

# Quantum Processes + Computation

Model solutions, sheet 4  
Oxford MT 2022

Ex 4.1

Let  $\begin{array}{|c|} \hline \Phi \\ \hline \end{array} // \begin{array}{|c|} \hline \Phi \\ \hline \end{array} = \begin{array}{|c|} \hline \hat{f} \\ \hline \end{array}$  and  $\begin{array}{|c|} \hline \Psi \\ \hline \end{array} // \begin{array}{|c|} \hline \Psi \\ \hline \end{array} = \begin{array}{|c|} \hline \hat{g} \\ \hline \end{array}$ .

Then  $\Phi \circ \Phi$  purifies as  $\begin{array}{|c|} \hline \hat{f} \\ \hline \end{array} // \begin{array}{|c|} \hline \hat{f} \\ \hline \end{array}$ , since:

$$\begin{array}{|c|} \hline \Phi \\ \hline \end{array} // \begin{array}{|c|} \hline \Phi \\ \hline \end{array} = \begin{array}{|c|} \hline \hat{f} \\ \hline \end{array}$$

And similarly,  $\Psi \circ \Psi$  purifies as  $\begin{array}{|c|} \hline \hat{g} \\ \hline \end{array} // \begin{array}{|c|} \hline \hat{g} \\ \hline \end{array}$ , since:

$$\begin{array}{|c|} \hline \Psi \\ \hline \end{array} // \begin{array}{|c|} \hline \Psi \\ \hline \end{array} = \begin{array}{|c|} \hline \hat{g} \\ \hline \end{array}$$

# Ex 4.2

Let  $\Phi = \begin{array}{c} | \quad | \\ \hline \Phi \\ \hline | \quad | \end{array} = \begin{array}{c} | \quad | \\ \hline \Phi_2 \quad \Phi_3 \\ \hline \Phi \\ \hline | \quad | \end{array}$  for causal g.maps  $\Phi_1, \Phi_2, \Phi_3$ .

Then:

$$\begin{array}{c} \dot{\vdots} \quad | \\ \hline \Phi \\ \hline | \quad | \end{array} = \begin{array}{c} \dot{\vdots} \quad | \quad | \\ \hline \Phi_2 \quad \Phi_3 \\ \hline \Phi \\ \hline | \quad | \end{array} = \begin{array}{c} \dot{\vdots} \quad \dot{\vdots} \quad | \\ \hline \Phi_3 \\ \hline \Phi \\ \hline | \quad | \end{array} = \begin{array}{c} \dot{\vdots} \\ \hline \Phi' \\ \hline | \quad | \end{array}$$

where  $\Phi' := \begin{array}{c} | \\ \hline \Phi_3 \\ \hline \Phi \\ \hline | \quad | \end{array}$ .

and:

$$\begin{array}{c} | \quad \dot{\vdots} \\ \hline \Phi \\ \hline | \quad | \end{array} = \begin{array}{c} | \quad | \quad \dot{\vdots} \\ \hline \Phi_2 \quad \Phi_3 \\ \hline \Phi \\ \hline | \quad | \end{array} = \begin{array}{c} | \quad | \quad \dot{\vdots} \quad \dot{\vdots} \\ \hline \Phi_2 \\ \hline \Phi \\ \hline | \quad | \end{array} = \begin{array}{c} | \\ \hline \Phi'' \\ \hline | \quad | \end{array} \dot{\vdots}$$

