Discrete Convexity in Joint Winner Property

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Abstract

In this paper, we reveal a relation between joint winner property (JWP) in the field of valued constraint satisfaction problems (VCSPs) and M$^\natural$-convexity in the field of discrete convex analysis (DCA). We introduce the M$^\natural$-convex completion problem, and show that a function f satisfying the JWP is Z-free if and only if a certain function $\overline{f}$ associated with f is M$^\natural$-convex completable. This means that if a function is Z-free, then the function can be minimized in polynomial time via M$^\natural$-convex intersection algorithms. Furthermore we propose a new algorithm for Z-free function minimization, which is faster than previous algorithms for some parameter values.

Keywords: valued constraint satisfaction problems, discrete convex analysis, M-convexity

1 Introduction

A *valued constraint satisfaction problem (VCSP)* is a general framework for discrete optimization (see [19] for details). Informally, the VCSP framework deals with the minimization problem of a function represented as the sum of "small" arity functions. It is known that various kinds of combinatorial optimization problems can be formulated in the VCSP framework. In general, the VCSP is NP-hard. An important line of research is to investigate which classes of instances are solvable in polynomial time, and why these classes ensure polynomial time solvability. Cooper–Živný [2] showed that if a function represented as the sum of unary or binary functions satisfies the *joint winner property (JWP)*, then the function can be minimized in polynomial time. This gives an example of a class of instances that are solvable in polynomial time.

In this paper, we present the reason why JWP ensures polynomial time solvability via *discrete convex analysis (DCA)* [10], particularly, *M$^\natural$-convexity* [13]. DCA is a theory of convex functions on discrete structures, and M$^\natural$-convexity is one of the important convexity concepts in DCA. M$^\natural$-convexity appears in many areas such as operations research, economics, and game theory (see e.g., [10, 11, 12]).

The results of this paper are summarized as follows:

- We reveal a relation between JWP and M$^\natural$-convexity. That is, we give a DCA interpretation of polynomial-time solvability of JWP.

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- To describe the connection of JWP and M$^\natural$-convexity, we introduce the *M$^\natural$-convex completion problem*, and give a characterization of M$^\natural$-convex completability.
- By utilizing a DCA interpretation of JWP, we propose a new algorithm for Z-free function minimization, which is faster than previous algorithms for some parameter values.

This study will hopefully be the first step towards fruitful interactions between VCSPs and DCA.

Notations. Let $\mathbf{R}$ and $\mathbf{R}_+$ denote the sets of reals and nonnegative reals, respectively. In this paper, functions can take the infinite value $+\infty$, where $a < +\infty$, $a + \infty = +\infty$ for $a \in \mathbf{R}$, and $0 \cdot (+\infty) = 0$. Let $\overline{\mathbf{R}} := \mathbf{R} \cup \{+\infty\}$ and $\overline{\mathbf{R}}_+ := \mathbf{R}_+ \cup \{+\infty\}$. For a function $f : \{0,1\}^n \to \overline{\mathbf{R}}$, the effective domain is denoted as $\mathrm{dom}\, f := \{x \in \{0,1\}^n \mid f(x) < +\infty\}$. For a positive integer k, we define $[k] := \{1, 2, \ldots, k\}$. For $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, we define $\mathrm{supp}^+(x) := \{i \in [n] \mid x_i > 0\}$.

2 Preliminaries

Joint Winner Property. Let $d_i \geq 2$ be a positive integer and $D_i := [d_i]$ for $i \in [r]$. We consider a function $f : D_1 \times D_2 \times \cdots \times D_r \to \overline{\mathbf{R}}_+$ represented as the sum of unary or binary functions as

$$f(x_1, x_2, \ldots, x_r) = \sum_{i \in [r]} c_i(x_i) + \sum_{1 \leq i < j \leq r} c_{ij}(x_i, x_j), \tag{1}$$

where $c_i : D_i \to \mathbf{R}_+$ is a unary function for $i \in [r]$ and $c_{ij} : D_i \times D_j \to \overline{\mathbf{R}}_+$ is a binary function for $1 \leq i < j \leq r$. Furthermore we assume $c_{ij} = c_{ji}$ for distinct $i, j \in [r]$. A function f of the form (1) is said [2] to satisfy the *joint winner property (JWP)* if it holds that

$$c_{ij}(a, b) \geq \min\{c_{jk}(b, c), c_{ik}(a, c)\} \tag{2}$$

for all distinct $i, j, k \in [r]$ and all $a \in D_i, b \in D_j, c \in D_k$. A function f of the form (1) satisfying the JWP is said to be *Z-free* if it satisfies that

$$|\operatorname{argmin}\{c_{ij}(a, c), c_{ij}(a, d), c_{ij}(b, c), c_{ij}(b, d)\}| \geq 2 \tag{3}$$

for any $i, j \in [r]$ $(i \neq j)$, $\{a, b\} \subseteq D_i$ $(a \neq b)$, and $\{c, d\} \subseteq D_j$ $(c \neq d)$.

Cooper–Živný [2] showed that if f of the form (1) satisfies the JWP, then f can be minimized in polynomial time. In fact, they showed that if f satisfies the JWP, then f can be transformed into a certain Z-free function f' in polynomial time such that a minimizer of f' is also a minimizer of f. Moreover they showed that a Z-free function can be minimized in polynomial time.

JWP appears in many contexts. For example, JWP identifies a tractable class of the MAX-2SAT problem, which is a well-known NP-hard problem [6]. Indeed, for a 2-CNF formula ψ, we can represent the MAX-2SAT problem for ψ as a Boolean binary $\{0,1\}$-valued VCSP instance. In this binary VCSP instance, JWP is equivalent to the following condition on ψ: if clauses $(x_1 \vee x_2)$ and $(x_1 \vee x_3)$ are contained in ψ, then so is $(x_2 \vee x_3)$. In addition, the AllDifferent constraint [15] and SoftAllDiff constraint [14] can be regarded as special cases of the JWP, and certain scheduling problems introduced in [1, 8] satisfy the JWP. See also Examples 5–8 in [2] for details.

