Logic and Proof Hilary 2024

Examples of Ground Resolution Proofs

James Worrell

In this lecture we show how to use the Ground Resolution Theorem, proved in the last lecture, to do some deduction in first-order logic.

1 Ground Resolution Theorem

Recall that the process of eliminating existential quantifiers by introducing extra function and constant symbols is called *Skolemisation.* The extra symbols introduced are called *Skolem functions.* We begin with a slight generalisation of a theorem that was stated in the previous lecture. In this generalisation we consider Skolemising a collection of formulas rather than a single formula.

Theorem 1. Let $F_1, \ldots, F_n$ be closed rectified formulas in prenex form with respective Skolem forms $G_1, \ldots, G_n$. Assume that each G_i is constructed using a different set of Skolem functions. Then $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ is satisfiable if and only if $G_1 \wedge G_2 \wedge \ldots \wedge G_n$ is satisfiable.

Recall that a ground term is a term that does not contain any variables. Given a quantifier-free formula F, a *ground instance* of F is a formula obtained by replacing all the variables in F with ground terms.

The following is a slight generalisation of the version of the Ground Resolution Theorem proved in the last lecture. Before we considered only a single formula in Skolem form. Here we consider a conjunction of such formulas, which is more convenient for the applications below.

Theorem 2 (Ground Resolution Theorem)**.** Let $F_1, \ldots, F_n$ be closed formulas in Skolem form whose respective matrices $F_1^* \wedge \ldots \wedge F_n^*$ are in **CNF**. Then $F_1 \wedge \ldots \wedge F_n$ is unsatisfiable if and only if there is a propositional resolution proof of $\Box$ from the set of ground instances of clauses from $F_1^*, \ldots, F_n^*$.

2 Examples

In this section we give two examples of the use of the Ground Resolution Theorem.

Example 3. We would like to formalise the following statements in first-order logic and to use ground resolution to show that (a), (b) and (c) together entail (d).

(a) Everyone at Oriel is either lazy, a rower or a drunk.

(b) All rowers are lazy.

(c) Someone at Oriel is not drunk.

(d) Someone at Oriel is lazy.

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\neg R(a), L(a)\} \qquad \{\neg O(a), L(a), R(a), D(a)\}}{\{L(a), \neg O(a), D(a)\}} \qquad \{\neg O(a), \neg L(a)\}}{\{\neg O(a), D(a)\}} \qquad \{\neg D(a)\}}{\{\neg O(a)\}} \qquad \{O(a)\}}{\Box}$$

Figure 1: The nature of Oriel students

We translate (a), (b), (c) and the negation of (d) into closed formulas of first-order logic as follows.

$$\begin{aligned} F_1 &= \forall x\,(O(x) \to (L(x) \vee R(x) \vee D(x))) \\ F_2 &= \forall x\,(R(x) \to L(x)) \\ F_3 &= \exists x\,(O(x) \wedge \neg D(x)) \\ F_4 &= \neg \exists x\,(O(x) \wedge L(x))\,. \end{aligned}$$

Next we translate F_1, F_2, F_3 and F_4 to Skolem form. To do this we bring all quantifiers to the outside, eliminate existential quantifiers by introducing Skolem functions and finally bring the matrix of each formula into **CNF**. This yields

$$\begin{aligned} G_1 &= \forall x\,(\neg O(x) \vee L(x) \vee R(x) \vee D(x)) \\ G_2 &= \forall x\,(\neg R(x) \vee L(x)) \\ G_3 &= O(a) \wedge \neg D(a) \\ G_4 &= \forall x\,(\neg O(x) \vee \neg L(x))\,. \end{aligned}$$

where a is a fresh constant symbol.

Now we deduce the empty clause $\Box$ from ground instances of clauses in the respective matrices of the Skolem-form formulas $G_1, \ldots, G_4$. Note that these formulas are defined over a signature with a single constant symbol a, which is therefore the only ground term. The proof is shown in Figure 1.

Example 4. Using ground resolution we show that

$$\forall x\, \exists y\, (P(x) \to Q(y)) \to \exists y\, \forall x\, (P(x) \to Q(y))$$

is a valid sentence.

We can show this by showing that the negation is unsatisfiable. The negation can be written:

$$\forall x\, \exists y\, (P(x) \to Q(y)) \wedge \neg \exists y\, \forall x\, (P(x) \to Q(y))\,.$$

We bring each conjunction to Skolem form, yielding

$$\begin{aligned} F_1 &= \forall x\,(\neg P(x) \vee Q(f(x))) \\ F_2 &= \forall y\,(P(g(y)) \wedge \neg Q(y))\,. \end{aligned}$$

Note that F_1 and F_2 are defined over a signature with no constants and so there are no ground terms. We remedy this problem by introducing a single new constant symbol a. Now the set of ground terms is $\{a, f(a), g(a), f(f(a)), f(g(a), \ldots\}$. We can now derive $\Box$ by the propositional resolution proof in Figure 2 which every leaf is a ground instance of a clause from the respective matrices of F_1 and F_2.

$$\dfrac{\dfrac{\{P(g(a))\} \qquad \{\neg P(g(a)), Q(f(g(a)))\}}{\{Q(f(g(a)))\}} \qquad \{\neg Q(f(g(a)))\}}{\square}$$

Figure 2: Ground Resolution proof for Example 4