

Lecture 12

Classical-quantum interaction

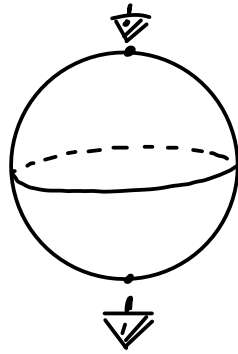
WRITE A Q. PROCESS $\{ \square, \dots, \square \}$ AS $\underbrace{\left(\square_i \right)_i}_{\sum_i \square_i = \mathbb{I}}$.

ONB measurements. :=

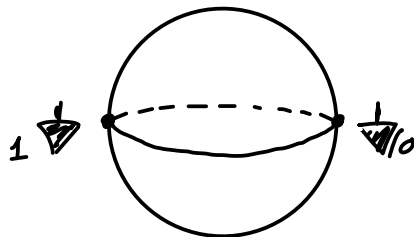
q. processes of the form $\left(\square_i \right)_i$ ← classical data
 where $\{ \square_i \}_i$ is an ONB. ← q. system

For QUBITS:

Z-basis := $\{ \downarrow_0, \downarrow_1 \}$



X-basis := $\{ \downarrow_0 := \frac{1}{\sqrt{2}}(\downarrow_0 + \downarrow_1), \downarrow_1 := \frac{1}{\sqrt{2}}(\downarrow_0 - \downarrow_1) \}$



Inner products: $\langle \downarrow_0 | \downarrow_0 \rangle = \frac{1}{\sqrt{2}} \left[\langle \downarrow_0 | \downarrow_0 \rangle + \langle \downarrow_0 | \downarrow_1 \rangle \right] = \frac{1}{\sqrt{2}}$

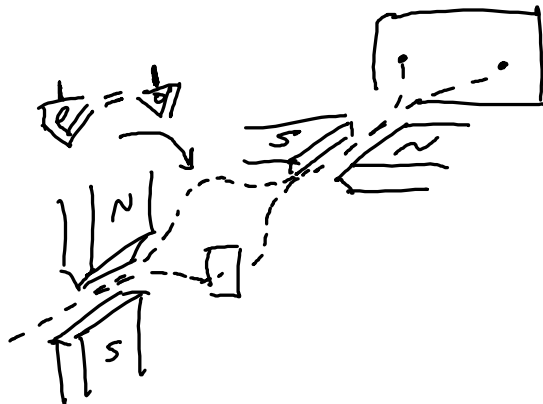
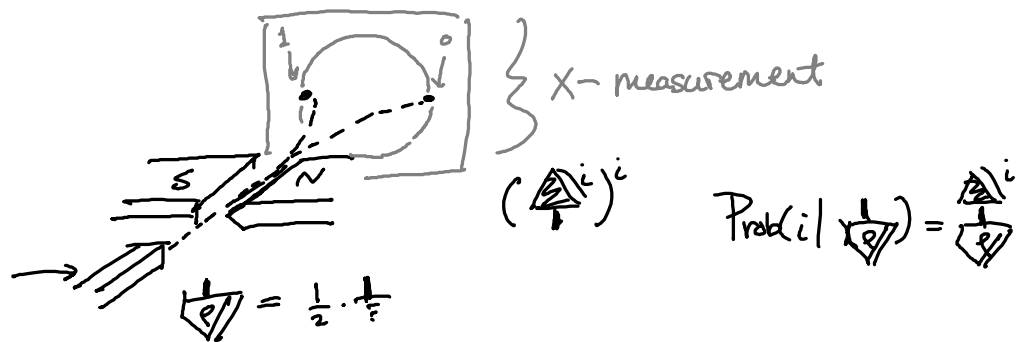
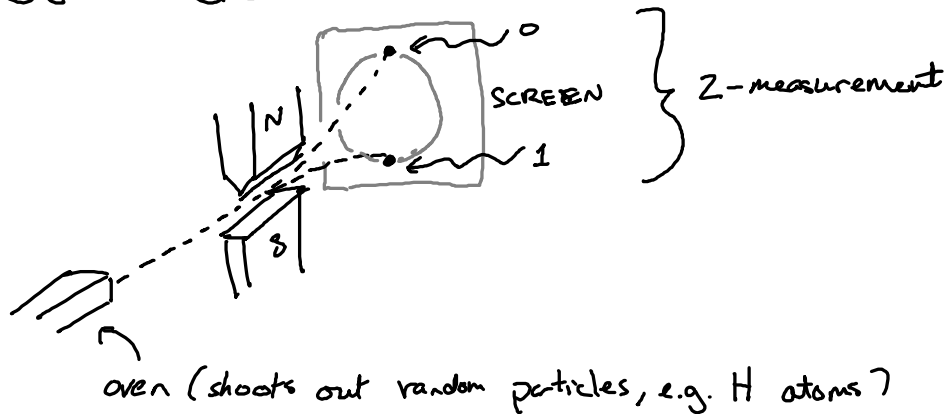
X-meas. of state \downarrow_i :

$$\text{Prob}(x=0 | \downarrow_i) = \left| \langle \downarrow_0 | \downarrow_i \rangle \right|^2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$\forall i, j: \langle \downarrow_j | \downarrow_i \rangle = \pm \frac{1}{\sqrt{2}} \Rightarrow \text{Prob}(x=j | \downarrow_i) = \left| \langle \downarrow_j | \downarrow_i \rangle \right|^2 = \frac{1}{2}$$

mutually unbiased bases.

EXAMPLE: Stern-Gerlach device.



6.4.2 Getting any q-map as part of a non-det. quantum process.

LEM (6.94) For any \hat{A} , there exists a quantum process

$$\left(\begin{array}{c} \hat{\phi}_i \\ \downarrow \\ \downarrow \end{array} \right)^i \text{ where } \begin{array}{c} \hat{\phi}_i \\ \downarrow \\ \downarrow \end{array} = \frac{1}{D} \cdot \text{cup} \text{ and } D = \dim A.$$

Proof First, let $\frac{1}{\sqrt{D}} \cdot U$ be the normalised cup.

$$\frac{1}{\sqrt{D}} \begin{array}{c} \text{cup} \\ \downarrow \\ \downarrow \end{array} = \frac{1}{D} \begin{array}{c} \text{cup} \\ \downarrow \\ \downarrow \end{array} \stackrel{\text{eg. (5.17)}}{=} \frac{1}{D} \cdot D = 1.$$

Then, any normalised state is part of an ONB (Prop. 5.79)

$$\text{let } \mathcal{B} = \left\{ \begin{array}{c} \downarrow \\ \hat{\phi}_1 \\ \downarrow \end{array} = \frac{1}{\sqrt{D}} U, \begin{array}{c} \downarrow \\ \hat{\phi}_2 \\ \downarrow \end{array}, \dots, \begin{array}{c} \downarrow \\ \hat{\phi}_D \\ \downarrow \end{array} \right\}.$$

Then: $\left(\begin{array}{c} \hat{\phi}_i \\ \downarrow \\ \downarrow \end{array} \right)^i$ is an ONB meas, hence a quantum process. \square

Ex In 2D, $\hat{A} = \mathbb{C}^2$, we can use the Bell basis.

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \begin{array}{c} \downarrow \\ \hat{\phi}_1 \\ \downarrow \end{array}, \frac{1}{\sqrt{2}} \begin{array}{c} \downarrow \\ \hat{\phi}_2 \\ \downarrow \end{array}, \frac{1}{\sqrt{2}} \begin{array}{c} \downarrow \\ \hat{\phi}_3 \\ \downarrow \end{array}, \frac{1}{\sqrt{2}} \begin{array}{c} \downarrow \\ \hat{\phi}_4 \\ \downarrow \end{array} \right\}$$

Bell matrices: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

So $\left(\frac{1}{2} \int \boxed{\Phi_i}\right)^i$ is a Bell measurement, and $\int \boxed{\Phi_i} = \int$.

Thm For any q. map $\boxed{\Phi}$ (not nec. causal), there exists a q. process $(\boxed{\Phi_i})^i$ such that $\boxed{\Phi} = r \cdot \boxed{\Phi}$.

Proof $\boxed{\Phi}$ q. map \rightsquigarrow $\int \boxed{\Phi}$ q. state.

Suppose: $\int \boxed{\Phi} = k$, then $\frac{1}{k} \int \boxed{\Phi} = 1 \Rightarrow \frac{1}{k} \int \boxed{\Phi}$ causal quantum state.