M$^\natural$-Convexity. A function $f : \{0,1\}^n \to \overline{\mathbf{R}}$ is said [10, 11] to be *M$^\natural$-convex* if for all $x, y \in \{0,1\}^n$ and all $i \in \mathrm{supp}^+(x-y)$ there exists $j \in \mathrm{supp}^+(y-x) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j), \tag{4}$$

where χ_i is the ith unit vector and χ_0 is the zero vector. A function $f : \{0,1\}^n \to \overline{\mathbf{R}}$ is said [10] to be *M$_2^\natural$-convex* if f can be represented as the sum of two M$^\natural$-convex functions. It is well known that M$^\natural$-convex functions can be minimized in polynomial time. Furthermore if we are given two M$^\natural$-convex functions g and h, we can minimize an M$_2^\natural$-convex function $f = g + h$ in polynomial time by solving the so-called "M$^\natural$-convex intersection problem."

Theorem 1 ([16, Theorem 10][18, Theorem 6.5]; see also [12, Theorem 3.3])**.** *A function $f : \{0,1\}^n \to \overline{\mathbf{R}}$ with the zero vector in* dom f *is M$^\natural$-convex if and only if f satisfies the following two conditions:*

Condition 1: *For all distinct $i, j, k \in [n]$ and all $z \in \{0,1\}^n$ with* $\mathrm{supp}^+(z) \subseteq [n] \setminus \{i, j, k\}$*, it holds that*

$$f(z + \chi_i + \chi_j) + f(z + \chi_k) \geq \min\{f(z + \chi_j + \chi_k) + f(z + \chi_i), f(z + \chi_i + \chi_k) + f(z + \chi_j)\}. \tag{5}$$

Condition 2: *For all distinct $i, j \in [n]$ and all $z \in \{0,1\}^n$ with* $\mathrm{supp}^+(z) \subseteq [n] \setminus \{i, j\}$*, it holds that*

$$f(z + \chi_i + \chi_j) + f(z) \geq f(z + \chi_i) + f(z + \chi_j).$$

We pay special attention to quadratic M$^\natural$-convex functions. Using Theorem 1, we provide a necessary and sufficient condition for the M$^\natural$-convexity of a function $f : \{0,1\}^n \to \overline{\mathbf{R}}$ of the form

$$f(x_1, x_2, \ldots, x_n) := \sum_{i \in [n]} h_i x_i + \sum_{1 \leq i < j \leq n} h_{ij} x_i x_j \qquad ((x_1, x_2, \ldots, x_n) \in \{0,1\}^n), \tag{6}$$

where we assume $h_{ij} = h_{ji}$ and $h_i < +\infty$ for $i, j \in [n]$.

Lemma 2. *A function f of the form* (6) *is M$^\natural$-convex if and only if it satisfies the following:*

- $h_{ij} \geq \min\{h_{ik}, h_{jk}\}$ $\quad$ *(i, j, k : distinct).*
- $h_{ij} \geq 0$ $\quad$ *(i, j : distinct).*

In Lemma 2, h_{ij} can take the infinite value $+\infty$, whereas all h_{ij}'s are assumed to be finite in the characterization in [7] and [11]. In particular, we refer to the first condition $h_{ij} \geq \min\{h_{ik}, h_{jk}\}$ (i, j, k : distinct) as the *anti-ultrametric property.* Note that no conditions are imposed on h_i. The proof of Lemma 2 is in Section 5

By Lemma 2, we know that M$^\natural$-convexity of a function of the form (6) depends only on quadratic coefficients $(h_{ij})_{i,j \in [n]}$. We say that a function f of the form (6) is defined by $(h_{ij})_{i,j \in [n]}$ if the quadratic coefficients of f is equal to $(h_{ij})_{i,j \in [n]}$.

3 M$^\natural$-Convexity in Joint Winner Property

M$^\natural$-Convex Completion Problem. We introduce the *M$^\natural$-convex completion problem*, and give a characterization of an M$^\natural$-convex completable function on $\{0,1\}^n$ defined by $(h_{ij})_{i,j \in [n]}$. The M$^\natural$-convex completion problem is the following:

Given: $(h_{ij})_{i,j\in[n]}$ such that $h_{ij} \in \overline{\mathbf{R}}$ or h_{ij} is undefined for every distinct $i, j \in [n]$.

Question: By assigning appropriate values in $\overline{\mathbf{R}}$ to "undefined" elements of $(h_{ij})_{i,j\in[n]}$, can we construct an M$^\natural$-convex function $f : \{0,1\}^n \to \overline{\mathbf{R}}$ of the form (6)?

It should be clear that a defined element can be equal to $+\infty$ and the infinite value $(+\infty)$ may be assigned to undefined elements. If there is an appropriate assignment of $(h_{ij})_{i,j\in[n]}$, then $(h_{ij})_{i,j\in[n]}$ is said to be *M$^\natural$-convex completable*. If $h_{ij} < 0$ or $h_{ij} < \min\{h_{jk}, h_{ik}\}$ holds for some defined elements h_{ij}, h_{jk}, h_{ik}, then we obviously know that $(h_{ij})_{i,j\in[n]}$ is not M$^\natural$-convex completable. Hence in considering the M$^\natural$-convex completion problem, we assume that

$$h_{ij} \geq 0, \tag{7}$$

$$h_{ij} \geq \min\{h_{jk}, h_{ik}\} \tag{8}$$

for all defined elements h_{ij}, h_{jk}, h_{ik}.

For quadratic coefficients $H := (h_{ij})_{i,j\in[n]}$ containing undefined elements, we define the *assignment graph* of H as a graph $G_H = ([n], E_H; w)$, where $E_H := \{\{i,j\} \mid i \neq j \text{ and } h_{ij} \text{ is defined}\}$ and $w : E_H \to \overline{\mathbf{R}}_+$ is defined by $w(\{i,j\}) := h_{ij}$ for $\{i,j\} \in E_H$. Then the following theorem holds.

