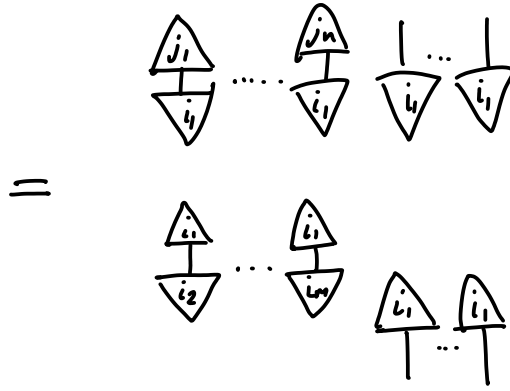
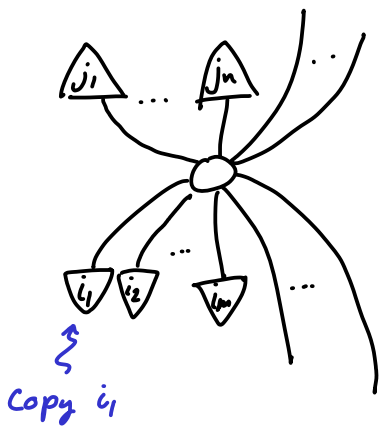


Quantum Processes + Computation

Model solutions, sheet 5
Oxford MT 2022

Ex 5.1

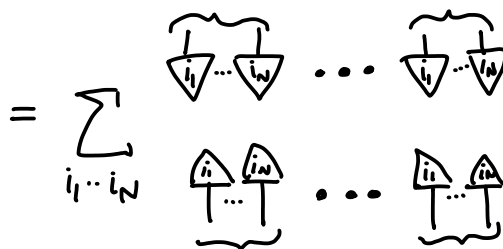
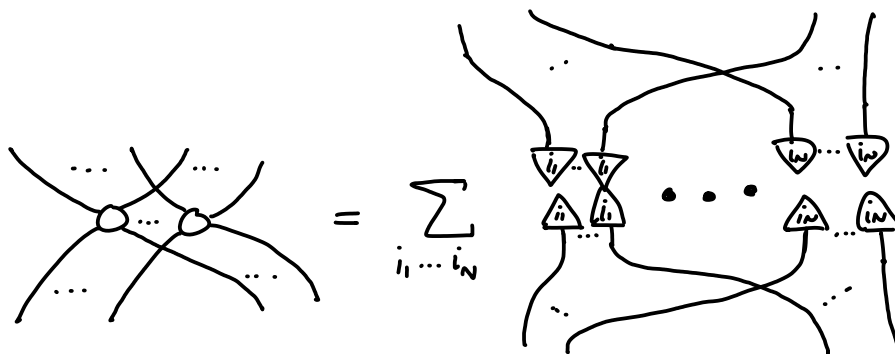


$$= \int_{i_1}^{i_2} \dots \int_{i_1}^{i_m} \int_{i_1}^{j_1} \dots \int_{i_1}^{j_n} \begin{array}{c} \downarrow \dots \downarrow \\ i_1 \dots i_1 \\ \uparrow \dots \uparrow \end{array}$$

$$= \int_{i_1 \dots i_m}^{j_1 \dots j_n} \begin{array}{c} \downarrow \dots \downarrow \\ i_1 \dots i_1 \\ \uparrow \dots \uparrow \end{array}$$

Ex 5.2

This follows just from expanding each of the spiders:



Ex 5.3

(i) Using $\cap = \cap$ & $\cup = \cup$, we have:

$$\bigcirc = \bigcirc = 0$$

(ii) We just showed $0 = \bigcirc = D$. There are 3 cases:

(1) 1-input spiders  are causal:

$$\text{1-input spider} = \text{1-input spider}$$

(2) 0-input spiders  can be renormalised to be causal:

$$\frac{1}{D} \text{0-input spider} = \frac{1}{D} 0 = \frac{1}{D} \cdot D = \text{dashed box}$$

