Higher linear algebra in topology and quantum information theory

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To my partner, Theresa,
and my family, Anette, Hans-Dieter, Sophia and Lea.

Abstract

We investigate categorifications of linear algebra, and their applications to the construction of 4-manifold invariants, to the construction of a variety of linear algebraic structures in quantum information theory, and to the classification of certain instances of ‘quantum pseudo-telepathy’, a phenomenon in quantum physics where two non-communicating parties can use pre-shared entanglement to perform tasks classically impossible without communicating.

This thesis is divided into four chapters, closely following `arXiv:1812.11933`, `arXiv:1609.07775`, and `arXiv:1801.09705`.

In the first chapter, we introduce semisimple 2-categories, fusion 2-categories and spherical fusion 2-categories. We prove that every finite semisimple 2-category is the 2-category of finite semisimple module categories of a multifusion category, and give examples of fusion 2-categories.

In the second chapter, we construct, for each spherical fusion 2-category, a state-sum invariant of oriented singular piecewise-linear 4-manifolds, and show that these invariants generalize various previous 4-manifold invariants, including the Crane-Yetter invariant and a recent invariant of Cui.

In the third chapter, we use biunitary connections in the 2-category of 2-Hilbert spaces to generate many new construction schemes for linear algebraic quantities of relevance to quantum information, including complex Hadamard matrices and unitary error bases, and we use these techniques to construct a unitary error basis which cannot be built using any previously known method.

In the fourth chapter, we classify quantum isomorphic graphs in terms of Morita equivalence classes of algebras in certain monoidal categories, give examples of such algebras arising from groups of central type, and discuss various applications to the study of quantum pseudo-telepathy in the graph isomorphism game.

Acknowledgements

Lists of names are ordered alphabetically.

First and foremost, I am grateful to my supervisor and mentor, Jamie Vicary, for introducing me to the world of research and the world of higher categories, for many shared memories (I am especially fond of many hours in the Oxford hackspace building little qubit simulators), and for his tireless support and encouragement throughout these four years.

I am also grateful to my mentor and collaborator Chris Douglas, who all but acted as my unofficial second supervisor, for his advice and guidance, both mathematical and personal, and for many very enjoyable and productive meetings, in Oxford and in California.

I would like to thank my collaborators, Ben Musto and Dominic Verdon for a very fruitful and enjoyable cooperation, and for their friendship.

I owe much of my experience at Oxford to the architects of the quantum group, Samson Abramsky and Bob Coecke. Thank you, for creating such a unique environment where one can study topological field theories in a computer science department and discuss and share ideas with mathematicians, physicists and computer sciencists working on topics ranging from quantum foundations to linguistics, all brought together by their shared passion for category theory. I count myself very lucky to have been part of this amazing group of people.

My work and thinking have been heavily influenced by conversations with many people, including (the ones named above and) Niel de Beaudrap, Shawn Cui, Ross Duncan, Giovanni de Felice, Tobias Fritz, André Henriques, Corey Jones, Andre Kornell, Dan Marsden, Dave Penneys, Paweł Sobocínski, Lucy Zhang, and most notably my colleagues and friends Christoph Dorn, Stefano Gogioso, Amar Hadzihasanovic and Matthew McMillan.

My DPhil would not have been nearly as much fun without the rest of the quantum group, especially Antonin, Carlo Maria, Konstantinos, Linde, Marietta, Nicola, Robin, Subhayan, and Vojta. I am looking forward to many more Thursday evenings, whenever we are all back together in Oxford!

Most importantly, I want to thank my family; my parents Anette and Hans-Dieter Reutter and my sisters Sophia and Lea, for making me who I am today and for being, quite simply, the best family in the world.

Finally, I thank my partner, Theresa Westphal, for more than can be said.

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Contributions

Up to minor rewordings and changes of convention, this thesis is composed of the papers [DR18, RV19b, MRV19]. Chapters 1 and 2 and Appendices B and C are taken from the following joint paper with Christopher Douglas:

> *Fusion 2-categories and a state-sum invariant for 4-manifolds.* With Christopher Douglas. Submitted (2019). pp 1–110. `arXiv:1812.11933`.

