (4.92f,g) follow immediately from (4.85a,b) the second bounds in (4.84a,b), (2.25) and (4.92e). The desired result (4.92h) follows immediately from (4.92a), (2.3) and (3.15). Finally, the nonnegativity of $\widehat{\psi}$ follows from (4.92e) and the second bound in (4.84a).

It remains to prove that $\{\underset{\sim}{u}, \widehat{\psi}\}$ solve (P). It follows from (4.5), (4.84a,b), (4.91a–d), (4.92h), (4.31b), (3.2) and (4.8) that we may pass to the limit, δ, h, $\Delta t \to 0_+$, in (4.82) to obtain that $\underset{\sim}{u} \in L^\infty(0,T;\underset{\sim}{L}^2(\Omega)) \cap L^2(0,T;\underset{\sim}{V}) \cap$

$W^{1,\frac{4}{\vartheta}}(0,T;\underset{\sim}{V}')$ and $\underset{\approx}{C}(M\,\widehat{\psi}) \in L^\infty(0,T;\underset{\approx}{L}^2(\Omega))$ satisfy (4.93a). It also follows from (4.28a), (4.5), (4.84a) and (4.91d) that $\underset{\sim}{u}(\cdot,0) = \underset{\sim}{u}^0(\cdot)$ in the required sense; recall Remark 3.1.

It follows from (4.92a–g), (4.91b,d), (3.12), (4.59a–c), (4.64), (4.84a,b), (4.48a,b), (4.49) and (4.50a,b) that we may pass to the limit δ, h, $\Delta t \to 0_+$ in (4.83) with $\widehat{\varphi}_h = \pi_h\,\widehat{\varphi}$ to obtain (4.93b) for any $\widehat{\varphi} \in C_0^\infty(0,T;C(\overline{\Omega\times D}))$. In order to pass to the limit on the first term in (4.83), we note that

$$\int_0^T \int_{\Omega\times D} M\,\pi_h \left[\frac{\partial \widehat{\psi}^{\Delta t}_{\delta,h}}{\partial t}\,[\pi_h\,\widehat{\varphi}]\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$
$$= \int_0^T \int_{\Omega\times D} M\,\frac{\partial \widehat{\psi}^{\Delta t}_{\delta,h}}{\partial t}\,[\pi_h\,\widehat{\varphi}]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t + \int_0^T \int_{\Omega\times D} M\,(I-\pi_h)\left[\widehat{\psi}^{\Delta t}_{\delta,h}\,\frac{\partial[\pi_h\,\widehat{\varphi}]}{\partial t}\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t. \tag{4.97}$$

The desired result (4.93b) then follows from noting that $C_0^\infty(0,T;C(\overline{\Omega\times D}))$ is dense in $L^2(0,T;\widehat{X})$, on recalling (3.8). Finally, it follows from (4.28b), (4.59c), (4.51c), (4.53a), (4.50a,b), (3.8), (4.84b) and (4.92e) that $\widehat{\psi}(\cdot,\cdot,0) = \widehat{\psi}^0(\cdot,\cdot)$ in the required sense; recall Remark 3.1. □

Remark 4.11. We note that (P), (4.93a,b), differs slightly from (P^L), (3.16a,b), in that $\underset{\sim}{u} \in W^{1,\frac{4}{\vartheta}}(0,T,\underset{\sim}{V}')$ for the stated value of ϑ, recall (4.73), is slightly weaker than $\underset{\sim}{u}^L \in W^{1,\frac{4}{d}}(0,T,\underset{\sim}{V}')$ in the case $d=2$ with the subsequent slight strengthening of the regularity of the test functions in (4.93a). In addition, $\widehat{\psi}^L$ in the convective term in (3.16b) is replaced by $\beta^L(\widehat{\psi})$ in (4.93b). It does not appear possible to construct a variation of the finite element approximation (P$^{h,\Delta t}_\delta$) that converges to the former version of the convective term, and at the same time converges to the other terms in (4.93b). The presence of the cut-off $\beta^L(\cdot)$ in this convective term improves the regularity in time of $\widehat{\psi}$ in (4.93a,b), to that in (3.16a,b), and hence the weakening of the regularity in time of the test functions in (4.93b).

Remark 4.12. Finally, it follows from (4.84a) and (4.91a,b) that

$$\sup_{t\in(0,T)} \left[\|\underset{\sim}{u}\|^2_{L^2(\Omega)}\right] + \int_0^T \|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}\|^2_{L^2(\Omega)}\,\mathrm{d}t \leq C. \tag{4.98}$$

Hence, although we have introduced a cut-off $L \gg 1$ to $\widehat{\psi}$ in the drag and convective terms, and added diffusion in the $\underset{\sim}{x}$ direction with a positive coefficient $\varepsilon \ll 1$ in the Fokker–Planck equation compared to the standard polymer model; the bound (4.98) on $\underset{\sim}{u}$, the variable of real physical interest, is independent of the parameters L and ε.

Appendix A. Quasi-interpolation in Maxwellian-weighted Sobolev norms

The aim of this Appendix is to prove the stability result (4.68). To do so, we first need to show certain quasi-interpolation results in Maxwellian weighted Sobolev spaces. The starting point for the construction of the relevant quasi-interpolation operators is the Brascamp–Lieb inequality stated below.

Suppose that D is a convex open set, $D \subset \mathbb{R}^d$ (*e.g.*, a bounded open ball in $\mathbb{R}^d$ centred at the origin; or, more specifically, in the case of the FENE model, $D = B(\underset{\sim}{0};b^{\frac{1}{2}})$, $b>2$). Consider a probability measure μ supported on D with density $\mathrm{e}^{-V(\underset{\sim}{q})}$, $\underset{\sim}{q} \in D$, with respect to the Lebesgue measure $\mathrm{d}\underset{\sim}{q}$ on $\mathbb{R}^d$, where V is a convex function on D; μ is usually referred to as a *Gibbs measure*. In particular,

$$\mu(B) = \int_B \mathrm{d}\mu = \int_B \mathrm{e}^{-V(\underset{\sim}{q})}\,\mathrm{d}\underset{\sim}{q},$$

for any μ-measurable set $B \subset D$, with $\mu(D) = 1$. The following geometric functional inequality comes from the paper of Bobkov & Ledoux [8].

Theorem A.1. (Brascamp–Lieb inequality) *Assume that V is a twice continuously differentiable and convex function on a convex open set $D \subset \mathbb{R}^d$, such that, for each $\underset{\sim}{q} \in D$, the Hessian*

$$\underset{\approx}{H}(\underset{\sim}{q}) := \left(\frac{\partial^2 V(\underset{\sim}{q})}{\partial q_i\, \partial q_j} \right)$$

is positive definite. Then, for any sufficiently smooth function f,

$$\mathrm{Var}_\mu[f] := \mathbb{E}_\mu[(f - \mathbb{E}_\mu[f])^2] \leq \int_D [\underset{\approx}{H}^{-1}(\underset{\sim}{q})\, \underset{\sim}{\nabla}_q f] \cdot \underset{\sim}{\nabla}_q f \,\mathrm{d}\mu, \qquad \textit{where} \quad \mathbb{E}_\mu[f] = \int_D f \,\mathrm{d}\mu.$$

In terms of simpler notation, the Brascamp–Lieb inequality can be restated as follows:

$$\int_D \left[f(\underset{\sim}{q}) - \int_D f(\underset{\sim}{p})\, \mathrm{e}^{-V(\underset{\sim}{p})}\, \mathrm{d}\underset{\sim}{p} \right]^2 \mathrm{e}^{-V(\underset{\sim}{q})}\, \mathrm{d}\underset{\sim}{q} \leq \int_D [\underset{\approx}{H}^{-1}(\underset{\sim}{q})\, \underset{\sim}{\nabla}_q f] \cdot \underset{\sim}{\nabla}_q f \;\mathrm{e}^{-V(\underset{\sim}{q})}\, \mathrm{d}\underset{\sim}{q},$$

for any sufficiently smooth function f.

A.1. **The univariate case**

Suppose that $d = 1$, $D := (0, q_1) \subset \mathbb{R}$, and $V(q) := \ln \left[\frac{q_1}{\alpha+1} \left(\frac{q_1}{q} \right)^\alpha \right]$ with $\alpha > 0$. Clearly, $\int_D \mathrm{e}^{-V(q)}\,\mathrm{d}q = 1$. By the Brascamp–Lieb inequality,

$$\int_0^{q_1} \left[f(q) - \frac{\alpha+1}{q_1^{\alpha+1}} \int_0^{q_1} f(p)\, p^\alpha \,\mathrm{d}p \right]^2 q^\alpha \,\mathrm{d}q \leq \frac{1}{\alpha} \int_0^{q_1} |f'(q)|^2 q^{\alpha+2} \,\mathrm{d}q \leq \frac{q_1^2}{\alpha} \int_0^{q_1} |f'(q)|^2 q^\alpha \,\mathrm{d}q. \tag{A.1}$$

Let us consider the nonuniform partition $0 = q_0 < q_1 < \cdots < q_N = 1$ of the interval $[0, 1]$, with $h_q := \max_{k=1\to N}(q_k - q_{k-1})$, and let $\widehat{X}_h^q$ denote the set of all continuous piecewise linear functions defined on this partition. For $m \in \mathbb{Z}_{\geq 0}$ and a nonempty open interval $(a, b) \subset \mathbb{R}_{>0}$, let

$$H^m((a,b); q^\alpha) := \{ \widehat{\varphi} \in H^m_{\mathrm{loc}}(a,b) \;:\; \|\widehat{\varphi}\|^2_{H^m((a,b);q^\alpha)} := \sum_{k=0}^m \int_a^b |\widehat{\varphi}^{(k)}(q)|^2 q^\alpha \,\mathrm{d}q < \infty \}.$$

When $m = 0$, we write $L^2((a,b); q^\alpha)$ instead of $H^0((a,b); q^\alpha)$.

For $\widehat{\psi} \in H^1((0,1); q^\alpha)$, let $I_h^q \widehat{\psi} \in \widehat{X}_h^q$ denote the continuous piecewise linear (quasi-)interpolant of $\widehat{\psi}$, defined by

$$(I_h^q \widehat{\psi})(q) := \begin{cases} \widehat{\psi}(q_1) + (q - q_1)\, \frac{\alpha+1}{q_1^{\alpha+1}} \int_0^{q_1} \widehat{\psi}'(p)\, p^\alpha \,\mathrm{d}p, & q \in [0, q_1], \\ \frac{\widehat{\psi}(q_k) - \widehat{\psi}(q_{k-1})}{q_k - q_{k-1}}\, (q - q_{k-1}) + \widehat{\psi}(q_{k-1}), & q \in [q_{k-1}, q_k], \quad k = 2 \to N. \end{cases}$$

We note that since $H^1((0,1); q^\alpha) \subset C(0,1]$, the definition is meaningful. Observe, further, that $(I_h^q \widehat{\psi})(q_k) = \widehat{\psi}(q_k)$, $k = 1 \to N$; i.e. the function $I_h^q \widehat{\psi}$ interpolates $\widehat{\psi}$ at $q = q_k$, $k = 1 \to N$, but not at $q = q_0 = 0$. In the interval $[0, q_1]$ the function $I_h^q \widehat{\psi}$ has been chosen so as to ensure that $(I_h^q \widehat{\psi})'(q) = \frac{\alpha+1}{q_1^{\alpha+1}} \int_0^{q_1} \widehat{\psi}'(p)\, p^\alpha \,\mathrm{d}p$ and $(I_h^q \widehat{\psi})(q_1) = \widehat{\psi}(q_1)$. Hence, on applying the inequality (A.1),

$$\int_0^{q_1} \left[\widehat{\psi}'(q) - (I_h^q \widehat{\psi})'(q) \right]^2 q^\alpha \,\mathrm{d}q \leq \frac{q_1^2}{\alpha} \int_0^{q_1} |\widehat{\psi}''(q)|^2 q^\alpha \,\mathrm{d}q.$$

On the remaining subintervals in the partition, using $q_{k-1}^{\alpha} \leq q^{\alpha} \leq q_k^{\alpha}$ and a standard error bound for the linear interpolant of $\widehat{\psi} \in H^2(q_{k-1}, q_k)$, $k = 2 \to N$, we have that

$$\int_{q_{k-1}}^{q_k} \left[\widehat{\psi}'(q) - (I_h^q \widehat{\psi})'(q)\right]^2 q^{\alpha}\,\mathrm{d}q \leq \left(\frac{q_k}{q_{k-1}}\right)^{\alpha} \frac{(q_k - q_{k-1})^2}{\pi^2} \int_{q_{k-1}}^{q_k} |\widehat{\psi}''(q)|^2\, q^{\alpha}\,\mathrm{d}q, \qquad k = 2 \to N.$$

On summing our bounds through $k = 1 \to N$ and noting that $q_1 \leq h_q$ and $q_k - q_{k-1} \leq h_q$ for $k = 1 \to N$, we obtain

$$\int_0^1 \left[\widehat{\psi}'(q) - (I_h^q \widehat{\psi})'(q)\right]^2 q^{\alpha}\,\mathrm{d}q \leq \max\left(\frac{h_q^2}{\alpha}\,,\, \max_{k=2\to N}\left(\frac{q_k}{q_{k-1}}\right)^{\alpha} \frac{h_q^2}{\pi^2}\right) \int_0^1 |\widehat{\psi}''(q)|^2\, q^{\alpha}\,\mathrm{d}q.$$

