

Lecture 23 | Grover's search algorithm.

PROBLEM Search

GIVEN $F: \{0,1\}^n \rightarrow \{0,1\}$

FIND $x \in \{0,1\}^n$ s.t. $F(x) = 1$

Simpler version: assume $N_1 = \#\{x \mid F(x) = 1\}$ is known.

Let $N_0 = \#\{x \mid F(x) = 0\}$, so $N_0 + N_1 = N = 2^n$.

A very useful state is: $\frac{1}{\sqrt{N}} \sum_x \frac{1}{\sqrt{2}} (|x\rangle + |F(x)\rangle)$

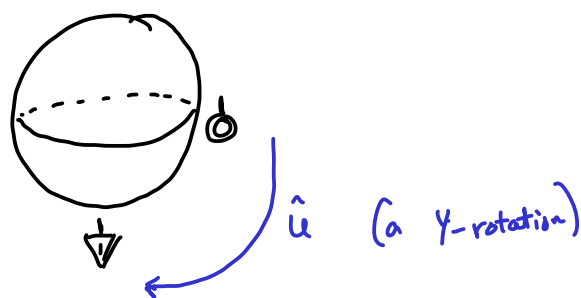
Why? $\text{Prob}(x \mid \hat{\Psi}) = \frac{1}{N} \sum_x \frac{1}{\sqrt{2}} (|x\rangle + |F(x)\rangle) = \begin{cases} 0 & F(x) = 0 \\ \frac{1}{N_1} & F(x) = 1 \end{cases}$

Q: How can we prep. $\hat{\Psi}$?


Recall $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $\frac{1}{\sqrt{2}}(|x\rangle + |F(x)\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} |x\rangle \\ |F(x)\rangle \end{pmatrix} \leftarrow \text{easy to prepare}$

$\left(\frac{1}{\sqrt{2}} \otimes \frac{1}{\sqrt{2}} \otimes \dots \otimes \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \approx \frac{1}{\sqrt{2^n}} \begin{pmatrix} |0\rangle \\ |1\rangle \\ \vdots \\ |1\rangle \end{pmatrix} \right)$

The goal: $\frac{1}{\sqrt{2^n}} \begin{pmatrix} |0\rangle \\ |1\rangle \\ \vdots \\ |1\rangle \end{pmatrix} = \frac{1}{\sqrt{2^n}} \begin{pmatrix} |x\rangle \\ |F(x)\rangle \end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2^n}} \begin{pmatrix} |x\rangle \\ |F(x)\rangle \end{pmatrix}$



$$\begin{array}{c} \boxed{\hat{f}} \\ \downarrow \\ \boxed{\hat{u}} \\ \odot \end{array} \approx \begin{array}{c} \boxed{f} \\ \downarrow \\ \nabla \end{array} \quad \text{⋮}$$

The problem: how to "do" a unitary here \rightarrow 

Prop For any isometry $\begin{array}{c} \mathbb{K} \\ \boxed{V} \\ \mathbb{H} \end{array}$ and unitary $\begin{array}{c} \mathbb{H} \\ \boxed{U} \\ \mathbb{H} \end{array}$, exists unitary $\begin{array}{c} \mathbb{K} \\ \boxed{U'} \\ \mathbb{K} \end{array}$ s.t.

$$\begin{array}{c} \boxed{U'} \\ \downarrow \\ \boxed{V} \end{array} = \begin{array}{c} \boxed{V} \\ \downarrow \\ \boxed{U} \end{array} .$$

Pf : $\left\{ \begin{array}{c} \boxed{V} \\ \downarrow \\ \nabla \end{array} \right\}_i$ and $\left\{ \begin{array}{c} \boxed{V} \\ \downarrow \\ \boxed{U} \\ \downarrow \\ \nabla \end{array} \right\}_i$ are orthonormal sets in \mathbb{K} .

