Discrete Models of Categorical Quantum Computational Semantics

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Abstract

Dagger compact closed categories are consider to be the abstract categorical framework for quantum computation and quantum mechanics. They also provide a description of classical structures and quantum measurements with no additional assumptions, but relying only on dagger compact closed structure. One advantage of this framework is that it can be applied in many different models. In this disertation we study discrete models of categorical quantum computation.

First we review how this categorical framework is obtained and how the underlying graphical calculus can be a concise representation of the categorical quantum semantics. The main body of work is an investigation into discrete models of categorical quantum semantics, namely **FRel**, the category of finite sets, relations and the cartesian product, and **Spek**, a subcategory of the former which formalizes Rob Spekken's toy model. In particular, we characterize the classical structures and the quantum measurements within these models. Finally, the quantum state transfer computational model is described and its application on the discrete models **FRel** and **Spek** is explored.

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Chapter 1

Introduction

Three quarters of a century have passed since the discovery of quantum mechanics and half a century since the birth of information theory, but only a quarter of a century before people finally realized that quantum mechanics dramatically alter the character of information processing and digital computation. The research that started at mid '90s on quantum mechanics provides new powerful computation paradigms, which can be applied to the processing of knowledge in an entirely new manner. Nowadays, quantum computation offers a conceptual arena of a better understanding of quantum oddness and at the same time gives a new perspective on interpretational questions. Therefore, the idea of using the possibilities and potentials of quantum mechanics in computation looks more and more appealing. Furthermore, experimental work has already begun. However, the physical realization of quantum computers appears at the moment very unclear and it could take some decades to achieve essential progress. Hence, the fundamental accomplishments in this area are mainly of purely theoretical and mathematical nature.

Since John von Neumann's quantum mechanical formalism in terms of Hilbert spaces, there have been many discoveries, insights and developments towards a more high-level formalism of quantum mechanics. A recent publication by S. Abramsky and B. Coecke [1] has developed the categorical foundations of quantum mechanics using dagger compact closed categories as the core of categorical semantics.

Category theory as a pure mathematical abstraction provides a new environment of economy of thought and expression. Also, it allows easier communication among basic ideas of various theorems and constructions and helps to determine and delineate the exact deepness and power of classical results [2]. Although, it is a relatively new area of pure mathematics, it plays a significant role in theoretical computer science in the areas, where operations and processes play a principal role. Examples can be found in programming and semantic models of programming languages, constructive logic, automata and proof theory and development of algorithms. Therefore, the combination of category theory and quantum computation provides a new context in which quantum mechanics are revealed under a new formalism.

Quantum mechanics involve measurements as well as operations, which depend on measurements. Dagger compact closed categories can embody them by conditioning on classical data. Hence, various quantum protocols can be handled by dagger compact closed categories. However, viewing classical data only as a category structure forces us to go beyond dagger compactness. In their article [1] S. Abramsky and B. Coecke used dagger compact closed categories with biproducts. They proved that this categorical structure can describe compound systems and allows preparations and measurements of entangled states. Additionally, biproducts capture the classical communication, indeterministic branching and superposition [1]. Nevertheless, the additive structure of direct sums, i.e. biproducts does not yield a simple graphical calculus and does not capture the essential decoherence component of quantum measurements. Therefore, biproducts end up to be unusable in many cases.

To overcome this problem P. Selinger [3] introduced the "category of positive maps"(**CPM**), which applies to every dagger compact closed category. Also B. Coecke [4] resolved this matter by introducing density operators. Furthermore, B. Coecke and D. Pavlovic [5] described quantum measurements explicitly by using the multiplicative tensor structure of the dagger compact closed categories. In the present dissertation we use this approach in order to define measurements. A classical structure (or basis structure as presented in [6, 7]) is a special dagger compact closed Frobenius algebra that comes with coping and deleting operators and connects the classical capabilities of copying and deleting with the mechanism of quantum measurement. More precisely, classical structures exploit the fact that quantum data can not be copied or deleted, while only classical data can do so.

Moreover, B. Coecke, D. Pavlovic and J.Vicary [8] showed that classical structures are in one-to-one correspondence with orthonormal bases and B. Coecke, E.O. Paquette, S. Perdrix [6] elaborated the classical structure axiomatization by introducing dagger dual Frobenius structures.

Additionally, since classical structures are in one-to-one correspondence with orthonormal bases, they correspond to non-degenerate observables [7,9]. We are particularly interested in incompatible observables. Therefore, dagger symmetric monoidal categories that have enough incompatible classical structures are suitable for describing various features of quantum mechanics. Such categories are **FdHilb**, but also discrete ones such as **FRel** and **Spek**. However, in these discrete models one expects to be able to express less quantum features than in **FdHilb**. Hence, by examining them we can identify what mathematical structures are required for full description of quantum mechanics and quantum computation.

1.1 Outline of dissertation

The purpose of this dissertation is to study discrete models of categorical quantum computation. Also, we present how quantum mechanics can be expressed using categories and especially dagger compact closed categories.

More explicitly:

- discrete models of categorical quantum computational semantics are presented (i.e. **FRel** and **Spek**),
- we investigate what features of these models are essential, i.e. the existence of complementary observables,
- we observe if these models can hold abstractly, i.e their soundness and completeness with respect to dagger compact closed categories,

- the behavior of these models is examined in the quantum state transfer protocol.

In Chapter 2 we introduce the basic concepts towards the definition of dagger compact closed category. Also, we provide the corresponding graphical language, which is essential for the pictorial interpretation of categorical semantics and for providing justification about several issues in the following chapters. Next, in Chapter 3 we give a description of classical structures and provide an "approximate" definition of quantum measurements.

The original work in this dissertation is the investigation of discrete models **FRel** and **Spek** and the application of the quantum state transfer protocol in them. The importance of classical structures in our categorical quantum computation framework is revealed in Chapter 4, where we present the discrete models **FRel** and **Spek** and investigate the quantum spectra of these models. In Chapter 5 we provide a description of the quantum state transfer protocol and its application to **FRel** and **Spek**. Finally, Chapter 6 draws the conclusions from this performed research regarding these models and suggests directions for further work.

Chapter 2

Categorical Semantics and Preliminaries

In this chapter we present the basic categorical definitions and semantics around dagger compact closed categories (originally presented in [1] as strongly compact closed categories). The structure of dagger compact closed categories is the central mathematical object of categorical quantum computational semantics.

The beauty of this structure is that it admits sound and complete graphical representations. In this sense, equations that use the categorical tensor calculus are provable from the dagger compact closed categories axioms if and only if the corresponding graphical representation is valid in the graphical language.

Also, another important aspect of dagger compact closed categories is that they provide a simple and complete axiomatic framework for essential structures of the quantum mechanical formalism such as unitarity, self-adjointness, trace, scalars, bell-states, Dirac notation. Therefore we are able to design and explain various quantum protocols like quantum teleportation, quantum entanglement and dense coding.

2.1 Categorical concepts

We give some basic categorical definitions as introduced by S. Eilenberg and S. MacLane [10] which are essential for defining dagger compact closed categories.

Definition 2.1. *A category $\mathcal{C}$ is a collection of:*

- *objects $A, B, C, \ldots$,*
- *morphisms (often called arrows) $f, g, h \in \mathcal{C}(A, B)$ for each pair of A, B (here $\mathcal{C}(A, B)$ is the collection of all morphisms $f : A \to B$),*
- *a composition operation for each pair of morphisms $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, resulting in $g \circ f \in \mathcal{C}(A, C)$ such that the following associativity law is satisfied:*

  $$(h \circ g) \circ f = h \circ (g \circ f)$$

  for each morphism $f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, C), h \in \mathcal{C}(C, D)$,
- *identity morphisms $id_A \in \mathcal{C}(A, A)$ for each object A which satisfy the following identity law:*

  $$f \circ id_A = id_B \circ f = f$$

  for any morphism $f \in \mathcal{C}(A, B)$.

Typical examples of categories arise from various mathematical structures. For example **Set** is the category with sets as objects and total functions as morphisms, **Rel** has sets as objects and relations as morphisms, **Group** has groups and group homomorphisms, **Top** has topological spaces and continuous maps and **FDHilb** has finite dimensional Hilbert spaces and linear maps.

Definition 2.2. *A morphism $f : A \to B$ is an isomorphism (iso) if it has an inverse $f^{-1} : B \to A$ such that*

$$f^{-1} \circ f = id_A \quad \text{and} \quad f \circ f^{-1} = id_B.$$

Therefore, a group G is a category with a single object in which elements are the isomorphisms of this object to itself. A monoid $(M, \cdot, e)$ can be represented as a category with a single object, in which the elements of M are the morphisms from M to itself, the binary operation $\cdot$ is represented as composition of morphisms such that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in M$ and the identity element e is the identity morphism such that $e \cdot x = x \cdot e = x$ for $x \in M$. In that sense, a monoid is just a group without inverses. Also we can consider the category **Mon** which has monoids as objects and monoid homomorphisms as morphisms, i.e. functions $f : M \to M'$ from monoid $(M, \cdot, e)$ to $(M', *, e')$ such that $f(e) = e'$ and $f(x \cdot y) = f(x) * f(y)$ for $x, y \in M$. Composition of morphisms, associativity and identical laws are as in category **Set**.

Moreover, we can consider the category **Cat** which has categories as objects and functors as morphisms. Below we give the definition of functors as morphisms between categories and natural transformations as morphisms between functors, as introduced by S. Eilenberg and S. MacLane [10].

Definition 2.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is a morphism which preserves the structure of categories, i.e. it takes each $\mathcal{C}$-object A to a $\mathcal{D}$-object $F(A)$ and each morphism $f \in \mathcal{C}(A, B)$ to $F(f) \in \mathcal{D}(F(A), F(B))$, such that for each $\mathcal{C}$-object A and $\mathcal{C}$-morphisms f, g we have:*

$$F(g \circ f) = F(g) \circ F(f) \quad \textit{and} \quad F(id_A) = id_{F(A)}.$$

For example in the category **Group** a group homomorphism can be consider as a functor between groups, since if G_1, G_2 are groups then $F : G_1 \to G_2$ satisfies $F(x \cdot y) = F(x) * F(y)$ for $x, y \in G_1$ and $F(e) = e'$ for e, e' being the identity morphisms of G_1 and G_2 respectively. Furthermore functor F preserves the inverses since:

$$\begin{aligned} a^{-1} \cdot a = e = a \cdot a^{-1} &\Rightarrow F(a^{-1} \cdot a) = F(e) = F(a \cdot a^{-1}) \\ &\Rightarrow F(a^{-1}) * F(a) = e' = F(a) * F(a^{-1}) \\ &\Rightarrow (F(a))^{-1} = F(a^{-1}). \end{aligned}$$

Definition 2.4. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors for categories $\mathcal{C}, \mathcal{D}$. Then a natural transformation $\xi : F \to G$ is a family $\{\xi_A : F(A) \to G(A)\}_A$ of morphisms in $\mathcal{D}$ that* *assigns to every $\mathcal{C}$-object A a $\mathcal{D}$-morphism $\xi_A : F(A) \to G(A)$ such that for every $\mathcal{C}$-morphism $f : A \to B$ the following diagram commutes in $\mathcal{D}$:*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \Big\downarrow{\scriptstyle \xi_A} & & \Big\downarrow{\scriptstyle \xi_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

If each component ξ_A of ξ is an isomorphism in $\mathcal{D}$ then ξ is called natural isomorphism.

For instance if $F : \mathcal{C} \to \mathcal{D}$ is a functor then the family of identity natural transformations $\{\iota : F(A) \to F(A)\}_A$ are the identity morphisms of the objects in the image of F, i.e. $F(A)$. Therefore $\iota_A = id_{F(A)}$ and since $id_{F(A)}$ is an isomorphism for every $\mathcal{D}$-object $F(A)$, then $\iota : F \to F$ is a natural isomorphism.

Definition 2.5. *A monoidal category is a structure $(\mathcal{C}, \otimes, I)$ where $\mathcal{C}$ is a category which comes with a monoidal tensor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as multiplication and a distinguished neutral object I as the multiplication unit. The monoidal tensor is an assignment on both pairs of objects and pairs of morphisms such that:*

$$(A, B) \mapsto A \otimes B$$
$$(A \xrightarrow{f} B, C \xrightarrow{g} D) \mapsto A \otimes C \xrightarrow{f \otimes g} B \otimes D.$$

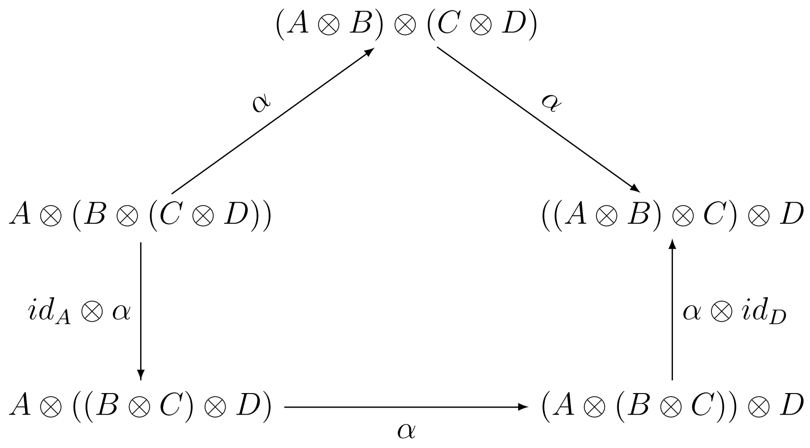
Moreover the monoidal category $(\mathcal{C}, \otimes, I)$ is equipped with left and right natural isomorphisms:

$$\lambda_A : I \otimes A \xrightarrow{\cong} A \quad \textit{and} \quad \rho_A : A \otimes I \xrightarrow{\cong} A$$

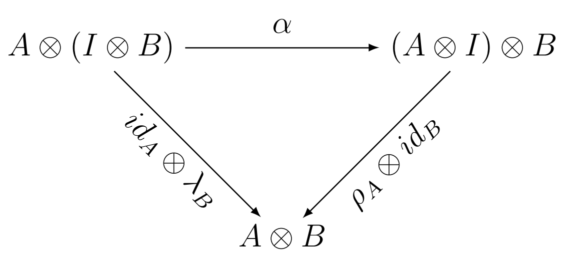
and an associativity natural isomorphism:

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C,$$

such that for all objects A, B, C, D the following diagrams commute (ensuring coherence):



(2.1.1)



(2.1.2)

Finally a monoidal category is called strict if the isomorphisms α, λ, ρ are all identities.

The monoidal tensor $\otimes$ is a bifunctor, i.e. a functor that is bifunctorial. In quantum mechanics bifunctoriality stands for:

$$(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2), \tag{2.1.3}$$

where $f_1 : A_1 \to B_1,\ f_2 : A_2 \to B_2,\ g_1 : B_1 \to C_1,\ g_2 : B_2 \to C_2$.

Conceptually, the above means that it does not matter if we consider the sequential composition of $f_1 \otimes f_2$ and $g_1 \otimes g_2$ or the parallel composition of the pairs (g_1, f_1) and (g_2, f_2) along the tensor. Actually, what the monoidal tensor $\otimes$ does is to conceive two systems A_1 and A_2 to a compound one $A_1 \otimes A_2$ and then considering the compound morphism $f_1 \otimes f_2$ inherited from the morphisms f_1 and f_2 on the individual systems. Also we require that:

$$id_{A_1} \otimes id_{A_2} \otimes \cdots \otimes id_{A_n} = id_{A_1 \otimes A_2 \otimes \cdots \otimes A_n}.$$

From bifunctoriality of the tensor it easily follows that:

$$\begin{aligned}(id_{B_1} \otimes g) \circ (f \otimes id_{A_2}) &= (id_{B_1} \circ f) \otimes (g \circ id_{A_2}) \\ &= (f \circ id_{A_1}) \otimes (id_{B_2} \circ g) = (f \otimes id_{B_2}) \circ (id_{A_1} \otimes g)\end{aligned} \tag{2.1.4}$$

for morphisms $f : A_1 \to B_1$, $g : A_2 \to B_2$, which is expressed by the following commutative diagram:

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f \otimes id_{A_2}} & B_1 \otimes A_2 \\ {\scriptstyle id_{A_1} \otimes g}\downarrow & & \downarrow{\scriptstyle id_{B_1} \otimes g} \\ A_1 \otimes B_2 & \xrightarrow[f \otimes id_{B_2}]{} & B_1 \otimes B_2 \end{array}$$

Hence it does not matter if we apply first the morphism f to the first system and later the morphism g to the other system, or vice versa. This express some notion of locality of space-liked separated systems, since what is at the left of the tensor does not inflect or temporally compare with what is at the right.

Finally, consider the functor $F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, such that $F(A, B) = A \otimes B$ and $F(f, g) = f \otimes g$, where $\mathcal{C} \times \mathcal{C}$ is the category with objects the pairs (A, B) and morphisms the pairs (f, g) for $f, g \in \mathcal{C}(A, B)$, for each $\mathcal{C}$-objects A, B. The composition is pairwise defined, i.e. $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, g_2 \circ f_2)$ and the identity morphisms are the pairs (id_A, id_B). F is indeed a functor since from bifunctoriality we have:

$$F(g_1 \circ f_1, g_2 \circ f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2) = F(g_1, g_2) \circ F(f_1, f_2)$$

and

$$F(id_A, id_B) = id_A \otimes id_B = id_{A \otimes B} = id_{F(A,B)}.$$

But since $F(-, -) = - \otimes -$, functor F is nothing else but the monoidal tensor $\otimes$.

Proposition 2.6. *In a monoidal category the equality*

$$\lambda_I = \rho_I : I \otimes I \xrightarrow{\cong} I$$

holds and the following diagrams commute:

$$\begin{array}{ccc} A \otimes (B \otimes I) & \xrightarrow{\alpha} & (A \otimes B) \otimes I \\ & {\scriptstyle id_A \oplus \rho_B} \searrow \qquad \swarrow {\scriptstyle \rho_{A \oplus B}} & \\ & A \otimes B & \end{array} \tag{2.1.5}$$

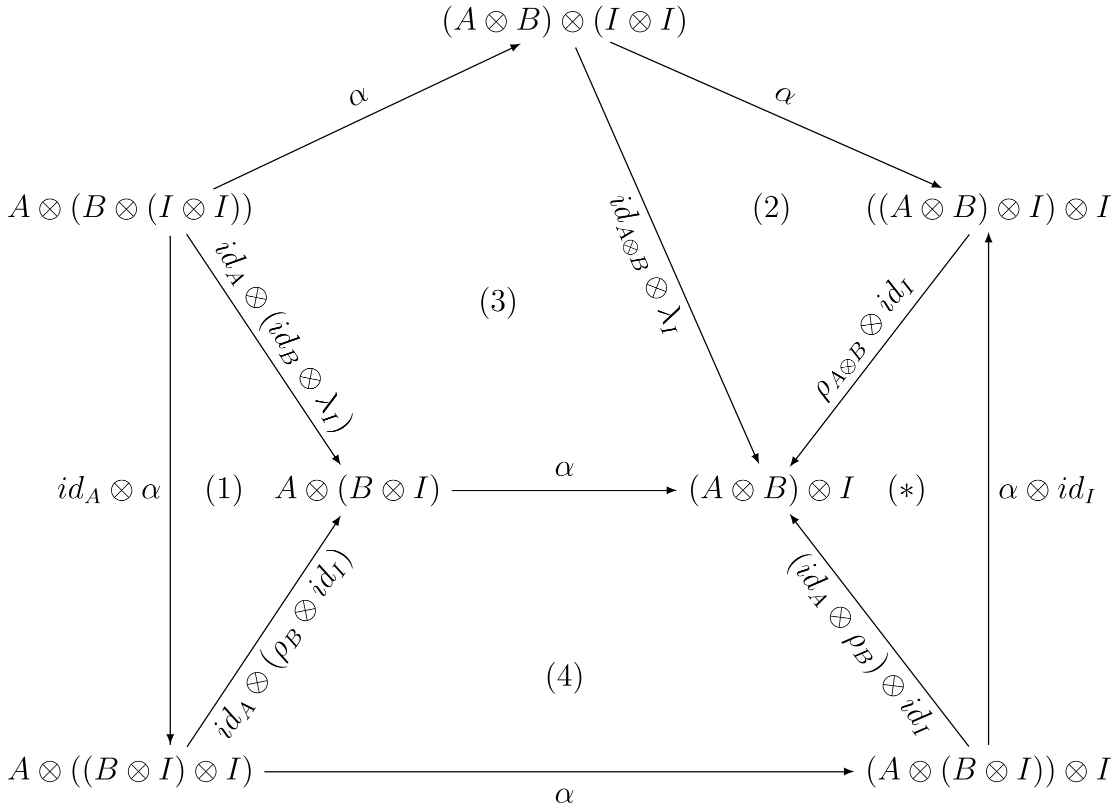
$$\begin{array}{ccc} I \otimes (A \otimes B) & \xrightarrow{\alpha} & (I \otimes A) \otimes B \\ & {\scriptstyle \lambda_{A \oplus B}} \searrow \qquad \swarrow {\scriptstyle \lambda_A \oplus id_B} & \\ & A \otimes B & \end{array} \tag{2.1.6}$$

Proof. Since $\lambda_I : I \otimes I \xrightarrow{\cong} I$ and $\rho_I : I \otimes I \xrightarrow{\cong} I$ are natural isomorphisms then the following diagrams:

$$\begin{array}{ccc} I \otimes (I \otimes I) & \xrightarrow{id_I \otimes \lambda_I} & I \otimes I \\ {\scriptstyle \lambda_{I \otimes I}} \downarrow & & \downarrow {\scriptstyle \lambda_I} \\ I \otimes I & \xrightarrow[\lambda_I]{} & I \end{array} \qquad \begin{array}{ccc} I \otimes (I \otimes I) & \xrightarrow{id_I \otimes \rho_I} & I \otimes I \\ {\scriptstyle \lambda_{I \otimes I}} \downarrow & & \downarrow {\scriptstyle \lambda_I} \\ I \otimes I & \xrightarrow[\lambda_I]{} & I \end{array}$$

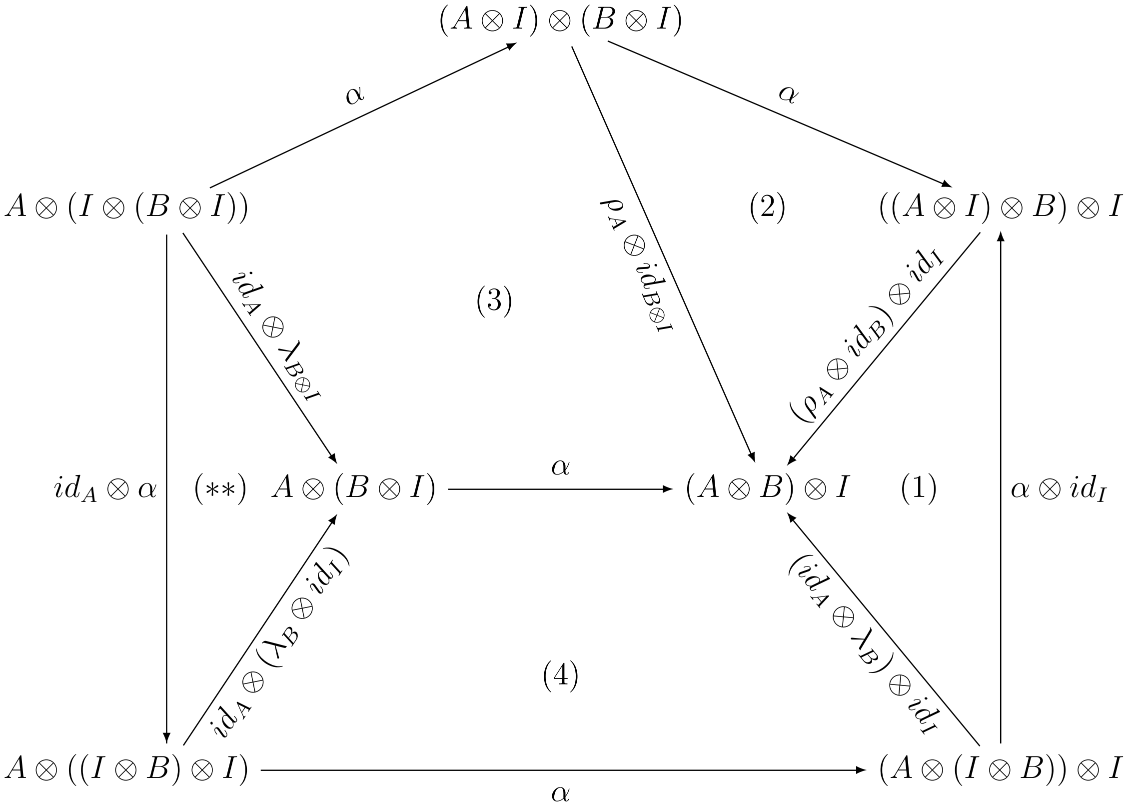
both commute, so $id_I \otimes \lambda_I = id_I \otimes \rho_I$. By naturality of λ_I and ρ_I we deduce that $\lambda_I = \rho_I$.

