UNIVERSITY COLLEGE LONDON

MRES THESIS

The ZX calculus, Non-Locality and Spekkens toy theory

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"Science is a great game. It is inspiring and refreshing. The playing field is the universe itself."

Isaac Rabi

University College London

Abstract

Faculty of Sciences
Department of Physics and Astronomy

The ZX calculus, Non-Locality and Spekkens toy theory

by Maria STASINOU

This thesis is about a graphical calculus, a toy theory and the concept of non-locality. Although, at a first glance, they seem three totally different topics, we will see that they make up a very nice and enjoyable story all together. The main research result of this thesis is related to the foundations of quantum mechanics and it can constitute an evidence of the great usefulness of quantum information tools in studying the underlying *pieces of structure* of quantum theory.

Acknowledgements

I would like to thank the UCL-CDT and all its co-ordinators who opened the path of the scientific research to me. I would like to thank in particular, Dan Browne, who was offering his guidance and advice throughout the year. I feel very grateful of Bob Coecke, the supervisor of this thesis, who is giving me the chance for a PhD in his group. Finally, many thanks to my family and friends for their support.

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Chapter 1

Background

1.1 Process theories

A physicist could define a *process theory* as a theory that focuses on processes, namely all the operations that an experimentalist could perform in the lab, such as transformations, preparations and measurements. The peculiarity of process theories is that we treat those kind of operations and the state of the system on equal footing, while traditionally in quantum mechanics our main focus is on states. States are then considered as a special kind of a preparation process. The process theory framework also admits a diagrammatic language, which happens to be more intuitive in nature than our traditional Hilbert space formalism: we can actually visualize the processes that take place. We aim at establishing diagrams for reasoning about all the essential features of quantum mechanics, setting therefore a new foundation for quantum theory.

1.2 A bit of category theory

The mathematical language of the process theory framework lies within the context of category theory. In particular, categorical quantum mechanics (CQM) was initiated by Coecke and Ambramsky in (S.Ambramsky and B.Coecke, 2004). CQM has solved open problems in Quantum Information and Computation, e.g.(B.Coecke and A.Kissinger, 2010; R.Duncan and S.Pedrix, 2010; Horsman, 2011). It has also been accepted by leading researchers in the field such as(G.Chiribella and P.Perinotti, 2010).

We begin by providing the formal definition of a category: A category C consists of

- A family $|C|$ of objects.
- For any A and $B \in |C|$, a set of $C(A,B)$ of morphisms.
- For any $A, B, C \in |C|$ and any $f \in C(A,B)$ and $g \in C(B,C)$, a composite $g \circ f \in C(A,C)$, i.e. there is a composition operation

$$- \circ - : C(A,B) \times C(B,C) \rightarrow C(A,C) :: (f,g) \rightarrow g \circ f$$

- For any $f \in C(A,B)$, $g \in C(B,C)$ and $h \in C(C,D)$ we have

$$h \circ (f \circ g) = (h \circ f) \circ g$$

- For any morphism $A \in |C|$ there exists a morphism $1_A \in C(A,A)$ called the identity.

This definition was proposed by Samuel Eilenberg and Saunders Mac Lane in 1945 with the aim to unify a variety of mathematical constructions(S.Eilenberg and S.MacLane, 1945). In this context the objects are the mathematical structures and the morphisms, structure-preserving maps between these. For our intuition, we mention the category **FdVectK**: It comprises of

- Finite dimensional vector spaces over K as objects
- Linear maps between these as morphisms
- ordinary composition of functions
- identity functions.

Of particular interest to us are the *strict monoidal categories* that come with an additional structure:

- For all objects A, B, C, D there exists an operation

  $$- \otimes - : C(A, B) \times C(C, D) :\to C(A \otimes C, B \otimes D) : (f, g) \to f \otimes g$$

  which is associative and has a unit, i.e.

  $$f \otimes (g \otimes h) = (f \otimes g) \otimes h$$

  and $1_I \otimes f = f \otimes 1_I = f$.

- For all morphisms f, g, h, k we have

  $$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

- For all objects $A, B \in |C|$

  $$1_A \otimes 1_B = 1_{A \otimes B}$$

  .

The $\otimes$ symbol is not necessary the tensor product (well, in our case, it is). In the context of category theory it provides a means for dealing with situations where several systems are involved. The most powerful characteristic of strict symmetric monoidal categories (strict monoidal categories endowed with a symmetry operation) is that they admit a graphical calculus. It is worth noting that the graphical language is typically simpler than its formal counterpart cause it has the potential to trivialize some complicated algebraic manipulations.

We will now indicate the graphical counterparts to strict symmetric monoidal structure(B.Coecke and E.O.Paquette, 2009):

- The identity 1_A:

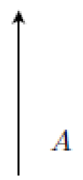


FIGURE 1.1: (B.Coecke and E.O.Paquette, 2009)

- A morphism $f : A \to B$.

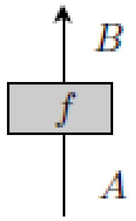


FIGURE 1.2: (B.Coecke and E.O.Paquette, 2009)

- Composition of morphisms $f : A \to B$, $g : B \to C$, which is represented by connecting the output of f to the input of g:

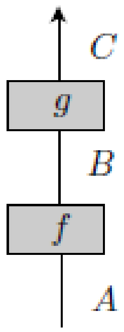


FIGURE 1.3: (B.Coecke and E.O.Paquette, 2009)

- tensor product of morphisms, $f : A \to B$, $g : C \to D$, which is depicted by aligning the representations of f and g side by side:

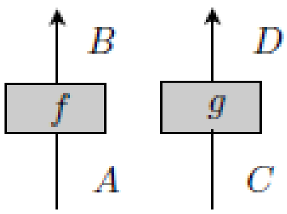


FIGURE 1.4: (B.Coecke and E.O.Paquette, 2009)

- Symmetry:

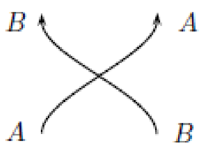


FIGURE 1.5: (B.Coecke and E.O.Paquette, 2009)

- Morphisms depicted as $\psi : I \to A$, $\phi : A \to I$, $s : I \to I$

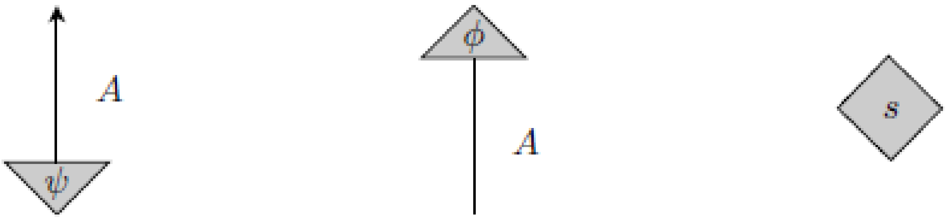


FIGURE 1.6: (B.Coecke and E.O.Paquette, 2009)

- The diamond shape arises when we compose two triangles:

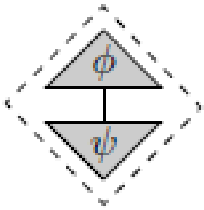


FIGURE 1.7: (B.Coecke and E.O.Paquette, 2009)

If we look at those morphisms a bit carefully we see that they represent our familiar 'bras' and 'kets' of quantum mechanics. In particular, the morphism on the left is a 'ket', the morphism in the middle is a 'bra', while the diamond shape is related to probabilities.

1.3 A distinction between classical and quantum systems

Based upon this formalism, we aim to construct a theory than includes the interaction of classical and quantum systems. Those two kind of systems admit fundamental differences, and therefore, we should distinguish between them. We account for two different types of wires: Quantum systems are represented by thick, black wires and classical systems by single ones.

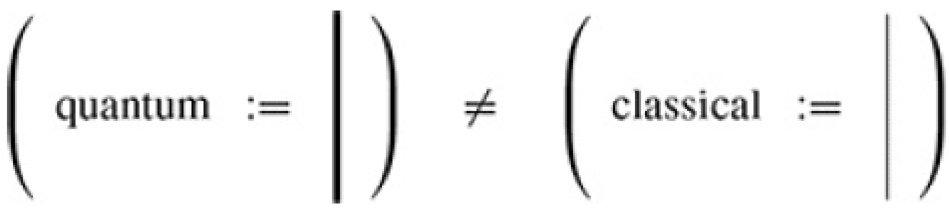


FIGURE 1.8: (Coecke and Kissinger, 2017)

The transitions from the classical to quantum world is achieved by a process called 'doubling': We can think of it as the transition of the state to the density matrix formalism in quantum mechanics. In the case of plain wires, if we 'unfold' the quantum wire, we will find inside two classical wires:

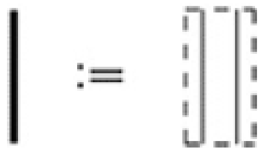


FIGURE 1.9: (Coecke and Kissinger, 2017)

Classical wires allow us to encode classical values as basis states of an orthonormal basis. We adopt the following representation for the states of an orthonormal basis:

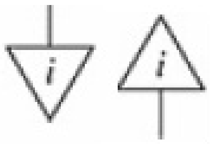


FIGURE 1.10: (Coecke and Kissinger, 2017)

The state on the left *provides* the classical value 'i', while the state on the right *tests* for the classical value 'i'. The orthonormality condition is represented as

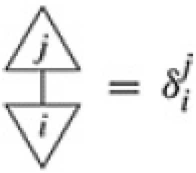


FIGURE 1.11: (Coecke and Kissinger, 2017)

1.4 Particularly important maps

We continue by defining some types of linear maps that constitute the core of the ZX calculus, the graphical language that this thesis is about. We first provide the notation of the **measurement** and **encoding** maps respectively:

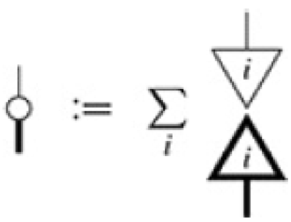


FIGURE 1.12: (Coecke and Kissinger, 2017)

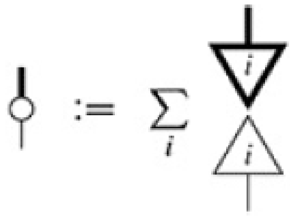


FIGURE 1.13: (Coecke and Kissinger, 2017)

We see what these maps do, by feeding them with suitable inputs. We thus apply the 'measurement map' to an arbitrary quantum state:

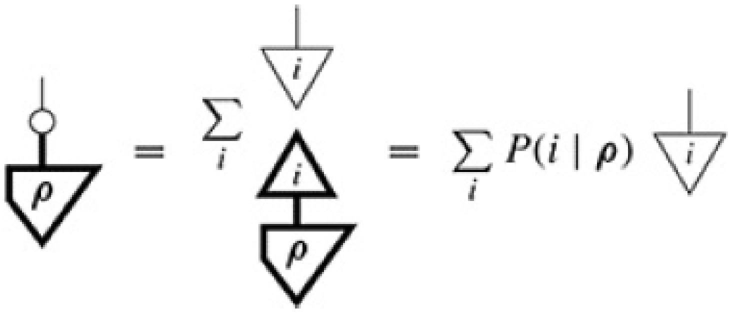


FIGURE 1.14: (Coecke and Kissinger, 2017)

We infer that the 'measurement' linear map sends a quantum state to a probability distribution. The 'encoding' map on the other hand, which is considered the adjoint of 'measurement', encodes a probability distribution to a quantum state:

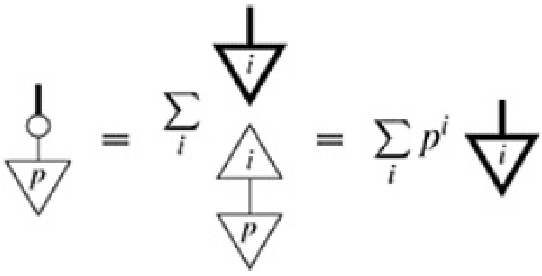


FIGURE 1.15: (Coecke and Kissinger, 2017)

Hence, we infer that the above maps provide a means of transition from the classical to quantum world:

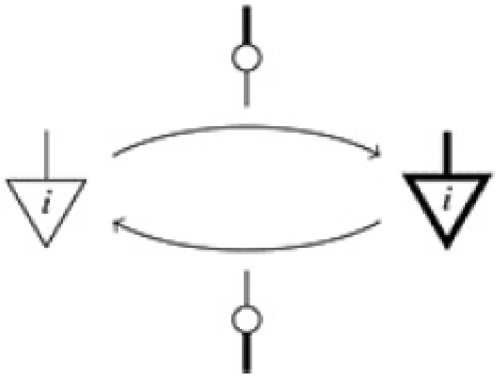


FIGURE 1.16: (Coecke and Kissinger, 2017)

We continue by introducing a process called 'deleting':

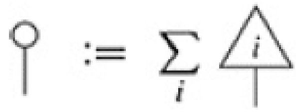


FIGURE 1.17: (Coecke and Kissinger, 2017)

When we apply 'deleting' to a classical value, the value vanishes. That is,

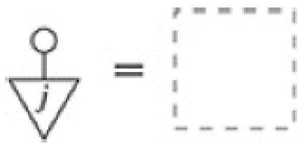


FIGURE 1.18: (Coecke and Kissinger, 2017)

The adjoint of 'deleting' is the map

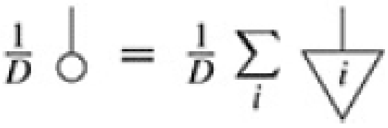


FIGURE 1.19: (Coecke and Kissinger, 2017)

which represents the uniform probability distribution.

