Lecture 23 | Grover's search algorithm.

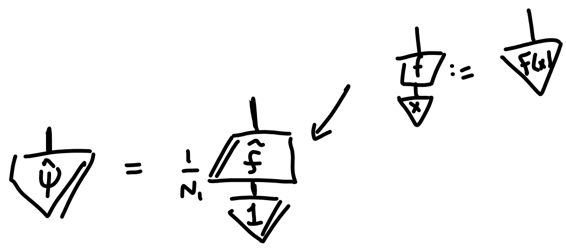
PROBLEM Search

GIVEN $F : \{0,1\}^n \to \{0,1\}$
FIND $x \in \{0,1\}^n$ s.t. $F(x) = 1$

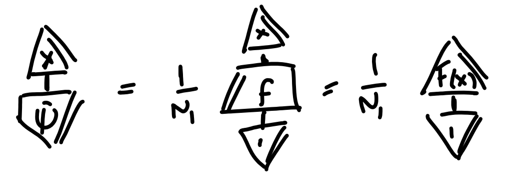
Simpler version: assume $N_1 = \#\{x \mid F(x) = 1\}$ is known.

Let $N_0 = \#\{x \mid F(x) = 0\}$, so $N_0 + N_1 = N = 2^n$.

A very useful state is:



Why? $\mathrm{Prob}(x \mid \hat{\varphi}) =$

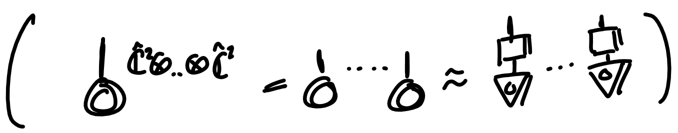


$= \begin{cases} 0 & F(x) = 0 \\ \frac{1}{N_1} & F(x) = 1 \end{cases}$.

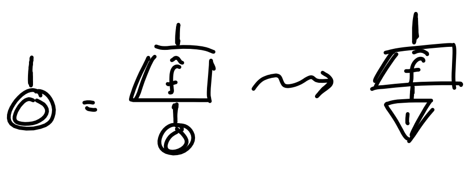
Q: How can we prep. $\hat{\varphi}$?

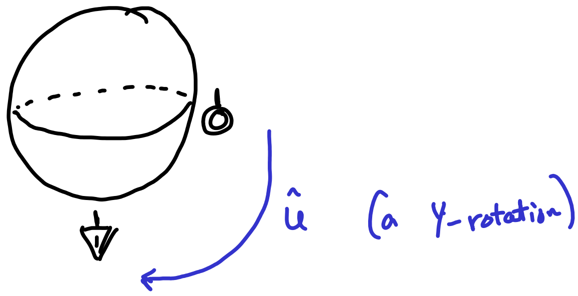
Recall

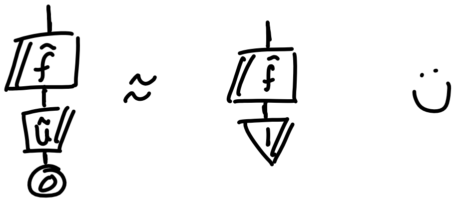
$\leftarrow$ easy to prepare



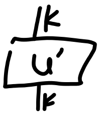
The goal:

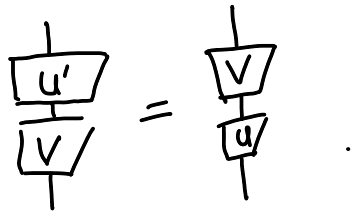


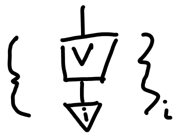
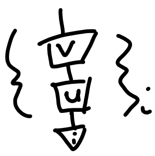


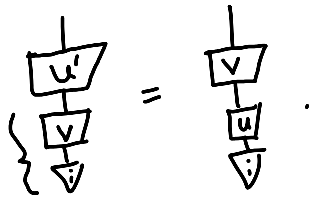


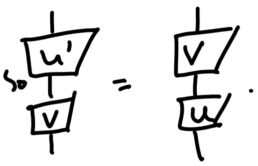
The problem: how to "do" a unitary here 

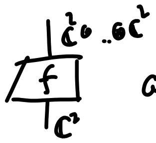
Prop For any isometry  and unitary , exists unitary  s.t.