Then, let: $\boxed{\Phi} = \frac{1}{k} \int \boxed{\Phi_i}$ (from Lem 6.94)

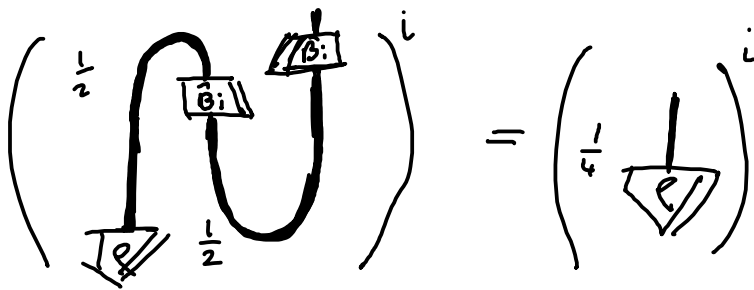
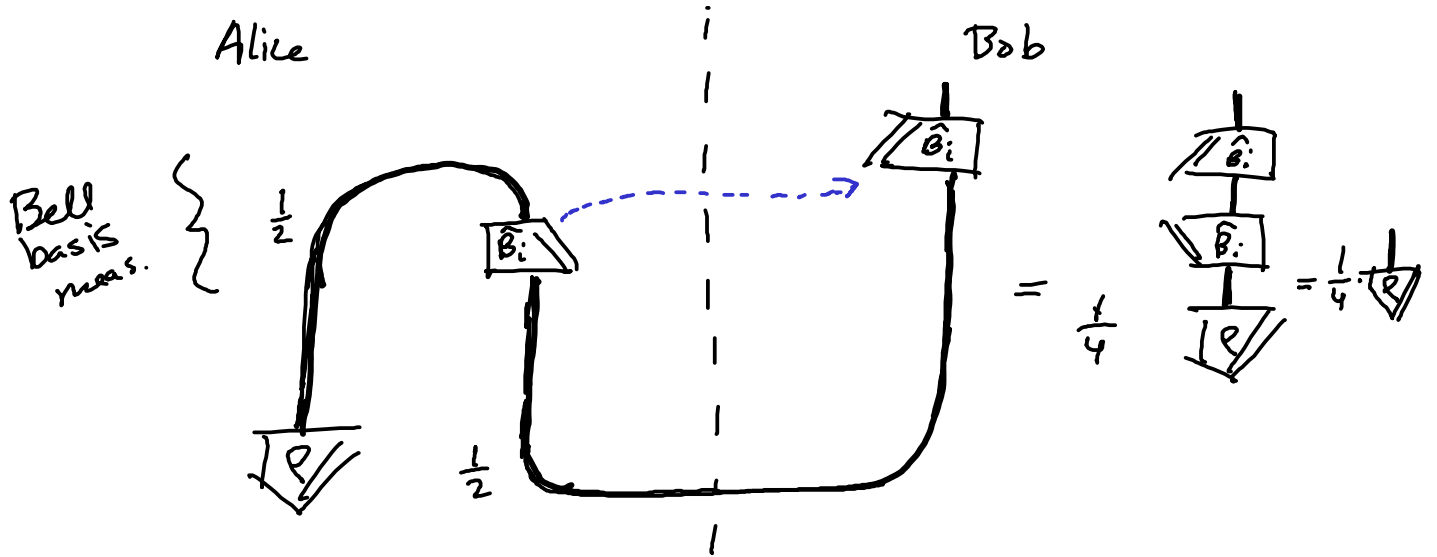
So:

$$\sum_i \int \boxed{\Phi_i} = \sum_i \frac{1}{k} \int \boxed{\Phi_i} = \frac{1}{k} \int \left(\sum_i \boxed{\Phi_i} \right) = \frac{1}{k} \int \int \boxed{\Phi_i} = \int$$

$\Rightarrow (\int \boxed{\Phi_i})^i$ are a quantum process.

$$\text{and } \boxed{\Phi} = \frac{1}{k} \int \int \boxed{\Phi_i} = \frac{1}{kD} \int \boxed{\Phi} = r \cdot \boxed{\Phi} \quad \square$$

EXAMPLE QUANTUM TELEPORTATION.



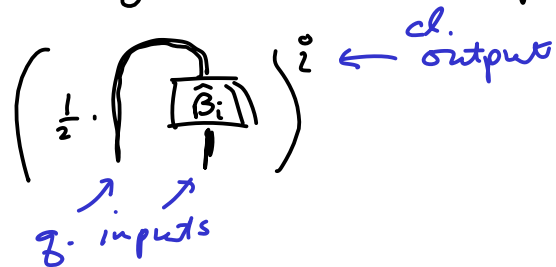
Don't care about $i \Rightarrow \sum_{i=0}^3 \frac{1}{4} \cdot |e\rangle = |e\rangle$

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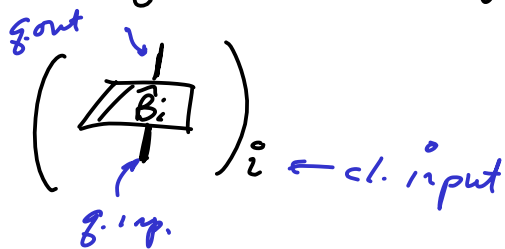
Ch. 8: Classical-quantum processes.

In teleportation, classical data plays 2 different roles:

— An output of a non-det proc:



— An input of a classically-controlled process.



We write these differently because the causality cond'n is different.

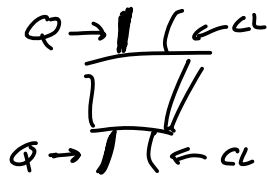
$$\text{CL. OUTPUT: } \sum_i \frac{1}{2} \left(\beta_i \right)_i = \dot{\tau} \dot{\tau}$$

$$\text{CL. INPUT } \forall i, \left(\beta_i \right)_i = \dot{\tau} \quad \left(\beta_i \right)_i = \dot{\tau}$$

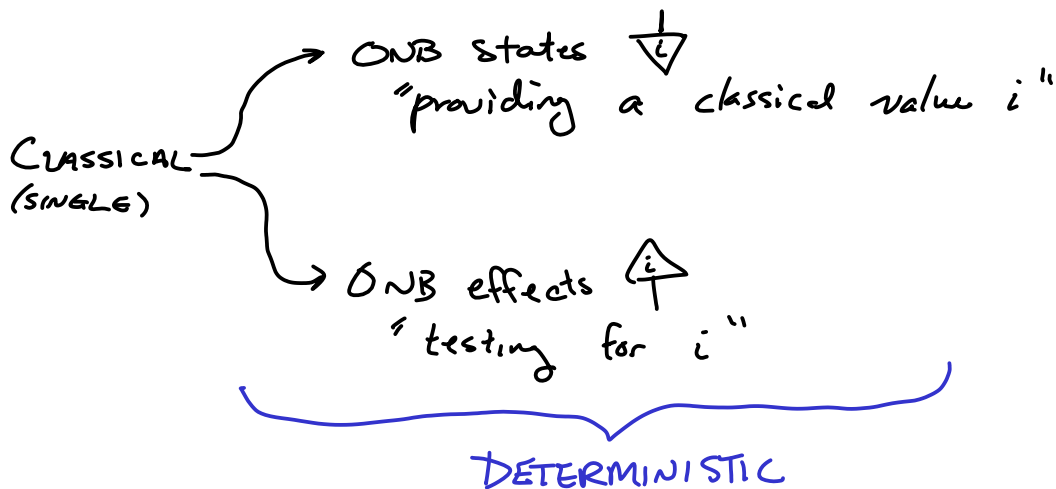
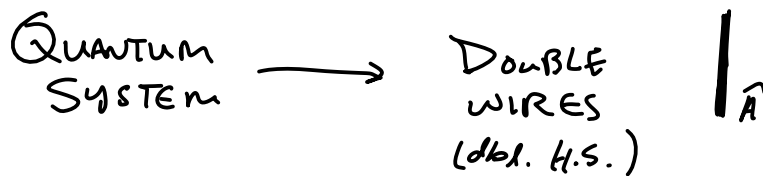
IN GENERAL: $\left(\beta_{ij} \right)_i$ where $\forall i, \sum_j \left(\beta_{ij} \right)_i = \dot{\tau}$

The diagram for $\left(\beta_{ij} \right)_i$ shows a box with a vertical line through it. Two arrows labeled "q. input" point into the box from the bottom. A vertical arrow labeled "q. output" points out of the top of the box. A horizontal arrow labeled "cl. output" points out of the right side of the box. The entire diagram is enclosed in large parentheses with a subscript i .

Q: Why can't everything be wires?



A: It can be!



MORE GENERALLY: WE CAN HAVE PROBABILISTIC STATES.

$$\begin{array}{c} \downarrow A \\ \triangle P \end{array} := \sum_i p_i \begin{array}{c} \downarrow A \\ \triangle i \end{array} \longleftrightarrow \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

probabilities.

$$\{ \frac{1}{2} \begin{array}{c} \downarrow \\ \square \end{array}, \frac{1}{2} \begin{array}{c} \downarrow \\ \square \end{array} \} \rightsquigarrow \sum_i \frac{1}{2} \begin{array}{c} \downarrow \\ \square \end{array} \downarrow i$$

$$\left(\begin{array}{c} \downarrow \\ \square \end{array} \right)^i \rightsquigarrow \begin{array}{c} \downarrow \\ \square \end{array} := \sum_j \begin{array}{c} \downarrow \\ \square \end{array} \downarrow j$$

Then $\begin{array}{c} \triangle \\ \downarrow \\ \square \end{array} = \sum_j \begin{array}{c} \downarrow \\ \square \end{array} \downarrow j \triangle = \begin{array}{c} \downarrow \\ \square \end{array}$

$$\sum_i \begin{array}{c} \downarrow \\ \square \end{array} = \downarrow$$

//

$$\sum_i \begin{array}{c} \downarrow \triangle \\ \square \end{array} = \begin{array}{c} \downarrow \triangle \\ \square \end{array} =: \begin{array}{c} \downarrow \circ \\ \square \end{array}$$

CAUSALITY:

$$\begin{array}{c} \downarrow \circ \\ \square \end{array} = \downarrow$$

WHERE $\circ := \sum_i \triangle$ is called deleting. $\left(\begin{array}{c} \downarrow \circ \\ \downarrow j \end{array} = \sum_i \begin{array}{c} \downarrow \triangle \\ \downarrow j \end{array} = 1 = \square \right)$

