Full Abstraction for Nominal General References

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Full Abstraction for Nominal General References – Overview

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We will be talking about:

- Nominal Sets (Gabbay, Pitts)
- A functional higher-order language with nominal general references (Pitts, Stark, NT), the $\nu\rho$-calculus
- Nominal Games (Abramsky, Ghica, Murawski, Ong, Stark, NT)

Nominal Sets [GP99, Pit03]

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Moreover, all $x \in X$ have finite support $\mathtt{S}(x)$,

$$\mathtt{S}(x) \triangleq \{\alpha \in \mathsf{N} \mid \text{for infinitely many } \beta.\ (\alpha\ \beta) \circ x \neq x\}$$

For $x \in X$ and $\alpha \in$N, *α is fresh for x*, written $\alpha \# x$, iff $\alpha \notin \mathtt{S}(x)$.

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N is a nominal set, and so is $\mathrm{N}^{\#}$ –the set of *finite lists* of distinct names.

Constructions in Nominal Sets

- If X, Y nominal sets then $X \times Y$ a nominal set.
- If Y a nominal set, $X \subseteq Y$, X closed under permutations then X is a *nominal subset* of Y.
- $R \subseteq X \times Y$ is a *nominal relation* iff $xRy \iff (\pi \circ x)R(\pi \circ y)$.
- $f : X \to Y$ is a *nominal function* iff $f(\pi \circ x) = \pi \circ f(x)$.
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- If $\alpha \in \mathsf{N}$ and $x \in X$ then define $\boxed{\mathtt{S}(\langle\alpha\rangle x) = \mathtt{S}(x) \setminus \{\alpha\}}$

$$\langle\alpha\rangle x \triangleq \{(\beta, y) \in \mathsf{N} \times X \mid (\beta = \alpha \vee \beta \# x) \wedge y = (\alpha\ \beta) \circ x\}$$

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- If $x \in X$ and $\vec{\alpha} \in \mathsf{N}^{\#}$ then define $\boxed{\mathtt{S}([x]_{\vec{\alpha}}) = \mathtt{S}(x) \cap \mathtt{S}(\vec{\alpha})}$

$$[x]_{\vec{\alpha}} \triangleq \{y \in X \mid \exists \pi.\ \pi \circ \vec{\alpha} = \vec{\alpha} \wedge y = \pi \circ x\}$$

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commands naturals references functions pairs

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We need to work in Nominal Sets over *a collection of sets of names*,

$$\mathsf{N} \triangleq \biguplus_{A \in \mathrm{TY}} \mathsf{N}_A \qquad \mathsf{PERM}(\mathsf{N}) = \bigoplus_{A \in \mathrm{TY}} \mathsf{PERM}(\mathsf{N}_A)$$

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Let $\mathbf{Nom}_{\mathrm{TY}}$ be the category of nominal sets (on N) and nominal functions.

$\rightsquigarrow$ we denote names by $\mathtt{a}^A, \mathtt{b}^B$,... or α, β,...,
and finite lists of distinct names by $\vec{\alpha}, \vec{\beta}, \ldots$.

The $\nu\rho$-calculus

The $\nu\rho$-calculus is a functional calculus with nominal references.

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$\mathsf{TE} \ni M, N ::=$	$x \mid \lambda x.M \mid M\,N$	λ-term
	$\mid \texttt{skip}$	return
	$\mid \tilde{n} \mid \texttt{pred}\,M \mid \texttt{succ}\,N$	arithmetic
	$\mid \texttt{if0}\ M\ \texttt{then}\ N_1\ \texttt{else}\ N_2$	if_then_else
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$\mid \alpha$	name, $\alpha = \mathtt{a}^A \in \mathsf{N}_A$
$\mid \nu\alpha.M$	ν-abstraction
$\mid [M = N]$	name-equality test
$\mid M := N \mid\ !M$	update / dereferencing

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$$\mathsf{VA} \ni V, W ::= \tilde{n} \mid \texttt{skip} \mid \alpha \mid x \mid \lambda x.M \mid \langle V, W \rangle$$

The $\nu\rho$-calculus: Typed Terms

Terms are typed in environments $(\Gamma, \vec{\alpha})$ consisting of:

- a set Γ of variable-type pairs
- *a list* $\vec{\alpha}$ of distinct names ($\vec{\alpha} \in \mathsf{N}^{\#}$)

$\vec{\alpha} \mid \Gamma \vdash M : A$
$\rightsquigarrow$ free vars in Γ
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$$\frac{}{\vec{\alpha} \mid \Gamma, x : A \vdash x : A} \qquad \frac{}{\vec{\alpha} \mid \Gamma \vdash \alpha : [A]}\ \alpha = \mathtt{a}^A \not\#\, \vec{\alpha}$$

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$$\frac{\vec{\alpha} \mid \Gamma \vdash M : [A]}{\vec{\alpha} \mid \Gamma \vdash !M : A} \qquad \frac{\vec{\alpha} \mid \Gamma \vdash M : [A] \qquad \vec{\alpha} \mid \Gamma \vdash N : A}{\vec{\alpha} \mid \Gamma \vdash M := N : \mathbb{1}}$$

The $\nu\rho$-calculus: Reduction

The reduction calculus is defined in store environments S:

$$S ::= \epsilon \mid \alpha, S \mid \alpha :: V, S$$

with their domains being lists of distinct names.

$$\text{EQ}\ \frac{}{S \vDash [\alpha = \beta] \longrightarrow S \vDash \tilde{n}}\ \begin{matrix} n{=}1 \text{ if } \alpha\#\beta \\ n{=}0 \text{ if } \alpha{=}\beta \end{matrix}$$

$$\text{NEW}\ \frac{}{S \vDash \nu\alpha.M \longrightarrow S, \beta \vDash (\alpha\ \beta) \circ M}\ \beta\#S$$

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The $\nu\rho$-calculus: An Example

$$M \triangleq \nu\alpha.\ \alpha := (\lambda x^{\mathbb{N}}, y^{\mathbb{N}}.\,\texttt{if0}\ x\ \texttt{then}\ y\ \texttt{else}\ (!\alpha)(\texttt{pred}\,x)(\texttt{succ}\,y))\,;\,!\alpha$$

– *What does it do?*

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&\longrightarrow S, \beta :: V \models V\,(n \tilde{-} 1)\,(m \tilde{+} 1) \longrightarrow \dots \\
&\longrightarrow S, \beta :: V \models V\,\tilde{0}\,(m \tilde{+} n) \longrightarrow S, \beta :: V \models m \tilde{+} n
\end{aligned}$$

$\nu\rho$-calculus : Observational Equivalence

The semantics yields the following notion of equivalence.

For typed terms $\vec{\alpha} \mid \Gamma \vdash M : A$ and $\vec{\alpha} \mid \Gamma \vdash N : A$,

$$\vec{\alpha} \mid \Gamma \vdash M \lessapprox N \iff$$
$$\forall C[\_] : \mathbb{N}.\, (\exists S'.\ \models C[M] \longrightarrow\!\!\!\!\!\rightarrow S' \models \tilde{0})$$
$$\implies (\exists S''.\ \models C[N] \longrightarrow\!\!\!\!\!\rightarrow S'' \models \tilde{0})$$

where $C[\_]$ is a variable- and name-closing context.

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For example,

$$\nu\alpha.\nu\beta.\lambda f^{\mathsf{N}_A \to \mathbb{N}}. \, (\texttt{zero}(f\alpha) \Leftrightarrow \texttt{zero}(f\beta)) \quad \not\lessapprox \quad \lambda f^{\mathsf{N}_A \to \mathbb{N}}. \, \tilde{0}$$

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e.g. $\quad C \triangleq \nu\gamma.\, \gamma := \tilde{2}\,;[\_]\,\lambda x.(\gamma := \texttt{pred}\,(!\gamma)\,;!\gamma)$

The Adventure - Fully Abstract Semantics

The goal:

$$M \lessapprox N \iff [\![M]\!] \lesssim [\![N]\!]$$

The Adventure - Fully Abstract Semantics

The goal:

$$M \lessapprox N \iff [\![M]\!] \lesssim [\![N]\!]$$

The plan:

- Rectify nominal games of [AGM^{+}04];
- Define a store monad in the category of nominal games –solve a *domain equation*;
- Show soundness;
- Restrict games –obtain *tidy strategies*;
- Show definability.

