Lecture 12

Classical-quantum interaction

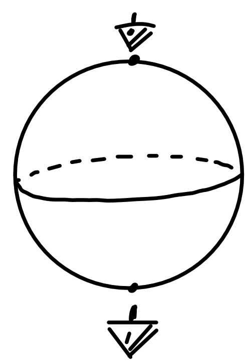
WRITE A Q. PROCESS $\{\Phi_1, \ldots, \Phi_n\}$ AS $\left(\Phi_i\right)^i$, where $\sum_i \Phi_i = \bar{\top}$.

ONB measurements := q. processes of the form $\left(i\right)^i$ where $\{i\}_i$ is an ONB.

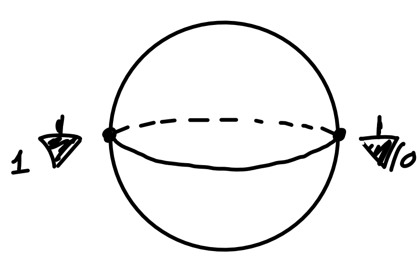
(classical data: i; q. system)

For QUBITS:

$$Z\text{-basis} := \{0, 1\}$$



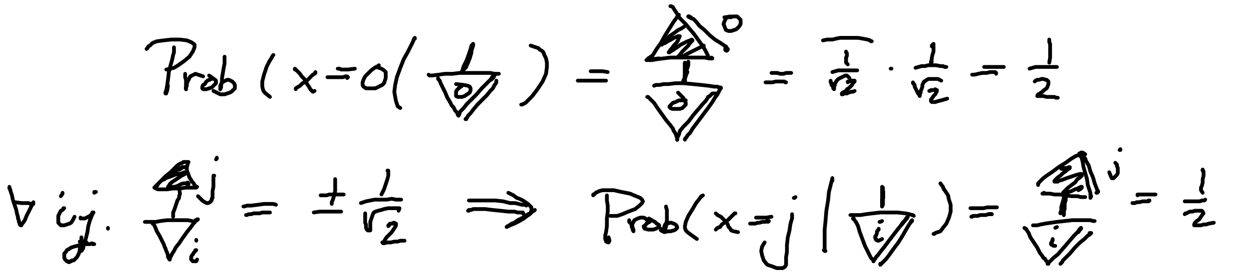
$$X\text{-basis} := \left\{ +_0 := \tfrac{1}{\sqrt{2}}(0 + 1),\ +_1 := \tfrac{1}{\sqrt{2}}(0 - 1) \right\}$$



Inner products: $\langle +_0 | 0 \rangle = \frac{1}{\sqrt{2}} \left[\langle 0|0\rangle + \langle 1|0\rangle \right] = \frac{1}{\sqrt{2}}$

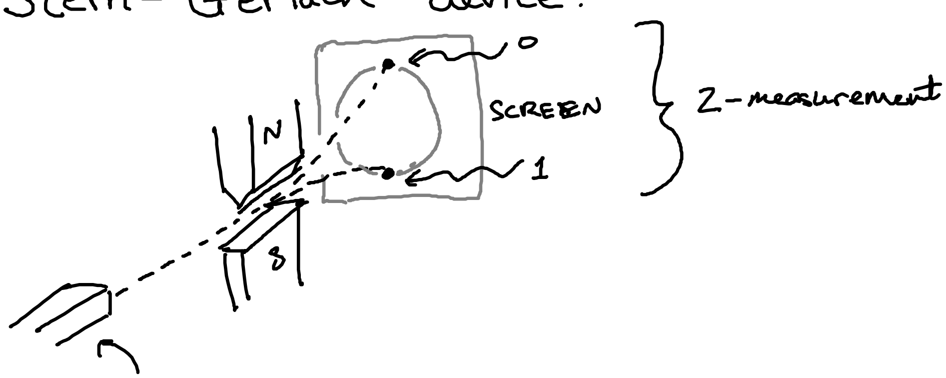
(with $\langle 0|0\rangle = 1$ and $\langle 1|0\rangle = 0$)

X-meas. of state  :

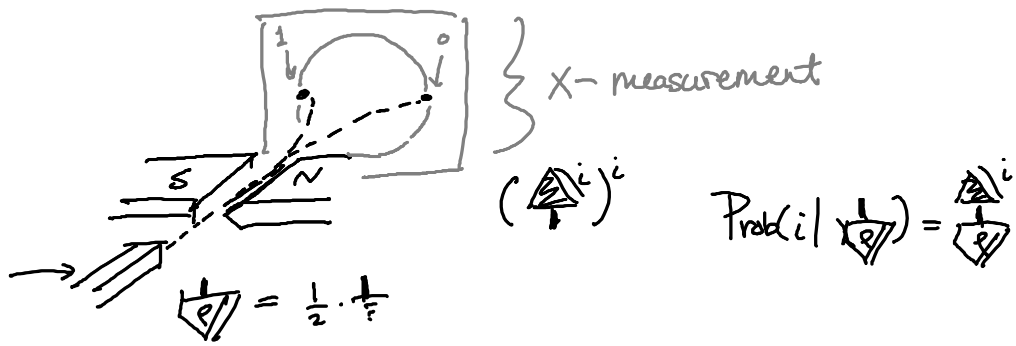


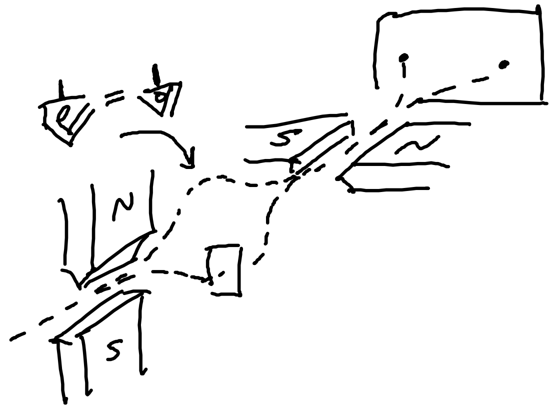
mutually unbiased bases.

EXAMPLE: Stern-Gerlach device.



oven (shoots out random particles, e.g. H atoms)

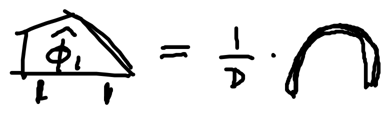




6.4.2 Getting *any* q-map as part of a non-det. quantum process.

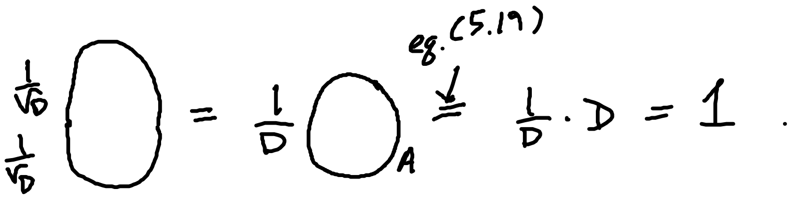
Lem (6.94) For any $\hat{A}$, there exists a quantum process

where



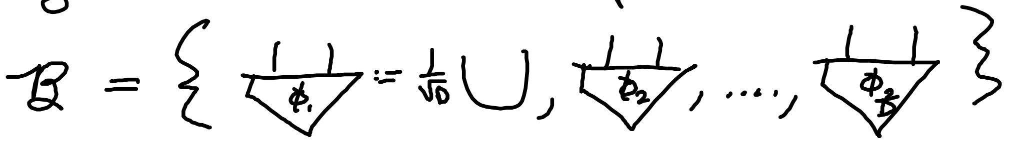
& $D = \dim A$.

Proof First, let $\frac{1}{\sqrt{D}} \cdot$ ∪ be the *normalised* cup.



Then, any normalised state is part of an ONB (Prop. 5.79)

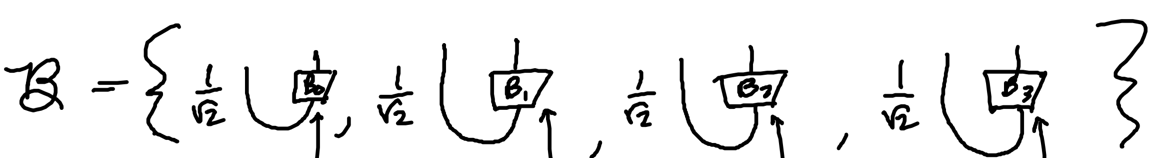
let



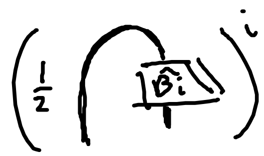
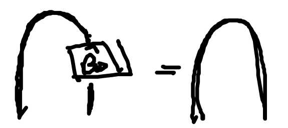
Then:

is an ONB meas, hence a quantum process. ∎

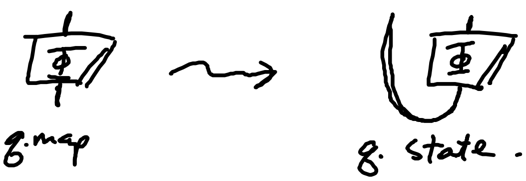
Ex In 2D, $\hat{A} = \hat{\mathbb{C}^2}$, we can use the Bell basis.

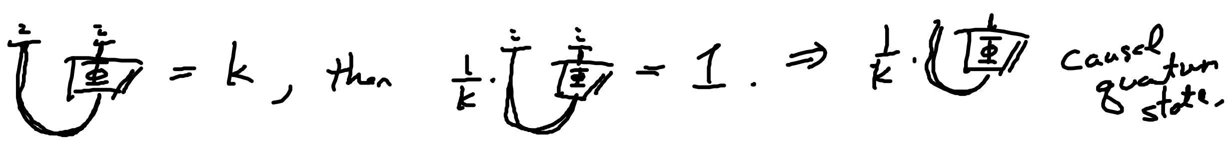


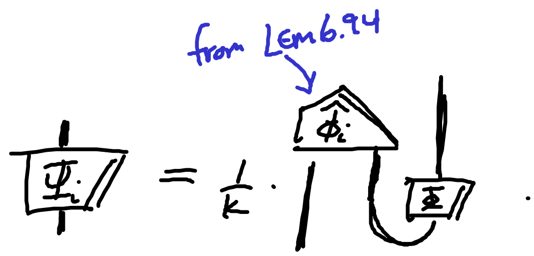
Bell matrices: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

So  is a Bell measurement, and .

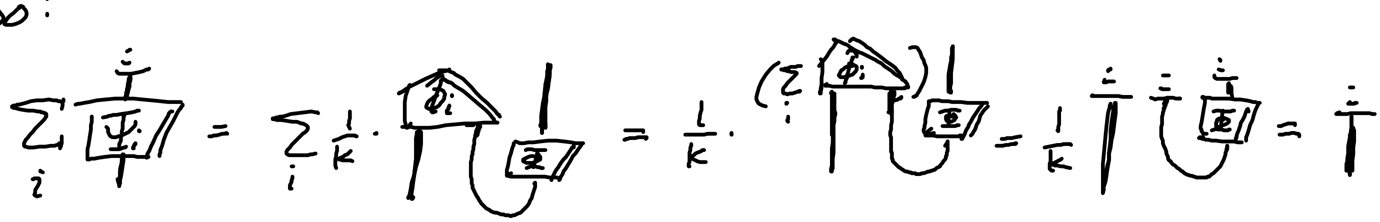
Thm For any q. map Φ (not nec. causal), there exists a q. process $(\Phi_i)^i$ such that $\Phi_i = r \cdot \Phi$.