Theorem 3. *$H := (h_{ij})_{i,j\in[n]}$ is M$^\natural$-convex completable if and only if $|\operatorname{argmin}_{e\in C} w(e)| \geq 2$ holds for every chordless cycle C of G_H.*

The proof of Theorem 3 is in Section 5.

Remark 4. Farach–Kannan–Warnow [4] introduced the *matrix sandwich problem for ultrametric property*, which contains the M$^\natural$-convex completion problem as a special case. They also constructed an $O(m + n \log n)$-time algorithm for the matrix sandwich problem for ultrametric property, where m is the number of defined elements. In our setting, $m = O(n^2)$. Hence, by using this algorithm, we can obtain an appropriate M$^\natural$-convex completion in $O(n^2)$ time if one exists. An $O(n^2)$-time algorithm based on Farach–Kannan–Warnow's algorithm is the following: Suppose that all h_{ij} are finite (if there exists h_{ij} with $h_{ij} = +\infty$, then we can redefine the value of h_{ij} as a sufficiently large finite value M). Take any maximum forest F of G_H. Let $\alpha_1 > \alpha_2 > \cdots > \alpha_p$ be the distinct values of defined elements of $(h_{ij})_{\{i,j\}\in F}$. For $k = 1, \ldots, p-1$, let F^{α_k} be the subgraph of F induced by the edges with weight at least α_k, i.e., $F^{\alpha_k} := \{\{i,j\} \in F \mid h_{ij} \geq \alpha_k\}$. Then, for each $\{i,j\} \notin E_H$ with i,j connected in G_H, set h_{ij} to α_k, where k is the minimum number such that i,j is connected in F^{α_k}. For each $\{i,j\} \notin E_H$ with i,j disconnected in G_H, set h_{ij} to α_p.

In this paper, we present a graphic characterization of M$^\natural$-convex completability. With this characterization, we provide a DCA interpretation of polynomial-time solvability of JWP.

Transformation into a Function over $\{0,1\}$. To connect JWP and M$^\natural$-convexity, we introduce a transformation of a function $f : D_1 \times D_2 \times \cdots \times D_r \to \overline{\mathbf{R}}$ into a function $\hat{f} : \{0,1\}^U \to \overline{\mathbf{R}}$, where U is the set of all assignments to variables, that is,

$$U := \{(1,1), (1,2), \ldots, (1,d_1), (2,1), (2,2), \ldots, (2,d_2), \ldots, (r,1), (r,2), \ldots, (r,d_r)\}.$$

We consider the following correspondence between $x = (x_1, x_2, \ldots, x_r) \in D_1 \times D_2 \times \cdots \times D_r$ and $\hat{x} = (\hat{x}_{(1,1)}, \ldots, \hat{x}_{(1,d_1)}, \hat{x}_{(2,1)}, \ldots, \hat{x}_{(2,d_2)}, \ldots, \hat{x}_{(r,1)}, \ldots, \hat{x}_{(r,d_r)}) \in \{0,1\}^U$:

$$(x_1, x_2, \ldots, x_r) \mapsto (\underbrace{0, \ldots, 0, \overset{(1,x_1)}{\check{1}}, 0, \ldots, 0}_{d_1}, \underbrace{0, \ldots, 0, \overset{(2,x_2)}{\check{1}}, 0, \ldots, 0}_{d_2}, \ldots, \underbrace{0, \ldots, 0, \overset{(r,x_r)}{\check{1}}, 0, \ldots, 0}_{d_r}). \tag{9}$$

That is, $\hat{x}_{(i,a)} = 1$ means that we assign a to x_i, and $\hat{x}_{(i,a)} = 0$ means that we do not. In view of (9), define a function $\hat{f}$ by

$$\hat{f}(\hat{x}) := \begin{cases} f(x) & \text{if there exists } x \text{ satisfying (9)}, \\ +\infty & \text{otherwise} \end{cases} \qquad (\hat{x} \in \{0,1\}^U).$$

Note that minimizing f is equivalent to minimizing $\hat{f}$.

Now we consider the transformation of f of the form (1) into $\hat{f}$, where f is given in terms of c_i for $i \in [r]$ and c_{ij} for $i, j \in [r]$. We define $\overline{f} : \{0,1\}^U \to \overline{\mathbf{R}}_+$ by

$$\overline{f}(\hat{x}) := \sum_{(i,a)\in U} c_i(a)\hat{x}_{(i,a)} + \sum_{(i,a),(j,b)\in U,\ (i,a)\neq(j,b)} h_{(i,a),(j,b)}\hat{x}_{(i,a)}\hat{x}_{(j,b)} \qquad (\hat{x} \in \{0,1\}^U), \tag{10}$$

where

$$h_{(i,a),(j,b)} := \begin{cases} c_{ij}(a,b) & \text{if } i \neq j, \\ \text{undefined} & \text{if } i = j. \end{cases} \tag{11}$$

We also define $\delta_U : \{0,1\}^U \to \overline{\mathbf{R}}$ by

$$\delta_U(\hat{x}) := \begin{cases} 0 & \text{if there exists } x \text{ satisfying (9)}, \\ +\infty & \text{otherwise} \end{cases} \qquad (\hat{x} \in \{0,1\}^U),$$

which is the indicator function for the feasible assignments. Then we have

$$\hat{f}(\hat{x}) = \overline{f}(\hat{x}) + \delta_U(\hat{x}) \qquad (\hat{x} \in \{0,1\}^U),$$

where arbitrary values in $\overline{\mathbf{R}}$ may be assigned to the undefined elements $h_{(i,a),(i,b)}$ in $\overline{f}$ without affecting the value of $\hat{f}$. Indeed, if $\hat{x} \in \mathrm{dom}\ \delta_U$, then $\hat{x}_{(i,a)}\hat{x}_{(i,b)} = 0$ for all $i \in [r]$ and all distinct $a, b \in D_i$. Hence $h_{(i,a),(i,b)}\hat{x}_{(i,a)}\hat{x}_{(i,b)} = 0$ holds for each undefined element $h_{(i,a),(i,b)}$ by the definition (11). In particular, the set of minimizers of both $\hat{f}(\hat{x})$ and $\overline{f}(\hat{x}) + \delta_U(\hat{x})$ are the same.