We shall henceforth assume that the partition $0 = q_0 < q_1 < \cdots < q_N = 1$ is such that there exists a fixed constant $C_0 > 1$ such that

$$\max_{k=2\to N} \frac{q_k}{q_{k-1}} \leq C_0. \tag{A.2}$$

Now, letting $C_\alpha := \max\left(\frac{1}{\alpha}\,,\, \frac{1}{\pi^2} C_0^{\alpha}\right)$, we get

$$\int_0^1 \left[\widehat{\psi}'(q) - (I_h^q \widehat{\psi})'(q)\right]^2 q^{\alpha}\,\mathrm{d}q \leq C_\alpha\, h_q^2 \int_0^1 |\widehat{\psi}''(q)|^2\, q^{\alpha}\,\mathrm{d}q. \tag{A.3}$$

We note the weighted Poincaré inequality for all $\widehat{v} \in H^1((0,1); q^{\alpha})$ with $\widehat{v}(1) = 0$

$$\begin{aligned}
\int_0^1 |\widehat{v}(q)|^2\, q^{\alpha}\,\mathrm{d}q &= \int_0^1 \left(\int_q^1 \widehat{v}'(t)\, t^{\frac{\alpha}{2}}\, t^{-\frac{\alpha}{2}}\,\mathrm{d}t\right)^2 q^{\alpha}\,\mathrm{d}q \\
&\leq \left(\int_0^1 q^{\alpha}\left(\int_q^1 t^{-\alpha}\,\mathrm{d}t\right)\mathrm{d}q\right) \int_0^1 |\widehat{v}'(q)|^2\, q^{\alpha}\,\mathrm{d}q = \frac{1}{2(\alpha+1)} \int_0^1 |\widehat{v}'(q)|^2\, q^{\alpha}\,\mathrm{d}q,
\end{aligned} \tag{A.4}$$

which, in fact, holds for any $\alpha > -1$. Applying (A.4) with $\widehat{v} = \widehat{\psi} - I_h^q \widehat{\psi}$, and noting (A.3) we deduce that

$$\int_0^1 \left[\widehat{\psi}(q) - (I_h^q \widehat{\psi})(q)\right]^2 q^{\alpha}\,\mathrm{d}q \leq \frac{C_\alpha}{2(\alpha+1)}\, h_q^2 \int_0^1 |\widehat{\psi}''(q)|^2\, q^{\alpha}\,\mathrm{d}q,$$

and therefore

$$\|\widehat{\psi} - I_h^q \widehat{\psi}\|^2_{H^1((0,1);q^{\alpha})} \leq C_\alpha \left(1 + \frac{1}{2(\alpha+1)}\right) h_q^2\, \|\widehat{\psi}''\|^2_{L^2((0,1);q^{\alpha})}. \tag{A.5}$$

Let P_h^q denote the orthogonal projector from $H^1((0,1); q^{\alpha})$ onto $\widehat{X}_h^q$ with respect to the q^{α}-weighted $H^1(0,1)$ inner product

$$a(\widehat{\psi}, \widehat{\varphi}) := \int_0^1 \widehat{\psi}'(q)\, \widehat{\varphi}'(q)\, q^{\alpha}\,\mathrm{d}q + \int_0^1 \widehat{\psi}(q)\, \widehat{\varphi}(q)\, q^{\alpha}\,\mathrm{d}q,$$

where $\alpha > 0$. That is,

$$a(\widehat{\psi} - P_h^q \widehat{\psi}, \widehat{\varphi}_h) = 0 \qquad \forall \widehat{\varphi}_h \in \widehat{X}_h^q. \tag{A.6}$$

Now, consider the following boundary-value problem: Find $\widehat{z} \in H^1((0,1); q^{\alpha})$ such that

$$a(\widehat{\varphi}, \widehat{z}) = \ell(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in H^1((0,1); q^{\alpha}), \tag{A.7}$$

where

$$\ell(\widehat{\varphi}) := \int_0^1 \widehat{g}(q)\, \widehat{\varphi}(q)\, q^{\alpha}\,\mathrm{d}q, \qquad \text{with} \qquad \widehat{g} := \widehat{\psi} - P_h^q \widehat{\psi}.$$

The existence of a unique weak solution $\widehat{z} \in H^1((0,1);q^\alpha)$ to (A.7) follows from the Lax–Milgram theorem. Hence, on taking $\widehat{\varphi} = \widehat{z}$ in (A.7), we obtain

$$\|\widehat{z}\|^2_{H^1((0,1);q^\alpha)} = a(\widehat{z},\widehat{z}) \le \|\widehat{z}\|_{L^2((0,1);q^\alpha)} \|\widehat{\psi} - P^q_h\widehat{\psi}\|_{L^2((0,1);q^\alpha)} \le \|\widehat{z}\|_{H^1((0,1);q^\alpha)} \|\widehat{\psi} - P^q_h\widehat{\psi}\|_{L^2((0,1);q^\alpha)},$$

and therefore

$$\|\widehat{z}\|_{H^1((0,1);q^\alpha)} \le \|\widehat{\psi} - P^q_h\widehat{\psi}\|_{L^2((0,1);q^\alpha)}.$$

Problem (A.7) is the weak form of the following boundary value problem:

$$-\widehat{z}'' - \frac{\alpha}{q}\,\widehat{z}' + \widehat{z} = \widehat{\psi} - P^q_h\widehat{\psi}, \quad q \in (0,1), \qquad \lim_{q\to 0_+} q^\alpha\, \widehat{z}'(q) = 0, \quad \widehat{z}'(1) = 0.$$

Formally differentiating this equation, multiplying the resulting equation by $\widehat{z}' q^\alpha$, integrating over $q \in (0,1)$ and integrating by parts in the first term on the left-hand side and on the right-hand side yields

$$\int_0^1 |\widehat{z}''|^2\, q^\alpha\,\mathrm{d}q + \alpha\int_0^1 |\widehat{z}'|^2\, q^{\alpha-2}\,\mathrm{d}q + \int_0^1 |\widehat{z}'|^2\, q^\alpha\,\mathrm{d}q = -\int_0^1 (\widehat{\psi} - P^q_h\widehat{\psi})\,\widehat{z}''\, q^\alpha\,\mathrm{d}q - \alpha\int_0^1 (\widehat{\psi} - P^q_h\widehat{\psi})\,\widehat{z}'\, q^{\alpha-1}\,\mathrm{d}q.$$

This formal argument can be made rigorous by replacing q^α with $(q+\delta)^\alpha$, $\delta > 0$, in the definitions of $a(\cdot,\cdot)$ and $\ell(\cdot)$ above, and passing to the limit $\delta \to 0_+$; we refer to Section A.2.4 for the details of an analogous, but rigorous, multidimensional argument. Hence,

$$\|\widehat{z}''\|^2_{L^2((0,1);q^\alpha)} + \alpha\,\|\widehat{z}'\|^2_{L^2((0,1);q^{\alpha-2})} + 2\,\|\widehat{z}'\|^2_{L^2((0,1);q^\alpha)} \le (1+\alpha)\,\|\widehat{\psi} - P^q_h\widehat{\psi}\|^2_{L^2((0,1);q^\alpha)}. \tag{A.8}$$

Now, by (A.7) with $\widehat{\varphi} = \widehat{\psi} - P^q_h\widehat{\psi}$, the definition (A.6) of the projector P^q_h, and the bound (A.5),

$$\begin{aligned}
\|\widehat{\psi} - P^q_h\widehat{\psi}\|^2_{L^2((0,1);q^\alpha)} &= a(\widehat{\psi} - P^q_h\widehat{\psi}, \widehat{z}) = a(\widehat{\psi} - P^q_h\widehat{\psi}, \widehat{z} - P^q_h\widehat{z}) \\
&\le \|\widehat{\psi} - P^q_h\widehat{\psi}\|_{H^1((0,1);q^\alpha)}\, \|\widehat{z} - P^q_h\widehat{z}\|_{H^1((0,1);q^\alpha)} \\
&\le \|\widehat{\psi} - P^q_h\widehat{\psi}\|_{H^1((0,1);q^\alpha)}\, \|\widehat{z} - I^q_h\widehat{z}\|_{H^1((0,1);q^\alpha)} \\
&\le \|\widehat{\psi} - P^q_h\widehat{\psi}\|_{H^1((0,1);q^\alpha)} \left[C_\alpha\left(1 + \frac{1}{2(\alpha+1)}\right)\right]^{1/2} h_q\, \|\widehat{z}''\|_{L^2((0,1);q^\alpha)}.
\end{aligned}$$

Thus, by (A.8),

$$\|\widehat{\psi} - P^q_h\widehat{\psi}\|_{L^2((0,1);q^\alpha)} \le \left[C_\alpha\left(\tfrac{3}{2} + \alpha\right)\right]^{1/2} h_q\, \|\widehat{\psi} - P^q_h\widehat{\psi}\|_{H^1((0,1);q^\alpha)}; \tag{A.9}$$

and, denoting by Q^q_h the orthogonal projection in the inner product of $L^2((0,1);q^\alpha)$ onto $\widehat{X}^q_h$, trivially

$$\|\widehat{\psi} - Q^q_h\widehat{\psi}\|_{L^2((0,1);q^\alpha)} \le \left[C_\alpha\left(\tfrac{3}{2} + \alpha\right)\right]^{1/2} h_q\, \|\widehat{\psi} - P^q_h\widehat{\psi}\|_{H^1((0,1);q^\alpha)}. \tag{A.10}$$

Now,

$$\|\widehat{\psi}' - (Q^q_h\widehat{\psi})'\|_{L^2((0,1);q^\alpha)} \le \|\widehat{\psi}' - (P^q_h\widehat{\psi})'\|_{L^2((0,1);q^\alpha)} + \|(P^q_h\widehat{\psi})' - (Q^q_h\widehat{\psi})'\|_{L^2((0,1);q^\alpha)}.$$

Let us, at this point, strengthen the mesh-regularity hypothesis (A.2) by assuming that the partition $0 = q_0 < q_1 < \cdots < q_N = 1$ is quasiuniform. Then, by the inverse inequality

$$\int_{q_{k-1}}^{q_k} |(\widehat{\varphi}_h)'|^2\, q^\alpha\,\mathrm{d}q \le C^2_{\mathrm{inv}}\, h_q^{-2} \int_{q_{k-1}}^{q_k} |\widehat{\varphi}_h|^2\, q^\alpha\,\mathrm{d}q \qquad \forall \widehat{\varphi}_h \in \widehat{X}^q_h,$$

whose proof is identical to that of the first inequality stated in (4.53a), we have that

$$\begin{aligned}\|\widehat{\psi}' - (Q_h^q \widehat{\psi})'\|_{L^2((0,1);q^\alpha)} &\leq \|\widehat{\psi}' - (P_h^q \widehat{\psi})'\|_{L^2((0,1);q^\alpha)} + C_{\rm inv}\, h_q^{-1} \, \|P_h^q \widehat{\psi} - Q_h^q \widehat{\psi}\|_{L^2((0,1);q^\alpha)} \\ &\leq \|\widehat{\psi}' - (P_h^q \widehat{\psi})'\|_{L^2((0,1);q^\alpha)} + 2C_{\rm inv}\, h_q^{-1} \, \|\widehat{\psi} - P_h^q \widehat{\psi}\|_{L^2((0,1);q^\alpha)}.\end{aligned}$$

This, together with (A.9) and (A.10) yields

$$\|\widehat{\psi} - Q_h^q \widehat{\psi}\|^2_{H^1((0,1);q^\alpha)} \leq \left[2 + (h_q^2 + 8C_{\rm inv}^2)\, C_\alpha \left(\tfrac{3}{2} + \alpha\right)\right] \|\widehat{\psi} - P_h^q \widehat{\psi}\|^2_{H^1((0,1);q^\alpha)}, \tag{A.11}$$

which in turn implies, by the triangle inequality and the fact that $\|P_h^q \widehat{\psi}\|_{H^1((0,1);q^\alpha)} \leq \|\widehat{\psi}\|_{H^1((0,1);q^\alpha)}$, the existence of a positive constant C, independent of h, such that

$$\|Q_h^q \widehat{\psi}\|_{H^1((0,1);q^\alpha)} \leq C\, \|\widehat{\psi}\|_{H^1((0,1);q^\alpha)} \qquad \forall \widehat{\psi} \in \widehat{H}^1((0,1);q^\alpha).$$

This is the univariate counterpart of the desired stability result (4.68).

Remark A.2. Supposing that $\widehat{\psi} \in H^2((0,1);q^\alpha)$, we have that

$$\|\widehat{\psi} - P_h^q \widehat{\psi}\|^2_{H^1((0,1);q^\alpha)} \leq \|\widehat{\psi} - I_h^q \widehat{\psi}\|^2_{H^1((0,1);q^\alpha)} \leq C_\alpha \left(1 + \frac{1}{2(\alpha+1)}\right) h_q^2 \int_0^1 |\widehat{\psi}''(q)|^2 \, q^\alpha \, {\rm d}q.$$

Thus, (A.11) implies that an analogous bound holds for $\widehat{\psi} - Q_h^q \widehat{\psi}$ in the $\|\cdot\|_{H^1((0,1);q^\alpha)}$ norm.

A.2. **Multiple dimensions**

In multiple space dimensions the proof of the stability result (4.68) proceeds in a similar manner as in the univariate case discussed above, except for two technical complications. The first is that D is ball, and therefore D has a curved boundary ∂D; the second is that an open (possibly, curved) simplex κ_q in the partition of D, whose closure has nonempty intersection with ∂D, may intersect ∂D in d different configurations: with exactly one curved $(d-k)$-dimensional face contained in ∂D, $k = 1 \to d-1$, accounting for $d-1$ different configurations, and with exactly one vertex contained in ∂D, accounting for the $d^{\rm th}$ configuration. Each of the d possible configurations necessitates a different local definition of the quasi-interpolation operator I_h^q, which we use in the proof of the stability result (4.68). Since the two-dimensional case is sufficiently representative of the general argument, we shall restrict ourselves to showing (4.68) in the bivariate case. The proof in the case of $d = 3$ is identical; in Section A.2.3 we shall indicate the essential alterations that have to be made to the arguments presented herein to obtain the corresponding bounds in the case of $d = 3$.