To prove that the diagram (2.1.5) commutes consider the following diagram:



This diagram commutes since the inside regions $(1), (2)$ commute from Definition 2.5, the regions $(3), (4)$ from naturality of α and the outside diagram from Definition 2.5. Therefore, the region indicated by $(*)$ commutes and hence, from naturality and invertibility of ρ and α we deduce commutation of diagram (2.1.5).

Similarly for the diagram (2.1.6) we consider the following diagram:

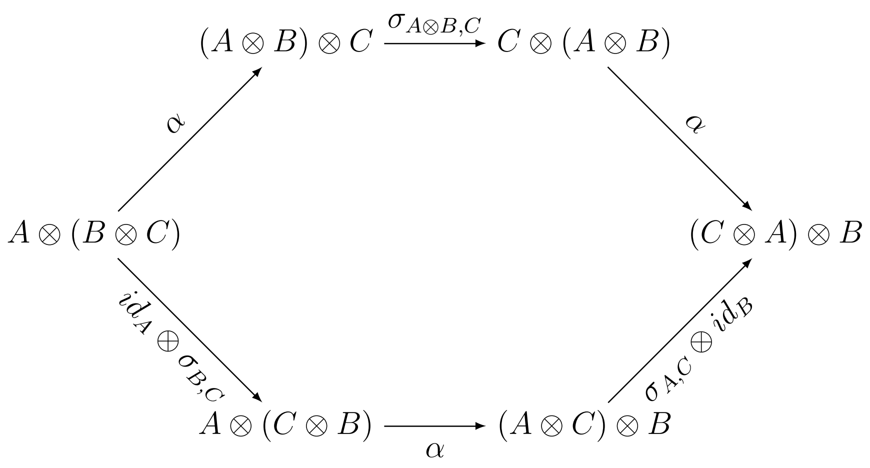


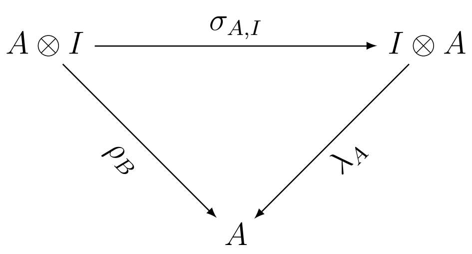
In the same vein the above diagram commutes, therefore the diagram indicated by $(**)$ commutes and hence, from naturality and invertibility of λ and α the diagram (2.1.6) commutes. $\square$

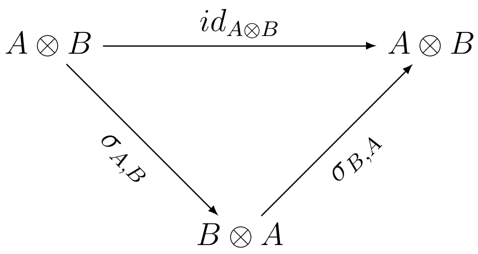
Definition 2.7. *A symmetric monoidal category is a monoidal category* $(\mathcal{C}, \otimes, I)$ *with an additional natural isomorphism (symmetry):*

$$\sigma_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A,$$

such that for all $\mathcal{C}$*-objects* A, B, C *the following diagrams commute:*







The diagrams of Definitions 2.5 and 2.7 and of Proposition 2.6 ensure coherence such that all the natural isomorphisms introduced above do coexist peacefully. Therefore, as described by S. MacLane [11] any formal diagram involving the natural isomorphisms α, λ, ρ and σ must commute.

Remark: S. MacLane [11] states that for every monoidal category there is an equivalent strict monoidal category. Hence, any commutative diagram of monoidal categories can be replaced by an equivalent commutative diagram of strict monoidal categories. Given this, we will assume from now on that any monoidal category under discussion is strict, unless otherwise stated. The reader may consider [11] and [12] for further reading regarding strict monoidal categories.

2.2 Dagger compact closed categories

As introduced by S. Abramsky and B. Coecke [1] by the name "strong compact closed categories", dagger compact closed categories play a crucial role in quantum mechanics axiomatization. Many of the structural properties of **FdHilb** can be axiomatized by dagger compact closed categories, so we can form a suitable and complete framework for quantum computation and information. Originally, compact closed categories were introduced by G.M. Kelly and M.L. Laplaza [13] and there are an enrichment of symmetric monoidal categories where every object A has its dual (or adjoint) object. Furthermore, dagger compact closed categories extend compact closed categories with a linear algebra notion of adjointness on the morphisms.

Definition 2.8. *A compact closed category is a symmetric monoidal category where every object A is assigned by a dual (or adjoint) object A^*, together with a unit morphism (or bell-state):*

$$n_A : I \to A^* \otimes A$$

and a counit morphism:

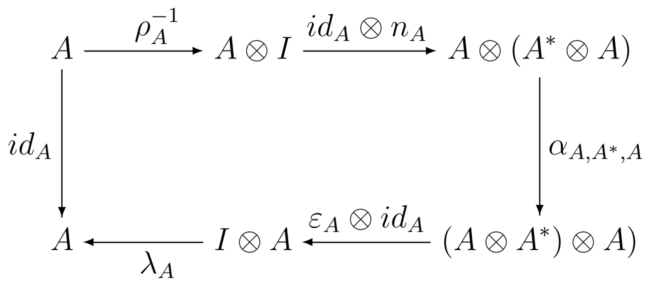
$$\varepsilon_A : A \otimes A^* \to I,$$

such that the following equations hold:[1]

$$\lambda_A \circ (\varepsilon_A \otimes id_A) \circ \alpha_{A,A^*,A} \circ (id_A \otimes n_A) \circ \rho_A^{-1} = id_A \tag{2.2.1}$$

$$\rho_{A^*} \circ (id_{A^*} \otimes \varepsilon_A) \circ \alpha^{-1}_{A^*,A,A^*} \circ (n_A \otimes id_{A^*}) \circ \lambda^{-1}_{A^*} = id_{A^*}, \tag{2.2.2}$$

i.e. diagrammatically:



[1] Assuming strict monoidal categories, equations (2.2.1) and (2.2.2) are written $(\varepsilon_A \otimes id_A) \circ (id_A \otimes n_A) = id_A$ and $(id_{A^*} \otimes \varepsilon_A) \circ (n_A \otimes id_{A^*}) = id_{A^*}$, respectively.

$$
\begin{array}{ccccc}
A^* & \xrightarrow{\lambda^{-1}_{A^*}} & I \otimes A^* & \xrightarrow{n_A \otimes id_{A^*}} & (A^* \otimes A) \otimes A^* \\
\downarrow{\scriptstyle id_{A^*}} & & & & \downarrow{\scriptstyle \alpha^{-1}_{A^*,A,A^*}} \\
A^* & \xleftarrow{\rho_{A^*}} & A^* \otimes I & \xleftarrow{id_{A^*} \otimes \varepsilon_A} & A^* \otimes (A \otimes A^*)
\end{array}
$$

From the previous definition it follows that the dual of A is unique up to isomorphism. In fact we have the following proposition:

Proposition 2.9. *[12] If A' and A'' are duals of A then $A' \cong A''$ and this isomorphism is natural in the sense that if B' and B'' are also duals of B and $f : A \to B$, then there exist $f' : B' \to A'$ and $f'' : B'' \to A''$ duals to f such that the following diagram commutes:*

$$
\begin{array}{ccc}
B' & \xrightarrow{f'} & A' \\
\downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_A} \\
B'' & \xrightarrow[f'']{} & A''
\end{array}
$$

where $\{d_A : A' \to A''\}_A$ is a family of natural isomorphisms.

Proof. Consider that A' and A'' are duals of A. Then we have unit and counit morphisms n_A, ε_A and n'_A, ε'_A for each dual respectively. Let

$$d_1 := \rho_{A''} \circ (id_{A''} \otimes \varepsilon_A) \circ \alpha^{-1}_{A'',A,A'} \circ (n'_A \otimes id_{A'}) \circ \lambda^{-1}_{A'} : A' \to A''$$

and

$$d_2 := \rho_{A'} \circ (id_{A'} \otimes \varepsilon'_A) \circ \alpha^{-1}_{A',A,A''} \circ (n_A \otimes id_{A''}) \circ \lambda^{-1}_{A''} : A'' \to A'.$$

Assuming a strict monoidal category, we then have:

$$
\begin{aligned}
d_1 \circ d_2 &= (id_{A''} \otimes \varepsilon_A) \circ (n'_A \otimes id_{A'}) \circ (id_{A'} \otimes \varepsilon'_A) \circ (n_A \otimes id_{A''}) \\
&= (id_{A''} \otimes \varepsilon_A) \circ (id_{A'' \otimes A \otimes A'} \otimes \varepsilon'_A) \circ (n'_A \otimes id_{A' \otimes A \otimes A''}) \circ (n_A \otimes id_{A''}) \\
&= (id_{A''} \otimes \varepsilon'_A) \circ (id_{A''} \otimes \varepsilon_A \otimes id_{A \otimes A''}) \circ (id_{A'' \otimes A} \otimes n_A \otimes id_{A''}) \circ (n'_A \otimes id_{A''}) \\
&= (id_{A''} \otimes \varepsilon'_A) \circ (id_{A''} \otimes ((\varepsilon_A \otimes id_A) \circ (id_A \otimes n_A)) \otimes id_{A''}) \circ (n'_A \otimes id_{A''}) \\
&= (id_{A''} \otimes \varepsilon'_A) \circ (id_{A''} \otimes id_A \otimes id_{A''}) \circ (n'_A \otimes id_{A''}) \\
&= (id_{A''} \otimes \varepsilon'_A) \circ (n'_A \otimes id_{A''}) \\
&= id_{A''}.
\end{aligned}
$$

Similarly $d_2 \circ d_1 = id_{A'}$ and therefore d_1 and d_2 are isomorphisms making $A' \cong A''$. To show now that this isomorphism is natural notice that:

$$
\begin{aligned}
f'' \circ d_B &= f'' \circ (id_{B''} \otimes \varepsilon_B) \circ (n'_B \otimes id_{B'}) \\
&= (f'' \otimes id_I) \circ (id_{B''} \otimes \varepsilon_B) \circ (n'_B \otimes id_{B'}) \\
&= (id_{A''} \otimes \varepsilon_B) \circ (f'' \otimes id_{B \otimes B'}) \circ (n'_B \otimes id_{B'}) \\
&= (id_{A''} \otimes \varepsilon_B) \circ (id_{A''} \otimes f \otimes id_{B'}) \circ (n'_A \otimes id_{B'})
\end{aligned}
$$

and

$$
\begin{aligned}
d_A \circ f' &= (id_{A''} \otimes \varepsilon_A) \circ (n'_A \otimes id_{A'}) \circ f' \\
&= (id_{A''} \otimes \varepsilon_A) \circ (n'_A \otimes id_{A'}) \circ (id_I \otimes f') \\
&= (id_{A''} \otimes \varepsilon_A) \circ (id_{A'' \otimes A} \otimes f') \circ (n'_A \otimes id_{B'}) \\
&= (id_{A''} \otimes \varepsilon_B) \circ (id_{A''} \otimes f \otimes id_{B'}) \circ (n'_A \otimes id_{B'})
\end{aligned}
$$

as required. □

Moreover, every compact closed category comes with the contravariant[2] functor $(-)^* : \mathcal{C}^{op} \to \mathcal{C}$, which preserves the structure of the symmetric monoidal categories and maps objects A to A^* and morphisms $f : A \to B$ to $f^* : B^* \to A^*$ such that:

$$
f^* = (id_{A^*} \otimes \varepsilon_B) \circ (id_{A^*} \otimes f \otimes id_{B^*}) \circ (n_A \otimes id_{B^*}), \tag{2.2.3}
$$

[2] A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is a functor that turns morphisms around and reverse the direction of composition, i.e. for every $\mathcal{C}$-morphism $f : A \to B$ we have $F(f) : F(B) \to F(A)$ and for $\mathcal{C}$-morphisms f, g we have $F(g \circ f) = F(f) \circ F(g)$.

i.e. diagrammatically:

$$
\begin{array}{ccccc}
B^* & \xrightarrow{\cong} & I \otimes B^* & \xrightarrow{n_A \otimes id_{B^*}} & A^* \otimes A \otimes B^* \\
\downarrow f^* & & & & \downarrow id_{A^*} \otimes f \otimes id_{B^*} \\
A^* & \xleftarrow[\cong]{} & A^* \otimes I & \xleftarrow[id_{A^*} \otimes \varepsilon_B]{} & A^* \otimes B \otimes B^*
\end{array}
$$

The morphism f^* is called the transpose of f.

Proposition 2.10. *The* $(-)^* : \mathcal{C}^{op} \to \mathcal{C}$ *defines a contravariant functor.*

Proof. We will first show that $(f^* \otimes id_B) \circ n_B = (id_{A^*} \otimes f) \circ n_A$, where $f^* : B^* \to A^*$ and $f : A \to B$. Assuming a strict monoidal category, we have

$$
\begin{aligned}
f^* = f^* \circ id_{B^*} &= f^* \circ (id_{B^*} \otimes \varepsilon_B) \circ (n_B \otimes id_{B^*}) \\
&= (f^* \otimes id_I) \circ (id_{B^*} \otimes \varepsilon_B) \circ (n_B \otimes id_{B^*}), \\
&= (id_{A^*} \otimes \varepsilon_B) \circ (f^* \otimes id_{B \otimes B^*}) \circ (n_B \otimes id_{B^*}), \qquad (2.2.4)
\end{aligned}
$$

where the second line follows by naturality of ρ and the third by bifunctoriality. From (2.2.3) and (2.2.4) it follows that:

$$
(f^* \otimes id_B) \circ n_B = (id_{A^*} \otimes f) \circ n_A.
$$

Now

$$
\begin{aligned}
f^* \circ g^* &= f^* \circ (id_{B^*} \otimes \varepsilon_C) \circ (id_{B^*} \otimes g \otimes id_{C^*}) \circ (n_B \otimes id_{C^*}) \\
&= (f^* \otimes id_I) \circ (id_{B^*} \otimes \varepsilon_C) \circ (id_{B^*} \otimes g \otimes id_{C^*}) \circ (n_B \otimes id_{C^*}) \\
&= (id_{A^*} \otimes \varepsilon_C) \circ (f^* \otimes id_{C \otimes C^*}) \circ (id_{B^*} \otimes g \otimes id_{C^*}) \circ (n_B \otimes id_{C^*}) \\
&= (id_{A^*} \otimes \varepsilon_C) \circ (id_{A^*} \otimes g \otimes id_{C^*}) \circ (f^* \otimes id_{B \otimes C^*}) \circ (n_B \otimes id_{C^*}) \\
&= (id_{A^*} \otimes \varepsilon_C) \circ (id_{A^*} \otimes g \otimes id_{C^*}) \circ (id_{A^*} \otimes f \otimes id_{C^*}) \circ (n_A \otimes id_{C^*}) \\
&= (id_{A^*} \otimes \varepsilon_C) \circ ((id_{A^*} \otimes g) \circ (id_{A^*} \otimes f)) \otimes (id_{C^*} \circ id_{C^*}) \circ (n_A \otimes id_{C^*}) \\
&= (id_{A^*} \otimes \varepsilon_C) \circ (id_{A^*} \otimes (g \circ f) \otimes id_{C^*}) \circ (n_A \otimes id_{C^*}) \\
&= (g \circ f)^*.
\end{aligned}
$$

Finally

$$
\begin{aligned}
(id_A)^* &= (id_{A^*} \otimes \varepsilon_A) \circ (id_{A^*} \otimes id_A \otimes id_{A^*}) \circ (n_A \otimes id_{A^*}) \\
&= (id_{A^*} \otimes \varepsilon_A) \circ id_{A^* \otimes A \otimes A^*} \circ (n_A \otimes id_{A^*}) \\
&= (id_{A^*} \otimes \varepsilon_A) \circ \alpha^{-1}_{A^*,A,A^*} \circ (n_A \otimes id_{A^*}) \\
&= id_{A^*}.
\end{aligned}
$$

□

Lemma 2.11. *In a compact closed category the following natural isomorphisms exist:*

1. $(A \otimes B)^* \cong B^* \otimes A^*$
2. $A^{**} \cong A$
3. $I^* \cong I$.

Proof. We need to find for each case unit and counit morphisms such that the equations (2.2.1) and (2.2.2) hold.

1. Consider

$$
\begin{aligned}
n_{A \otimes B} &:= (id_{B^*} \otimes n_A \otimes id_B) \circ n_B : I \to B^* \otimes A^* \otimes A \otimes B \\
\varepsilon_{A \otimes B} &:= \varepsilon_A \circ (id_A \otimes \varepsilon_B \otimes id_{A^*}) : A \otimes B \otimes B^* \otimes A^* \to I.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(\varepsilon_{A \otimes B} \otimes id_{A \otimes B}) \circ (id_{A \otimes B} \otimes n_{A \otimes B}) \\
&\quad = \big((\varepsilon_A \circ (id_A \otimes \varepsilon_B \otimes id_{A^*})) \otimes id_{A \otimes B}\big) \circ \big(id_{A \otimes B} \otimes ((id_{B^*} \otimes n_A \otimes id_B) \circ n_B)\big) \\
&\quad = (\varepsilon_A \otimes id_{A \otimes B}) \circ (id_A \otimes \varepsilon_B \otimes id_{A^*} \otimes id_{A \otimes B}) \circ (id_{A \otimes B} \otimes id_{B^*} \otimes n_A \otimes id_B) \circ (id_{A \otimes B} \otimes n_B) \\
&\quad = (\varepsilon_A \otimes id_{A \otimes B}) \circ (id_A \otimes n_A \otimes id_B) \circ (id_A \otimes \varepsilon_B \otimes id_B) \circ (id_{A \otimes B} \otimes n_B) \\
&\quad = (id_A \otimes id_B) \circ (id_A \otimes id_B) \\
&\quad = id_{A \otimes B}.
\end{aligned}
$$

Similarly

$$
(id_{B^* \otimes A^*} \otimes \varepsilon_{A \otimes B}) \circ (n_{A \otimes B} \otimes id_{B^* \otimes A^*}) = id_{B^* \otimes A^*}.
$$

2. Consider

$$
\begin{aligned}
n_{A^*} &:= \sigma_{A^*,A} \circ n_A : I \to A \otimes A^* \\
\varepsilon_{A^*} &:= \varepsilon_A \circ \sigma_{A^*,A} : A^* \otimes A \to I.
\end{aligned}
$$

Then

$$
\begin{aligned}
(\varepsilon_{A^*} \otimes id_{A^*}) \circ (id_{A^*} \otimes n_{A^*}) &= ((\varepsilon_A \circ \sigma_{A^*,A}) \otimes id_{A^*}) \circ (id_{A^*} \otimes (\sigma_{A^*,A} \circ n_A)) \\
&= (\varepsilon_A \otimes id_{A^*}) \circ (\sigma_{A^*,A} \otimes id_{A^*}) \circ (id_{A^*} \otimes \sigma_{A^*,A}) \circ (id_{A^*} \otimes n_A) \\
&= (id_{A^*} \otimes \varepsilon_A) \circ (n_A \otimes id_{A^*}) \\
&= id_{A^*}.
\end{aligned}
$$

Similarly

$$
(id_A \otimes \varepsilon_{A^*}) \circ (n_{A^*} \otimes id_A) = id_A.
$$

3. Consider

$$
\begin{aligned}
n_I &:= \lambda_I^{-1} : I \to I \otimes I \\
\varepsilon_I &:= \lambda_I : I \otimes I \to I.
\end{aligned}
$$

Then clearly

$$
(\varepsilon_I \otimes id_I) \circ (id_I \otimes n_I) = (id_I \otimes \varepsilon_I) \circ (n_I \otimes id_I) = id_I.
$$

By Proposition 2.9 duals are naturally isomorphic and that completes the proof. □

Lemma 2.12. *In a strict compact closed category the following are equivalent:*

1. $(id_{A^*} \otimes f) \circ n_A = (f^* \otimes id_B) \circ n_B$
2. $\varepsilon_B \circ (f \otimes id_{B^*}) = \varepsilon_A \circ (id_A \otimes f^*)$
3. $f = (id_B \otimes \varepsilon_{A^*}) \circ (id_B \otimes f^* \otimes id_A) \circ (n_{B^*} \otimes id_A)$
4. $f^* = (id_{A^*} \otimes \varepsilon_B) \circ (id_{A^*} \otimes f \otimes id_{B^*}) \circ (n_A \otimes id_{B^*})$,

where $f : A \to B$.

Proof. See [13]. □

Recall that the reason for defining compact closed categories is to address a complete framework for quantum mechanics and computation. However, taking in account that the John von Neumann's formalism takes place in **FdHilb**, the language of compact closed categories does not provide a complete description of **FdHilb** structure. What is missing is the inner product, which is essential for many parts of quantum mechanics. Therefore, the definition of dagger compact closed categories comes to fulfill this lacuna.

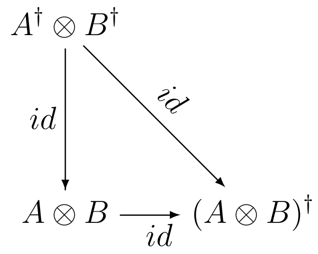
Definition 2.13. *A dagger symmetric monoidal category (†-symmetric monoidal category) is a symmetric monoidal category with a contravariant functor* $(-)^\dagger : \mathcal{C}^{op} \to \mathcal{C}$, *which is identity on the objects and involutive on the morphisms, i.e. it maps every morphism* $f : A \to B$ *to* $f^\dagger : B \to A$, *where* $f^\dagger$ *is called the adjoint of* f. *Also for every* $f : A \to B$ *and* $g : B \to C$ *the following hold:*

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger : C \to A$$
$$f^{\dagger\dagger} = f : A \to B$$
$$id_A^\dagger = id_A : A \to A.$$

Finally the contravariant functor $(-)^\dagger : \mathcal{C}^{op} \to \mathcal{C}$ *must coherently preserve the structure of the symmetric monoidal category, such that for every* $f : A \to B$ *and* $g : C \to D$:

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger : B \otimes D \to A \otimes C$$
$$\alpha^\dagger_{A,B,C} = \alpha^{-1}_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
$$\lambda^\dagger_A = \lambda^{-1}_A : A \to I \otimes A$$
$$\rho^\dagger_A = \rho^{-1}_A : A \to A \otimes I$$
$$\sigma^\dagger_{A,B} = \sigma^{-1}_{A,B} : B \otimes A \to A \otimes B.$$

Remark: Since $(-)^\dagger : \mathcal{C}^{op} \to \mathcal{C}$ is identity on objects and preserves the symmetric monoidal structure we have that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$, namely the following diagram commutes:



Similarly to Proposition 2.10 the adjoint functor $(-)^\dagger : \mathcal{C}^{op} \to \mathcal{C}$ indeed defines a contravariant functor by definition.

Definition 2.14. *In a †-symmetric monoidal category a morphism* $f : A \to B$ *is called unitary if it is an isomorphism and* $f^{-1} = f^\dagger$. *A morphism* $f : A \to B$ *is called self-adjoint if* $f^\dagger = f$.