Proof 

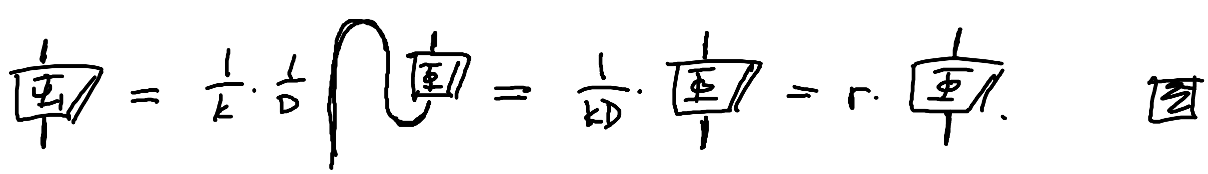
Suppose: 

Then, let: 

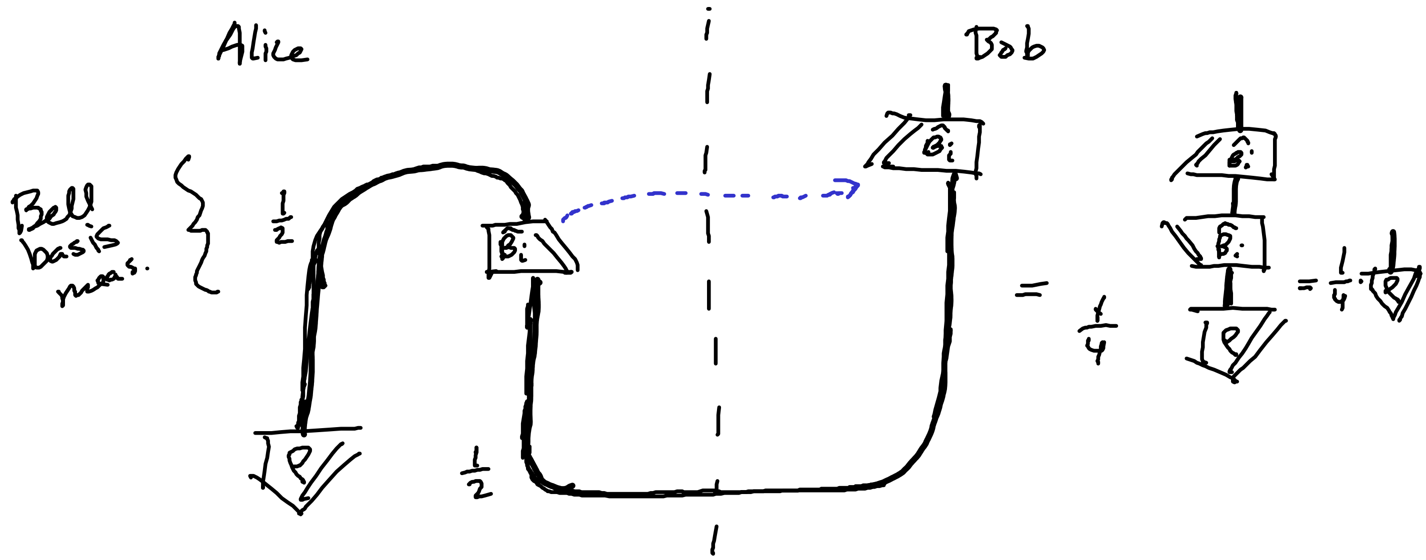
So:

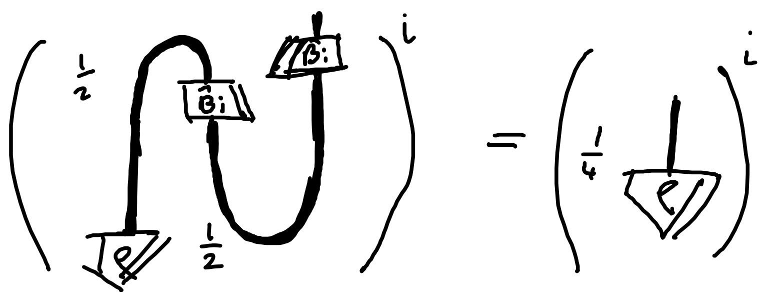


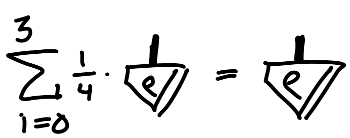
$\Rightarrow$ $(\Phi_i)^i$ are a quantum process.

and 

EXAMPLE QUANTUM TELEPORTATION.





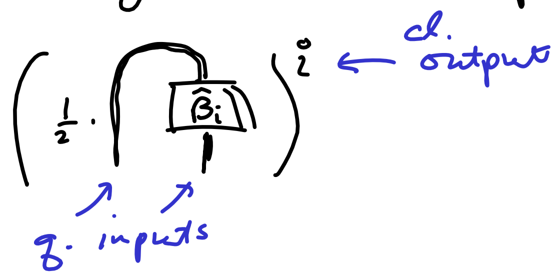
Don't care about i $\Longrightarrow$ 

Lecture 13

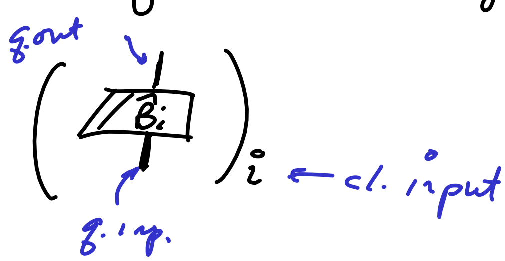
Ch. 8: Classical-quantum processes.

In teleportation, classical data plays 2 different roles:

- An <u>output</u> of a non-det proc:

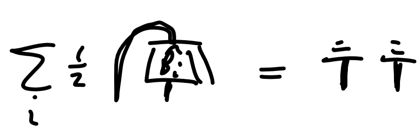


- An <u>input</u> of a classically-controlled process.

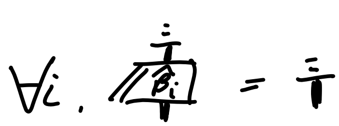


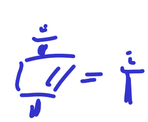
We write these differently because the <u>causality</u> cond'n is different.

CL. OUTPUT:

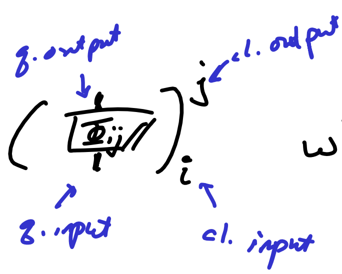


CL. INPUT

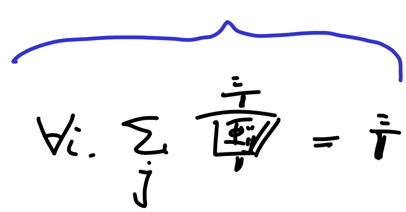




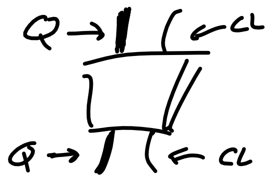
IN GENERAL:



where



Q: Why can't everything be wires?



A: It can be!

QUANTUM SYSTEM $\longrightarrow$ DOUBLE WIRES (dbl. H.S.)

CLASSICAL SYSTEM $\longrightarrow$ SINGLE WIRES

CLASSICAL (SINGLE)

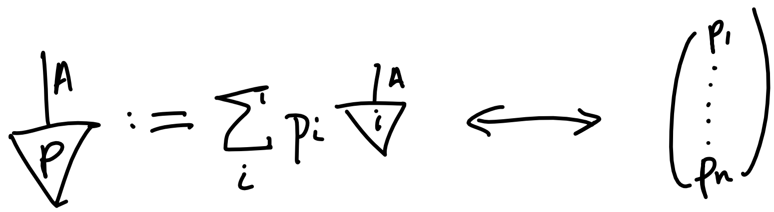
- ONB states 

"providing a classical value i"
- ONB effects 

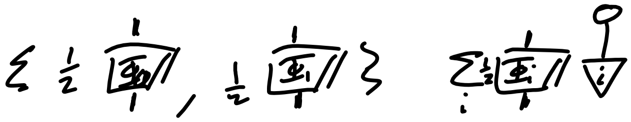
"testing for i"

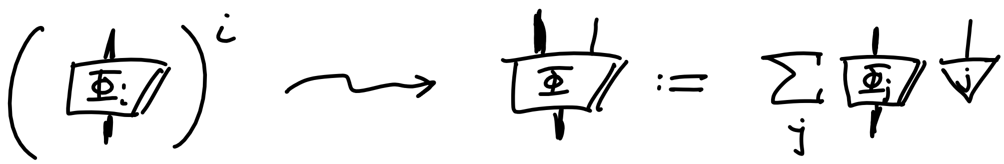
DETERMINISTIC

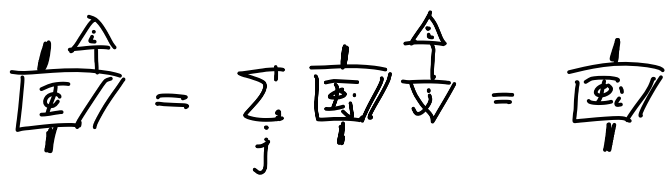
MORE GENERALLY: WE CAN HAVE PROBABILISTIC STATES.

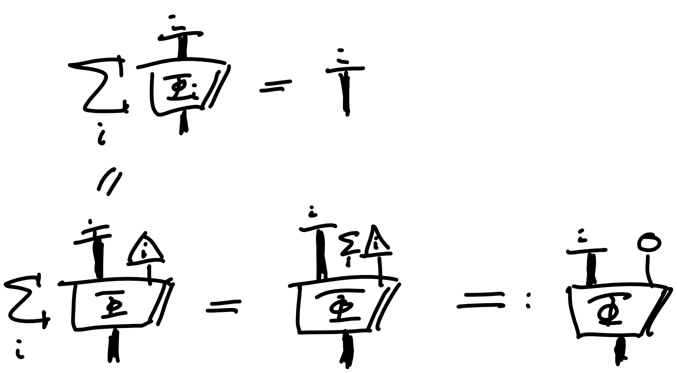


probabilities.

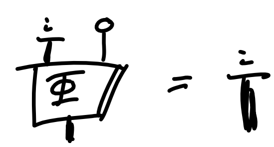


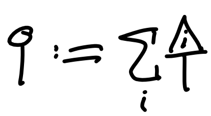
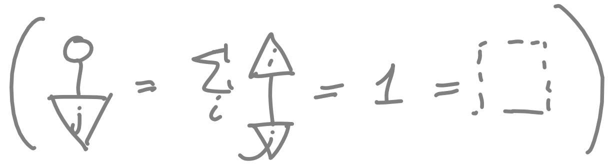


Then 

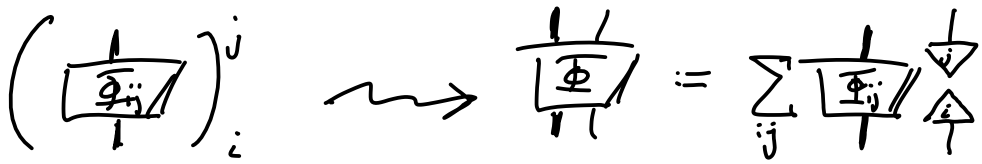


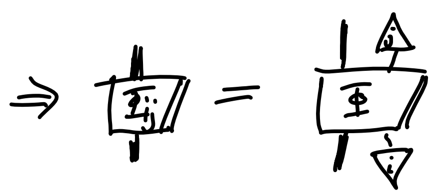
CAUSALITY:



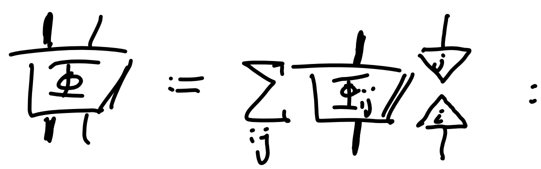
WHERE  is called <u>deleting</u>. 