Nominal Games

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– But now we have fixed it using lists instead.

$$\begin{aligned} \text{Nominal Games} = {} & \text{CBV games of [HY99]} \\ & + \text{moves with state of [Ong02]} \\ & + \text{Nominal Sets } \textit{and strong support} \end{aligned}$$

For X a nominal set, $x \in X$, x has strong support iff

$$\forall \pi.\ (\pi \circ x = x \iff \forall \alpha \in \mathtt{S}(x).\, \pi(\alpha) = \alpha)$$

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Basic Arenas, Prearenas

$\mathbf{1}$ $M_1 \triangleq \{*\}$ $\lambda_1(*) \triangleq PA$ $\vdash_1 \triangleq \{(\dagger, *)\}$

$\mathbb{N}$ $M_{\mathbb{N}} \triangleq \mathbb{N}$ $\lambda_{\mathbb{N}}(m) \triangleq PA$ $\vdash_{\mathbb{N}} \triangleq \{(\dagger, m)\}$

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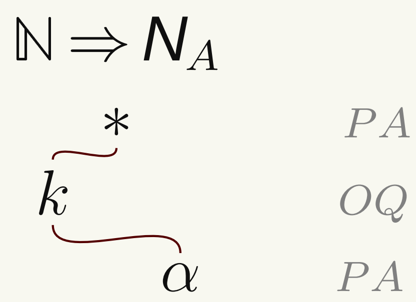

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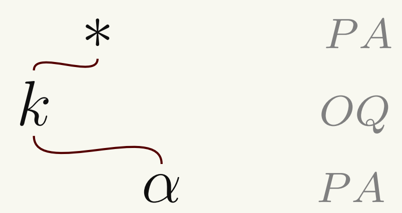
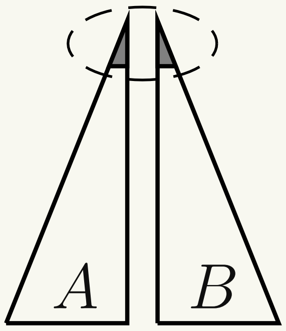


A *prearena* is an arena with its initial moves labeled OQ.

Arena Constructions

For nominal arenas A, B, define $A \otimes B$, $A_\perp$, $A \mathrel{\tilde{\Rightarrow}} B$ and $A \Rightarrow B$:



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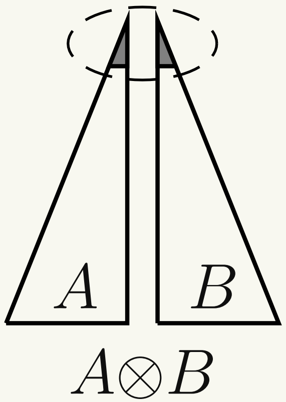

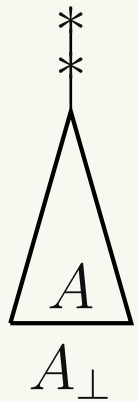

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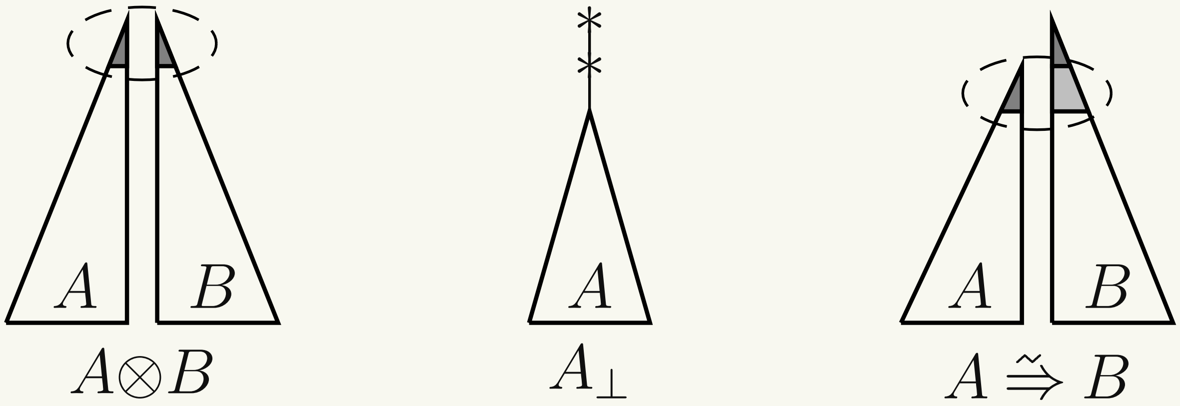

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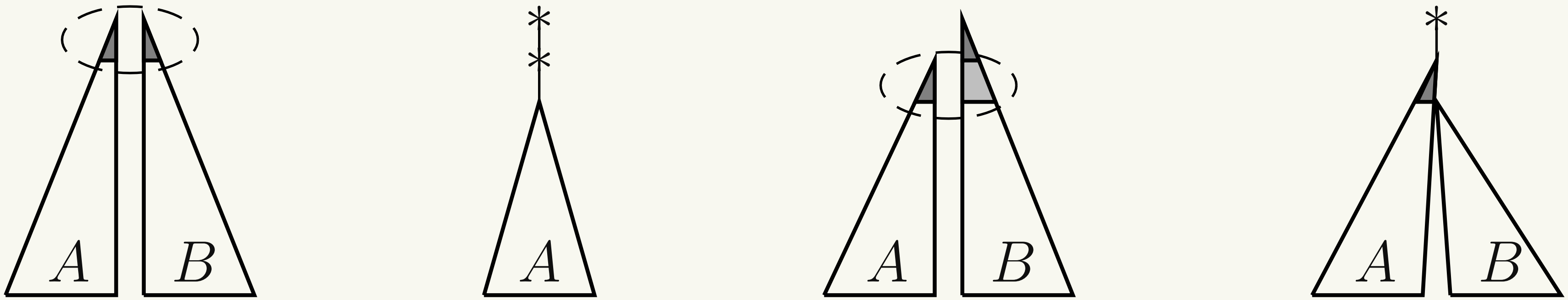

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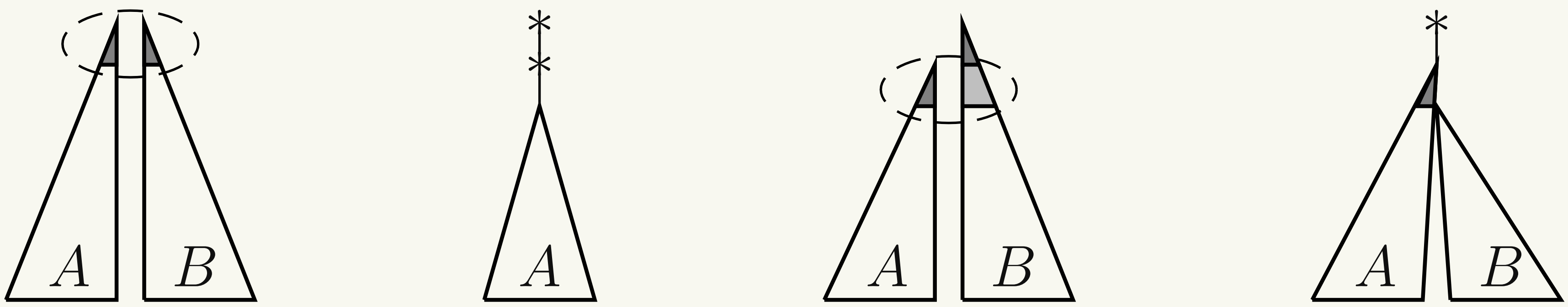

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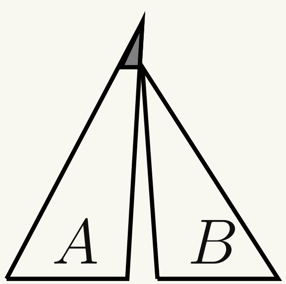
Also, the prearena $A \to B$:



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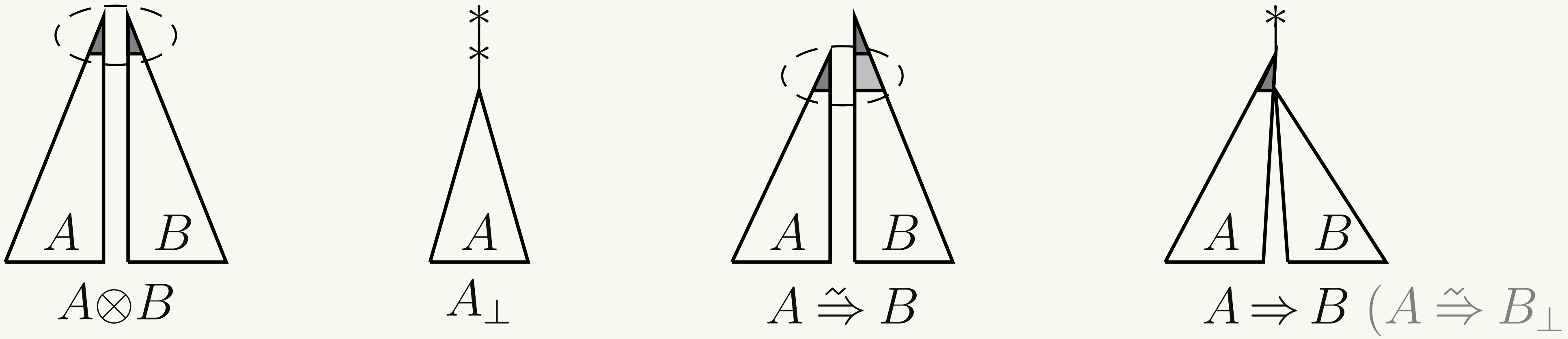

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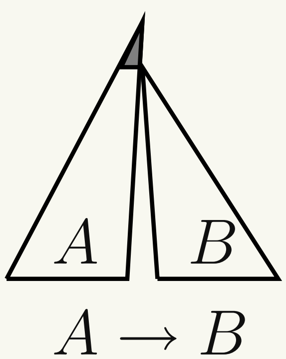

$$M_{A \to B} \triangleq M_A + M_B$$

$$\lambda_{A \to B} \triangleq [(i_A \mapsto OQ)\ ,\ m_A \mapsto \overline{\lambda_A}(m_A))\ ,\ \lambda_B]$$

$$\vdash_{A \to B} \triangleq \{(\dagger, i_A), (i_A, i_B)\} \cup \{(m, n) \mid m \vdash_{A,B} n\}$$

Sequences of Moves

For a prearena A, a sequence s of moves from A is:

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Moreover, the *P-view*, $\ulcorner s \urcorner$, of a justified sequence s is given by:

$\ulcorner sx \urcorner \triangleq \ulcorner s \urcorner x$	if x a P-move
$\ulcorner x \urcorner \triangleq x$	if x is initial
$\ulcorner s \overset{\frown}{x s' y} \urcorner \triangleq \ulcorner s \urcorner \overset{\frown}{xy}$	if y an O-move justified by x

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A *play* is a legal sequence of moves-with-names s satisfying:

(NC1) P-moves may (only) add fresh names to the local state;

(NC2) If a P-move x contains in its support a name α that is fresh for the previous P-view then α must appear in $\mathsf{nlist}(x)$;

(NC3) O-moves don't change the local state even if they contain fresh names in their supports.

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A *play* is a legal sequence of moves-with-names s satisfying:

(NC1) If x a P-move in s preceded by y then $\mathsf{nlist}(y) \leq \mathsf{nlist}(x)$;
if $\alpha \not\# \mathsf{nlist}(x)$ and $\alpha \# \mathsf{nlist}(y)$ then $\alpha \# s_{<x}$ *(α introduced by P)*.

(NC2) If x a P-move, $\alpha \not\# x$ and $\alpha \# \ulcorner s_{<x} \urcorner$ then $\alpha \not\# \mathsf{nlist}(x)$.

(NC3) If y an O-move justified by z then $\mathsf{nlist}(y) = \mathsf{nlist}(z)$.

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$$N_A \longrightarrow N_A \Rightarrow N_A$$

β $\quad OQ$

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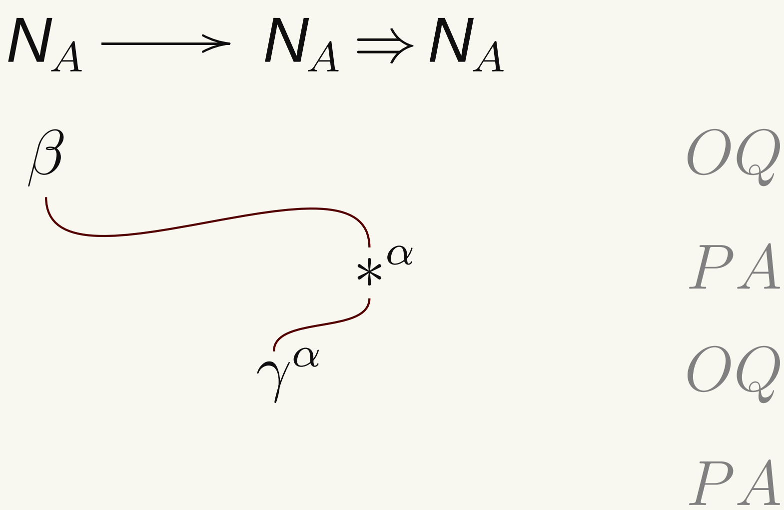

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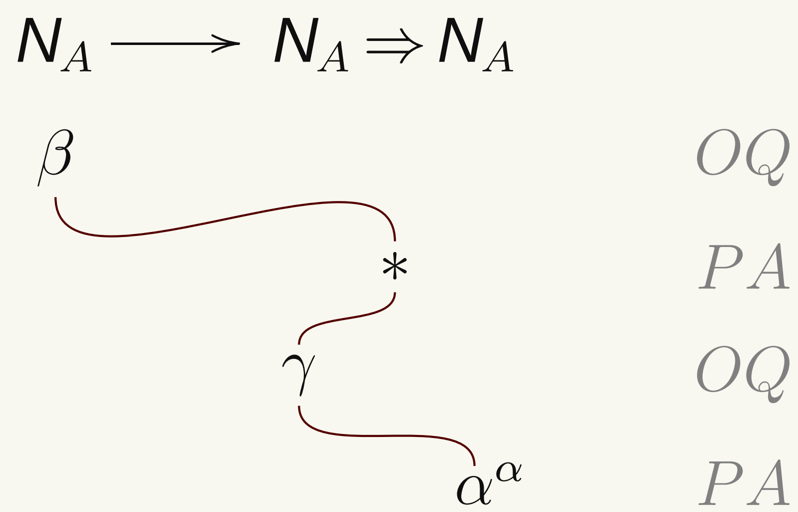

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An $\vec{\alpha}$-strategy σ on $A \to B$ is written $\sigma : A \to B$.