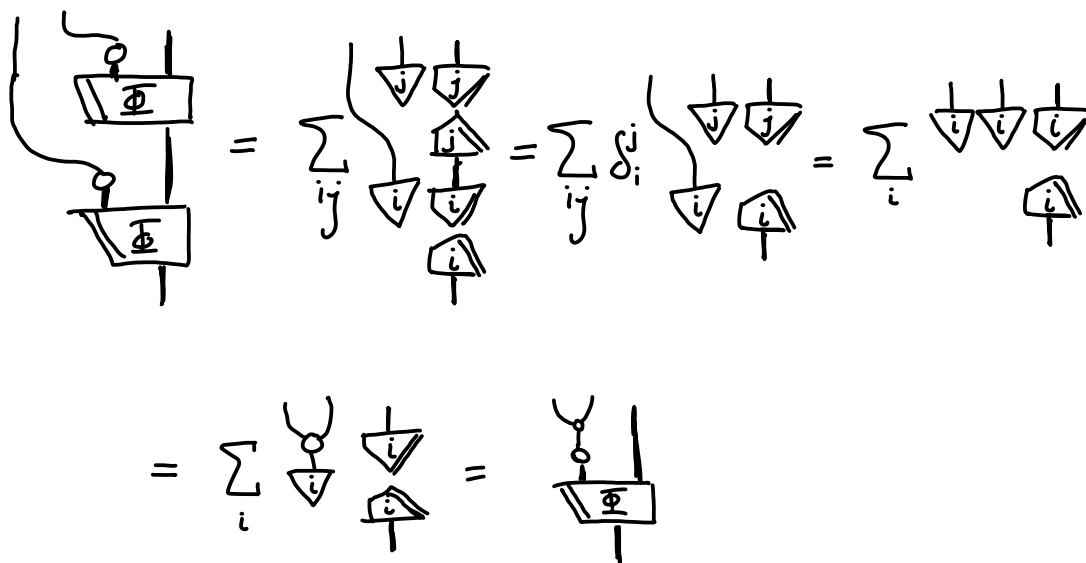
where  $\Phi'' := \begin{array}{c} | \\ \hline \Phi_2 \\ \hline \Phi \\ \hline | \quad | \end{array}$ .

Ex 4.3

Let  $\downarrow e = \text{double}(\sum_j \downarrow j) = \sum_{jk} \downarrow j \downarrow k$ .

Then  $\sum_i \downarrow i = \sum_{ijk} \begin{array}{c} \downarrow i \\ \uparrow i \\ \downarrow j \downarrow k \end{array} = \sum_i \downarrow i \downarrow i \neq \downarrow e$ .