In addition to deleting, we can also copy classical data by defining the 'copying' map:

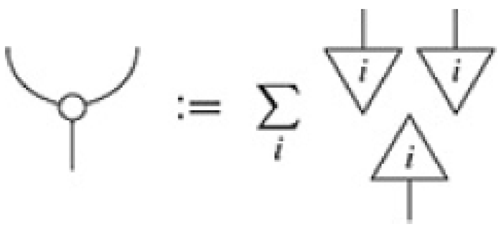


FIGURE 1.20: (Coecke and Kissinger, 2017)

When we feed in a classical basis state, we get a doubled version of it:

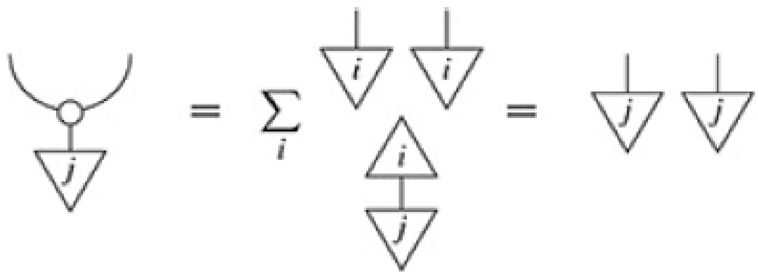


FIGURE 1.21: (Coecke and Kissinger, 2017)

The adjoint of ‘copying’ is ‘matching’:

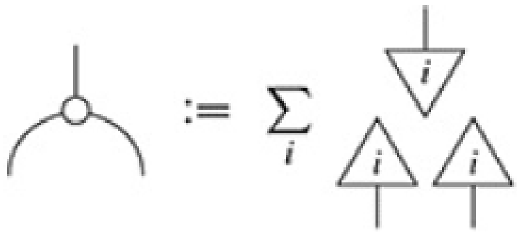


FIGURE 1.22: (Coecke and Kissinger, 2017)

This map takes in two states and if they are the same, it sends one of them out:

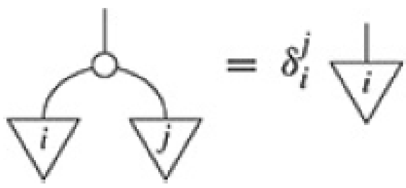


FIGURE 1.23: (Coecke and Kissinger, 2017)

We can also represent states of perfect correlation:

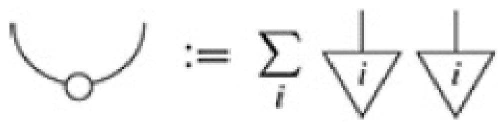


FIGURE 1.24: (Coecke and Kissinger, 2017)

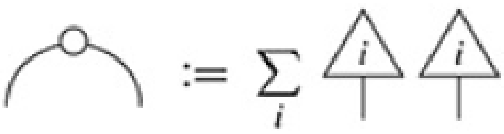


FIGURE 1.25: (Coecke and Kissinger, 2017)

1.5 Spiders

The above were special cases of more general types of maps called 'spiders'. Spiders are linear maps of the form:

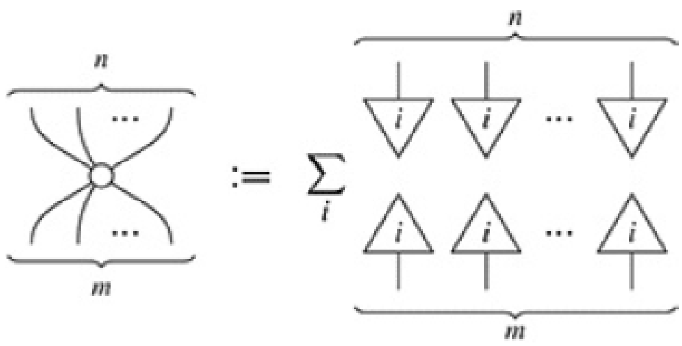


FIGURE 1.26: (Coecke and Kissinger, 2017)

Spiders force the inputs and outputs being in the same basis. If we are to compute the matrix form of a spider then it takes the form of a 'big' Kronecker delta.

If two spiders touch, they fuse together. This can be depicted as:

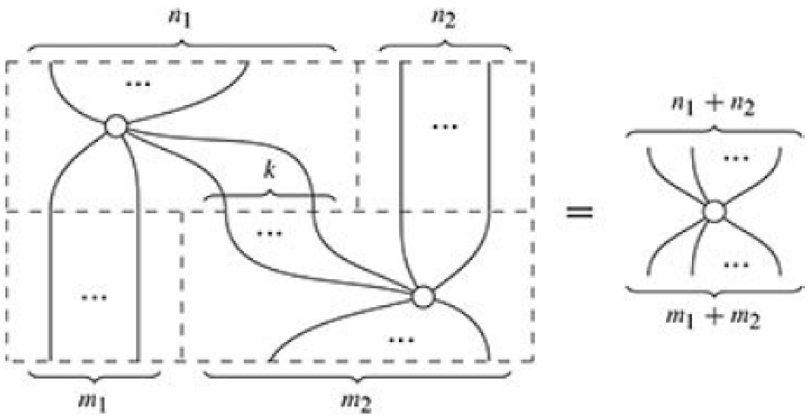


FIGURE 1.27: (Coecke and Kissinger, 2017)

Moreover, we are able to define quantum spiders (that admit exclusively quantum inputs and outputs) by doubling the classical spiders:

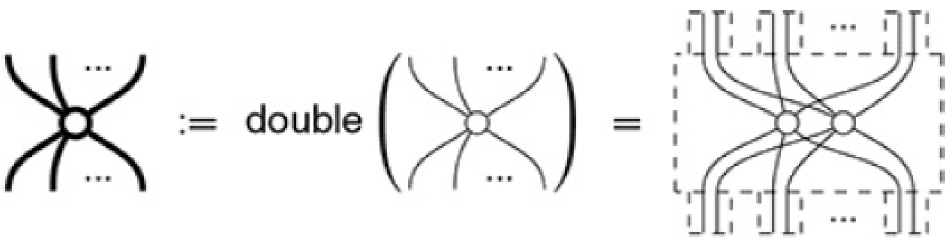


FIGURE 1.28: (Coecke and Kissinger, 2017)

Quantum spiders also fuse together:

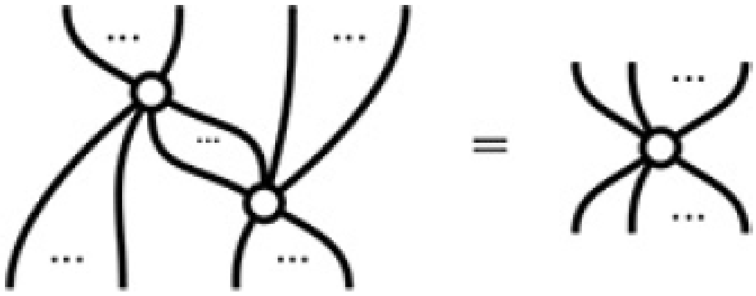


FIGURE 1.29: (Coecke and Kissinger, 2017)

Via quantum spiders we can represent the two and three system spider states, which correspond to our familiar Bell and GHZ states respectively:

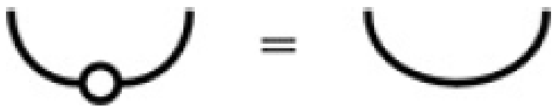


FIGURE 1.30: (Coecke and Kissinger, 2017)

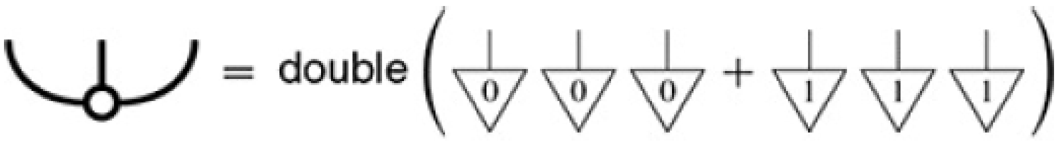


FIGURE 1.31: (Coecke and Kissinger, 2017)

Spiders that are characterized by both quantum and classical legs are known as 'bastard spiders':

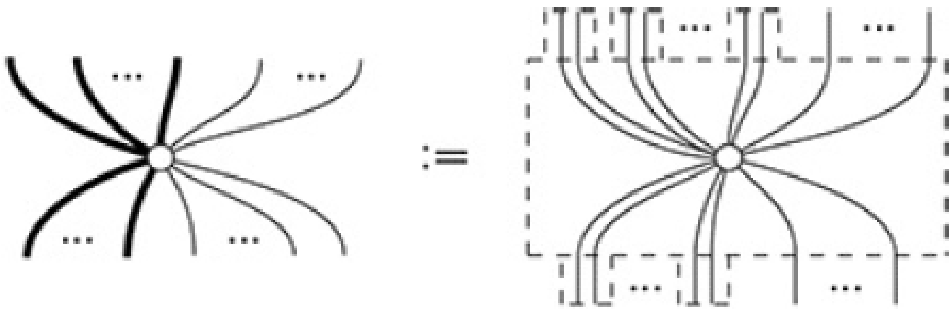


FIGURE 1.32: (Coecke and Kissinger, 2017)

Bastard spiders also fuse together:

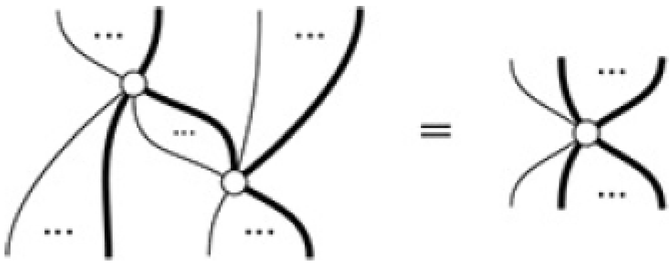


FIGURE 1.33: (Coecke and Kissinger, 2017)

1.6 Phases and the notion of complementarity

Spiders can be decorated by phases:

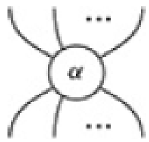


FIGURE 1.34: (Coecke and Kissinger, 2017)

When phase spiders touch, they fuse together:

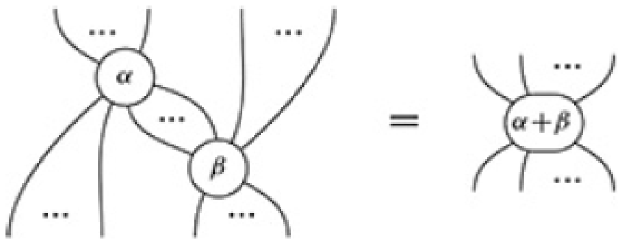


FIGURE 1.35: (Coecke and Kissinger, 2017)

Furthermore, when a quantum phase spider attempts to make contact with the classical world, its decoration vanishes:

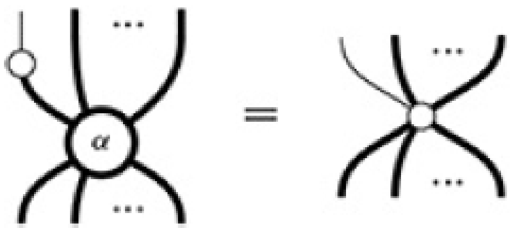


FIGURE 1.36: (Coecke and Kissinger, 2017)

The phase data in this context, are the analogue of the relative phases in quantum mechanics. That is, phases represent maximally quantum data.