WITH CL. CONTROL:

$$\left(\begin{array}{|c|} \hline \Phi_{ij} \\ \hline \end{array} \right)_i^j \rightsquigarrow \begin{array}{|c|} \hline \Phi \\ \hline \end{array} = \sum_{ij} \begin{array}{|c|} \hline \Phi_{ij} \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{|c|} \hline \Phi_{ij} \\ \hline \end{array} = \begin{array}{|c|} \hline \Phi \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array}$$


Prop (8.9) For $\begin{array}{|c|} \hline \Phi \\ \hline \end{array} = \sum_{ij} \begin{array}{|c|} \hline \Phi_{ij} \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \end{array}$:



$$\underbrace{\forall i \sum_j}_{\downarrow} \begin{array}{|c|} \hline \Phi_{ij} \\ \hline \end{array} = \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \iff \begin{array}{|c|} \hline \Phi \\ \hline \end{array} = \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \left(\begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \right)$$

8.13 Measurement + encoding:



MEASURE: $\left(\begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \right)_i^i \rightsquigarrow \sum_i \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} = \begin{array}{|c|} \hline \downarrow \\ \hline \end{array}$

$$\begin{array}{|c|} \hline \downarrow \\ \hline \end{array} = \sum_i \underbrace{\begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \begin{array}{|c|} \hline \downarrow \\ \hline \end{array}}_{\text{Born rule probs}} = \sum_i \text{Prob}(i|e) \cdot \begin{array}{|c|} \hline \downarrow \\ \hline \end{array} \iff \begin{pmatrix} \text{Prob}(1|e) \\ \vdots \\ \text{Prob}(n|e) \end{pmatrix}$$

 \therefore Q-STATES \longrightarrow Prob. DIST FOR $(\uparrow)_i$ OUB meas.

In particular:  =  \leftarrow point distr. at i .
 $\text{Prob}(i|p) = 1$
 $\text{Prob}(j \neq i|p) = 0$.

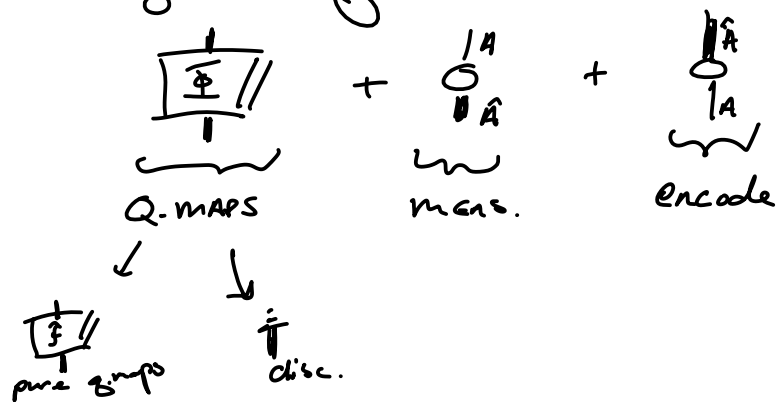
ENCODE: $(\downarrow_i)_e \rightsquigarrow \sum_i \downarrow_i \uparrow_i =: \uparrow$

 = 

DEF The process theory of classical-quantum maps has:

types $\begin{cases} \rightarrow \text{Q. systems (dbl. H.s. } \hat{A} \text{)} \\ \rightarrow \text{CL. systems (H.s. } A \text{)} \end{cases}$

procs: diagrams of:



PROP: $\boxed{\Phi}$ ARE CQ-MAPS!

PF:

$$\boxed{\Phi} := \sum_j \boxed{\Phi_j} \begin{array}{c} \downarrow \\ \uparrow \end{array} = \underbrace{\sum_j \boxed{\Phi_j} \begin{array}{c} \downarrow \\ \uparrow \end{array}}_{\text{Q.MAP}} = \boxed{\Phi'} \quad \square$$

PREFERRED NOTATION FOR CQ-MAPS

CAUSAL QUANTUM MAPS

$$\boxed{\Phi} = \dot{\downarrow}_A$$

QUANTUM PROCESSES := CAUSAL CQ-MAPS

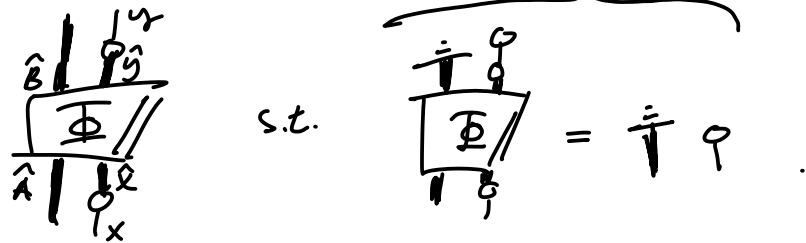
$$\boxed{\Phi} \text{ s.t. } \boxed{\Phi} = \dot{\downarrow} \circ \uparrow$$

DELETING $\rightarrow \dot{\downarrow} \circ \uparrow = \circ \quad \circ \circ = \dot{\downarrow}$

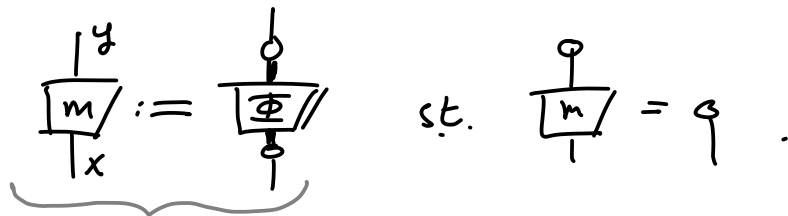
$$\begin{array}{l} \dot{\downarrow} \circ \uparrow = \sum_i \uparrow \circ \downarrow = \sum_i \downarrow \uparrow = \circ \\ \circ \circ = \sum_i \downarrow \circ \uparrow = \sum_i \uparrow \downarrow = \dot{\downarrow} \end{array}$$

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Quantum processes := causal cq-maps

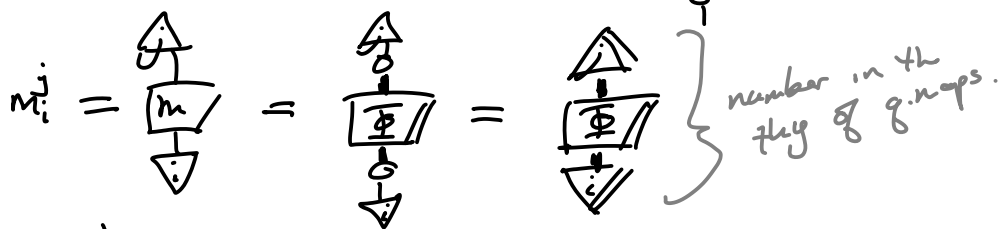


Special case: Classical processes := causal cq-maps w/ no quantum inputs/outputs.



LEM $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \text{---} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} := \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \Phi \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$ for some q.map Φ if & only if $m_i^j \geq 0$.

Pf For (\Rightarrow) assume \exists q.map Φ . $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \text{---} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \Phi \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$. Then:



$\Rightarrow m_i^j$ is a positive real number.
(non-negative)

For (\Leftarrow), assume $\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \text{---} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$ is a linear map where $m_i^j \geq 0 \forall i, j$.

Then $\exists \lambda_i^j \in \mathbb{C}$ such that $m_i^j = \overline{\lambda_i^j} \lambda_i^j$. $\lambda_i^j = \sqrt{m_i^j}$
 $\overline{\lambda_i^j} = \lambda_i^j$

Let \boxed{f} be a linear map w/ matrix (λ_i^j) .

$$\boxed{f} = \lambda_i^j$$

Then:

$$\begin{array}{c} \triangle \\ \downarrow \\ \circ \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \circ \\ \downarrow \\ \triangle \end{array} = \begin{array}{c} \triangle \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \triangle \end{array} = \begin{array}{c} \triangle \quad \triangle \\ \downarrow \quad \downarrow \\ \underbrace{\triangle} \quad \underbrace{\triangle} \\ \downarrow \quad \downarrow \\ \lambda_i^j \quad \lambda_i^j \end{array} = \overline{\lambda_i^j} \cdot \lambda_i^j = m_i^j := \begin{array}{c} \triangle \\ \downarrow \\ \boxed{m} \\ \downarrow \\ \triangle \end{array}$$

Since $\{ \triangle \} + \{ \triangle \}$ are bases for $A+B$

$$\Rightarrow \boxed{\hat{f}} = \boxed{m}$$

□

Thm \boxed{m} is a causal classical map (= classical process) if

and only if its matrix has:

- (i) positive entries $m_i^j \geq 0$.
- (ii) each column summing to 1. $\forall i. \sum_j m_i^j = 1$. $\left(\begin{array}{c} \triangle \\ \downarrow \\ \square \\ \downarrow \\ \triangle \end{array} \right)^{\sum=1}$

Pf (i) is lemma. For (ii), assume $\boxed{m} \neq \rho$, then:

$$\forall i. \sum_j m_i^j := \sum_j \boxed{m} \rightarrow \begin{array}{c} \triangle \\ \downarrow \\ \boxed{m} \\ \downarrow \\ \triangle \end{array} \neq \begin{array}{c} \circ \\ \downarrow \\ \triangle \end{array} = 1$$

For the converse, assume $\forall i: \sum_j m_i^j = 1$. Then:

$$\forall i: \begin{array}{c} \circ \\ \square \\ m \\ \downarrow \\ \triangle \\ i \end{array} = \sum_j \begin{array}{c} \circ \\ \square \\ m \\ \uparrow \\ \triangle \\ i \end{array} =: \sum_j m_i^j = 1 = \begin{array}{c} \circ \\ \square \\ 1 \\ \downarrow \\ \triangle \\ i \end{array}.$$

Since $\{\begin{array}{c} \circ \\ \square \\ 1 \\ \downarrow \\ \triangle \\ i \end{array}\}_i$ is a basis, $\begin{array}{c} \circ \\ \square \\ m \\ \downarrow \\ \triangle \\ i \end{array} = \begin{array}{c} \circ \\ \square \\ 1 \\ \downarrow \\ \triangle \\ i \end{array}$. □

(i) positive entries $m_i^j \geq 0$.
 (ii) each column summing to 1. $\forall i: \sum_j m_i^j = 1$. $\left(\begin{array}{c} \circ \\ \square \\ \vdots \\ \square \\ \downarrow \\ \triangle \\ i \end{array} \right)$

Stochastic maps

Conditional prob. distr.
 $m_i^j = P(j|i)$
 (input i , output j)

Ex Classical states $\begin{array}{c} \circ \\ \square \\ p \\ \downarrow \\ \triangle \\ i \end{array} := \begin{array}{c} \circ \\ \square \\ p \\ \downarrow \\ \triangle \\ i \end{array}$ s.t. $\begin{array}{c} \circ \\ \square \\ 1 \\ \downarrow \\ \triangle \\ i \end{array} = 1$ are
 prob. distributions.
 $p^i = P(i)$.