Chapter 3 and Appendix E are taken from the following joint paper with Jamie Vicary:

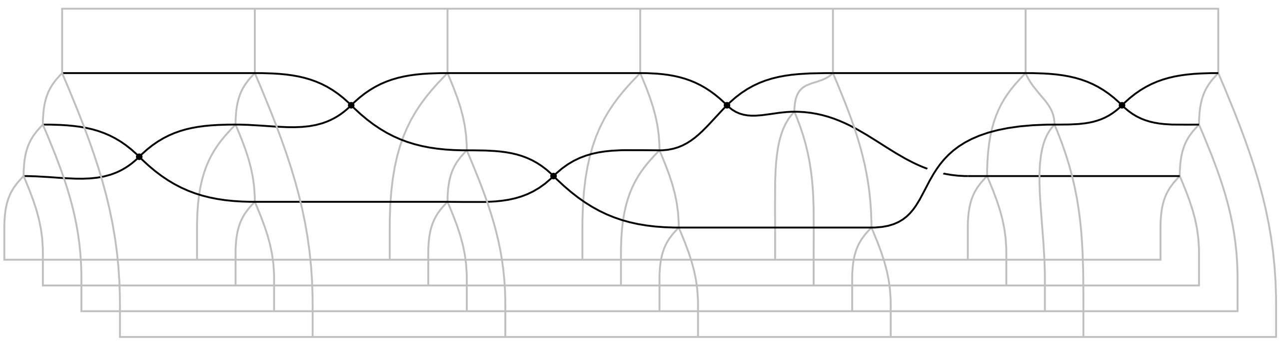
> *Biunitary constructions in quantum information.* With Jamie Vicary. Higher structures 3(1):109–154, 2019. `arXiv:1609.07775`

Chapter 4 and Appendix D are taken from the following joint paper with Benjamin Musto and Dominic Verdon:

> *The Morita theory of quantum graph isomorphisms.* With Benjamin Musto and Dominic Verdon. Comm. Math. Phys., 365(2):797–845, 2019. `arXiv:1801.09705`

Some of the introductory material in Chapter 4 overlaps with introductory material in Dominic Verdon's thesis [Ver19] (which builds on the constructions in Chapter 4 and rewrites some of them in representation-theoretic terms).

For similar applications of low-dimensional higher category theory to questions in linear algebra and quantum information, and for a combinatorial model of higher string diagrams, we refer the interested reader to the author's other papers [RV19d, RV19a, MRV18, RV19c].



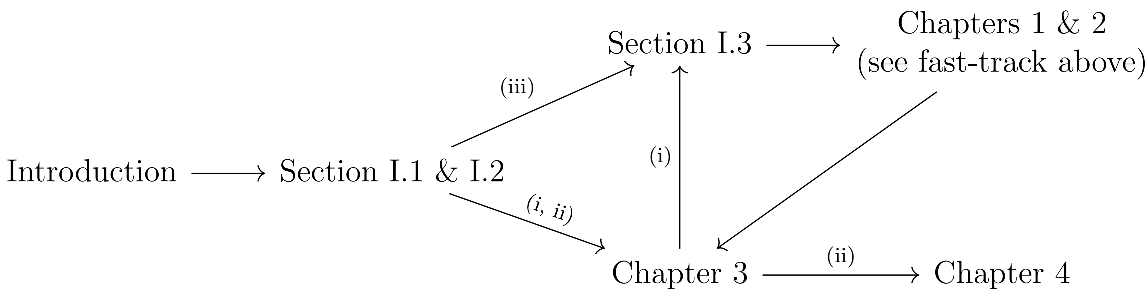
Guide

The introductory Sections I.1, I.2 and I.3 are mostly expositional and not strictly necessary for the mathematical development of later sections, but convey some of the key motivations and ideas guiding our categorification in Chaper 1.