It is clear that δ_U is M$^\natural$-convex (dom δ_U is the base family of a partition matroid, which is a direct sum of matroids of rank 1). Hence if $(h_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$ has an M$^\natural$-convex completion $(\tilde{h}_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$, then $\overline{f}$ defined by $(\tilde{h}_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$ is M$^\natural$-convex and $\hat{f} = \overline{f} + \delta_U$ is M$_2^\natural$-convex. This means that $\hat{f}$ can be minimized in polynomial time. We need the values of $(\tilde{h}_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$ in a minimization algorithm of M$_2^\natural$-convex functions.

A function of the form (1) satisfies the JWP if and only if $h_{(i,a),(j,b)} \geq \min\{h_{(j,b),(k,c)}, h_{(i,a),(k,c)}\}$ holds for defined elements $h_{(i,a),(j,b)}, h_{(j,b),(k,c)}, h_{(i,a),(k,c)}$ given in (10). Hence $(h_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$ satisfies the assumptions (7) and (8) for the M$^\natural$-convex completion problem. Theorem 3 implies the following theorem (the proof is in Section 5).

Theorem 5. *For a function f of the form* (1)*, let $(h_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$ be defined by* (11)*. Then $(h_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$ is M$^\natural$-convex completable if and only if f (has the JWP and) is Z-free.*

4 Algorithm

By using a general algorithm for the M$^\natural$-convex intersection (minimization of M$_2^\natural$-convex functions), we can minimize Z-free functions of the form (1) in polynomial time. Suppose that we are given $c_i : D_i \to \mathbf{R}_+$ for $i \in [r]$, $c_{ij} : D_i \times D_j \to \overline{\mathbf{R}}_+$ for $1 \leq i < j \leq r$, and a Z-free function

f defined as (1). We can minimize f by minimizing $\hat{f} = \overline{f} + \delta_U$ with an M$^\natural$-convex intersection algorithm.

Here we take advantage of the fact that all the vectors in dom δ_U have a constant component sum, i.e., $\sum_{(i,a)\in U} \hat{x}_{(i,a)} = r$ for all $\hat{x} \in$ dom δ_U. This implies that δ_U is an M-convex function [10] and we can use an *M-convex intersection* algorithm. An M-convex intersection algorithm is easier to describe than an M$^\natural$-convex intersection algorithm, though the time complexity is the same. Therefore we devise a minimization algorithm for Z-free functions via an M-convex intersection algorithm. Since the functions are defined on $\{0,1\}^n$, the proposed algorithm is actually a variant of valuated matroid intersection algorithms [9]. Specifically, let $\overline{f}|_r$ denote the restriction of $\overline{f}$ to the hyperplane containing dom δ_U, i.e.,

$$\overline{f}|_r(\hat{x}) := \begin{cases} \overline{f}(\hat{x}) & \text{if } \displaystyle\sum_{(i,a)\in U} \hat{x}_{(i,a)} = r, \\ +\infty & \text{otherwise.} \end{cases}$$

Then minimizing $\overline{f} + \delta_U$ is equivalent to minimizing $\overline{f}|_r + \delta_U$, where $\overline{f}|_r$ and δ_U are M-convex functions.

The proposed algorithm consists of three steps.

Step 1: On the basis of Theorem 5, we construct an M$^\natural$-convex function $\overline{f} : \{0,1\}^U \to \overline{\mathbf{R}}$ in (10) through an M$^\natural$-convex completion of $(h_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$ in (11).

Step 2: We find a minimizer of $\overline{f}|_r$, to be used as an initial solution in Step 3.

Step 3: We find a minimizer of $\overline{f}|_r + \delta_U$ by the successive shortest path algorithm with potentials for the M-convex intersection [10] (see also [9, Section 5.2]).

In Step 3 of the algorithm, we use the *auxiliary graph* $G_{\hat{x},\hat{y}} = (V, E_{\hat{x},\hat{y}})$ defined for $\hat{x} \in$ dom $\overline{f}|_r$ and $\hat{y} \in$ dom δ_U by

$$V := \{s,t\} \cup U, \tag{12}$$

$$E_{\hat{x}} := \{((i,a),(j,b)) \mid (i,a),(j,b) \in U,\ \hat{x} + \chi_{(j,b)} - \chi_{(i,a)} \in \mathrm{dom}\ \overline{f}|_r\}, \tag{13}$$

$$E_{\hat{y}} := \{((i,a),(j,b)) \mid (i,a),(j,b) \in U,\ \hat{y} + \chi_{(i,a)} - \chi_{(j,b)} \in \mathrm{dom}\ \delta_U\}, \tag{14}$$

$$E^+ := \{(s,(i,a)) \mid (i,a) \in \mathrm{supp}^+(\hat{x} - \hat{y})\}, \tag{15}$$

$$E^- := \{((j,b),t) \mid (j,b) \in \mathrm{supp}^+(\hat{y} - \hat{x})\}, \tag{16}$$

$$E_{\hat{x},\hat{y}} := E_{\hat{x}} \cup E_{\hat{y}} \cup E^+ \cup E^- \tag{17}$$

with the arc length function $\ell = \ell_{\hat{x},\hat{y}} : E_{\hat{x},\hat{y}} \to \mathbf{R}$ given by

$$\ell(u,v) := \begin{cases} \overline{f}|_r(\hat{x} + \chi_v - \chi_u) - \overline{f}|_r(\hat{x}) & \text{if } (u,v) \in E_{\hat{x}}, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Note that, by the definition of δ_U, we can also describe $E_{\hat{y}}$ as $E_{\hat{y}} = \{((i,a),(i,b)) \mid i \in [r],\ a,b \in D_i,\ (i,a) \notin \mathrm{supp}^+(\hat{y}),\ (i,b) \in \mathrm{supp}^+(\hat{y})\}$.