Ex Suppose a process takes a bit as input and flips it w/ prob $\frac{1}{3}$.

Matrix: $f_0^0 = \frac{2}{3}$ $f_0^1 = \frac{1}{3}$ $f_1^0 = \frac{1}{3}$ $f_1^1 = \frac{2}{3}$

$$\boxed{f} \leftrightarrow \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{matrix} 0 \\ 1 \end{matrix}$$

Suppose I have a bit that is 0 w/ prob. $\frac{1}{4}$ and 1 w/ prob $\frac{3}{4}$. Then:

prior $\left\{ \begin{matrix} \downarrow \\ P \end{matrix} \right\} \leftrightarrow \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} \begin{matrix} 0 \\ 1 \end{matrix}$

$$\begin{aligned} \frac{2}{12} + \frac{3}{12} &= \frac{5}{12} \\ \frac{1}{12} + \frac{6}{12} &= \frac{7}{12} \end{aligned}$$

Posterior $\left\{ \begin{matrix} \downarrow \\ \boxed{f} \\ \downarrow \\ P \end{matrix} \right\} \leftrightarrow \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{12} \\ \frac{7}{12} \end{pmatrix}$

$$\begin{aligned} &\sum_i f_i^j p^i \\ &\hookrightarrow \sum_i P(j|i)P(i) \end{aligned}$$

DEF A classical process is deterministic if \exists a

function $F: \mathcal{B}_A \rightarrow \mathcal{B}_B$
 $\{1, \dots, m\} \quad \{1, \dots, n\}$

where $\begin{matrix} \downarrow \\ \boxed{f} \\ \downarrow \\ P \end{matrix} = \begin{matrix} \downarrow \\ \boxed{F(i)} \\ \downarrow \\ P \end{matrix}$

\boxed{f} is a det. classical proc

$$\Leftrightarrow \forall i. \boxed{f} = \boxed{F(i)}$$

$$\Leftrightarrow \boxed{f} = \sum_i \boxed{F(i)}$$

\Leftrightarrow matrix of f has one 1 in every column.

e.g.

$$\boxed{f} \leftrightarrow \begin{matrix} 00 & 01 & 10 & 11 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix}$$

$$\boxed{f} = \boxed{F(i,j)}$$

where $F(i,j) := \text{Xor}(i,j)$

$$F: \{ \downarrow\downarrow, \downarrow\downarrow, \downarrow\downarrow, \downarrow\downarrow \} \rightarrow \{ \downarrow\downarrow \}$$

8.2.2 Copying + deleting.

$$F: \mathcal{B}_A \rightarrow \mathcal{B}_{A \otimes A}$$

$$F(i) = (i,i)$$

$$\boxed{\text{Copy}} = \downarrow_i \downarrow_i$$

NEW NOTATION: $\text{Y} := \text{copy}$

NOTE: $\left(\text{Y} = \downarrow_i \uparrow_i = \downarrow_i \right) \Rightarrow \text{Y} = |$


SIMILARLY: $\text{Y} = |$

Sim. $\text{Y} = \text{Y} \quad \left(\text{Y} = \text{Y} = \downarrow_i \downarrow_i \downarrow_i \right)$

$\text{Y} = \sum_i \downarrow_i \uparrow_i \quad (\text{Y})^T = \text{Y} = \sum_i \downarrow_i \uparrow_i$
 $(\text{Y})^+ = \text{Y} = \sum_i \downarrow_i$

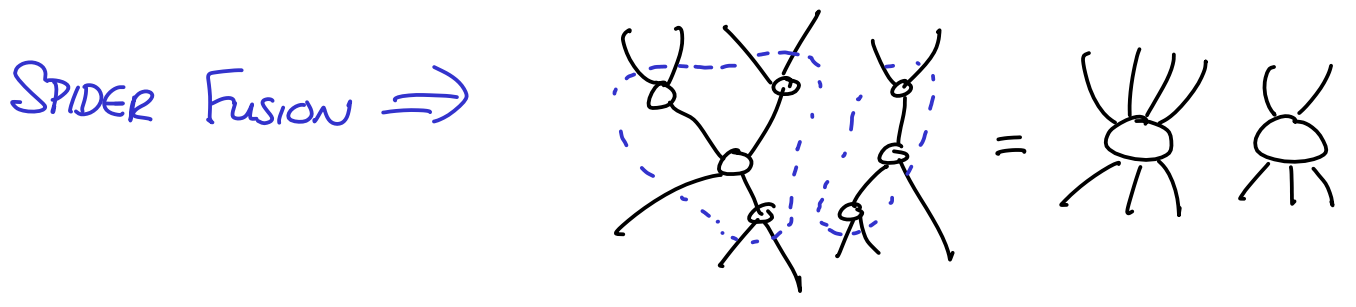
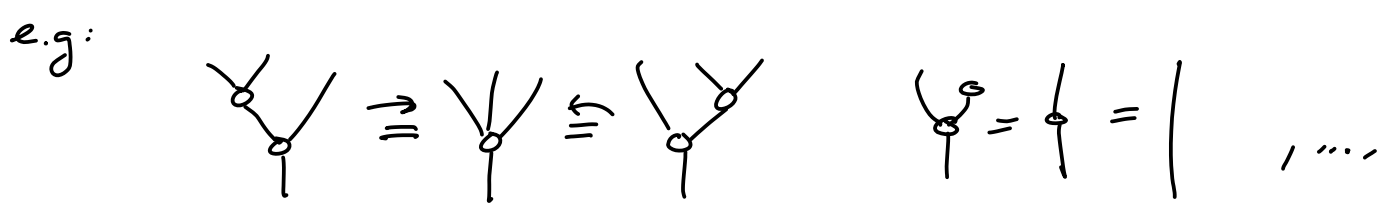
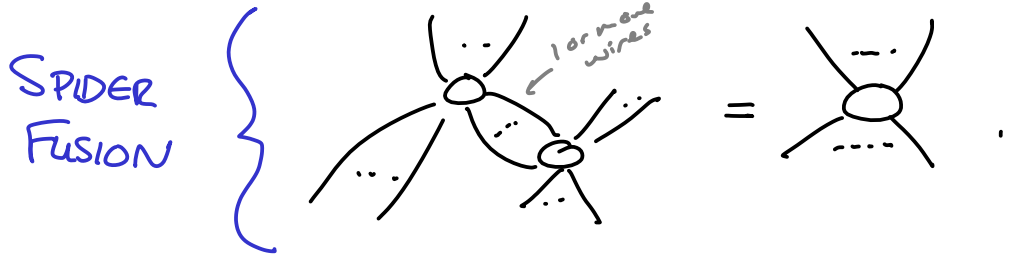
Sim.: $\left[\text{Y} \right] = \sum_{ij} \downarrow_i \uparrow_j = \sum_{ij} \downarrow_i \uparrow_j = \sum_{ij} \downarrow_i \uparrow_j = \sum_{ij} \downarrow_i \uparrow_j$

Sim.: $\text{Y} = | = \text{Y} \quad \text{Y} = \text{Y}$

DEF A spider is a map:  = $\sum_i \begin{matrix} \downarrow & \dots & \downarrow \\ \uparrow & \dots & \uparrow \end{matrix}$.

(e.g. $\gamma, \delta, \rho, \sigma, \delta = \sum_i \begin{matrix} \downarrow \\ \uparrow \end{matrix} = |, \cup = \sum_i \begin{matrix} \downarrow \\ \uparrow \end{matrix} = U, \mathcal{R} = \cap$)

The only eqn. we need:



Lecture 15

LAST TIME:

deleting

$$\epsilon^1 = \sum_i \uparrow \leftrightarrow \overbrace{(11 \dots 1)}^1$$

copy

$$\epsilon^2 = \sum_i \downarrow \uparrow \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

classical process
a.k.a. stochastic map

$$\downarrow \downarrow + \downarrow \uparrow \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \downarrow \\ \downarrow \\ \uparrow \end{matrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

↑ ↑
2 copies

SPECIAL CASES OF SPIDERS

$$\text{Spider} = \sum_i \downarrow \dots \downarrow \uparrow \dots \uparrow$$

SPIDER FUSION

$$\text{Fusion of two spiders} = \text{Single spider} \quad \downarrow = | \quad \wedge = \cap \quad \vee = \cup$$

$$\text{Spider with multiple legs} = \text{Single spider} = \text{Spider with multiple legs}$$

TODAY USE SPIDERS TO BUILD OTHER CQ-MAPS, NOT JUST CLASSICAL MAPS.