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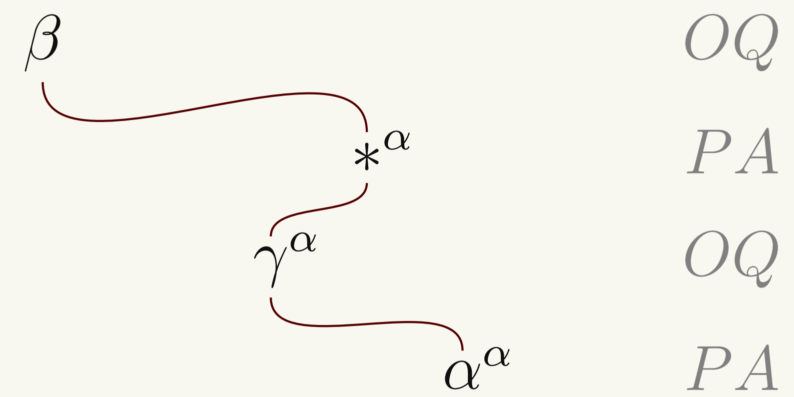

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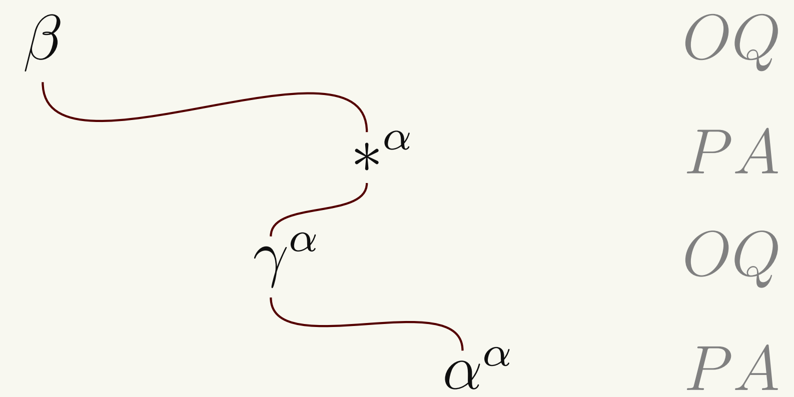

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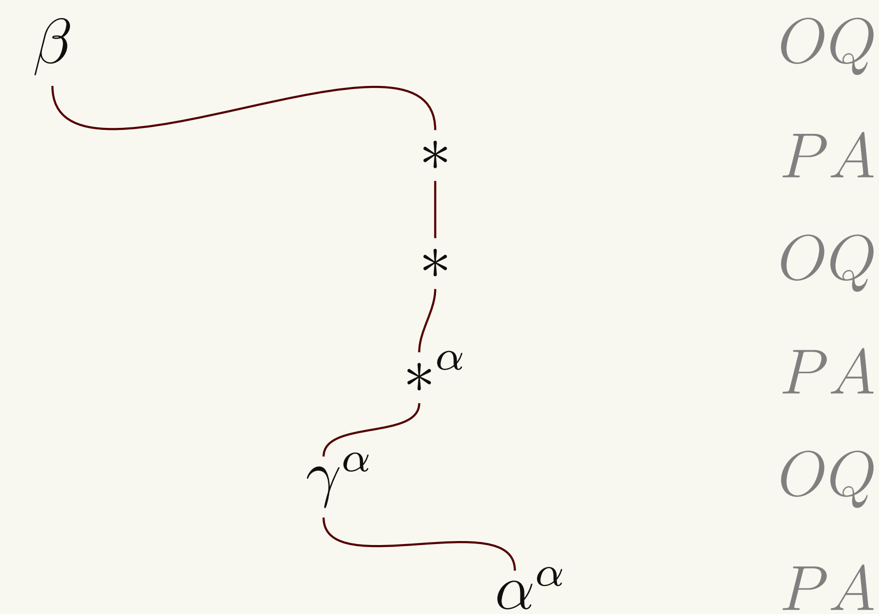
$N_A \longrightarrow (N_A \Rightarrow N_A)_\perp$



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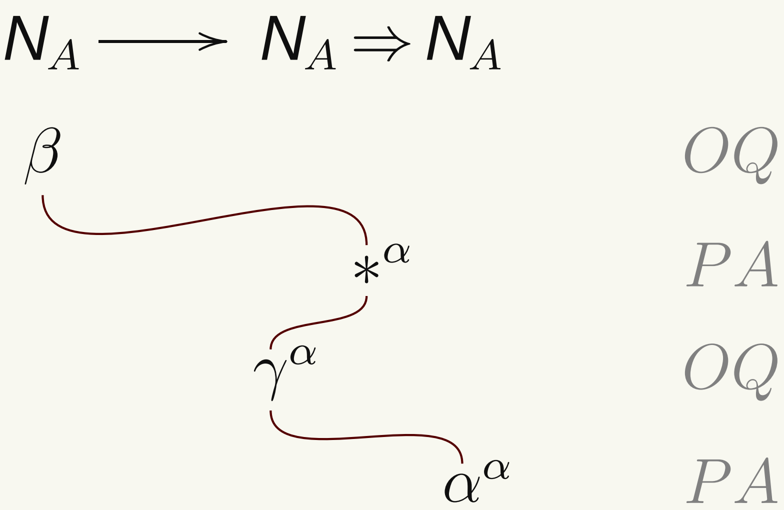

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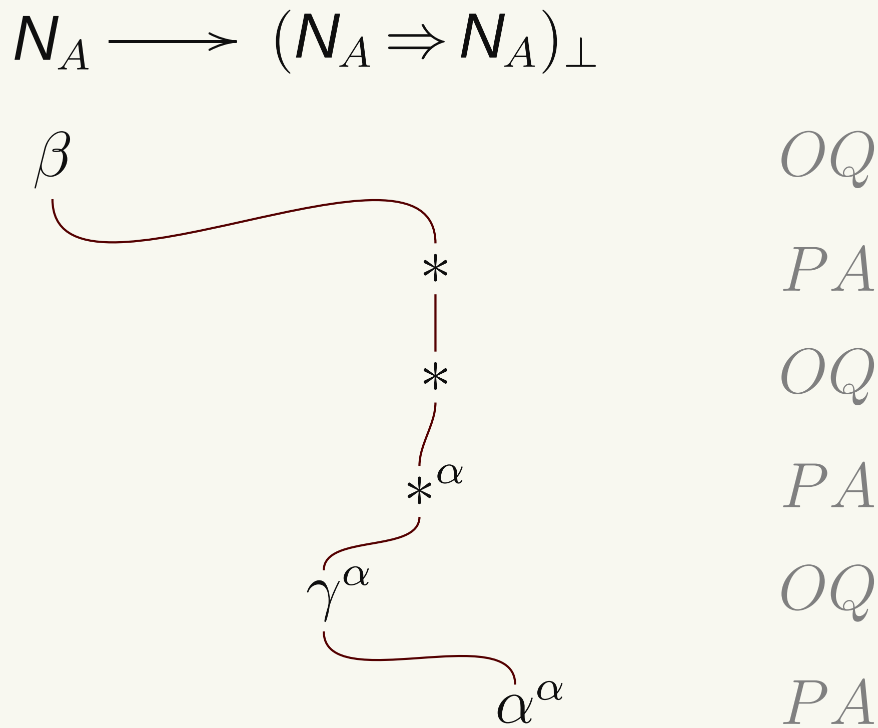

$([\![\mid x : [A] \vdash \nu\alpha.\lambda y.\alpha]\!])$

The category $\mathcal{V}_{\mathtt{t}}^{\vec{\alpha}}$

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Note: Our intention is to translate each typed term $\vec{\alpha} \mid \Gamma \vdash M : A$ to an arrow $[\![\Gamma]\!] \longrightarrow T[\![A]\!]$ in $\mathcal{V}_{\mathtt{t}}^{\vec{\alpha}}$.
$\rightsquigarrow$ Accommodate *name-addition* and *name-abstraction*.

Properties of $\mathcal{V}_{\mathtt{t}}^{\vec{\alpha}}$

$\mathcal{V}_{\mathtt{t}}^{\vec{\alpha}}$ is a symmetric monoidal category under $\otimes$, and is partially closed in the following sense.