Definition 2.15. *A dagger compact closed category (†-compact closed category) is a dagger symmetric monoidal category that is also compact closed. In addition we require the following coherence conditions:*

- *every natural isomorphism ξ that derives from the symmetric monoidal category must be unitary.*
- $n_{A^*} = \varepsilon_A^\dagger = \sigma_{A^*,A} \circ n_A$*, i.e. the following diagram commutes:*

$$
\begin{array}{ccc}
I & \xrightarrow{\;n_{A^*} = \varepsilon_A^\dagger\;} & A \otimes A^* \\
& \searrow^{n_A} & \uparrow{\scriptstyle \sigma_{A^*,A}} \\
& & A^* \otimes A
\end{array}
$$

Note that we can now replace $f : A \to B$ by $f^\dagger : B \to A$ in equation (2.2.3) extending in this way the duality assignment on objects $A \mapsto A^*$ by the morphism assignment $f \mapsto f_*$. Therefore we have:

$$f_* = (id_{B^*} \otimes \varepsilon_A) \circ (id_{B^*} \otimes f^\dagger \otimes id_{A^*}) \circ (n_B \otimes id_{A^*}),$$

i.e. diagrammatically:

$$
\begin{array}{ccccc}
A^* & \xrightarrow{\;\cong\;} & I \otimes A^* & \xrightarrow{\;n_B \otimes id_{A^*}\;} & B^* \otimes B \otimes A^* \\
\downarrow{\scriptstyle f_*} & & & & \downarrow{\scriptstyle id_{B^*} \otimes f^\dagger \otimes id_{A^*}} \\
B^* & \xleftarrow[\;\cong\;]{} & B^* \otimes I & \xleftarrow[\;id_{B^*} \otimes \varepsilon_A\;]{} & B^* \otimes A \otimes A^*
\end{array}
$$

giving rise to the following definition.

Definition 2.16. *In a †-compact closed category the covariant functor $(-)_* : \mathcal{C} \to \mathcal{C}$ maps objects A to A^* and morphisms $f : A \to B$ to $f_* : A^* \to B^*$, such that $f^\dagger = (f_*)^* = (f^*)_*$ for every morphism $f : A \to B$. The morphism f_* is called the conjugate of f.*

Similarly to Lemma 2.12 we then have the following lemma:

Lemma 2.17. *In a strict †-compact closed category the following are equivalent:*

1. $(id_{B^*} \otimes f^\dagger) \circ n_B = (f_* \otimes id_A) \circ n_A$

2. $\varepsilon_A \circ (f^\dagger \otimes id_{A^*}) = \varepsilon_B \circ (id_B \otimes f_*)$

3. $f^\dagger = (id_A \otimes \varepsilon_{B^*}) \circ (id_A \otimes f_* \otimes id_B) \circ (n_{A^*} \otimes id_B)$

4. $f_* = (id_{B^*} \otimes \varepsilon_A) \circ (id_{B^*} \otimes f^\dagger \otimes id_{A^*}) \circ (n_B \otimes id_{A^*})$,

where $f : A \to B$*.*

2.3 Graphical language

Graphical calculi have been an important part of computation in quantum mechanics and a protuberant research area in category theory. The categorical graphical calculus of †-compact closed categories admits a sound and complete interpretation of the axioms in quantum mechanics and computation, such that every equation that can be proved in the categorical calculus can also be proved in the graphical calculus and vive versa. The graphical language of symmetric monoidal categories and compact closed categories were introduced by G.M. Kelly, A. Joyal and R. Street in [13–15]. An extension to †-compact closed categories was given by P. Selinger [3] and also by B. Coecke [16, 17] and in [5] by B. Coecke and D. Pavlovic.

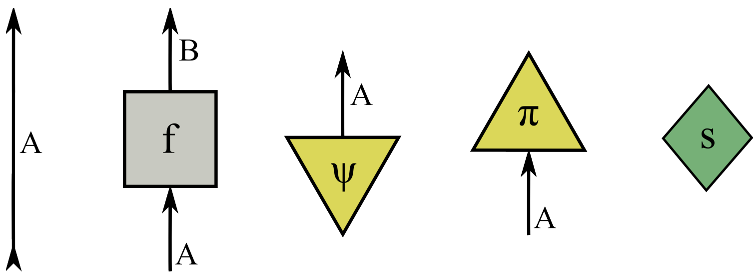
The graphical language consists of pictures with some primitive data, in which two kinds of composition take place, namely the sequential composition for the concatenation in time and the parallel composition for conceiving two systems to a compound one. Analogously to the categorical semantics these compositions correspond to usual composition and tensor product respectively. The primitive data consists of:

- lines which may carry a symbol referring to the kind or type of system (i.e. one qubit, n-qubits, classical data, quantum data e.t.c),

- input/output boxes which depict morphisms (i.e. operations, physical processes),

- triangles with only an output which correspond to states or preparation procedures,
- triangles with only an input which correspond to costates or measurement branches,
- diamonds without input or output lines which correspond to values or probabilities or weights.

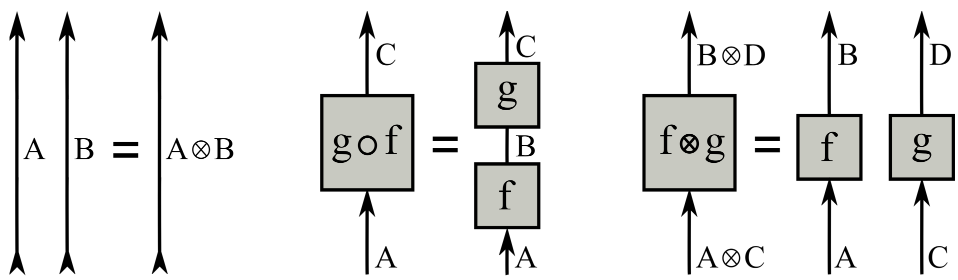
The above can be depicted as follows:

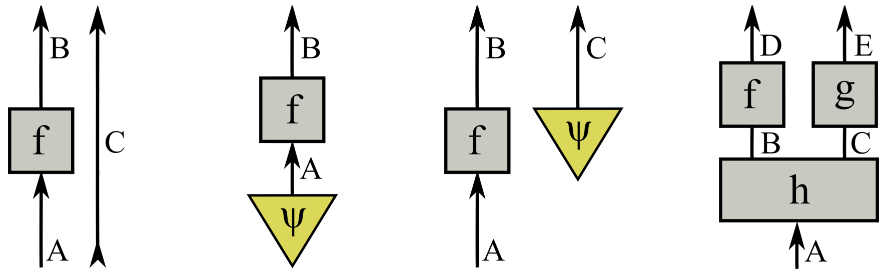


which respectively correspond to $id_A : A \to A$, $f : A \to B$, $\psi : I \to A$, $\pi : A \to I$, $s : I \to I$. Here we assume that in all pictures the sequence of time is from the bottom to the top, as denoted by the arrows.

Remark: Note that a single line denoted by a letter may correspond to the type of the system or the identity morphism. Also the multiplicative unit I corresponds to the "no system", so is depicted by the empty line. Finally, the state $\psi : I \to A$ and the costate $\pi : A \to I$ are called "ket"and "bra" in Dirac's notation [18].

Sequential composition is obtained by connecting the inputs and the outputs of the boxes (if there exist any) by lines and parallel composition is obtained by placing two boxes side by side. For example the following pictures:

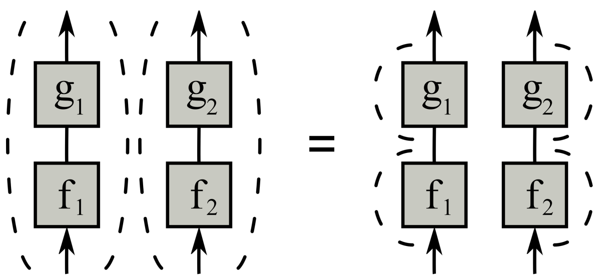




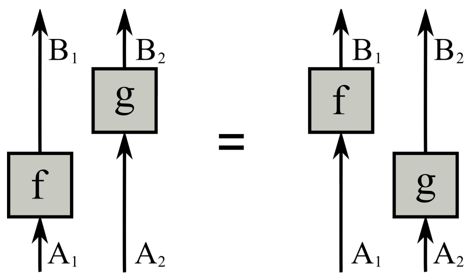
correspond to:

$$
\begin{aligned}
id_A \otimes id_B = id_{A\otimes B} &: A \otimes B \to A \otimes B \\
g \circ f &: A \to C \\
f \otimes g &: A \otimes C \to B \otimes D \\
f \otimes id_C &: A \otimes C \to B \otimes C \\
f \circ \psi &: I \to B \\
f \otimes \psi &: A \otimes I \to B \otimes C \\
(f \otimes g) \circ h &: A \to D \otimes E.
\end{aligned}
$$

Recall the Definition 2.7 of symmetric monoidal categories. Based on the definition of graphical calculus, we can now depict what bifunctoriality and natural isomorphisms stand for. Therefore we have for bifunctoriality (equation (2.1.3)) that:



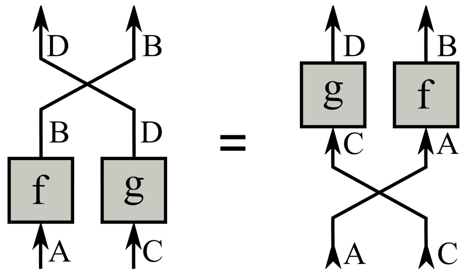
and hence equation (2.1.4) is depicted as:



Similarly the equation

$$\sigma_{B,D} \circ (f \otimes g) = (g \otimes f) \circ \sigma_{A,C},$$

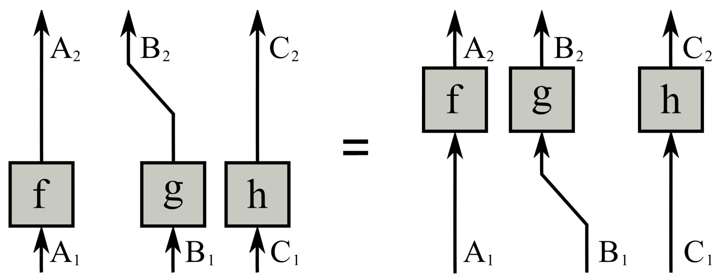
where $f : A \to B$, $g : C \to D$, that stands for swapping the systems, can be depicted as:



and the equation

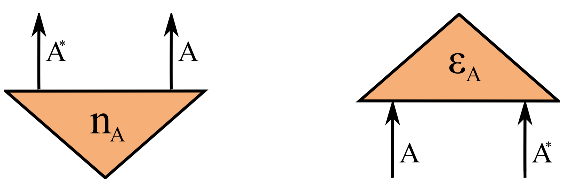
$$\alpha_{A_2,B_2,C_2} \circ (f \otimes (g \otimes h)) = ((f \otimes g) \otimes h) \circ \alpha_{A_1,B_1,C_1},$$

where $f : A_1 \to A_2$, $g : B_1 \to B_2$, $h : C_1 \to C_2$, that stands for associating the systems is in a picture:

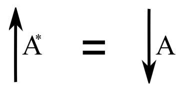


Thus, every equation is depicted in the graphical language in the sense that the graphical representation of the left-hand-side and the right-hand-side are isomorphic as graphs with respect to a fixed orientation of input and output.

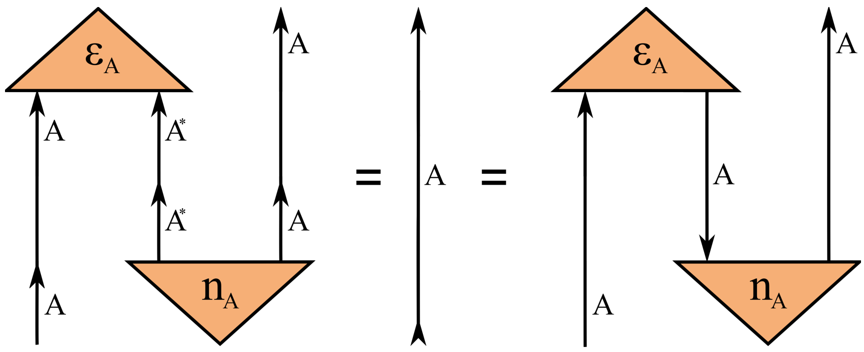
Extending the graphical language of symmetric monoidal categories we can have a graphical structure for compact closed categories and hence for †-compact closed categories. The unit (bell-state) $n_A : I \to A^* \otimes A$ and the counit $\varepsilon_A : A \otimes A^* \to I$ are represented as follows:



where the dual A^* of A can be represented by changing the orientation of the arrow, that is:

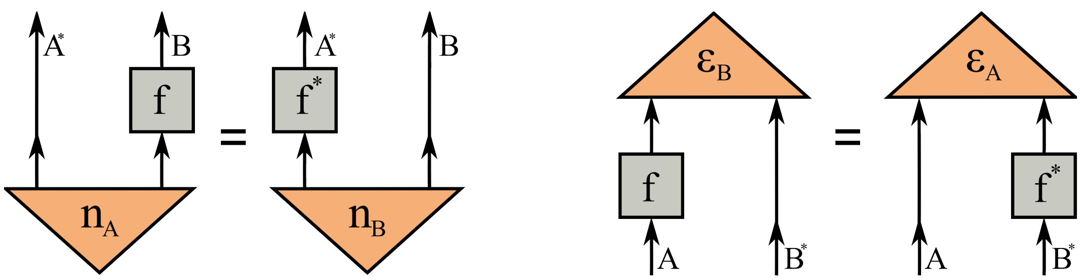


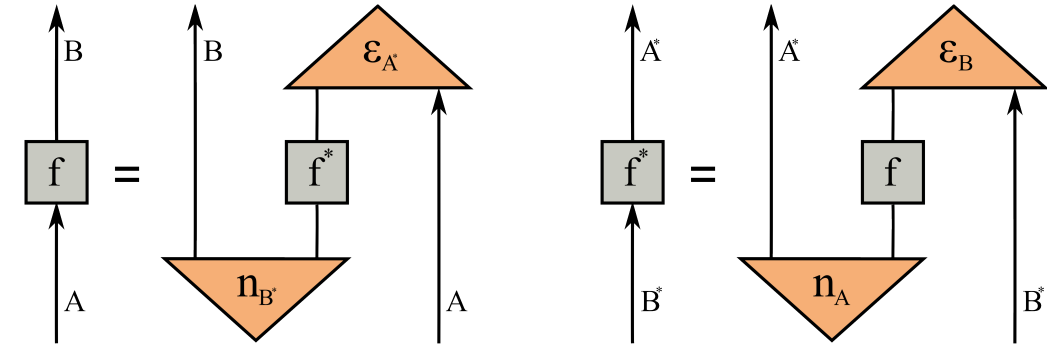
In that sense equation (2.2.1) is depicted as:



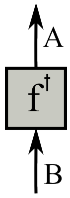
Remark: Note that in the previous picture the left-hand-side represents the physical flow of information with respect to casual ordering and the right-hand-side the logical flow of information. Using the second interpretation we actually viewing a transfer of the input through the counit and unit procedures and finally receiving it as an output at the other end. Also from now on we will omit the type of the system on a line when it is straightforward.

Having the graphical representation of n and ε we can now depict the equations in Lemma 2.12:

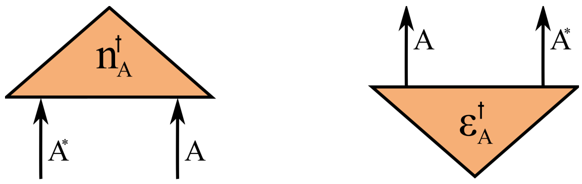




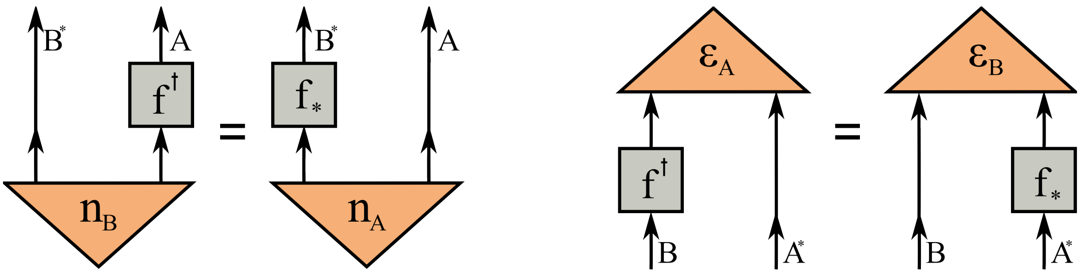
Finally for the †-compact closed categories if $f : A \to B$ is a morphism then $f^\dagger : B \to A$ is depicted by reversing the picture vertically, that is:

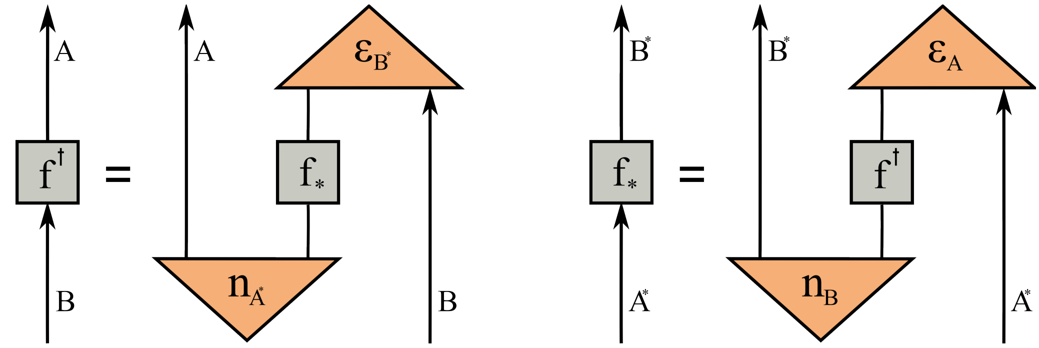


and hence we introduce pictures for $n^\dagger$ and $\varepsilon^\dagger$:

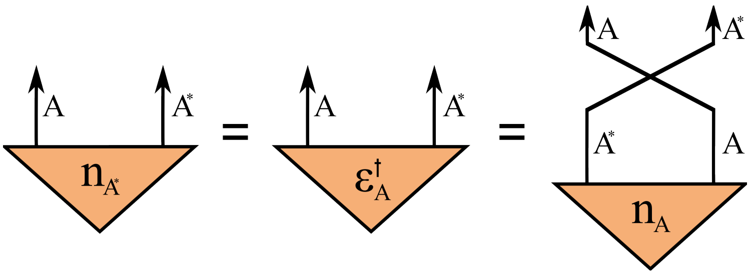


Notice that indeed the equations $n_A^\dagger = \varepsilon_{A^*}$ and $\varepsilon_A^\dagger = n_{A^*}$ both hold in the graphical language. Therefore, we depict the equations of Lemma 2.17 as follows:





The equation $n_{A^*} = \varepsilon_A^\dagger = \sigma_{A^*,A} \circ n_A$ of the Definition 2.15 is depicted as:



To sum up everything, we present the following theorem [3] which illustrates the coherence for the graphical language of †-compact closed categories. Similarly, we have equivalent theorems for the symmetric monoidal categories and compact closed categories.

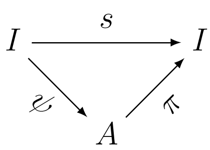
Theorem 2.18. *A well-typed equation between morphisms in the language of †-compact closed categories follows from the axioms of †-compact closed categories if and only if it holds, up to graphical isomorphism, in the graphical language.*

Proof. See [3]. □

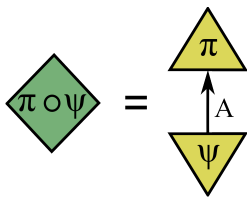
2.4 Scalars and trace

Definition 2.19. *Given a monoidal category $\mathcal{C}$ we define $\mathcal{C}(I, A)$ to be the state space, $\mathcal{C}(A, I)$ the costate space of a system A and $\mathcal{C}(I, I)$ the scalar monoid.*

As mentioned in section 2.3 diamonds are endomorphisms of type $s : I \to I$, called scalars and can arise by composition of a state with a costate, that is:



and in a picture:



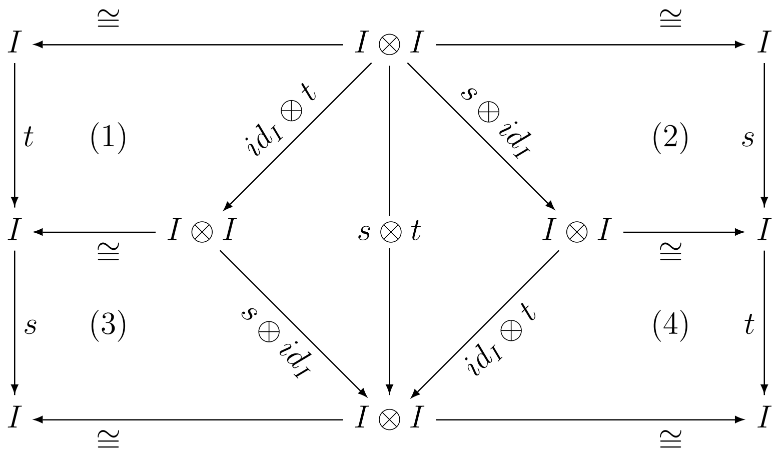
The composition $\pi \circ \psi$ is called "bra-ket" in Dirac's notation [18]. What is remarkable about scalars is that the scalar monoid is always commutative. That is for $s, t \in \mathcal{C}(I, I)$ we have $s \circ t = t \circ s$. In fact we have an even stronger result presented in the following lemma:

Lemma 2.20. *[12] Given scalars $s, t \in \mathcal{C}(I, I)$ then $s \circ t = t \circ s$ and the composite*

$$s \otimes t : I \cong I \otimes I \to I \otimes I \cong I$$

is equal to $s \circ t = t \circ s$.

Proof. Let $s, t \in \mathcal{C}(I, I)$. Then the following diagram commutes due to naturality of λ and ρ in left and right hand-side diagrams and diagrams (1)-(4) and due to bifunctoriality of the middle diagram.



Therefore, $s \circ t = t \circ s$. Furthermore it is noticed that the left-hand-side diagram gives $s \circ t = s \otimes t : I \cong I \otimes I \to I \otimes I \cong I$. $\square$

Definition 2.21. *Given a monoidal category $\mathcal{C}$, a scalar multiplication is the composition:*

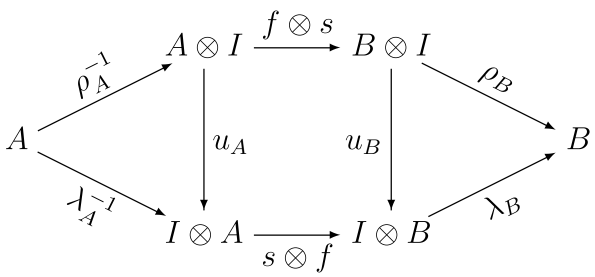
$$s \bullet f := \rho_B \circ (f \otimes s) \circ \rho_A^{-1} : A \to B,$$

where $s : I \to I$, $f : A \to B$.

Remark: An equivalent definition of the scalar multiplication could have been the following:

$$s \bullet f := \lambda_B \circ (s \otimes f) \circ \lambda_A^{-1} : A \to B,$$

i.e. defining the multiplication on the left rather than on the right. It appears that the two definitions are equivalent [12], since $\{u_A = \lambda_A^{-1} \circ \rho_A\}_A$ is a natural isomorphism, making the following diagram to commute:



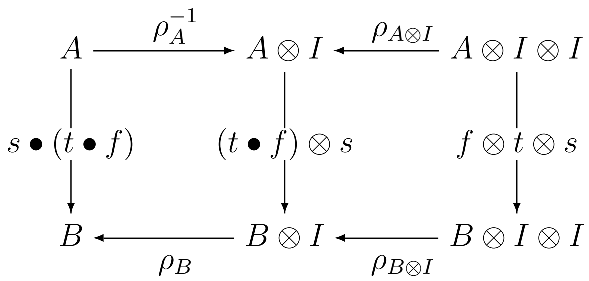
Lemma 2.22. *For scalars $s, t \in \mathcal{C}(I, I)$ the following hold:*

1. $s \bullet (t \bullet f) = (s \circ t) \bullet f$, *where* $f : A \to B$

2. $(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$, *where* $g : A \to B$ *and* $f : B \to C$

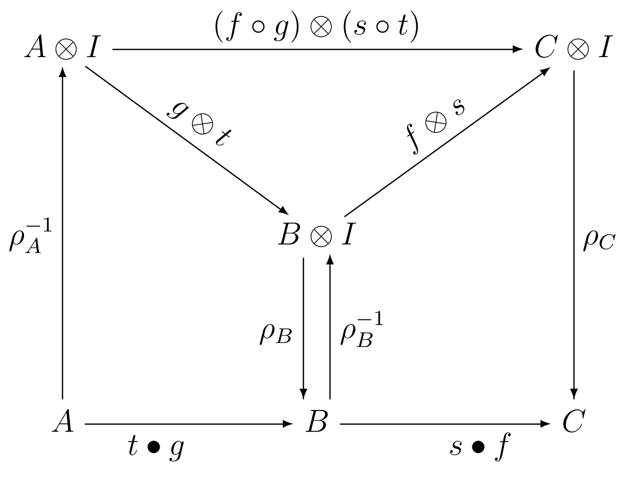
3. $(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$, *where* $f : A \to C$ *and* $g : B \to D$.