For Chapters 1 and 2, readers exclusively interested in the theory of fusion 2-categories can restrict attention to Sections 1.2 and 1.3 and Appendices B and C. Readers primarily interested in the state sum construction will want to focus on Sections 2.1 and 2.2 and Appendix C, but will still need to at least skim Sections 1.2 and 1.3. Many readers, whether expert or novice, will want to merely skim Section 1.2.3, which is both more motivational and more technical, and most readers will want to skip Appendix B (containing the proofs about idempotent completion) and Appendix C (containing the proofs of the dimension formulas). Experts can skim or skip Sections 2.1.1, 2.1.2, and 2.1.4, and only the hardiest souls will want to get into the combinatorial invariance proof in Section 2.1.3. The recommended fast-track for Chapters 1 and 2 is therefore Sections 1.2.1, 1.2.2, 1.2.4, 1.3.1, 1.3.2, 1.3.3, 2.1.3, and then only as much of the proof in 2.2.1, 2.2.2, and 2.2.3 as required for one's purposes.

Chapters 3 and 4 can be read completely independently from each other and from Chapters 1 and 2.

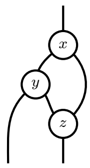
Depending on taste and background, the recommended overall fast-track looks as follows for: (i) Readers new to the ideas of higher linear algebra;
(ii) Readers mainly interested in applications to quantum information;
(iii) Readers mainly interested in the categorification of semisimple categories.

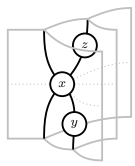


Contents

Introduction

From a certain vantage point, higher category theory may be thought of as a theory of 'higher dimensional algebra'. Here, 'algebra' is understood (admittedly very reductionistically) as describing the composition of symbols on a line, dealing with expressions such as 'xyx^2z'. In contrast, 'higher dimensional algebra' is about manipulating symbols in higher dimensions, with expressions such as the following:





The mathematical theory underlying the compositional behaviour of such 'higher dimensional string diagrams' in n-dimensional space is (conjectured to be) n-category theory[1,2]. Explicitly, every appropriately "progressive" (roughly speaking, a progressive diagram does not include any 'bends' or other Morse singularities) n-dimensional diagram, whose singularities are labeled by morphisms of an n-category $\mathcal{C}$, may be understood as encoding a composite n-morphism in $\mathcal{C}$.

Dropping the requirement of progressiveness corresponds to the introduction of so called 'dualizability' conditions — an n-category which allows arbitary non-progressive (but still appropriately framed and stratified) n-diagrams is called 'fully dualizable'. For example, given an object in a symmetric monoidal fully dualizable n-category, we may use the graphical calculus of higher string diagrams to evaluate an arbitrary closed framed[3] n-manifold to a 'scalar diagram' in $\mathcal{C}$, that is to an n-morphism on the $(n-1)$-fold identity of the monoidal unit[4]. As an example, a symmetric monoidal

[1] Here and throughout, we use the term 'n-category' to refer to the fully weak structure.

[2] We hasten to remark that this perspective on higher category theory has only been made rigorous in dimensions $1, 2$ and 3 [JS91, BMS12, Hum12] and remains conjectural in all other dimensions. A recent, purely combinatorial approach to higher string diagrams is outlined in [Dor18, RV19c].

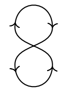
[3] A framed n-manifold is an n-manifold equipped with a trivialization of its tangent bundle.

[4] More generally, objects of fully dualizable symmetric monoidal n-categories $\mathcal{C}$ give rise to symmetric monoidal n-functors $\mathrm{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ from an appropriate symmetric monoidal n-category of n-framed bordisms to $\mathcal{C}$. This consequence of the graphical calculus of higher string diagrams is

1-category $\mathcal{C}$ is fully dualizable if every object admits a right dual; for every object X there is an object X^* together with evaluation and coevaluation morphisms $\mathrm{ev}_X : X \otimes X^* \to I$ and $\mathrm{coev}_X : I \to X^* \otimes X$ depicted as a 'cap' and a 'cup' (also sometimes known as 'folds') fulfilling the following 'cusp equations' (here, diagrams are read from bottom to top):

(1)

In particular, every object X has an associated scalar diagram corresponding to the 1-framed circle evaluated on X, and often thought of as a categorical notion of the dimension of X:



Conversely, given a symmetric monoidal n-category $\mathcal{C}$, we may use this 'graphical calculus' and the resulting geometric insight to study the behaviour and properties of morphisms in $\mathcal{C}$. This leads to an interesting interplay between algebra and geometry — we may use our favourite symmetric monoidal fully dualizable n-category $\mathcal{C}$ as a means to construct n-manifold invariants, or conversely, we may use the geometry of said n-manifolds to study the n-category $\mathcal{C}$.