We also define spiders of different colors,

FIGURE 1.37: (Coecke and Kissinger, 2017)

that correspond to two different orthonormal basis. In particular, the white spiders admit inputs and outputs in the Z basis, while the gray spiders in the X. While spiders of the same color fuse together, spiders of different colors do not:

FIGURE 1.38: (Coecke and Kissinger, 2017)

Essentially,

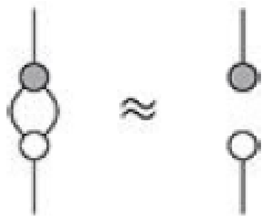


FIGURE 1.39: (Coecke and Kissinger, 2017)

or

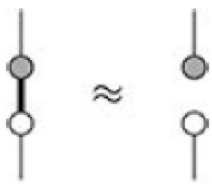


FIGURE 1.40: (Coecke and Kissinger, 2017)

which is known as the 'complementarity relation.' There is a clear physical meaning behind the last equation: *If we measure in the Z basis and then encode in the X, then there is no data flow.* The notion of complementarity applies to classical, quantum as well as bastard spiders:

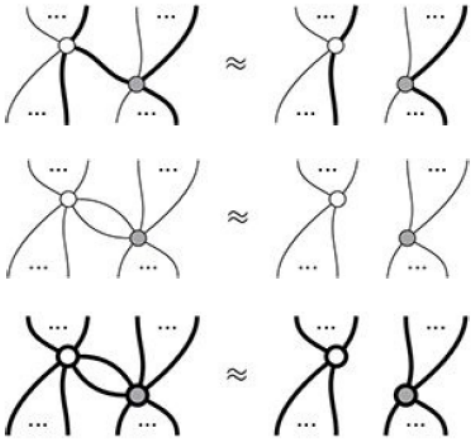


FIGURE 1.41: (Coecke and Kissinger, 2017)

We will now state the definition of an 'unbiased state:'

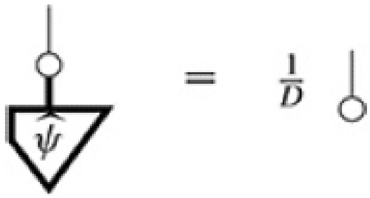


FIGURE 1.42: (Coecke and Kissinger, 2017)

We call a state 'unbiased', if when we perform a measurement on it, we get the uniform probability distribution. The state has no bias towards a particular measurement outcome, and therefore it is 'unbiased.' In terms of matrices, we can associate to every state, a phase:

$$\begin{pmatrix} \psi^0 \\ \vdots \\ \psi^{D-1} \end{pmatrix} = \begin{pmatrix} e^{i\alpha_0} \\ \vdots \\ e^{i\alpha_{D-1}} \end{pmatrix}$$

FIGURE 1.43: (Coecke and Kissinger, 2017)

We call the above state a *phase state.* From now on we will symbolize phase states as

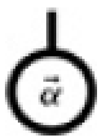


FIGURE 1.44: (Coecke and Kissinger, 2017)

The defining equality for phase states now is

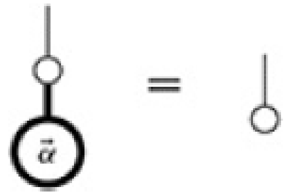


FIGURE 1.45: (Coecke and Kissinger, 2017)

As we have already mentioned, the 'phase' represents maximally quantum data, which are destroyed by a quantum-classical passage. For our intuition let's look at the form of the phase state in two dimensions

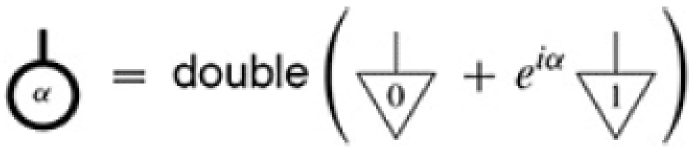


FIGURE 1.46: (Coecke and Kissinger, 2017)

and its representation in the Bloch sphere:

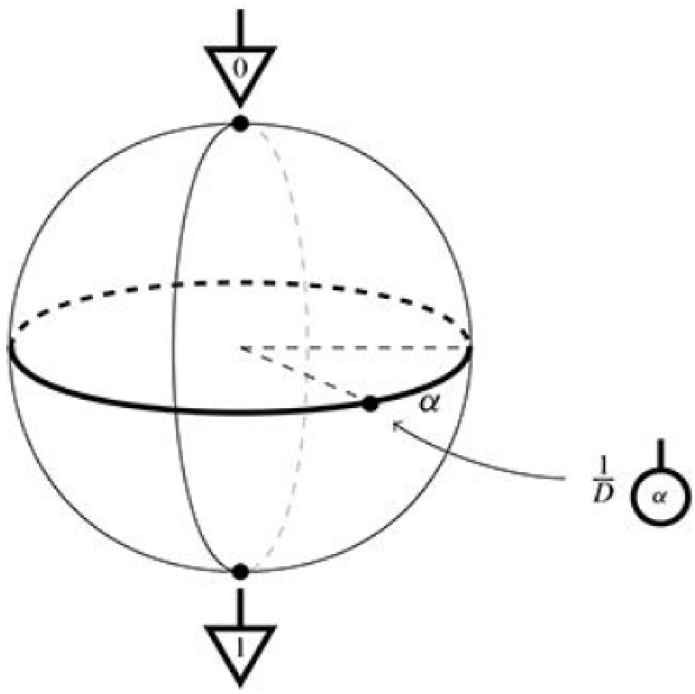


FIGURE 1.47: (Coecke and Kissinger, 2017)

Phase states admit a rigorous definition in the diagrammatic formalism. This is the reason why they play an important role in other process theories as well. One example is the Spekkens toy theory, which we will examine in Chapter 4.

Since phases constitute maximally quantum data, they decorate quantum spiders. Phase spiders are quantum maps of the form:

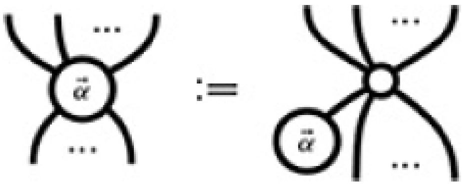


FIGURE 1.48: (Coecke and Kissinger, 2017)

Phase spiders also fuse together:

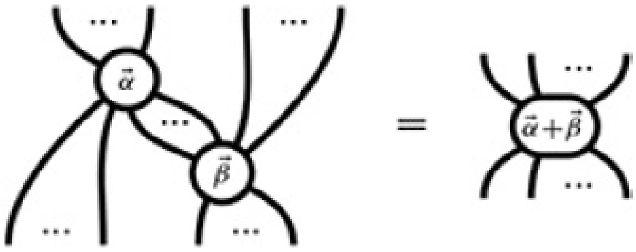


FIGURE 1.49: (Coecke and Kissinger, 2017)

We can furthermore define phase gates, i.e. quantum processes of the form:

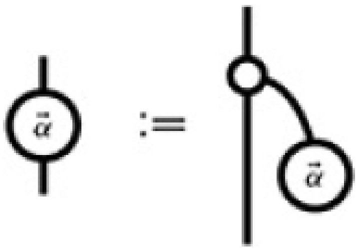


FIGURE 1.50: (Coecke and Kissinger, 2017)

Phase gates are in fact unitary operators that correspond to rotations around the Z axis in the Bloch sphere.

Having defined what a phase state is, we can state two equations equivalent to complementarity:

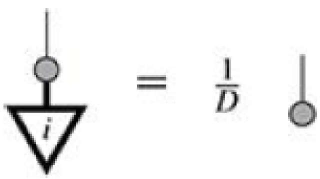


FIGURE 1.51: (Coecke and Kissinger, 2017)

and

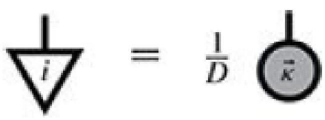


FIGURE 1.52: (Coecke and Kissinger, 2017)

The first equation states that if we measure a Z-basis state with respect to the X basis, we get the uniform probability distribution in the X basis. The second equations states that we can associate a basis state with a phase state of the complementary color. The above equations are valid with the colors reversed as well. We present below the Bloch sphere in terms of the basis states (on the left) and in terms of the phase states (on the right):

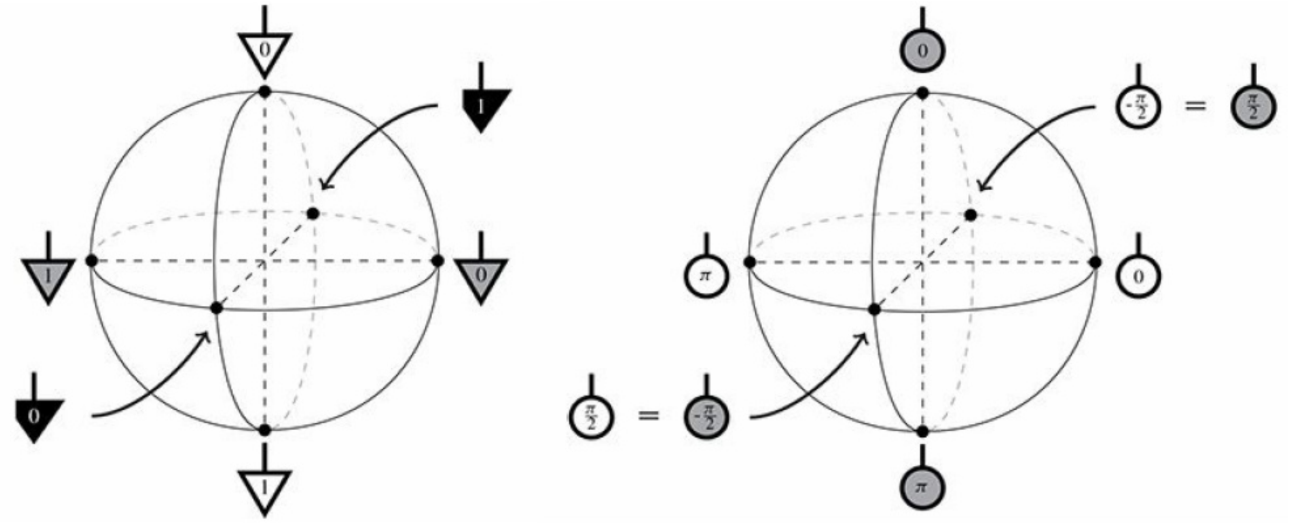


FIGURE 1.53: (Coecke and Kissinger, 2017)

1.7 Strong complementarity

We refer now to the notion of strong complementarity, which plays an important role when proving diagrammatically that quantum mechanics is a non-local theory. We will present the particular proof in Chapter 3. This notion is considered stronger than complementarity, as strongly complementary observables can be completely classified. This has not been yet achieved for ordinary complementarity. More precisely, we do not know how many pairwise complementary measurements are just in the sixth dimension. On the other hand, we know everything about strongly complementary observables, as they are classified by commutative groups. Strong complementarity was first introduced by Bob Coecke et.al(B.Coecke and Q.Wang, 2012).

We can simply define 'strong complementarity' as a set of additional rules that arise naturally in the diagrammatic formalism:

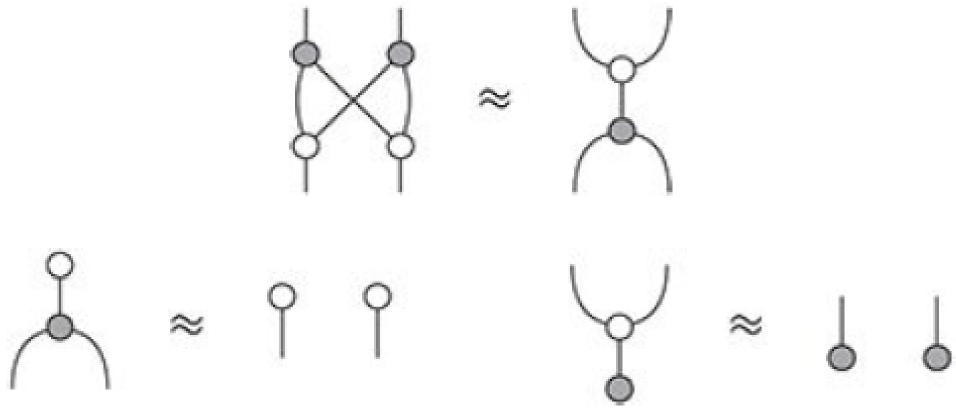


FIGURE 1.54: (Coecke and Kissinger, 2017)

We will make use of them when *depicting* non-locality.

Chapter 2

The ZX calculus

2.1 The basics

We now turn our attention to a graphical language, which exploits the framework presented in Chapter 1. ZX calculus is a formalism for pure state qubit quantum mechanics with post-selected measurements(B.Coecke and R.Duncan, 2011). Having acquired the ability to represent states and post-selected measurements, ZX calculus can simulate measurement based quantum computing in a more efficient way than circuits(Duncan and Perdrix, 2010).

We impose two main questions:

- Which classical-quantum maps can we express using just phase-Z and X spiders?
- Which equations can we prove using ZX calculus?

The answer to the first question is positive. It turns out that we can build any linear map just using phase-Z and X spiders. The answer to the second question is again positive and related to the notion of completeness. We say that a graphical language is *complete* when two diagrams are equal and we can move from one diagram to another by using the graphical rules. ZX calculus has recently been completed for the Clifford+T group in quantum mechanics(E.Jeandel and R.Vilmart, 2017).

In our case we have focused on the study of the ZX calculus for Clifford maps, a subtheory of quantum maps with phases restricted to multiples of $\frac{\pi}{2}$. The theory of Clifford maps is adequate for proving that quantum theory is non-local. It relies on four families of equations, sufficient to prove everything. The first two show us how spiders of different colors fuse together:

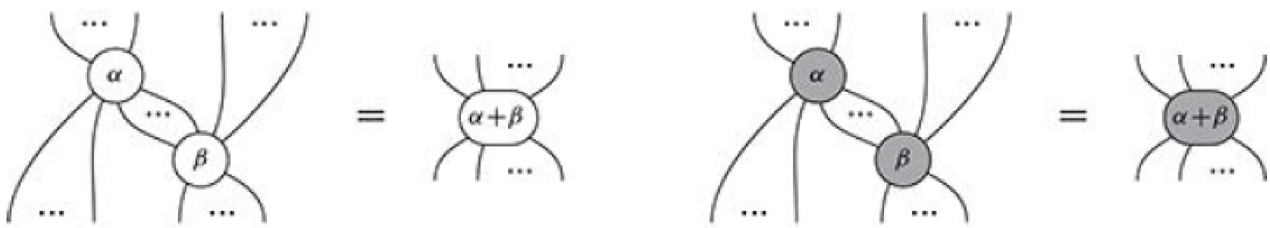


FIGURE 2.1: (Coecke and Kissinger, 2017)

The third equation shows how spiders of different colors can commute past each other:

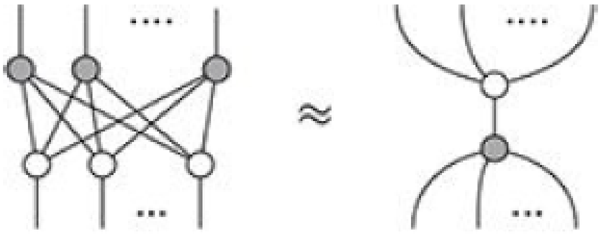


FIGURE 2.2: (Coecke and Kissinger, 2017)

This is a diagrammatic equation equivalent to strong complementarity. The fourth equation is:

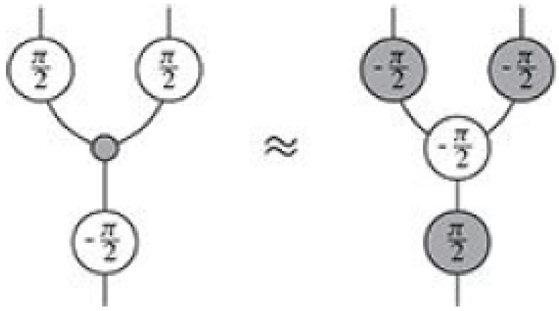


FIGURE 2.3: (Coecke and Kissinger, 2017)

It is connected to the geometry of the Bloch sphere and we will prove it in the following.