MEASURE $\downarrow \circlearrowleft = \sum_i \downarrow \uparrow \triangle = \sum_i \downarrow \uparrow \uparrow = \downarrow \circlearrowleft$ (CLASSICAL / QUANTUM)

ENCODE $\downarrow \circlearrowleft = \sum_i \downarrow \triangle \uparrow = \downarrow \circlearrowleft$ (NOT SELF-CONS: $\cap \neq \rho$)

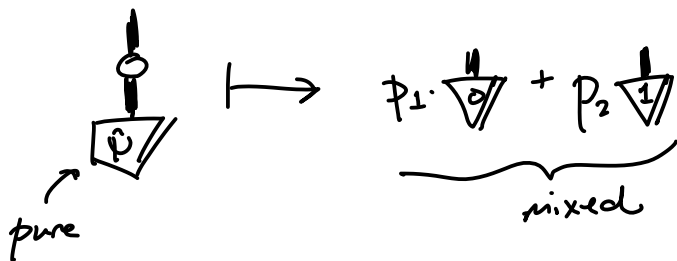
USING THESE DEFⁿS OF MEAS + ENC, WE SEE:

* $\downarrow \circlearrowleft \downarrow \triangle = \downarrow \circlearrowleft \downarrow \triangle = \downarrow \downarrow \downarrow \triangle = \downarrow \downarrow \triangle$

* $\downarrow \circlearrowleft \downarrow \circlearrowleft = \downarrow \circlearrowleft \downarrow \circlearrowleft \stackrel{\text{S.F.}}{=} \downarrow \circlearrowleft \downarrow \circlearrowleft = \downarrow \circlearrowleft \downarrow \circlearrowleft$

* $\downarrow \circlearrowleft \downarrow \circlearrowleft = \downarrow \circlearrowleft \downarrow \circlearrowleft \stackrel{\text{S.F.}}{=} \downarrow \circlearrowleft \downarrow \circlearrowleft = \downarrow \circlearrowleft \downarrow \circlearrowleft$

* $\downarrow \circlearrowleft \downarrow \circlearrowleft = \downarrow \circlearrowleft \downarrow \circlearrowleft = \downarrow \circlearrowleft \downarrow \circlearrowleft \neq \downarrow \circlearrowleft \downarrow \circlearrowleft = \downarrow \circlearrowleft \downarrow \circlearrowleft$ decoherence



* $\text{circle} = \text{circle} \Rightarrow \text{circle} \text{ is causal}$

* $\text{circle} =: \text{circle} = \text{circle} \stackrel{\text{SF}}{=} \text{circle} \Rightarrow \text{circle} \text{ is causal}$

DEF A quantum spider is:

$$\text{spider} := \text{double}(\text{circle}) = \text{spider} \quad \text{pure quantum map}$$

EX "Plus" STATE (a.k.a. \downarrow_0) X-basis state

$$\downarrow_0 = \frac{1}{2} \cdot \text{circle}$$

$$\text{double}(\frac{1}{2} [\downarrow_0 + \downarrow_1])$$

$$\frac{1}{2} \cdot \text{double}(\downarrow_0 + \downarrow_1) = \text{double}(\sum_i \downarrow_i)$$

EX GHZ state: $\frac{1}{2} \cdot \text{spider} := \frac{1}{2} \text{double}(\sum_{i=0}^1 \downarrow \downarrow \downarrow)$

Ex "coherent" copy:

$$\begin{array}{c} \text{Spider} \\ \downarrow \\ \text{Triangle} \end{array} = \begin{array}{c} \downarrow \\ \text{Triangle} \end{array} \begin{array}{c} \downarrow \\ \text{Triangle} \end{array} \quad \left(\text{but } \begin{array}{c} \downarrow \\ \text{Triangle} \end{array} \notin \{ \begin{array}{c} \downarrow \\ \text{Triangle} \end{array} \} \Rightarrow \begin{array}{c} \text{Spider} \\ \downarrow \\ \text{Triangle} \end{array} \neq \begin{array}{c} \downarrow \\ \text{Triangle} \end{array} \begin{array}{c} \downarrow \\ \text{Triangle} \end{array} \right)$$

in particular, for $\begin{array}{c} \downarrow \\ \text{Triangle} \end{array} \approx \begin{array}{c} \text{Spider} \\ \downarrow \\ \text{Triangle} \end{array}$:

$$\begin{array}{c} \text{Spider} \\ \downarrow \\ \text{Triangle} \end{array} \approx \begin{array}{c} \text{Spider} \\ \downarrow \\ \text{Triangle} \end{array} =: \begin{array}{c} \text{Spider} \\ \downarrow \\ \text{Triangle} \end{array} = \cup =: \cup$$

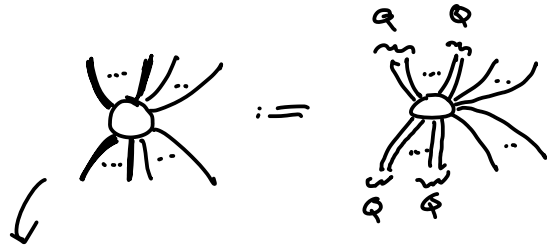
Thm Quantum spiders satisfy spider-fusion:

$$\begin{array}{c} \dots \\ \text{Spider} \\ \downarrow \\ \text{Spider} \\ \downarrow \\ \dots \end{array} = \begin{array}{c} \dots \\ \text{Spider} \\ \downarrow \\ \dots \end{array}$$

Pf 2x applications of normal sp. fusion.

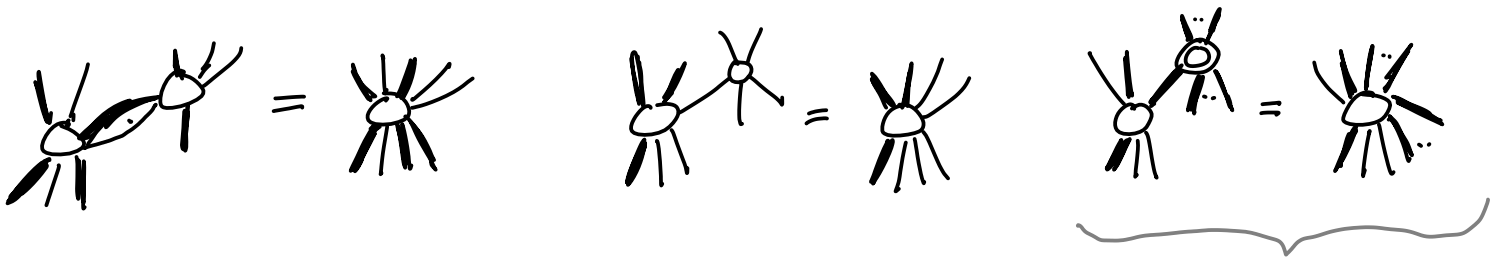
$$\begin{array}{c} \dots \\ \text{Spider} \\ \downarrow \\ \text{Spider} \\ \downarrow \\ \dots \end{array} = \begin{array}{c} \dots \\ \text{Spider} \\ \downarrow \\ \text{Spider} \\ \downarrow \\ \dots \end{array} = \begin{array}{c} \dots \\ \text{Spider} \\ \downarrow \\ \dots \end{array} =: \begin{array}{c} \dots \\ \text{Spider} \\ \downarrow \\ \dots \end{array}$$

DEF A bastard spider is:



$$\hat{A} \otimes \hat{A} \dots \otimes \hat{A} \otimes A \otimes A \rightarrow A \otimes \hat{A} \otimes \dots \otimes \hat{A}$$

BASTARD SPIDER FUSION:



"CONTACT W/ THE CLASSICAL WORLD DESTROYS QUANTUM-NESS OF SPIDERS."

Ex Delete, copy, etc. classical (\Rightarrow bastard)

Ex Measure, encode or b. spiders.

Ex Decoding: $\dagger = \cap = \cap =: \dagger$

Ex Dehorence: $\circ = \circ \neq |$

Lecture 16

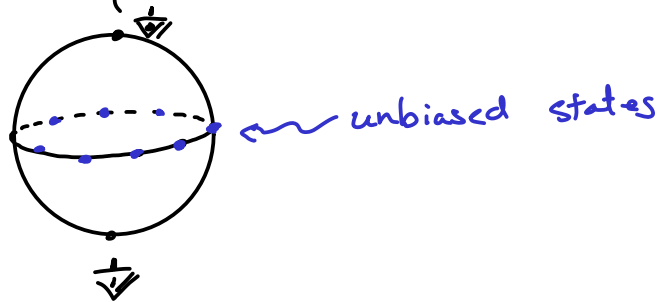
Q: What is "quantumness"?

An unbiased state $\hat{\psi}$ is a state where:

$$\langle \hat{\psi} | b \rangle = \frac{1}{D} \cdot b \iff \left(\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right) \Bigg\}^D$$

equivalently, for all i : $\langle \hat{\psi} | i \rangle = \frac{1}{D}$.

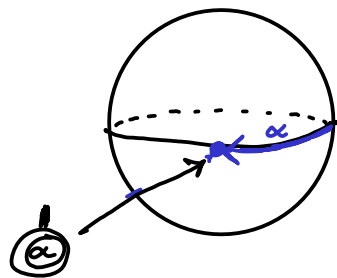
e.g. on the Bloch sphere:



DEF A phase state is a state that satis $\langle \hat{\psi} | b \rangle = b$.