A.2.1. *Two dimensions: flat boundary*

We begin by assuming that the boundary of $D \subset \mathbb{R}^2$ is flat, e.g. that it is the straight line $q_1 = 0$ in the $\underset{\sim}{q} = (q_1, q_2)$-plane. For ease of exposition we shall, intermittently, write x and y instead of q_1 and q_2, i.e. $x := q_1$ and $y := q_2$.

Two dimensions: configuration 1-flat. Consider a nonobtuse open triangle $\kappa = \triangle$ABC, as in Figure 1, with A$=(h,0)$, B$=(0,k)$, C$=(0,l)$, contained in the rectangle $R(\kappa) := (0,h) \times (l,k) = \square$B$'BCC'$, with B$' = (h,k)$ and C$' = (h,l)$, where $l \leq 0 \leq k$, $k - l > 0$ and $h > 0$. Here, B and C belong to the line $x = 0$ along which the weight-function $(x,y) \mapsto x^\alpha$ vanishes; $\alpha > 0$. We define,

$$\widehat{\Phi}(0,k) := \widehat{\varphi}(h,k) - h\, \frac{\alpha+1}{h^{\alpha+1}} \int_0^h \widehat{\varphi}_x(x,k)\, x^\alpha \, {\rm d}x \quad \text{and} \quad \widehat{\Phi}(0,l) := \widehat{\varphi}(h,l) - h\, \frac{\alpha+1}{h^{\alpha+1}} \int_0^h \widehat{\varphi}_x(x,l)\, x^\alpha \, {\rm d}x.$$

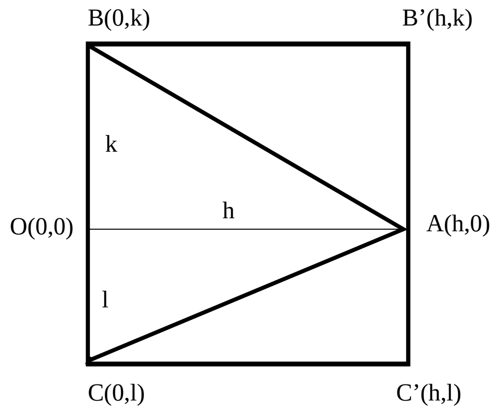

Figure 1. The nonobtuse open triangle $\kappa = \triangle$ABC in the $(x,y) := (q_1,q_2)$-plane, with A=$(h,0)$, B=$(0,k)$, C=$(0,l)$, in configuration 1-flat, that is with two points, B and C, on the line $x=0$ along which the weight function $(x,y) \mapsto x^\alpha$ vanishes.

We then define $p_{\widehat{\varphi}}$ as the affine function whose values at the points A, B and C are, respectively, $\widehat{\varphi}(h,0)$, $\widehat{\Phi}(0,k)$ and $\widehat{\Phi}(0,l)$. Thus, $p_{\widehat{\varphi}}$ interpolates $\widehat{\varphi}$ at A, while at the points B and C the values of $p_{\widehat{\varphi}}$ are based on extrapolating from the points B$'$ and C$'$, respectively, by means of the univariate quasi-interpolant I_h^q. Thus,

$$p_{\widehat{\varphi}}(x,y) := \widehat{\varphi}(h,0)\,\frac{x}{h} + \widehat{\Phi}(0,k)\left(1-\frac{x}{h}-\frac{y}{l}\right)\frac{l}{l-k} + \widehat{\Phi}(0,l)\left(1-\frac{x}{h}-\frac{y}{k}\right)\frac{k}{k-l},$$

which implies that the partial derivatives of $p_{\widehat{\varphi}}$ with respect to x and y are:

$$(p_{\widehat{\varphi}})_x(x,y) = \widehat{\varphi}(h,0)\,\frac{1}{h} + \widehat{\Phi}(0,k)\left(-\frac{1}{h}\right)\frac{l}{l-k} + \widehat{\Phi}(0,l)\left(-\frac{1}{h}\right)\frac{k}{k-l},$$

and

$$(p_{\widehat{\varphi}})_y(x,y) = \widehat{\Phi}(0,k)\left(-\frac{1}{l}\right)\frac{l}{l-k} + \widehat{\Phi}(0,l)\left(-\frac{1}{k}\right)\frac{k}{k-l}.$$

We define the linear functionals

$$L_1(\widehat{\varphi}) := \widehat{\varphi}_x - (p_{\widehat{\varphi}})_x \qquad \text{and} \qquad L_2(\widehat{\varphi}) := \widehat{\varphi}_y - (p_{\widehat{\varphi}})_y.$$

By direct computation, $\widehat{\Phi}(0,k) = \widehat{\varphi}(0,k)$ and $\widehat{\Phi}(0,l) = \widehat{\varphi}(0,l)$ all $\widehat{\varphi} \in \mathbb{P}_1$, and hence $p_{\widehat{\varphi}} \equiv \widehat{\varphi}$ and $L_i(\widehat{\varphi}) \equiv 0$ for all $\widehat{\varphi} \in \mathbb{P}_1$, $i=1,2$. Further,

$$|\widehat{\Phi}(0,k)| \le \frac{\alpha+1}{h^{\alpha+1}} \int_0^h |\widehat{\varphi}(h,k) - h\,\widehat{\varphi}_x(x,k)|\, x^\alpha \,\mathrm{d}x. \tag{A.12}$$

Now,

$$\widehat{\varphi}(h,k)\,h^\alpha = \widehat{\varphi}(x,k)\,x^\alpha + \int_x^h \frac{\mathrm{d}}{\mathrm{d}t}(\widehat{\varphi}(t,k)\,t^\alpha)\,\mathrm{d}t = \widehat{\varphi}(x,k)\,x^\alpha + \int_x^h \widehat{\varphi}_x(t,k)\,t^\alpha\,\mathrm{d}t + \alpha\int_x^h \widehat{\varphi}(t,k)\,t^{\alpha-1}\,\mathrm{d}t.$$

Therefore, by integration over the interval $x \in [0,h]$, integration by parts in the third integral on the right-hand side, and applying the Cauchy–Schwarz inequality,

$$
\begin{aligned}
|\widehat{\varphi}(h,k)|\, h^{\alpha+1} &\le \int_0^h |\widehat{\varphi}(x,k)|\, x^\alpha \,\mathrm{d}x + \int_0^h \int_x^h |\widehat{\varphi}_x(t,k)|\, t^\alpha \,\mathrm{d}t\,\mathrm{d}x + \alpha \int_0^h \int_x^h |\widehat{\varphi}(t,k)|\, t^{\alpha-1} \,\mathrm{d}t\,\mathrm{d}x \\
&= (\alpha+1) \int_0^h |\widehat{\varphi}(x,k)|\, x^\alpha \,\mathrm{d}x + \int_0^h \int_x^h |\widehat{\varphi}_x(t,k)|\, t^\alpha \,\mathrm{d}t\,\mathrm{d}x \\
&\le (\alpha+1) \left(\frac{h^{\alpha+1}}{\alpha+1}\right)^{1/2} \left(\int_0^h |\widehat{\varphi}(x,k)|^2\, x^\alpha \,\mathrm{d}x\right)^{1/2} + h \left(\frac{h^{\alpha+1}}{\alpha+1}\right)^{1/2} \left(\int_0^h |\widehat{\varphi}_x(x,k)|^2\, x^\alpha \,\mathrm{d}x\right)^{1/2}.
\end{aligned}
$$

Thus,

$$
|\widehat{\varphi}(h,k)| \le \left(\frac{\alpha+1}{h^{\alpha+1}}\right)^{1/2} \left(\int_0^h |\widehat{\varphi}(x,k)|^2\, x^\alpha \,\mathrm{d}x\right)^{1/2} + h^{-\alpha} \left(\frac{h^{\alpha+1}}{\alpha+1}\right)^{1/2} \left(\int_0^h |\widehat{\varphi}_x(x,k)|^2\, x^\alpha \,\mathrm{d}x\right)^{1/2}. \tag{A.13}
$$

To bound the first term on the right-hand side, note that, for any $y \in [l,k]$,

$$
|\widehat{\varphi}(x,k)|^2 = |\widehat{\varphi}(x,y)|^2 + 2 \int_y^k \widehat{\varphi}(x,s)\, \widehat{\varphi}_y(x,s) \,\mathrm{d}s,
$$

and hence

$$
\begin{aligned}
\int_0^h |\widehat{\varphi}(x,k)|^2\, x^\alpha \,\mathrm{d}x &= \int_0^h |\widehat{\varphi}(x,y)|^2\, x^\alpha \,\mathrm{d}x + 2 \int_0^h \left(\int_y^k \widehat{\varphi}(x,s)\, x^{\frac{\alpha}{2}}\, \widehat{\varphi}_y(x,s)\, x^{\frac{\alpha}{2}} \,\mathrm{d}s \right) \mathrm{d}x \\
&\le \int_0^h |\widehat{\varphi}(x,y)|^2\, x^\alpha \,\mathrm{d}x + 2 \int_0^h \int_l^k |\widehat{\varphi}(x,y)|\, x^{\frac{\alpha}{2}}\, |\widehat{\varphi}_y(x,y)|\, x^{\frac{\alpha}{2}} \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Thus, on integrating over all $y \in [l,k]$ (recall that $l \le 0 \le k$, $k - l > 0$ and $h > 0$),

$$
\begin{aligned}
(k-l) \int_0^h |\widehat{\varphi}(x,k)|^2 x^\alpha \,\mathrm{d}x \le{}& \int_0^h \int_l^k |\widehat{\varphi}(x,y)|^2\, x^\alpha \,\mathrm{d}x\,\mathrm{d}y \\
&+ 2(k-l) \left(\int_0^h \int_l^k |\widehat{\varphi}(x,y)|^2\, x^\alpha \,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \left(\int_0^h \int_l^k |\widehat{\varphi}_y(x,y)|^2\, x^\alpha \,\mathrm{d}x\,\mathrm{d}y \right)^{1/2},
\end{aligned}
$$

which then implies that

$$
\begin{aligned}
\int_0^h |\widehat{\varphi}(x,k)|^2\, x^\alpha \,\mathrm{d}x \le{}& \frac{1}{k-l} \int_0^h \int_l^k |\widehat{\varphi}(x,y)|^2\, x^\alpha \,\mathrm{d}x\,\mathrm{d}y \\
&+ 2 \left(\int_0^h \int_l^k |\widehat{\varphi}(x,y)|^2\, x^\alpha \,\mathrm{d}x\,\mathrm{d}y \right)^{1/2} \left(\int_0^h \int_l^k |\widehat{\varphi}_y(x,y)|^2\, x^\alpha \,\mathrm{d}x\,\mathrm{d}y \right)^{1/2}.
\end{aligned}
$$

Analogously,

$$\int_0^h |\widehat{\varphi}_x(x,k)|^2\, x^\alpha\, \mathrm{d}x \le \frac{1}{k-l}\int_0^h\int_l^k |\widehat{\varphi}_x(x,y)|^2\, x^\alpha\, \mathrm{d}x\, \mathrm{d}y + 2\left(\int_0^h\int_l^k |\widehat{\varphi}_x(x,y)|^2\, x^\alpha\, \mathrm{d}x\, \mathrm{d}y\right)^{1/2}\left(\int_0^h\int_l^k |\widehat{\varphi}_{xy}(x,y)|^2\, x^\alpha\, \mathrm{d}x\, \mathrm{d}y\right)^{1/2}. \tag{A.14}$$

Substituting the last two bounds into (A.13) it follows that

$$|\widehat{\varphi}(h,k)| \le C(h,k-l)\, \|\widehat{\varphi}\|_{H^2((0,h)\times(l,k);x^\alpha)}.$$

Further, (A.14) implies that

$$\int_0^h |\widehat{\varphi}_x(x,k)|\, x^\alpha\, \mathrm{d}x \le \left(\frac{h^{\alpha+1}}{\alpha+1}\right)^{1/2}\left(\int_0^h |\widehat{\varphi}_x(x,k)|^2\, x^\alpha\, \mathrm{d}x\right)^{1/2} \le C(h,k-l)\|\widehat{\varphi}\|_{H^2((0,h)\times(l,k);x^\alpha)}.$$

Substituting the last two bounds into (A.12), we deduce that

$$|\widehat{\Phi}(0,k)| \le C(h,k-l)\, \|\widehat{\varphi}\|_{H^2((0,h)\times(l,k);x^\alpha)}.$$

Analogously,

$$|\widehat{\varphi}(h,l)| \le C(h,k-l)\, \|\widehat{\varphi}\|_{H^2((0,h)\times(l,k);x^\alpha)} \quad \text{and} \quad |\widehat{\Phi}(0,l)| \le C(h,k-l)\, \|\widehat{\varphi}\|_{H^2((0,h)\times(l,k);x^\alpha)},$$

as well as

$$|\widehat{\varphi}(h,0)| \le C(h,k-l)\, \|\widehat{\varphi}\|_{H^2((0,h)\times(l,k);x^\alpha)}.$$

These inequalities imply that, for $i=1,2$,

$$\|L_i(\widehat{\varphi})\|_{L^2(\kappa;x^\alpha)} \le \|L_i(\widehat{\varphi})\|_{L^2((0,h)\times(l,k);x^\alpha)} \le \left[1+\max\left(\frac{3}{h},\frac{2}{k-l}\right)\left(\frac{h^{\alpha+1}}{\alpha+1}(k-l)\right)^{\frac{1}{2}} C(h,k-l)\right]\|\widehat{\varphi}\|_{H^2((0,h)\times(l,k);x^\alpha)}.$$

Recall that $L_i(\widehat{\varphi}) \equiv 0$ for all $\widehat{\varphi} \in \mathbb{P}_1$, $i=1,2$.