2.2 Universality

We begin by giving the definition of a ZX diagram: A ZX diagram is simply a diagram consisting only of Z and X spiders:

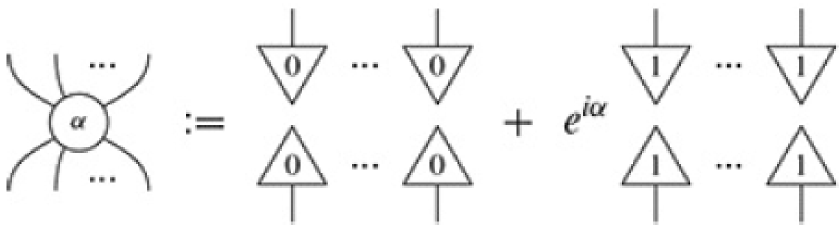


FIGURE 2.4: (Coecke and Kissinger, 2017)

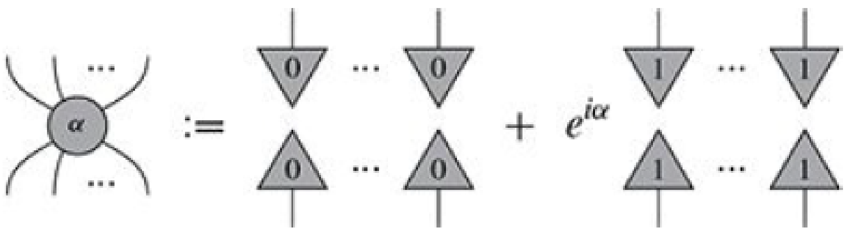


FIGURE 2.5: (Coecke and Kissinger, 2017)

The property of universality requires that every process can be represented graphically. Therefore, we allow processes that are built using only Z and X diagrams. We begin by recalling that every rotation can be decomposed as three rotations about a pair of orthogonal axis. Rotations are expressed through unitary operators. We can choose our pair of axis being the Z and X axis. Then, every unitary can be represented as a ZX diagram with phases (corresponding to the Euler angles) α, β and γ :

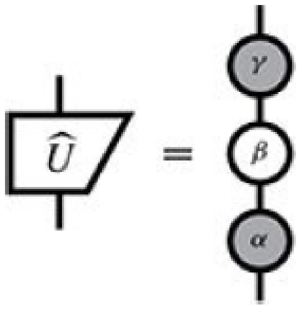


FIGURE 2.6: (Coecke and Kissinger, 2017)

We can now apply the unitary and recover any state we wish. Therefore, states admit a ZX diagram representation as well:

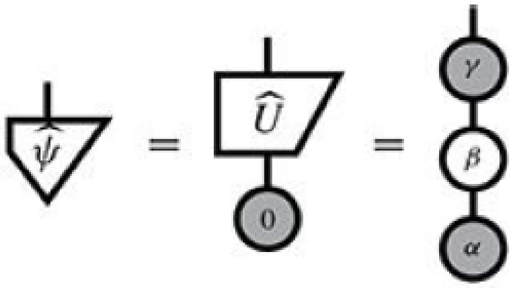


FIGURE 2.7: (Coecke and Kissinger, 2017)

By applying the process-state duality theorem, we can recover as well, any linear map as a ZX diagram.

2.3 ZX calculus and Clifford Diagrams

Stabilizer quantum mechanics plays a central role in error correcting codes and measurement-based quantum computing. Operationally, it can be described in the fragment of quantum theory where the allowed transformations, preparations and measurements belong to the Clifford group. ZX calculus has been adopted in this context(M.Backens, 2014). ZX calculus is complete for pure qubit stabilizer quantum mechanics.

We define a Clifford diagram as a ZX diagram with phases restricted to multiples of $\pi/2$. We depict them in the Bloch sphere,

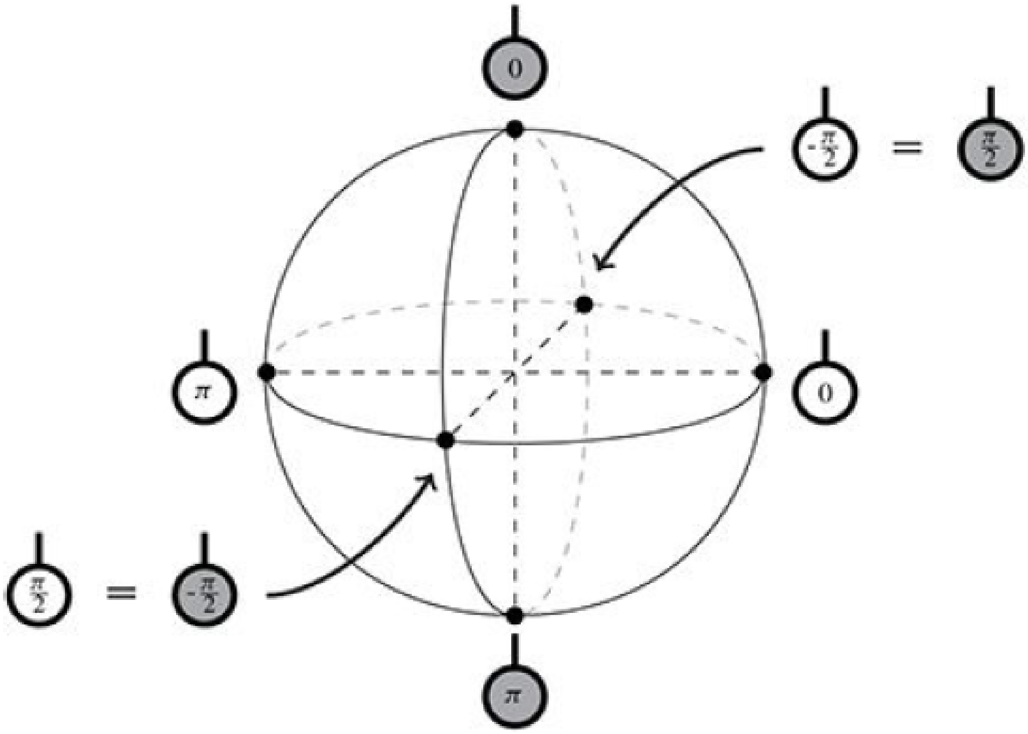


FIGURE 2.8: (Coecke and Kissinger, 2017)

The corresponding phase group is the Z4 of U(1), which we depict as four little wheels for our intuition:

FIGURE 2.9: (Coecke and Kissinger, 2017)

We gave previously the definition of a Clifford diagram. We now define Clifford maps being quantum maps that arise by doubling those linear maps that can be expressed as Clifford diagrams. Clifford maps are complete in the context of the graphical calculus that applies for Clifford diagrams. This can be translated explicitly as when two Clifford maps are equal, the rules of graphical language can be used for moving from one diagram to the other.

In order to have a complete theory for qubits, Y spiders should somehow participate in the formalism. Indeed, this is the case, as the Y states are expressed with respect to the Z and X phase states via the relation:

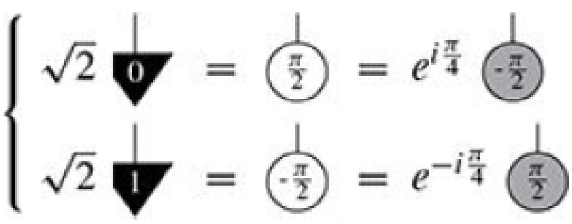


FIGURE 2.10: (Coecke and Kissinger, 2017)

We now have two ways to copy the Y spiders:

1. First, we perform a $-\pi/2$ rotation around the Z axis, sending the Y-basis to the X basis. We then copy the X-basis via the X-copy map and by performing a $\pi/2$ rotation, we send the X basis back to the Y basis.

2. The second way to perform the same task, is doing the same procedure with the colors reversed.

We depict the first and second procedure respectively:

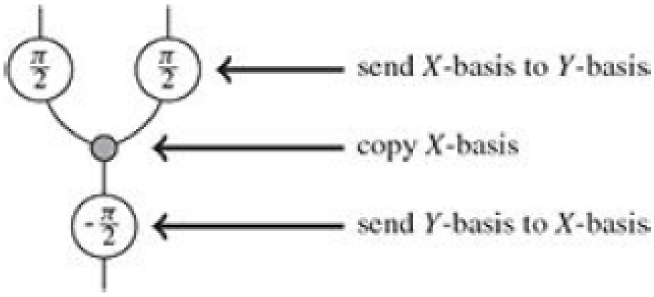


FIGURE 2.11: (Coecke and Kissinger, 2017)

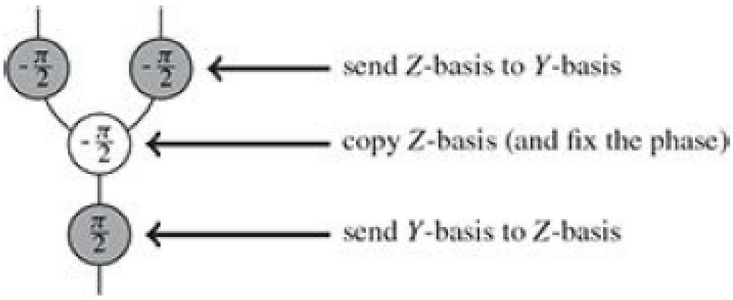


FIGURE 2.12: (Coecke and Kissinger, 2017)

In the second case we account for the phase difference between the phase states. The phases are $e^{i\pi/4}$ and $e^{-i\pi/4}$. Thus, the overall phase difference is $\pi/2$. We account for this by placing a phase of $-\pi/2$ in the white spider. Hence, we arrive at the following diagrammatic relation which constitutes also the 4th rule:

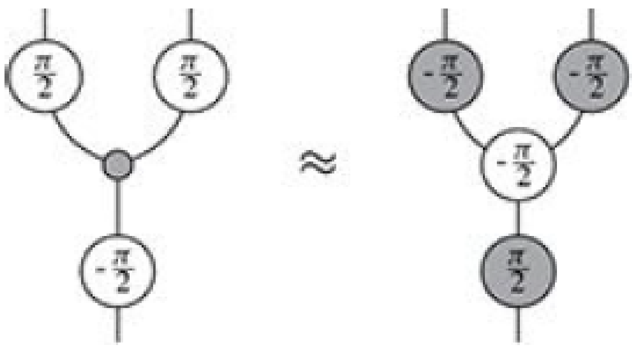


FIGURE 2.13: (Coecke and Kissinger, 2017)

2.4 Where we stand with the ZX calculus?

ZX calculus is an incomplete graphical language. Incompleteness means that if we have two diagrams representing the same process, there is no way of expressing one diagram into the other using the rewrite rules. There is very little hope that it can be completed with a finite set of rules. The incompleteness of the full ZX calculus stems from the fact that we allow arbitrary processes and phase angles. However, there exist completeness results for different fragments of the language ($\frac{\pi}{2}$-fragment (M.Backens, 2014); π-fragment (R.Duncan and S.Predrix, 2013); $\frac{\pi}{4}$-fragment (M., 2014)). The main obstacle so far was the completeness result for a universal fragment in quantum mechanics, which guarantees that any true property can be proved using the ZX calculus. The fragments that we mentioned above are not universal and in fact every quantum algorithm in the context of them, can be efficiently simulated by a classical computer.

Therefore, there were efforts that aimed to provide a completeness result for the Clifford+T fragment in quantum mechanics, which is approximately universal and widely used in quantum computing. The first successful attempt has been made by M.Backens(M., 2014) for single qubits. The many qubit case was considered an open question to this day. Recently, this question has been answered (E.Jeandel and R.Vilmart, 2017) using ZW-calculus, a graphical calculus that existed prior to the ZX-calculus and is based on the interactions of GHZ and W states (Coecke and Kissinger, 2010).

2.5 Author's suggestions for moving forward

We propose the following ways for moving forward:

- Can we treat the Y-spiders on equal footing with the Z and X spiders, therefore forming a concise extension from the ZX calculus to the ZXY calculus?
- What would be the implications in other process theories such as Spekken's toy theory? (Chapter 4 of the thesis). In Chapter 2, we present the depiction of non-locality using ZX-calculus. How this would be modified if we used a ZXY calculus?
- Can we model rigorously other fundamental notions of quantum mechanics, such as contextuality, using ZX calculus?
- ZX calculus is a graphical language for pure state qubit quantum mechanics. Is there a way to generalize it to mixed states?
- What about infinite Hilbert space? Can we formulate notions such as strong complementarity in this case?