Proof. We prove the three equations diagrammatically.
1. For $f : A \to B$, $s, t \in \mathcal{C}(I, I)$ the following diagram commutes:



due to naturality of ρ and Definition 2.21. Note that from Lemma 2.20 $(t \otimes s) = t \circ s = s \circ t$. Since $\rho \circ \rho$ is a natural transformation then by Definition 2.21 we have $s \bullet (t \bullet f) = (s \circ t) \bullet f$.

2. For $g : A \to B$ and $f : B \to C$, $s, t \in \mathcal{C}(I, I)$ the following diagram commutes:



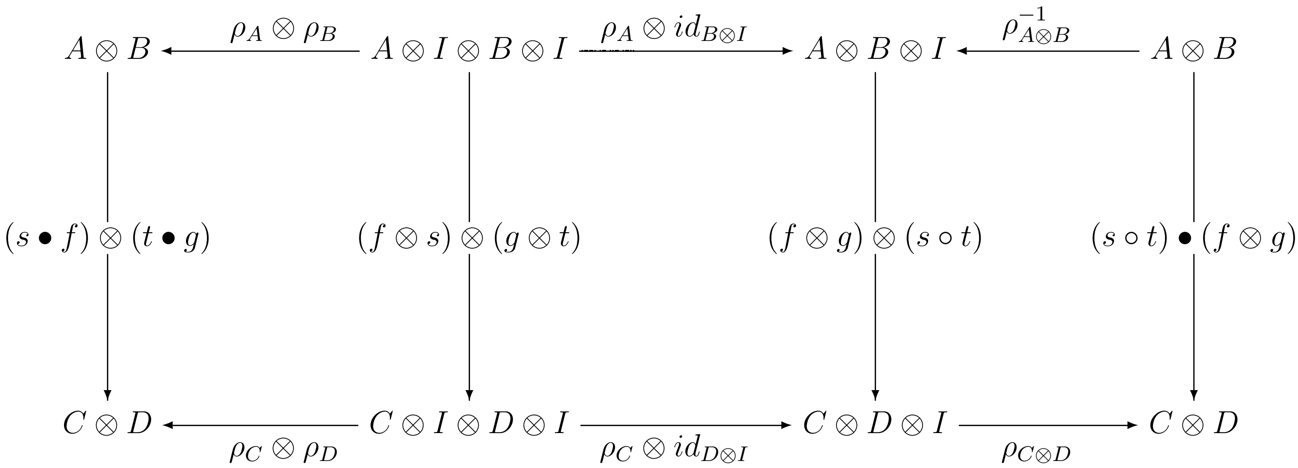
due to Definition 2.21 and to bifunctoriality for the upper diagram. Now by Definition 2.21

$$(s \circ t) \bullet (f \circ g) = \rho_C \circ ((f \circ g) \otimes (s \circ t)) \circ \rho_A^{-1},$$

therefore

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g).$$

3. For $g : B \to D$ and $f : A \to C$, $s, t \in \mathcal{C}(I, I)$ the following diagram commutes:

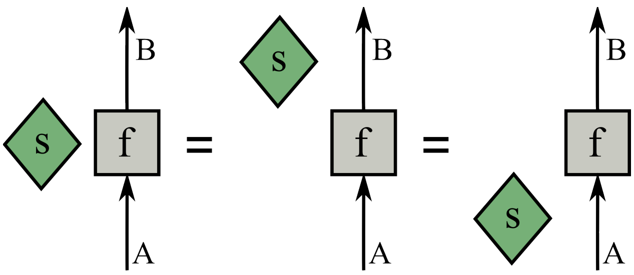


since it uses naturality of ρ and Definition 2.21 for the right-hand-side diagram. Therefore

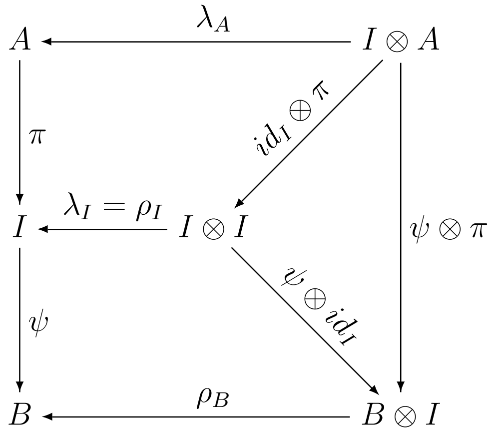
$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g).$$

□

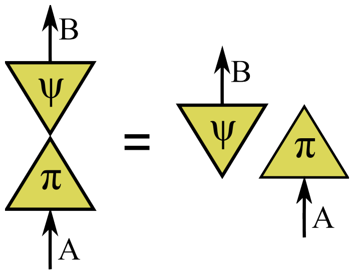
Scalars are viewed as probabilities or weights and hence, they can move freely in "time"and "space", that is:



We have the same property for $\psi \circ \pi : A \to B$, since the following diagram commutes, using naturality of λ and ρ, $\lambda_I = \rho_I$ and bifunctoriality:



Hence, $\psi \circ \pi = \rho_B \circ (\psi \otimes \pi) \circ \lambda_A^{-1}$ and in a picture:



The notion of trace was first introduced by A. Joyal and R. Street [15]. The definition relevant to compact closed categories is the following:

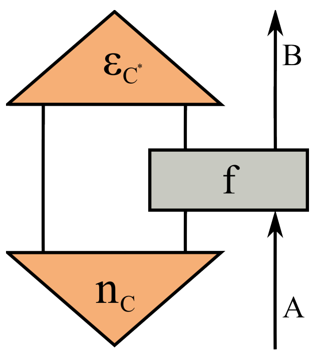
Definition 2.23. *Given a symmetric monoidal category $\mathcal{C}$ a trace is a family of morphisms $tr^C_{A,B} : \mathcal{C}(C \otimes A, C \otimes B) \to \mathcal{C}(A, B)$, such that for every morphism $f : C \otimes A \to C \otimes B$:*

$$tr^C_{A,B}(f) = \lambda_B \circ (\varepsilon_{C^*} \otimes id_B) \circ (id_{C^*} \otimes f) \circ (n_C \otimes id_A) \circ \lambda_A^{-1},$$

i.e. diagrammatically:

$$\begin{array}{ccccc} A & \xrightarrow{\lambda_A^{-1}} & I \otimes A & \xrightarrow{n_C \otimes id_A} & C^* \otimes C \otimes A \\ {\scriptstyle tr^C_{A,B}(f)}\downarrow & & & & \downarrow{\scriptstyle id_{C^*} \otimes f} \\ B & \xleftarrow[\lambda_B]{} & I \otimes B & \xleftarrow[\varepsilon_{C^*} \otimes id_B]{} & C^* \otimes C \otimes B \end{array}$$

We can depict $tr^C_{A,B}$ in a picture:



Similarly to Definition 2.23 we can define the partial transpose for a morphism $f : C \otimes A \to D \otimes B$.

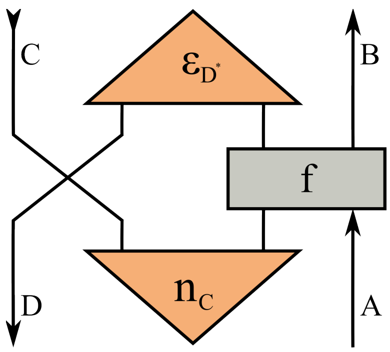
Definition 2.24. *Given a symmetric monoidal category* $\mathcal{C}$ *a partial transpose is a family of morphisms* $pt^{C,D}_{A,B} : \mathcal{C}(C \otimes A, D \otimes B) \to \mathcal{C}(D^* \otimes A, C^* \otimes B)$*, such that for every morphism* $f : C \otimes A \to D \otimes B$*:*

$$pt^{C,D}_{A,B}(f) = (id_{C^*} \otimes \lambda_B) \circ (id_{C^*} \otimes \varepsilon_{D^*} \otimes id_B) \circ (\sigma_{D^*,C^*} \otimes f) \circ (id_{D^*} \otimes n_C \otimes id_A) \circ (id_{D^*} \otimes \lambda_A^{-1}),$$

i.e. diagrammatically:

$$\begin{array}{ccccc}
D^* \otimes A & \xrightarrow{id_{D^*} \otimes \lambda_A^{-1}} & D^* \otimes I \otimes A & \xrightarrow{id_{D^*} \otimes n_C \otimes id_A} & D^* \otimes C^* \otimes C \otimes A \\
\Big\downarrow{\scriptstyle pt^{C,D}_{A,B}(f)} & & & & \Big\downarrow{\scriptstyle \sigma_{D^*,C^*} \otimes f} \\
C^* \otimes B & \xleftarrow{id_{C^*} \otimes \lambda_B} & C^* \otimes I \otimes B & \xleftarrow{id_{C^*} \otimes \varepsilon_{D^*} \otimes id_B} & C^* \otimes D^* \otimes D \otimes B
\end{array}$$

In a picture $pt^{C,D}_{A,B}(f)$ is:



Note that in the above picture the arrows present the logical flow of information. Therefore, partial transpose can be viewed as a swapping and transposition of an input and an output.

Chapter 3

Classical Structures and Measurements

Our aim in this chapter is to present what classical structures stand for and to define the quantum measurement. Abstractly, a quantum measurement can be described as an operation or physical procedure that takes a quantum state as an input and produces a measurement outcome, together with a quantum state. Due to the fundamental property of collapse during the measurement, the outcome quantum state is typically different from the input one, therefore the quantum measurement performs a change of the input state. To distinguish between quantum and classical data we introduce the notion of classical structure as a special †-compact closed Frobenius algebra as presented in [5] and we define quantum measurements based on that categorical concept.

3.1 Classical structures

The definition of a quantum measurement is based on the representation of classical data in the categorical concept of †-compact closed categories. This leads us to define classical data as a structured object (X, δ, γ) where X is a classical object, $\delta : X \to X \otimes X$ is a copying operation and $\gamma : X \to I$ a deleting operation. The axiomatization of classical structure is based on the particular axioms that morphisms δ and γ satisfy, yielding the definition of special †-compact closed Frobenius algebra [5].

3.1.1 Axiomatization of classical structures

Definition 3.1. *Given a monoidal category $\mathcal{C}$ an internal monoid is a structure (X, μ_X, ν_X), where*

$$\mu_X : X \otimes X \to X$$
$$\nu_X : I \to X,$$

such that

$$\mu_X \circ (id_X \otimes \mu_X) = \mu_X \circ (\mu_X \otimes id_X)$$

and

$$\mu_X \circ (\nu_X \otimes id_X) = \lambda_X$$
$$\mu_X \circ (id_X \otimes \nu_X) = \rho_X.$$

The morphisms μ and ν are called multiplication and multiplication unit, respectively.

Dually to Definition 3.1 we can define an internal comonoid (X, δ, γ) as follows:

Definition 3.2. *Given a monoidal category $\mathcal{C}$ an internal comonoid is a structure (X, δ_X, γ_X), where*

$$\delta_X : X \to X \otimes X$$
$$\gamma_X : X \to I,$$

such that

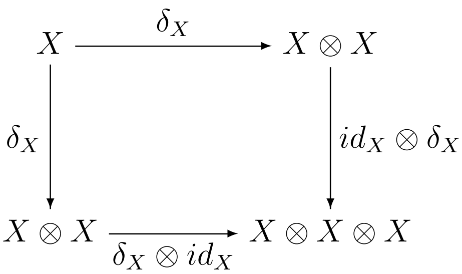
$$(id_X \otimes \delta_X) \circ \delta_X = (\delta_X \otimes id_X) \circ \delta_X$$

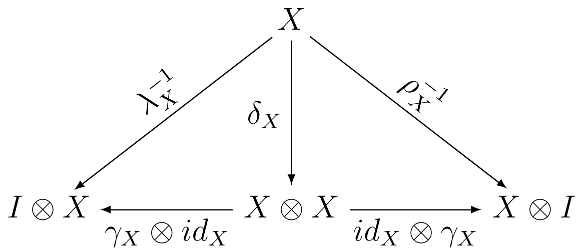
and

$$(\gamma_X \otimes id_X) \circ \delta_X = \lambda_X^{-1}$$
$$(id_X \otimes \gamma_X) \circ \delta_X = \rho_X^{-1}.$$

The morphisms δ and γ are called comultiplication and comultiplication unit, respectively.

Diagrammatically the above equations can be presented by the following commutative diagrams:





Definition 3.3. *Given a symmetric monoidal category $\mathcal{C}$ the internal monoid and the internal comonoid are commutative if and only if*

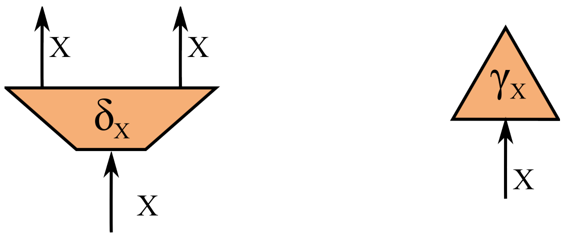
$$\mu_X \circ \sigma_{X,X} = \mu_X$$

and

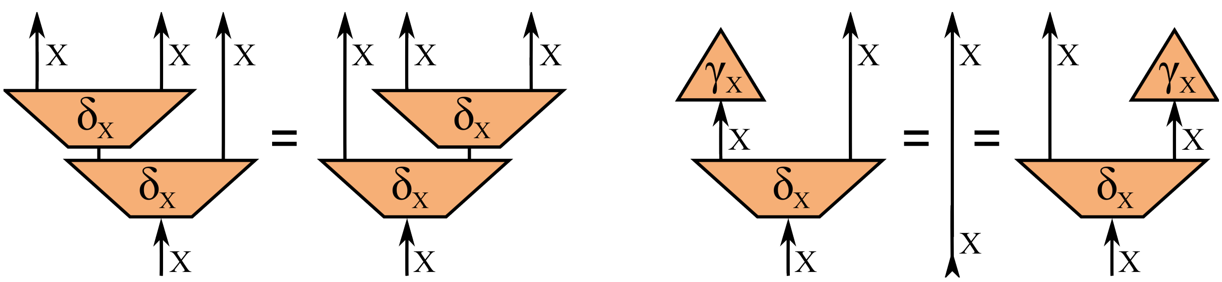
$$\sigma_{X,X} \circ \delta_X = \delta_X,$$

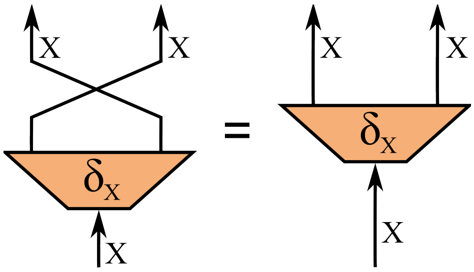
respectively.

In the graphical language we can depict the morphisms δ_X and γ_X as:



and hence the equations of Definition 3.2 and 3.3 for the internal comonoid (X, δ_X, γ_X) are in pictures:





Note that the above pictures express exactly what is expected from a copying and deleting operation. Explicitly, the first picture expresses that it does not matter which object is copied twice, the second expresses that copying and then deleting one of the copied objects is equivalent with doing nothing[1] and the last one that swapping the two copied objects does not alter something, i.e. the objects in the output of the copying operation are exactly the same.

Definition 3.4. *A symmetric Frobenius algebra is a structure that combines an internal commutative monoid* (X, μ_X, ν_X) *and the internal commutative comonoid* (X, δ_X, γ_X) *such that:*

$$\delta_X \circ \mu_X = (\mu_X \otimes id_X) \circ (id_X \otimes \delta_X), \tag{3.1.1}$$

that is diagrammatically:

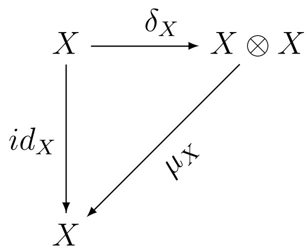
$$\begin{array}{ccc} X \otimes X & \xrightarrow{\mu_X} & X \\ {\scriptstyle id_X \otimes \delta_X}\downarrow & & \downarrow{\scriptstyle \delta_X} \\ X \otimes X \otimes X & \xrightarrow[\mu_X \otimes id_X]{} & X \otimes X \end{array}$$

Moreover a symmetric Frobenius algebra is special if

$$\mu_X \circ \delta_X = id_X,$$

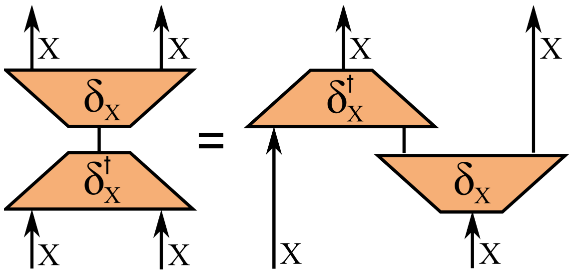
[1] Here we assume a strict symmetric monoidal category since we require $\lambda_X = \rho_X = id_X$.

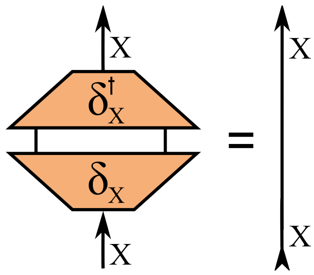
i.e.:



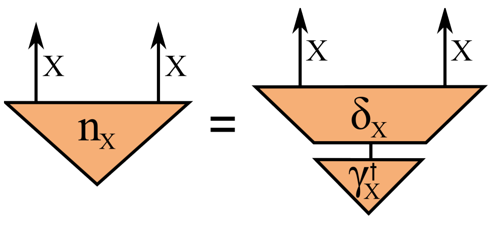
Equation (3.1.1) *is known as the Frobenius identity.*

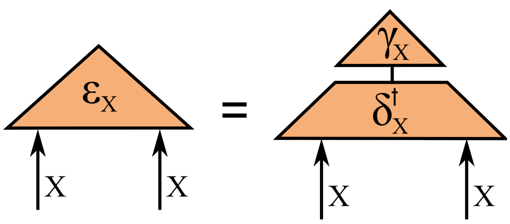
When we have a †-symmetric monoidal category then a †-Frobenius algebra can be constructed, since every internal commutative comonoid (X, δ_X, γ_X) defines an internal commutative monoid $(X, \delta_X^\dagger, \gamma_X^\dagger)$. Therefore, the equations of Definition 3.4 can be respectively depicted by:



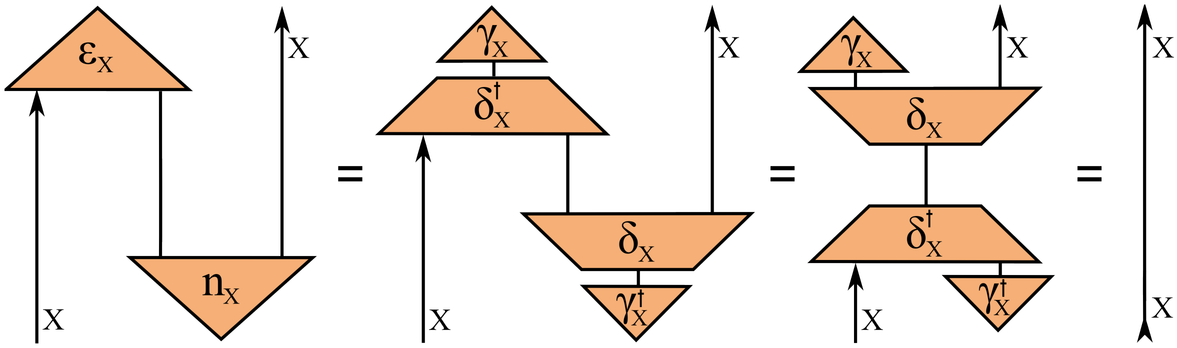


In [5] it is showed that $\delta_X \circ \gamma_X^\dagger : I \to X \otimes X$ and $\gamma_X \circ \delta_X^\dagger : X \otimes X \to I$ satisfy the equations of Definition 2.8, therefore they provide a unit morphism $n_X = \delta_X \circ \gamma_X^\dagger$ and a counit morphism $\varepsilon_X = \gamma_X \circ \delta_X^\dagger$, provided that $X = X^*$. In a picture that stands for:





and hence equation (2.2.1) can be depicted as:



We are now in a position to give the exact definition of a classical structure as presented in [5].

Definition 3.5. *In a* †*-compact closed category a classical structure* (X, δ_X, γ_X) *is defined to be a special* †*-compact closed Frobenius algebra, i.e. a special* †*-Frobenius algebra where* $n_X = \delta_X \circ \gamma_X^\dagger$*, with* $X = X^*$.

3.1.2 Classical structures in FdHilb

Recall that in **FdHilb** a classical structure corresponds to the space $\mathbb{C}^{\oplus n}$ and therefore is the structure $(\mathbb{C}^{\oplus n}, \delta^{(n)}, \gamma^{(n)})$. Fixing a basis $\{|i\rangle\}_i$ in $\mathbb{C}^{\oplus n}$ we have:

$$\delta^{(n)} : \mathbb{C}^{\oplus n} \to \mathbb{C}^{\oplus n} \otimes \mathbb{C}^{\oplus n} :: |i\rangle \mapsto |ii\rangle$$
$$\gamma^{(n)} : \mathbb{C}^{\oplus n} \to \mathbb{C} :: |i\rangle \mapsto 1$$

In **FdHilb** the copying map $\delta^{(n)}$ is indeed a copying operation of classical data. Notice that $\delta^{(n)}$ can copy only the base vectors, namely $|i\rangle$, but not arbitrary states $|\psi\rangle = \sum_i^n a_i|i\rangle$:

$$|\psi\rangle = \sum_i^n a_i|i\rangle \overset{\delta^{(n)}}{\longmapsto} \sum_i^n a_i|ii\rangle \neq |\psi\rangle \otimes |\psi\rangle = \Big(\sum_i^n a_i|i\rangle\Big) \otimes \Big(\sum_i^n a_i|i\rangle\Big).$$

As mentioned in [5] δ is base dependent as it captures the base $\{|i\rangle\}_i$, since

$$\delta\Big(\sum_{i\in I} a_i|i\rangle\Big) = \sum_{i\in I} a_i|ii\rangle$$

and hence, the set I has to be a singleton in order for $\sum_{i\in I} a_i|ii\rangle$ to be a disentangled state. Therefore $\sum_{i\in I} a_i|i\rangle$ boils down to base vector $|i\rangle$. The fact that the copying operation δ is base dependent prevents it from being a natural transformation, even if it is diagonal. This result is also derived from the No-Cloning theorem [5]. Since δ is restricted to copy only base vectors, is a copying operation of classical data only.

Note also that $\delta^\dagger$ is defined by:

$$\delta^\dagger : \mathbb{C}^{\oplus n} \otimes \mathbb{C}^{\oplus n} \to \mathbb{C}^{\oplus n} :: \begin{cases} |ij\rangle \mapsto \vec{o} & , i \neq j \\ |ii\rangle \mapsto |i\rangle & , \text{else} \end{cases}$$

and hence

$$\delta \circ \delta^\dagger :: \begin{cases} |ij\rangle \mapsto \vec{o} \mapsto \vec{o} & , i \neq j \\ |ii\rangle \mapsto |i\rangle \mapsto |ii\rangle & , \text{else} \end{cases}$$

i.e. $\delta \circ \delta^\dagger$ erase the non-diagonal elements. Classical data are deleted by γ, hence

$$id_{\mathbb{C}^{\oplus n}} \otimes \gamma :: |ij\rangle \mapsto |i\rangle$$

and finally

$$\delta^\dagger \circ \delta = (id_{\mathbb{C}^{\oplus n}} \otimes \gamma) \circ \delta = (\gamma \otimes id_{\mathbb{C}^{\oplus n}}) \circ \delta :: |i\rangle \mapsto |ii\rangle \mapsto |i\rangle.$$

3.1.3 Self-adjointness with respect to a classical structure

For simplicity reasons we will denote from now on a classical structure (X, δ_X, γ_X) by its classical object X, where this is clear by the context, meaning that whenever an object X is referred as classical, then it posses a classical structure (X, δ_X, γ_X) and is not an unstructured quantum object.