Let $\widetilde{\mathrm{A}}=(1,0)$, $\widetilde{\mathrm{B}}=(0,b)$, $\widetilde{\mathrm{C}}=(0,c)$ denote the counterparts of A, B and C, respectively, with $c<0<b$, in the open reference triangle $\widetilde{\kappa}$, obtained by rescaling the open triangle $\kappa = \triangle\mathrm{ABC}$ by h, i.e. $b=k/h$ and $c=l/h$, and let $\rho := (k-l)/h = b-c$ (>0). We define $\widetilde{x}=x/h$ and $\widetilde{y}=y/h$, $\widetilde{\varphi}(\widetilde{x},\widetilde{y}) := \widehat{\varphi}(x,y)$, $p_{\widetilde{\varphi}}(\widetilde{x},\widetilde{y}) := p_{\widehat{\varphi}}(x,y)$. Finally, we define $\widetilde{L}_i$ by

$$\widetilde{L}_i(\widetilde{\varphi})(\widetilde{x},\widetilde{y}) := h\, L_i(\widehat{\varphi})(x,y), \qquad i=1,2.$$

Thus,

$$\widetilde{L}_1(\widetilde{\varphi})(\widetilde{x},\widetilde{y}) = \widetilde{\varphi}_{\widetilde{x}}(\widetilde{x},\widetilde{y}) - (p_{\widetilde{\varphi}})_{\widetilde{x}}(\widetilde{x},\widetilde{y}), \qquad \widetilde{L}_2(\widetilde{\varphi})(\widetilde{x},\widetilde{y}) = \widetilde{\varphi}_{\widetilde{y}}(\widetilde{x},\widetilde{y}) - (p_{\widetilde{\varphi}})_{\widetilde{y}}(\widetilde{x},\widetilde{y}).$$

Then, $\widetilde{L}_i(\widetilde{\varphi}) \equiv 0$ for all $\widetilde{\varphi} \in \mathbb{P}_1$. In addition, repeating the bounds above with h, k and l replaced by 1, b and c, noting that all constants in the bounds depend continuously on $\rho = b-c$, we deduce the existence of a positive constant $C(\rho)$, which depends continuously on ρ, such that

$$\|\widetilde{L}_i(\widetilde{\varphi})\|_{L^2(\widetilde{\kappa};\widetilde{x}^\alpha)} \le \|\widetilde{L}_i(\widetilde{\varphi})\|_{L^2((0,1)\times(c,b);\widetilde{x}^\alpha)} \le C(\rho)\, \|\widetilde{\varphi}\|_{H^2((0,1)\times(c,b);\widetilde{x}^\alpha)}, \qquad i=1,2.$$

Note that ρ depends only on the shape of κ; in particular, it is independent of the size of κ.

Let us recall the following generalization of the Bramble–Hilbert Lemma, due to Tartar (cf. Ciarlet [12], Section 3.1, Exercise 3.1.1).

Lemma A.3. *(**L. Tartar**) Let V be a Banach space, and let V_1, V_2 and W be three normed linear spaces. Suppose that $A_i \in \mathcal{L}(V; V_i)$, $i = 1, 2$, and that A_1 is compact. Suppose, further, that there exists a positive constant c_0 such that*

$$\|v\|_V \le c_0 \left(\|A_1 v\|_{V_1} + \|A_2 v\|_{V_2}\right) \qquad \forall v \in V.$$

Finally, suppose that $L \in \mathcal{L}(V; W)$ is such that

$$v \in \ker A_2 \Longrightarrow Lv = 0.$$

Then, the following statements hold.

(i) *$\mathbb{P} := \ker A_2$ is a finite-dimensional linear space.*

(ii) *There exists a positive constant c_1 such that*

$$\inf_{p\in\mathbb{P}} \|v - p\|_V \le c_1 \|A_2 v\|_{V_2} \qquad \forall v \in V.$$

(iii) *There exists a positive constant C such that*

$$\|Lv\|_W \le C \|A_2 v\|_{V_2} \qquad \forall v \in V.$$

We shall apply this result with $\alpha \ge 1$, $V := H^2((0,1) \times (c,b); \widetilde{x}^\alpha)$, $V_1 := H^1((0,1) \times (c,b); \widetilde{x}^\alpha)$, $V_2 := [L^2((0,1) \times (c,b); \widetilde{x}^\alpha)]^4$, $W := L^2((0,1) \times (c,b); \widetilde{x}^\alpha)$, $A_2 : \widetilde{v} \in H^2((0,1) \times (c,b); \widetilde{x}^\alpha) \mapsto (\widetilde{v}_{\widetilde{x}\widetilde{x}}, \widetilde{v}_{\widetilde{x}\widetilde{y}}, \widetilde{v}_{\widetilde{y}\widetilde{x}}, \widetilde{v}_{\widetilde{y}\widetilde{y}})$, $A_1 := \mathrm{Id}$, and $L = \widetilde{L}_i$, $i = 1, 2$, together with the compact embedding

$$H^2((0,1) \times (c,b); \widetilde{x}^\alpha) \hookrightarrow H^1((0,1) \times (c,b); \widetilde{x}^\alpha),$$

which requires the restriction $\alpha \ge 1$ (cf. Lemma 5.2 in Antoci [1])

Thus, we deduce that

$$\|\widetilde{\varphi}_{\widetilde{x}} - (\widetilde{p}_{\widetilde{\varphi}})_{\widetilde{x}}\|_{L^2((0,1)\times(c,b);\widetilde{x}^\alpha)} \le C(\rho)\, |\widetilde{\varphi}|_{H^2((0,1)\times(c,b);\widetilde{x}^\alpha)}$$

and

$$\|\widetilde{\varphi}_{\widetilde{y}} - (\widetilde{p}_{\widetilde{\varphi}})_{\widetilde{y}}\|_{L^2((0,1)\times(c,b);\widetilde{x}^\alpha)} \le C(\rho)\, |\widetilde{\varphi}|_{H^2((0,1)\times(c,b);\widetilde{x}^\alpha)},$$

where $|\cdot|_{H^2((0,1)\times(c,b);\widetilde{x}^\alpha)}$ is the semi-norm on $H^2((0,1) \times (c,b); \widetilde{x}^\alpha)$.

After returning from the scaled variables $\widetilde{x}$ and $\widetilde{y}$ to the original variables $x = h\,\widetilde{x}$ and $y = h\,\widetilde{y}$ and combining the resulting inequalities into a single inequality, we obtain

$$\|\underset{\sim}{\nabla}(\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2((0,h)\times(l,k);x^\alpha)} \le C(\rho)\, h\, |\widehat{\varphi}|_{H^2((0,h)\times(l,k);x^\alpha)}.$$

In other words,

$$\|\underset{\sim}{\nabla}(\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2(R(\kappa);x^\alpha)} \le C(\rho)\, h\, |\widehat{\varphi}|_{H^2(R(\kappa);x^\alpha)}, \tag{A.15}$$

whereupon

$$\|\underset{\sim}{\nabla}(\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2(\kappa;x^\alpha)} \le C(\rho)\, h\, |\widehat{\varphi}|_{H^2(R(\kappa);x^\alpha)}, \tag{A.16}$$

where $R(\kappa) := (0,h) \times (l,k)$, $\rho := (k-l)/h$ and $\alpha \ge 1$.

Using that, for $(x,y) \in \kappa$, $0 \le x/h \le 1$ and $|y|/(k-l) \le 1$, one can obtain a similar bound on $\widehat{\varphi} - p_{\widehat{\varphi}}$ in the x^α-weighted L^2 norm on κ. The only difference is that then

$$L(\widehat{\varphi}) := \widehat{\varphi} - p_{\widehat{\varphi}} \qquad \text{and} \qquad \widetilde{L}(\widetilde{\varphi})(\widetilde{x}, \widetilde{y}) := L(\widehat{\varphi})(x,y),$$

with the same definitions of $p_{\widehat{\varphi}}$, $\widetilde{\varphi}$, $\widetilde{p}_{\widetilde{\varphi}}$, $\widetilde{x}$ and $\widetilde{y}$ as before. We recall that $p_{\widehat{\varphi}} \equiv \widehat{\varphi}$ for all $\widehat{\varphi} \in \mathbb{P}_1$, and hence $L(\widehat{\varphi}) \equiv 0$ for all $\widehat{\varphi} \in \mathbb{P}_1$ and therefore $\widetilde{L}(\widetilde{\varphi}) \equiv 0$ for all $\widetilde{\varphi} \in \mathbb{P}_1$. We still have that

$$\|\widetilde{L}(\widetilde{\varphi})\|_{L^2(\widetilde{\kappa};\widetilde{x}^\alpha)} \leq \|\widetilde{L}(\widetilde{\varphi})\|_{L^2((0,1)\times(c,b);\widetilde{x}^\alpha)} \leq C(\rho)\, \|\widetilde{\varphi}\|_{H^2((0,1)\times(c,b);\widetilde{x}^\alpha)}.$$

Hence, Lemma A.3, with the same choice of V, V_1, V_2, W, A_1 and A_2 as before, and $\alpha \geq 1$, implies that

$$\|\widetilde{\varphi} - \widetilde{p}_{\widetilde{\varphi}}\|_{L^2((0,1)\times(c,b);\widetilde{x}^\alpha)} \leq C(\rho)\, |\widetilde{\varphi}|_{H^2((0,1)\times(c,b);\widetilde{x}^\alpha)}.$$

After returning from the scaled variables $\widetilde{x} = x/h$ and $\widetilde{y} = y/h$ to the original variables x and y, we obtain that

$$\|\widehat{\varphi} - p_{\widehat{\varphi}}\|_{L^2((0,h)\times(l,k);x^\alpha)} \leq C(\rho)\, h^2\, |\widehat{\varphi}|_{H^2((0,h)\times(l,k);x^\alpha)}.$$

In other words,

$$\|\widehat{\varphi} - p_{\widehat{\varphi}}\|_{L^2(R(\kappa);x^\alpha)} \leq C(\rho)\, h^2\, |\widehat{\varphi}|_{H^2(R(\kappa);x^\alpha)},$$

whereupon

$$\|\widehat{\varphi} - p_{\widehat{\varphi}}\|_{L^2(\kappa;x^\alpha)} \leq C(\rho)\, h^2\, |\widehat{\varphi}|_{H^2(R(\kappa);x^\alpha)}, \tag{A.17}$$

with $R(\kappa) := (0,h) \times (l,k)$, $\rho := (k-l)/h$ and $\alpha \geq 1$. The constant $C(\rho)$ is a continuous function of ρ in each of these bounds.

Two dimensions: configuration 2-flat. The alternative configuration of the triangle $\kappa = \triangle$ABC is: A=$(0,0)$, B=(h,k) and C=(h,l), with only one point, A, on the line $x = 0$ along which the weight-function $(x,y) \mapsto x^\alpha$ vanishes. In this case, we define $p_{\widehat{\varphi}}$ as the affine function that interpolates $\widehat{\varphi}$ at B and C, and has the value

$$\widehat{\Phi}(0,0) = \widehat{\varphi}(h,0) - h\,\frac{\alpha+1}{h^{\alpha+1}} \int_0^h \widehat{\varphi}_x(x,0)\, x^\alpha\, \mathrm{d}x$$

at A=$(0,0)$, extrapolated from $(h,0)$ using the univariate quasi-interpolation operator. Thus,

$$\begin{aligned}
p_{\widehat{\varphi}}(x,y) &= \widehat{\Phi}(0,0)\left(1 - \frac{x}{h}\right) + \widehat{\varphi}(h,k)\left(y - \frac{l}{h}x\right)\frac{1}{k-l} + \widehat{\varphi}(h,l)\left(y - \frac{k}{h}x\right)\frac{1}{l-k},\\
(p_{\widehat{\varphi}})_x(x,y) &= -\widehat{\Phi}(0,0)\,\frac{1}{h} + \widehat{\varphi}(h,k)\left(-\frac{l}{h}\right)\frac{1}{k-l} + \widehat{\varphi}(h,l)\left(-\frac{k}{h}\right)\frac{1}{l-k},\\
(p_{\widehat{\varphi}})_y(x,y) &= \widehat{\varphi}(h,k)\,\frac{1}{k-l} + \widehat{\varphi}(h,l)\,\frac{1}{l-k}.
\end{aligned}$$

Again, we define

$$L_1(\widehat{\varphi}) := \widehat{\varphi}_x - (p_{\widehat{\varphi}})_x \qquad \text{and} \qquad L_2(\widehat{\varphi}) := \widehat{\varphi}_y - (p_{\widehat{\varphi}})_y,$$

and we observe that $\widehat{\Phi}(0,0) = \widehat{\varphi}(0,0)$ for all $\widehat{\varphi} \in \mathbb{P}_1$, and hence $p_{\widehat{\varphi}} \equiv \widehat{\varphi}$ and $L_i(\widehat{\varphi}) \equiv 0$ for all $\widehat{\varphi} \in \mathbb{P}_1$, $i = 1,2$.