Chapter 3

Depicting Non-Locality

The notion of non-locality has been concerning physicists for years. It is a phenomenon that contradicts a collection of notions known as **local realism**, which intuitively hold in classical physics. In particular, our intuition forces us to assume the following notions for every physical theory:

- Realism: The properties of the systems are characterized by values that exist before a measurement is made.
- Locality: An object is influenced directly only by its immediate surroundings.

It is a well-known fact that quantum theory is a non-local realistic theory. In this chapter we present a proof in the diagrammatic formalism, using ZX calculus as a tool. In particular, we establish a contradiction between quantum theory and local hidden variable models. For this purpose, we should make a careful choice of measurement scenarios. This proof was first presented in (B.Coecke and Q.Wang, 2012).

3.1 GHZ Mermin Scenarios

We start by considering measurements on the GHZ state:

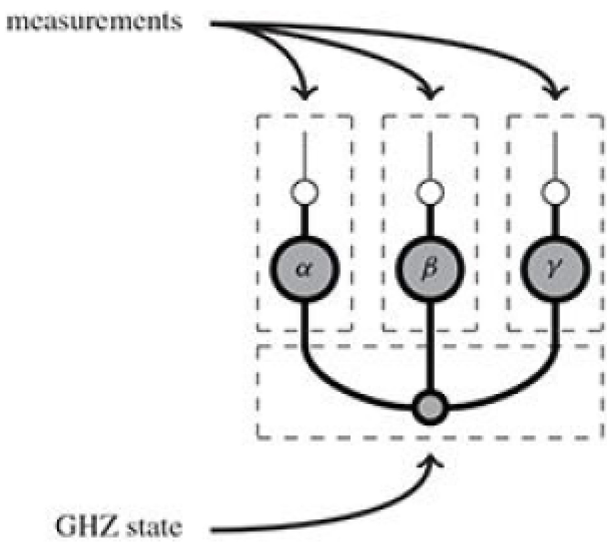


FIGURE 3.1: (Coecke and Kissinger, 2017)

Using phase spider fusion and strong complementarity we obtain:

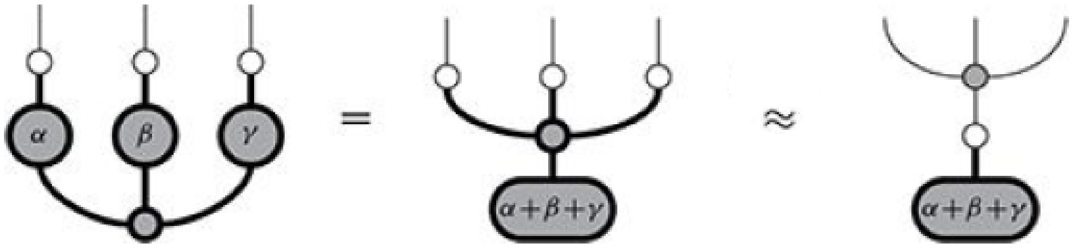


FIGURE 3.2: (Coecke and Kissinger, 2017)

We focus on the cases here $\alpha + \beta + \gamma$ is either 0 or π. Then, we get

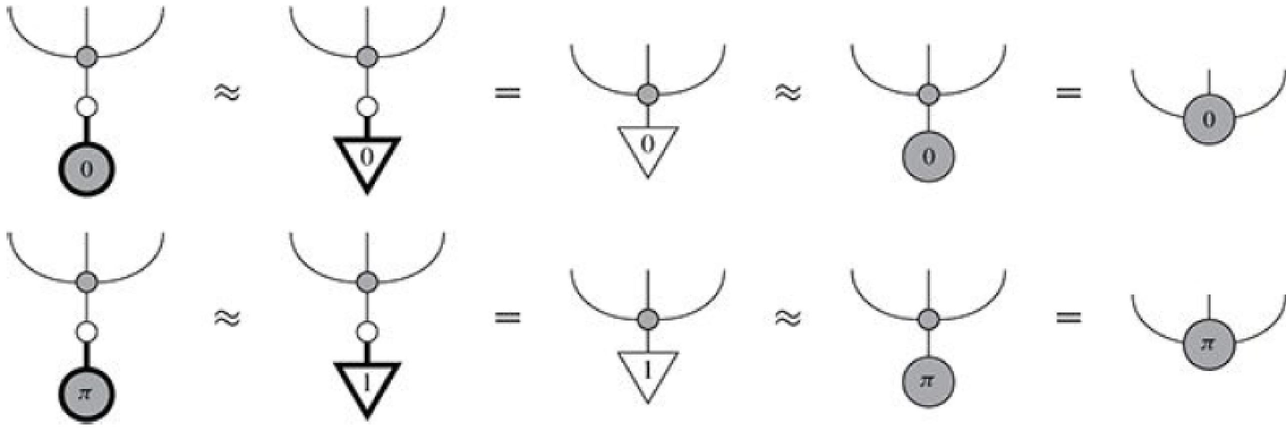


FIGURE 3.3: (Coecke and Kissinger, 2017)

We now fix the phases so as to impose two kinds of measurements:

FIGURE 3.4: (Coecke and Kissinger, 2017)

We consider four measurement scenarios that will help us establish the contradiction. More precisely, we consider one measurement scenario that gives the even parity state,

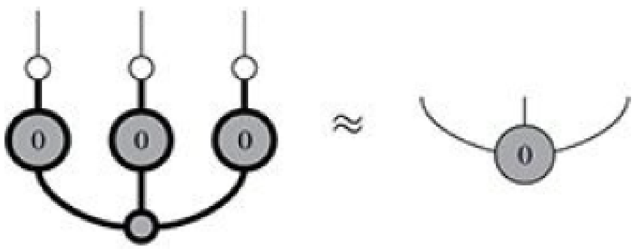


FIGURE 3.5: (Coecke and Kissinger, 2017)

and three measurement scenarios that yield the odd parity state:

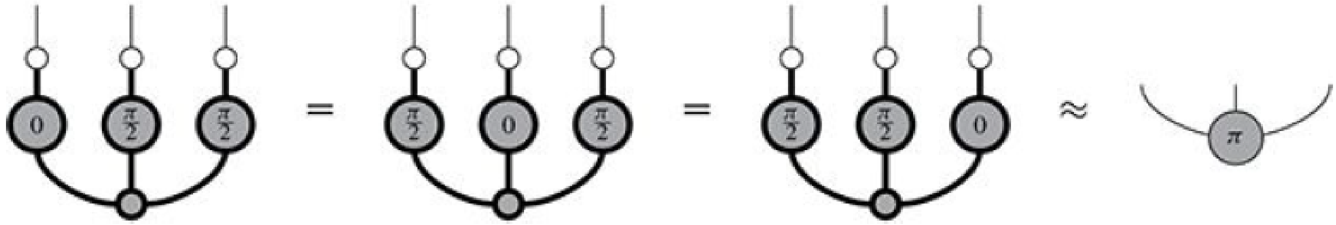


FIGURE 3.6: (Coecke and Kissinger, 2017)

The four scenarios are gathered in the following table:

	system A	system B	system C
scenario 1	Z	Z	Z
scenario 2a	Z	Y	Y
scenario 2b	Y	Z	Y
scenario 2c	Y	Y	Z

FIGURE 3.7: (Coecke and Kissinger, 2017)

The property that will allow us to draw the contradiction with the hidden variable model is the 'overall parity'. More specifically, the map

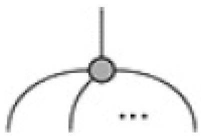


FIGURE 3.8: (Coecke and Kissinger, 2017)

functions as the 'parity map'. It returns the 0-state if the number of 1-states is even and the 1-state if it is odd.

We measure the overall parity of our set-up:

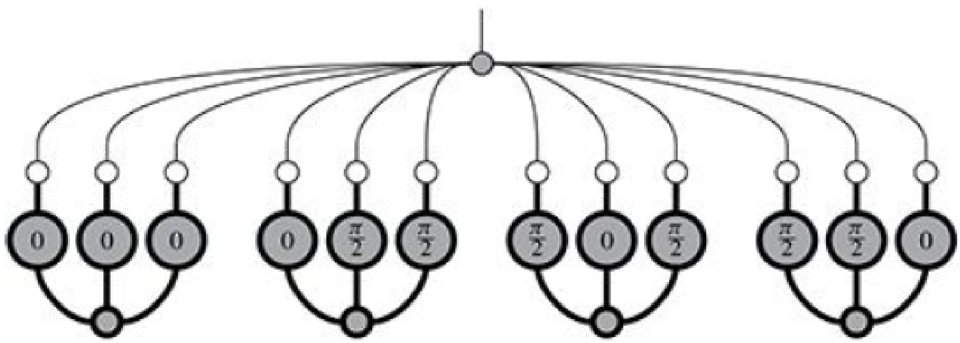


FIGURE 3.9: (Coecke and Kissinger, 2017)

We now substitute the parities of the individual scenarios:

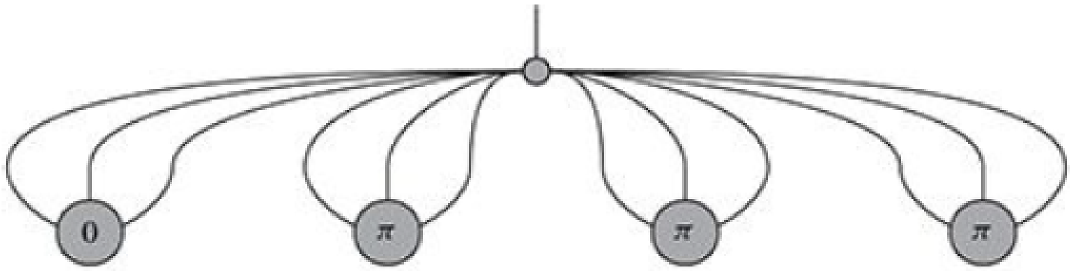


FIGURE 3.10: (Coecke and Kissinger, 2017)

We perform spider fusion to obtain:

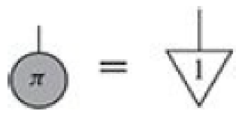


FIGURE 3.11: (Coecke and Kissinger, 2017)

We came up with the odd parity state.

3.2 Drawing a contradiction

We now construct a model in which the classical values are known before we perform any measurement:

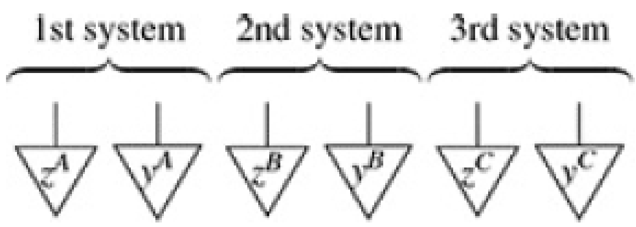


FIGURE 3.12: (Coecke and Kissinger, 2017)

For example if we measure the variable Z on the first system we get outcome z_A or if we measure the variable Y in the second system we get the outcome y_B. A state then of this model is a uniform probability distribution over the possible outcomes:

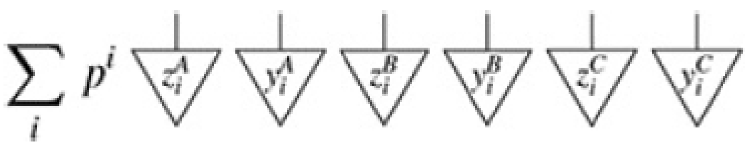


FIGURE 3.13: (Coecke and Kissinger, 2017)

We know that the overall parity in the quantum case is odd. Hence, in order to be consistent, the possible values in the above probability distributions should result in an odd parity as well. However, this is not the case. Each value gives specific measurement outcomes for the four measurement cases, depicted below:

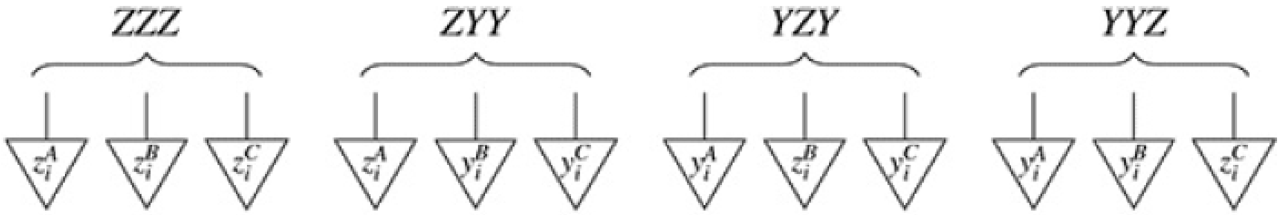


FIGURE 3.14: (Coecke and Kissinger, 2017)

Combining the results via copy spiders, we get:

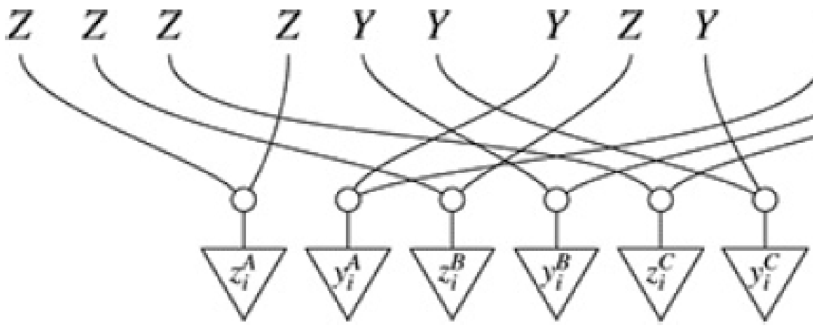


FIGURE 3.15: (Coecke and Kissinger, 2017)