Ex 4.4



# Ex 4.5

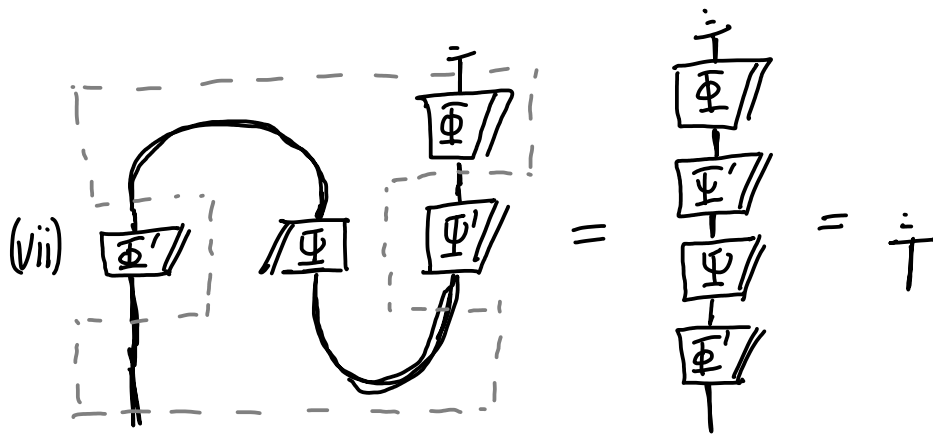
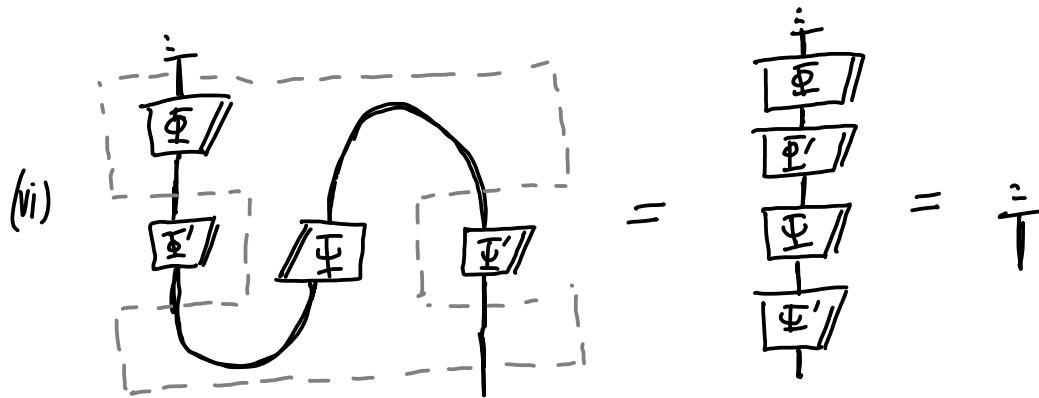
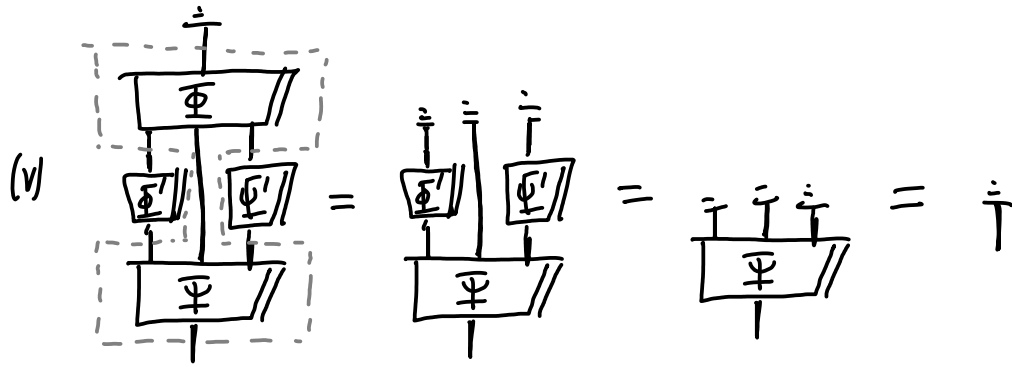
(a) Let  $\boxed{\Phi'}$ ,  $\boxed{\Psi'}$ ,  $\boxed{\Xi'}$  and  $\boxed{\rho'}$  be any causal processes, then:

(i) 
$$\boxed{\boxed{\Phi'} \boxed{\Psi'}} = \boxed{\boxed{\Phi'} \boxed{\Psi'}} = \boxed{\boxed{\Phi'}} = \dot{\uparrow}$$

(ii) 
$$\boxed{\boxed{\Phi'} \boxed{\rho'}} = \boxed{\boxed{\rho'}} \dot{\uparrow} = \dot{\uparrow}$$

(iii) 
$$\boxed{\boxed{\Phi} \boxed{\Xi'} \boxed{\Psi}} = \boxed{\boxed{\Xi'} \boxed{\Psi}} = \boxed{\boxed{\Psi}} = \dot{\uparrow} \dot{\uparrow}$$

(iv) 
$$\boxed{\boxed{\rho'}} = 1$$



(b) Assume all wires are of type  $\hat{A}$ , where

$D := \dim A \neq 1$ . Then:

(i)  $\boxed{\Phi} := ||$  is causal but:

$$\text{[Diagram: a box with } \Phi \text{ and a wire looping around it]} = \text{[Diagram: a circle with a wire]} = D^2 \dot{\uparrow} \neq \dot{\uparrow}$$

↑ double circle = double(D) =  $D^2$

so:  $\text{[Diagram: a box with } \Phi \text{ and a wire looping around it]}$  is not causal.