(equiv. a phase state is $D \cdot$ (unbiased state))

NEW NOTATION write phase states (in 2D) as $\textcircled{\alpha}$ for $\alpha \in [0, 2\pi)$.

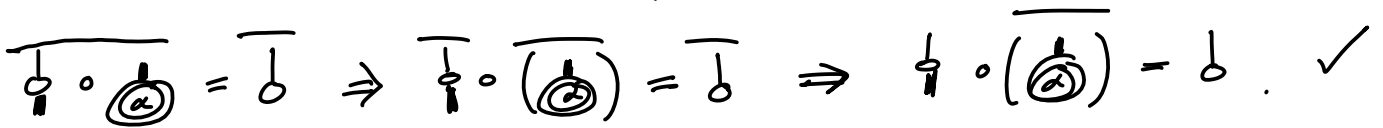
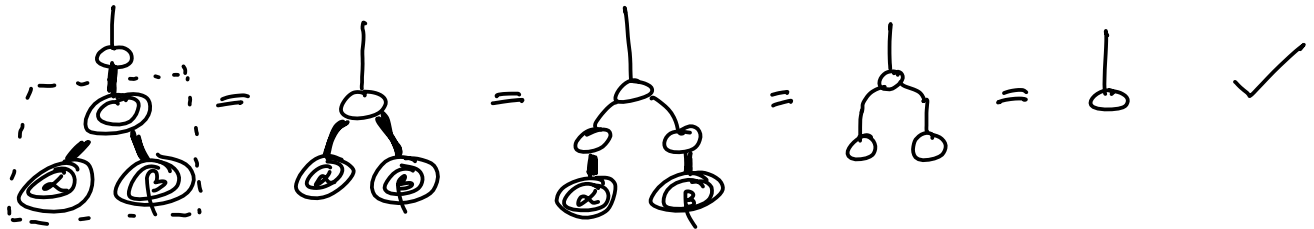


$$\left(\textcircled{\alpha} = b \right)$$

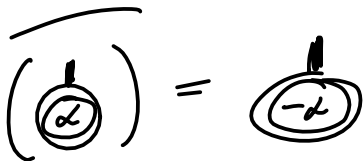
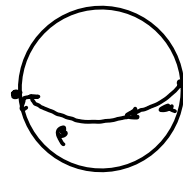
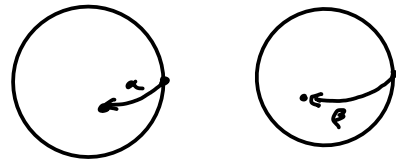
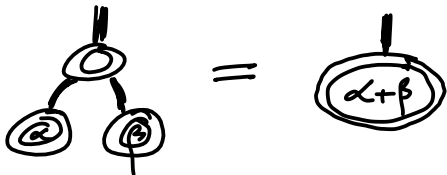
Thm If α and β are phase states, then:

$\alpha \cup \beta$ and $\overline{(\alpha)}$ are also phase states.

PF



Prop In 2D:

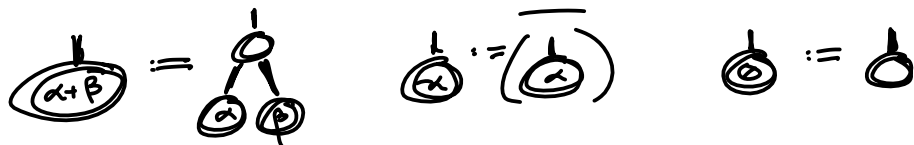


$$\alpha := \text{double} \left(\downarrow_0 + e^{i\alpha} \downarrow_1 \right)$$

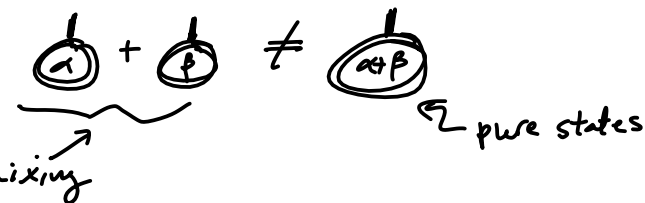
$$\therefore \circ := \text{double} \left(\downarrow_0 + \downarrow_1 \right) = \text{double}(\circ) = \circ$$

COR 


⇒ Phase states form a group called the phase group.



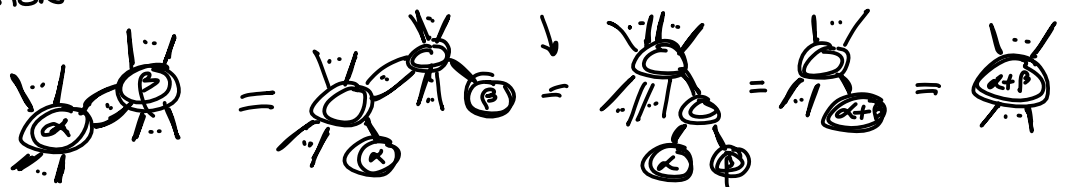
For \mathbb{Z}_2 , this group is the circle group. (aka. S_1 or $U(1)$)
1+1 unitary matrices (e^{ix})

NOTE: 

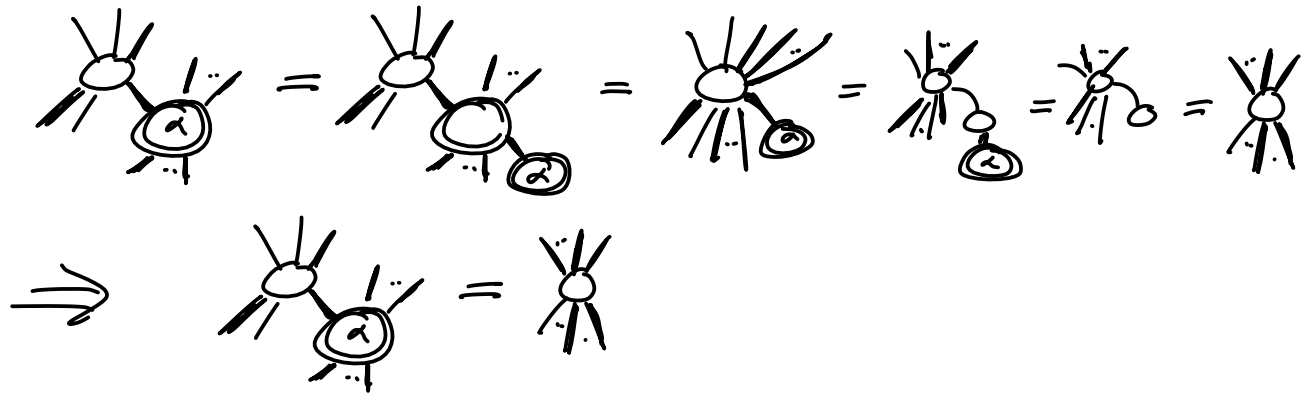
Another species of spider:

DEF A phase spider is a pure g.map: 

PHASE SPIDER FUSION:



MOTTO: Contact w/ classical world destroys ^(q. data) phases.



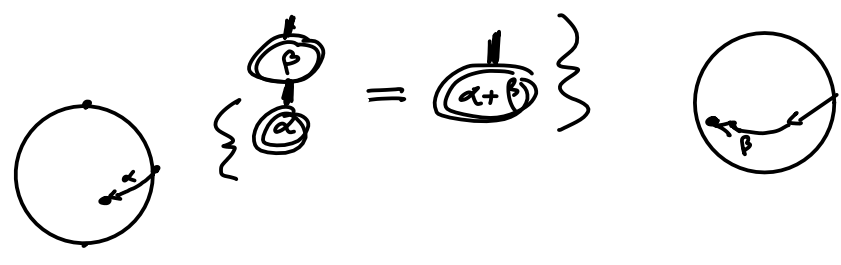
nb. PHASE SPIDERS SAT: $\left(\begin{matrix} \alpha \\ \alpha \end{matrix} \right)^T = \begin{matrix} \alpha \\ \alpha \end{matrix} \Rightarrow \left(\begin{matrix} \alpha \\ \alpha \end{matrix} \right)^\dagger = \begin{matrix} \alpha \\ \alpha \end{matrix}$

PHASE GATES: $\begin{matrix} \alpha \\ \alpha \end{matrix} := \begin{matrix} \alpha \\ \alpha \end{matrix}$

... are unitary: $\begin{matrix} \alpha \\ \alpha \end{matrix} \circ \left(\begin{matrix} \alpha \\ \alpha \end{matrix} \right)^\dagger = \begin{matrix} \alpha \\ \alpha \end{matrix} \circ \begin{matrix} -\alpha \\ -\alpha \end{matrix} = \begin{matrix} \alpha \\ -\alpha \end{matrix} = \begin{matrix} \alpha \\ \alpha \end{matrix} = |$

$\begin{matrix} -\alpha \\ \alpha \end{matrix} = \begin{matrix} \alpha \\ \alpha \end{matrix} = |$