(3) Spiders with >1 input are never causal:

$$\text{N>1 spider} = \text{N>1 spider} \neq \text{1...1 spiders}$$

Ex 5.4

$$\begin{array}{c} | \\ \hline \square \\ \hline f \\ \hline \square \\ \hline | \end{array} \leftrightarrow \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Means $\begin{array}{c} \triangle \\ | \\ \square \\ | \\ \triangle \end{array} = \begin{array}{c} \triangle \\ | \\ \square \\ | \\ \nabla \end{array} = \frac{2}{3}$ and $\begin{array}{c} \triangle \\ | \\ \square \\ | \\ \triangle \end{array} = \begin{array}{c} \triangle \\ | \\ \square \\ | \\ \nabla \end{array} = \frac{1}{3}$

↖ Z-basis ↗

But if we change to the basis $\begin{array}{c} | \\ \hline \square \\ \hline \phi_0 \\ \hline | \end{array} = \begin{array}{c} \triangle \\ | \\ \square \\ | \\ \triangle \end{array}$, $\begin{array}{c} | \\ \hline \square \\ \hline \phi_1 \\ \hline | \end{array} = -\begin{array}{c} \triangle \\ | \\ \square \\ | \\ \triangle \end{array}$

$$\begin{array}{c} \phi_0 \\ | \\ \square \\ | \\ \phi_0 \end{array} = \begin{array}{c} \triangle \\ | \\ \square \\ | \\ \triangle \end{array} = \frac{2}{3}$$

$$\begin{array}{c} \phi_1 \\ | \\ \square \\ | \\ \phi_0 \end{array} = \begin{array}{c} \triangle \\ | \\ \square \\ | \\ \triangle \end{array} = -\frac{1}{3}$$

$$\begin{array}{c} \phi_0 \\ | \\ \square \\ | \\ \phi_1 \end{array} = \begin{array}{c} \triangle \\ | \\ \square \\ | \\ \nabla \end{array} = -\frac{1}{3}$$

$$\begin{array}{c} \phi_1 \\ | \\ \square \\ | \\ \phi_1 \end{array} = \begin{array}{c} \triangle \\ | \\ \square \\ | \\ \nabla \end{array} = \frac{2}{3}$$

So $\begin{array}{c} | \\ \hline \square \\ \hline f \\ \hline \square \\ \hline | \end{array} \leftrightarrow \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

↖ w.r.t. $\{\phi_i\}$ ↗

Ex 5.5

(i) Let $\boxed{h} = \begin{array}{c} | \\ \text{---} \\ \boxed{f} \text{---} \boxed{g} \\ \text{---} \\ | \end{array}$

$$h_i^j := \begin{array}{c} \uparrow j \\ | \\ \boxed{h} \\ | \\ \downarrow i \end{array} = \begin{array}{c} \uparrow j \\ | \\ \boxed{f} \text{---} \boxed{g} \\ | \\ \downarrow i \end{array} = \begin{array}{c} \uparrow j \\ | \\ \boxed{f} \\ | \\ \downarrow i \end{array} \begin{array}{c} \uparrow j \\ | \\ \boxed{g} \\ | \\ \downarrow i \end{array} = f_i^j g_i^j$$

Hence: $\begin{array}{c} | \\ \text{---} \\ \boxed{f} \text{---} \boxed{g} \\ \text{---} \\ | \end{array}$ has matrix $\begin{pmatrix} h_i^i & \dots & h_i^D \\ \vdots & \ddots & \vdots \\ h_1^i & \dots & h_1^D \end{pmatrix} = \begin{pmatrix} f_i^i g_i^i & \dots & f_i^D g_i^D \\ \vdots & \ddots & \vdots \\ f_1^i g_1^i & \dots & f_1^D g_1^D \end{pmatrix}$