The rest of the argument is the same as in the case of configuration 1-flat, and leads to the same final bound:

$$\|\underset{\sim}{\nabla}(\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2(\kappa;x^\alpha)} \leq C(\rho)\, h\, |\widehat{\varphi}|_{H^2(R(\kappa);x^\alpha)}, \tag{A.18}$$

where again $R(\kappa) := (0,h) \times (l,k)$, $\rho := (k-l)/h$ and $\alpha \geq 1$. Also, as in the case of configuration 1-flat,

$$\|\widehat{\varphi} - p_{\widehat{\varphi}}\|_{L^2(\kappa;x^\alpha)} \leq C(\rho)\, h^2\, |\widehat{\varphi}|_{H^2(R(\kappa);x^\alpha)}, \tag{A.19}$$

with $R(\kappa) := (0,h) \times (l,k)$, $\rho := (k-l)/h$ and $\alpha \geq 1$. The constant $C(\rho)$ is a continuous function of ρ in each of these bounds.

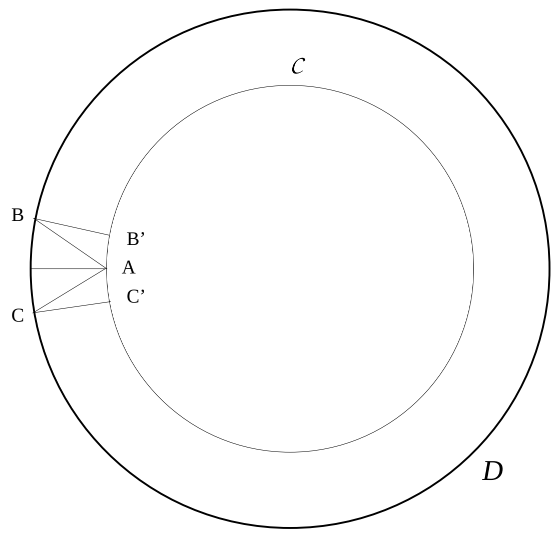

FIGURE 2. The domain D, the circle $\mathcal{C} \subset D$, with $\operatorname{dist}(\partial D, \mathcal{C}) = h$, and C', A, $\mathrm{B}' \in \mathcal{C}$ and the open curved nonobtuse triangle $\kappa_q = \triangle\mathrm{ABC}$ in configuration 1-curved.

A.2.2. *Two dimensions: curved boundary*

Now suppose that D is an open disc in $\mathbb{R}^2$ of radius $r_D \in \mathbb{R}_{>0}$, centred at the origin. Suppose, further, that $\{\mathcal{T}_h^q\}_{h>0}$ is a quasiuniform family of partitions of D (in the sense of Hypothesis (A1) from Section 4, with $d = 2$,) into disjoint open nonobtuse triangles κ_q, with possibly one curved edge on ∂D. We focus our attention on elements κ_q that are in contact with ∂D. There are again two possible configurations, which will be considered separately. We shall assume throughout the section that the potential U and the associated Maxwellian M satisfy on D the assumptions stated at the start of Section 2.3, including (2.9a), with $\zeta \geq 1$, and (2.9b).

Two dimensions: configuration 1-curved. We consider a circle $\mathcal{C} \subset D$, concentric with ∂D, which is a distance h away from ∂D; cf. Figure 2. The analogue of configuration 1-flat is an open curved nonobtuse triangle $\kappa_q := \triangle\mathrm{ABC}$, with one curved edge $\mathrm{BC} \subset \partial D$ and with $\mathrm{A} \in \mathcal{C}$. Let B' and C' be points on $\mathcal{C}$ such that BB' and CC' are aligned with the directions of the normal vectors to ∂D at B and C, respectively. We mimic the construction of the quasi-interpolant $p_{\widehat{\varphi}}$ of $\widehat{\varphi}$ described in the previous subsection.

Note that, for $\widehat{\varphi} \in H^2_M(D)$ and any pair of points Q_1 and Q_2 in D,

$$\widehat{\varphi}(\mathrm{Q}_2) = \widehat{\varphi}(\mathrm{Q}_1) - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\, \widehat{\varphi}((1-\tau)\,\mathrm{Q}_2 + \tau\,\mathrm{Q}_1)\,\mathrm{d}\tau.$$

Motivated by this identity, for $\mathrm{Q}_1 \in D$ and $\mathrm{Q}_2 \in \overline{D}$, we define

$$\widehat{\Phi}(\mathrm{Q}_2) := \widehat{\varphi}(\mathrm{Q}_1) - \frac{\int_0^1 M((1-\tau)\,\mathrm{Q}_2 + \tau\,\mathrm{Q}_1)\ \frac{\mathrm{d}}{\mathrm{d}\tau}\,\widehat{\varphi}((1-\tau)\,\mathrm{Q}_2 + \tau\,\mathrm{Q}_1)\,\mathrm{d}\tau}{\int_0^1 M((1-\tau)\,\mathrm{Q}_2 + \tau\,\mathrm{Q}_1)\,\mathrm{d}\tau}. \tag{A.20}$$

Remark A.4. In one space dimension, with $M(q) = q^\alpha$, $q \in [0, h]$, $\mathrm{Q}_2 = 0$, $\mathrm{Q}_1 = h$, and performing the change of variable $q = \tau h$, (A.20) yields our univariate extrapolation operator:

$$\widehat{\Phi}(0) := \widehat{\varphi}(h) - h\,\frac{\int_0^h q^\alpha\, \widehat{\varphi}'(q)\,\mathrm{d}q}{\int_0^h q^\alpha\,\mathrm{d}q} = \widehat{\varphi}(h) - h\,\frac{\alpha+1}{h^{\alpha+1}} \int_0^h q^\alpha\, \widehat{\varphi}'(q)\,\mathrm{d}q.$$

In multiple space dimensions the formula (A.20), after performing the τ-differentiation under the integral sign, becomes

$$\widehat{\Phi}(\mathrm{Q}_2) := \widehat{\varphi}(\mathrm{Q}_1) - (\mathrm{Q}_1 - \mathrm{Q}_2) \cdot \frac{\int_0^1 M((1-\tau)\,\mathrm{Q}_2 + \tau\,\mathrm{Q}_1)\ (\underset{\sim}{\nabla}_q\,\widehat{\varphi})((1-\tau)\,\mathrm{Q}_2 + \tau\,\mathrm{Q}_1)\,\mathrm{d}\tau}{\int_0^1 M((1-\tau)\,\mathrm{Q}_2 + \tau\,\mathrm{Q}_1)\,\mathrm{d}\tau}.$$

In particular, in the two-dimensional setting considered here, and with reference to Figure 2,

$$\widehat{\Phi}(\mathrm{B}) := \widehat{\varphi}(\mathrm{B}') - (\mathrm{B}' - \mathrm{B}) \cdot \frac{\int_0^1 M((1-\tau)\,\mathrm{B} + \tau\,\mathrm{B}')\ (\underset{\sim}{\nabla}_q\,\widehat{\varphi})((1-\tau)\,\mathrm{B} + \tau\,\mathrm{B}')\,\mathrm{d}\tau}{\int_0^1 M((1-\tau)\,\mathrm{B} + \tau\,\mathrm{B}')\,\mathrm{d}\tau}$$

and

$$\widehat{\Phi}(\mathrm{C}) := \widehat{\varphi}(\mathrm{C}') - (\mathrm{C}' - \mathrm{C}) \cdot \frac{\int_0^1 M((1-\tau)\,\mathrm{C} + \tau\,\mathrm{C}')\ (\underset{\sim}{\nabla}_q\,\widehat{\varphi})((1-\tau)\,\mathrm{C} + \tau\,\mathrm{C}')\,\mathrm{d}\tau}{\int_0^1 M((1-\tau)\,\mathrm{C} + \tau\,\mathrm{C}')\,\mathrm{d}\tau}.$$

We then define the affine function $p_{\widehat{\varphi}}$ on $\kappa_q = \triangle\mathrm{ABC}$ by

$$p_{\widehat{\varphi}}(\underset{\sim}{q}) := \widehat{\varphi}(\mathrm{A})\,\psi_{\mathrm{A}}(\underset{\sim}{q}) + \widehat{\Phi}(\mathrm{B})\,\psi_{\mathrm{B}}(\underset{\sim}{q}) + \widehat{\Phi}(\mathrm{C})\,\psi_{\mathrm{C}}(\underset{\sim}{q}), \qquad \underset{\sim}{q} = (q_1, q_2) \in \kappa_q, \tag{A.21}$$

where $\{\psi_{\mathrm{A}}, \psi_{\mathrm{B}}, \psi_{\mathrm{C}}\}$ is the $\mathbb{P}_1^q$ local (nodal/Lagrange) basis associated with the triangle $\triangle\mathrm{ABC}$.

Let $R(\kappa_q)$ denote the curvilinear rectangle $\mathrm{B'BCC'}$ depicted in Figure 2. Our aim is to show that, in analogy with (A.15),

$$\|\underset{\sim}{\nabla}_q\,(\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2_M(R(\kappa_q))} \leq C(\rho)\,h\,|\widehat{\varphi}|_{H^2_M(R(\kappa_q))},$$

where ρ is a positive constant dependent only on the shape of κ_q; this will in turn imply that

$$\|\underset{\sim}{\nabla}_q\,(\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2_M(\kappa_q)} \leq C(\rho)\,h\,|\widehat{\varphi}|_{H^2_M(R(\kappa_q))}.$$

Using polar co-ordinates, the curvilinear rectangle $R(\kappa_q)$ in the $\underset{\sim}{q} := (q_1, q_2)$ domain can be mapped into the rectangular domain

$$R_{\mathrm{polar}}(\kappa_q) := \{(r,\theta) \,:\, -r_D < r < -r_D + h, \quad \theta_{\mathrm{C}} < \theta < \theta_{\mathrm{B}}\}.$$

Let us therefore perform the following change of independent variables:

$$q_1 = r\cos\theta, \quad q_2 = r\sin\theta, \qquad r \in (-r_D, -r_D + h), \quad \theta \in (\theta_{\mathrm{C}}, \theta_{\mathrm{B}}); \tag{A.22}$$

thus, $r = -|\underset{\sim}{q}|$. Naturally, $0 < h \ll 1 < r_D$, and we can therefore assume without loss of generality that $-r_D + h \leq -\frac{1}{2}$; therefore, $r = 0$ is, uniformly in h, separated from the range $(-r_D, -r_D + h)$ of r, whereby the change of variables (A.22) is a smooth bijective diffeomorphism from $R(\kappa_q)$ to $R_{\mathrm{polar}}(\kappa_q)$.

By virtue of (2.9a) we may assume without loss of generality that $M(\underset{\sim}{q}) = (r_D - |\underset{\sim}{q}|)^\alpha$, with $\alpha = \zeta \geq 1$ and ζ as in (2.9a), and $|\underset{\sim}{q}| \in (r_D - h, r_D)$. In polar co-ordinates, with $|\underset{\sim}{q}| = -r$, we therefore define $N(r) := (r_D + r)^\alpha$ for $r \in (-r_D, -r_D + h)$, where $\alpha = \zeta \geq 1$.

Now, on noting that $M(\underset{\sim}{q}) = N(r)$ with $r = -|\underset{\sim}{q}| \in (-r_D, -r_D + h)$, we have that

$$\widehat{\Phi}(\mathrm{B}) = \widehat{\Phi}(-r_D, \theta_{\mathrm{B}}) = \widehat{\varphi}(-r_D + h, \theta_{\mathrm{B}}) - h\,\frac{\int_0^1 N(-r_D + \tau\,h)\,\widehat{\varphi}_r(-r_D + \tau\,h, \theta_{\mathrm{B}})\,\mathrm{d}\tau}{\int_0^1 N(-r_D + \tau\,h)\,\mathrm{d}\tau}.$$

Hence,

$$\widehat{\Phi}(\mathrm{B}) = \widehat{\Phi}(-r_D, \theta_{\mathrm{B}}) = \widehat{\varphi}(-r_D + h, \theta_{\mathrm{B}}) - h\,\frac{\alpha+1}{h^{\alpha+1}} \int_0^h t^\alpha\,\widehat{\varphi}_r(-r_D + t, \theta_{\mathrm{B}})\,\mathrm{d}t. \tag{A.23}$$

Analogously,
$$\widehat{\Phi}(\mathrm{C}) = \widehat{\Phi}(-r_D, \theta_{\mathrm{C}}) = \widehat{\varphi}(-r_D + h, \theta_{\mathrm{C}}) - h\,\frac{\alpha+1}{h^{\alpha+1}} \int_0^h t^\alpha\, \widehat{\varphi}_r(-r_D + t, \theta_{\mathrm{C}})\,\mathrm{d}t, \tag{A.24}$$
while
$$\widehat{\varphi}(\mathrm{A}) = \widehat{\varphi}(-r_D + h, \theta_{\mathrm{A}}). \tag{A.25}$$
It is clear from (A.23) that if the restriction of $\widehat{\varphi}$ to the closed line segment connecting B′ to B is a linear function, and therefore $\widehat{\varphi}_r$ is constant along this line segment, then $\widehat{\Phi}(\mathrm{B}) = \widehat{\varphi}(-r_D, \theta_{\mathrm{B}}) = \widehat{\varphi}(\mathrm{B})$. Analogously, (A.24) implies that if the restriction of $\widehat{\varphi}$ to the closed line segment connecting C′ to C is a linear function, then $\widehat{\Phi}(\mathrm{C}) = \widehat{\varphi}(-r_D, \theta_{\mathrm{C}}) = \widehat{\varphi}(\mathrm{C})$.