In this thesis, we investigate various consequences of this interplay for appropriate fully dualizable categories of 'higher vector spaces' over a field k, categorifying the symmetric monoidal 1-category of finite-dimensional k-vector spaces. In Chapter 1, after recalling the well established theory of '2-vector spaces', we introduce 'semisimple 2-categories' and 'fusion 2-categories' as candidate definitions for 3- and 4-vector spaces. In Chapter 2, we show that fusion 2-categories may indeed be used to construct linear algebraic 4-manifold invariants.

Conversely, in Chapter 3, we show how 2-vector spaces, and the insights gained from the associated 2-dimensional graphical calculus may be used to uncover new constructions of concrete and basic linear algebraic quantities which play important roles in quantum information theory. Finally, in Chapter 4, we show how the classification of module categories over certain fusion categories and the splitting of separable algebras — both ideas of central importance to the definition of 3-vector spaces — can be used to classify instances of 'quantum pseudo-telepathy', a phenomenon in quantum information theory related to Bell's theorem. These latter two chapters may serve

known as the 'cobordism hypothesis' [BD95], a proof of which was recently sketched [Lur09] in the setting of (∞, n)-categories modelled on complete Segal spaces.

as an advertisement for the essential role that higher category theory can play in quantum information theory.

The (unreasonable?) effectiveness of higher linear algebra

Before diving into the technical developements of this thesis, let us paint a 'big picture' on the use and effectiveness of higher linear algebra in topology and — on the other hand — in areas of quantum information theory not usually associated with topology, and on how these applications are related.

As sketched above, every object in a fully dualizable symmetric monoidal n-category gives rise to a framed n-manifold invariant. The easier it is to compute in the n-category, the easier it is to compute the corresponding manifold invariant. This immediately justifies the demand for a theory of higher vector spaces: An appropriate symmetric monoidal n-category of finite-dimensional n-vector spaces should lead to n-manifold invariants which can be computed by essentially linear algebraic means. And indeed, in Chapter 2, we define a state-sum 4-manifold invariant arising from a summation of linear algebraic data — encoded in our notion of 'fusion 2-category' (or '4-vector space') — over the simplices of a combinatorial 4-manifold.

On the other hand, and maybe somewhat surprisingly, higher vector spaces also play a role in quantum information theory. This manifests most prominently in various applications of topological field theories[5], but there are also other — largely unexplored — applications of higher categories to areas of quantum information theory not usually associated with topology. In this thesis, we discuss two such applications.

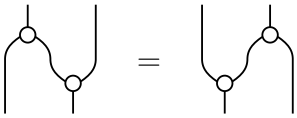
In Chapter 3, we show that 'biunitaries' — certain 2-categorical algebraic structures usually encountered in the operator algebraic classification of planar algebras and subfactors — encode important 'quantum combinatorial' objects, including complex Hadamard matrices (unitary matrices in which every entry has the same modulus), quantum Latin squares (square grids of vectors in a finite-dimensional Hilbert space, such that every row and every column forms an orthonormal basis), and unitary error bases (bases of unitary operators on a finite-dimensional Hilbert space, orthogonal with respect to the trace inner product). These quantum combinatorial objects play a critical role in many quantum procedures — from teleportation to key distribution —, and are notoriously hard to construct. Using the two-dimensional

[5]The connection between topological field theory and quantum information theory is well established: Topological field theories provide the mathematical framework for topological quantum computation. They also play an essential role in the design of quantum error-correction protocols in ordinary quantum computation (these protocols essentially 'simulate' topological quantum computers on 'ordinary' quantum computers to exploit their excellent fault tolerance properties).

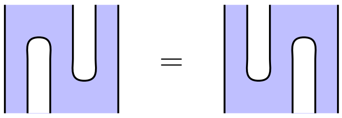
graphical calculus of the symmetric monoidal 2-category of 2-Hilbert spaces, we uncover many new constructions of such quantities which are not readily accessible from a more conventional linear algebraic perspective.