(ii) Similarly,  $\boxed{\Phi} := \times$  is causal, but:

$$\text{[Diagram: a box with } \Phi \text{ and a wire crossing it]} = \text{[Diagram: a circle with a wire]} = D^2 \cdot | \text{ is } \underline{\text{not}} \text{ causal.}$$



(iii)  $\frac{1}{\mathbb{D}} \frac{1}{\mathbb{I}} = \frac{1}{\mathbb{D}} \frac{1}{\mathbb{I}}$  is causal, because  $\mathbb{I} = \overset{\text{single circle.}}{\mathbb{O}} = \mathbb{D}$   
 and hence  $\frac{\mathbb{I}}{\mathbb{D}} = \frac{1}{\mathbb{D}} \frac{\mathbb{I}}{\mathbb{I}} = \frac{\mathbb{I}}{\mathbb{I}}$

But  $\frac{\mathbb{I}}{\mathbb{D}} = \frac{\mathbb{P}}{\mathbb{I}} = 1$ , so:

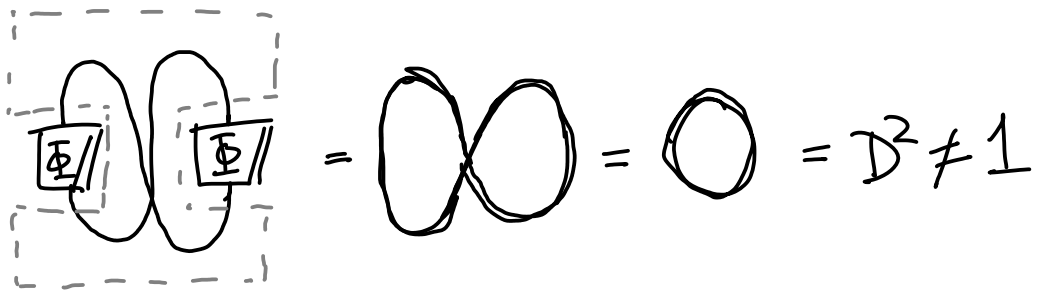
$$\frac{\mathbb{P}}{\mathbb{D}} = \frac{\mathbb{P}}{\mathbb{I}} = \frac{1}{\mathbb{D}} \neq 1$$

Hence  $\frac{\mathbb{P}}{\mathbb{D}}$  is not causal, because 1 is the only causal number.

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Note  $\frac{1}{\mathbb{D}} := |$  does not give a counter-example, because  $\mathbb{P}$  might be a pure state, in which case  $\frac{\mathbb{P}}{\mathbb{P}} = 1$ .

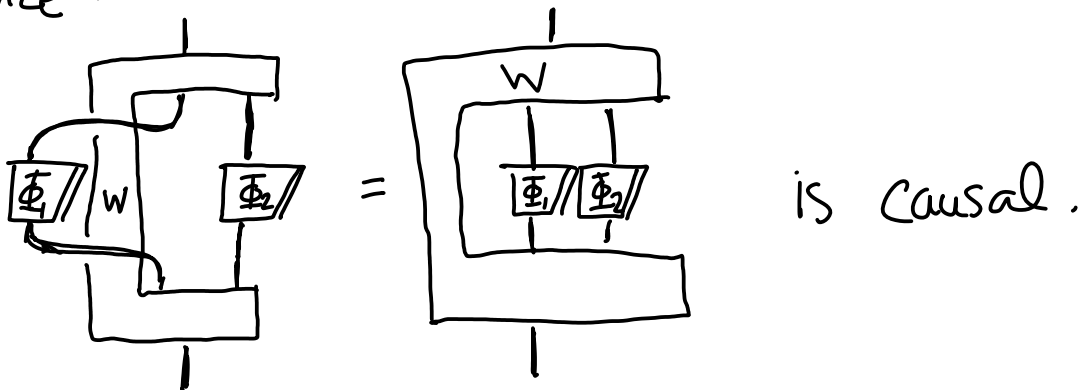
(iv)  $\boxed{\Phi} // \mid$  is causal but



is not causal.