For any object B, for any object A and any *pointed object* C there exists a bijection

$$\Lambda^{B}_{A,C} : \mathcal{V}_{\mathtt{t}}^{\vec{\alpha}}(A \otimes B, C) \xrightarrow{\cong} \mathcal{V}_{\mathtt{t}}^{\vec{\alpha}}(A, B \mathrel{\tilde{\Rightarrow}} C)$$

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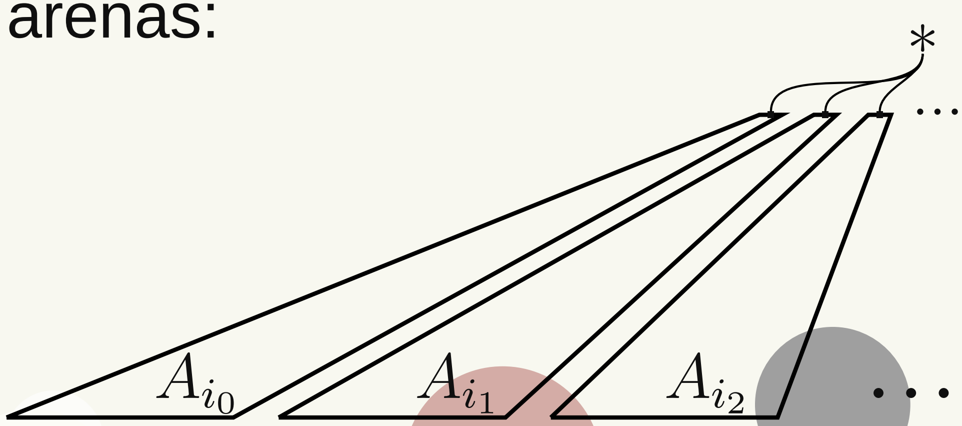
We can also extend $\otimes$ to an infinite tensor product of pointed arenas:



$$= \bigotimes_{i \in I} A_i$$

The Store Equation

We view general references as an effect and formulate a monadic semantics for $\nu\rho$. If types A are translated to $[\![A]\!]$ then we require:

$$[\![\mathbb{1}]\!] = 1 \qquad [\![\mathbb{N}]\!] = \mathbb{N} \qquad [\![[A]]\!] = \mathsf{N}_A \qquad [\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$$

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1. Use the categorical machinery of [SP82] for solving recursive domain equations, as adapted to games in [McC00].

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Having solved (SE) we obtain a strong monad (T, η, μ, τ):

$$TA = \xi \Rightarrow (A \otimes \xi)$$

$\mathcal{V}_{\mathtt{t}}$ is a model of $\nu\rho$

The previous reasoning applies for all $\vec{\alpha}$, so we obtain a model

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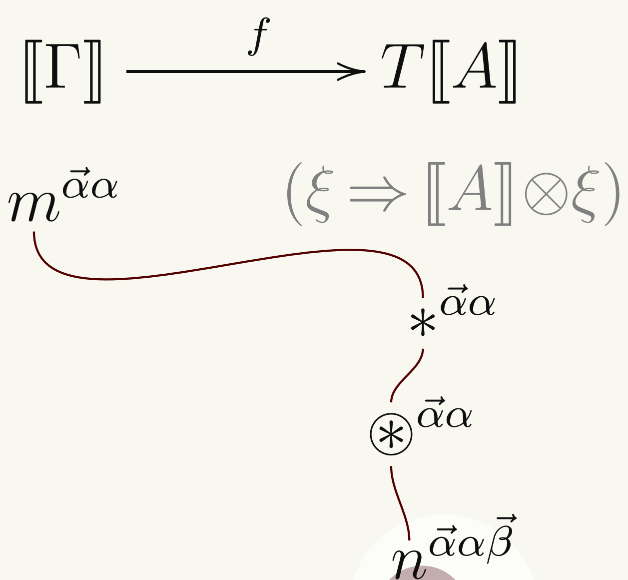

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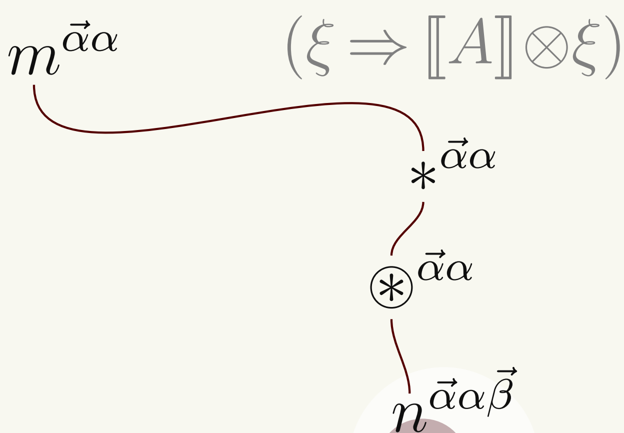

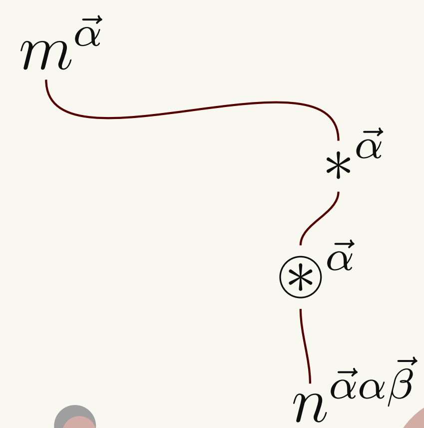

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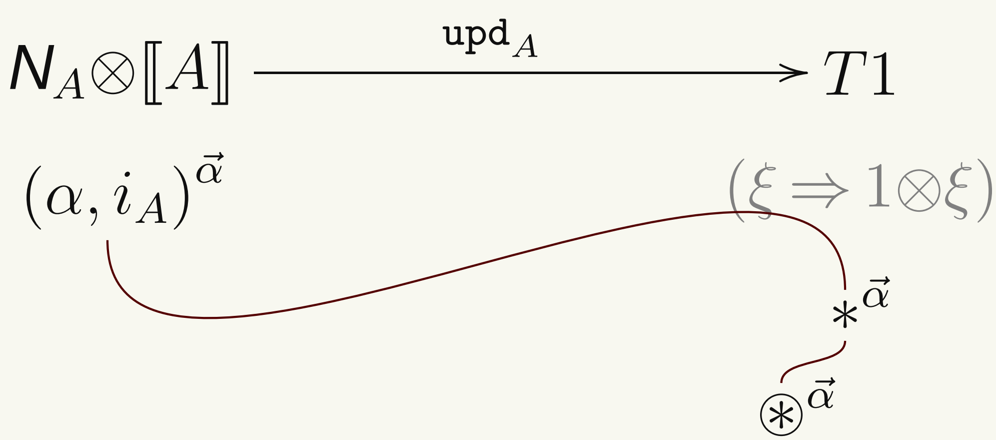

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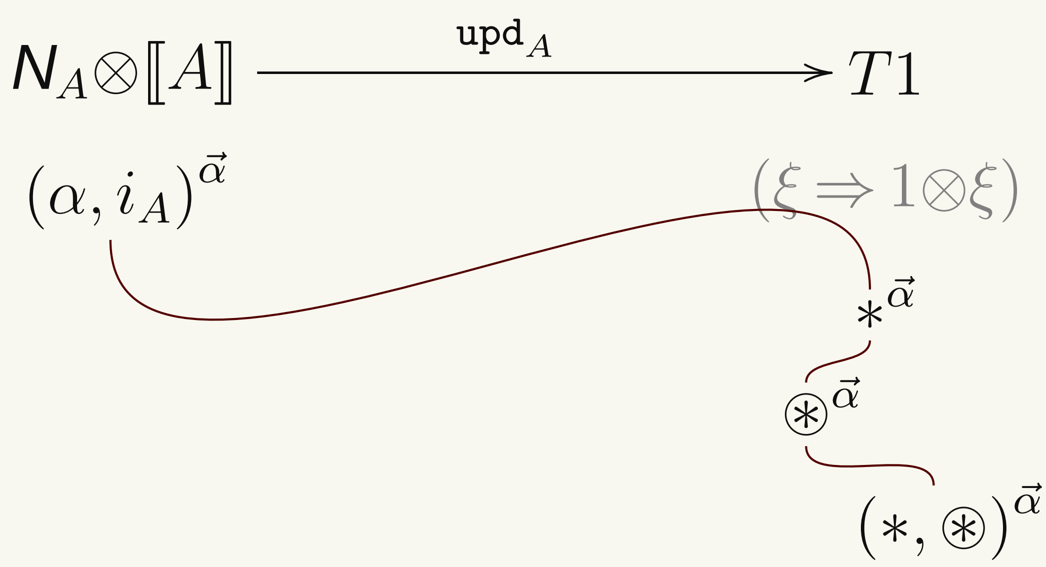