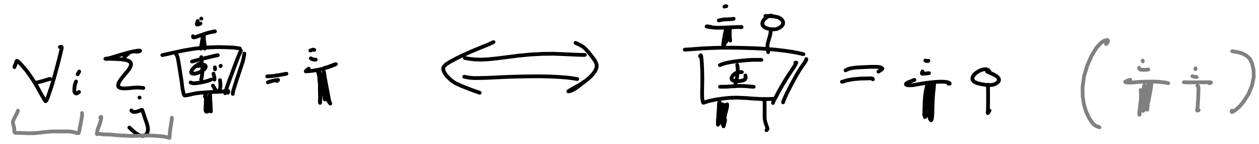
WITH CL. CONTROL:





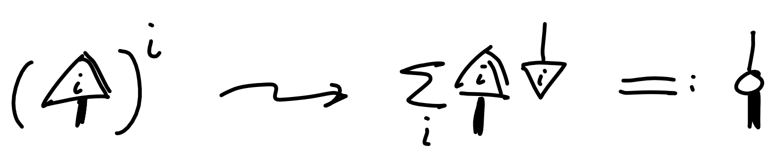
Prop (8.9) For

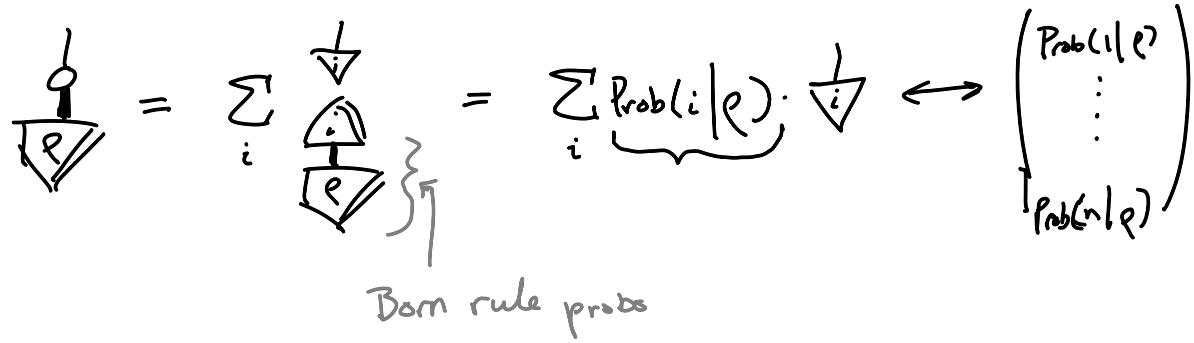




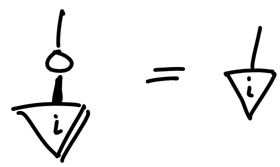
8.13 Measurement + encoding.

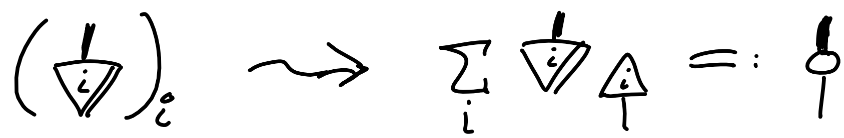
MEASURE:

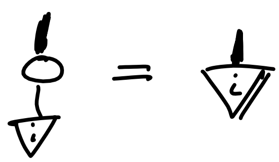




 :: Q-STATES ⟶ PROB DIST FOR  ONB meas.

In particular:  ← point distr. at i.
Prob$(i|\rho) = 1$
Prob$(j \neq i|\rho) = 0$.

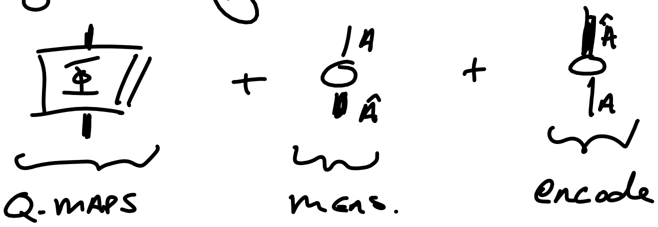
ENCODE: 

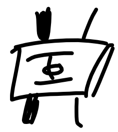


DEF The process theory of classical-quantum maps has:

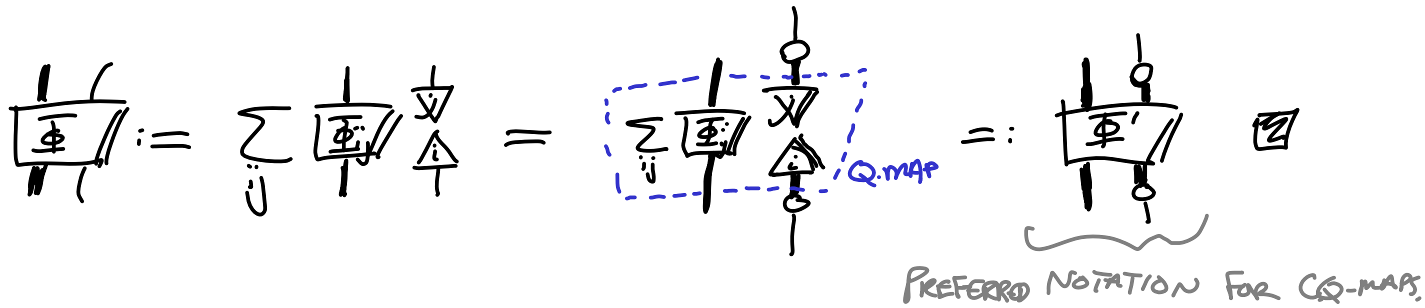
types → Q. systems (dbl. H.S. $\hat{A}$)
types → CL. systems (H.S. A)

procs: diagrams of:

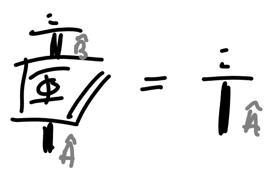


PROP:  ARE CQ-MAPS!

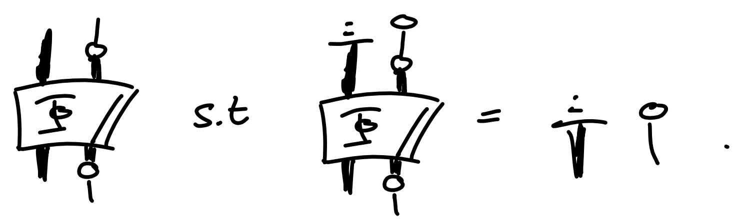
PF:



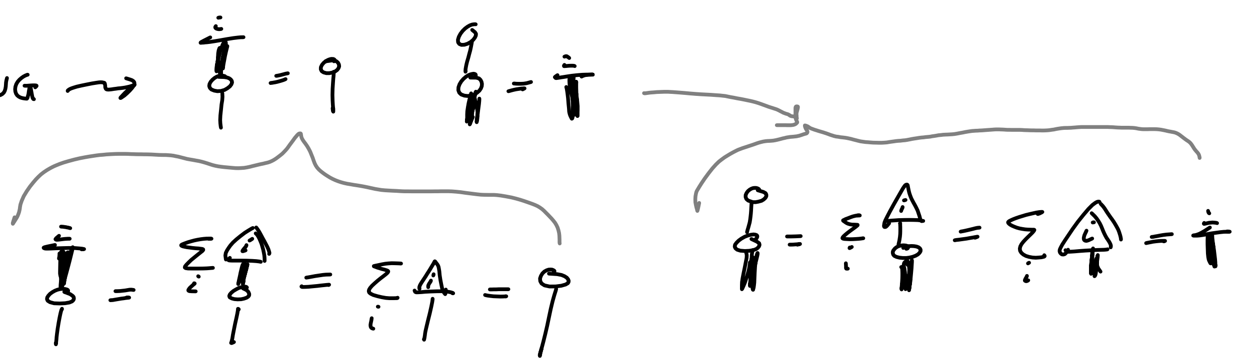
CAUSAL QUANTUM MAPS



QUANTUM PROCESSES := CAUSAL CQ-MAPS

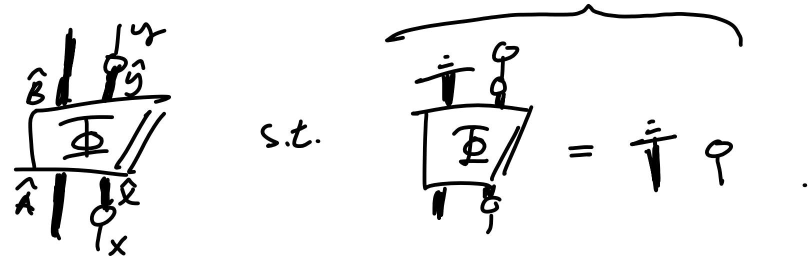


DELETING ⤳

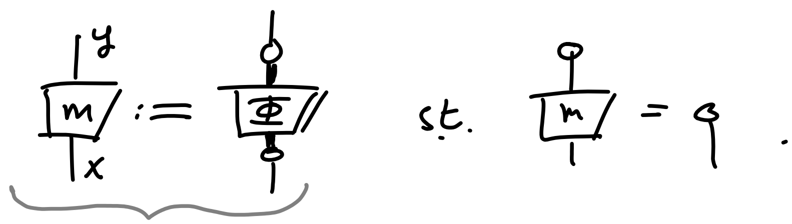


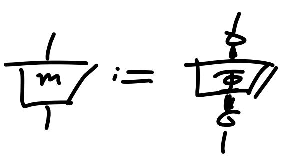
Lecture 14

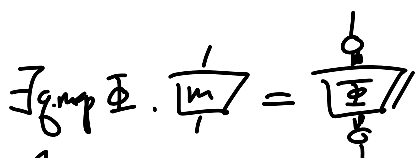
Quantum processes := causal cq-maps

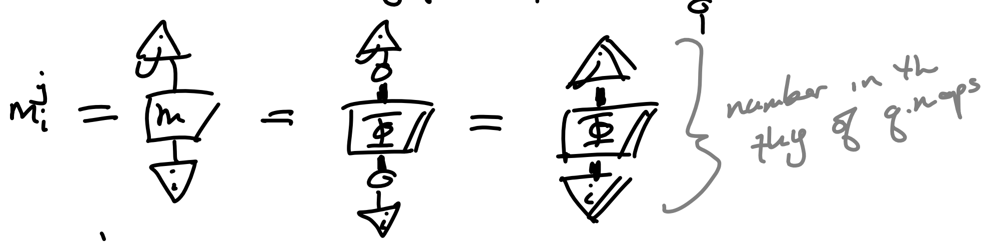


Special case: Classical processes := causal cq-maps w/ no quantum inputs/outputs.

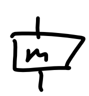


<u>Lem</u>  for some q.map Φ if & only if $m_i^j \geq 0$.

<u>Pf</u> For (⇒) assume  Then:



⇒ m_i^j is a positive real number. (non-negative)

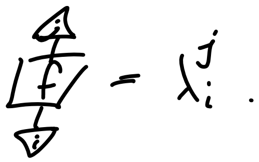
For (⇐), assume  is a linear map where $m_i^j \geq 0 \ \forall i,j$.

Then $\exists \lambda_i^j \in \mathbb{C}$ such that $m_i^j = \overline{\lambda_i^j}\lambda_i^j$.

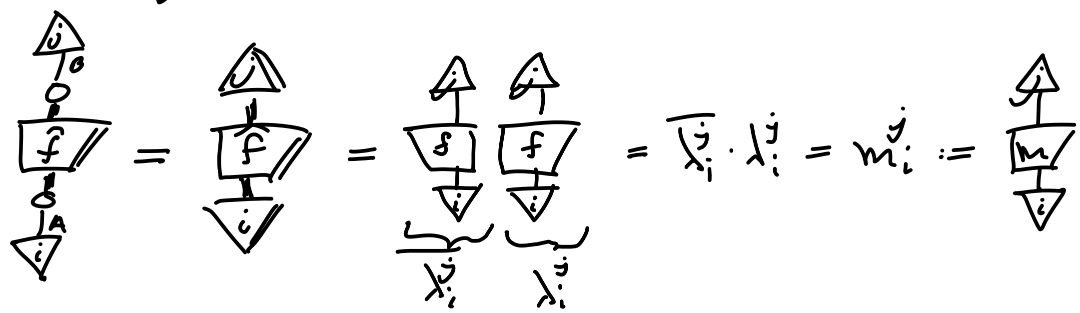
$\lambda_i^j := \sqrt{m_i^j}$

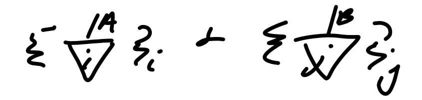
$\overline{\lambda_i^j} = \lambda_i^j$

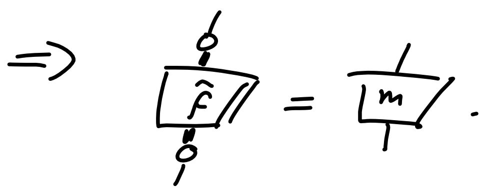
Let  be a linear map w/ matrix (λ^j_i).