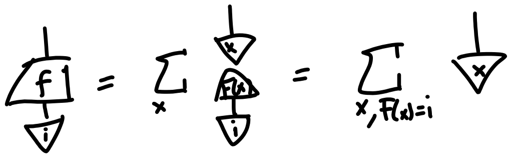


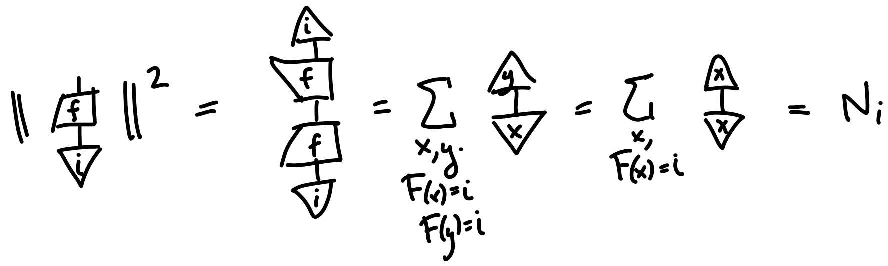
Pf .  and  are orthonormal sets in K.

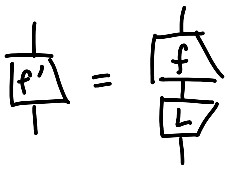
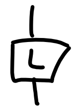
Let U' be any unitary such that: 

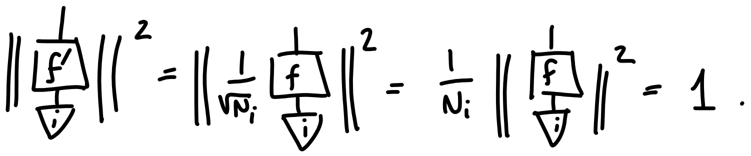
Then  is an ONB, so  ∎

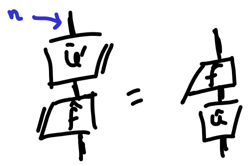
Q: Is  an isometry?



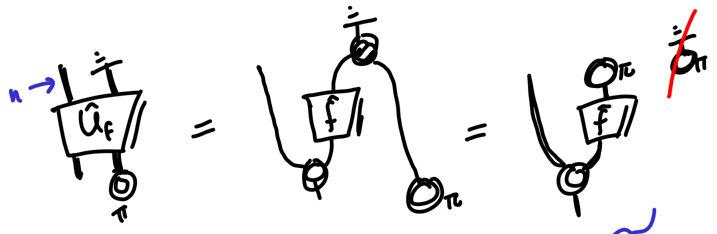


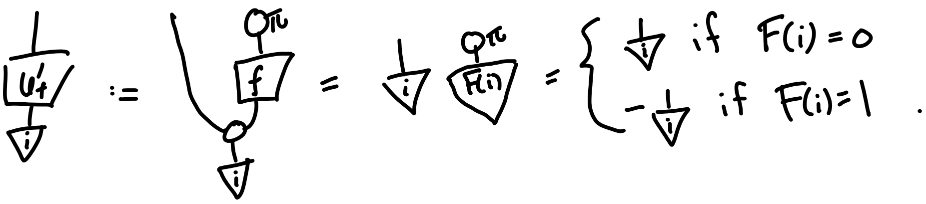
No, but let  where  $\leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{N_0}} & 0 \\ 0 & \frac{1}{\sqrt{N_1}} \end{pmatrix}$.

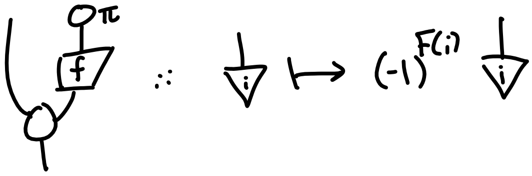
Now:  $= 1$. $\Rightarrow f'$ is an isom.