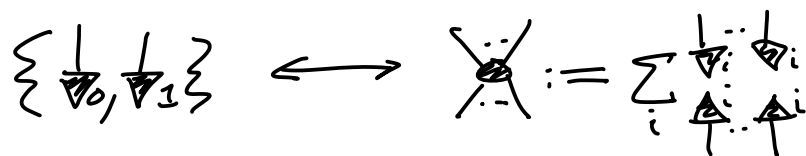
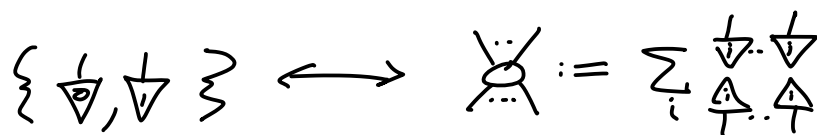
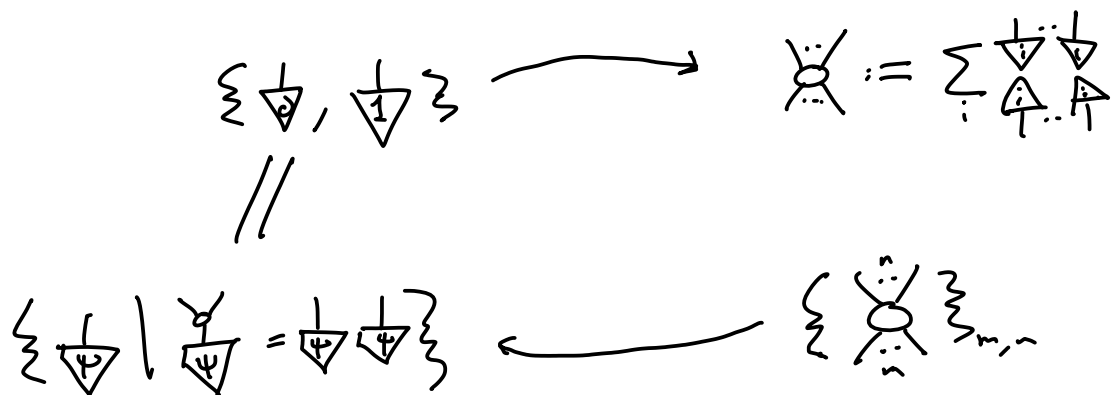
⇒ In 2D phase gates are rotations of the Bloch sphere.



∴ $\begin{matrix} \beta \end{matrix}$ is a rotation by β of the Bloch sphere around the Z axis.

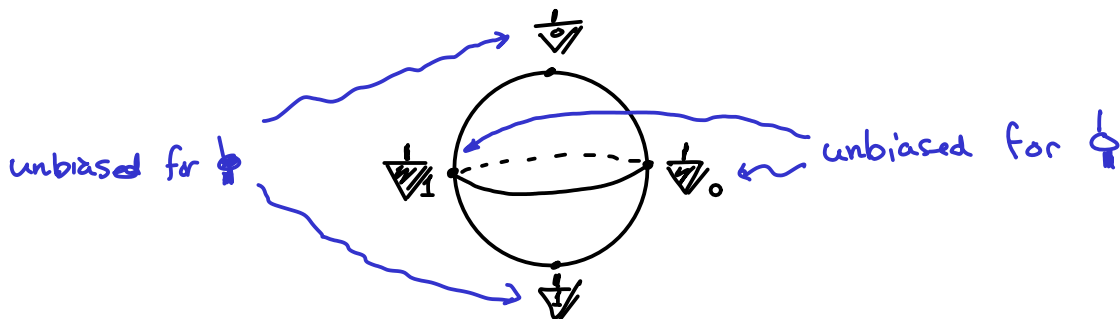
9.2 Complementarity.

Thm (8.41) Families of spiders are in 1-to-1 correspondence with ONB's.



$\bullet \rightsquigarrow Z\text{-meas.}$

$\bullet \rightsquigarrow X\text{-meas.}$



DEF Two ONB's are mutually unbiased (m.u.b.) if

$$\left(\forall i. \text{spider}_i^{\bullet} = \frac{1}{D} \cdot \text{dot}_i \right) \text{ and } \left(\forall j. \text{spider}_j^{\circ} = \frac{1}{D} \cdot \text{dot}_j \right)$$

equiv. $\forall ij. \text{spider}_{ij}^{\bullet} = \frac{1}{D}$

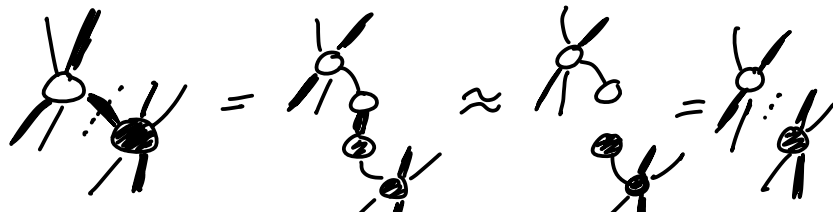
DEF Two families of spiders \circ and \bullet are complementary if:

$$\text{spider}_{\circ\bullet} = \frac{1}{D} \cdot \text{dot}_{\circ}$$

equivalently:

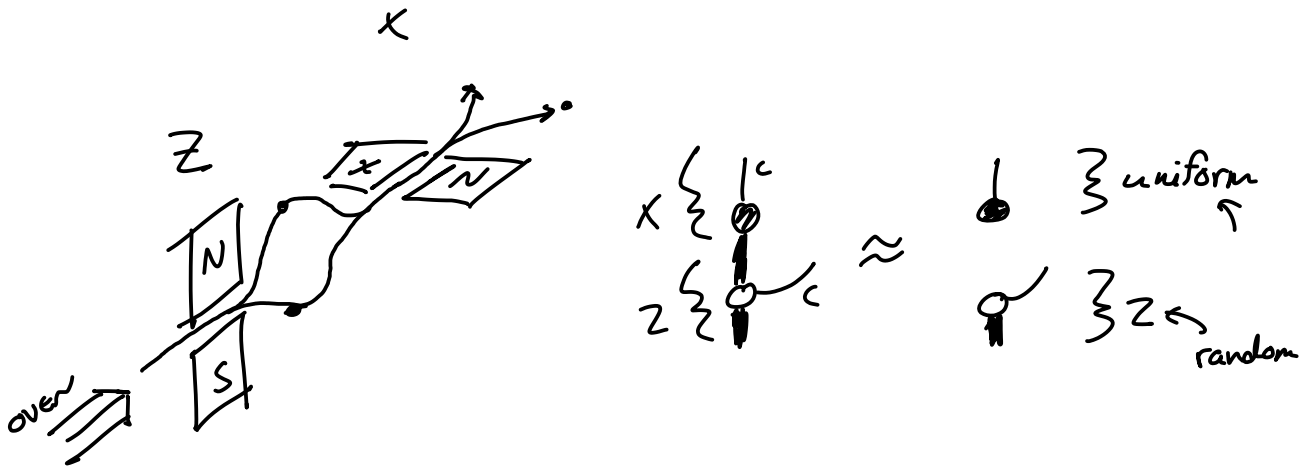
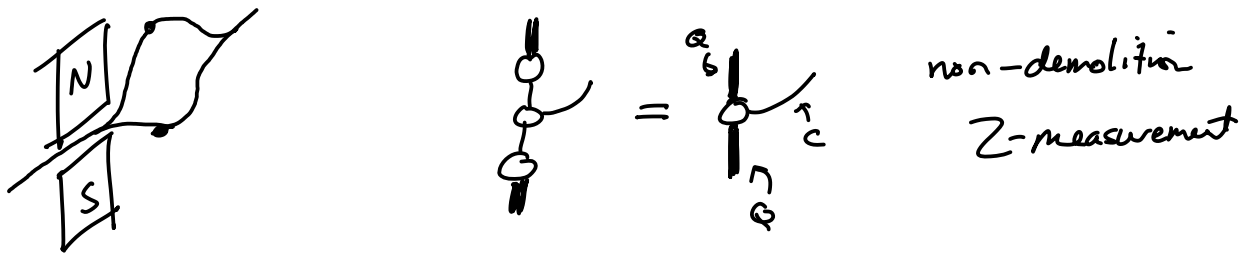
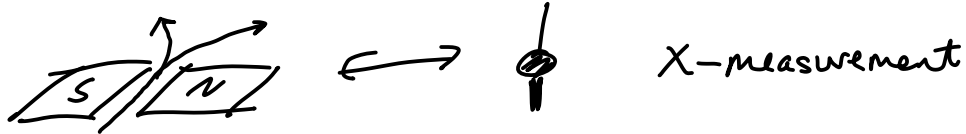
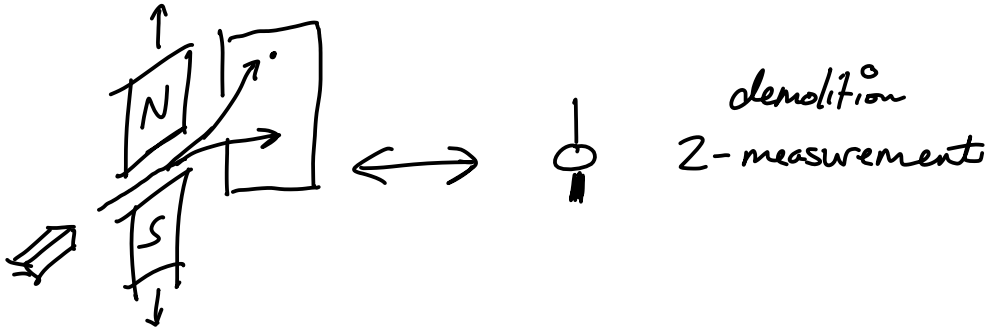
$$\left. \begin{array}{l} \text{measure in } \bullet \left\{ \text{spider}_{\bullet\circ} \right\} = \frac{1}{D} \text{dot}_{\bullet} \left\{ \text{uniform for } \bullet \right\} \\ \text{encode in } \circ \left\{ \text{spider}_{\circ\bullet} \right\} = \text{dot}_{\circ} \left\{ \text{delete } \circ \right\} \end{array} \right\} \text{ "no information flow"}$$

THM Two ONB's are mut. unbiased iff their spiders are complementary.