Thus, the overall parity is given by

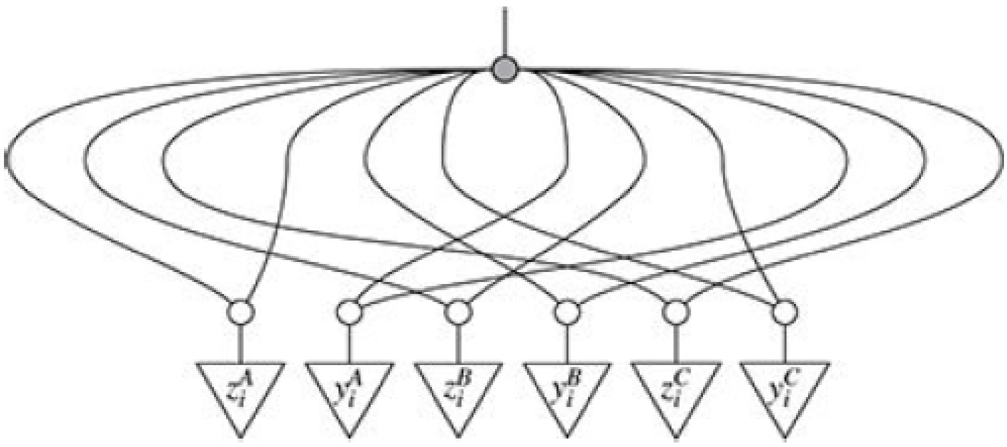


FIGURE 3.16: (Coecke and Kissinger, 2017)

We observe that every white spider admits exactly two connections with a gray spider. Therefore, we apply the complementarity relation to get:

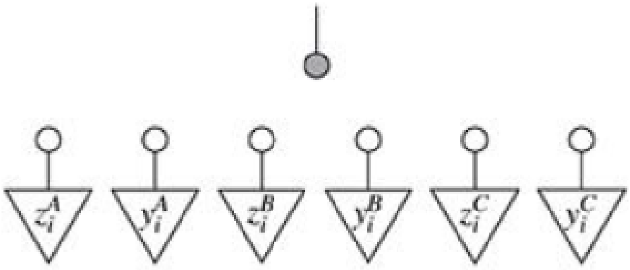


FIGURE 3.17: (Coecke and Kissinger, 2017)

Therefore, we have

FIGURE 3.18: (Coecke and Kissinger, 2017)

The result is the even parity state. This is in contradiction with quantum mechanics and therefore we infer that quantum theory is non-local.

Chapter 4

Spekkens toy theory

4.1 Basic notions

In this chapter we will describe a toy theory that aims to demonstrate that many features that are considered typically quantum, can be exhibited by a classical system as long us there is some restriction on our knowledge about the system(R.W.Spekkens, 2004). Even though the toy theory closely resembles quantum mechanics, it is by no means equivalent. The toy theory arises from classical probabilistic theories. However, it can provide new insights when compared to quantum mechanics and help us understand the differences between a classical and a quantum world from a foundational point of view.

In the theory we have one kind of system that aims to resemble a qubit. We call it *elementary system*. The system can be in one of four different states, which we call *ontic states.* Ontic states will be labeled as 1, 2, 3 and 4. The ontic state space is the set IV:={1,2,3,4}. Spekkens states that there are restrictions on how well we know the state of the system.

We now define a *canonical set of questions* as a set of yes-no questions with which we can identify what the ontic state of the system is. For example, we may ask: 'Is the ontic state one of {1,2} or not?' The amount of knowledge we have equals the amount of questions that we know the answer to. The restriction that we impose in our knowledge about the system is related to the *knowledge-balance* principle. The *knowledge-balance* principle states that *The amount of knowledge one has, equals the amount of knowledge one lacks.*

A canonical set of questions that applies to an elementary system consists of two questions, but we know the answer to one of them. Therefore, we know that the system occupies one of two ontic states. The state of our knowledge about the system is called *epistemic state.*

Spekkens refers to the dichotomy between ontic and epistemic states: Ontic states are states of reality, while epistemic states are states of knowledge. In particular, for many physicists pure quantum states are ontic states. Epistemic states correspond to the mixed states, because we have incomplete knowledge of the specific pure state our system is in. Spekkens, on the other hand argues that all quantum states are states of incomplete knowledge. His toy aims at defending the epistemic view of quantum states.

We now turn to define suitable representations for the ontic and epistemic states. An ontic state of an elementary system can be represented as

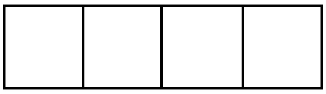


FIGURE 4.1: (W.Edwards, 2009)

We then depict the epistemic states by shading those ontic states, that our system might be in:

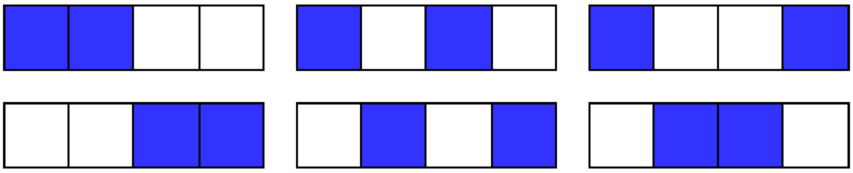


FIGURE 4.2: (W.Edwards, 2009)

Any two states now that have an empty intersection are called *disjoint*. These are the analogue of orthogonal quantum states. It is also apparent from the above figure that these states fall into three family of states, where the states of each family are disjoint and when combined we get the full ontic space. Such a family is the analogue of an orthogonal basis in quantum mechanics. In this context, we can also define states of knowledge less than maximal. This means that we know the answer to no questions. These correspond to mixed states:

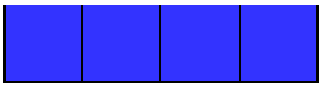


FIGURE 4.3: (W.Edwards, 2009)

We can place the epistemic states in a sphere, which we call 'spek-sphere'. This is the analogue of the Bloch sphere:

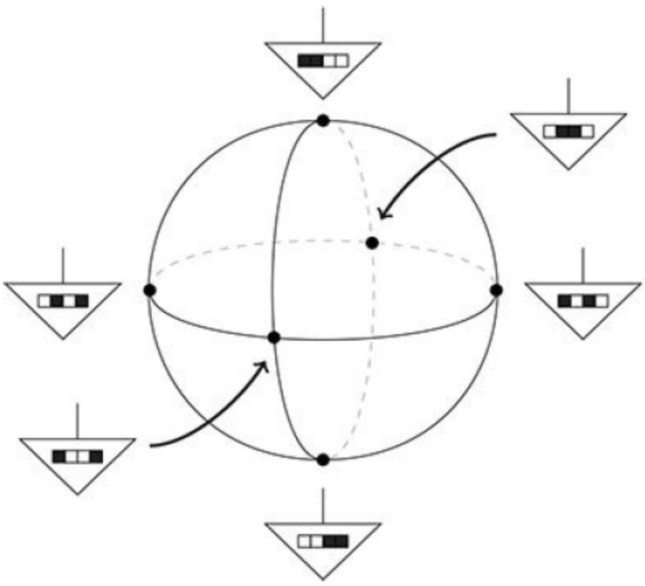


FIGURE 4.4: (Coecke and Kissinger, 2017)

4.2 Dynamics

The next step is to consider the dynamics of our system. Spekkens sets the question: ‘What transformations of the epistemic states are allowed by the knowledge-balance principle?’ The only allowed transformations are the permutations of the four ontic states. In order to better understand this, let’s imagine a transformation that takes the ontic states 1 and 2 in an ontic state 3. Whereas before applying the transformation the epistemic state was the set {1,2}, after the transformation we are sure that the system is in the ontic state 3. This indicates a violation of the knoweledge-balance principle, therefore this particular transformation is not allowed. Bearing in mind the knowledge-balance principle we come up with three types of measurements that one can perform:

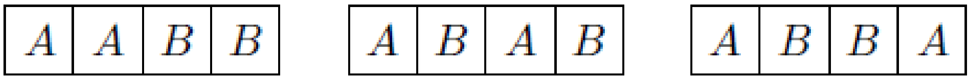


FIGURE 4.5: (W.Edwards, 2009)

We denote with A and B the possible measurement outcomes. For example for the first measurement pattern we ask ’Is the ontic state one of {1,2}(outcome A) or is it one of {3,4} (outcome B)’? Let’s provide an example: Suppose that we apply the first measurement pattern in the following states:

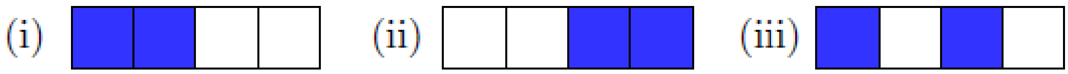


FIGURE 4.6: (W.Edwards, 2009)

If the system is described by (i) we will get outcome A; If it is described by (ii), we will get outcome B and if the system is in state (iii), we get either outcome A or outcome B.

We should make an important remark: Suppose that initially our system is in epistemic state {1,2}. We perform the second measurement pattern two times: We first measure in the set {1,3} (outcome A) and then in {2,4} (outcome B). Suppose also that we get outcome A. Then, we can infer that prior to measurement we had the ontic state 1. This does not violate the knowledge balance principle. The knowledge balance principle does not prevent us from knowing at a given time, what the state was at an earlier time. The restriction that it imposes is about the information that we know at a given time, about the state of that particular time.

4.3 Composite systems

In the context of the toy theory we can also define composite systems. The simplest composite system consists of two elementary systems. We mentioned before that an elementary system consists of four ontic states. Therefore, the pair of them will contain sixteen ontic states. We represent the ontic state space as

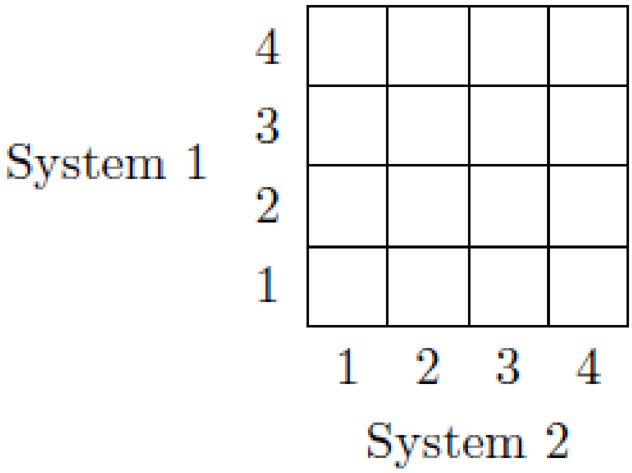


FIGURE 4.7: (W.Edwards, 2009)

The canonical set of questions contains now four questions, since this is the number of questions we need in order to identify an ontic state. According to the knowledge-balance principle, we know the answer to two of them. However, the knowledge-balance principle should apply not only for the composite system as a whole, but also to its constituents. For instance, the following two states are not allowed:

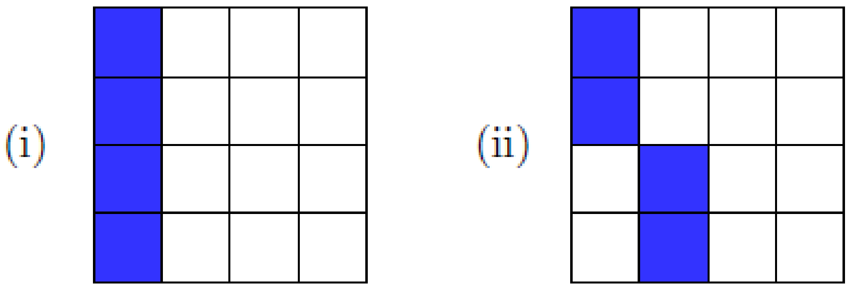


FIGURE 4.8: (W.Edwards, 2009)

The state (i), although it satisfies the principle for the composite system, it violates it when we apply it to system 2: We are sure that system 2 is in ontic state 1. The epistemic state in (ii) satisfies the knowledge-balance principle for the composite system and its constituents. However, if we apply a measurement of the first type, and the outcome is, say, B, we will end up with the state

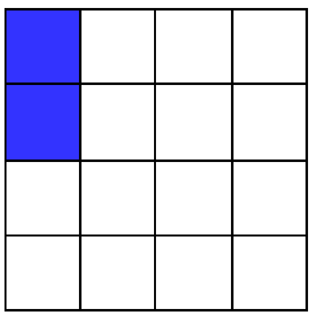


FIGURE 4.9: (W.Edwards, 2009)

This state violates the knowledge-balance principle both for the composite system and for system 2 alone.

Accounting for the above, Spekkens provides two types of allowed states:

- This state and all the states that can be obtained from it by permuting the rows and the columns:

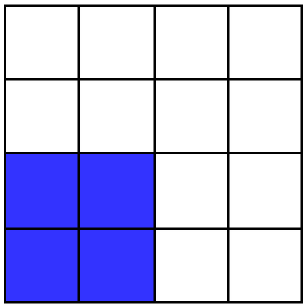


FIGURE 4.10: (W.Edwards, 2009)

In this case, we have maximal knowledge of the constituents but we know nothing about the relationship between them. These states are the analogue of separable states in quantum mechanics.