Algorithm for Z-free function minimization:

Step 1: Find an M$^\natural$-convex completion $(\tilde{h}_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$ of $(h_{(i,a),(j,b)})_{(i,a),(j,b)\in U}$, and define $\overline{f} : \{0,1\}^U \to \overline{\mathbf{R}}$ by

$$\overline{f}(\hat{x}) = \sum_{(i,a)\in U} c_i(a)\hat{x}_{(i,a)} + \sum_{(i,a),(j,b)\in U,\ (i,a)\neq(j,b)} \tilde{h}_{(i,a),(j,b)}\hat{x}_{(i,a)}\hat{x}_{(j,b)} \qquad (\hat{x} \in \{0,1\}^U).$$

Step 2: Let $\hat{x}^* \in \{0,1\}^U$ be the zero vector. While $\sum_{(i,a)\in U} \hat{x}^*_{(i,a)} < r$, do the following:

Step 2-1: Obtain $(i,a)^* \in \operatorname{argmin}\{\overline{f}(\hat{x}^* + \chi_{(i,a)}) \mid (i,a) \in U \setminus \operatorname{supp}^+(\hat{x}^*)\}$.

Step 2-2: $\hat{x}^* \leftarrow \hat{x}^* + \chi_{(i,a)^*}$.

Step 3: Let $p : V \to \mathbf{R}$ be a potential defined by $p(v) := 0$ for $v \in \{s,t\} \cup U$. Take any $\hat{y}^* \in \operatorname{dom} \delta_U$. While $\hat{x}^* \neq \hat{y}^*$, do the following:

Step 3-1: Make the auxiliary graph $G_{\hat{x}^*,\hat{y}^*}$. Define the modified arc length $\ell_p : E_{\hat{x}^*,\hat{y}^*} \to \mathbf{R}$ by $\ell_p(u,v) := \ell(u,v) + p(u) - p(v)$ for $(u,v) \in E_{\hat{x}^*,\hat{y}^*}$.

Step 3-2: For each $v \in V$, compute the length $\Delta p(v)$ of an s-v shortest path in $G_{\hat{x}^*,\hat{y}^*}$ with respect to the modified arc length ℓ_p. Let P be an s-t shortest path having the smallest number of arcs in $G_{\hat{x}^*,\hat{y}^*}$ with respect to the modified arc length ℓ_p.

Step 3-3: For $(i,a) \in U$,

$$\hat{x}^*_{(i,a)} \leftarrow \begin{cases} \hat{x}^*_{(i,a)} - 1 & \text{if } ((i,a),(j,b)) \in P \cap E_{\hat{x}^*}, \\ \hat{x}^*_{(i,a)} + 1 & \text{if } ((j,b),(i,a)) \in P \cap E_{\hat{x}^*}, \\ \hat{x}^*_{(i,a)} & \text{otherwise,} \end{cases}$$

$$\hat{y}^*_{(i,a)} \leftarrow \begin{cases} \hat{y}^*_{(i,a)} + 1 & \text{if } ((i,a),(j,b)) \in P \cap E_{\hat{y}^*}, \\ \hat{y}^*_{(i,a)} - 1 & \text{if } ((j,b),(i,a)) \in P \cap E_{\hat{y}^*}, \\ \hat{y}^*_{(i,a)} & \text{otherwise.} \end{cases}$$

For $v \in V$, $p(v) \leftarrow p(v) + \Delta p(v)$. □

At the end of Step 2, we obtain a minimizer of $\overline{f}|_r$. The validity of Step 2 is given in [13, Theorem 3.2]. The time complexity of this algorithm is as follows, where $n := |U| = \sum_{i\in[r]} d_i$ (the proof is in Section 5).

Theorem 6. *The proposed algorithm runs in $O(nr^3 + nr\log n + n^2)$ time.*

By improving the algorithm of running time $O(n^3)$ given in [2], Cooper–Živný [3] gave an $O(n^2 \log n \log r)$-time algorithm for minimizing Z-free functions of the form (1). Our proposed algorithm is faster than Cooper–Živný's for some r (e.g., $r = O(n^{1/3})$).

Remark 7. In the VCSP framework, we assume that the function f of the form (1) is explicitly given. This means that the input size is proportional to

$$\sum_{i\in[r]} d_i + \sum_{i\in[r]} \sum_{d\in D_i} \log c_i(d) + \sum_{1\le i<j\le r} \sum_{d\in D_i} \sum_{e\in D_j} \log c_{ij}(d,e),$$

and then the running time in Theorem 6 is strongly polynomial in the input size. On the other hand, if we assume that f is given by the value oracles for the functions c_i and c_{ij}, the input size of f is proportional to

$$r + \sum_{i\in[r]} \log d_i + \sum_{i\in[r]} \sum_{d\in D_i} \log c_i(d) + \sum_{1\le i<j\le r} \sum_{d\in D_i} \sum_{e\in D_j} \log c_{ij}(d,e).$$

In this case, the running time in Theorem 6 is pseudo-polynomial in the input size.

5 Proofs

In this section, we give the proofs of Lemma 2, Theorem 3, Theorem 5, and Theorem 6.