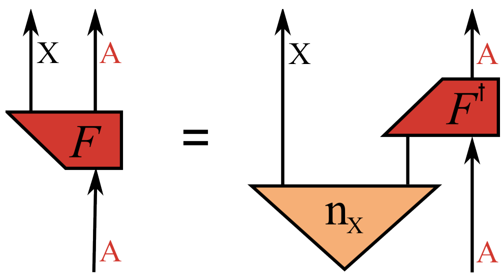
Definition 3.6. *Given a classical structure X and a quantum object A, a morphism $\mathcal{F} : A \to X \otimes A$ is called self-adjoint with respect to X (or X-self-adjoint) if*

$$\mathcal{F} = (id_X \otimes \mathcal{F}^\dagger) \circ (n_X \otimes id_A) \circ \lambda_A^\dagger,$$

i.e. if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{F}} & X \otimes A \\ {\scriptstyle \lambda_A^\dagger}\downarrow & & \uparrow{\scriptstyle id_X \otimes \mathcal{F}^\dagger} \\ I \otimes A & \xrightarrow[n_X \otimes id_A]{} & X \otimes X \otimes A \end{array}$$

We can depict the above by the following picture, where quantum data is denoted by red colour and classical by black.



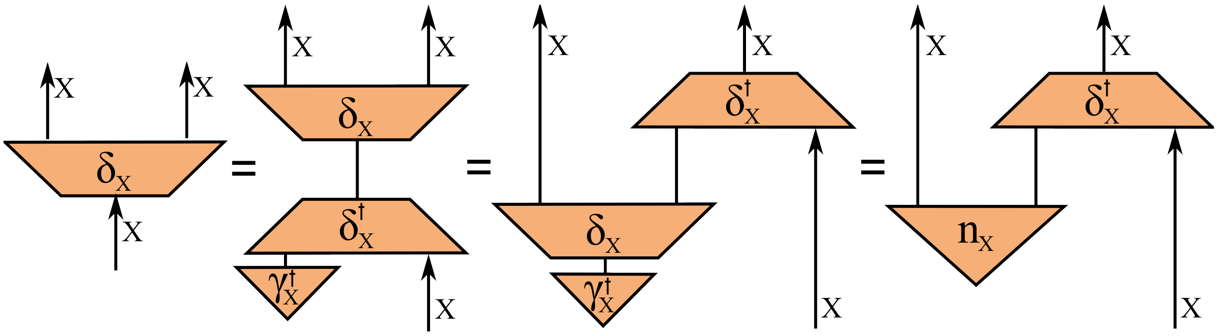
Proposition 3.7. *[5] Given a classical structure* (X, δ_X, γ_X) *then* δ_X *and* γ_X *are always* X*-self-adjoint.*

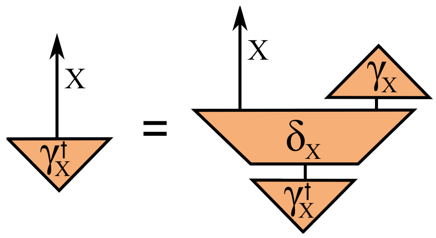
Proof. We assume a strict †-compact closed category, therefore we need to show that $\delta_X = (id_X \otimes \delta_X^\dagger) \circ (n_X \otimes id_X)$ and $\gamma_X^\dagger = (id_X \otimes \gamma_X) \circ n_X$. Note that whenever $\mathcal{F}$ is X-self-adjoint for a morphism $\mathcal{F} : A \to X \otimes A$ then $\mathcal{F}^\dagger$ is. Therefore if $\gamma^\dagger : I \to X \cong X \otimes I$ is X-self-adjoint, then $\gamma^{\dagger\dagger} = \gamma$ is. We then have:

$$\begin{aligned} \delta = \delta \circ id_X &= \delta \circ \delta^\dagger \circ (\gamma^\dagger \otimes id_X) \\ &= (id_X \otimes \delta_X^\dagger) \circ (\delta_X \otimes id_X) \circ (\gamma_X^\dagger \otimes id_X) \\ &= (id_X \otimes \delta_X^\dagger) \circ ((\delta_X \circ \gamma_X^\dagger) \otimes id_X) \\ &= (id_X \otimes \delta_X^\dagger) \circ (n_X \otimes id_X) \end{aligned}$$

$$\begin{aligned} \gamma_X^\dagger = id_X \circ \gamma_X^\dagger &= (id_X \otimes \gamma_X) \circ \delta_X \circ \gamma_X^\dagger \\ &= (id_X \otimes \gamma_X) \circ n_X \end{aligned}$$

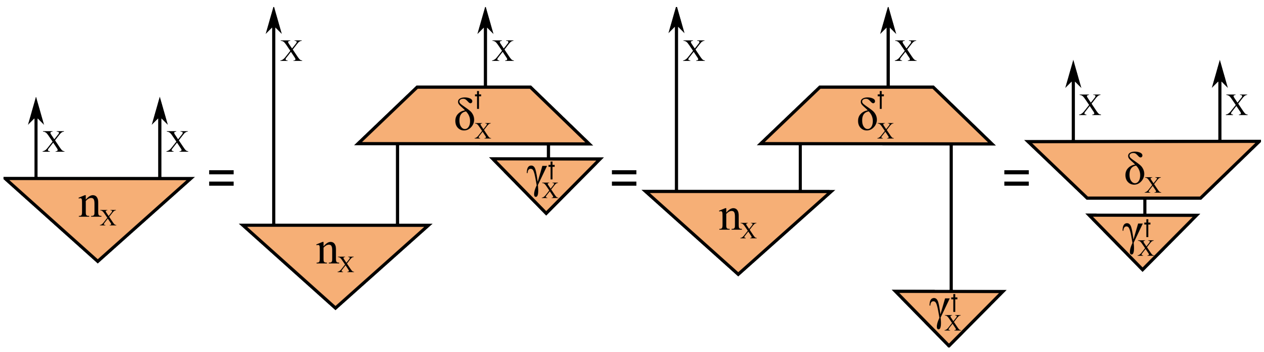
and these can be also viewed in pictures:





□

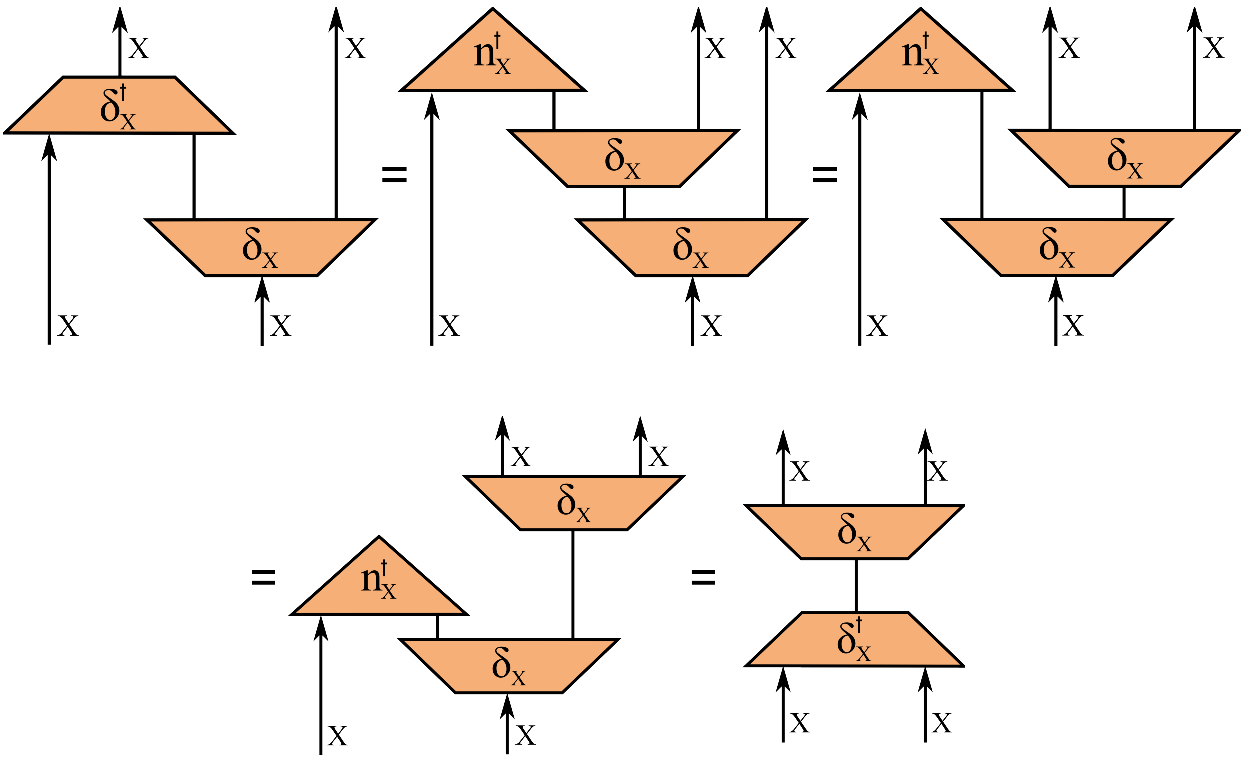
Also, given an internal commutative comonoid (X, δ_X, γ_X), then Definition 3.6 implies the self-duality of X and X-self-adjointness of δ_X and γ_X imply that $n_X = \delta_X \circ \gamma_X^\dagger$ which can be proved by:



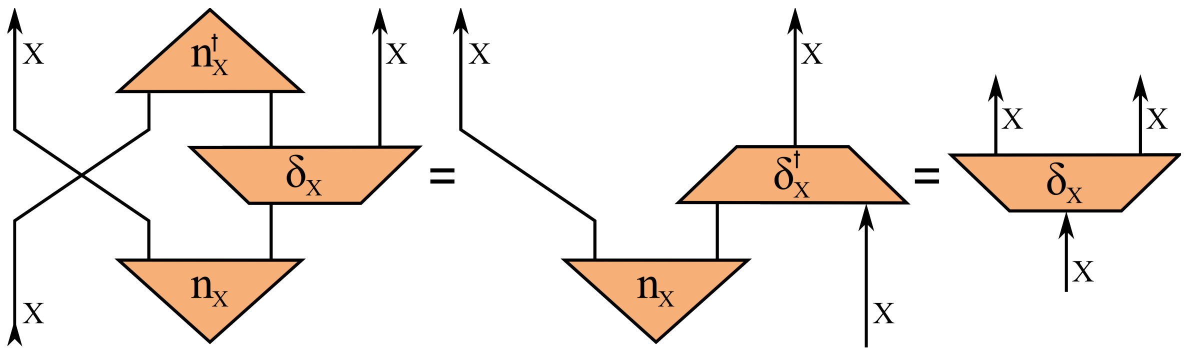
Hence in a †-compact closed category we can define an X-self-adjoint internal comonoid (X, δ_X, γ_X) and therefore we have the following lemma:

Lemma 3.8. *Given an X-self-adjoint internal comonoid (X, δ_X, γ_X), then δ_X satisfies the Frobenius identity (equation (3.1.1)), is invariant under partial transposition, i.e. $pt_{I,X}^{X,X}(\delta_X) = \delta_X$ and is self-dual, i.e. $(\delta_X)_* = \delta_X$.*

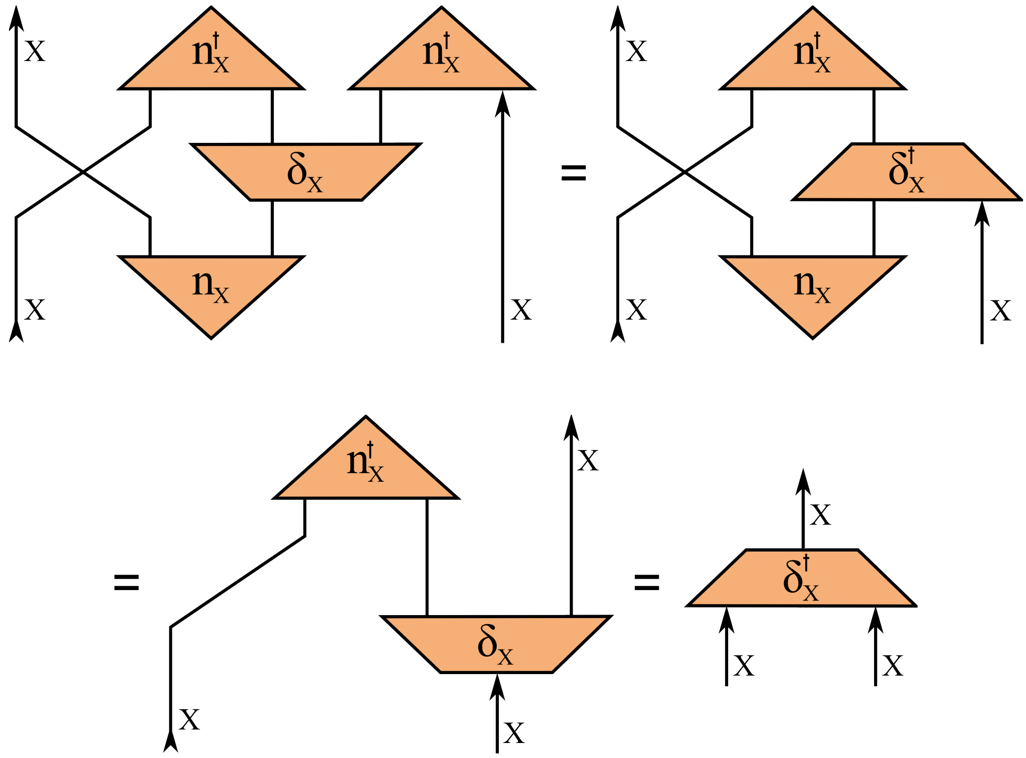
Proof. We prove the lemma graphically. For the Frobenius identity we have:



and for the partial-transpose-invariance:



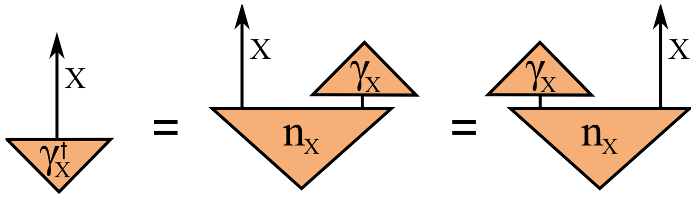
For the self-duality just consider:



Note that the arrows in the above pictures represent the physical flow of information. □

Theorem 3.9. *For a classical structure* (X, δ_X, γ_X) *we have* $(\delta_X)_* = \delta_X$ *and* $(\gamma_X)_* = \gamma_X$.

Proof. From Lemma 3.8 we have that $(\delta_X)_* = \delta_X$. For the comultiplication unit notice that the self-adjointness of γ_X and Definition 3.2 imply that $(\gamma_X)_* = \gamma_X$, since



□

We can therefore have an equivalent definition of a classical structure based on self-adjointness [5]:

Theorem 3.10. *[5] A classical structure is equivalently defined as a special X-self-adjoint internal commutative comonoid* (X, δ_X, γ_X).

3.2 Quantum measurements

Before we describe a quantum measurement we introduce some essential concepts.

Definition 3.11. *Given a classical structure X and a quantum object A, a morphism $\mathcal{F} : A \to X \otimes A$ is called X-idempotent if*

$$(id_X \otimes \mathcal{F}) \circ \mathcal{F} = (\delta_X \otimes id_A) \circ \mathcal{F},$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{F}} & X \otimes A \\ {\scriptstyle \mathcal{F}}\downarrow & & \downarrow{\scriptstyle id_X \otimes \mathcal{F}} \\ X \otimes A & \xrightarrow[\delta_X \otimes id_A]{} & X \otimes X \otimes A \end{array}$$

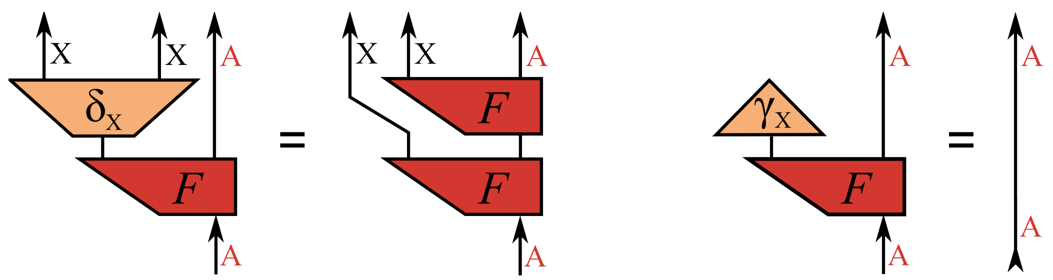
Moreover $\mathcal{F}$ is called complete if

$$\lambda_A \circ (\gamma_X \otimes id_A) \circ \mathcal{F} = id_A,$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{F}} & X \otimes A \\ {\scriptstyle id_A}\downarrow & & \downarrow{\scriptstyle \gamma_X \otimes id_A} \\ A & \xleftarrow[\lambda_A]{} & I \otimes A \end{array}$$

In pictures the above are shown respectively:



Definition 3.12. *A morphism $\mathcal{F} : A \to X \otimes A$ is said to be an X-projector if it is X-self-adjoint and X-idempotent. A morphism $\mathcal{F} : A \to X \otimes A$ is an X-projector-valued spectrum if it is an X-projector and moreover if it is X-complete.*

Hence, in **FdHilb** we have the following theorem:

Theorem 3.13. *[5] In **FdHilb** the projector-valued spectra relative to $\mathbb{C}^{\oplus n}$ exactly correspond to the complete family of mutually orthogonal projectors $\{\mathcal{P}_i\}_i$, $i \in \{1, \ldots, n\}$.*

Proof. In **FdHilb** a $\mathbb{C}^{\oplus n}$-projector-valued spectrum is a morphism $\mathcal{P} : \mathcal{H} \to \mathbb{C}^{\oplus n} \otimes \mathcal{H}$. where $dim(\mathcal{H}) = n$, i.e $\mathcal{H} \cong \mathbb{C}^{\oplus n}$.

Generally for a classical structure X a morphism $\mathcal{F} : A \to X \otimes A$ can be thought as an X-indexed family of morphisms $\{\mathcal{F}_X : A \to A\}_X$. Therefore, that means we can view X-self-adjointness of morphism $\mathcal{F}$ as self-adjointness of the X-indexed family $\{\mathcal{F}_X : A \to A\}_X$, that is $\mathcal{F}_X^\dagger = \mathcal{F}_X$. In the same sense X-idempotent is viewed as $\mathcal{F}_X \circ \mathcal{F}_X = \mathcal{F}_X$.

Back to **FdHilb**, the family of $\mathbb{C}^{\oplus n}$-indexed projectors $\{\mathcal{P}_i : \mathcal{H} \to \mathcal{H}\}_i$ corresponds to morphism $\mathcal{P} : \mathcal{H} \to \mathbb{C}^{\oplus n} \otimes \mathcal{H}$. Therefore, $\mathbb{C}^{\oplus n}$-self-adjointness of $\mathcal{P}$ yields self-adjointness of projectors $\mathcal{P}_i^\dagger = \mathcal{P}_i$ and $\mathbb{C}^{\oplus n}$-idempotence yields idempotence $\mathcal{P}_i \circ \mathcal{P}_i = \mathcal{P}_i$ and mutual orthogonality since $\mathcal{P}_i \circ \mathcal{P}_j = \mathcal{O}$, for $i \neq j$. Finally $\mathbb{C}^{\oplus n}$-completeness gives $\sum_i^n \mathcal{P}_i = 1_{\mathcal{H}}$ and that completes the proof. □

We are now in a position to define the abstract notion of quantum measurement. In fact projector-valued spectra $\mathcal{F} : A \to X \otimes A$ are actually a composition type of quantum measurements.

However as stated in [5] projector-valued spectra are an approximate notions of quantum measurements. To realize this first notice that the comultiplication morphism δ_X is indeed a projector-valued spectrum, since X-self-adjointness follows from Proposition 3.7 and X-idempotence and X-completeness from Definition 3.2. Therefore in **FdHilb** the canonical projector-valued spectrum $\delta^{(n)} : A \to X \otimes A$ yields

$$\delta^{(n)}\Big(\sum_i^n a_i|i\rangle_A\Big) = \sum_i^n a_i(|i\rangle_X \otimes |i\rangle_A),$$

where $A = X := \mathbb{C}^{\oplus n}$ and $|i\rangle_X$ is the measurement outcome, $|i\rangle_A$ the resulting quantum state and a_i the probability amplitudes captured in the outcome. Hence, projector-valued spectra in **FdHilb** maintain the relative phases present in probability amplitudes a_i, so we do not have a fully abstract quantum measurement. To solve this, P. Selinger introduced the category of complete positive maps (**CPM**) [3]. He proved that for every †-compact closed category $\mathcal{C}$ there is a construction of a correspondent **CPM**$(\mathcal{C})$ and hence, the approximate measurements turn into exact quantum measurements [5]. However, for many practical reasons the approximate notion of quantum measurements suffices and is the one used in this dissertation.

Note also that in **FdHilb** a measurement is described by a self-adjoint operator

$$H = \sum_i \lambda_i \mathcal{P}_i,$$

where $\mathcal{P}_i = |i\rangle\langle i|$ and hence the action of the above spectra decomposition on the state $|\psi\rangle$ is

$$\sum_i \lambda_i \mathcal{P}_i |\psi\rangle = \sum_i \lambda_i^{'} |i\rangle,$$

where $\lambda_i^{'} = \lambda_i \langle i|\psi\rangle$ is the measurement outcome and $|i\rangle$ the resulting state. Therefore the above description of measurement concises with the approximate description of quantum measurement, i.e. that of projector-valued spectra.

Chapter 4

Discrete Models

In this chapter we present the discrete models **FRel** and **Spek** as introduced by B. Coecke and B. Edwards in [7]. As shown in the previous chapters quantum mechanics can be abstractly expressed by †-compact closed categories. Actually, is sufficient to have a †-symmetric monoidal category with enough classical structures that enable compact closure. Since $n = \delta \circ \gamma^{\dagger}$, as mentioned in section 3.1 and in [5,7], then every morphism of a †-symmetric monoidal category can be formed using a classical structure. Hence, if all objects of a †-symmetric monoidal category have basis structures then we get a †-compact closed category.

4.1 The category FRel

The category **FRel** consists of finite sets and relations as morphisms. One can easily verify that $(\mathbf{FRel}, \times)$ is a †-symmetric monoidal category, where $\times$ is the cartesian product and the identity object I is the singleton $\{*\}$. The functor $(-)^{\dagger}$ corresponds to the relation converse, i.e. if $R \subseteq X \times Y$ and $R = \{(x, y) : x \in X, y \in Y\}$ then $R^{\dagger} \subseteq Y \times X$ and $R^{\dagger} = \{(y, x) : x \in X, y \in Y\}$. Also we have $R^{*} = R^{\dagger}$ and $R_{*} = R$. Finally for a finite set X the compact closure is captured by $n_X := \{(*, (x, x)) : x \in X\}$ and $\varepsilon_X = n_X^{\dagger} := \{((x, x), *) : x \in X\}$. Therefore, $(\mathbf{FRel}, \times)$ is indeed a compact closed category.

Now let X to be a set with n elements. The following relations constitute a classical structure:

$$\delta \subseteq X \times (X \times X) :: i \sim (i,i) \qquad\qquad \gamma \subseteq X \times I :: i \sim *$$

therefore, (X, δ, γ) is a classical structure in **FRel**.

Recall that an observable in **FdHilb** is represented by a self-adjoint operator H in the spectra decomposition:

$$H = \sum_i a_i \mathcal{P}_i,$$

where a_i are the eigenvalues of H and $\mathcal{P}_i$ are projectors. In particular the eigenvalues a_i are real since H is self-adjoint and the projectors $\mathcal{P}_i$ are mutually orthogonal. If the set $\{\mathcal{P}_i\}_i$ has dimension equal to the dimension of the state space, i.e. $\sum_i \mathcal{P}_i = 1$ then we have a non-degenerate observable.

Recently has been proved that in **FdHilb** classical structures are in one-to-one correspondence with orthonormal bases [8]. If we have orthonormal bases we then have non-degenerate measurements and non-degenerate spectral decompositions, so classical structures correspond to non-degenerate observables. However, more interesting are the observables that are complementary, i.e. the observables whose operators are the most incompatible possible [9]. We state below the basic definitions of the abstract characterization of complementary classical structures given by B. Coecke and R. Duncan in [9].