- This state and all the states that can be obtained from it via permutations of the rows and columns:

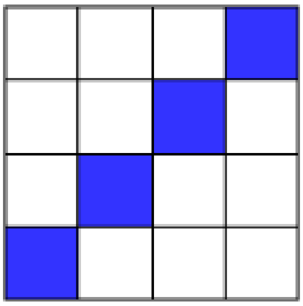


FIGURE 4.11: (W.Edwards, 2009)

In this case, we know nothing about the constituents but we can identify the relationship between them. These are the analogues of the entangled states in quantum mechanics.

4.4 Spek phase states

We can now proceed by trying to think the points on the equator of the Bloch sphere as phase states, in analogy with the theory of Clifford maps:

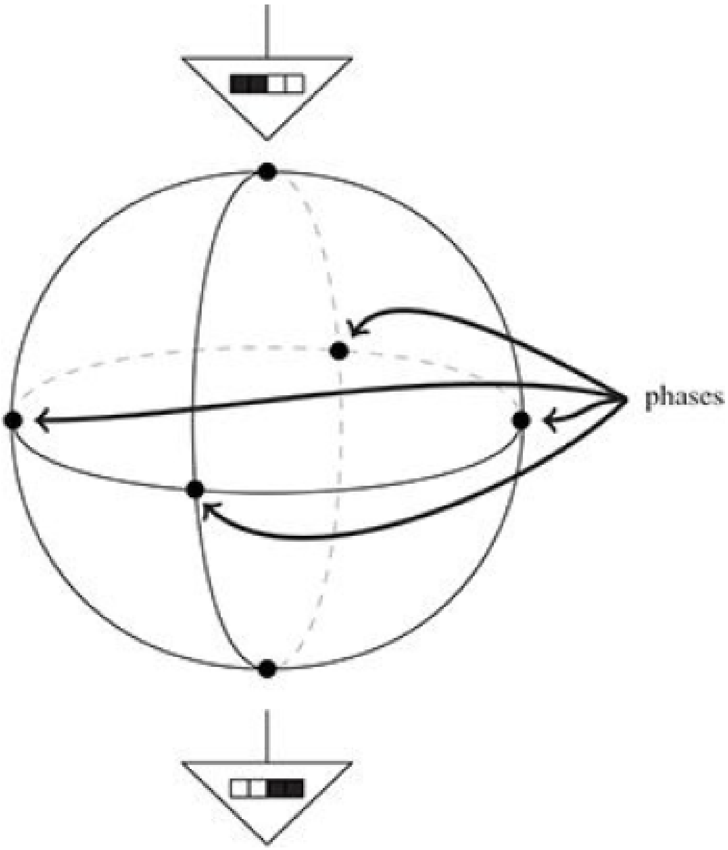


FIGURE 4.12: (Coecke and Kissinger, 2017)

In order to acquire a phase state for the spek-Z states, the following equation must hold:

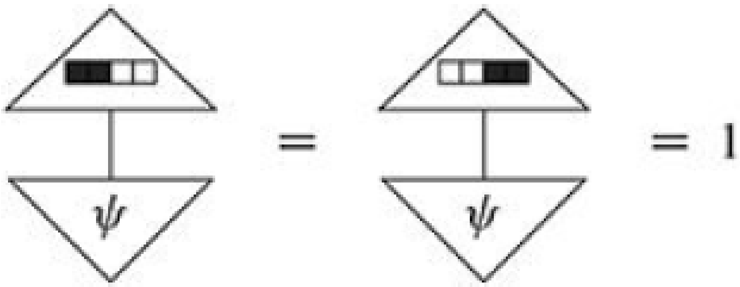


FIGURE 4.13: (Coecke and Kissinger, 2017)

Therefore, for a phase state we must pick one element from {1,2} and one of {3,4}. The corresponding phase group is $Z_2 \times Z_2$. For our intuition, we picture it with two cycles:

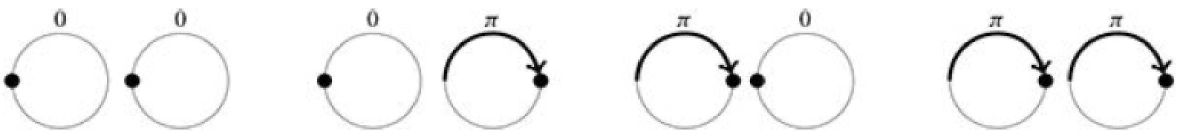


FIGURE 4.14: (Coecke and Kissinger, 2017)

The correspondence of the phase states and the group elements, is just a matter of convention. We simply choose to color the boxes where the black dots land:

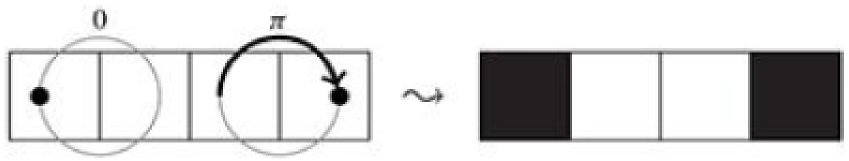


FIGURE 4.15: (Coecke and Kissinger, 2017)

Thus,

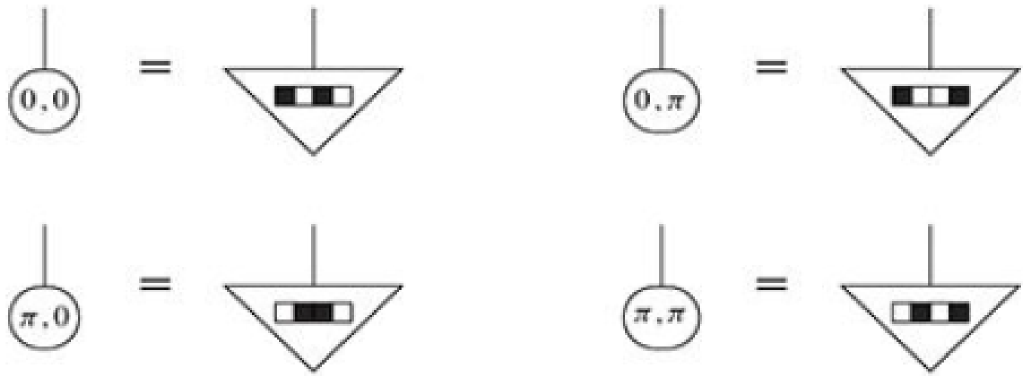


FIGURE 4.16: (Coecke and Kissinger, 2017)

In this context we can go further and define spiders maps as well.

we have encountered many similarities between the theory of Clifford maps and Spekkens toy theory. To sum up, in the case of Clifford maps

- The systems consist of n copies of C^2.
- The processes are described through quantum spiders

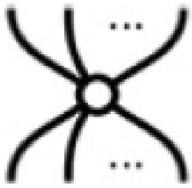


FIGURE 4.17: (Coecke and Kissinger, 2017)

and Clifford unitaries

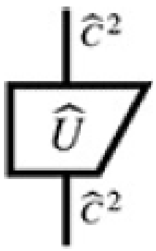


FIGURE 4.18: (Coecke and Kissinger, 2017)

In the case of Spekkens theory

- The system consist of n copies of IV.
- The processes are described via spiders

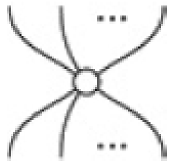


FIGURE 4.19: (Coecke and Kissinger, 2017)

and permutations that cause rotations in the spek sphere and consequently constitute the analogue of Clifford unitaries:

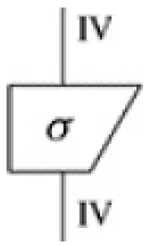


FIGURE 4.20: (Coecke and Kissinger, 2017)

The two theories look indeed very similar. It seems like the only difference between them is the phase group: In the case of Clifford maps the corresponding phase group is Z_4, while in Spekkens theory it is $Z_2 \times Z_2$. Does it have any consequences? Of course it does, and we will find out using ZX calculus. Below, we provide the Bloch and spek spheres respectively:

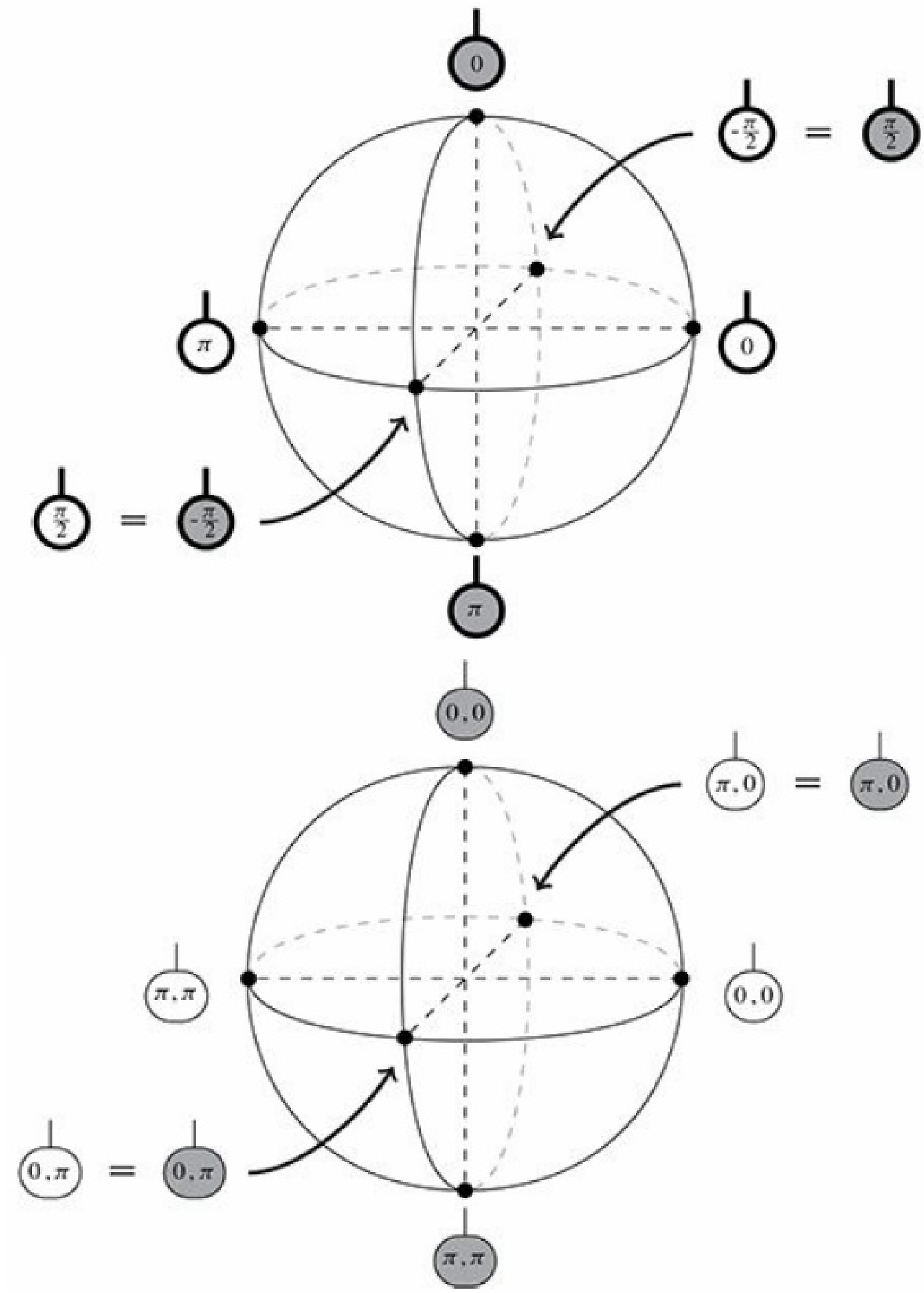


FIGURE 4.21: (Coecke and Kissinger, 2017)

4.5 ZX calculus for spek

Spekkens toy theory admits a graphical interpretation in the same manner Clifford maps do (M.Backens and A.N.Duman, 2014). The particular graphical language is very similar to ZX calculus and it is also complete.