Proof of Lemma 2. (only-if part). Suppose that there exist distinct $i, j, k \in [n]$ such that $h_{ij} < \min\{h_{jk}, h_{ik}\}$. Note that $h_{ij} < +\infty$ holds. Then

$$f(\chi_i + \chi_j) + f(\chi_k) = h_i + h_j + h_k + h_{ij} < h_i + h_j + h_k + h_{jk} = f(\chi_j + \chi_k) + f(\chi_i),$$
$$f(\chi_i + \chi_j) + f(\chi_k) = h_i + h_j + h_k + h_{ij} < h_i + h_j + h_k + h_{ik} = f(\chi_i + \chi_k) + f(\chi_j)$$

hold since $h_i, h_j, h_k, h_{ij} < +\infty$. By Condition 1 of Theorem 1, f is not M$^\natural$-convex.

Suppose that there exist distinct $i, j \in [n]$ such that $h_{ij} < 0 (< +\infty)$. Then

$$f(\chi_i + \chi_j) + f(\chi_0) = h_i + h_j + h_{ij} < h_i + h_j = f(\chi_i) + f(\chi_j)$$

holds since $h_i, h_j < +\infty$. By Condition 2 of Theorem 1, f is not M$^\natural$-convex.

(if part). Take arbitrary distinct $i, j, k \in [n]$ and $z \in \{0, 1\}^n$ with $\mathrm{supp}^+(z) \subseteq [n] \setminus \{i, j, k\}$. If $f(z + \chi_i + \chi_j) = +\infty$ or $f(z + \chi_k) = +\infty$ holds, then Condition 1 of Theorem 1 obviously holds. We assume $f(z + \chi_i + \chi_j) < +\infty$ and $f(z + \chi_k) < +\infty$.

It holds that

$$f(z + \chi_i + \chi_j) = f(z) + h_i + h_j + \sum_{p \in \mathrm{supp}^+(z)} h_{ip} + \sum_{p \in \mathrm{supp}^+(z)} h_{jp} + h_{ij}, \tag{19}$$

$$f(z + \chi_k) = f(z) + h_k + \sum_{p \in \mathrm{supp}^+(z)} h_{kp}. \tag{20}$$

Note that all terms appearing in (19) and (20) have finite values since $f(z + \chi_i + \chi_j) < +\infty$ and $f(z + \chi_k) < +\infty$ hold. Then we have

$$\begin{aligned}
& f(z + \chi_i + \chi_j) + f(z + \chi_k) \geq f(z + \chi_j + \chi_k) + f(z + \chi_i) \\
\Leftrightarrow\ & 2f(z) + h_i + h_j + h_k + \sum_{p \in \mathrm{supp}^+(z)} h_{ip} + \sum_{p \in \mathrm{supp}^+(z)} h_{jp} + \sum_{p \in \mathrm{supp}^+(z)} h_{kp} + h_{ij} \\
& \geq 2f(z) + h_j + h_k + h_i + \sum_{p \in \mathrm{supp}^+(z)} h_{jp} + \sum_{p \in \mathrm{supp}^+(z)} h_{kp} + \sum_{p \in \mathrm{supp}^+(z)} h_{ip} + h_{jk} \\
\Leftrightarrow\ & h_{ij} \geq h_{jk}.
\end{aligned}$$

Also we have

$$\begin{aligned}
& f(z + \chi_i + \chi_j) + f(z + \chi_k) \geq f(z + \chi_i + \chi_k) + f(z + \chi_j) \\
\Leftrightarrow\ & h_{ij} \geq h_{ik}.
\end{aligned}$$

By the assumption, it holds that $h_{ij} \geq \min\{h_{jk}, h_{ik}\}$. Hence we obtain

$$f(z + \chi_i + \chi_j) + f(z + \chi_k) \geq \min\{f(z + \chi_j + \chi_k) + f(z + \chi_i), f(z + \chi_i + \chi_k) + f(z + \chi_j)\}.$$

By the assumption of $h_{ij} \geq 0$, we also obtain

$$f(z + \chi_i + \chi_j) + f(z) \geq f(z + \chi_i) + f(z + \chi_j)$$

for all distinct $i, j \in [n]$.

Proof of Theorem 3. First we give a graphical interpretation for the anti-ultrametric property. For $H := (h_{ij})_{i,j\in[n]}$ and $\alpha \in \overline{\mathbf{R}}$, let us define E_H^α and V_H^α by

$$E_H^\alpha := \{\{i,j\} \in E_H \mid h_{ij} \geq \alpha\}, \tag{21}$$
$$V_H^\alpha := \{i \mid \exists e \in E_H^\alpha \text{ such that } i \in e\}. \tag{22}$$

Let $G_H^\alpha := (V_H^\alpha, E_H^\alpha)$. Then the following lemma holds:

Lemma 8. *$H := (h_{ij})_{i,j\in[n]}$ satisfies the anti-ultrametric property if and only if each connected component of G_H^α is a complete graph for every $\alpha \in \overline{\mathbf{R}}$.*

Proof. (only-if part). We show the contraposition. Suppose that for some $\alpha \in \overline{\mathbf{R}}$ there exists a non-complete graph among the connected components of G_H^α. Then there exist distinct $i, j, k \in [n]$ with $\{i,j\}, \{j,k\} \in E_H^\alpha \not\ni \{i,k\}$. By the definition of E_H^α, it holds that $\min\{h_{ij}, h_{jk}\} \geq \alpha > h_{ik}$. This means that $\{h_{ij}, h_{jk}, h_{ik}\}$ does not satisfy the anti-ultrametric property.