In Chapter 4, we discuss another application of 2-category theory to a completely different aspect of quantum information theory: the study of 'pseudo-telepathy'. Pseudo-telepathy is a phenomenon in quantum information theory, similar in spirit to Bell's theorem, where two non-communicating parties can use pre-shared entanglement to perform a task classically impossible without communication. Such tasks are usually formulated as games, where isolated players are provided with inputs, and must return outputs satisfying some winning condition. One such game is the graph isomorphism game [AMR$^+$19], whose instances correspond to pairs of graphs Γ and Γ', and whose winning classical strategies are exactly graph isomorphisms $\Gamma \to \Gamma'$. Winning quantum strategies are called quantum graph isomorphisms. Quantum pseudo-telepathy is exhibited by graphs that are quantum but not classically isomorphic. In Chapter 4, we use a 2-categorical framework for noncommutative set and graph theory to give an explicit classification of instances of pseudo-telepathy in the graph isomorphism game.

These applications raise the question of why and how higher category theory — and in particular higher linear algebra — is able to shed light on such apparently non-topological phenomena. One possible explanation is that many algebraic structures of relevance to various aspects of quantum information theory seem to intrinsically behave 'geometrically'. For example, a central observation of [AC04], which initiated the program of Categorical Quantum Mechanics, was the realization that 'maximal entanglement' corresponds to duality: Given two finite-dimensional Hilbert spaces V and W, a maximally entangled state between V and W is precisely a vector $\eta \in V \otimes W$ which is the coevaluation morphism $\mathbb{C} \to V \otimes W$ of a duality between V and W, and hence may be graphically represented as a 'cup' (1). The cusp equations (1) then immediately lead to a variant of the seminal quantum teleportation protocol [BBC$^+$93]. Hence, one of the most central concepts of quantum information theory is mathematically expressed by 1-categorical dualizability, 'explaining' the appearance of 1-dimensional geometry in quantum information theory. Such geometrical re-interpretations of algebraic structures are not restricted to dimension one. For example, in Categorical Quantum Mechanics, the interaction between classical and quantum information is modelled [CP08] by Frobenius algebras. A Frobenius algebra in a monoidal category is a pair of a monoid $(A \otimes A \to A, I \to A)$ and a comonoid ($A \to A \otimes A, A \to I$) fulfilling the following compatibility condition:



Passing to a 2-categorical setting we may think of a Frobenius algebra A as being composed from a pair of (ambidextrously) adjoint 1-morphisms $A \cong R \circ L$, in which case the above equations — and all other defining equations of the monoid and comonoid — become planar isotopies:



In fact, every Frobenius algebra in a monoidal category arises in this way from a dualizable 1-morphism in some 2-category in which the monoidal category fully faithfully embeds [Lau06]. In this sense, a Frobenius algebra in a monoidal category may always be understood as a 'shadow', or remnant, of an underlying genuinely two-dimensional structure. (A version of this principle underlies much of the development in Chapters 1 and 4, see Section 1.2.3 and Appendix D.) Although the applications of Chapters 3 and 4 are based on 2-dimensional structures, there are several indications that higher-dimensional structures should also play a role in quantum information theory[6].

In conclusion, by passing to higher dimensional categories, we have turned a non-trivial algebraic structure into something 'purely topological' which is essentially hard-wired into the graphical calculus. In some sense, we have traded a 'non-trivial' algebraic structure in a low dimension against a 'trivial' algebraic structure in a higher dimension (here, 'non-trivial' should be read as 'fulfilling equations that are not just isotopies in the graphical calculus'). Following this line of reasoning, one might ultimately imagine an approach to quantum information theory in which one develops protocols, answers foundational questions, or simply proves equations between algebraic expressions by comparing and studying the diffeomorphism classes of certain manifolds or higher-dimensional string diagrams.

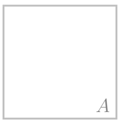
[6]For example, in Categorical Quantum Mechanics, the concept of 'complementarity' is modelled [CD08] by Hopf algebras, which are inherently 3-dimensional/3-categorical structures.