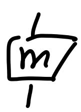
Then:



Since  are bases for A & B

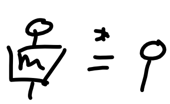


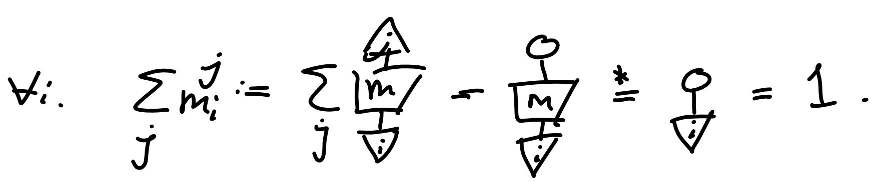
∎

THM  is a causal classical map (=: classical process) if and only if its matrix has:

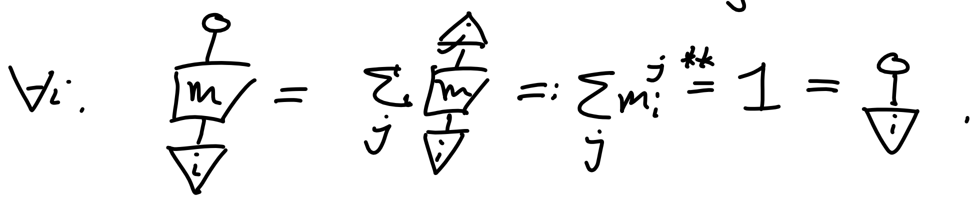
(i) positive entries $m^j_i \geq 0$.

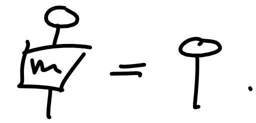
(ii) each column summing to 1. $\forall i. \sum_j m^j_i = 1$. $\left([\,] \right)$ ← $\Sigma = 1$

Pf (i) is lemma. For (ii), assume , then:



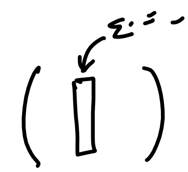
For the converse, assume $\forall i: \sum_j m_i^j \overset{*}{=} 1$. Then:



Since $\{\triangledown_i\}_i$ is a basis,  . $\boxed{\Sigma}$

(i) positive entries $m_i^j \geq 0$.

(ii) each column summing to 1. $\forall i. \sum_j m_i^j = 1$.

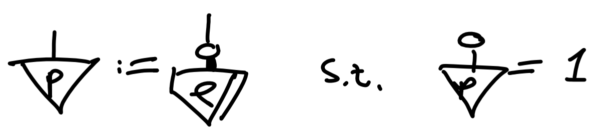


Stochastic maps

Conditional prob. distr.

$$m_i^j = P(j|i)$$

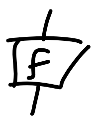
(output: j; input: i)

Ex Classical states  are prob. distributions.

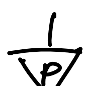
$$p^i = P(i).$$

Ex Suppose a process takes a bit as input and flips it w/ prob $\frac{1}{3}$.

Matrix: $f_0^0 = \frac{2}{3}$ $f_0^1 = \frac{1}{3}$ $f_1^0 = \frac{1}{3}$ $f_1^1 = \frac{2}{3}$

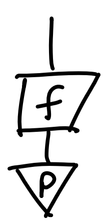
 $\leftrightarrow \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ (columns 0, 1; rows 0, 1.)

Suppose I have a bit that is 0 w/ prob. $\frac{1}{4}$ and 1 w/ prob $\frac{3}{4}$. Then:

prior {  $\leftrightarrow \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$ (rows 0, 1)

$\frac{2}{12} + \frac{3}{12} = \frac{5}{12}$

$\frac{1}{12} + \frac{6}{12} = \frac{7}{12}$

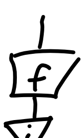
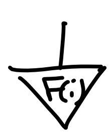
posterior {  $\leftrightarrow \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{12} \\ \frac{7}{12} \end{pmatrix}$

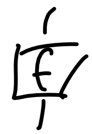
$\sum_i f_i^j p^i$

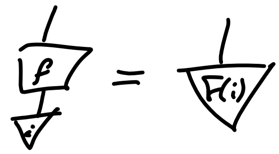
$\hookrightarrow \sum_i P(j|i)P(i)$

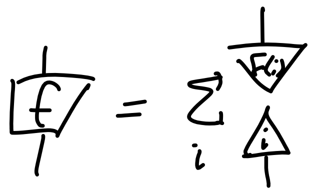
DEF A classical process is deterministic if $\exists$ a function $F: \mathcal{B}_A \to \mathcal{B}_B$

$\{1, \ldots, m\}$ $\{1, \ldots, n\}$

where  =  .

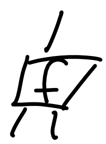
 is a det. classical proc

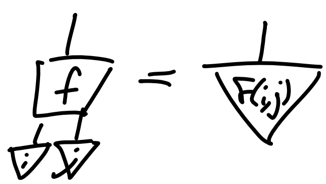
$\iff$ $\forall i.$ 

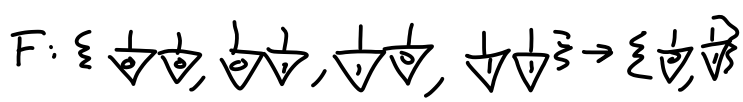
$\iff$ 

$\iff$ matrix of f has one 1 in every column.

e.g.

 $\longleftrightarrow$ $\begin{array}{c} \\ 0 \\ 1 \end{array}\!\!\begin{array}{cccc} 00 & 01 & 10 & 11 \\ \end{array}$ $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

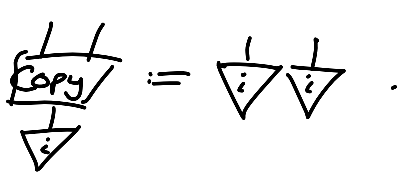
 where $F(i,j) := XOR(i,j)$



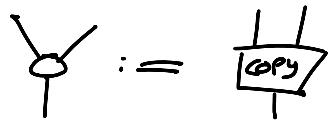
8.2.2 Copying + deleting.

$F : \mathcal{B}_A \longrightarrow \mathcal{B}_{A\otimes A}$

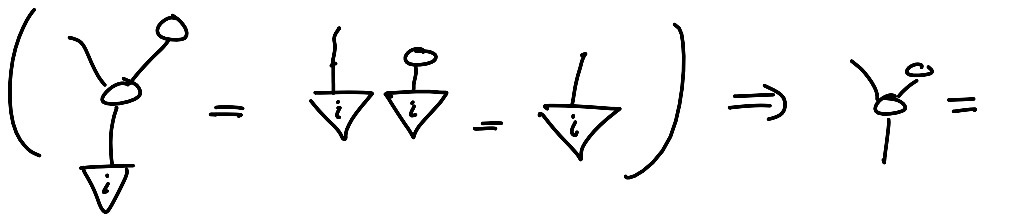
$F(i) = (i,i)$



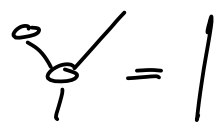
NEW NOTATION:



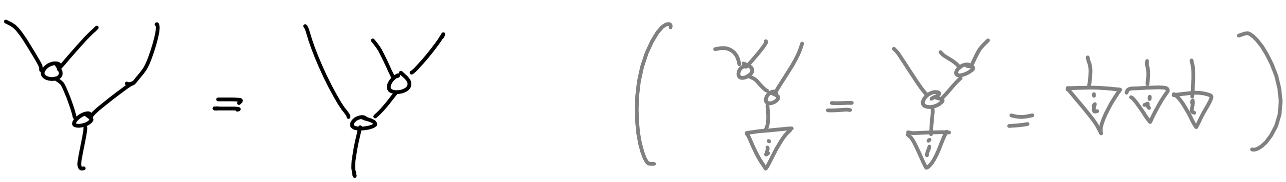
NOTE:

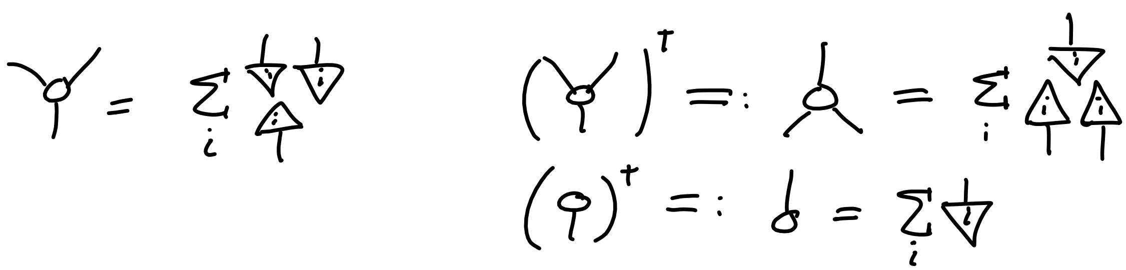


SIMILARLY:

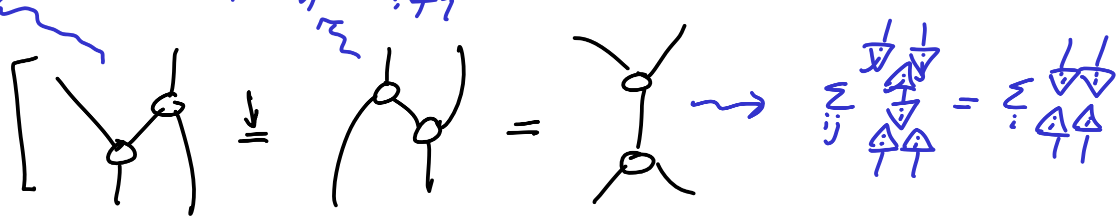


Sim.

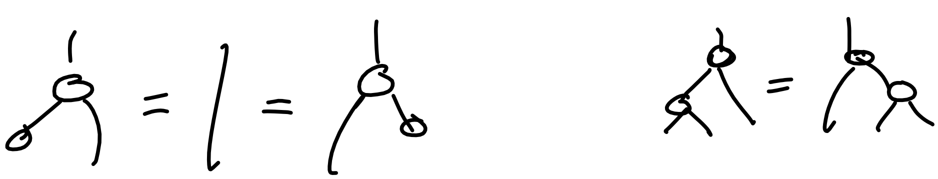




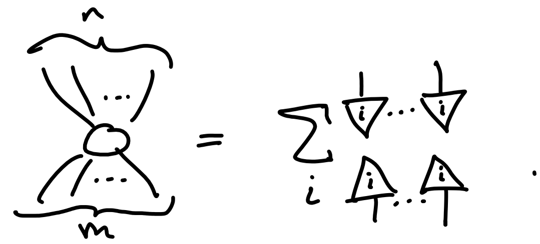
Sim.:

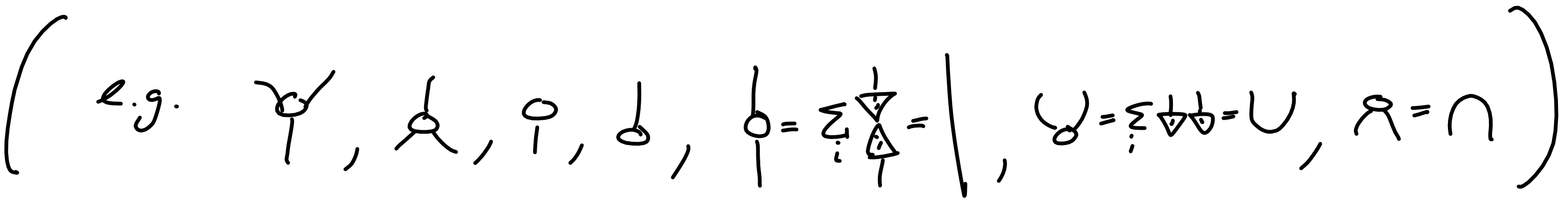


Sim.:



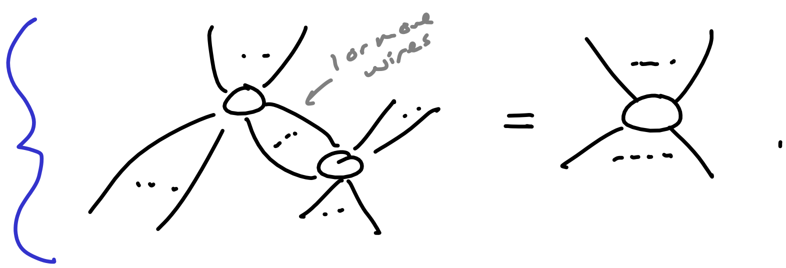
DEF A *spider* is a map:



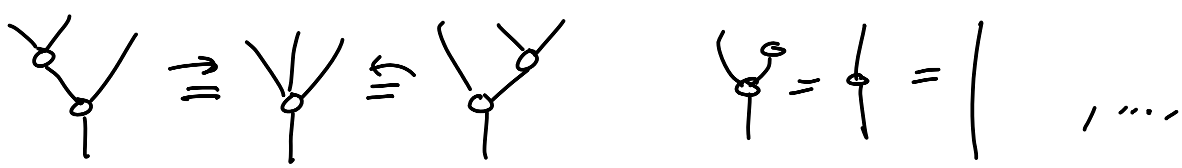


The only eqn. we need:

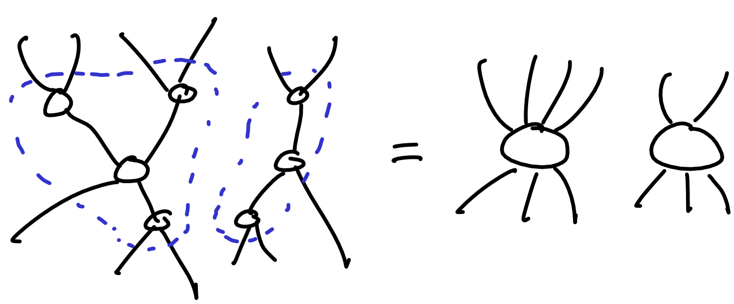
SPIDER FUSION



e.g.:

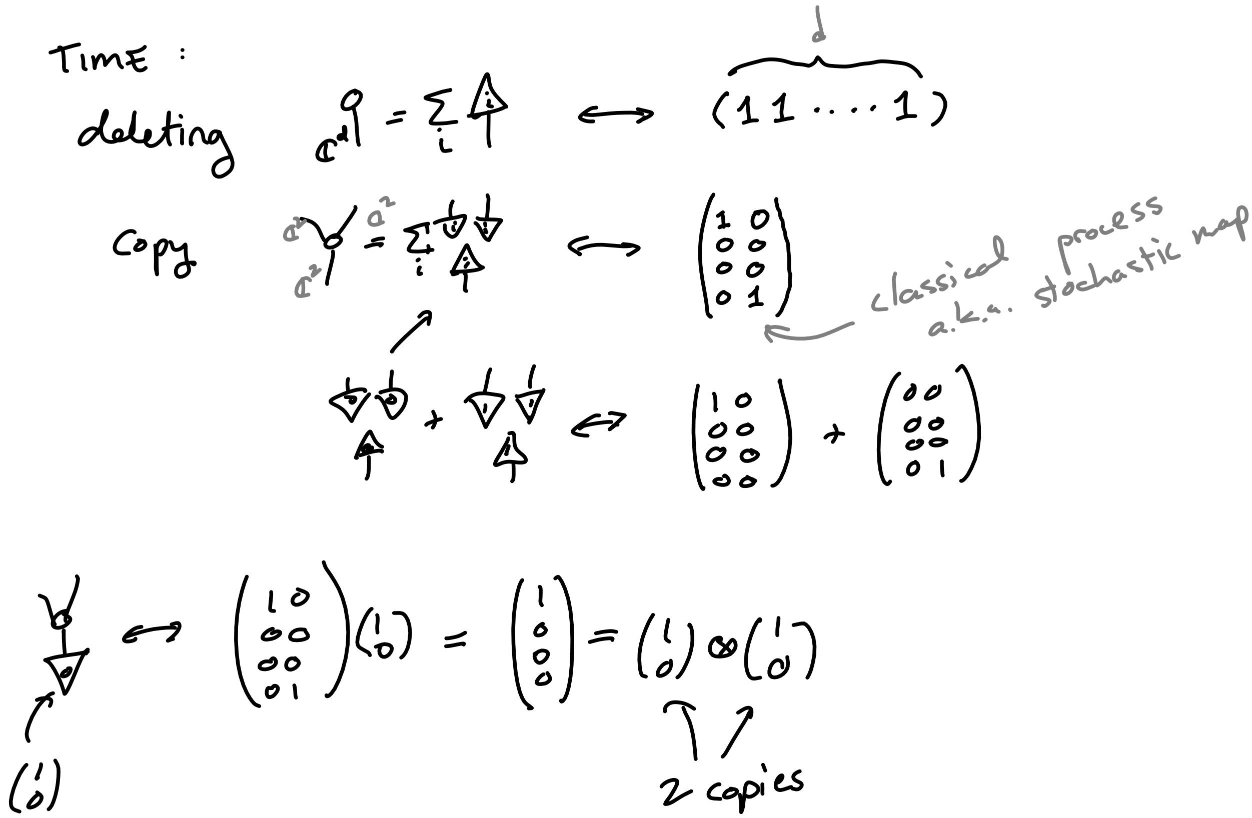


SPIDER FUSION ⟹

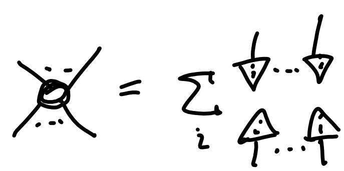


Lecture 15

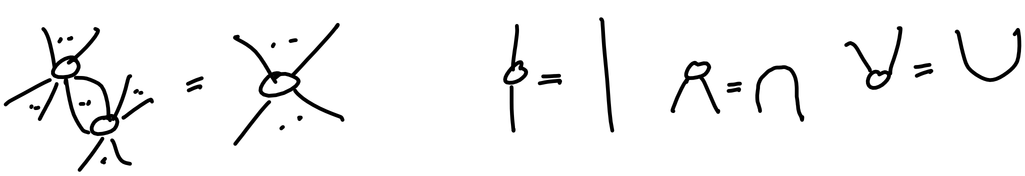
LAST TIME:

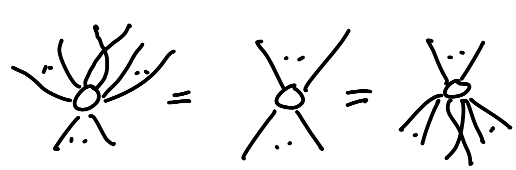


SPECIAL CASES OF SPIDERS



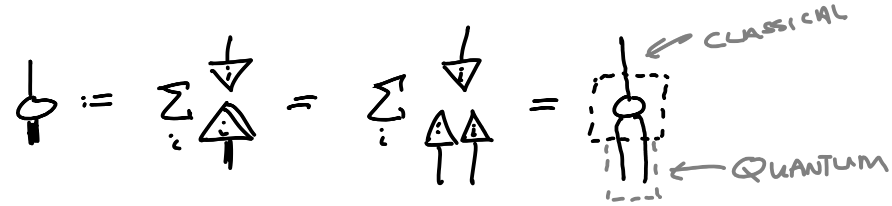
SPIDER FUSION



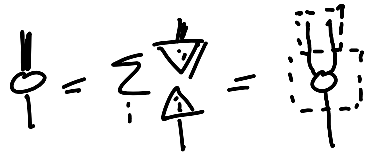


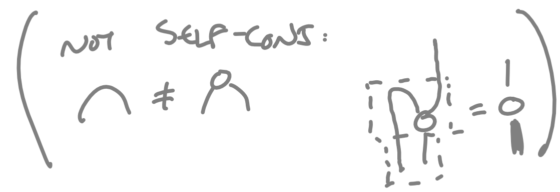
TODAY USE SPIDERS TO BUILD OTHER CQ-MAPS, NOT JUST CLASSICAL MAPS.

MEASURE



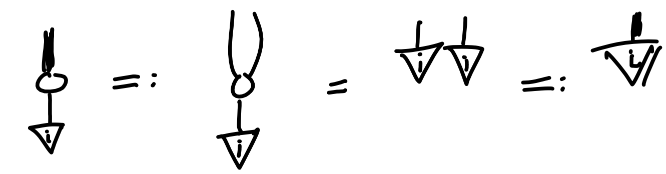
ENCODE



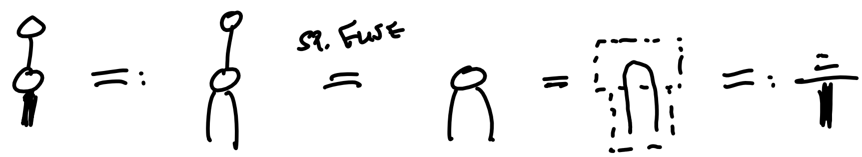


USING THESE DEF^N S OF MEAS & ENC, WE SEE:

*

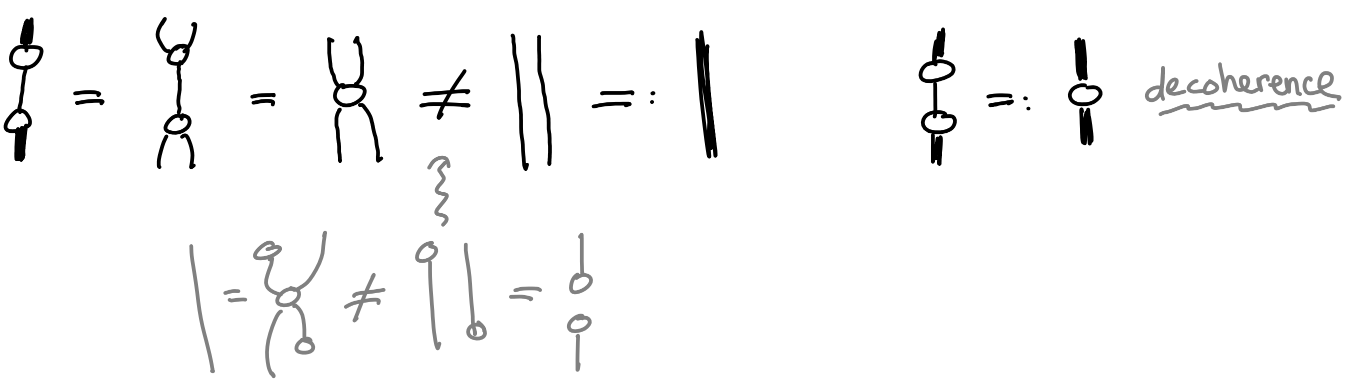


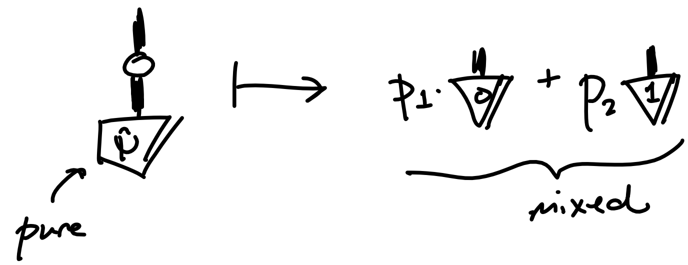
*

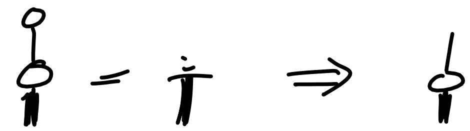


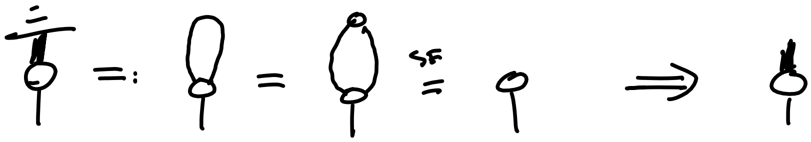
*

*

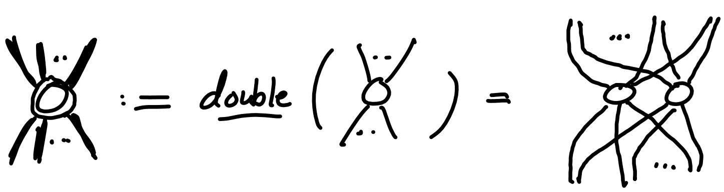




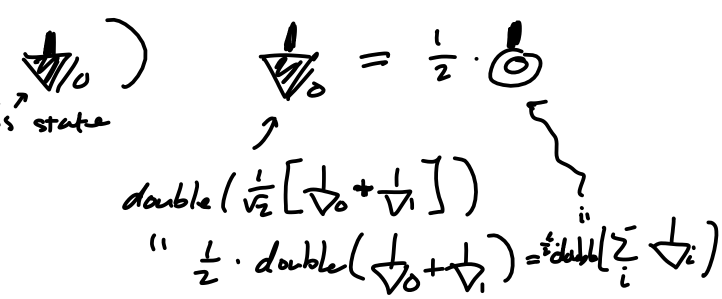
*  is causal

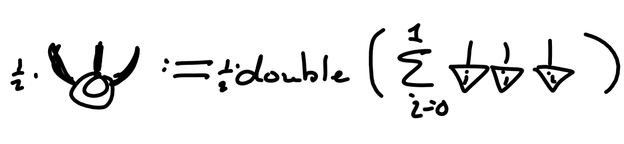
*  is causal

DEF A quantum spider is:

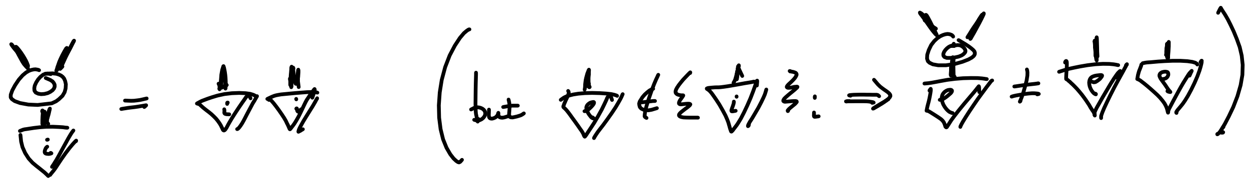


for pure quantum map

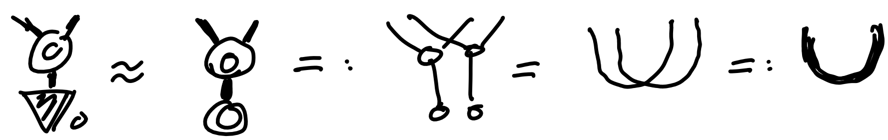
Ex "PLUS" STATE (a.k.a. )

Ex GHZ state: 

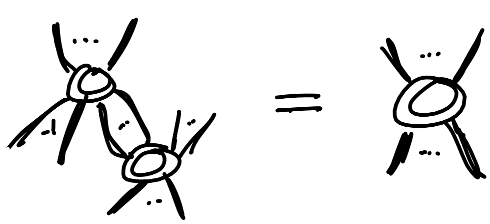
Ex "coherent" copy: 



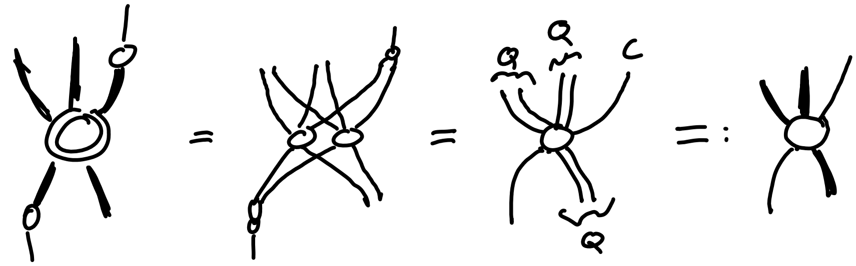
in particular, for  :



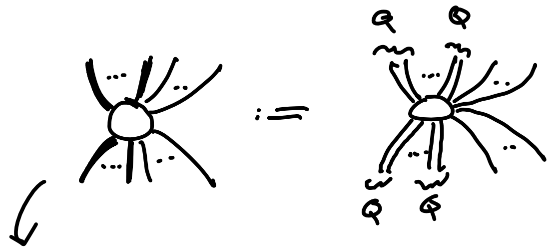
THM Quantum spiders satisfy spider-fusion:



Pf 2x applications of normal sp. fusion.

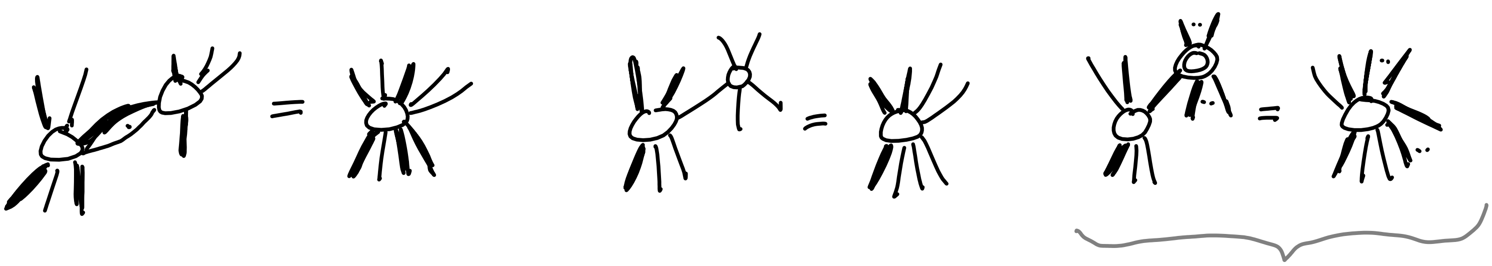


Def A <u>bastard</u> <u>spider</u> is:



$$\hat{A}\otimes\hat{A}\cdots\otimes\hat{A}\otimes A\otimes A \to A\otimes\hat{A}\otimes\cdots\otimes\hat{A}$$

BASTARD SPIDER FUSION:

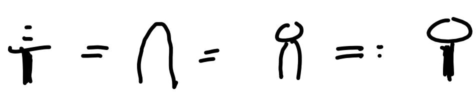


"CONTACT W/ THE CLASSICAL WORLD DESTROYS QUANTUM-NESS OF SPIDERS."

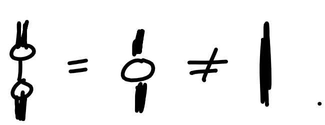
Ex Delete, copy, etc. classical ($\Rightarrow$ bastard)

Ex Measure, encode are b. spiders.

Ex Discarding:



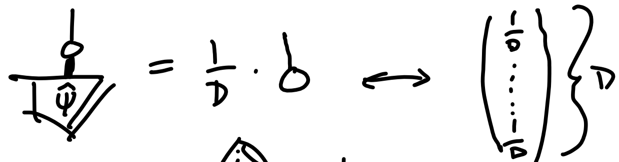
Ex Decoherence:



Lecture 16

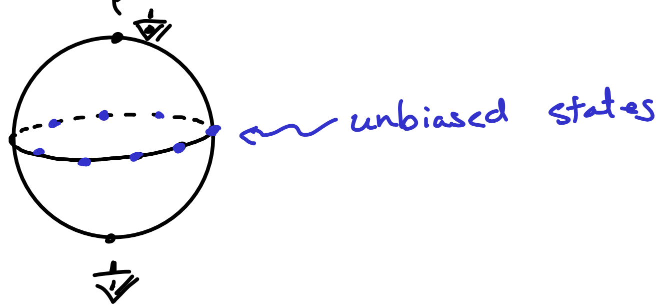
Q: What is "quantumness"?

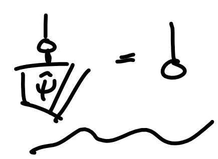
An <u>unbiased</u> state $\hat{\psi}$ is a state where:



equivalently, for all i:  $= \frac{1}{D}$.

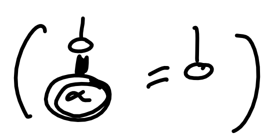
e.g. on the Bloch sphere:

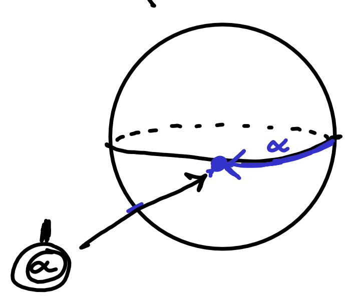


<u>DEF</u> A <u>phase state</u> is a state that sats  .

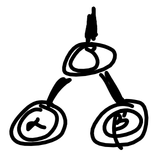
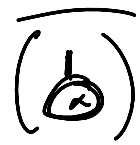
(equiv. a phase state is D · (unbiased state))

<u>NEW NOTATION</u> write phase states (in 2D) as  for $\alpha \in [0, 2\pi)$.

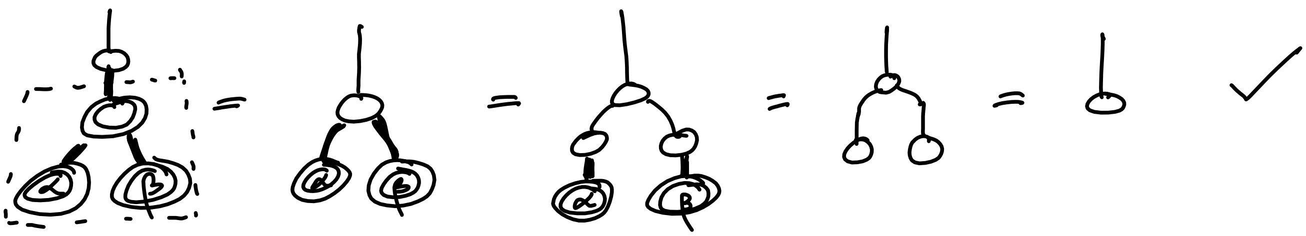


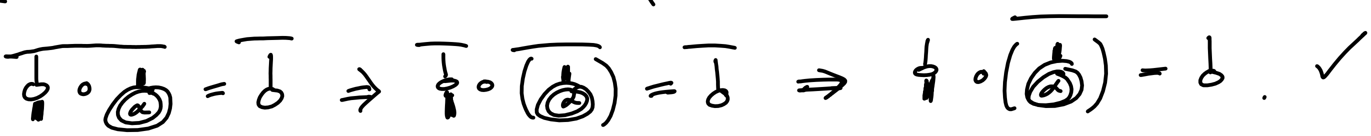


Thm If  &  are phase states, then:

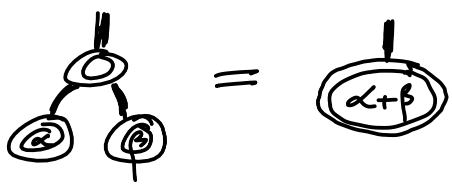
 and  are also phase states.

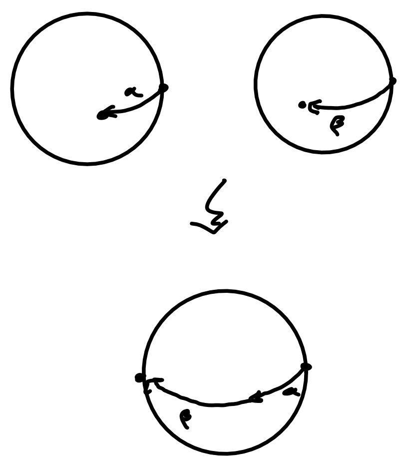
Pf

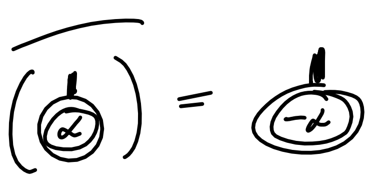


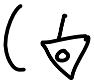


Prop In 2D:

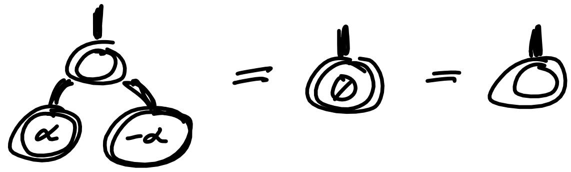






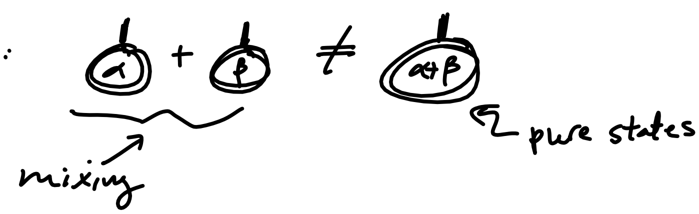
 := double( + $e^{i\alpha}$ )

∴  := double( + ) = double() = 

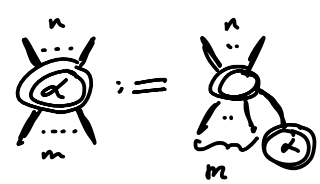
COR 

⇒ Phase states form a group, called the phase group.