(if part). Suppose that each connected component of G_H^α is a complete graph for all $\alpha \in \overline{\mathbf{R}}$. To show the anti-ultrametric property of $(h_{ij})_{i,j\in[n]}$, it suffices to prove $h_{jk} = h_{ik}$ for all distinct i, j, k satisfying $h_{ij} > h_{jk}$. If $h_{ik} \geq h_{ij}$, then there exists a non-complete graph among the connected components of G_H^α for $\alpha = h_{ij}$, which is a contradiction. If $h_{ij} > h_{ik} > h_{jk}$, then there exists a non-complete graph among the connected components of G_H^α for $\alpha = h_{ik}$, which is a contradiction. If $h_{jk} > h_{ik}$, then there exists a non-complete graph among the connected components of G_H^α for $\alpha = h_{jk}$, which is a contradiction. Therefore we must have $h_{jk} = h_{ik}$. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. (only-if part). Suppose to the contrary that $H := (h_{ij})_{i,j\in[n]}$ is M$^\natural$-convex completable and that there exists a chordless cycle C of G_H with $|\operatorname{argmin}_{e\in C} w(e)| = 1$. Let $C = \{\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_m,i_1\}\}$, and consider the corresponding entries $\{h_{i_1i_2}, h_{i_2i_3}, \ldots, h_{i_mi_1}\}$ of H. Note that $h_{i_pi_q}$ is undefined for $p, q \in [m]$ with $|p-q| \neq 1 \mod m$. We may assume $\alpha := h_{i_1i_2} = \min\{h_{i_1i_2}, h_{i_2i_3}, \ldots, h_{i_mi_1}\}$. By the assumption of $|\operatorname{argmin}_{e\in C} w(e)| = 1$, we have $\min\{h_{i_2i_3}, \ldots, h_{i_mi_1}\} > \alpha$. Since $\{h_{i_1i_2}, h_{i_2i_3}, h_{i_1i_3}\}$ should satisfy the anti-ultrametric property, we have to assign α to $h_{i_1i_3}$ to obtain an M$^\natural$-convex completion. Since $\{h_{i_1i_3}, h_{i_3i_4}, h_{i_1i_4}\}$ should satisfy the anti-ultrametric property, we have to assign α to $h_{i_1i_3}$ to obtain an M$^\natural$-convex completion. By repeating this procedure, we arrive at $h_{i_1i_{m-1}} = \alpha$. This is a contradiction, since $h_{i_1i_{m-1}} < \min\{h_{i_{m-1}i_m}, h_{i_1i_m}\}$ and hence the anti-ultrametric property fails for $\{h_{i_1i_{m-1}}, h_{i_{m-1}i_m}, h_{i_mi_1}\}$.

(if part). For $\alpha \in \overline{\mathbf{R}}_+$, define S_H^α by

$$S_H^\alpha := \{\{i,j\} \notin E_H \mid i,j \in V_H^\alpha,\ i \text{ and } j \text{ are connected in } G_H^\alpha\}.$$

Let $\tilde{G}_H^\alpha := (V_H^\alpha, E_H^\alpha \cup S_H^\alpha)$. Recall that E_H^α and V_H^α are defined in (21) and (22).

First we show that if each connected component of $\tilde{G}_H^\alpha$ is a complete graph for every $\alpha \in \overline{\mathbf{R}}_+$, then H is M$^\natural$-convex completable. Let $\alpha_1 > \alpha_2 > \cdots > \alpha_p$ be the distinct values of defined elements of $(h_{ij})_{i,j\in[n]}$ (α_1 can be the infinite value). We assign α_1 to each undefined element h_{ij} such that $\{i,j\} \in S_H^{\alpha_1}$, α_k to each h_{ij} such that $\{i,j\} \in S_H^{\alpha_k} \setminus S_H^{\alpha_{k-1}}$ for $k = 2, \ldots, p-1$, and α_p to each h_{ij} such that $\{i,j\} \notin S_H^{\alpha_{p-1}}$. Then we obtain a certain completion $\tilde{H} := (\tilde{h}_{ij})_{i,j\in[n]}$ of $(h_{ij})_{i,j\in[n]}$. It is clear that each connected component of $G_{\tilde{H}}^\alpha$ is a complete graph for every $\alpha \in \overline{\mathbf{R}}$. By Lemma 8, $\tilde{H}$ satisfies the anti-ultrametric property. This means that H is M$^\natural$-convex completable.

Next we show that if $|\operatorname{argmin}_{e\in C} w(e)| \geq 2$ holds for every chordless cycle C of G_H, then each connected component of $\tilde{G}_H^\alpha$ is a complete graph for every $\alpha \in \overline{\mathbf{R}}_+$. Take arbitrary $\alpha \in \overline{\mathbf{R}}_+$

and i and j which are connected in $\tilde{G}_H^\alpha$ (Note that vertex sets of connected components of $\tilde{G}_H^\alpha$ are the same as those of G_H^α). It suffices to prove that $\{i,j\} \in E_H^\alpha$ or $\{i,j\} \in S_H^\alpha$ holds. Suppose to the contrary that there exist i and j such that $\{i,j\} \notin E_H^\alpha$ and $\{i,j\} \notin S_H^\alpha$ hold. Let I be the set of such $\{i,j\}$. Let $\{i_0,j_0\} \in I$ be a pair of vertices such that the number of edges of a shortest i_0-j_0 path on G_H^α is minimum in I. Since i_0 and j_0 are connected in G_H^α and $\{i_0,j_0\} \notin S_H^\alpha$, we have $\{i_0,j_0\} \in E_H$. Moreover since $\{i_0,j_0\} \notin E_H^\alpha$, $h_{i_0j_0} < \alpha$ holds. Take a i_0-j_0 shortest path P_0. Then $P_0 \cup \{i_0,j_0\}$ is a chordless cycle of G_H. Indeed, if $P_0 \cup \{i_0,j_0\}$ has a chord in G_H, there exist i' and j' satisfying $\{i',j'\} \neq \{i_0,j_0\}$ in $P_0 \cup \{i_0,j_0\}$ such that $E_H^\alpha \not\ni \{i',j'\} \in E_H$ by the minimality of $|I|$. Then $\{i',j'\} \in I$ and the number of edges of a shortest i'-j' path is smaller that those of P_0. However this is a contradiction to the minimality of $\{i_0,j_0\}$. Hence $P_0 \cup \{i_0,j_0\}$ is a chordless cycle of G_H. It holds that $h_{ij} \geq \alpha$ for $\{i,j\} \in P_0$ and $h_{i_0j_0} < \alpha$. Therefore we obtain $|\operatorname{argmin}_{e \in P_0 \cup \{i_0,j_0\}} w(e)| = 1$. This contracts the assumption of $|\operatorname{argmin}_{e \in C} w(e)| \geq 2$. Hence we have $\{i,j\} \in E_H^\alpha$ or $\{i,j\} \in S_H^\alpha$. □