For the group sum in the toy theory, we just have to add the angle elements together. For instance,

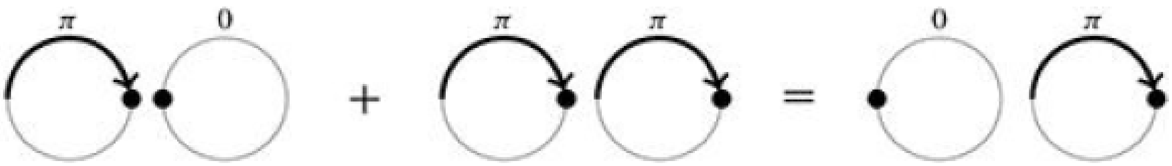


FIGURE 4.22: (Coecke and Kissinger, 2017)

We can therefore define a version of spider fusion in the context of toy theory:

FIGURE 4.23: (Coecke and Kissinger, 2017)

Hence, we can establish a full ZX calculus for spek. We have again four families of equations:

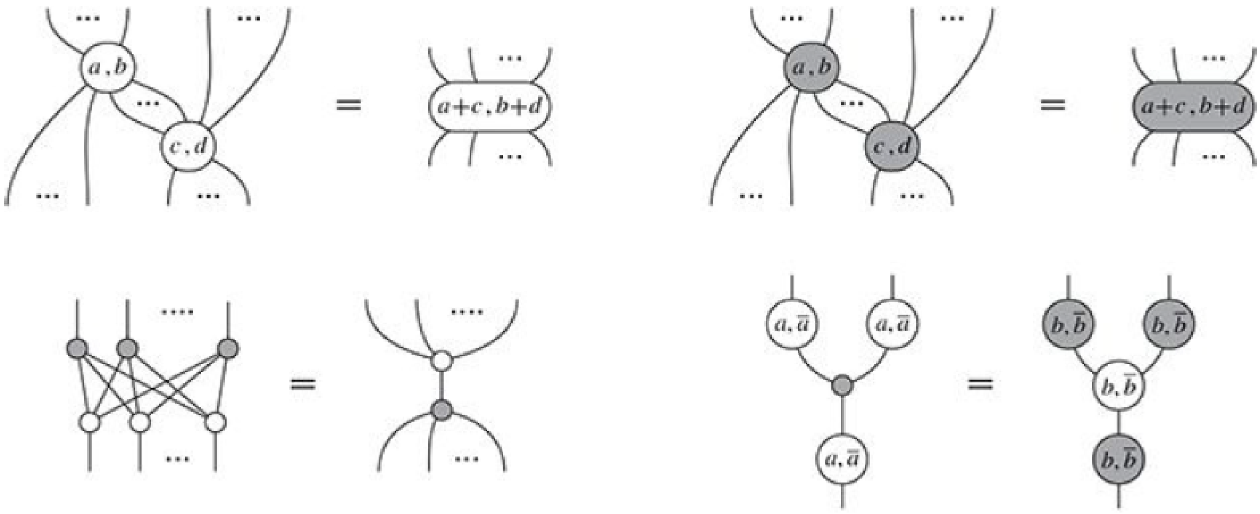


FIGURE 4.24: (Coecke and Kissinger, 2017)

where $\alpha, b, c, d \in \{0, \pi\}$. Also, $\bar{\alpha}$ stands for $\pi + \alpha$.

4.6 Non-locality in Spek

We will now see why the difference in the phase group is of fundamental importance by examining the concept of non-locality in Spekkens theory. In particular, we will reproduce the GHZ- Mermin scenario using spek-ZX calculus, in the same manner that we used ZX calculus to depict non-locality in Chapter 3. Therefore, we define Y and Z measurements as:

FIGURE 4.25: (Coecke and Kissinger, 2017)

Previously, we came up with an odd parity result for quantum theory and an even parity result for a local hidden variable theory. In our case, because the phase group is $Z_2 \times Z_2$, we have cancellations of all pairs of phases, for example

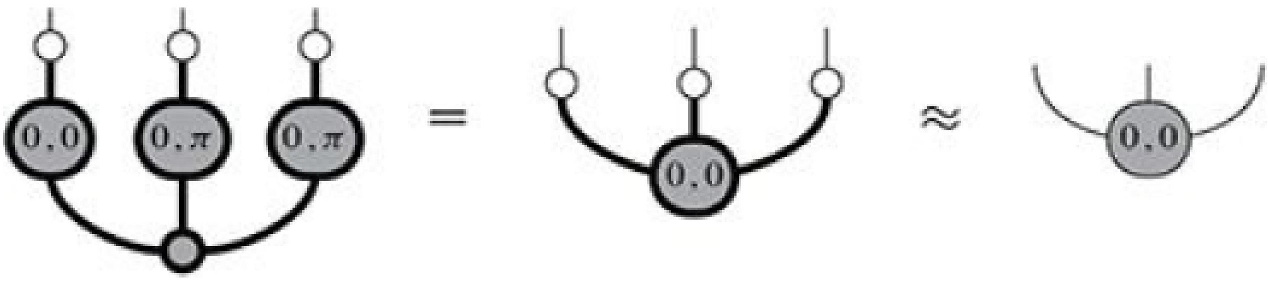


FIGURE 4.26: (Coecke and Kissinger, 2017)

Thus,

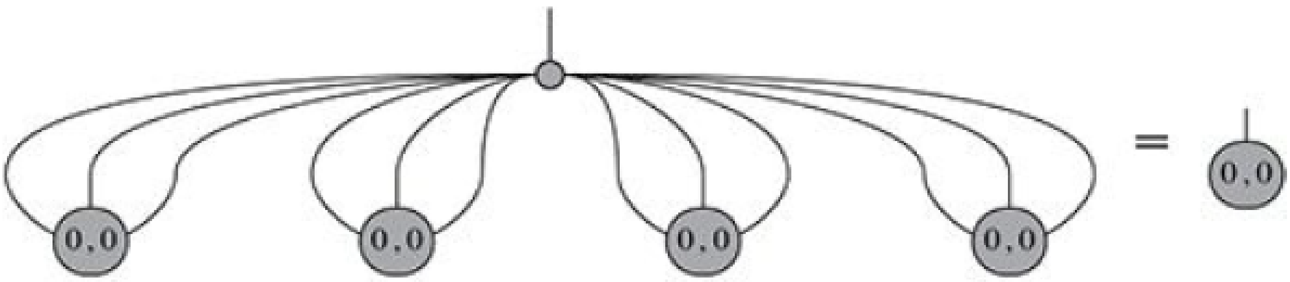


FIGURE 4.27: (Coecke and Kissinger, 2017)

We infer that there is no non-locality argument in Spekkens theory.

4.7 Author' s suggestions of moving forward

The toy theory that we just presented refers to states of maximal knowledge (the analogue of pure states in stabilizer quantum mechanics). In Chapter 2, we had set the question 'Can we extend ZX calculus to mixed states?'. Here we ask 'Can we have a rigorous formulation of the toy theory for states with knowledge, less than maximal?' Furthermore, what about higher dimensions?

Chapter 5

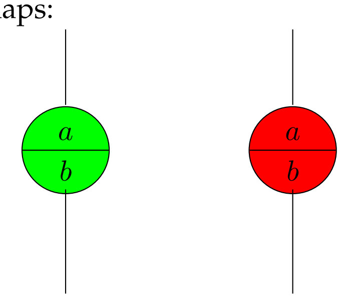
New Results

In the previous chapters we described a framework in which we could compare 'quantum-like' theories with the actual quantum theory. In particular, we compared the qubit stabilizer theory and Spekkens toy theory. We referred to the piece of structure that is responsible for the fundamental difference of the two theories, which is the phase group. The phase group arises quite naturally in both cases. For the stabilizer theory it is the group Z_4, while for the Spekkens theory, it is the group $Z_2 \times Z_2$. We saw that the group structure is the key, when we ask if our theory can be modeled by a local hidden variable model or not.

In this thesis, we move a step forward by considering the case of qutrits. Phase groups, in this case, have nine elements. There are two nine element groups, namely Z_9 and $Z_3 \times Z_3$. What theories correspond to these groups? Spekkens includes a 'qutrit' version of his toy theory, with the phase group being $Z_3 \times Z_3$. However, we do not yet have an answer in the type of theory that corresponds to the group Z_9.

Here, we employ the philosophy of the diagrammatic proof of Chapter 3 and by 'drawing analogies' we prove that the concept of non-locality exists in Z_9, while this is not the case for $Z_3 \times Z_3$. The result provides new insight regarding the theory that might correspond to Z_9.

Before proceeding, we should give some basic rewrite rules for qutrit ZX calculus(Q.Wang and X.Bian, 2014). We begin by indicating the formula for the spiderlike maps:



For a change, we symbolize the X spiders with red and the Z spiders with green. The ZX calculus is known in the bibliography as the Red-Green calculus as well. We will write down explicitly the correspondence of the above spiders to the Hilbert space formalism:

- For the red spider we have

$$[RED] = |+\rangle\langle+| + e^{i\alpha}|\omega\rangle\langle\omega| + e^{i\beta}|\bar{\omega}\rangle\langle\bar{\omega}|$$

  where $\omega = e^{i\pi\frac{2}{3}}$ and $\bar{\omega} = e^{i\pi\frac{4}{3}}$.

- For the green spider we have

$$[GREEN] = |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1| + e^{i\beta}|2\rangle\langle 2|.$$

We can generalize to spiders with m inputs and n outputs as in the qubit case. Qutrit spiders can also fuse together:

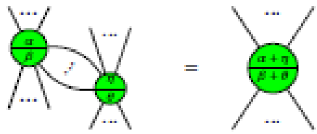


FIGURE 5.1: (Q.Wang and X.Bian, 2014)

In qutrit calculus we have to be careful cause the corresponding relation of the qubit complementarity does not hold for qutrits. For qutrits we have:

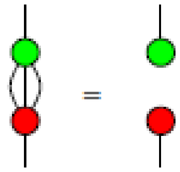


FIGURE 5.2: (Q.Wang and X.Bian, 2014)

that is, in order for the red and green dot to disconnect, they should be combined with three wires (and not two as in the qubit case).

We now proceed by constructing a measurement scenario for four parties. From now on, thick wires are equivalent to three classical wires. In the case of Clifford maps, where the corresponding group was Z_4, we chose a Z measurement with 0 phase and a Y measurement with a $\frac{\pi}{2}$ phase. Therefore, now that we are in the group Z_9 we choose a Z operation with $(0/0)$ phases and a Y operation with phases $(2\frac{\pi}{9}/0)$. It is also worth noting, that the strong complementarity relation that we used in order to rearrange the dots in the case of non-locality proof, holds also in the qutrit case. Furthermore, we impose the following restrictions on the possible measurement scenarios, which can make their specification a bit tricky:

- The outcome of the measurement of each scenario separately, should belong in Z_3, in the same manner that the outcome of each scenario in Z_4 belonged in Z_2.
- The resulting state, after the spider fusion, should also belong in Z_3, in analogy again with the theory of Clifford maps.
- In order to be able to draw a contradiction with a version of a Spekkens theory, we need to make sure that there exist three copies of each possible result in a local hidden variable model. This is because, we need three connections of each green dot with the red in order to have our diagram disconnected and be left with the (0,0) state.

Having made this sequence of thoughts, we came up with the following measurement patterns:

SystemA	SystemB	SystemC	SystemD
Z	Z	Z	Z
Z	Y	Y	Y
Z	Z	Z	Z
Y	Z	Y	Y
Y	Y	Z	Y
Y	Y	Y	Z

The outcome of each measurement scenario is either (0/0), or ($\frac{2\pi}{3}$/0). After the final spider fusion the resulting state is ($\frac{2\pi}{3}$/0). This is in accordance with our prerequisites and in perfect analogy with the Clifford map scenarios. Moreover, we can notice that in the pattern we propose the contradiction with a local theory can be easily seen. There are precisely three copies for each variable (for example we have three Z_A , three Y_C etc.). Therefore, the corresponding diagram will be disconnected, in the same manner that the diagram for the toy theory for qubits was disconnected. We will then be left with the state (0,0) which is in contrast to the result of the Z_9 group.

We would also like to note that the figures in both cases remain exactly the same, except from the spider representation, the different nature of the connection wires and the fact that we now have four parties instead of three. We attempted to use the Tikzit software in order to draw them, but due to lack of time, and some incompatibilities with overleaf, in which this thesis was written, we could not provide them.

Finally, we made everything in analogy with the Clifford maps because as a first attempt this feels the safest path to take. Whether the prerequisites that we impose are necessary or not, would be something that requires further investigation. Furthermore, if we maintain the same prerequisites with the Clifford maps, we do not seem to infer anything for the qutrit case for three parties. Thus, this constitutes another open question worth investigating.

Chapter 6

Conclusions

We begun this thesis by providing the basic mathematical foundation of the framework that we use. We argued that adopting a diagrammatic formulation of quantum mechanics, is far more intuitive than the Hilbert space formalism. We described how the analogues of fundamental concepts, such as the interaction of classical and quantum systems, can be translated into diagrams. From this framework, ZX calculus arises, to provide us with tools to prove that quantum theory is non-local. ZX calculus is now complete for the Clifford+T group, which is an approximatively universal set and widely used in quantum computing. We can therefore be ready for entirely pictorial quantum computing. Furthermore, we proposed new ways to move forward.

In this thesis, we did not exploit the ZX calculus for quantum computing but rather we examined how this quantum information tool, sheds light to the very foundations of quantum theory, a necessary knowledge for revolutionizing quantum technologies. We show how non-locality is depicted and how ZX calculus is even adopted by a 'quantum-like' theory, Spekkens toy theory, to distinguish it from the actual quantum theory. With spek-ZX calculus we see how a non-locality argument for the toy theory, is doomed. The reason turns out to be the difference in the phase groups of the two theories. While, in the theory of quantum maps, the underlying phase group is Z_4, in toy theory it is $Z_2 \times Z_2$.

We observed that group structure should be of fundamental importance. Therefore, we proceeded by identifying our own results, which might constitute a small continuation of the whole story. We went further from qubits and we turned our attention to qutrits. Spekkens has provided a version of his theory for qutrits, the corresponding group being $Z_3 \times Z_3$. Therefore, we looked at the group Z_9, in the same manner that we looked at the group Z_4, when the toy theory was described by $Z_2 \times Z_2$. We do not know yet what the theory of Z_9 might be, but we proved that the concept of non-locality exists there. This was achieved by 'drawing analogies' with the stabilizer theory. The obvious generalization of the above is the qudits case. Therefore, there are still much to be done!

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