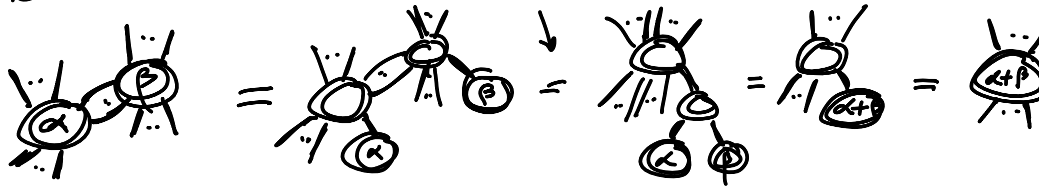
For 2D, this group is the circle group. (aka. S_1 or $U(1)$ — 1×1 unitary matrices ($e^{i\alpha}$))

NOTE: 

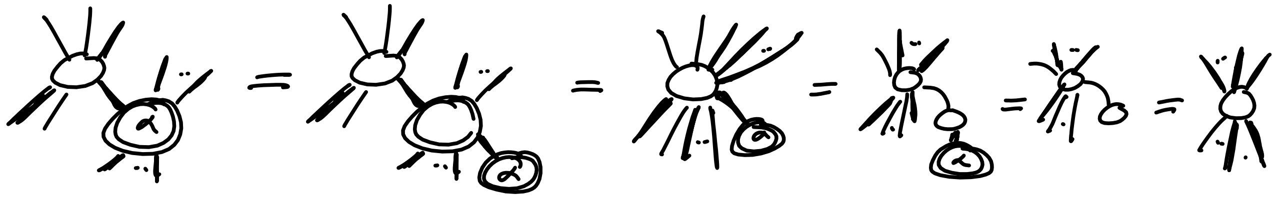
Another species of spider:

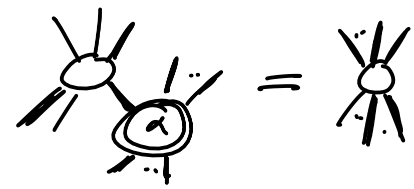
DEF A phase spider is a pure q-map: 

PHASE SPIDER FUSION:

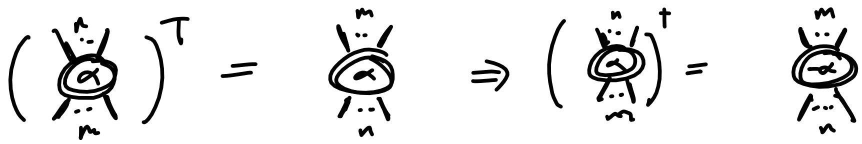


MOTTO: Contact w/ classical world destroys phases (q. data).

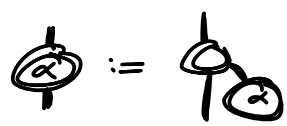


⇒ 

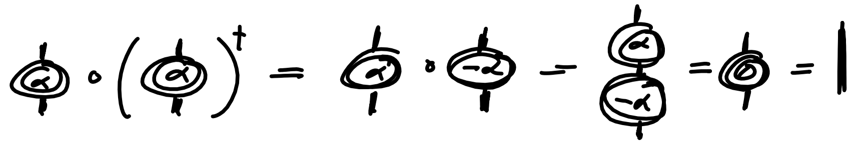
n.b. PHASE SPIDERS SAT:

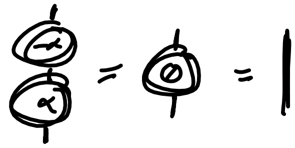


PHASE GATES:

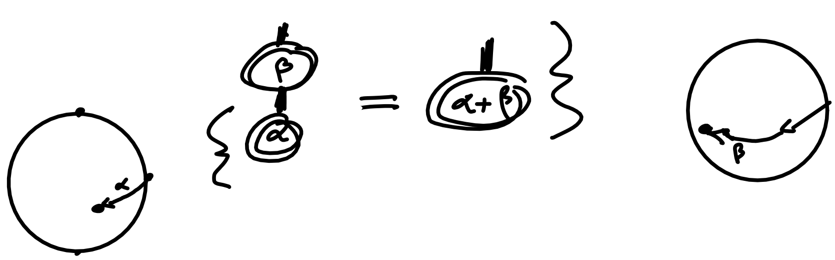


... are unitary:





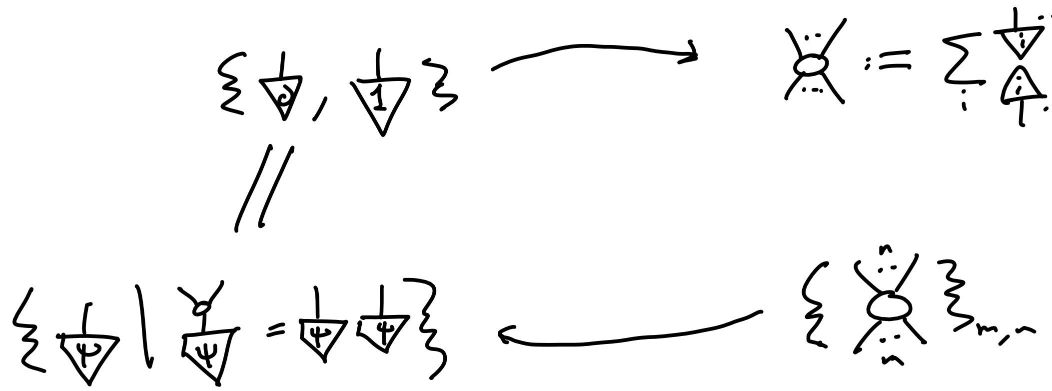
⇒ In 2D phase gates are rotations of the Bloch sphere.

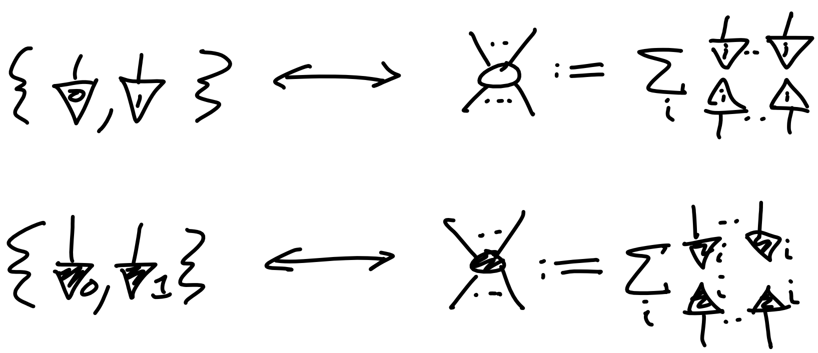


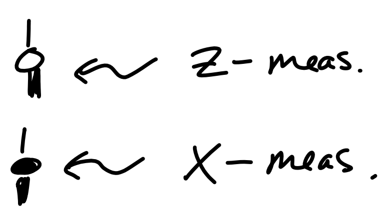
∴  is a rotation by β of the Bloch sphere around the Z axis.

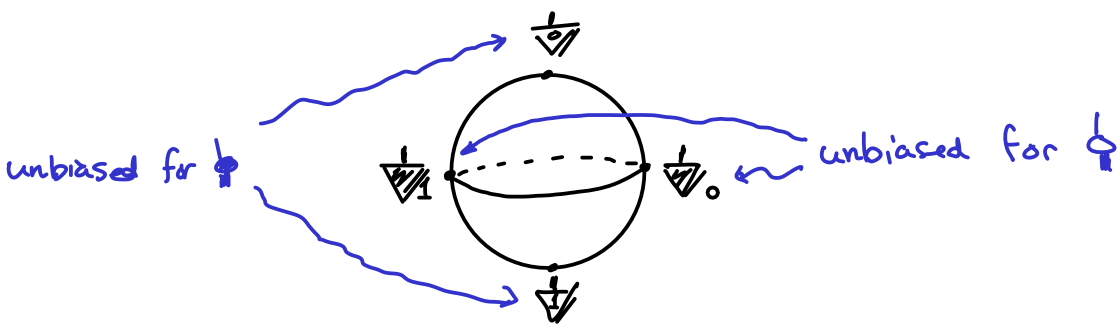
9.2 Complementarity.

<u>Thm</u> (8.41) Families of spiders are in 1-to-1 correspondence with ONB's.

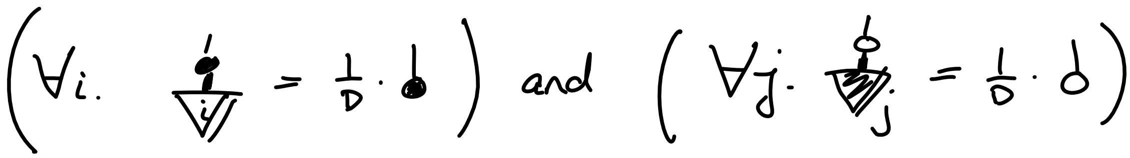


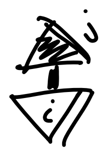




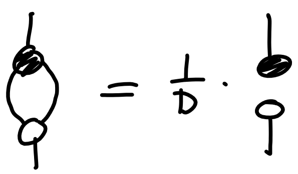


DEF Two ONB's are mutually unbiased (m.u.b.'s) if

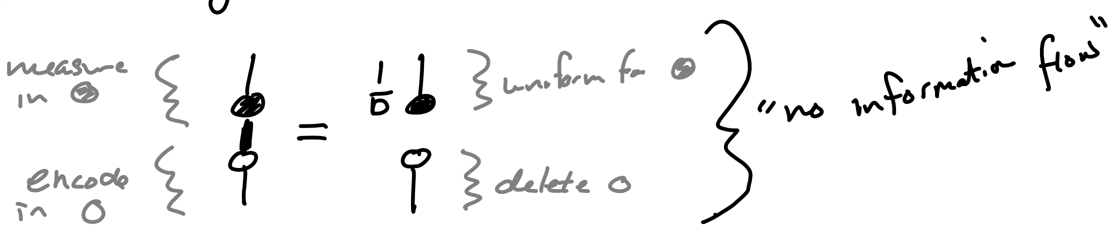


equiv. ∀ij.  $= \frac{1}{D}$.

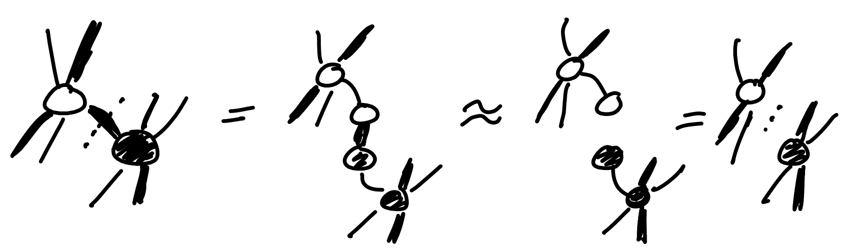
DEF Two families of spiders ○ and ● are <u>complementary</u> if:



equivalently:

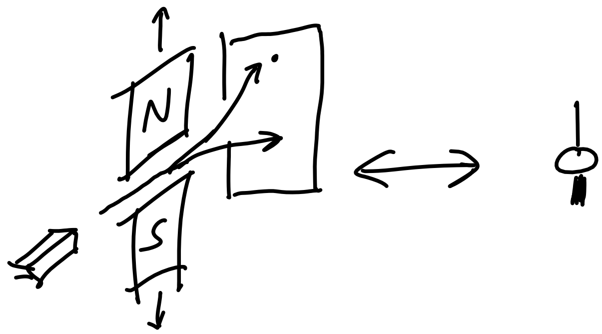


THM Two ONB's are <u>mut. unbiased</u> iff their spiders are <u>complementary</u>.