Proof of Theorem 5. Let $H := (h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$, where the entries $h_{(i,a),(j,b)}$ with $i = j$ are undefined. Recall that $G_H = (U, E_H; w)$ is the assignment graph of H. By the definition of E_H and $h_{(i,a),(j,b)}$ in (11), we have $E_H = \{\{(i,a),(j,b)\} \mid i \neq j,\ a \in D_i,\ b \in D_j\}$. By Theorem 3, H is M$^\natural$-convex completable if and only if every chordless cycle C satisfies the condition $|\operatorname{argmin}_{e \in C} w(e)| \geq 2$.

First we show that chordless cycles in G_H have length 3 or 4. Take any chordless cycle $C = \{\{(i_1,a_1),(i_2,a_2)\},\{(i_2,a_2),(i_3,a_3)\},\ldots,\{(i_k,a_k),(i_1,a_1)\}\}$ of G_H. Since C is chordless, we have $i_1 = i_p$ for $3 \leq p \leq k-1$ and $i_2 = i_q$ for $4 \leq q \leq k$. This implies $k \leq 4$, since otherwise we obtain $i_1 = i_4 = i_2$, contradicting the existence of an edge between (i_1,a_1) and (i_2,a_2).

For a (chordless) cycle of length 3, say, $C = \{\{(i_1,a_1),(i_2,a_2)\},\{(i_2,a_2),(i_3,a_3)\},\{(i_3,a_3),(i_1,a_1)\}\}$ with $i_1 \neq i_2 \neq i_3 \neq i_1$, the condition $|\operatorname{argmin}_{e \in C} w(e)| \geq 2$ is equivalent to (2) for JWP. For a chordless cycle of length 4, say, $C = \{\{(i_1,a_1),(i_2,a_2)\},\{(i_2,a_2),(i_3,a_3)\},\{(i_3,a_3),(i_4,a_4)\},\{(i_4,a_4),(i_1,a_1)\}\}$ we have $i_1 \neq i_2$, $i_3 \neq i_4$, $i_1 = i_3$, $i_2 = i_4$, $a_1 \neq a_3$, $a_2 \neq a_4$, and then the condition $|\operatorname{argmin}_{e \in C} w(e)| \geq 2$ is equivalent to (3) for Z-freeness.

Proof of Theorem 6. We investigate each step in turn.

(Step 1). Since the number of defined elements of $(h_{(i,a),(j,b)})_{(i,a),(j,b) \in U}$ is $O(n^2 - \sum_{i=1}^r d_i^2) = O(n^2)$, we can find an M$^\natural$-convex completion in $O(n^2 + n \log n)$ time (recall Remark 4).

(Step 2). If we have the value of $\overline{f}(\hat{x}^*)$, we can compute the value of $\overline{f}(\hat{x}^* + \chi_{(i,a)})$ in $O(r)$ time since $\overline{f}(\hat{x}^* + \chi_{(i,a)}) = \overline{f}(\hat{x}^*) + c_i(a) + \sum_{(j,b) \in \operatorname{supp}^+(\hat{x}^*)} \tilde{h}_{(i,a),(j,b)}$. Hence the time complexity of Step 3 is $O(nr^2)$ time.

(Step 3). Recall the definition of $G_{\hat{x}^*,\hat{y}^*}$ in (12)–(18). We have $|E_{\hat{x}^*}| = O(r(n-r)) = O(nr)$, $|E_{\hat{y}^*}| = O(n)$, $|E^+| = O(r)$, and $|E^-| = O(r)$. Hence $|E_{\hat{x}^*,\hat{y}^*}| = O(nr)$. Furthermore we need to compute ℓ only on $E_{\hat{x}^*}$, since ℓ is equal to zero on other arcs. If we have the value of $\overline{f}(\hat{x}^*)$ at hand, we can compute the value of $\overline{f}(\hat{x}^* + \chi_{(j,b)} - \chi_{(i,a)})$ in $O(r)$ time since

$$\begin{aligned}&\overline{f}(\hat{x}^* + \chi_{(j,b)} - \chi_{(i,a)})\\&= \overline{f}(\hat{x}^*) - \left(c_i(a) + \sum_{(k,c) \in \operatorname{supp}^+(\hat{x}^*)} \tilde{h}_{(i,a),(k,c)}\right) + \left(c_j(b) + \sum_{(k,c) \in \operatorname{supp}^+(\hat{x}^* - \chi_{(i,a)})} \tilde{h}_{(j,b),(k,c)}\right).\end{aligned}$$

Therefore we can construct the auxiliary graph $G_{\hat{x}^*,\hat{y}^*}$ in $O(nr^2)$ time.

The modified arc length ℓ_p is nonnegative [9, Section 5.2]. Hence we can compute $\Delta p(v)$ for $v \in V$ and a shortest path P in Step 3-2 in $O(nr + n \log n)$ time by using Dijkstra's algorithm with Fibonacci heaps [5] (see also [17, Section 7.4]). We can update $\hat{x}^*$, $\hat{y}^*$, and p in Step 3-3 in

$O(nr)$ time. By one iteration of Step 3, the value of $\|\hat{x}^* - \hat{y}^*\|_1$ is decreased by two. Hence the number of iterations of Step 3 is bounded by $O(r)$. Therefore the time complexity of Step 3 is $O(nr^3 + nr \log n)$.

By the above argument, we see that the proposed algorithm runs in $O(nr^3 + nr \log n + n^2)$ time.

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