Let U' be any unitary such that: $\left\{ \begin{array}{c} \boxed{U'} \\ \downarrow \\ \boxed{V} \\ \downarrow \\ \nabla \end{array} \right\}_i = \left\{ \begin{array}{c} \boxed{V} \\ \downarrow \\ \boxed{U} \\ \downarrow \\ \nabla \end{array} \right\}_i$.

Then $\left\{ \begin{array}{c} \boxed{V} \\ \downarrow \\ \nabla \end{array} \right\}_i$ is an O.B., so $\begin{array}{c} \boxed{U'} \\ \downarrow \\ \boxed{V} \end{array} = \begin{array}{c} \boxed{V} \\ \downarrow \\ \boxed{U} \end{array}$.

□

Q: Is $\begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \end{array}$ an isometry?

$$\begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \end{array} = \sum_x \begin{array}{c} |x\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |x\rangle \end{array} = \sum_{x, F(x)=i} \begin{array}{c} |x\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |x\rangle \end{array}$$

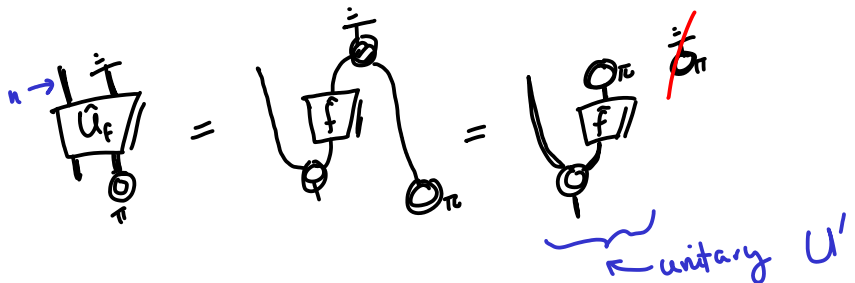
$$\left\| \begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \end{array} \right\|^2 = \begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \end{array} = \sum_{\substack{x,y \\ F(x)=i \\ F(y)=i}} \begin{array}{c} |x\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |x\rangle \end{array} = \sum_{\substack{x \\ F(x)=i}} \begin{array}{c} |x\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |x\rangle \end{array} = N_i$$

No, but let $\begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f'} \\ \text{---} \\ |i\rangle \end{array} = \begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |L\rangle \end{array}$ where $|L\rangle \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{N_i}} & 0 \\ 0 & \frac{1}{\sqrt{N_i}} \end{pmatrix}$.

