Logic and Proof	Hilary 2024

Exercise Sheet 4

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1. Formalise the following as sentences of first-order logic. Use $B(x)$ for "x is a barber" and $S(x, y)$ for "x shaves y".

(a) Every barber shaves all persons who do not shave themselves.

(b) No barber shaves any person who shaves himself.

Convert your answers to Skolem form and use ground resolution to show that (c) below is a consequence of (a) and (b).

(c) There are no barbers.

2. Consider the unification algorithm from the lecture notes.

(a) Apply the algorithm to the set of literals

$$\mathbf{L} = \{P(x, y), P(f(a), g(x)), P(f(z), g(f(z)))\}.$$

(b) Suppose we omit the *occurs check* "does x occur in t?" to improve efficiency. Exhibit literals L_1 and L_2 with no variable in common such that the unification algorithm fails to terminate on $\{L_1, L_2\}$.

3. Express the following by formulas of first-order logic, using predicate $H(x)$ for "x is happy", $R(x)$ for "x is rich", $G(x)$ for "x is a graduate", and $C(x, y)$ for "y is a child of x".

(a) Any person is happy if all their children are rich.

(b) All graduates are rich.

(c) Someone is a graduate if they are a child of a graduate.

(d) All graduates are happy.

Use first-order resolution to show that (d) is entailed by (a), (b) and (c). Indicate the substitutions in each resolution step.

4. Give an example of a finite set of clauses F in first-order logic such that $Res^*(F)$ is infinite.

5. Give an example of a signature σ that has at least one constant symbol and a σ-formula F (that does not mention equality) such that F is satisfiable but does not have a Herbrand model.

6. A closed formula is in the class $\exists^*\forall^*$ if it has the form $\exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_n F$, where F is quantifier-free and $m, n \geq 0$.

(a) Prove that if an $\exists^*\forall^*$-formula over a signature with no function symbols has a model then it has a finite model.

(b) Suggest an algorithm for deciding whether a given $\exists^*\forall^*$-formula over a signature with no function symbols has a model.

(c) Argue that the satisfiability problem for the class of $\forall^*$-formulas that may mention function symbols is undecidable.