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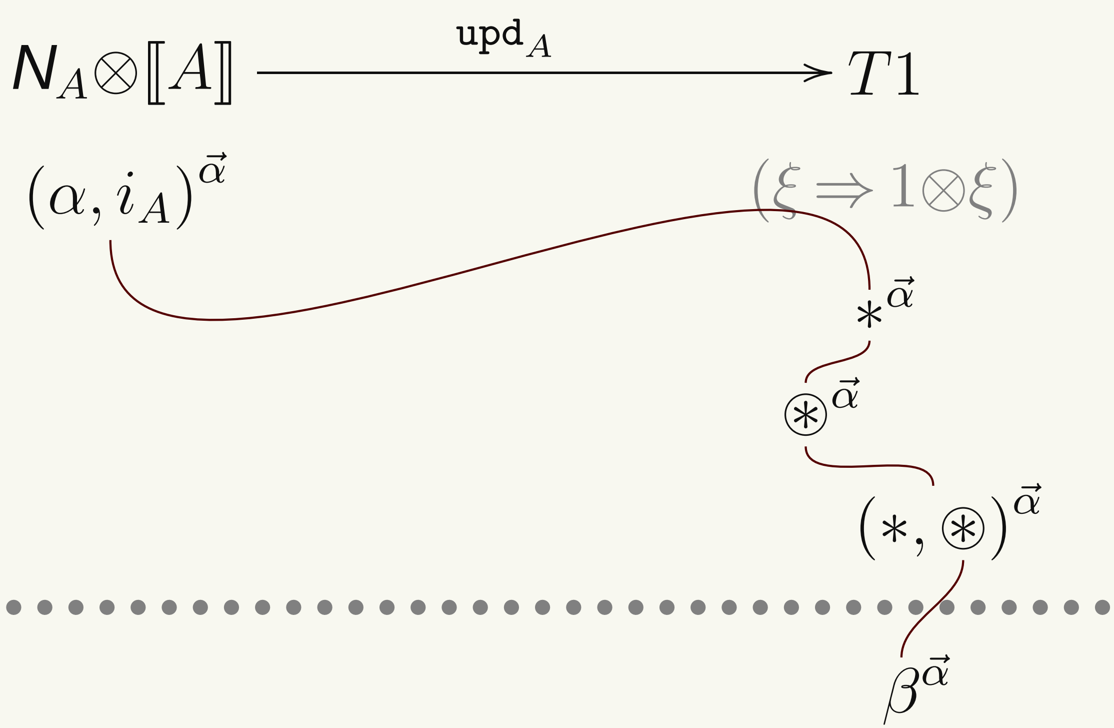

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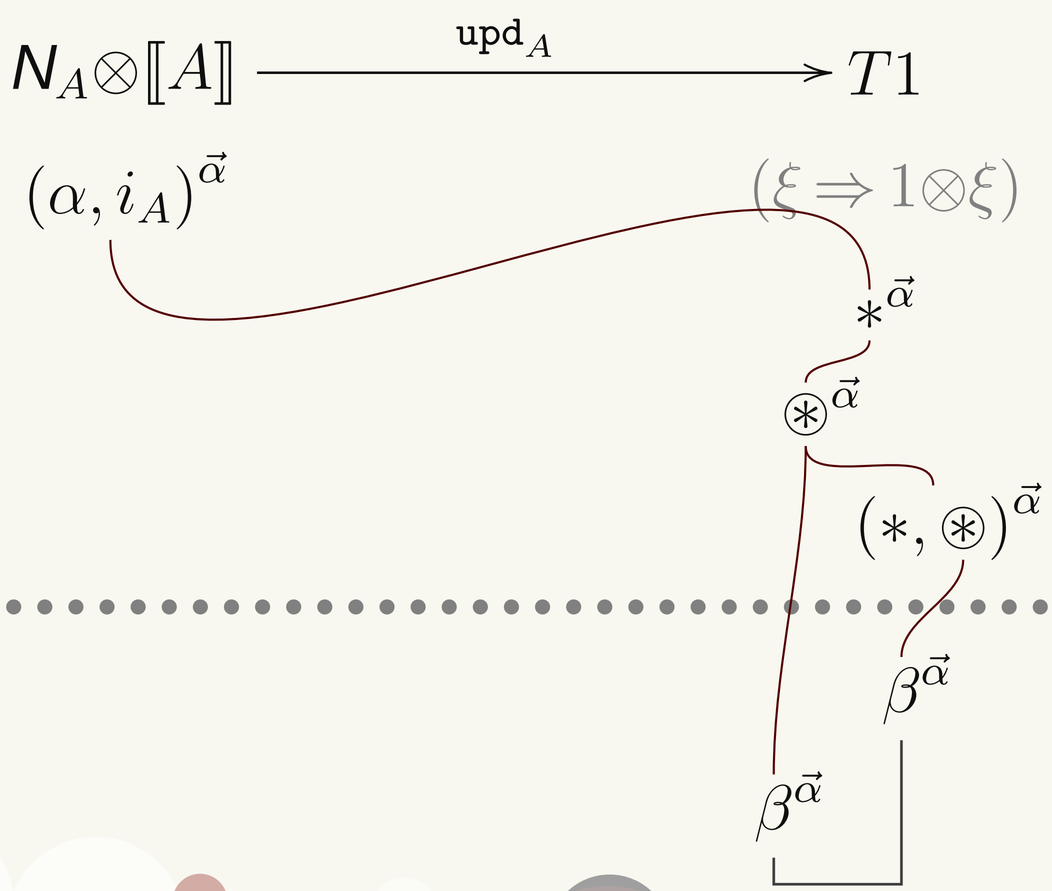

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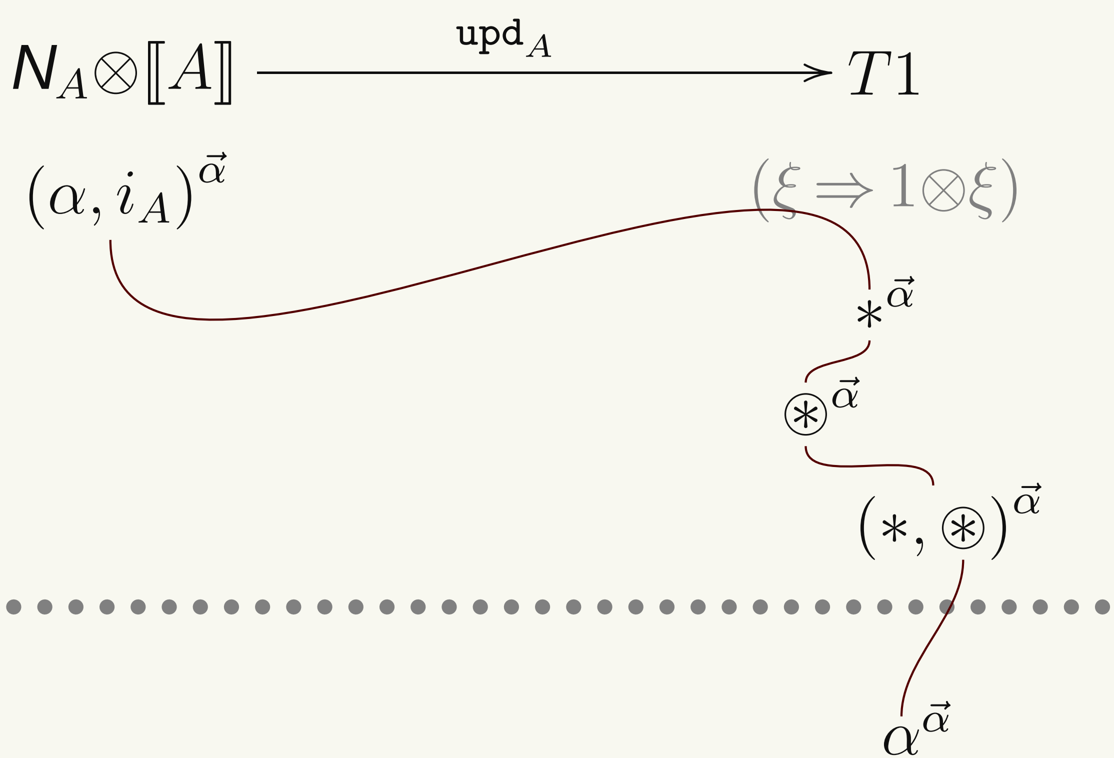