Hence, if $\widehat{\varphi} \in \mathbb{P}_1^q$, then (A.21) implies that $p_{\widehat{\varphi}}(\underset{\sim}{q}) = \widehat{\varphi}(\mathrm{A})\,\psi_{\mathrm{A}}(\underset{\sim}{q}) + \widehat{\varphi}(\mathrm{B})\,\psi_{\mathrm{B}}(\underset{\sim}{q}) + \widehat{\varphi}(\mathrm{C})\,\psi_{\mathrm{C}}(\underset{\sim}{q})$, the standard linear nodal/Lagrange interpolant of $\widehat{\varphi}$, whereby $\underset{\sim}{\nabla}_q(\widehat{\varphi} - p_{\widehat{\varphi}}) \equiv 0$. Equivalently, letting
$$L_1(\widehat{\varphi}) = (\widehat{\varphi})_{q_1} - (p_{\widehat{\varphi}})_{q_1}, \qquad L_2(\widehat{\varphi}) = (\widehat{\varphi})_{q_2} - (p_{\widehat{\varphi}})_{q_2},$$
we have that $L_i(\widehat{\varphi}) \equiv 0$ for all $\widehat{\varphi} \in \mathbb{P}_1^q$, $i = 1, 2$.

Since the formulae (A.23), (A.24), (A.25) are essentially the same as those corresponding to $\widehat{\Phi}(B) = \widehat{\Phi}(0, k)$, $\widehat{\Phi}(C) = \widehat{\Phi}(0, l)$ and $\widehat{\varphi}(A) = \widehat{\varphi}(h, 0)$ in the case of configuration 1-flat in the previous section, defining $\rho := (\theta_{\mathrm{B}} - \theta_{\mathrm{C}})/h$, changing variables to the rectangular region $R_{\mathrm{polar}}(\kappa_q)$, rescaling this by $1/h$ as in the previous section, applying Lemma A.3, and then rescaling by h to return from $R_{\mathrm{polar}}(\kappa_q)$ to $R(\kappa_q)$ yields
$$\|\underset{\sim}{\nabla}_q\,(\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2_M(R(\kappa_q))} \leq C(\rho)\,h\,|\widehat{\varphi}|_{H^2_M(R(\kappa_q))}.$$
Hence,
$$\|\underset{\sim}{\nabla}_q\,(\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2_M(\kappa_q)} \leq C(\rho)\,h\,|\widehat{\varphi}|_{H^2_M(R(\kappa_q))}, \tag{A.26}$$
with $\rho := (\theta_{\mathrm{B}} - \theta_{\mathrm{C}})/h$.

Next, we prove that
$$\|\widehat{\varphi} - p_{\widehat{\varphi}}\|_{L^2_M(\kappa_q)} \leq C(\rho)\,h^2\,|\widehat{\varphi}|_{H^2_M(R(\kappa_q))}. \tag{A.27}$$
This time, we define $L(\widehat{\varphi}) := \widehat{\varphi} - p_{\widehat{\varphi}}$ where, again, $p_{\widehat{\varphi}}(\underset{\sim}{q}) = \widehat{\varphi}(\mathrm{A})\,\psi_{\mathrm{A}}(\underset{\sim}{q}) + \widehat{\Phi}(\mathrm{B})\,\psi_{\mathrm{B}}(\underset{\sim}{q}) + \widehat{\Phi}(\mathrm{C})\,\psi_{\mathrm{C}}(\underset{\sim}{q})$. Once again, if $\widehat{\varphi} \in \mathbb{P}_1^q$, then $p_{\widehat{\varphi}}$ is just the standard linear nodal/Lagrange interpolant of $\widehat{\varphi}$ and therefore $L(\widehat{\varphi}) \equiv 0$. The rest of the argument is the same as in the case of the error estimate in the M-weighted H^1 seminorm above. Thus, on applying Lemma A.3 and a scaling argument in the same way as before,
$$\|\widehat{\varphi} - p_{\widehat{\varphi}}\|_{L^2_M(\kappa_q)} \leq C(\rho) h^2 |\widehat{\varphi}|_{H^2_M(R(\kappa_q))}, \tag{A.28}$$
where, again, $\rho := (\theta_{\mathrm{B}} - \theta_{\mathrm{C}})/h$. The constant $C(\rho)$ is a continuous function of ρ in each of these bounds.

Two dimensions: configuration 2-curved. The alternative configuration of the triangle $\kappa_q = \triangle\mathrm{ABC}$ is that A$\in \partial D$ while $\mathrm{B}, \mathrm{C} \in \mathcal{C}$. In this case, we define $p_{\widehat{\varphi}}$ as the affine function that interpolates $\widehat{\varphi}$ at B and C, and has the value
$$\widehat{\Phi}(\mathrm{A}) := \widehat{\varphi}(\mathrm{A}') - (\mathrm{A}' - \mathrm{A}) \cdot \frac{\int_0^1 M((1-\tau)\,\mathrm{A} + \tau\,\mathrm{A}')\ (\underset{\sim}{\nabla}\widehat{\varphi})((1-\tau)\ \mathrm{A} + \tau\,\mathrm{A}')\,\mathrm{d}\tau}{\int_0^1 M((1-\tau)\,\mathrm{A} + \tau\,\mathrm{A}')\,\mathrm{d}\tau}$$
at A. Here A′ is the point on $\mathcal{C}$ where the line segment, normal to ∂D, connecting A to the centre of the disc D intersects $\mathcal{C}$; thus the segment AA′ is orthogonal to ∂D. The value $\widehat{\Phi}(\mathrm{A})$ is therefore obtained by extrapolating $\widehat{\varphi}$ from A′. Thus,
$$p_{\widehat{\varphi}}(\underset{\sim}{q}) = \widehat{\Phi}(\mathrm{A})\,\psi_{\mathrm{A}}(\underset{\sim}{q}) + \widehat{\varphi}(\mathrm{B})\,\psi_{\mathrm{B}}(\underset{\sim}{q}) + \widehat{\varphi}(\mathrm{C})\,\psi_{\mathrm{C}}(\underset{\sim}{q}).$$
Again, we define,
$$L_1(\widehat{\varphi}) := \widehat{\varphi}_{q_1} - (p_{\widehat{\varphi}})_{q_1} \qquad \text{and} \qquad L_2(\widehat{\varphi}) := \widehat{\varphi}_{q_2} - (p_{\widehat{\varphi}})_{q_2},$$

and we observe that $L_i(\widehat{\varphi}) \equiv 0$, $i = 1, 2$, for all $\widehat{\varphi} \in \mathbb{P}_1^q$. The rest of the argument is the same as in the case of configuration 1-curved, and leads to the same final bound:

$$\|\nabla_q (\widehat{\varphi} - p_{\widehat{\varphi}})\|_{L^2_M(\kappa_q)} \leq C(\rho)\, h\, |\widehat{\varphi}|_{H^2_M(R(\kappa_q))}, \tag{A.29}$$

where now $R(\kappa_q)$ is the curvilinear rectangle $\mathrm{BB'C'C}$, whose curved edges $\mathrm{B'C'} \subset \partial D$, $\mathrm{BC} \subset \mathcal{C}$; here $\mathrm{B'}$ and $\mathrm{C'}$ are the points on ∂D where the line segments passing through the centre of the disc D and the points B and C, respectively, extended beyond B and C, respectively, intersect ∂D. Clearly, each of the line segments $\mathrm{BB'}$ and $\mathrm{CC'}$ is orthogonal to ∂D as in the case of configuration 1-curved. The definition of ρ is the same as in the case of configuration 1-curved, i.e. $\rho := (\theta_{\mathrm{B}} - \theta_{\mathrm{C}})/h$.

Arguing in the same way as in the case of the M-weighted L^2 norm bound derived above in the case of configuration 1-curved, we also have that, with $\rho := (\theta_{\mathrm{B}} - \theta_{\mathrm{C}})/h$,

$$\|\widehat{\varphi} - p_{\widehat{\varphi}}\|_{L^2_M(\kappa_q)} \leq C(\rho)\, h^2\, |\widehat{\varphi}|_{H^2_M(R(\kappa_q))}. \tag{A.30}$$

The constant $C(\rho)$ is a continuous function of ρ in each of these bounds.

Two dimensions: global interpolation bound. Let h_q denote the maximum diameter of any triangle κ_q in the quasiuniform and nonobtuse family of partitions $\{\mathcal{T}_h^q\}_{h>0}$ of D. Each triangle $\kappa_q \in \mathcal{T}_h^q$ whose closure intersects ∂D is either in configuration 1-curved or in configuration 2-curved; on such triangles we define $p_{\widehat{\varphi}}$ as above. Any triangle $\kappa_q \in \mathcal{T}_h^q$ that is neither in configuration 1-curved or configuration 2-curved is such that the closure of κ_q is contained in the open disc D; on such triangles, referred to as being in configuration 0, we define $p_{\widehat{\varphi}}$ as the standard nodal interpolant of $\widehat{\varphi}$. For $\widehat{\varphi} \in H^2_M(D)$, we then define the global quasi-interpolant $I_h^q\widehat{\varphi} := p_{\widehat{\varphi}}$. Note, in particular, that $I_h^q\widehat{\varphi}$ is a continuous piecewise linear function on $\overline{D}$ with the following properties: suppose that P is a vertex of a triangle $\kappa_q \in \mathcal{T}_h^q$; if $\mathrm{P} \in D$, then $(I_h^q\widehat{\varphi})(\mathrm{P}) = \widehat{\varphi}(\mathrm{P})$; if, on the other hand, $\mathrm{P} \in \partial D$, then $(I_h^q\widehat{\varphi})(\mathrm{P}) = \widehat{\Phi}(\mathrm{P})$, the value extrapolated from $\mathrm{P'} \in D$ using the formula

$$\widehat{\Phi}(\mathrm{P}) := \widehat{\varphi}(\mathrm{P'}) - (\mathrm{P'} - \mathrm{P}) \cdot \frac{\int_0^1 M((1-\tau)\,\mathrm{P} + \tau\,\mathrm{P'})\ (\nabla_q\, \widehat{\varphi})((1-\tau)\,\mathrm{P} + \tau\,\mathrm{P'})\,\mathrm{d}\tau}{\int_0^1 M((1-\tau)\,\mathrm{P} + \tau\,\mathrm{P'})\,\mathrm{d}\tau},$$

where $\mathrm{P'}$ is the unique point of intersection of the line segment that connects $\mathrm{P} \in \partial D$ to the centre of D with the circle $\mathcal{C} \subset D$ concentric with ∂D and such that $\mathrm{dist}(\partial D, \mathcal{C}) = h$ and $0 < h \ll r_D$.

By virtue of (A.26) (on triangles $\kappa_q \subset D$ in configuration 1-curved), (A.29) (on triangles $\kappa_q \in D$ in configuration 2-curved), and classical interpolation results on the remaining triangles $\kappa_q \in \mathcal{T}_h$ (in configuration 0) whose closure does not intersect ∂D, together with upper and lower bounds on M on triangles in configuration 0 and recalling (4.54), to relate the M-weighted L^2, H^1 and H^2 norms to standard (nonweighted) L^2, H^1 and H^2 norms, we deduce that

$$\|\nabla_q (\widehat{\psi} - I_h^q\widehat{\psi})\|_{L^2_M(D)} \leq C\, h_q\, |\widehat{\psi}|_{H^2_M(D)} \qquad \text{and} \qquad \|\widehat{\psi} - I_h^q\widehat{\psi}\|_{L^2_M(D)} \leq C\, h_q^2\, |\widehat{\psi}|_{H^2_M(D)},$$

whereby

$$\|\widehat{\psi} - I_h^q\widehat{\psi}\|_{H^1_M(D)} \leq C\, h_q\, |\widehat{\psi}|_{H^2_M(D)}. \tag{A.31}$$

Here we made use of the fact that the parameter ρ appearing in the bounds on the triangles $\kappa_q \in \mathcal{T}_h^q$ in configuration 1-curved and configuration 2-curved belongs to a compact subinterval of $\mathbb{R}_{>0}$, independent of h_q, due to our assumption that $\{\mathcal{T}_h^q\}_{h>0}$ is a quasiuniform family of nonobtuse partitions; since the constants $C(\rho)$ featuring in those bounds are continuous functions of ρ, it follows that the constant C in (A.31) depends only on the shape-regularity parameters of $\{\mathcal{T}_h^q\}_{h>0}$, which, in particular, fix the range of ρ.

A.2.3. *Three dimensions*

We briefly comment on the modifications that need to be made to our arguments above when $d = 3$. Consider a family of quasiuniform nonobtuse partitions $\{\mathcal{T}_h^q\}_{h>0}$, in the sense of (A1) in Section 4, of the ball $D = B(\underset{\sim}{0}, r_D) \subset \mathbb{R}^3$. Excluding the case of configuration 0, when the closure of a simplex $\kappa_q \in \mathcal{T}_h^q$ has empty intersection with ∂D, there are now three different configurations to consider, corresponding to the cases when the closure of κ_q has one, two or three vertices on ∂D.

Let us suppose, for example, that the open nonobtuse simplex $\kappa_q \in \mathcal{T}_h^q$ has three vertices A, B and C on the sphere ∂D, while the fourth vertex D is in the interior of the domain D, on a sphere $\mathcal{C}$ concentric with ∂D, that is a distance h away from ∂D. We raise the inward normals from A, B, C to ∂D, and consider the points A′, B′, C′ in the interior of the ball D that are on the respective normals to ∂D at A, B and C, and a distance h away from A, B and C, respectively; i.e. A′, B′, C′ are on the sphere $\mathcal{C}$. The tetrahedron $\kappa_q =$ ABCD is then contained in the curved triangular prismoid $R(\kappa_q) :=$ ABCA′B′C′, with curved faces ABC and A′B′C′.