Candidates for U': 

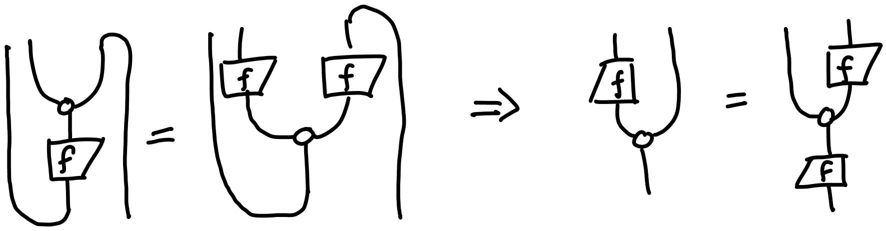
The reduced quantum oracle:



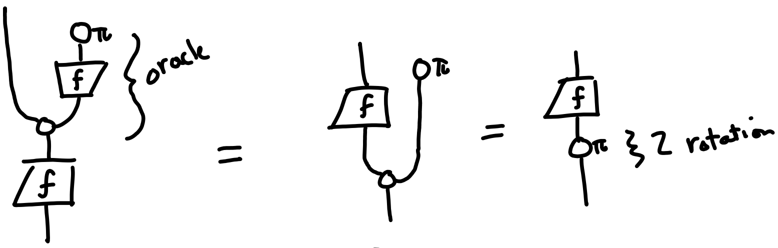




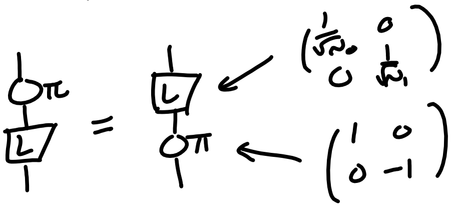
Note:



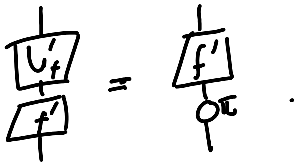
So:



And:

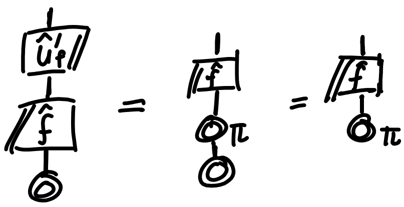


So:

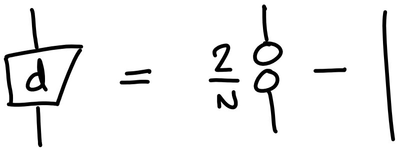


Is it enough?

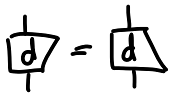
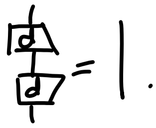
No →

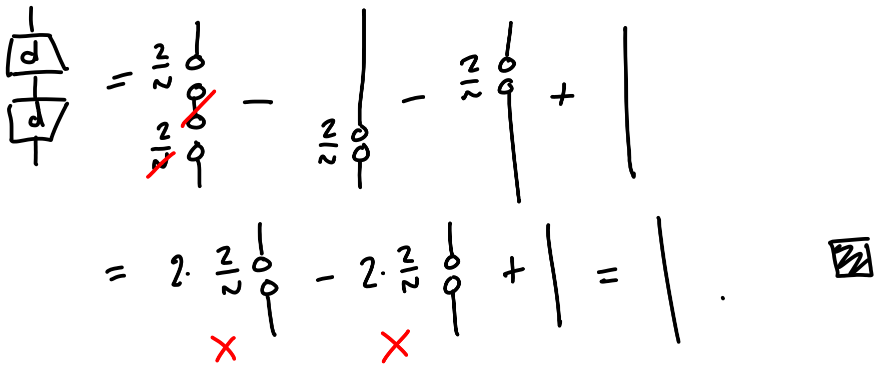


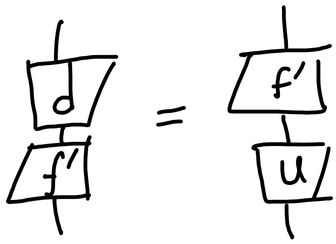
Candidate 2: the miracle "diffusion" operator.