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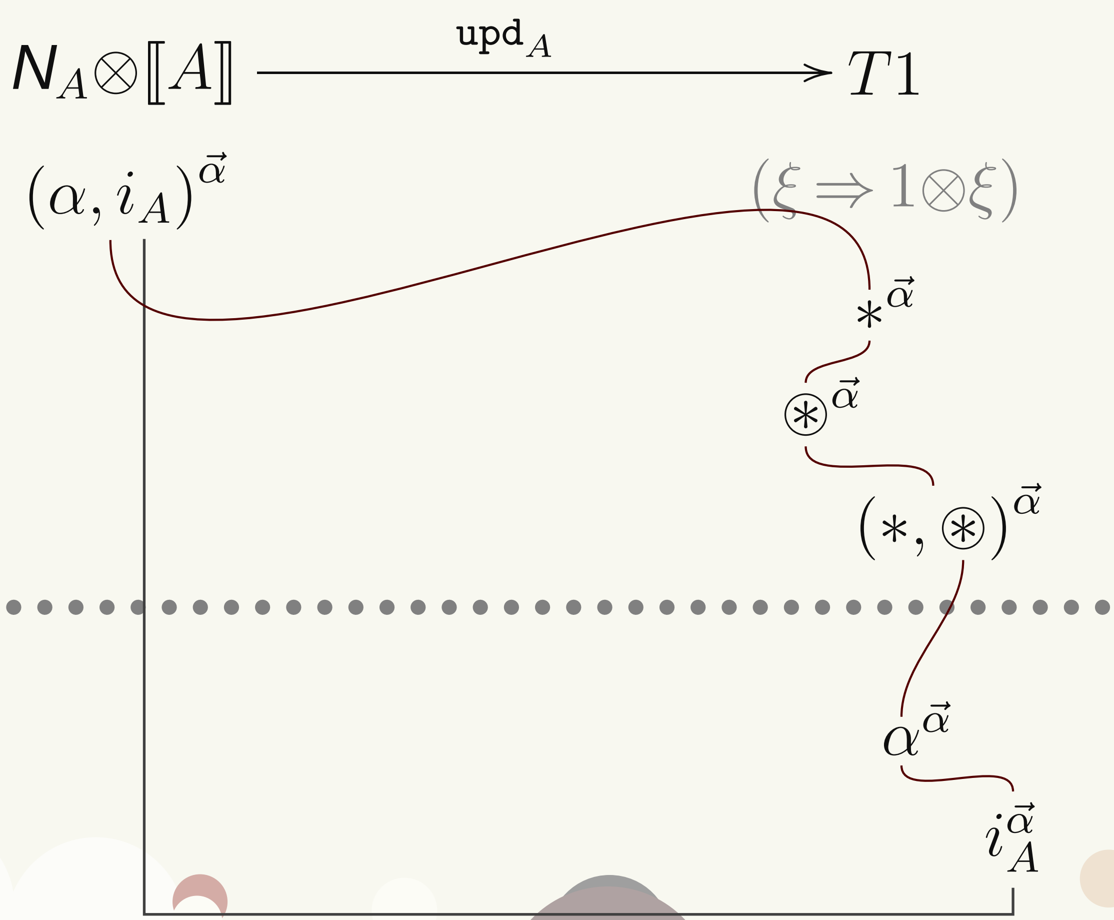

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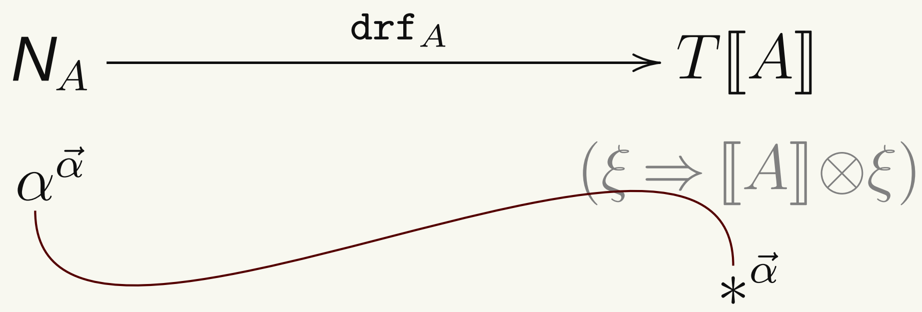

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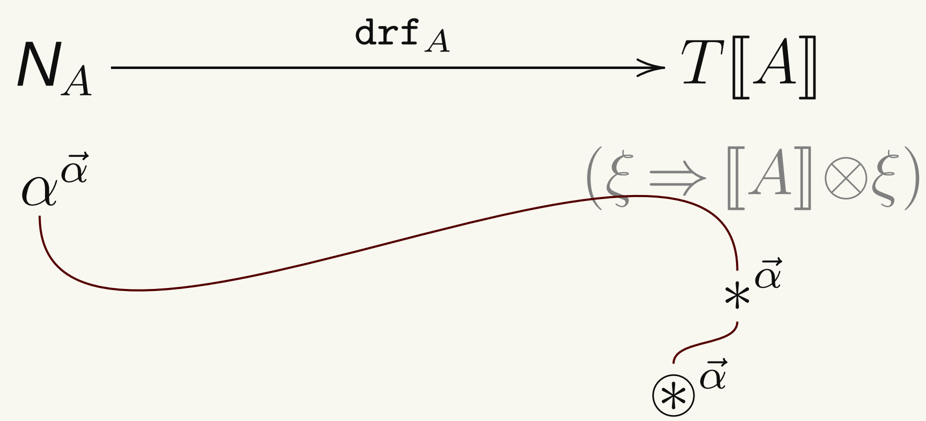

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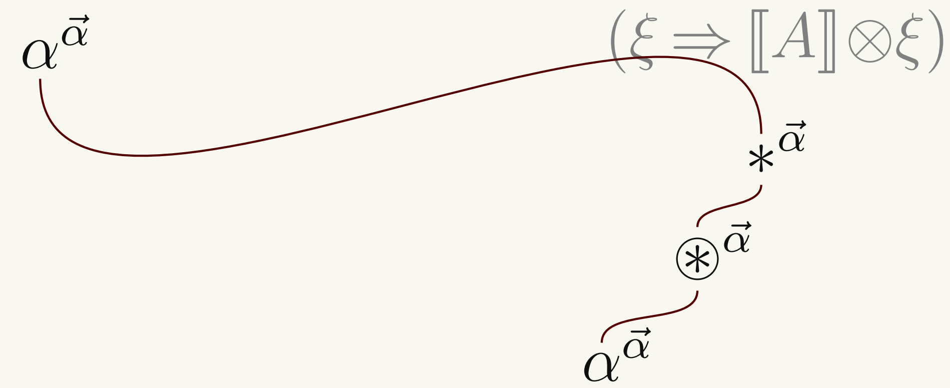

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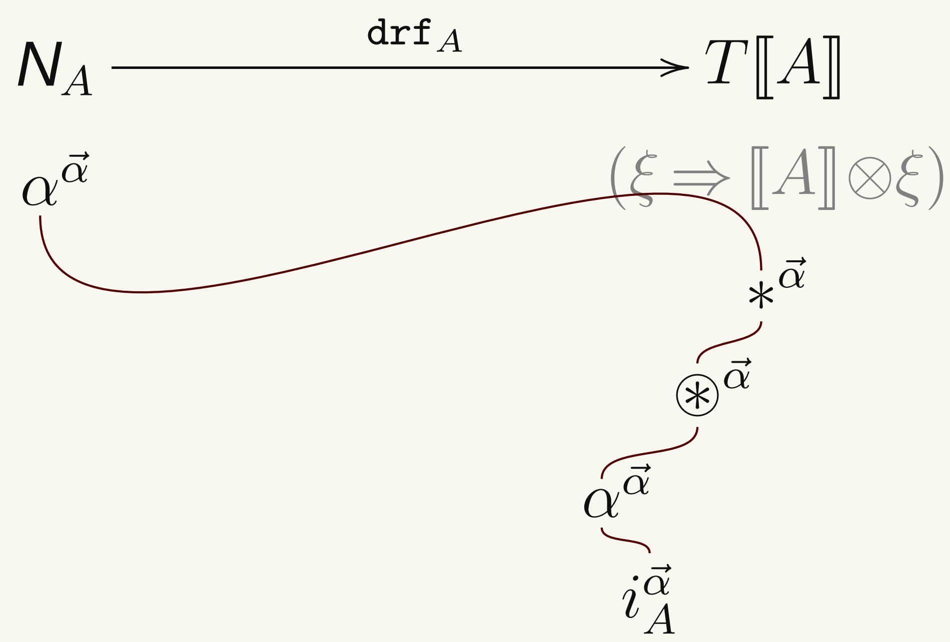