Given a function $\widehat{\varphi} \in H^2_M(D)$, we then extrapolate $\widehat{\varphi}$ from A′, B′ and C′ using (A.20) to define $\widehat{\Phi}(\mathrm{A})$, $\widehat{\Phi}(\mathrm{B})$ and $\widehat{\Phi}(\mathrm{C})$, and define $p_{\widehat{\varphi}}$ as the affine function of $\underset{\sim}{q}$ on the simplex ABCD whose nodal values are $\widehat{\Phi}(\mathrm{A})$, $\widehat{\Phi}(\mathrm{B})$, $\widehat{\Phi}(\mathrm{C})$ and $\widehat{\varphi}(\mathrm{D})$. We note in particular that if $\widehat{\varphi} \in \mathbb{P}_1^q$, then $p_{\widehat{\varphi}} = \widehat{\varphi}$. Using spherical polar co-ordinates we map the curved triangular prismoid $R(\kappa_q)$ containing the simplex $\kappa_q =$ ABCD into a right triangular prism $R_{\mathrm{polar}}(\kappa_q)$, and then argue as in the case of $d = 2$ above, using Lemma A.3, to deduce the analogue of (A.31) in the case of $d = 3$.

A.2.4. *Stability of the Maxwellian-weighted L^2 projector in the Maxwellian-weighted H^1 norm*

Now we are ready to discuss the question of stability, in the M-weighted H^1 norm, of the orthogonal projector in the M-weighted L^2 inner product on $D \subset \mathbb{R}^d$, $d = 2, 3$. We begin by considering the following auxiliary problem: Let $\widehat{g} \in L^2_M(D)$; find $\widehat{z} \in H^1_M(D)$ such that

$$a(\widehat{z}, \widehat{\varphi}) = \ell(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in H^1_M(D), \tag{A.32}$$

where, for $\widehat{\zeta}, \widehat{\varphi} \in H^1_M(D)$,

$$a(\widehat{\zeta}, \widehat{\varphi}) := \int_D M \left(\underset{\sim}{\nabla}_q \widehat{\zeta} \cdot \underset{\sim}{\nabla}_q \widehat{\varphi} + \widehat{\zeta}\, \widehat{\varphi} \right) \mathrm{d}\underset{\sim}{q} \qquad \text{and} \qquad \ell(\widehat{\varphi}) := \int_D M\, \widehat{g}\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}.$$

The existence of a unique solution $\widehat{z} \in H^1_M(D)$ to (A.32) follows by the Lax–Milgram theorem. Note that

$$\|\widehat{z}\|_{L^2_M(D)} \le \|\widehat{z}\|_{H^1_M(D)} \le \|\widehat{g}\|_{L^2_M(D)}.$$

We begin by showing the following elliptic regularity result for (A.32): $\widehat{z} \in H^2_D(M)$, and the bound stated in (A.41) below holds. To this end, for $\delta > 0$ we define

$$U_\delta(s) := U\left(\left(\frac{r_D}{r_D + \delta} \right)^2 s \right), \quad s \in \left[0, \tfrac{1}{2} r_D^2\right) \qquad \text{and} \qquad M_\delta(\underset{\sim}{q}) := Z^{-1} \exp(-U_\delta(\tfrac{1}{2}|\underset{\sim}{q}|^2)), \quad \underset{\sim}{q} \in D,$$

where, as in (1.6) (i.e. with no δ-dependence in the definition of Z),

$$Z := \int_D \exp(-U(\tfrac{1}{2}|\underset{\sim}{q}|^2))\, \mathrm{d}\underset{\sim}{q}.$$

Note that since $U'(s) > 0$ for $s \in \left[0, \tfrac{1}{2} r_D^2\right)$, we have $0 \le U_\delta(s) \le U(s)$ for all $s \in \left[0, \tfrac{1}{2} r_D^2\right)$, with strict inequalities for $s \ne 0$, and $M(\underset{\sim}{q}) \le M_\delta(\underset{\sim}{q})$ for $\underset{\sim}{q} \in D$, with strict inequality for $\underset{\sim}{q} \ne \underset{\sim}{0}$. The fact that, thereby, $\int_D M_\delta(\underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q}$

is strictly greater than 1 rather than equal to 1 is of no significance. For $\widehat{g} \in L^2_M(D)$ and $\delta > 0$, we define

$$\widehat{g}_\delta(\underset{\sim}{q}) := \left(\frac{M(\underset{\sim}{q})}{M_\delta(\underset{\sim}{q})}\right)^{\frac{1}{2}} \widehat{g}(\underset{\sim}{q}), \qquad \underset{\sim}{q} \in D,$$

and note that $\widehat{g}_\delta \in L^2_{M_\delta}(D)$ with $\|\widehat{g}_\delta\|_{L^2_{M_\delta}(D)} = \|\widehat{g}\|_{L^2_M(D)}$.

We consider the following problem: For $\widehat{g} \in L^2_M(D)$ and $\delta > 0$, and with M_δ and $\widehat{g}_\delta$ as defined above, find $\widehat{z}_\delta \in H^1_{M_\delta}(D)$ such that

$$a_\delta(\widehat{z}_\delta, \widehat{\varphi}) = \ell_\delta(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in H^1_{M_\delta}(D), \tag{A.33}$$

where, for $\widehat{\zeta}, \widehat{\varphi} \in H^1_{M_\delta}(D)$,

$$a_\delta(\widehat{\zeta}, \widehat{\varphi}) := \int_D M_\delta \left(\underset{\sim}{\nabla}_q \widehat{\zeta} \cdot \underset{\sim}{\nabla}_q \widehat{\varphi} + \widehat{\zeta}\,\widehat{\varphi}\right) \mathrm{d}\underset{\sim}{q} \qquad \text{and} \qquad \ell_\delta(\widehat{\varphi}) := \int_D M_\delta\, \widehat{g}_\delta\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}.$$

We note that, for $\delta > 0$ and $\underset{\sim}{q} \in D$, $0 < Z^{-1} \exp\left(-U_\delta(\frac{1}{2} r_D^2)\right) \leq M_\delta(\underset{\sim}{q}) \leq Z^{-1}$, and therefore $L^2_{M_\delta}(D)$ and $H^1_{M_\delta}(D)$ are homeomorphic to $L^2(D)$ and $H^1(D)$, respectively, with equivalent respective norms, so they can be identified with $L^2(D)$ and $H^1(D)$, respectively.

As in the case of (A.32), the existence of a unique solution $\widehat{z}_\delta \in H^1_{M_\delta}(D)$ to (A.33) follows by the Lax–Milgram theorem, and

$$\|\widehat{z}_\delta\|_{L^2_{M_\delta}(D)} \leq \|\widehat{z}_\delta\|_{H^1_{M_\delta}(D)} \leq \|\widehat{g}_\delta\|_{L^2_{M_\delta}(D)} = \|\widehat{g}\|_{L^2_M(D)}. \tag{A.34}$$

Also, by (standard) elliptic regularity theory, $\widehat{z}_\delta \in H^1_{M_\delta}(D) = H^1(D)$ belongs to $H^2(D) = H^2_{M_\delta}(D)$ for all $\delta > 0$.

Since $C^\infty_0(D) \subset H^1_{M_\delta}(D)$ for any $\delta > 0$, on choosing $\widehat{\varphi} \in C^\infty_0(D)$ in (A.33), it follows that

$$-\underset{\sim}{\nabla}_q \cdot (M_\delta \underset{\sim}{\nabla}_q \widehat{z}_\delta) + M_\delta\, \widehat{z}_\delta = M_\delta\, \widehat{g}_\delta \qquad \text{in } \mathcal{D}'(D), \tag{A.35}$$

i.e. in the sense of distributions on D. As $M_\delta \in C^\infty(D)$, multiplication by M_δ of elements of $\mathcal{D}'(D)$ is correctly defined; thus, by the Leibniz rule for differentiation of the product of a $C^\infty(D)$ function and an element of $\mathcal{D}'(D)$, (A.35) yields

$$-M_\delta\, \Delta_q\, \widehat{z}_\delta - \underset{\sim}{\nabla}_q M_\delta \cdot \underset{\sim}{\nabla}_q \widehat{z}_\delta + M_\delta\, \widehat{z}_\delta = M_\delta\, \widehat{g} \qquad \text{in } \mathcal{D}'(D). \tag{A.36}$$

Noting that M_δ and U'_δ satisfy an identity analogous to (2.5), and that since $M_\delta^{-1} \in C^\infty(D)$ multiplication by M_δ^{-1} in $\mathcal{D}'(D)$ is meaningful, multiplying (A.36) by M_δ^{-1} we deduce that

$$-\Delta_q\, \widehat{z}_\delta + U'_\delta\, \underset{\sim}{q} \cdot \underset{\sim}{\nabla}_q \widehat{z}_\delta + \widehat{z}_\delta = \widehat{g}_\delta \qquad \text{in } \mathcal{D}'(D). \tag{A.37}$$

As $\underset{\sim}{q} \mapsto U'_\delta(\frac{1}{2}|\underset{\sim}{q}|^2)\underset{\sim}{q}$ belongs to $[C^\infty(D)]^d$, the dot-product in the second term of (A.37) is meaningful as an operation in $[\mathcal{D}'(D)]^d$. Taking the partial derivative in $\mathcal{D}'(D)$ of (A.37) with respect to q_i, the ith component of $\underset{\sim}{q}$, gives

$$-\Delta_q \frac{\partial \widehat{z}_\delta}{\partial q_i} + q_i\, U''_\delta\, \underset{\sim}{q} \cdot \underset{\sim}{\nabla}_q \widehat{z}_\delta + U'_\delta \frac{\partial \widehat{z}_\delta}{\partial q_i} + U'_\delta\, \underset{\sim}{q} \cdot \underset{\sim}{\nabla}_q \frac{\partial \widehat{z}_\delta}{\partial q_i} + \frac{\partial \widehat{z}_\delta}{\partial q_i} = \frac{\partial \widehat{g}_\delta}{\partial q_i} \qquad \text{in } \mathcal{D}'(D), \quad i \in \{1, \dots, d\}. \tag{A.38}$$

For $\widehat{\varphi} \in C^\infty_0(D)$, we have $M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i} \in C^\infty_0(D)$, and therefore (A.38) implies that

$$\left\langle -\Delta_q \frac{\partial \widehat{z}_\delta}{\partial q_i}, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i} \right\rangle + \left\langle q_i\, U''_\delta\, \underset{\sim}{q} \cdot \underset{\sim}{\nabla}_q \widehat{z}_\delta, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i} \right\rangle + \left\langle U'_\delta \frac{\partial \widehat{z}_\delta}{\partial q_i}, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i} \right\rangle + \left\langle U'_\delta\, \underset{\sim}{q} \cdot \underset{\sim}{\nabla}_q \frac{\partial \widehat{z}_\delta}{\partial q_i}, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i} \right\rangle$$
$$+ \left\langle \frac{\partial \widehat{z}_\delta}{\partial q_i}, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i} \right\rangle = \left\langle \frac{\partial \widehat{g}_\delta}{\partial q_i}, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i} \right\rangle \qquad \forall \widehat{\varphi} \in C^\infty_0(D), \qquad i \in \{1, \dots, d\}; \tag{A.39}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality paring on $\mathcal{D}'(D)\times C_0^\infty(D)$. Writing $\Delta_q = \underset{\sim}{\nabla}_q \cdot \underset{\sim}{\nabla}_q$ in the first term on the left-hand side of (A.39), passing $\underset{\sim}{\nabla}_q$ to the test function in this term, using the Leibniz rule in $C^\infty(D)$, noting (2.5) and that $U_\delta' \in C^\infty(D)$, whereby multiplication in $\mathcal{D}'(D)$ by U_δ' is legitimate, and observing that one of the two terms that result upon the use of the Leibniz rule from the first term on the left-hand side of (A.39) cancels with the fourth term on the left-hand side of (A.39), gives

$$\left\langle \underset{\sim}{\nabla}_q \frac{\partial \widehat{z}_\delta}{\partial q_i}, M_\delta \underset{\sim}{\nabla}_q \frac{\partial \widehat{\varphi}}{\partial q_i}\right\rangle + \left\langle q_i\, U_\delta''\, \underset{\sim}{q}\cdot \underset{\sim}{\nabla}_q\, \widehat{z}_\delta, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i}\right\rangle + \left\langle U_\delta' \frac{\partial \widehat{z}_\delta}{\partial q_i}, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i}\right\rangle + \left\langle \frac{\partial \widehat{z}_\delta}{\partial q_i}, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i}\right\rangle = \left\langle \frac{\partial \widehat{g}_\delta}{\partial q_i}, M_\delta \frac{\partial \widehat{\varphi}}{\partial q_i}\right\rangle$$

for all $\widehat{\varphi} \in C_0^\infty(D)$, $i \in \{1,\dots,d\}$. Summing over $i = 1 \to d$, we deduce the identity

$$\begin{aligned}\mathcal{A}_\delta(\widehat{z}_\delta,\widehat{\varphi}) &:= \int_D M_\delta\, \underset{\sim}{\nabla}_q \underset{\sim}{\nabla}_q\, \widehat{z}_\delta : \underset{\sim}{\nabla}_q \underset{\sim}{\nabla}_q\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q} + \int_D M_\delta\, U_\delta''\, (\underset{\sim}{q}\cdot \underset{\sim}{\nabla}_q\, \widehat{z}_\delta)\,(\underset{\sim}{q}\cdot \underset{\sim}{\nabla}_q\, \widehat{\varphi})\, \mathrm{d}\underset{\sim}{q} + \int_D M_\delta\,(U_\delta'+1)\, \underset{\sim}{\nabla}_q\, \widehat{z}_\delta \cdot \underset{\sim}{\nabla}_q\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q}\\ &= -\int_D \widehat{g}_\delta\, \underset{\sim}{\nabla}_q \cdot \left(M_\delta\, \underset{\sim}{\nabla}_q\, \widehat{\varphi}\right) \mathrm{d}\underset{\sim}{q} = -\int_D M_\delta\, \widehat{g}_\delta\, \Delta_q \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q} + \int_D M_\delta\, \widehat{g}_\delta\, U_\delta'\, \underset{\sim}{q}\cdot \underset{\sim}{\nabla}_q\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q} =: \mathcal{L}_\delta(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in C_0^\infty(D).\end{aligned}$$