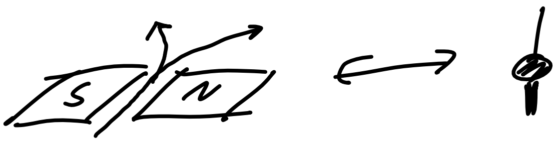
COMPLEMENTARITY := 

Lecture 17

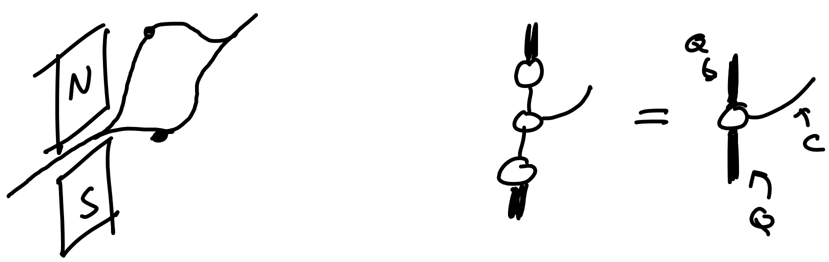
EXAMPLE Stern-Gerlach.



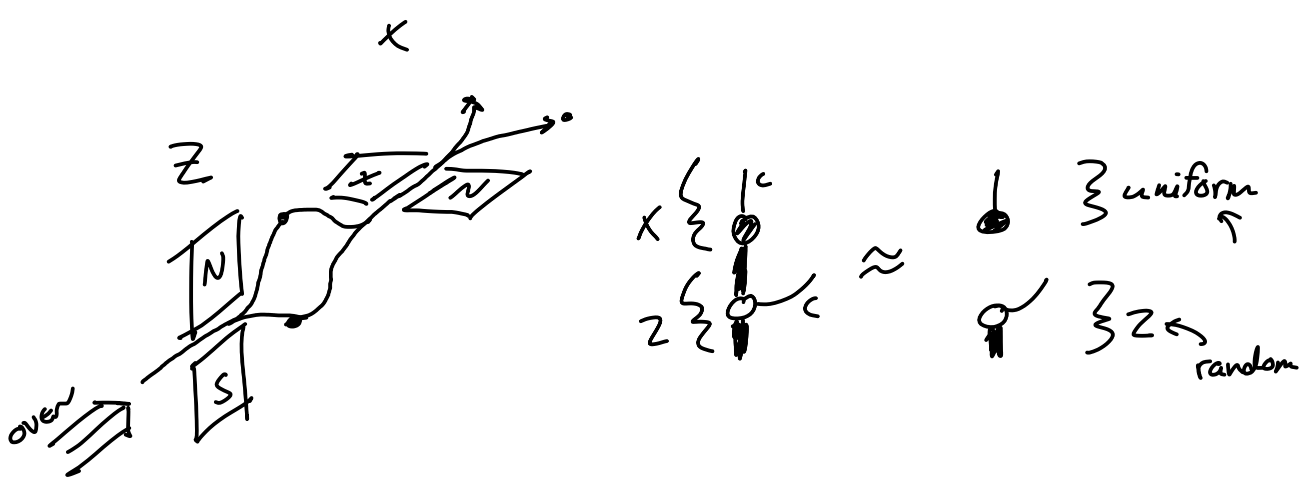
demolition Z-measurement



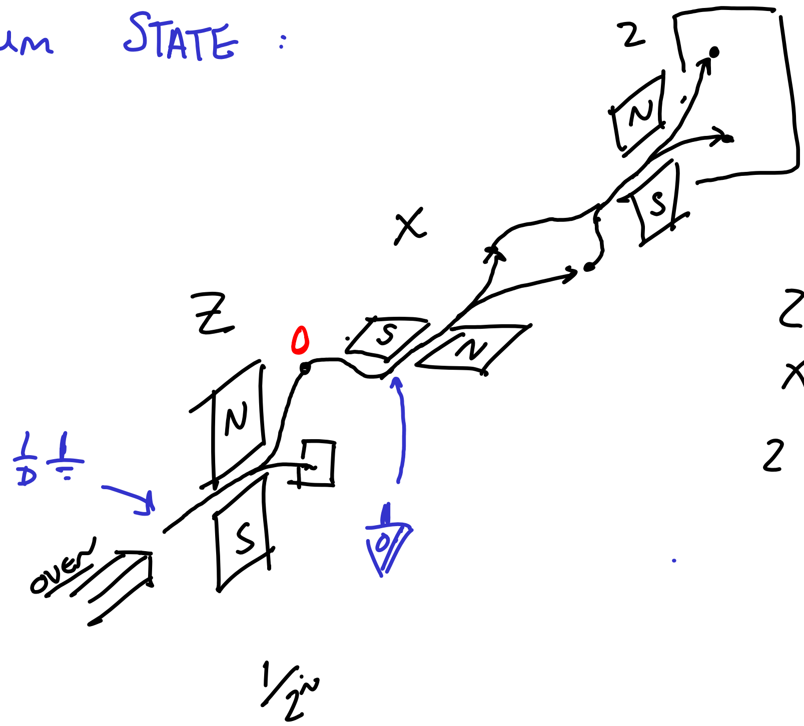
X-measurement

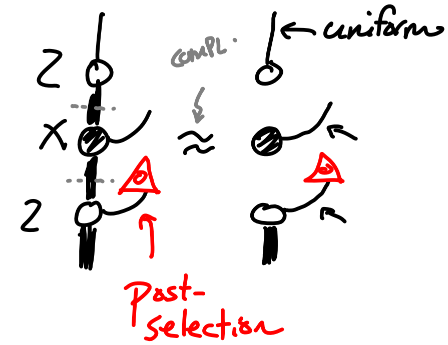


non-demolition Z-measurement

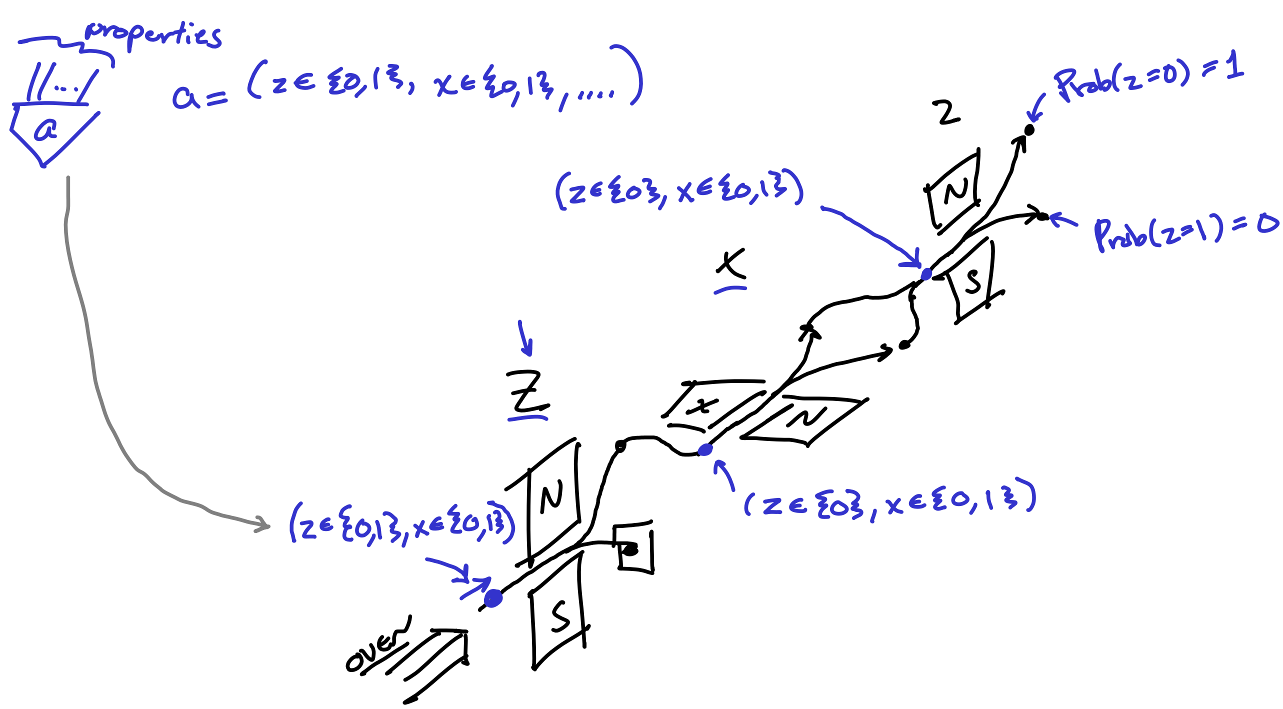


QUANTUM STATE:





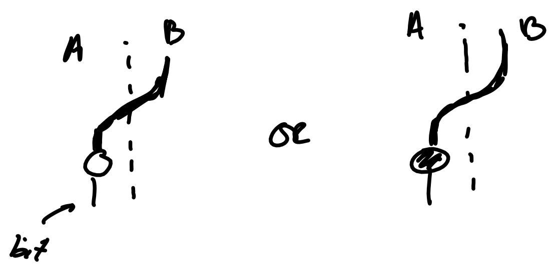
CLASSICAL STATE:



EXAMPLE Quantum key distribution (q. crypto)

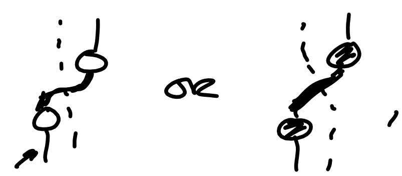
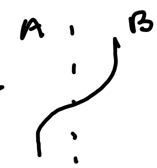
A wants to send a (private) bit to B.

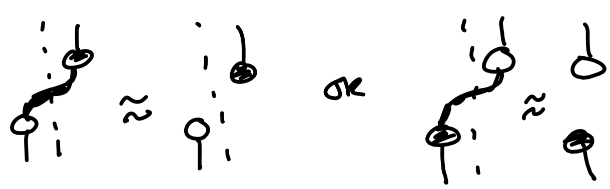
Step 1 A randomly chooses to do:



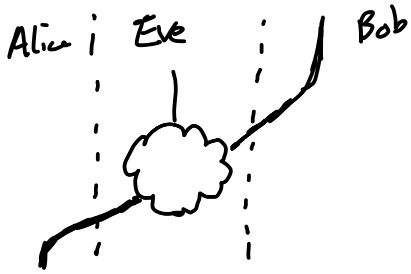
Step 2 B randomly chooses:

Step 3 A and B broadcast their basis choice.

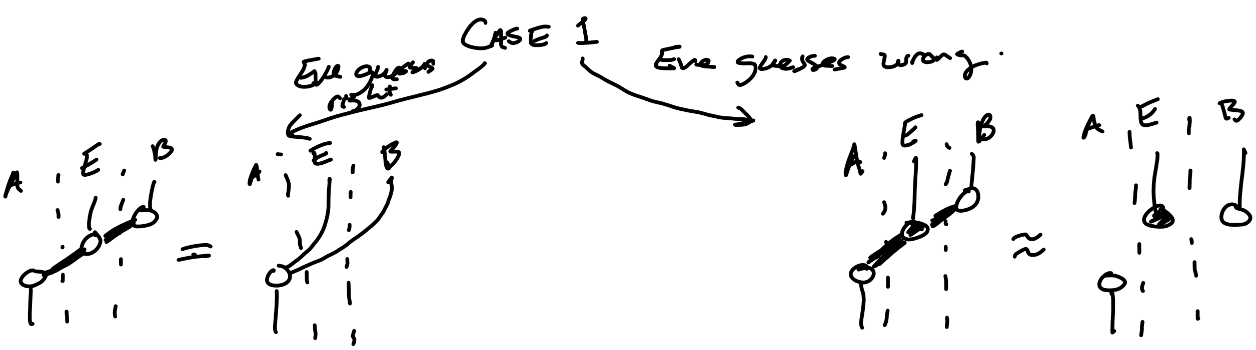
50% CASE 1 SAME CHOICE:  , which both = 

50% CASE 2 Different choice: 

Q What happens if ....



CASE 1 is the only one that matters.



E&B both get A's bit. E&B both get noise!

Step 4 A & B compare a random selection of their private (CASE 1) bits. Too much noise ⟹ eavesdropper!

[BB'84]

Bennett → B, Brassard → B