Definition 4.1. *Given a classical structure (A, δ, γ), a state $\psi : I \to A$ is unbiased relative to (A, δ, γ) if*

$$\delta^\dagger \circ (\psi \otimes id_A) = (\psi^\dagger \otimes id_A) \circ \delta.$$

Definition 4.2. *Given a classical structure (A, δ, γ), a state $\psi : I \to A$ is classical relative to (A, δ, γ) if ψ is a real comoinoid homomorphism, i.e.:*

$$\psi_* = \psi \qquad \delta \circ \psi = \psi \otimes \psi \qquad \gamma \circ \psi = id_I.$$

Definition 4.3. *[7] Two classical structures (A, δ_1, γ_1) and (A, δ_2, γ_2) are complementary if and only if:*

- *whenever $\psi : I \to A$ is classical for (A, δ_1, γ_1), it is unbiased for (A, δ_2, γ_2),*
- *whenever $\psi : I \to A$ is unbiased for (A, δ_1, γ_1), it is classical for (A, δ_2, γ_2),*
- *$\gamma_2^\dagger$ is classical for (A, δ_1, γ_1) and $\gamma_1^\dagger$ is classical for (A, δ_2, γ_2).*

4.2 The discrete model FRel

We are interested in the two element set, which we will denote by $\mathbb{I}\mathbb{I} := \{0, 1\}$. One can easily verify that the structure $(\mathbb{I}\mathbb{I}, \delta_Z, \gamma_Z)$ deduced by:

$$\delta_Z \subseteq \mathbb{I}\mathbb{I} \times (\mathbb{I}\mathbb{I} \times \mathbb{I}\mathbb{I}) :: \begin{cases} 0 \sim (0,0) \\ 1 \sim (1,1) \end{cases} \qquad \gamma_Z \subseteq \mathbb{I}\mathbb{I} \times \mathrm{I} :: \begin{cases} 0 \sim * \\ 1 \sim * \end{cases}$$

is a classical structure. In [7] B. Coecke and B. Edwards observed that besides this classical structure the set $\mathbb{I}\mathbb{I}$ has another one, that is $(\mathbb{I}\mathbb{I}, \delta_X, \gamma_X)$, where

$$\delta_X \subseteq \mathbb{I}\mathbb{I} \times (\mathbb{I}\mathbb{I} \times \mathbb{I}\mathbb{I}) :: \begin{cases} 0 \sim \{(0,0),(1,1)\} \\ 1 \sim \{(0,1),(1,0)\} \end{cases} \qquad \gamma_Z \subseteq \mathbb{I}\mathbb{I} \times \mathrm{I} :: 0 \sim *.$$

Clearly $(\mathbb{I}\mathbb{I}, \delta_X, \gamma_X)$ is an internal commutative comonoid, since

$$(\delta_X \times id_{\mathbb{I}\mathbb{I}}) \circ \delta_X = (id_{\mathbb{I}\mathbb{I}} \times \delta_X) \circ \delta_X :: \begin{cases} 0 \mapsto \{(0,0),(1,1)\} \mapsto \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\} \\ 1 \mapsto \{(0,1),(1,0)\} \mapsto \{(0,0,1),(1,1,1),(0,1,0),(1,0,0)\} \end{cases}$$

and trivially $(id_{\mathbb{I}\mathbb{I}} \times \gamma_X) \circ \delta_X = id_{\mathbb{I}\mathbb{I}} = (\gamma_X \times id_{\mathbb{I}\mathbb{I}}) \circ \delta_X = \{(0,0),(1,1)\}$ and $\sigma_{\mathbb{I}\mathbb{I},\mathbb{I}\mathbb{I}} \circ \delta_X = \delta_X$.

Finally, $(\mathbb{I}\mathbb{I}, \delta_X, \gamma_X)$ is a special †-compact closed Frobenius algebra, because

$$\delta_X \circ \delta_X^\dagger = (\delta_X^\dagger \times id_{\mathbb{I}\mathbb{I}}) \circ (id_{\mathbb{I}\mathbb{I}} \times \delta_X) :: \begin{cases} (0,0) \mapsto \{(0,0),(1,1)\} \\ (0,1) \mapsto \{(1,0),(0,1)\} \\ (1,0) \mapsto \{(0,1),(1,0)\} \\ (1,1) \mapsto \{(0,0),(1,1)\}, \end{cases}$$

$\delta_X^\dagger \circ \delta_X = id_{\mathbb{I}\mathbb{I}}$ and $n_{\mathbb{I}\mathbb{I}} = \delta_X \circ \gamma_X^\dagger = \{(*, \{(0,0),(1,1)\})\}$. Therefore, by Definition 3.5 $(\mathbb{I}\mathbb{I}, \delta_X, \gamma_X)$ is a classical structure.

The states over the set $\mathbb{I}$ are

$$z_0 \subseteq \mathrm{I} \times \mathbb{I} :: * \sim 0 \qquad z_1 \subseteq \mathrm{I} \times \mathbb{I} :: * \sim 1 \qquad x_0 \subseteq \mathrm{I} \times \mathbb{I} :: * \sim \{0,1\}$$

As stated in [7] we have the following theorem:

Theorem 4.4. *The classical structures* $(\mathbb{I}, \delta_Z, \gamma_Z)$ *and* $(\mathbb{I}, \delta_X, \gamma_X)$ *are complementary in the sense of the Definition 4.3.*

Proof. We prove that z_0, z_1 are classical for $(\mathbb{I}, \delta_Z, \gamma_Z)$ and x_0 is classical for $(\mathbb{I}, \delta_X, \gamma_X)$. We have

$$\delta_Z \circ z_0 :: * \mapsto 0 \mapsto \{(0,0)\} \qquad z_0 \times z_0 :: * \mapsto \{(0,0)\}$$
$$\delta_Z \circ z_1 :: * \mapsto 1 \mapsto \{(1,1)\} \qquad z_1 \times z_1 :: * \mapsto \{(1,1)\}$$

and

$$id_{\mathrm{I}} = \gamma_Z \circ z_0 :: * \mapsto 0 \mapsto *$$
$$id_{\mathrm{I}} = \gamma_Z \circ z_1 :: * \mapsto 1 \mapsto *.$$

Also

$$\delta_X \circ x_0 :: * \mapsto \{0,1\} \mapsto \{(0,0),(1,1),(0,1),(1,0)\},$$
$$x_0 \circ x_0 :: * \mapsto \{0,1\} \times \{0,1\} = \{(0,0),(1,1),(0,1),(1,0)\}$$

and

$$id_{\mathrm{I}} = \gamma_X \circ x_0 :: * \mapsto \{0,1\} \mapsto *.$$

Similarly it can be proved that z_0 and z_1 are unbiased for $(\mathbb{I}, \delta_X, \gamma_X)$ and x_0 is unbiased for $(\mathbb{I}, \delta_Z, \gamma_Z)$. Finally $\gamma_X^\dagger$ is classical for $(\mathbb{I}, \delta_Z, \gamma_Z)$ since

$$\delta_Z \circ \gamma_X^\dagger :: * \mapsto 0 \mapsto \{(0,0)\} \qquad \gamma_X^\dagger \times \gamma_X^\dagger :: * \mapsto \{(0,0)\}$$
$$id_{\mathrm{I}} = \gamma_Z \circ \gamma_X^\dagger :: * \mapsto 0 \mapsto *,$$

and $\gamma_Z^\dagger$ is classical for $(\mathbb{I}, \delta_X, \gamma_X)$ since

$$\delta_X \circ \gamma_Z^\dagger :: * \mapsto \{0,1\} \mapsto \{(0,0),(0,1),(1,0),(1,1)\} \qquad \gamma_Z^\dagger \times \gamma_Z^\dagger :: * \mapsto \{0,1\} \times \{0,1\}$$
$$id_{\mathrm{I}} = \gamma_X \circ \gamma_Z^\dagger :: * \mapsto \{0,1\} \mapsto *.$$

□

The previous theorem states that in **FRel** the set $\mathbb{I}$ represents a system with only two complementary observables. In **FdHilb** a standard qubit has a continuum of observables each with two classical objects and only three complementary observables can exist at the same time [7]. Hence, **FRel** with the set $\mathbb{I}$ is a more prefect model than a qubit in **FdHilb**. We state an important proposition mentioned in [7]:

Proposition 4.5. *The two-observable structure* $\{(\mathbb{I}, \delta_Z, \gamma_Z), (\mathbb{I}, \delta_X, \gamma_X)\}$ *in* ***FRel*** *is rich enough to simulate quantum teleportation and dense coding protocols.*

Proof. See [7]. □

4.3 Quantum spectra in FRel

Generally in **FRel** a $\mathbb{I}$-projector-valued spectrum corresponds to relation $R \subseteq A \times (\mathbb{I} \times A)$, where A is a finite set, such that the following equations are satisfied (see Definition 3.12):

$$R = (id_{\mathbb{I}} \times R^{\dagger}) \circ (n_{\mathbb{I}} \times id_A) \circ \lambda_A^{\dagger} \tag{4.3.1}$$

$$(\delta \times id_A) \circ R = (id_{\mathbb{I}} \times R) \circ R \tag{4.3.2}$$

$$id_A = \lambda_A \circ (\gamma \times id_A) \circ R \tag{4.3.3}$$

for a classical structure $(\mathbb{I}, \delta, \gamma)$ in **FRel**. As shown in the previous section there are two classical structures in **FRel**, namely $Z = (\mathbb{I}, \delta_Z, \gamma_Z)$ and $X = (\mathbb{I}, \delta_X, \gamma_X)$.

4.3.1 The Z classical structure

Fixing the classical structure $(\mathbb{I}, \delta_Z, \gamma_Z)$ then $\mathbb{I}$-projector-valued spectra in **FRel** are the identity relations:

$$R_0 \subseteq A \times (\mathbb{I} \times A) :: i \sim (0, i)$$

$$R_1 \subseteq A \times (\mathbb{I} \times A) :: i \sim (1, i),$$

where $A \subseteq \mathbb{N}$ is a finite set with $|A| = n$. We show that R_0 and R_1 satisfy equations (4.3.1), (4.3.2) and (4.3.3)[1]:

1. $\mathbb{I}$-self-adjointness:

$$\begin{array}{ccc} i & \xmapsto{R_0} & (0,i) \\ \lambda_A^\dagger \big\downarrow & & \big\uparrow id_{\mathbb{I}} \times R_0^\dagger \\ (*,i) & \xmapsto[n_{\mathbb{I}} \times id_A]{} & \{(0,0,i),(1,1,i)\} \end{array} \qquad \begin{array}{ccc} i & \xmapsto{R_1} & (1,i) \\ \lambda_A^\dagger \big\downarrow & & \big\uparrow id_{\mathbb{I}} \times R_1^\dagger \\ (*,i) & \xmapsto[n_{\mathbb{I}} \times id_A]{} & \{(0,0,i),(1,1,i)\} \end{array}$$

2. $\mathbb{I}$-idempotence:

$$\begin{array}{ccc} i & \xmapsto{R_0} & (0,i) \\ R_0 \big\downarrow & & \big\downarrow id_{\mathbb{I}} \times R_0 \\ (0,i) & \xmapsto[\delta_Z \times id_A]{} & (0,0,i) \end{array} \qquad \begin{array}{ccc} i & \xmapsto{R_1} & (1,i) \\ R_1 \big\downarrow & & \big\downarrow id_{\mathbb{I}} \times R_1 \\ (1,i) & \xmapsto[\delta_Z \times id_A]{} & (1,1,i) \end{array}$$

3. $\mathbb{I}$-completeness:

$$\begin{array}{ccc} i & \xmapsto{id_A} & i \\ R_0 \big\downarrow & & \big\uparrow \lambda_A \\ (0,i) & \xmapsto[\gamma_Z \times id_A]{} & (*,i) \end{array} \qquad \begin{array}{ccc} i & \xmapsto{id_A} & i \\ R_1 \big\downarrow & & \big\uparrow \lambda_A \\ (1,i) & \xmapsto[\gamma_Z \times id_A]{} & (*,i) \end{array}$$

Therefore, if $A := \mathbb{I}$ then R_0 and R_1 become:

$$R_0 \subseteq \mathbb{I} \times (\mathbb{I} \times \mathbb{I}) :: \begin{cases} 0 \sim (0,0) \\ 1 \sim (0,1) \end{cases} \qquad R_1 \subseteq \mathbb{I} \times (\mathbb{I} \times \mathbb{I}) :: \begin{cases} 0 \sim (1,0) \\ 1 \sim (1,1) \end{cases}$$

---

[1]Here due to associativity we have $((a,b),c) \cong (a,b,c) \cong (a,(b,c))$ for $a,b,c \in \mathbb{N}$.

Also, $\delta_Z \subseteq \mathbb{I} \times (\mathbb{I} \times \mathbb{I})$ is trivially a $\mathbb{I}$-projector-valued spectrum due to Proposition 3.7 and Definition 3.2. However, is not the only one. If we consider:

$$\delta_Z^{'} \subseteq \mathbb{I} \times (\mathbb{I} \times \mathbb{I}) :: \begin{cases} 0 \sim (1,0) \\ 1 \sim (0,1) \end{cases}$$

then the equations (4.3.1), (4.3.2) and (4.3.3) are satisfied. Respectively we have:

1. $\mathbb{I}$-self-adjointness:

$$\begin{array}{ccc} 0 & \xmapsto{\delta_Z^{'}} & (1,0) \\ {\scriptstyle \lambda_{\mathbb{I}}^{\dagger}}\downarrow & & \uparrow{\scriptstyle id_{\mathbb{I}} \times (\delta_Z^{'})^{\dagger}} \\ (*,0) & \xmapsto[n_{\mathbb{I}} \times id_{\mathbb{I}}]{} & \{(0,0,0),(1,1,0)\} \end{array} \qquad \begin{array}{ccc} 1 & \xmapsto{\delta_Z^{'}} & (0,1) \\ {\scriptstyle \lambda_{\mathbb{I}}^{\dagger}}\downarrow & & \uparrow{\scriptstyle id_{\mathbb{I}} \times (\delta_Z^{'})^{\dagger}} \\ (*,1) & \xmapsto[n_{\mathbb{I}} \times id_{\mathbb{I}}]{} & \{(0,0,1),(1,1,1)\} \end{array}$$

2. $\mathbb{I}$-idempotence:

$$\begin{array}{ccc} 0 & \xmapsto{\delta_Z^{'}} & (1,0) \\ {\scriptstyle \delta_Z^{'}}\downarrow & & \downarrow{\scriptstyle id_{\mathbb{I}} \times \delta_Z^{'}} \\ (1,0) & \xmapsto[\delta_Z \times id_{\mathbb{I}}]{} & (1,1,0) \end{array} \qquad \begin{array}{ccc} 1 & \xmapsto{\delta_Z^{'}} & (0,1) \\ {\scriptstyle \delta_Z^{'}}\downarrow & & \downarrow{\scriptstyle id_{\mathbb{I}} \times \delta_Z^{'}} \\ (0,1) & \xmapsto[\delta_Z \times id_{\mathbb{I}}]{} & (0,0,1) \end{array}$$

3. $\mathbb{I}$-completeness:

$$\begin{array}{ccc} 0 & \xmapsto{id_{\mathbb{I}}} & 0 \\ {\scriptstyle \delta_Z^{'}}\downarrow & & \uparrow{\scriptstyle \lambda_{\mathbb{I}}} \\ (1,0) & \xmapsto[\gamma_Z \times id_{\mathbb{I}}]{} & (*,0) \end{array} \qquad \begin{array}{ccc} 1 & \xmapsto{id_{\mathbb{I}}} & 1 \\ {\scriptstyle \delta_Z^{'}}\downarrow & & \uparrow{\scriptstyle \lambda_{\mathbb{I}}} \\ (0,1) & \xmapsto[\gamma_Z \times id_{\mathbb{I}}]{} & (*,1) \end{array}$$

Summarizing all the above, the relations $R_0, R_1, \delta_Z, \delta_Z^{'}$ of type $\mathbb{I} \to \mathbb{I} \times \mathbb{I}$ are $\mathbb{I}$-projector-valued spectra for the classical structure $(\mathbb{I}, \delta_Z, \gamma_Z)$. We prove that these are the only relations of type $\mathbb{II} \to \mathbb{II} \times \mathbb{II}$.

First of all, the relation

$$\delta_X \subseteq \mathbb{II} \times (\mathbb{II} \times \mathbb{II}) :: \begin{cases} 0 \sim \{(0,0),(1,1)\} \\ 1 \sim \{(0,1),(1,0)\} \end{cases}$$

does not satisfy the equation (4.3.2) since

$$(\delta_Z \times id_{\mathbb{II}}) \circ \delta_X :: 1 \mapsto \{(0,1),(1,0)\} \mapsto \{(0,0,1),(1,1,0)\}$$

and

$$(id_{\mathbb{II}} \times \delta_X) \circ \delta_X :: 1 \mapsto \{(0,1),(1,0)\} \mapsto \{(0,1,0),(0,0,1),(1,0,0),(1,1,1)\}.$$

We also observe that all five relations arising from δ_X by permutations fail to satisfy the equation (4.3.2). Furthermore, if we consider the relation

$$R \subseteq \mathbb{II} \times (\mathbb{II} \times \mathbb{II}) :: \begin{cases} 0 & \sim & (0,0) \\ 1 & \sim & \{(0,1),(1,0)\} \end{cases}$$

the equation (4.3.2) is not satisfied because

$$(\delta_Z \times id_{\mathbb{II}}) \circ R :: 1 \mapsto \{(0,1),(1,0)\} \mapsto \{(0,0,1),(1,1,0)\}$$

and

$$(id_{\mathbb{II}} \times R) \circ R :: 1 \mapsto \{(0,1),(1,0)\} \mapsto \{(0,0,1),(0,1,0),(1,0,0)\}.$$

Finally it can be shown that all permutations of R do not satisfy equation (4.3.2). Also the permutations of $R_0, R_1, \delta_Z, \delta_Z'$ fail to satisfy the required equations as well.

In the case that $A := \mathbb{III} = \{0,1,2\}$, there are eight relations $R_k \subseteq \mathbb{III} \times (\mathbb{II} \times \mathbb{III})$, $k \in \{1,2,\ldots,8\}$, namely:

$$R_1 :: \begin{cases} 0 \sim (0,0) \\ 1 \sim (0,1) \\ 2 \sim (0,2) \end{cases} \qquad R_2 :: \begin{cases} 0 \sim (0,0) \\ 1 \sim (0,1) \\ 2 \sim (1,2) \end{cases} \qquad R_3 :: \begin{cases} 0 \sim (0,0) \\ 1 \sim (1,1) \\ 2 \sim (0,2) \end{cases}$$

$$R_4 :: \begin{cases} 0 \sim (0,0) \\ 1 \sim (1,1) \\ 2 \sim (1,2) \end{cases} \qquad R_5 :: \begin{cases} 0 \sim (1,0) \\ 1 \sim (0,1) \\ 2 \sim (0,2) \end{cases} \qquad R_6 :: \begin{cases} 0 \sim (1,0) \\ 1 \sim (0,1) \\ 2 \sim (1,2) \end{cases}$$

$$R_7 :: \begin{cases} 0 \sim (1,0) \\ 1 \sim (1,1) \\ 2 \sim (0,2) \end{cases} \qquad R_8 :: \begin{cases} 0 \sim (1,0) \\ 1 \sim (1,1) \\ 2 \sim (1,2) \end{cases}$$

that satisfy equations (4.3.1), (4.3.2) and (4.3.3) and therefore there are $\mathbb{I}$-projector valued spectra relative to $(\mathbb{I}, \delta_Z, \gamma_Z)$. Note that further permutations of R_k can not be projector-valued spectra.

We can generalize this result, stating that the relations

$$R_k \subseteq A \times (\mathbb{I} \times A) :: i \sim (j, i),$$

where $|A| = n$, $i \in \{0, 1, \ldots, n-1\}$, $j \in \{0, 1\}$ and $k \in \{1, \ldots, 2^n\}$ are projector-valued spectra relative to the classical structure $(\mathbb{I}, \delta_Z, \gamma_Z)$.

4.3.2 The X classical structure

Fixing now the classical structure $(\mathbb{I}, \delta_X, \gamma_X)$ we observe that in opposition to the classical structure $(\mathbb{I}, \delta_Z, \gamma_Z)$ the relations

$$R_k \subseteq A \times (\mathbb{I} \times A) :: i \sim (j, i),$$

as stated previously are not projector-valued spectra relative to $(\mathbb{I}, \delta_X, \gamma_X)$. For example if $A := \mathbb{I}$ then for the relation

$$R_7 \subseteq \mathbb{I} \times (\mathbb{I} \times \mathbb{I}) :: \begin{cases} 0 \sim (1,0) \\ 1 \sim (1,1) \end{cases}$$

we have that

$$(\delta_X \times id_{\mathbb{I}}) \circ R_7 :: 0 \mapsto (1,0) \mapsto \{(0,1,0), (1,0,0)\}$$

but

$$(id_{\mathbb{I}} \times R_7) \circ R_7 :: 0 \mapsto (1,0) \mapsto (1,1,0).$$

We restrict our study to $A := \mathbb{I}$. Obviously the relation

$$\delta_X \subseteq \mathbb{I} \times (\mathbb{I} \times \mathbb{I}) :: \begin{cases} 0 \sim \{(0,0), (1,1)\} \\ 1 \sim \{(1,0), (0,1)\} \end{cases}$$

is a $\mathbb{I}$-projector-valued spectrum for $(\mathbb{I}, \delta_X, \gamma_X)$. Checking all the permutations of δ_X only one satisfies the equations (4.3.1), (4.3.2) and (4.3.3). Analytically the relations arising by permutations of δ_X are:

$$\delta_X^{(1)} :: \begin{cases} 0 \sim \{(0,0),(0,1)\} \\ 1 \sim \{(1,0),(1,1)\} \end{cases} \qquad \delta_X^{(2)} :: \begin{cases} 0 \sim \{(0,0),(1,0)\} \\ 1 \sim \{(0,1),(1,1)\} \end{cases} \qquad \delta_X^{(3)} :: \begin{cases} 0 \sim \{(0,1),(1,0)\} \\ 1 \sim \{(0,0),(1,1)\} \end{cases}$$

$$\delta_X^{(4)} :: \begin{cases} 0 \sim \{(0,1),(1,1)\} \\ 1 \sim \{(0,0),(1,0)\} \end{cases} \quad \delta_X^{(5)} :: \begin{cases} 0 \sim \{(1,0),(1,1)\} \\ 1 \sim \{(0,0),(0,1)\} \end{cases}$$

All the above relations except $\delta_X^{(2)}$ do not satisfy equation (4.3.3), therefore δ_X and $\delta_X^{(2)}$ are the only projector-valued spectra of type $\mathbb{I} \to \mathbb{I} \times \mathbb{I}$ relative to $(\mathbb{I}, \delta_X, \gamma_X)$. We demostrate that $\delta_X^{(2)}$ is a $\mathbb{I}$-projector-valued spectrum:

1. $\mathbb{I}$-self-adjointness:

$$\begin{array}{ccc} 0 & \xrightarrow{\delta_X^{(2)}} & \{(0,0),(1,0)\} \\ \lambda_{\mathbb{I}}^{\dagger} \downarrow & & \uparrow id_{\mathbb{I}} \times (\delta_X^{(2)})^{\dagger} \\ (*,0) & \xrightarrow[n_{\mathbb{I}} \times id_{\mathbb{I}}]{} & \{(0,0,0),(1,1,0)\} \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{\delta_X^{(2)}} & \{(0,1),(1,1)\} \\ \lambda_{\mathbb{I}}^{\dagger} \downarrow & & \uparrow id_{\mathbb{I}} \times (\delta_X^{(2)})^{\dagger} \\ (*,1) & \xrightarrow[n_{\mathbb{I}} \times id_{\mathbb{I}}]{} & \{(0,0,1),(1,1,1)\} \end{array}$$

2. $\mathbb{I}$-idempotence:

$$\begin{array}{ccc} 0 & \xrightarrow{\delta_X^{(2)}} & \{(0,0),(1,0)\} \\ \delta_X^{(2)} \downarrow & & \downarrow id_{\mathbb{I}} \times \delta_X^{(2)} \\ \{(0,0),(1,0)\} & \xrightarrow[\delta_X \times id_{\mathbb{I}}]{} & \{(0,0,0),(1,1,0),(0,1,0),(1,0,0)\} \end{array}$$

$$\begin{array}{ccc} 1 & \xrightarrow{\delta_X^{(2)}} & \{(0,1),(1,1)\} \\ \delta_X^{(2)} \downarrow & & \downarrow id_{\mathbb{I}} \times \delta_X^{(2)} \\ \{(0,1),(1,1)\} & \xrightarrow[\delta_X \times id_{\mathbb{I}}]{} & \{(0,0,1),(1,1,1),(0,1,1),(1,0,1)\} \end{array}$$

3. $\mathbb{II}$-completeness:

$$\begin{array}{ccc} 0 & \xmapsto{id_{\mathbb{II}}} & 0 \\ \delta_X^{(2)} \downarrow & & \uparrow \lambda_{\mathbb{II}} \\ \{(0,0),(1,0)\} & \xmapsto[\gamma_X \times id_{\mathbb{II}}]{} & (*,0) \end{array} \qquad \begin{array}{ccc} 1 & \xmapsto{id_{\mathbb{II}}} & 1 \\ \delta_X^{(2)} \downarrow & & \uparrow \lambda_{\mathbb{II}} \\ \{(0,1),(1,1)\} & \xmapsto[\gamma_X \times id_{\mathbb{II}}]{} & (*,1) \end{array}$$

4.4 The discrete model Spek

The category **Spek** (the name is derived by the Spekken's toy model of categorical quantum mechanics) is a sub-category of **FRel**, where we consider the four elements set $\mathbb{IV} := \{1, 2, 3, 4\}$.