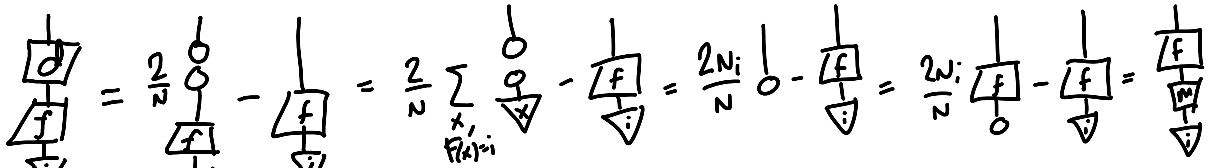


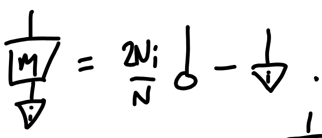
Prop d is unitary.

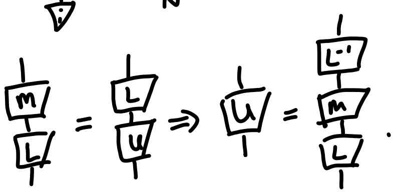
Pf n.b.  , so we only n.t.s  .

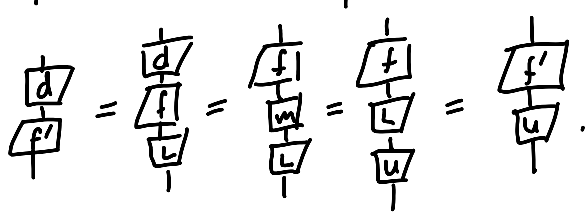


Prop  where $U \leftrightarrow \begin{pmatrix} \frac{N_0 - N_1}{N} & \frac{2\sqrt{N_0 N_1}}{N} \\ \frac{2\sqrt{N_0 N_1}}{N} & \frac{N_1 - N_0}{N} \end{pmatrix}$

Pf 

where  . so $m = \begin{pmatrix} \frac{2N_0}{N} - 1 & \frac{2N_1}{N} \\ \frac{2N_0}{N} & 2\frac{N_1}{N} - 1 \end{pmatrix} = \begin{pmatrix} \frac{N_0 - N_1}{N} & \frac{2N_1}{N} \\ \frac{2N_0}{N} & \frac{N_1 - N_0}{N} \end{pmatrix}$.

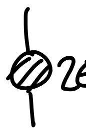
Now:  . $U = \begin{pmatrix} \sqrt{N_0} & 0 \\ 0 & \sqrt{N_1} \end{pmatrix} m \begin{pmatrix} \frac{1}{\sqrt{N_0}} & 0 \\ 0 & \frac{1}{\sqrt{N_1}} \end{pmatrix} = \begin{pmatrix} \frac{N_0 - N_1}{N} & \frac{2\sqrt{N_0 N_1}}{N} \\ \frac{2\sqrt{N_0 N_1}}{N} & \frac{N_1 - N_0}{N} \end{pmatrix}$

Hence:  . ∎

Columns of U are normalised $\Rightarrow \exists\, \theta$ s.t.

$$\cos\theta = \frac{N_0 - N_1}{N} \qquad \sin\theta = \frac{2\sqrt{N_0 N_1}}{N}$$

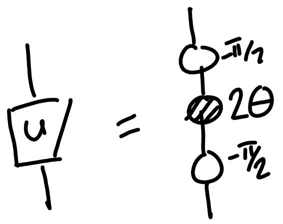
So:
$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

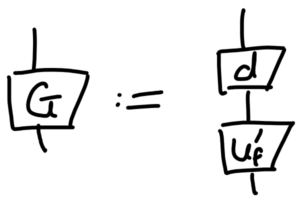
Fact: 

$$\approx \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}.$$

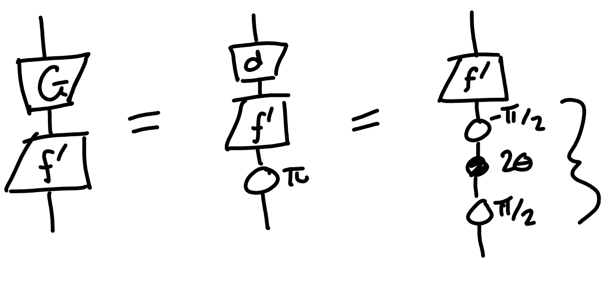
So:
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}$$