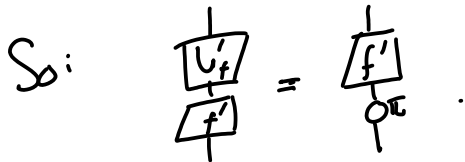
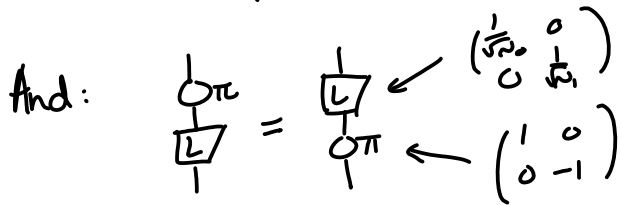
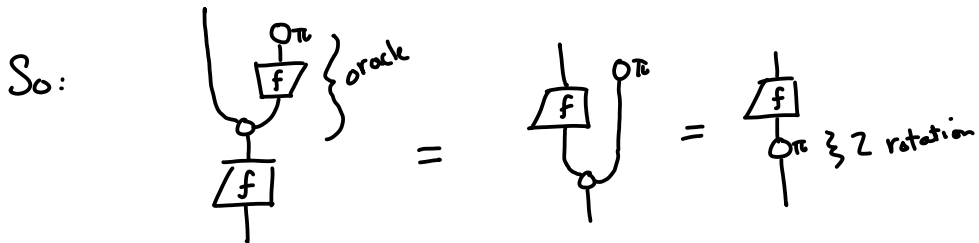
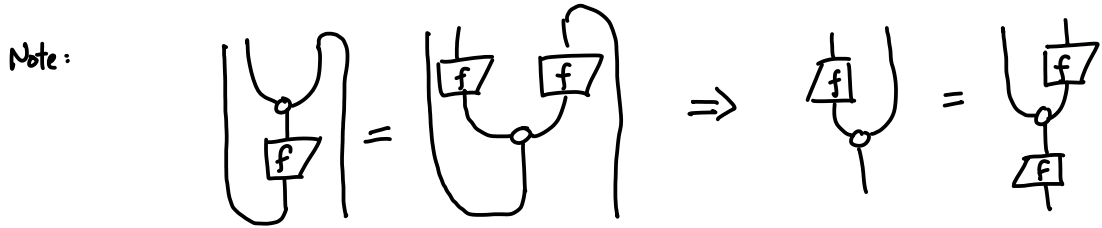
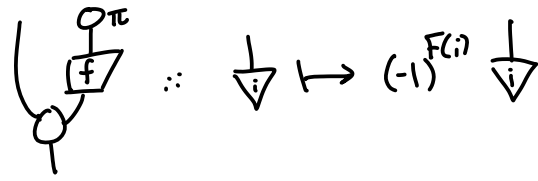
Now: $\left\| \begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f'} \\ \text{---} \\ |i\rangle \end{array} \right\|^2 = \left\| \frac{1}{\sqrt{N_i}} \begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \end{array} \right\|^2 = \frac{1}{N_i} \left\| \begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \end{array} \right\|^2 = 1 \Rightarrow f' \text{ is an isom.}$

Candidates for U' : $\begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{U'} \\ \text{---} \\ |i\rangle \end{array} = \begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |L\rangle \end{array}$

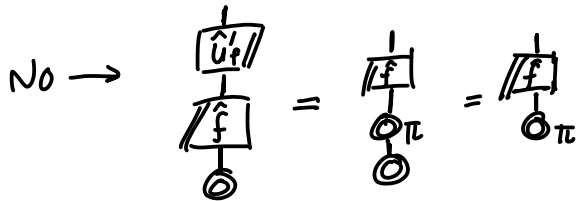
The ^{reduced} quantum oracle:



$$\begin{array}{c} |i\rangle \\ \text{---} \\ \boxed{U_f} \\ \text{---} \\ |i\rangle \end{array} = \begin{array}{c} |i\rangle \\ \text{---} \\ \text{---} \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \end{array} = \begin{array}{c} |i\rangle \\ \text{---} \\ \text{---} \\ \text{---} \\ \boxed{f} \\ \text{---} \\ |i\rangle \end{array} = \begin{cases} |i\rangle & \text{if } F(i)=0 \\ -|i\rangle & \text{if } F(i)=1 \end{cases}$$



Is it enough?



Candidate 2: the miracle "diffusion" operator.

$$\boxed{d} = \frac{2}{N} \begin{array}{c} | \\ \circ \\ | \end{array} - \begin{array}{c} | \\ | \end{array}$$

Prop d is unitary.

Pf n.b. $\boxed{d} = \boxed{d}$, so we only n.t.s $\boxed{d} \boxed{d} = |$.

$$\begin{aligned} \boxed{d} \boxed{d} &= \frac{2}{N} \begin{array}{c} | \\ \circ \\ | \end{array} - \begin{array}{c} | \\ | \end{array} - \frac{2}{N} \begin{array}{c} | \\ \circ \\ | \end{array} + \begin{array}{c} | \\ | \end{array} \\ &= 2 \cdot \frac{2}{N} \begin{array}{c} | \\ \circ \\ | \end{array} - 2 \cdot \frac{2}{N} \begin{array}{c} | \\ \circ \\ | \end{array} + \begin{array}{c} | \\ | \end{array} = \begin{array}{c} | \\ | \end{array} \quad \square \end{aligned}$$