Consider the norm $\|\cdot\|_{\mathcal{H}^2_{M_\delta}(D)}$ defined by

$$\|\widehat{\zeta}\|^2_{\mathcal{H}^2_{M_\delta}(D)} := \int_D M_\delta \left[|\underset{\sim}{\nabla}_q \underset{\sim}{\nabla}_q\, \widehat{\zeta}|^2 + U_\delta''\, |\underset{\sim}{q}\cdot \underset{\sim}{\nabla}_q\, \widehat{\zeta}|^2 + (U_\delta'+1)\, |\underset{\sim}{\nabla}_q\, \widehat{\zeta}|^2 + |\widehat{\zeta}|^2\right] \mathrm{d}\underset{\sim}{q}.$$

We observe that $\|\cdot\|_{\mathcal{H}^2_{M_\delta}(D)}$ is an equivalent norm on $H^2_{M_\delta}(D) = H^2(D)$ and, in particular, $\|\widehat{z}_\delta\|_{\mathcal{H}^2_{M_\delta}(D)} < \infty$. Next, we show that $\|\widehat{z}_\delta\|_{\mathcal{H}^2_{M_\delta}(D)}$ is, in fact, bounded, independent of $\delta > 0$. Recalling (A.34) we have that

$$\begin{aligned}\|\widehat{z}_\delta\|^2_{\mathcal{H}^2_{M_\delta}(D)} &= \mathcal{A}(\widehat{z}_\delta,\widehat{z}_\delta) + (M_\delta\, \widehat{z}_\delta, \widehat{z}_\delta)_D = \mathcal{L}_\delta(\widehat{z}_\delta) + (M_\delta\, \widehat{z}_\delta, \widehat{z}_\delta)_D = \mathcal{L}_\delta(\widehat{z}_\delta) + \|\widehat{z}_\delta\|^2_{L^2_{M_\delta}(D)}\\ &\le \|\widehat{g}_\delta\|_{L^2_{M_\delta}(D)}\, \|\Delta_q\, \widehat{z}_\delta\|_{L^2_{M_\delta}(D)} + \|\widehat{g}_\delta\|_{L^2_{M_\delta}(D)}\, \|U_\delta'\, \underset{\sim}{q}\cdot \underset{\sim}{\nabla}_q\, \widehat{z}_\delta\|_{L^2_{M_\delta}(D)} + \|\widehat{g}_\delta\|_{L^2_{M_\delta}(D)} \|\widehat{z}_\delta\|_{L^2_{M_\delta}(D)}.\end{aligned}$$

Since $\|\Delta_q\, \widehat{z}_\delta\|_{L^2_{M_\delta}(D)} \le d^{\frac12} \|\underset{\sim}{\nabla}_q \underset{\sim}{\nabla}_q\, \widehat{z}_\delta\|_{L^2_{M_\delta}(D)}$ and, thanks to (2.9b), $[U_\delta'(s)]^2 \le c_5\, U_\delta''(s)$, $s \in [0, \frac12 r_D^2)$, we thus have that

$$\|\widehat{z}_\delta\|^2_{\mathcal{H}^2_{M_\delta}(D)} \le (d + c_5 + 1)^{\frac12}\, \|\widehat{g}_\delta\|_{L^2_{M_\delta}(D)}\, \|\widehat{z}_\delta\|_{\mathcal{H}^2_{M_\delta}(D)},$$

which implies that

$$\|\widehat{z}_\delta\|^2_{H^2_{M_\delta}(D)} \le \|\widehat{z}_\delta\|^2_{\mathcal{H}^2_{M_\delta}(D)} \le (d + c_5 + 1)\, \|\widehat{g}_\delta\|^2_{L^2_{M_\delta}(D)} = (d + c_5 + 1)\, \|\widehat{g}\|^2_{L^2_M(D)}.$$

Since $M(\underset{\sim}{q}) \le M_\delta(\underset{\sim}{q})$ for all $\underset{\sim}{q} \in D$ and $\delta > 0$, we deduce that

$$\|\widehat{z}_\delta\|^2_{H^2_M(D)} \le (d + c_5 + 1)\, \|\widehat{g}\|^2_{L^2_M(D)}.$$

Since $\{\widehat{z}_\delta\}_{\delta>0}$ is bounded in $H^2_M(D)$, there exists $\widehat{z}_0 \in H^2_M(D)$ and a subsequence, still denoted $\{\widehat{z}_\delta\}_{\delta>0}$, such that $\widehat{z}_\delta \to \widehat{z}_0$ weakly in $H^2_M(D)$ as $\delta \to 0_+$. By the weak lower semicontinuity of the norm function $\widehat{\zeta} \mapsto \|\widehat{\zeta}\|_{H^2_M(D)}$,

$$|\widehat{z}_0|^2_{H^2_M(D)} \le \|\widehat{z}_0\|^2_{H^2_M(D)} \le (d + c_5 + 1)\, \|\widehat{g}\|^2_{L^2_M(D)}. \tag{A.40}$$

Since for $\zeta \ge 1$ (cf. (2.9a)) the space $H^2_M(D)$ is compactly embedded into $H^1_M(D)$ (see Lemma 5.2 in Antoci [1]), $\{\widehat{z}_\delta\}_{\delta>0}$ is strongly convergent to $\widehat{z}_0$ in $H^1_M(D)$ as $\delta \to 0_+$. Noting that $\{M_\delta\}_{\delta>0}$ converges to M uniformly on $\overline{D}$ as $\delta \to 0_+$ it follows that, as $\delta \to 0_+$,

$$\ell_\delta(\widehat{\varphi}) = \int_D M_\delta\, \widehat{g}_\delta\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q} = \int_D (M_\delta)^{\frac12}\, \widehat{g}\, M^{\frac12}\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q} \to \int_D M^{\frac12}\, \widehat{g}\, M^{\frac12}\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q} = \int_D M\, \widehat{g}\, \widehat{\varphi}\, \mathrm{d}\underset{\sim}{q} = \ell(\widehat{\varphi}) \qquad \forall \widehat{\varphi} \in C^\infty(\overline{D}),$$

and $a_\delta(\widehat{z}_\delta, \widehat{\varphi}) \to a(\widehat{z}_0, \widehat{\varphi})$ for all $\widehat{\varphi} \in C^\infty(\overline{D})$. Hence, passage to the limit $\delta \to 0_+$ in (A.33) yields $a(\widehat{z}_0, \widehat{\varphi}) = \ell(\widehat{\varphi})$ for all $\widehat{\varphi} \in C^\infty(\overline{D})$. Since $C^\infty(\overline{D})$ is dense in $H^1_M(D)$, also $a(\widehat{z}_0, \widehat{\varphi}) = \ell(\widehat{\varphi})$ for all $\widehat{\varphi} \in H^1_M(D)$. However, $\widehat{z} \in H^1_M(D)$ is the unique solution to (A.32), and therefore $\widehat{z} = \widehat{z}_0 \in H^2_M(D)$, and then by (A.40),

$$|\widehat{z}|^2_{H^2_M(D)} \le \|\widehat{z}\|^2_{H^2_M(D)} \le (d + c_5 + 1)\, \|\widehat{g}\|^2_{L^2_M(D)}. \tag{A.41}$$

That completes the proof of the elliptic regularity result that we need in order to proceed with the proof of stability, in the M-weighted H^1 norm, of the orthogonal projector in the M-weighted L^2 inner product on D.

Taking $g = \widehat{\psi} - P^q_h \widehat{\psi}$ in (A.32), where P^q_h denotes the orthogonal projector in the M-weighted H^1 inner product on D, we have from the symmetry of the bilinear form $a(\cdot, \cdot)$, the definitions of $\widehat{z}$ and P^q_h, the Cauchy–Schwarz inequality and (A.31) that

$$\begin{aligned}
\|\widehat{\psi} - P^q_h \widehat{\psi}\|^2_{L^2_M(D)} &\le a(\widehat{\psi} - P^q_h \widehat{\psi}, \widehat{z}) = a(\widehat{\psi} - P^q_h \widehat{\psi}, \widehat{z} - P^q_h \widehat{z}) \\
&\le \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{H^1_M(D)}\, \|\widehat{z} - P^q_h \widehat{z}\|_{H^1_M(D)} \\
&\le C\, h_q\, \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{H^1_M(D)}\, |\widehat{z}|_{H^2_M(D)}.
\end{aligned}$$

The elliptic regularity result (A.41) with $\widehat{g} = \widehat{\psi} - P^q_h \widehat{\psi}$ gives

$$|\widehat{z}|_{H^2_M(D)} \le (d + c_5 + 1)^{\frac{1}{2}}\, \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{L^2_M(D)}.$$

We thus have that

$$\|\widehat{\psi} - P^q_h \widehat{\psi}\|_{L^2_M(D)} \le C\, h_q\, \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{H^1_M(D)}. \tag{A.42}$$

Now, by the first inverse inequality in the M-weighted H^1 norm on D stated in (4.53a), and (A.42),

$$\begin{aligned}
\|\widehat{\psi} - Q^q_h \widehat{\psi}\|_{H^1_M(D)} &\le \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{H^1_M(D)} + \|P^q_h \widehat{\psi} - Q^q_h \widehat{\psi}\|_{H^1_M(D)} \\
&\le \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{H^1_M(D)} + C_{\rm inv}\, h_q^{-1} \|P^q_h \widehat{\psi} - Q^q_h \widehat{\psi}\|_{L^2_M(D)} \\
&\le \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{H^1_M(D)} + C_{\rm inv}\, h_q^{-1} \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{L^2_M(D)} + C_{\rm inv}\, h_q^{-1} \|\widehat{\psi} - Q^q_h \widehat{\psi}\|_{L^2_M(D)} \\
&\le \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{H^1_M(D)} + 2\, C_{\rm inv}\, h_q^{-1} \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{L^2_M(D)} \le (1 + C)\, \|\widehat{\psi} - P^q_h \widehat{\psi}\|_{H^1_M(D)}.
\end{aligned}$$

In particular the last inequality implies that

$$\|\widehat{\psi} - Q^q_h \widehat{\psi}\|_{H^1_M(D)} \le 2(1 + C)\, \|\widehat{\psi}\|_{H^1_M(D)} \qquad \forall \widehat{\psi} \in H^1_M(D)$$

and therefore also,

$$\|Q^q_h \widehat{\psi}\|_{H^1_M(D)} \le (3 + 2C)\, \|\widehat{\psi}\|_{H^1_M(D)} \qquad \forall \widehat{\psi} \in H^1_M(D). \tag{A.43}$$

It remains to prove that the projector $Q^M_h = Q^x_h\, Q^q_h = Q^q_h\, Q^x_h$, where Q^x_h is the orthogonal projector in $L^2(\Omega)$ onto X^x_h and Q^q_h is the orthogonal projector in $L^2_M(D)$ onto X^q_h, is stable in the norm of $\widehat{X} := H^1(\Omega \times D; M)$.

Indeed,

$$
\begin{aligned}
\|Q_h^M \widehat{\psi}\|_{\widehat{X}}^2 = \|Q_h^x\, Q_h^q \widehat{\psi}\|_{\widehat{X}}^2 &= \int_{\Omega\times D} M \left[|Q_h^x\, Q_h^q \widehat{\psi}|^2 + |\underset{\sim}{\nabla}_x\, (Q_h^x\, Q_h^q \widehat{\psi})|^2 + |\underset{\sim}{\nabla}_q\, (Q_h^x\, Q_h^q \widehat{\psi})|^2 \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x} \\
&\leq \int_D M\, \|Q_h^x\, (Q_h^q \widehat{\psi})(\cdot, \underset{\sim}{q})\|_{H^1(\Omega)}^2\, \mathrm{d}\underset{\sim}{q} + \int_\Omega \|Q_h^q\, (Q_h^x \widehat{\psi})(\underset{\sim}{x}, \cdot)\|_{H_M^1(D)}^2\, \mathrm{d}\underset{\sim}{x} \\
&\leq C \left[\int_D M\, \|Q_h^q \widehat{\psi}(\cdot, \underset{\sim}{q})\|_{H^1(\Omega)}^2\, \mathrm{d}\underset{\sim}{q} + \int_\Omega \|Q_h^x \widehat{\psi}(\underset{\sim}{x}, \cdot)\|_{H_M^1(D)}^2\, \mathrm{d}\underset{\sim}{x} \right] \\
&\leq C \left[\int_D M\, \|\widehat{\psi}(\cdot, \underset{\sim}{q})\|_{H^1(\Omega)}^2\, \mathrm{d}\underset{\sim}{q} + \int_\Omega \|\widehat{\psi}(\underset{\sim}{x}, \cdot)\|_{H_M^1(D)}^2\, \mathrm{d}\underset{\sim}{x} \right] \\
&= C\, \|\widehat{\psi}\|_{\widehat{X}}^2,
\end{aligned}
$$

where in the transition to the third line we used the stability of Q_h^x in the $H^1(\Omega)$ norm, and the stability of Q_h^q in the $H_M^1(D)$ norm stated in (A.43). In the transition to the penultimate line we used Fubini's theorem to exchange the order of integration, together with the fact that Q_h^q is a contraction in the norm of $L_M^2(D)$ and Q_h^x is a contraction in the norm of $L^2(\Omega)$.

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