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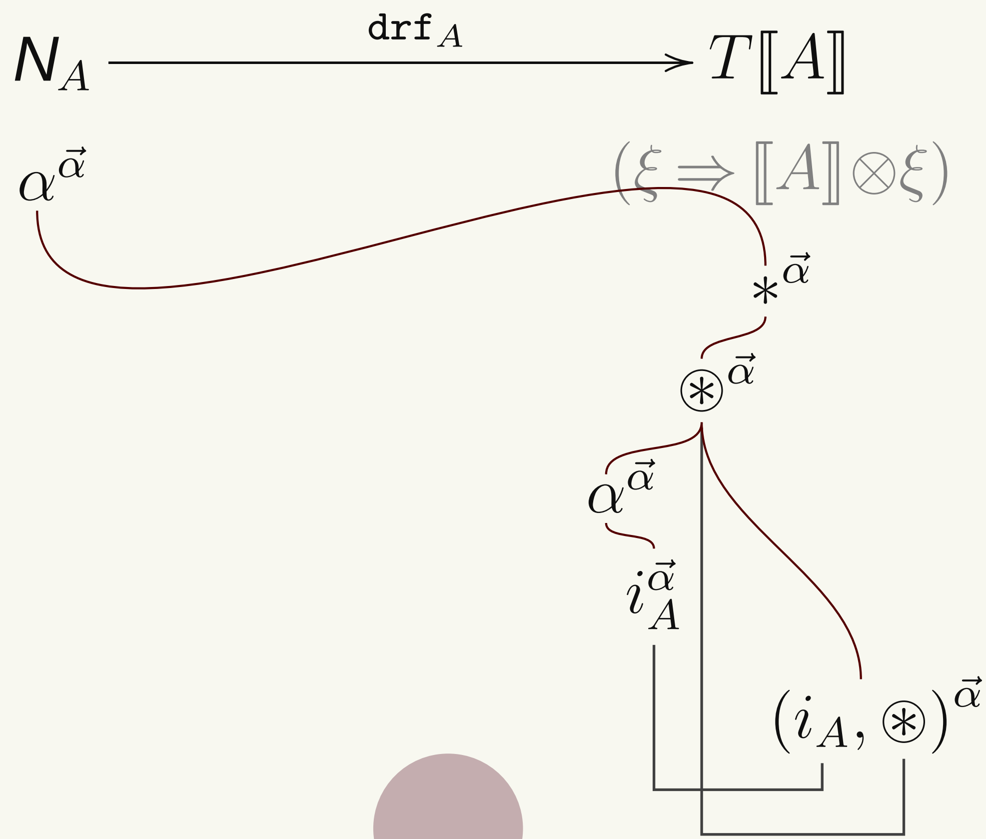

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  When Opponent asks the value of some name, Player is free to evade answering and play elsewhere.

Tidy strategies

We therefore restrict strategies by imposing *tidiness conditions*, obtaining thus *tidy strategies*.

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$\rightsquigarrow$ either a store-A to $\alpha^{\vec{\alpha}'}$ introducing no new names,
$\rightsquigarrow$ or a copy of $\alpha^{\vec{\alpha}'}$.
In particular, if $\alpha \# \ulcorner s \urcorner^-$ then the latter case holds.

(TD2) If $[s\alpha^{\vec{\alpha}'}] \in \sigma$ with $\alpha^{\vec{\alpha}'}$ a store-Q then $\alpha^{\vec{\alpha}'}$ is justified by last O-store-H in $\ulcorner s \urcorner$.

(TD3) If $\ulcorner s \urcorner = s' \alpha^{\vec{\alpha}'}_{(O)} \alpha^{\vec{\alpha}'}_{(P)} t\, y$ with $\alpha^{\vec{\alpha}'}$ a store-Q then $[sy] \in \sigma$ with $t\, y\, y$ forming a copycat.

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(TD1) If s ends in a store-Q $\alpha^{\vec{\alpha}'}$ then $[sx] \in \sigma$, with x being:
$\rightsquigarrow$ either a store-A to $\alpha^{\vec{\alpha}'}$ introducing no new names,
$\rightsquigarrow$ or a copy of $\alpha^{\vec{\alpha}'}$.
In particular, if $\alpha \# \ulcorner s \urcorner^-$ then the latter case holds.

(TD2) If $[s\alpha^{\vec{\alpha}'}] \in \sigma$ with $\alpha^{\vec{\alpha}'}$ a store-Q then $\alpha^{\vec{\alpha}'}$ is justified by last O-store-H in $\ulcorner s \urcorner$.

(TD3) If $\ulcorner s \urcorner = s' \alpha^{\vec{\alpha}'}_{(O)} \alpha^{\vec{\alpha}'}_{(P)} t\, y$ with $\alpha^{\vec{\alpha}'}$ a store-Q then $[sy] \in \sigma$ with $t\, y\, y$ forming a copycat.

Let $\mathcal{T}^{\vec{\alpha}}$ be the subcategory of $\mathcal{V}_{\mathrm{t}}^{\vec{\alpha}}$ with objects $[\![A]\!]$ and arrows tidy strategies

$\mathcal{T}$ is a FA model

All relevant structure passes from $\mathcal{V}_{\mathtt{t}}^{\vec{\alpha}}$ to $\mathcal{T}^{\vec{\alpha}}$.
Hence, $\mathcal{T} = \langle \mathcal{T}^{\vec{\alpha}}, T^{\vec{\alpha}} \rangle_{\vec{\alpha} \in \mathsf{N}^{\#}}$ is a sound model.

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We call an $\vec{\alpha}$-strategy $\sigma : 1 \longrightarrow T\mathbb{N}$ *observable* iff, for some $\vec{\beta}$,

$$[*^{\vec{\alpha}} *^{\vec{\alpha}} \circledast^{\vec{\alpha}}(0, \circledast)^{\vec{\alpha}\vec{\beta}}] \in \sigma$$

and define the *intrinsic preorder* $\lesssim^{\vec{\alpha}} \subseteq \mathcal{T}^{\vec{\alpha}}(A, TB)$ around it.

$$[\![M]\!] \lesssim [\![N]\!] \iff M \lessapprox N$$

References

[AGM+04] Samson Abramsky, Dan Ghica, Andrzej Murawski, Luke Ong, and Ian Stark. Nominal games and full abstraction for the nu-calculus. In *Proceedings of LICS '04*, 2004.

[GP99] M. J. Gabbay and A. M. Pitts. A new approach to abstract syntax involving binders. In *Proceedings of LICS '99*, 1999.

[HY99] Kohei Honda and Nobuko Yoshida. Game-theoretic analysis of call-by-value computation. *TCS*, 221(1–2):393–456, 1999.

[McC00] Guy McCusker. Games and full abstraction for FPC. *Information and Computation*, 160(1-2):1–61, 2000.

[Ong02] Luke Ong. Observational equivalence of third-order idealized algol is decidable. In *Proceedings of LICS '02*, 2002.

[Pit03] A. M. Pitts. Nominal logic, a first order theory of names and binding. *Information and Computation*, 186:165–193, 2003.

[PS93] A. M. Pitts and I. D. B. Stark. Observable properties of higher order functions that dynamically create local names, or: What's new? In *Proc. 18th MFCS*, 1993. LNCS 711.

[SP82] M. B. Smyth and G. D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal on Computing*, 11(4):761–783, 1982.

[Tze07] Nikos Tzevelekos. Full abstraction for nominal general references. 2007. To be presented in LICS '07.