(ii) For $\boxed{g} = \sum_{ij} p_i^j \begin{array}{c} \downarrow j \\ | \\ \triangle \\ | \\ \uparrow i \end{array}$, let $\boxed{f} = \sum_{ij} \sqrt{p_i^j} \begin{array}{c} \downarrow j \\ | \\ \triangle \\ | \\ \uparrow i \end{array} = \boxed{s}$

↑ positive ↑ real

Then, by (i) $\begin{array}{c} | \\ \text{---} \\ \boxed{f} \text{---} \boxed{f} \\ \text{---} \\ | \end{array}$ has matrix entries $\sqrt{p_i^j} \cdot \sqrt{p_i^j} = p_i^j$.

Hence $\boxed{g} = \begin{array}{c} | \\ \text{---} \\ \boxed{f} \text{---} \boxed{f} \\ \text{---} \\ | \end{array} = \begin{array}{c} | \\ \text{---} \\ \boxed{s} \\ \text{---} \\ | \end{array}$

(iii) If \boxed{f} is a function-map, then

$$\boxed{f} = \boxed{f} \quad \text{and} \quad \begin{array}{c} \cup \\ \circ \\ | \\ \boxed{f} \end{array} = \begin{array}{c} \boxed{f} \quad \boxed{f} \\ \cup \\ \circ \\ | \end{array}$$

So:

$$\boxed{f} = \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \boxed{f} \end{array} = \begin{array}{c} \circ \\ \cup \\ \boxed{f} \quad \boxed{f} \\ \cup \\ \circ \\ | \end{array} = \begin{array}{c} \circ \\ | \\ \circ \\ \cup \\ \boxed{f} \quad \boxed{f} \\ \cup \\ \circ \\ | \end{array} = \begin{array}{c} \circ \\ | \\ \boxed{f} \\ \circ \\ | \end{array} .$$

Ex 5.6

$\hat{\Psi}$ is unbiased for θ iff $\hat{\Psi} \circ \theta = \theta$.

For (\Rightarrow) , assume $\hat{\Psi}$ unbiased. Then:

$$\hat{\Psi} \circ \theta = \hat{\Psi} \circ \theta = \frac{1}{D} \theta = \theta.$$

For (\Leftarrow) , assume $\hat{\Psi} \circ \theta = \theta$ for all θ . Then:

$$\hat{\Psi} \circ \theta = \hat{\Psi} \circ \theta = \frac{1}{D} \theta = \theta$$

Hence $\hat{\Psi} \circ \theta = \theta$.

Ex 5.7

Unit.

$$\mathbb{1} = \text{double}(\mathbb{1}) = \text{double}\left(\sum_j \downarrow_j\right) = \text{double}\left(\sum_j e^{i\alpha_j} \downarrow_j\right)$$

for all $\alpha_j = 0$, i.e. $\vec{\alpha} = (0, \dots, 0)$.

$$\begin{aligned} \begin{array}{c} \text{---} \\ | \\ \circ \\ / \quad \backslash \\ \circ \quad \circ \\ \vec{\alpha} \quad \vec{\beta} \end{array} &= \sum_{jk} e^{i\alpha_j} e^{i\beta_k} \begin{array}{c} \text{---} \\ | \\ \circ \\ / \quad \backslash \\ \nabla_j \quad \nabla_k \end{array} = \sum_{jk} d_j^k e^{i(\alpha_j + \beta_k)} \downarrow_j \\ &= \sum_j e^{i(\alpha_j + \beta_j)} \downarrow_j \end{aligned}$$

$$\text{So } \vec{\alpha} + \vec{\beta} = (\alpha_1 + \beta_1, \dots, \alpha_D + \beta_D)$$

$$\overline{\left(\downarrow_{\vec{\alpha}}\right)} = \sum_j \overline{e^{i\alpha_j}} \downarrow_j = \sum_j e^{-i\alpha_j} \downarrow_j.$$

$$\text{So } -\vec{\alpha} = (-\alpha_1, \dots, -\alpha_D).$$