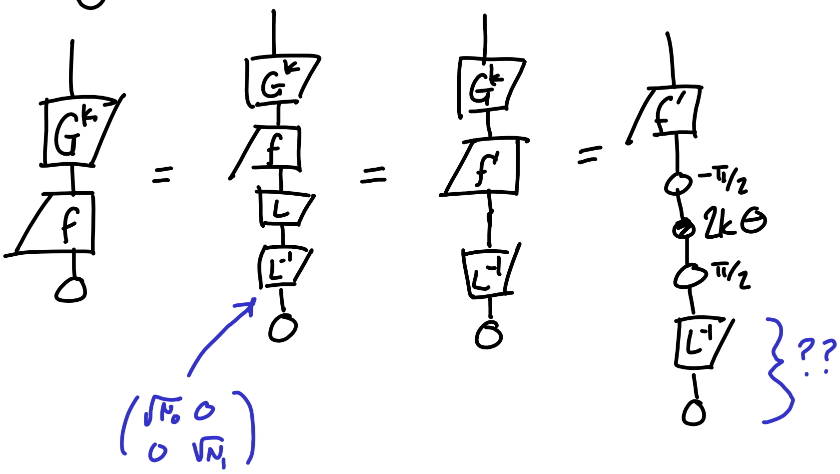
Now, let: 

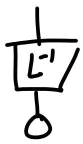
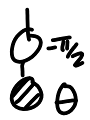
. Then:

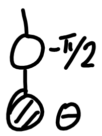


y-rotation!

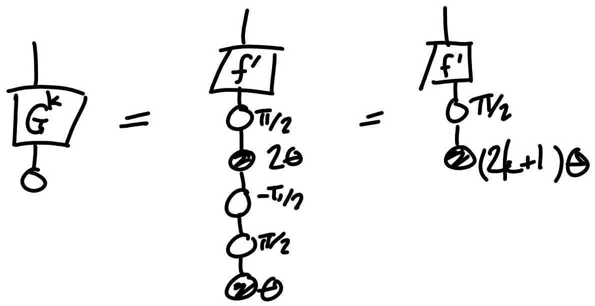
Putting it together:

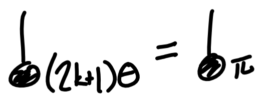


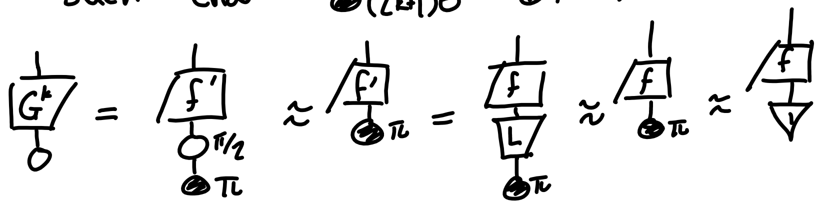
Prop  $= \begin{pmatrix}\sqrt{N_0}\\ \sqrt{N_1}\end{pmatrix} \approx$  } y phase state!

Pf  $= \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$. Let $\cos\frac{\theta}{2} = \sqrt{\frac{N_0}{N}}$ & $\sin\frac{\theta}{2} = \sqrt{\frac{N_1}{N}}$.

By dbl. angle ids, $\cos\theta = \frac{N_0 - N_1}{N}$ & $\sin\theta = 2\frac{\sqrt{N_0 N_1}}{N}$. $\boxtimes$

So: 

Find k such that 

Then  !

Q: What is k?

Assuming $N_1 = 1$, $\sin\theta = \frac{1}{\sqrt{N}}$ $\Rightarrow$ θ is approx. $\frac{1}{\sqrt{N}}$.

v. small

$$(2k+1)\theta = \pi \Rightarrow k = \frac{\pi/\theta - 1}{2} = \frac{\pi\sqrt{N} - 1}{2} \sim O(\sqrt{N})$$