Definition 4.6. *[7] The category **Spek** consists of objects of the form* $\mathbb{IV} \times \mathbb{IV} \times \cdots \times \mathbb{IV}$ *and the identity object is* $\mathrm{I} := \{*\}$. *For convenience we assume strictness of associativity and left and right unit isomorphisms. The morphisms in **Spek** are generated by relational composition, cartesian product of relations and relational converse from:*

- *permutations* $\{\sigma_i \subseteq \mathbb{IV} \times \mathbb{IV}\}_i$,
- *a copying relation* $\delta_Z \subseteq \mathbb{IV} \times (\mathbb{IV} \times \mathbb{IV})$ *defined by*

$$\delta_Z :: \begin{cases} 1 \sim \{(1,1),(2,2)\} \\ 2 \sim \{(1,2),(2,1)\} \\ 3 \sim \{(3,3),(4,4)\} \\ 4 \sim \{(3,4),(4,3)\} \end{cases}$$

- *and a deleting relation*

$$\gamma_Z \subseteq \mathbb{IV} \times \mathrm{I} :: \{1, 3\} \sim *.$$

We observe that the states of type $\mathrm{I} \to \mathrm{IV}$ in **Spek** correspond to permutations of $\gamma_Z^\dagger$, these are:

$$\begin{array}{lll} z_0 :: * \sim \{1,2\} & x_0 :: * \sim \{1,3\} & y_0 ::\sim \{1,4\} \\ z_1 :: * \sim \{3,4\} & x_1 :: * \sim \{2,4\} & y_1 ::\sim \{2,3\}. \end{array}$$

Here we set z_0, z_1 for the states that are copied by δ_Z. Therefore, z_0, z_1 are classical for the classical structure $(\mathrm{IV}, \delta_Z, x_0^\dagger)$, where $x_0^\dagger := \gamma_Z$, and hence x_0 is unbiased for $(\mathrm{IV}, \delta_Z, x_0^\dagger)$ as required. However, there are other three distinct states, namely x_1, y_0, y_1 that are also unbiased for the classical structure $(\mathrm{IV}, \delta_Z, x_0^\dagger)$.

As stated in [7] we can obtain further copying relations for each state x_1, y_0, y_1 by applying various permutations to δ_Z. Therefore, by setting

$$\begin{aligned} \delta_Z^{'} &:= (\sigma_{(12)(34)} \times \sigma_{(12)(34)}) \circ \delta_Z \circ \sigma_{(12)(34)} \\ \delta_Z^{''} &:= (\sigma_{(34)} \times \sigma_{(34)}) \circ \delta_Z \circ \sigma_{(34)} \\ \delta_Z^{'''} &:= (\sigma_{(12)} \times \sigma_{(12)}) \circ \delta_Z \circ \sigma_{(12)}, \end{aligned}$$

we get classical structures $(\mathrm{IV}, \delta_Z^{'}, x_1^\dagger)$, $(\mathrm{IV}, \delta_Z^{''}, y_0^\dagger)$ and $(\mathrm{IV}, \delta_Z^{'''}, y_1^\dagger)$. Since all of these structures share the same classical states as $(\mathrm{IV}, \delta_Z, x_0^\dagger)$ - these are z_0 and z_1 - then we can refer to this family of four structures as an observable [7]. Hence, we have the observable

$$Z := \{(\mathrm{IV}, \delta_Z, x_0^\dagger), (\mathrm{IV}, \delta_Z^{'}, x_1^\dagger), (\mathrm{IV}, \delta_Z^{''}, y_0^\dagger), (\mathrm{IV}, \delta_Z^{'''}, y_1^\dagger)\},$$

where

$$\delta_Z^{'} :: \begin{cases} 1 \sim \{(1,2),(2,1)\} \\ 2 \sim \{(1,1),(2,2)\} \\ 3 \sim \{(3,4),(4,3)\} \\ 4 \sim \{(3,3),(4,4)\} \end{cases} \qquad x_1^\dagger :: \{2,4\} \sim *$$

$$\delta_Z^{''} :: \begin{cases} 1 \sim \{(1,1),(2,2)\} \\ 2 \sim \{(1,2),(2,1)\} \\ 3 \sim \{(3,4),(4,3)\} \\ 4 \sim \{(3,3),(4,4)\} \end{cases} \qquad y_0^\dagger :: \{1,4\} \sim *$$

$$\delta_Z''' :: \begin{cases} 1 \sim \{(1,2),(2,1)\} \\ 2 \sim \{(1,1),(2,2)\} \\ 3 \sim \{(3,3),(4,4)\} \\ 4 \sim \{(3,4),(4,3)\} \end{cases} \qquad y_1^\dagger :: \{2,3\} \sim *$$

Clearly it can be verified that z_0, z_1 are classical for the observable Z, hence $\delta_Z, \delta_Z', \delta_Z'', \delta_Z'''$ copy z_0, z_1 and x_0, x_1, y_0, y_1 are unbiased for the observable Z.

Furthermore, it is shown in [7] that new observables can be found by applying permutations to the copying operations of the Z observable. Therefore, setting

$$\delta_X := (\sigma_{(23)} \times \sigma_{(23)}) \circ \delta_Z \circ \sigma_{(23)}$$

we obtain

$$\delta_X :: \begin{cases} 1 \sim \{(1,1),(3,3)\} \\ 2 \sim \{(2,2),(4,4)\} \\ 3 \sim \{(1,3),(3,1)\} \\ 4 \sim \{(2,4),(4,2)\} \end{cases}$$

Note that now x_0, x_1 can be copied by δ_X hence we can form the observable X:

$$X := \{(\mathrm{IV}, \delta_X, z_0^\dagger), (\mathrm{IV}, \delta_X', z_1^\dagger), (\mathrm{IV}, \delta_X'', y_0^\dagger), (\mathrm{IV}, \delta_X''', y_1^\dagger)\},$$

for which x_0, x_1 are classical and z_0, z_1, y_0, y_1 are unbiased. Similarly by setting

$$\delta_Y := (\sigma_{(24)} \times \sigma_{(24)}) \circ \delta_Z \circ \sigma_{(24)}$$

we obtain

$$\delta_Y :: \begin{cases} 1 \sim \{(1,1),(4,4)\} \\ 2 \sim \{(3,2),(2,3)\} \\ 3 \sim \{(2,2),(3,3)\} \\ 4 \sim \{(1,4),(4,1)\} \end{cases}$$

and hence, we can form the observable Y:

$$Y := \{(\mathrm{IV}, \delta_Y, x_0^\dagger), (\mathrm{IV}, \delta_Y', x_1^\dagger), (\mathrm{IV}, \delta_Y'', z_0^\dagger), (\mathrm{IV}, \delta_Y''', z_1^\dagger)\},$$

for which y_0, y_1 are classical and x_0, x_1, z_0, z_1 are unbiased. Further permutations of the copying relations yield no more observables.

Definition 4.7. *[7] Two observables A and B are called complementary if there exist classical structures $(X, \delta_A, \gamma_A) \in A$ and $(X, \delta_B, \gamma_B) \in B$ which are complementary.*

Based on the previous definition we have the following theorem:

Theorem 4.8. *The observables X, Z, Y are mutually complementary.*

Proof. As seen before classical structures of observable Z are complementary with the ones of observables X, Y since

- z_0, z_1 are classical for the observable Z and unbiased for observables X, Y,
- x_0, x_1 and y_0, y_1 are classical for the observables X, Y, respectively, and unbiased for observable Z.

Similarly observable X is complementary with observables Z, Y and observable Y is complementary with observables Z, X. □

Also we can form "bell-states":

$$n_{\mathrm{IV}} := \delta_Z \circ x_0^{\dagger} \subseteq \mathrm{I} \times (\mathrm{IV} \times \mathrm{IV}) :: * \sim \{(1,1), (2,2), (3,3), (4,4)\}$$

and the requirements for compact closure are inherited form **FRel**, therefore **Spek** is a †-compact closed category [7].

4.5 Quantum spectra in Spek

In **Spek** a IV-projector-valued spectrum corresponds to relations $R \subseteq A \times (\mathrm{IV} \times A)$, where A is a finite set and R satisfies equations (4.3.1), (4.3.2) and (4.3.3). We restrict set A, such that $A := \mathrm{IV}$.

It is clear that for every classical structure in each of the three observables Z, X, Y the corresponding copying relation is a $\mathbb{IV}$-projector-valued spectrum. However, it can be shown that for the classical structure $(\mathbb{IV}, \delta_Z, x_0^\dagger)$ the copying operations $\delta'_Z, \delta''_Z, \delta'''_Z$ fail to satisfy equation (4.3.3) and hence there are not $\mathbb{IV}$-projector-valued spectra with respect to $(\mathbb{IV}, \delta_Z, x_0^\dagger)$. For example we have for δ'''_Z:

$$\lambda_{\mathbb{IV}} \circ (x_0^\dagger \times id_{\mathbb{IV}}) \circ \delta'''_Z :: \{2,3\} \mapsto \{(2,2),(1,1),(3,3),(4,4)\} \mapsto \{(*,1),(*,3)\} \mapsto \{1,3\}$$

so

$$\lambda_{\mathbb{IV}} \circ (x_0^\dagger \times id_{\mathbb{IV}}) \circ \delta'''_Z \neq id_{\mathbb{IV}}.$$

Similarly all copying relations of observables X and Y are not $\mathbb{IV}$-projector-valued spectra for $(\mathbb{IV}, \delta_Z, x_0^\dagger)$. Likewise, this stands for every classical structure and for all three observables, hence for each classical structure only its copying relation is a $\mathbb{IV}$-projector-valued spectrum.

However, we can apply different permutations on δ_Z yielding other relations $R \subseteq \mathbb{IV} \times (\mathbb{IV} \times \mathbb{IV})$. For example consider the relation:

$$\delta := (id_{\mathbb{IV}} \times \sigma_{(23)}^{-1}) \circ \delta_Z \circ \sigma_{(23)} :: \begin{cases} 1 \sim \{(1,1),(2,3)\} \\ 2 \sim \{(3,2),(4,4)\} \\ 3 \sim \{(1,3),(2,1)\} \\ 4 \sim \{(3,4),(4,2)\} \end{cases}$$

We show that δ satisfies eqautions (4.3.1), (4.3.2) and (4.3.3):

1. $\mathbb{IV}$-self-adjointness:

$$\begin{array}{ccc} 1 & \xmapsto{\quad\delta\quad} & \{(1,1),(2,3)\} \\ \Big\downarrow \lambda_{\mathbb{IV}}^\dagger & & \Big\uparrow id_{\mathbb{IV}} \times \delta^\dagger \\ (*,1) & \xmapsto[n_{\mathbb{IV}} \times id_{\mathbb{IV}}]{} & \{(1,1,1),(2,2,1),(3,3,1)(4,4,1)\} \end{array}$$

$$\begin{array}{ccc} 2 & \xmapsto{\quad\delta\quad} & \{(3,2),(4,4)\} \\ \Big\downarrow {\scriptstyle \lambda^{\dagger}_{\mathbb{IV}}} & & \Big\uparrow {\scriptstyle id_{\mathbb{IV}} \times \delta^{\dagger}} \\ (*,2) & \xmapsto[n_{\mathbb{IV}} \times id_{\mathbb{IV}}]{} & \{(1,1,2),(2,2,2),(3,3,2)(4,4,2)\} \end{array}$$

$$\begin{array}{ccc} 3 & \xmapsto{\quad\delta\quad} & \{(1,3),(2,1)\} \\ \Big\downarrow {\scriptstyle \lambda^{\dagger}_{\mathbb{IV}}} & & \Big\uparrow {\scriptstyle id_{\mathbb{IV}} \times \delta^{\dagger}} \\ (*,3) & \xmapsto[n_{\mathbb{IV}} \times id_{\mathbb{IV}}]{} & \{(1,1,3),(2,2,3),(3,3,3)(4,4,3)\} \end{array}$$

$$\begin{array}{ccc} 4 & \xmapsto{\quad\delta\quad} & \{(3,4),(4,2)\} \\ \Big\downarrow {\scriptstyle \lambda^{\dagger}_{\mathbb{IV}}} & & \Big\uparrow {\scriptstyle id_{\mathbb{IV}} \times \delta^{\dagger}} \\ (*,4) & \xmapsto[n_{\mathbb{IV}} \times id_{\mathbb{IV}}]{} & \{(1,1,4),(2,2,4),(3,3,4)(4,4,4)\} \end{array}$$

2. $\mathbb{IV}$-idempotence:

$$\begin{array}{ccc} 1 & \xmapsto{\quad\delta\quad} & \{(1,1),(2,3)\} \\ \Big\downarrow {\scriptstyle \delta} & & \Big\downarrow {\scriptstyle id_{\mathbb{IV}} \times \delta} \\ \{(1,1),(2,3)\} & \xmapsto[\delta_Z \times id_{\mathbb{IV}}]{} & \{(1,1,1),(1,2,3),(2,1,3),(2,2,1)\} \end{array}$$

$$\begin{array}{ccc} 2 & \xmapsto{\quad\delta\quad} & \{(3,2),(4,4)\} \\ \Big\downarrow {\scriptstyle \delta} & & \Big\downarrow {\scriptstyle id_{\mathbb{IV}} \times \delta} \\ \{(3,2),(4,4)\} & \xmapsto[\delta_Z \times id_{\mathbb{IV}}]{} & \{(3,3,2),(4,4,2),(3,4,4),(4,3,4)\} \end{array}$$

$$\begin{array}{ccc}
3 & \xmapsto{\quad\delta\quad} & \{(1,3),(2,1)\} \\
\Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle id_{\mathbb{IV}}\times\delta} \\
\{(1,3),(2,1)\} & \xmapsto[\delta_Z\times id_{\mathbb{IV}}]{} & \{(1,1,3),(2,2,3),(1,2,1),(2,1,1)\}
\end{array}$$

$$\begin{array}{ccc}
4 & \xmapsto{\quad\delta\quad} & \{(3,4),(4,2)\} \\
\Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle id_{\mathbb{IV}}\times\delta} \\
\{(3,4),(4,2)\} & \xmapsto[\delta_Z\times id_{\mathbb{IV}}]{} & \{(3,3,4),(4,4,4),(3,4,2),(4,3,2)\}
\end{array}$$

3. $\mathbb{IV}$-completeness:

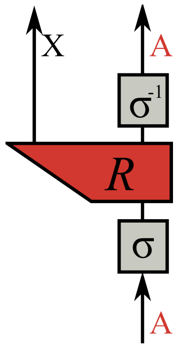
$$\begin{array}{ccc}
1 & \xmapsto{\quad id_{\mathbb{IV}}\quad} & 1 \\
\Big\downarrow{\scriptstyle\delta} & & \Big\uparrow{\scriptstyle\lambda_{\mathbb{IV}}} \\
\{(1,1),(2,3)\} & \xmapsto[x_0^\dagger\times id_{\mathbb{IV}}]{} & (*,1)
\end{array}
\qquad
\begin{array}{ccc}
2 & \xmapsto{\quad id_{\mathbb{IV}}\quad} & 2 \\
\Big\downarrow{\scriptstyle\delta} & & \Big\uparrow{\scriptstyle\lambda_{\mathbb{IV}}} \\
\{(3,2),(4,4)\} & \xmapsto[x_0^\dagger\times id_{\mathbb{IV}}]{} & (*,2)
\end{array}$$

$$\begin{array}{ccc}
3 & \xmapsto{\quad id_{\mathbb{IV}}\quad} & 3 \\
\Big\downarrow{\scriptstyle\delta} & & \Big\uparrow{\scriptstyle\lambda_{\mathbb{IV}}} \\
\{(1,3),(2,1)\} & \xmapsto[x_0^\dagger\times id_{\mathbb{IV}}]{} & (*,3)
\end{array}
\qquad
\begin{array}{ccc}
4 & \xmapsto{\quad id_{\mathbb{IV}}\quad} & 4 \\
\Big\downarrow{\scriptstyle\delta} & & \Big\uparrow{\scriptstyle\lambda_{\mathbb{IV}}} \\
\{(3,4),(4,2)\} & \xmapsto[x_0^\dagger\times id_{\mathbb{IV}}]{} & (*,4)
\end{array}$$

Generally, we observe that if $R \subseteq A \times (X \times A)$ is an X-projector valued spectrum, then by applying permutations $\{\sigma_i \subseteq A \times A\}_i$ on the quantum object A we get

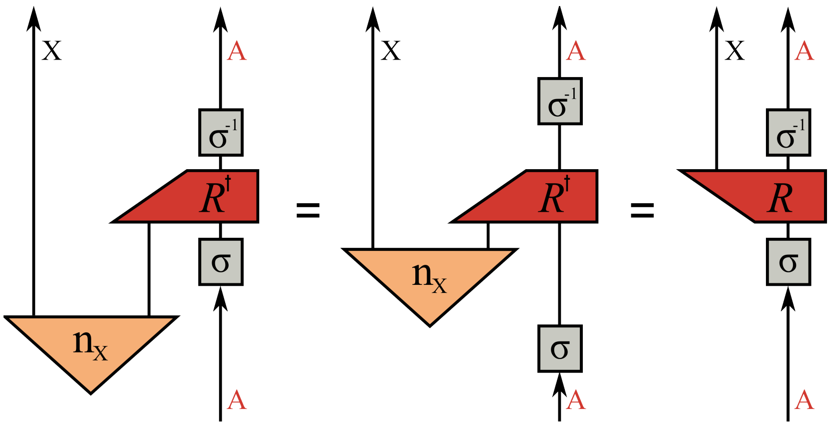
$$R' := (id_X \times \sigma_i^{-1}) \circ R \circ \sigma_i.$$

In a picture R' is:

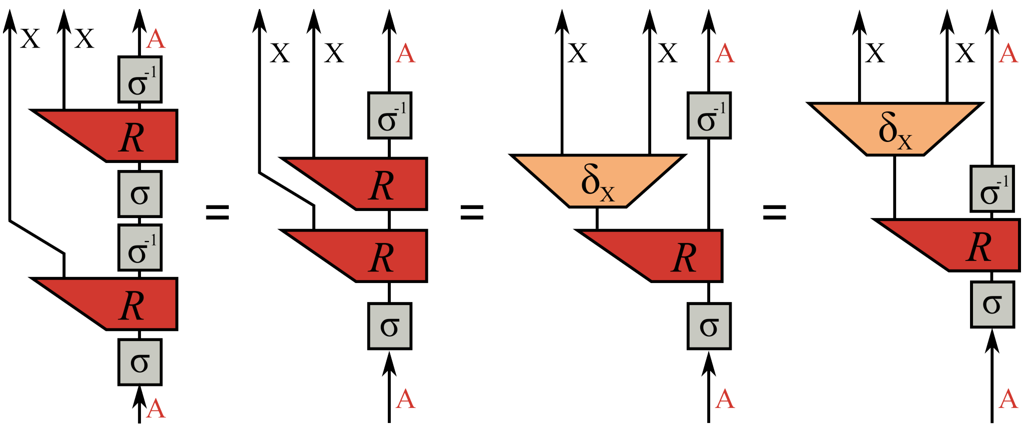


It can be proved that R' is also an X-projector valued spectrum. We prove this statement graphically: (note that X-self-adjointness is established since σ_i is unitary)

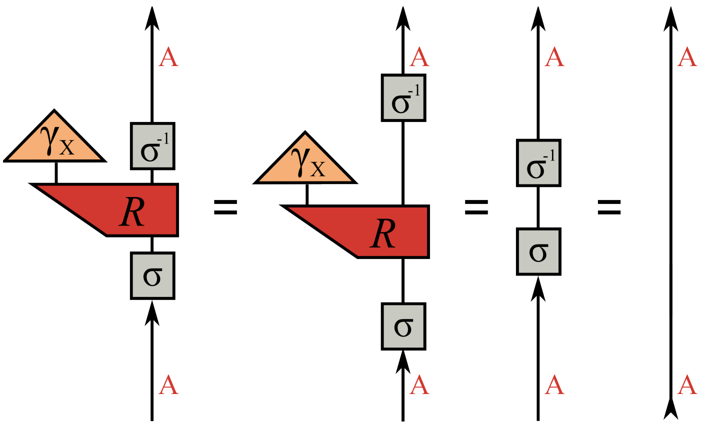
1. X-self-adjointness:



2. X-idempotence:



3. X-completeness:



Therefore, for relations $R \subseteq \mathbb{IV} \times (\mathbb{IV} \times \mathbb{IV})$ and classical structure $(\mathbb{IV}, \delta_Z, x_0^\dagger)$ we have that

$$\delta_i := (id_{\mathbb{IV}} \times \sigma_i^{-1}) \circ \delta_Z \circ \sigma_i,$$

are $\mathbb{IV}$-projector valued spectra for $(\mathbb{IV}, \delta_Z, x_0^\dagger)$, where $\{\sigma_i \subseteq \mathbb{IV} \times \mathbb{IV}\}_i$ are permutations over the set $\mathbb{IV}$. So we have 24 different $\mathbb{IV}$-projector valued spectra relative to $(\mathbb{IV}, \delta_Z, x_0^\dagger)$.

Generalizing the above, for each classical structure in the three observables Z, X, Y, we have 24 different $\mathbb{IV}$-projector valued spectra in each case.

Chapter 5

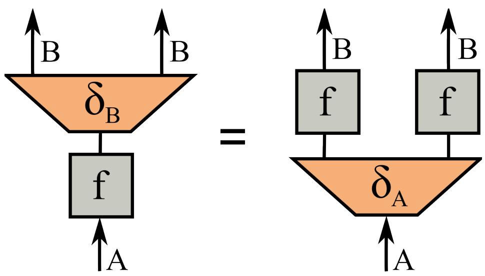
State transfer protocol

With the framework of †-symmetric monoidal categories with classical structures we are now able to expose and explain certain quantum protocols. In this dissertation we confine our interest to the state transfer protocol [6] and we study its application on the discrete models **FRel** and **Spek**. We begin by introducing some concepts necessary for the description of the protocol.

Definition 5.1. *[6] Given two classical structures* (A, δ_A, γ_A), (B, δ_B, γ_B) *then a morphism* $f : A \to B$ *is a partial map if*

$$\delta_B \circ f = (f \otimes f) \circ \delta_A,$$

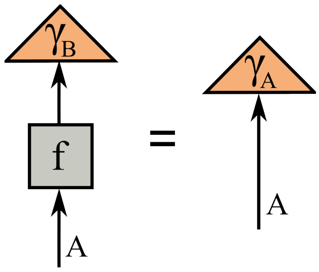
i.e. in a picture:



Furthermore morphism $f : A \to B$ *is a total map if also*

$$\gamma_B \circ f = \gamma_A,$$

which is depicted as:

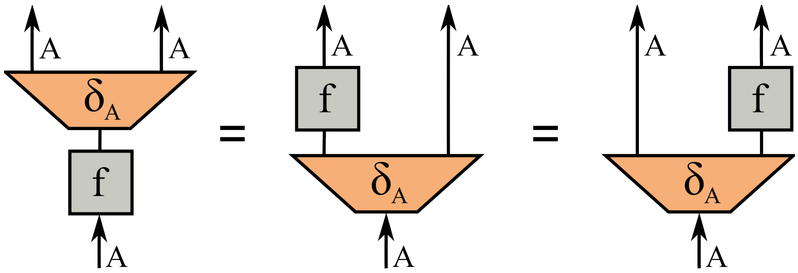


Finally, $f : A \to A$ is a permutation, if in addition to the previous, f is unitary.