Prop $\boxed{d} \boxed{f'} = \boxed{f'}$ where $U \leftrightarrow \begin{pmatrix} \frac{N_0 - N_1}{2} & \frac{2\sqrt{N_0 N_1}}{2} \\ \frac{2\sqrt{N_0 N_1}}{2} & \frac{N_1 - N_0}{2} \end{pmatrix}$

Pf $\boxed{d} \boxed{f} = \frac{2}{N} \begin{array}{c} | \\ \circ \\ | \end{array} - \begin{array}{c} | \\ | \end{array} = \frac{2}{N} \sum_{x, f(x)=i} \begin{array}{c} | \\ \circ \\ | \end{array} - \begin{array}{c} | \\ | \end{array} = \frac{2N_i}{N} \begin{array}{c} | \\ \circ \\ | \end{array} - \begin{array}{c} | \\ | \end{array} = \frac{2N_i}{2} \begin{array}{c} | \\ \circ \\ | \end{array} - \begin{array}{c} | \\ | \end{array} = \begin{array}{c} | \\ | \end{array}$

where $\boxed{M} = \frac{2N_i}{N} \begin{array}{c} | \\ \circ \\ | \end{array} - \begin{array}{c} | \\ | \end{array}$. So $M = \begin{pmatrix} \frac{2N_0}{2} - 1 & \frac{2N_1}{2} \\ \frac{2N_0}{2} & \frac{2N_1}{2} - 1 \end{pmatrix} = \begin{pmatrix} \frac{N_0 - N_1}{2} & \frac{2N_1}{2} \\ \frac{2N_0}{2} & \frac{N_1 - N_0}{2} \end{pmatrix}$.

Now: $\boxed{M} \begin{array}{c} | \\ | \end{array} = \begin{array}{c} | \\ | \end{array} \Rightarrow \boxed{U} = \begin{array}{c} | \\ | \end{array} \cdot U = \begin{pmatrix} \sqrt{N_0} & 0 \\ 0 & \sqrt{N_1} \end{pmatrix} M \begin{pmatrix} \frac{1}{\sqrt{N_0}} & 0 \\ 0 & \frac{1}{\sqrt{N_1}} \end{pmatrix} = \begin{pmatrix} \frac{N_0 - N_1}{2} & \frac{2\sqrt{N_0 N_1}}{2} \\ \frac{2\sqrt{N_0 N_1}}{2} & \frac{N_1 - N_0}{2} \end{pmatrix}$

Hence: $\boxed{d} \boxed{f'} = \boxed{f'} = \begin{array}{c} | \\ | \end{array} = \begin{array}{c} | \\ | \end{array} = \begin{array}{c} | \\ | \end{array} = \begin{array}{c} | \\ | \end{array} \quad \square$

Columns of U are normalised $\Rightarrow \exists \theta$ s.t.

$$\cos\theta = \frac{N_0 - N_1}{N} \quad \sin\theta = \frac{2\sqrt{N_0 N_1}}{N}$$

So:
$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Fact:
$$\phi_{2\theta} \approx \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

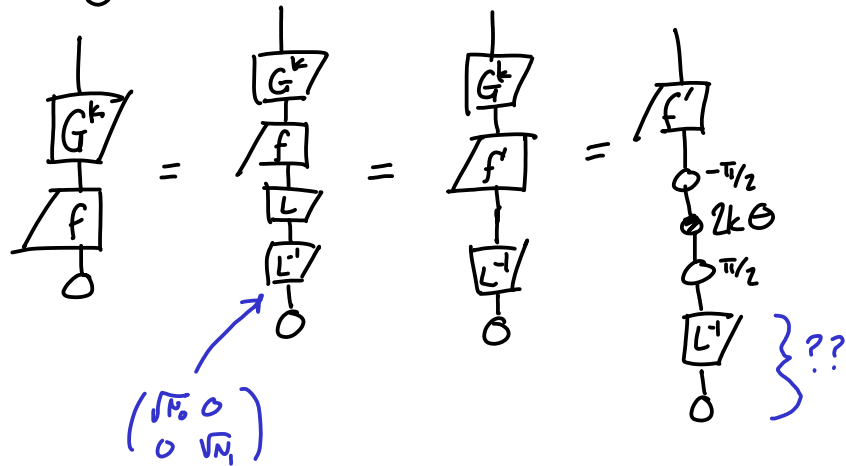
So:
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}}_{\phi_{-\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}}$$