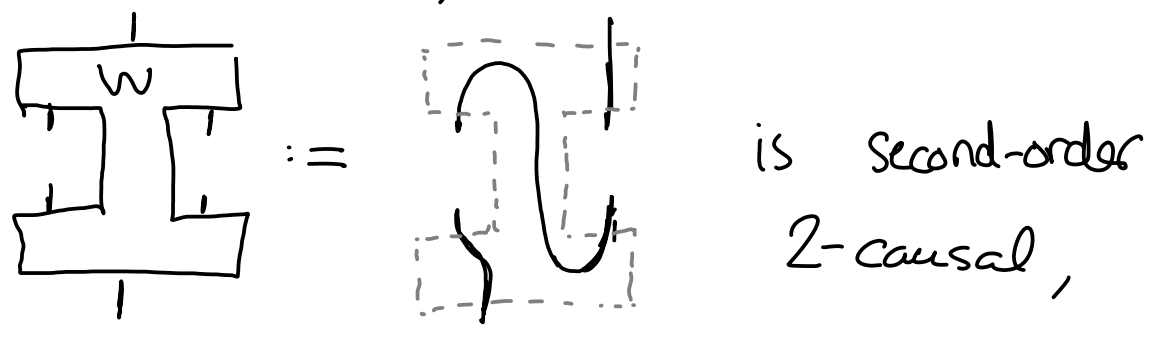
(c) Suppose  $W$  is second-order causal. Then, for any causal  $\Phi_1, \Phi_2$ ,  $\boxed{\Phi_1} // \boxed{\Phi_2}$  is causal.

Hence:

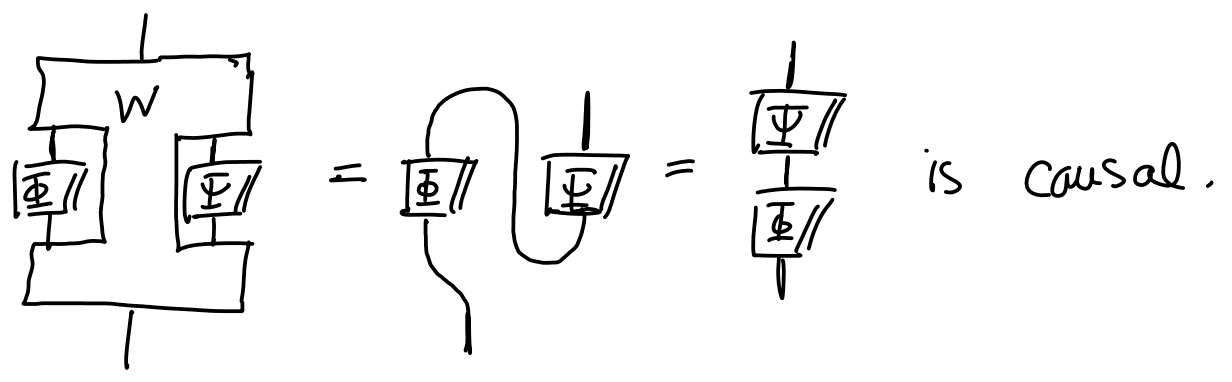


So  $W$  is second-order 2-causal.

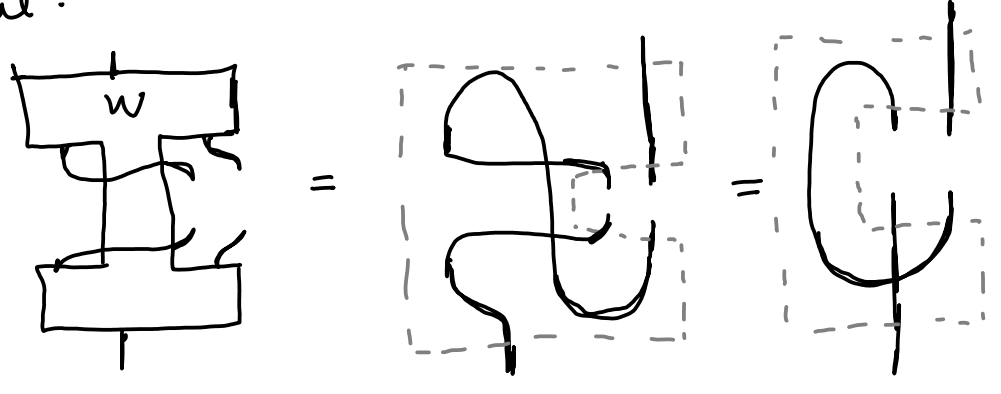
For the converse, note that



i.e for any causal  $\Phi, \Psi$  :



But:



which, as shown in (b)(ii), is not second-order causal.