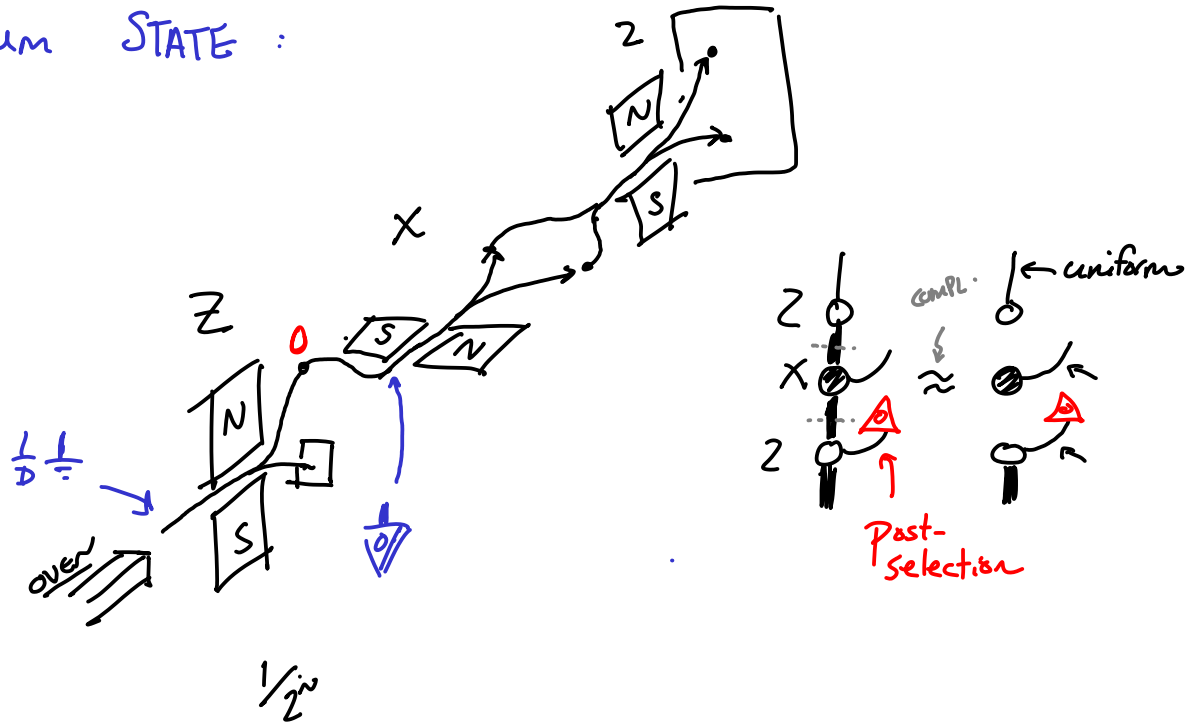
COMPLEMENTARITY := 

Lecture 17

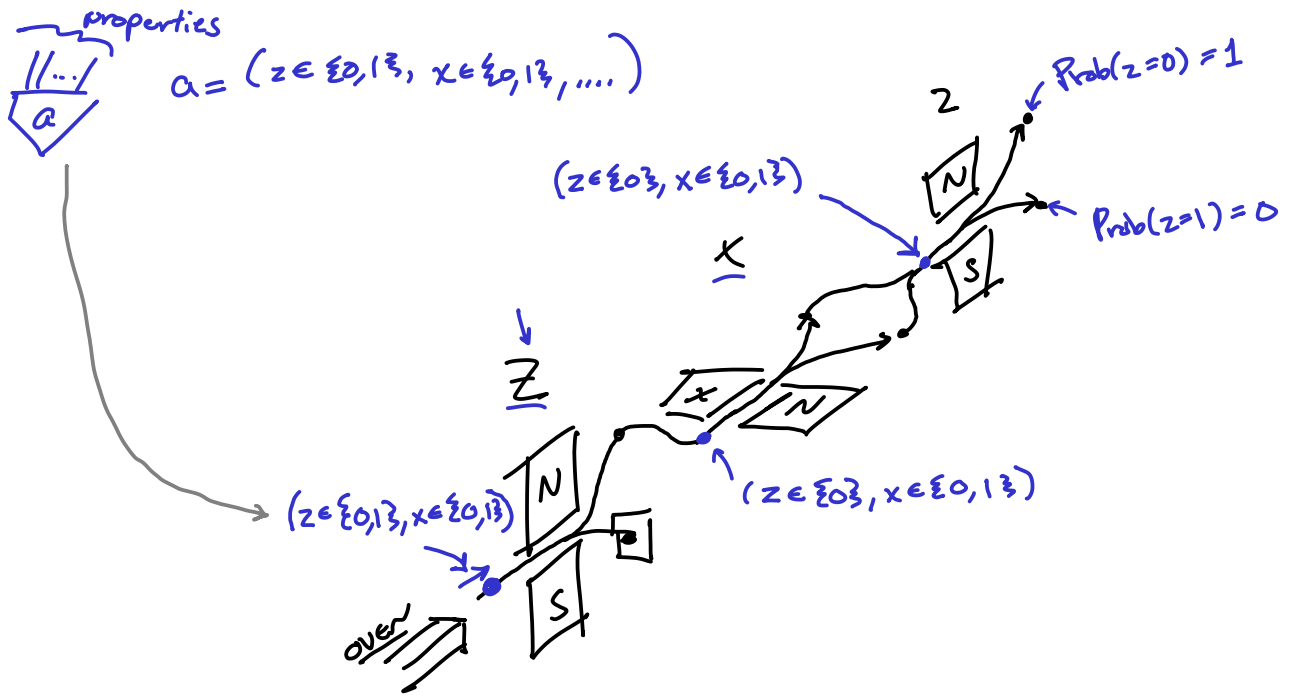
EXAMPLE Stern-Gerlach.



QUANTUM STATE :



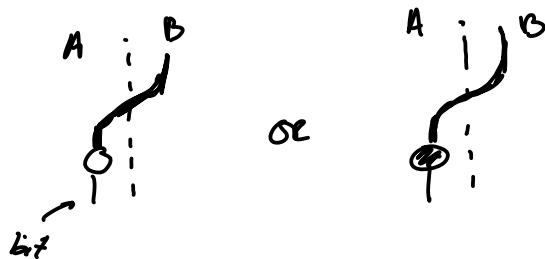
CLASSICAL STATE :



EXAMPLE Quantum key distribution (q. crypto)

A wants to send a (private) bit to B.

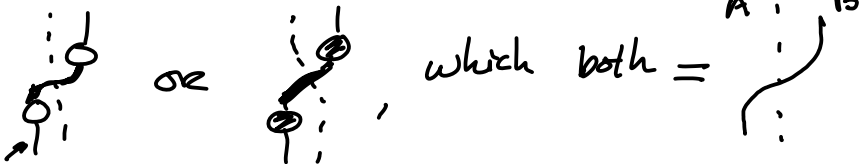
Step 1 A randomly chooses to do:




Step 2 B randomly chooses:

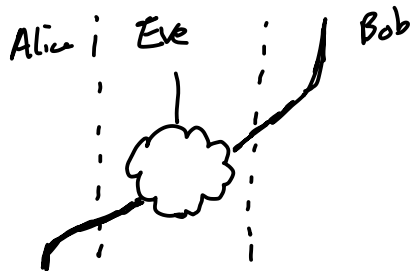


Step 3 A and B broadcast their basis choice.

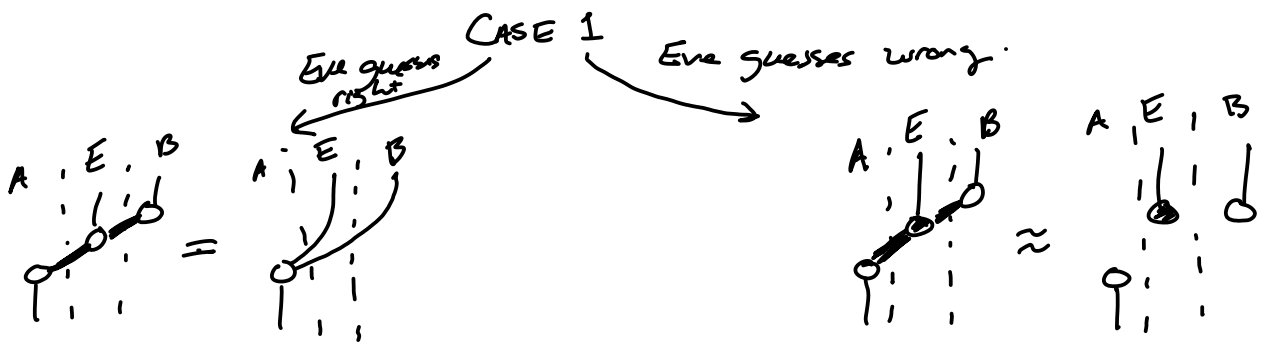
50% CASE 1 SAME CHOICE: 

50% CASE 2 Different choice: 

Q What happens if



CASE 1 is the only one that matters.



E + B both get A's bit.

E + B both get noise!

Step 4 A + B compare a random selection of their private ^(CASE 1) bits. Too much noise \Rightarrow eavesdropper!

[BB'84]
↑ ↓
Bennett Brassard