Definition 5.2. *[6] Given a classical structure (A, δ_A, γ_A), then a unitary morphism $f : A \to A$ is a phase map if*

$$\delta_A \circ f = (f \otimes id_A) \circ \delta_A = (id_A \otimes f) \circ \delta_A,$$

and it is depicted as:



Firstly, we describe the state transfer in **FdHilb**. Unlike the quantum teleportation protocol the quantum state transfer protocol involves only two qubits. At the beginning the first qubit is in an unknown state $|\psi\rangle$ and the other is in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Then the two qubits are measured according to the a partial measurement described by the projectors

$$\Pi_i = (id_{\mathcal{Q}} \otimes f_i) \circ \delta \circ \delta^\dagger \circ (id_{\mathcal{Q}} \otimes f_i^\dagger) \qquad i \in \{0, 1\},$$

where morphisms f_i, $i \in \{0, 1\}$ are permutations. Hence, we have a degenerate measurement. After a 1-qubit measurement is applied on the first qubit described by the projector

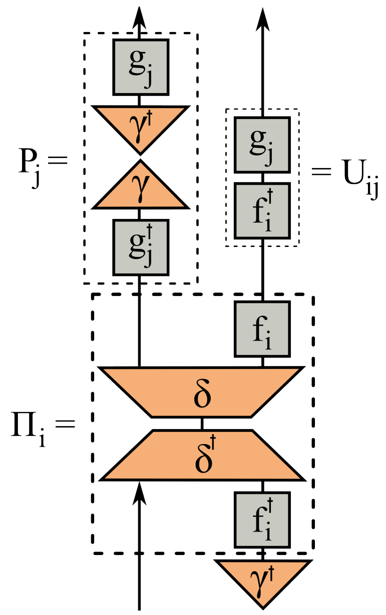
$$P_j = g_j \circ \gamma^\dagger \circ \gamma \circ g_j^\dagger \qquad j \in \{0, 1\},$$

where morphisms g_j, $j \in \{0, 1\}$ are phase maps. At the end we apply a correction

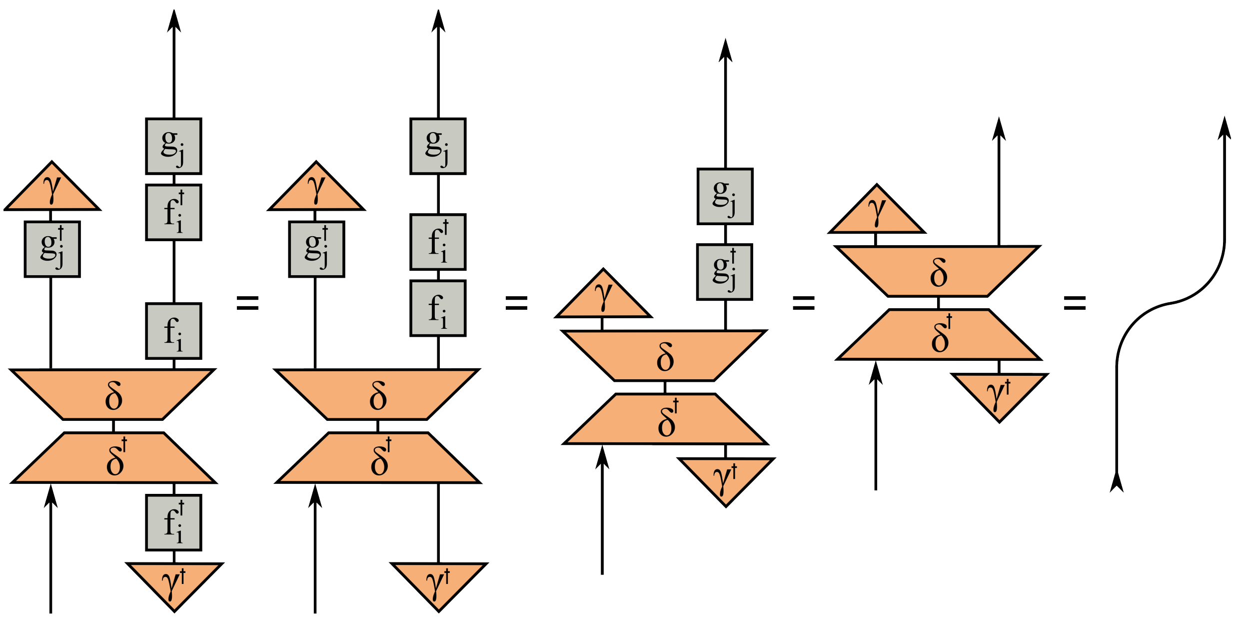
$$U_{ij} = g_j \circ f_i^\dagger \qquad i, j \in \{0, 1\},$$

on the second qubit, resulting the initial state $|\psi\rangle$ as an outcome.

In our graphical language of †-symmetric monoidal categories with classical structures the state transfer protocol can be depicted as:



Based an the properties of classical structures, permutations and phase maps, we provide a diagrammatic proof of state transfer [7]:



Remark: Note that in **FdHilb** the above equations describe the state transfer with $(\mathcal{Q}, \delta, \gamma)$ being the classical structure, in the standard computational basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{Q} := \mathbb{C} \oplus \mathbb{C}$, where

$$\delta : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q} :: |i\rangle \mapsto |ii\rangle$$
$$\gamma : \mathcal{Q} \to \mathbb{C} :: |i\rangle \mapsto 1.$$

Also the permutation maps f_i, $i \in \{0, 1\}$ correspond to the matrix operators

$$X_i = \begin{cases} I, & i = 0 \\ X - Pauli, & i = 1 \end{cases}$$

and the phase maps g_j, $j \in \{0, 1\}$ to

$$Z_j = \begin{cases} I, & j = 0 \\ Z - Pauli, & j = 1. \end{cases}$$

5.1 State transfer in FRel

In order to describe the state transfer protocol in **FRel** we need to find permutations f_i, $i \in \{0, 1\}$ and phase maps g_j, $j \in \{0, 1\}$ for each classical structure $(\mathbb{I}, \delta_Z, \gamma_Z)$ and $(\mathbb{I}, \delta_X, \gamma_X)$. Therefore we are looking for relations $\{\sigma_i \subseteq \mathbb{I} \times \mathbb{I}\}_i$ and these are only two, the identity relation $e = (0)(1)$ and $\sigma_1 = (01)$.

Fixing the classical structure $(\mathbb{I}, \delta_Z, \gamma_Z)$ we have that σ_1 is a permutation since

$$\delta_Z \circ \sigma_1 :: \begin{cases} 0 \mapsto 1 \mapsto (1, 1) \\ 1 \mapsto 0 \mapsto (0, 0) \end{cases}$$
$$(\sigma_1 \times \sigma_1) \circ \delta_Z :: \begin{cases} 0 \mapsto (0, 0) \mapsto (1, 1) \\ 1 \mapsto (1, 1) \mapsto (0, 0) \end{cases}$$

and

$$\gamma_Z \circ \sigma_1 = \gamma_Z :: \{0, 1\} \mapsto *.$$

Obviously e is both a permutation and a phase map, but σ_1 is not a phase map, since

$$\delta_Z \circ \sigma_1 = \{(0,(1,1)),(1,(0,0))\} \neq \{(0,(0,1)),(1,(1,0))\} = (id_{\mathbb{II}} \times \sigma_1) \circ \delta_Z.$$

Similarly for the classical structure $(\mathbb{II}, \delta_X, \gamma_X)$, σ_1 is a phase map:

$$\delta_X \circ \sigma_1 = (id_{\mathbb{II}} \times \sigma_1) \circ \delta_X = (\sigma_1 \times id_{\mathbb{II}}) \circ \delta_X = \Big\{ \Big(0, \big\{(0,1),(1,0)\big\}\Big), \Big(1, \big\{(0,0),(1,1)\big\}\Big) \Big\},$$

but σ_1 is not a permutation because

$$\delta_X \circ \sigma_1 :: 0 \mapsto 1 \mapsto \{(0,1),(1,0)\}$$

and

$$(\sigma_1 \times \sigma_1) \circ \delta_X :: 0 \mapsto \{(0,0),(1,1)\} \mapsto \{(1,1),(0,0)\}.$$

To conclude, in both classical structures $(\mathbb{II}, \delta_Z, \gamma_Z)$ and $(\mathbb{II}, \delta_X, \gamma_X)$ in **FRel** we do not have the degenerate measurements required for the state transfer protocol, so this quantum protocol is not stimulated in **FRel**.

5.2 State transfer in Spek

Working in the same way as in section 5.1 we search for relations $\{\sigma_i \subseteq \mathbb{IV} \times \mathbb{IV}\}_i$ for each observable Z, X, Y as candidates for permutations f_i, $i \in \{0,1\}$ and phase maps g_j, $j \in \{0,1\}$.

Having the classical structure $(\mathbb{IV}, \delta_Z, x_0^\dagger)$ we observe that the relations $e = (1)(2)(3)(4)$, $\sigma_1 = (13)(24)$ and $e = (1)(2)(3)(4)$, $\sigma_2 = (12)(34)$ are respectively permutations and phase maps with respect to $(\mathbb{IV}, \delta_Z, x_0^\dagger)$. We demonstrate this explicitly:

$$\delta_Z \circ \sigma_1 = (\sigma_1 \times \sigma_1) \circ \delta_Z = \Big\{ \Big(1, \big\{(3,3),(4,4)\big\}\Big), \Big(2, \big\{(3,4),(4,3)\big\}\Big), \Big(3, \big\{(1,1),(2,2)\big\}\Big), \Big(4, \big\{(1,2),(2,1)\big\}\Big) \Big\}$$

and

$$x_0^\dagger \circ \sigma_1 = x_0^\dagger = \{(\{1,3\},*)\}.$$

For $\sigma_2 = (12)(34)$ we have:

$$\delta_Z \circ \sigma_2 = (id_{\mathrm{IV}} \times \sigma_2) \circ \delta_Z = (\sigma_2 \times id_{\mathrm{IV}}) \circ \delta_Z = \Big\{ \Big(1, \big\{(1,2),(2,1)\big\}\Big), \Big(2, \big\{(1,1),(2,2)\big\}\Big), \Big(3, \big\{(3,4),(4,3)\big\}\Big), \Big(4, \big\{(3,3),(4,4)\big\}\Big) \Big\}.$$

Trivially e is both a permutation and a phase map. One can easily verify that the same permutations and phase maps apply for the classical structure $(\mathrm{IV}, \delta'_Z, x_1^\dagger)$. However, these do not apply for the classical structures $(\mathrm{IV}, \delta''_Z, y_0^\dagger)$ and $(\mathrm{IV}, \delta'''_Z, y_1^\dagger)$. For example, we have for $(\mathrm{IV}, \delta''_Z, y_0^\dagger)$ that σ_1 is not a permutation because

$$\delta''_Z \circ \sigma_1 :: 1 \mapsto 3 \mapsto \{(3,4),(4,3)\}$$

but

$$(\sigma_1 \times \sigma_1) \circ \delta''_Z :: 1 \mapsto \{(1,1),(2,2)\} \mapsto \{(3,3),(4,4)\}.$$

However we can prove that for classical structures $(\mathrm{IV}, \delta''_Z, y_0^\dagger)$ and $(\mathrm{IV}, \delta'''_Z, y_1^\dagger)$ the relation $\sigma_3 = (14)(23)$ is a permutation. For example for $(\mathrm{IV}, \delta''_Z, y_0^\dagger)$ we have:

$$\delta''_Z \circ \sigma_3 = (\sigma_3 \times \sigma_3) \circ \delta''_Z = \Big\{ \Big(1, \big\{(4,4),(3,3)\big\}\Big), \Big(2, \big\{(3,4),(4,3)\big\}\Big), \Big(3, \big\{(1,2),(2,1)\big\}\Big), \Big(4, \big\{(1,1),(2,2)\big\}\Big) \Big\}$$

and

$$y_0^\dagger \circ \sigma_3 = y_0^\dagger = \{(\{1,4\},*)\}.$$

The relations e and $(12)(34)$ are phase maps for classical structures $(\mathrm{IV}, \delta''_Z, y_0^\dagger)$ and $(\mathrm{IV}, \delta'''_Z, y_1^\dagger)$.

Therefore for observable Z we have that the phase maps are e and $(12)(34)$. For the classical structures $(\mathrm{IV}, \delta_Z, x_0^\dagger)$ and $(\mathrm{IV}, \delta'_Z, x_1^\dagger)$ the permutations are e and $(13)(24)$ and for $(\mathrm{IV}, \delta''_Z, y_0^\dagger)$ and $(\mathrm{IV}, \delta'''_Z, y_1^\dagger)$ the permutations are e and $(14)(23)$.

In the same way the phase maps for observable X are the relations e and $(13)(24)$. For the classical structures $(\mathrm{IV}, \delta_X, z_0^\dagger)$ and $(\mathrm{IV}, \delta'_X, z_1^\dagger)$ the permutations are the relations

e and $(12)(34)$ and for classical structures $(\mathrm{IV}, \delta''_X, y_0^\dagger)$ and $(\mathrm{IV}, \delta'''_X, y_1^\dagger)$ the permutations are e and $(14)(23)$.

Similarly for observable Y phase maps are the relations e and $(14)(23)$. For the classical structures $(\mathrm{IV}, \delta_Y, x_0^\dagger)$ and $(\mathrm{IV}, \delta'_Y, x_1^\dagger)$ the permutations are the relations e and $(13)(24)$ and for classical structures $(\mathrm{IV}, \delta''_Y, z_0^\dagger)$ and $(\mathrm{IV}, \delta'''_Y, z_1^\dagger)$ the permutations are e and $(12)(34)$. All the results are demonstrated in the following table:

<table>
<tr><th>Observable</th><th>Classical structures</th><th>Permutations</th><th>Phase maps</th></tr>
<tr><td rowspan="2">Z</td><td>$(\mathrm{IV}, \delta_Z, x_0^\dagger)$
$(\mathrm{IV}, \delta'_Z, x_1^\dagger)$</td><td>$e$
$(13)(24)$</td><td rowspan="2">e
$(12)(34)$</td></tr>
<tr><td>$(\mathrm{IV}, \delta''_Z, y_0^\dagger)$
$(\mathrm{IV}, \delta'''_Z, y_1^\dagger)$</td><td>$e$
$(14)(23)$</td></tr>
<tr><td rowspan="2">X</td><td>$(\mathrm{IV}, \delta_X, z_0^\dagger)$
$(\mathrm{IV}, \delta'_X, z_1^\dagger)$</td><td>$e$
$(12)(34)$</td><td rowspan="2">e
$(13)(24)$</td></tr>
<tr><td>$(\mathrm{IV}, \delta''_X, y_0^\dagger)$
$(\mathrm{IV}, \delta'''_X, y_1^\dagger)$</td><td>$e$
$(14)(23)$</td></tr>
<tr><td rowspan="2">Y</td><td>$(\mathrm{IV}, \delta_Y, x_0^\dagger)$
$(\mathrm{IV}, \delta'_Y, x_1^\dagger)$</td><td>$e$
$(13)(24)$</td><td rowspan="2">e
$(14)(23)$</td></tr>
<tr><td>$(\mathrm{IV}, \delta''_Y, z_0^\dagger)$
$(\mathrm{IV}, \delta'''_Y, z_1^\dagger)$</td><td>$e$
$(12)(34)$</td></tr>
</table>

Taking everything into account we proved that, in opposition to **FRel**, the quantum state transfer protocol can be stimulated in **Spek**.

Chapter 6

Conclusion

We conclude by summing up the main points we have covered about the discrete models **FRel** and **Spek**, outlining the results and suggesting topics for future work.

6.1 Discussion

The main aim of this dissertation is to present discrete models of categorical quantum computation. The discrete models that are illustrated are **FRel** and **Spek**. Categorical quantum computation semantics are based on the †-compact closed categories which are introduced in Chapter 1. Furthermore, the notion of classical structure is needed in order to provide a full description of the discrete models **FRel** and **Spek** and Chapter 2 presents this issue explicitly. In this study quantum measurements are considered to be projector-valued spectra with respect to a classical structure, i.e. we use an "approximate" definition. This is also shown in Chapter 2.

In Chapter 3, from the description of discrete models **FRel** and **Spek** it is denoted that features of categorical quantum computation and mechanics can be presented not only in **FdHilb**, but also in any †-symmetric monoidal category that has enough complementary classical structures. As mentioned in [5] this changes the current approach of establishing complementary classical structures for a †-symmetric monoidal category.

Until now †-symmetric monoidal categories with biproducts [1] were considered to provide all †-symmetric monoidal categories with enough complementary classical structures arising from the matrix calculus [5]. However, **Spek** is not a biproduct category and in **FRel** one of the two complementary observables do not arise from biproduct structure, even though **FRel** is a biproduct category. It is clear that classical structures equip a certain †-symmetric monoidal category with enough complementary classical structures and this enables the description of many features of quantum mechanics.

In **FRel** the existence of the two complementary structures is sufficient to provide us with a simpler and smaller quantum model than that of a qubit in **FdHilb**. As explained in [7] this occurs because of the matrix representation of relations in **FRel** and the fact that classical structures in **FdHilb** can be described by matrices. Actually, the matrix representation of the copying and deleting relations of the two classical structures $(\mathbb{II}, \delta_Z, \gamma_Z)$ and $(\mathbb{II}, \delta_X, \gamma_X)$ in **FRel** is exactly the same with that of the morphisms in **FdHilb** that copy the Z-basis and the X-basis, respectively. Thus, there is a one-to-one correspondence between $(\mathbb{II}, \delta_Z, \gamma_Z)$ in **FRel** and $(\mathcal{Q}, \delta_Z :: |i\rangle \mapsto |ii\rangle, \gamma_Z :: |i\rangle \mapsto 1)$ in **FdHilb** and also between $(\mathbb{II}, \delta_X, \gamma_X)$ in **FRel** and $(\mathcal{Q}, \delta_X :: |j\rangle \mapsto |jj\rangle, \gamma_X :: |j\rangle \mapsto 1)$ in **FdHilb**, where $i \in \{0, 1\}$ and $j \in \{+, -\}$.

Furthermore, we have seen that in **FRel** we have four $\mathbb{II}$-projector-valued spectra of type $\mathbb{II} \to \mathbb{II} \times \mathbb{II}$ relative to $(\mathbb{II}, \delta_Z, \gamma_Z)$. What it is interesting is that δ_X and $\delta_X^{(2)}$, as defined in Section 4.3.2, are $\mathbb{II}$-projector-valued spectra in **FRel** relative to $(\mathbb{II}, \delta_X, \gamma_X)$, but their equivalent morphisms in **FdHilb** are not. To demonstrate this explicitly notice that δ_X corresponds to $|0\rangle \otimes \mathbf{I} + |1\rangle \otimes \mathbf{X}$ and $\delta_X^{(2)}$ to $|0\rangle \otimes \mathbf{I} + |1\rangle \otimes \mathbf{I}$ in the standard basis of **FdHilb**, where $\mathbf{I}$ is the identity matrix and $\mathbf{X}$ the X-Pauli matrix. Both fail to satisfy equation (4.3.3), since

$$|0\rangle \otimes \mathbf{I} + |1\rangle \otimes \mathbf{I} \xrightarrow{\gamma_Z \otimes id_{\mathcal{Q}}} 1 \otimes \mathbf{I} + 1 \otimes \mathbf{I} = 2\mathbf{I} \neq \mathbf{I}$$

and

$$|0\rangle \otimes \mathbf{I} + |1\rangle \otimes \mathbf{X} \xrightarrow{\gamma_Z \otimes id_{\mathcal{Q}}} 1 \otimes \mathbf{I} + 1 \otimes \mathbf{X} = \mathbf{I} + \mathbf{X} \neq \mathbf{I},$$

where $\gamma_Z : \mathcal{Q} \to \mathbb{C} :: |i\rangle \mapsto 1$ for $i \in \{0, 1\}$.

Regarding the discrete model **Spek**, as it is shown in Section 4.4, it comes with three complementary observables enabling a full description of the quantum phenomena that are experienced in **FdHilb**. In [7] it is stated that there is no connection with biproducts since the classical structures on set $\mathbb{IV}$ are not inherited from **FRel**. Also, it is proved that for each classical structure in **Spek** we have 24 different $\mathbb{IV}$-projector-valued spectra arising by permutations of the corresponding copying relation (see Section 4.5). However, in every observable the copying relations of classical structures are not measurements relative to other classical structures of the observable. Extending the table presented in [7] we have the following results, where in **FdHilb** the Z-basis is $\{|0\rangle, |1\rangle\}$, the X-basis is $\{|+\rangle, |-\rangle\}$ and the Y-basis is $\{|i\rangle, |-i\rangle\}$:

FdHilb		**Matrix representation**	**FRel**		**Spek**	
$\|0\rangle$	classical for Z unbiased for X, Y	$\begin{pmatrix}1\\0\end{pmatrix}$	z_0	classical for Z unbiased for X	z_0	classical for Z unbiased for X, Y
$\|1\rangle$	classical for Z unbiased for X, Y	$\begin{pmatrix}0\\1\end{pmatrix}$	z_1	classical for Z unbiased for X	z_1	classical for Z unbiased for X, Y
$\|+\rangle$	classical for X unbiased for Z, Y	$\begin{pmatrix}1\\1\end{pmatrix}$	x_0	classical for X unbiased for Z	x_0	classical for X unbiased for Z, Y
$\|-\rangle$	classical for X unbiased for Z, Y	$\begin{pmatrix}1\\-1\end{pmatrix}$	none		x_1	classical for X unbiased for Z, Y
$\|i\rangle$	classical for Y unbiased for Z, X	$\begin{pmatrix}1\\i\end{pmatrix}$	none		y_0	classical for Y unbiased for Z, X
$\|-i\rangle$	classical for Y unbiased for Z, X	$\begin{pmatrix}1\\-i\end{pmatrix}$	none		y_1	classical for Y unbiased for Z, X

Finally, concerning the state transfer protocol we have shown in Chapter 5 that is not stimulated in **FRel**, but only in **Spek**. This is because stage transfer, while it requires less qubits than teleportation it needs more structural resources, i.e. the morphisms f_i and g_j as stated in Chapter 5 have to be permutations and phase maps respectively. We notice that for every observable in **Spek** these requirements are met. Hence, we can perform the protocol in **Spek**. On the other hand, in **FRel** this is not possible because the lack of negative and complex numbers prevents the generation of phases for $(\mathbb{II}, \delta_Z, \gamma_Z)$ and permutations for $(\mathbb{II}, \delta_X, \gamma_X)$. However, in **FRel** as Proposition 4.5 demonstrates we can stimulate quantum teleportation. As presented in [7] this is because quantum teleportation is based on the existence of a "Bell-basis", i.e. a structure $(A, Bell : A \otimes A \rightarrow B)$ relative to a basis structure (B, δ, γ). Also, in [7] it is proved that the complementary observables $(\mathbb{I}, \delta_Z, \gamma_Z)$ and $(\mathbb{I}, \delta_X, \gamma_X)$ are sufficient to construct such a structure. It is noteworthy, that while quantum teleportation relies on the existence of a "Bell-basis", that is in compact structure only, this does not happen in the state transfer protocol. The basic resource for state transfer is the classical structure, that is why this protocol is not present in **FRel**.

6.2 Future work

The study of discrete models in categorical quantum computation is obviously an important aspect of quantum computer science. The examination of discrete models allows a better understanding of the mathematical structures that are essential in describing the various phenomena of quantum mechanics. Therefore, we can clarify which mathematical features can describe and how, certain physical features. Moreover, discrete models have important computer science applications such as checking technics. In that sense if a property is violated in the discrete model then it can not hold in the abstract †-compact closed category. Hence, further investication of these discrete models is needed to expose their full capabilities.

Also, future work may involve a more abstract description of quantum measurement in discrete models as well as the investigation of other models, for example **Spek** model over the $4^n \times 4^m$ matrices in $\mathbb{Z}_2$. Finally, the connection of **Spek** with Spekken's toy theory [19, 20], already started in [7], worths more investigation. Possible topic could be a detailed description of the connection between **Spek** and the toy theory and the interpretation of the results following from toy theory in a categorical framework.

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