$$\boxed{U} = \begin{array}{c} \circlearrowleft^{-\pi/2} \\ \phi_{2\theta} \\ \circlearrowright^{-\pi/2} \end{array}$$

Now, let: $\boxed{G} := \begin{array}{c} \boxed{d} \\ \boxed{U_f'} \end{array}$. Then:

$$\begin{array}{c} \boxed{G} \\ \boxed{f'} \end{array} = \begin{array}{c} \boxed{d} \\ \boxed{f'} \\ \circlearrowright^{\pi} \end{array} = \begin{array}{c} \boxed{f'} \\ \circlearrowleft^{-\pi/2} \\ \phi_{2\theta} \\ \circlearrowright^{\pi/2} \end{array} \left. \vphantom{\begin{array}{c} \boxed{f'} \\ \circlearrowleft^{-\pi/2} \\ \phi_{2\theta} \\ \circlearrowright^{\pi/2} \end{array}} \right\} y\text{-rotation!}$$

Putting it together:



Prop $L^{-1} = \begin{pmatrix} \sqrt{N_0} \\ \sqrt{N_1} \end{pmatrix} \approx \begin{matrix} \text{phase shifter } -\pi/2 \\ \text{phase shifter } \theta \end{matrix} \}$ y phase state!

Pf $\begin{matrix} \text{phase shifter } -\pi/2 \\ \text{phase shifter } \theta \end{matrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}$. Let $\cos \frac{\theta}{2} = \sqrt{\frac{N_0}{N}}$ and $\sin \frac{\theta}{2} = \sqrt{\frac{N_1}{N}}$.

By dbl. angle ids, $\cos \theta = \frac{N_0 - N_1}{N}$ and $\sin \theta = \frac{2\sqrt{N_0 N_1}}{N}$. □

So: $G^k = \begin{matrix} \text{block } f' \\ \text{phase shifter } \pi/2 \\ \text{phase shifter } 2k\theta \\ \text{phase shifter } -\pi/2 \\ \text{phase shifter } \pi/2 \end{matrix} = \begin{matrix} \text{block } f' \\ \text{phase shifter } \pi/2 \\ \text{phase shifter } (2k+1)\theta \end{matrix}$

Find k such that $(2k+1)\theta = \pi$.

Then $G^k = \begin{matrix} \text{block } f' \\ \text{phase shifter } \pi/2 \\ \text{phase shifter } \pi \end{matrix} \approx \begin{matrix} \text{block } f' \\ \text{phase shifter } \pi \end{matrix} = \begin{matrix} \text{block } f \\ \text{phase shifter } \pi \end{matrix} \approx \begin{matrix} \text{block } f \\ \text{phase shifter } \pi \end{matrix} \approx \begin{matrix} \text{block } f \\ \text{phase shifter } \pi \end{matrix} \approx \begin{matrix} \text{block } f \\ \text{phase shifter } \pi \end{matrix} !$

Q: What is k ?

Assuming $N_1=1$, $\sin\theta = \frac{1}{\sqrt{N}}$ $\Rightarrow \theta$ is approx. $\frac{1}{\sqrt{N}}$.
v. small

$$(2k+1)\theta = \pi \Rightarrow k = \frac{\pi/\theta - 1}{2} = \frac{\pi\sqrt{N} - 1}{2} \sim O(\sqrt{N})$$