Outline

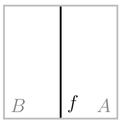
In the following introductory sections, we recall background material on string diagrams, linear categories and 2-vector spaces; Section I.1 recalls string diagrams, Section I.2 discusses various alternative characterizations of semisimplicity, and compares the 2-category of semisimple categories with the 2-category of semisimple algebras and the 2-category of Kapranov and Voevodsky's 2-vector spaces. Section I.3 re-examines the material of Section I.2 with an eye towards categorification, recalling that semisimple categories can be characterized as fully dualizable objects in a 2-category of profunctors, highlighting the special role played by Cauchy completeness.

I.1 Higher-dimensional string diagrams

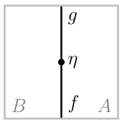
In this thesis, we make frequent use of the string diagram calculus for monoidal 1- and 2-categories. The diagrammatic calculus for monoidal 1-categories goes back to a calculus for the composition of multilinear maps proposed by Penrose [Pen71], and is today widely used in a range of areas [JS91, Sel11, BK01, AC09, Orú14]. Thinking of a monoidal category $\mathcal{M}$ as a 2-category $\mathrm{B}\mathcal{M}$ with only one object and endomorphism category $\mathcal{M}$, this calculus is a special case of the diagrammatic calculus of (strict) 2-categories:



An object A

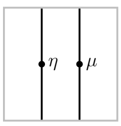


A 1-morphism $f : A \to B$

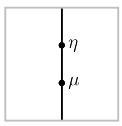


A 2-morphism $\eta : f \Rightarrow g$

Here, objects are depicted as regions in the plane, a 1-morphism $A \to B$ is depicted as a wire separating the region labeled A from the region labeled B, and a 2-morphism $f \Rightarrow g$ is depicted as a node separating the wire labeled f from the wire labeled g. (The gray bounding box simply indicates the extent of the picture.) We draw 1-morphism composition from right to left: that is, in a diagram for $g \circ f$, the wire labeled g appears to the left of the wire labeled f. Similarly, we draw horizontal and vertical composition of 2-morphisms from right to left and from bottom to top: that is, in a diagram for $\eta \circ \mu$, the node labeled η appears on the left of the node labeled μ, and in a diagram for $\eta \cdot \mu$, the node labeled η appears above the node labeled μ:



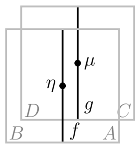
The horizontal composite $\eta \circ \mu$

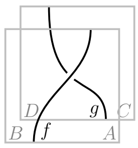


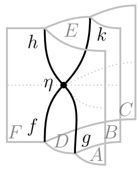
The vertical composite $\eta \cdot \mu$

In Chapter 3 and Appendix D, we use colors instead of labels to more clearly distinguish different regions.

In a monoidal 2-category, the monoidal structure is depicted by layering surfaces behind each other, resulting in a graphical calculus of 'surface diagrams' in 3-space, such as the following:







In Section 1.3.1 we give a more careful introduction to monoidal 2-categories — in their semistrict incarnation as 'Gray monoids' — and their graphical calculus. For now, we urge the reader to treat a monoidal 2-category as precisely the sort of data which gives rise to this calculus.

The 2-categories (and monoidal 1-categories) appearing in Chapters 3 and 4 are equipped with a dagger structure [Sel11, HK16]; given a 2-morphism $\eta : f \Rightarrow g$, we express its †-adjoint $\eta^\dagger : g \Rightarrow f$ as a reflection of the corresponding diagram across a horizontal axis.

I.2 On 2-vector spaces

Kapranov and Voevodsky's 2-vector spaces

The predominant notion of (finite-dimensional) 2-vector space in higher representation theory is due to Kapranov and Voevodsky [KV94]. Categorifying the category $\mathrm{Mat}(k)$ whose objects are natural numbers and whose morphisms $n \to m$ are $m \times n$ matrices, Kapranov and Voevodsky introduce a symmetric monoidal 2-category $\mathrm{Mat}(\mathrm{Vect}_k)$ whose objects are natural numbers, whose 1-morphisms $n \to m$ are $m \times n$ matrices of finite-dimensional vector spaces and whose 2-morphisms are matrices of linear maps (see Figure 1). The composition (and tensor product) of 1-morphisms is defined as the product (and Kronecker product) of matrices with multiplication and addition in k replaced by the tensor product and direct sum of k-vector spaces.

For an elementary description of the two-dimensional graphical calculus of $\mathrm{Mat}(\mathrm{Vect}_k)$ which can be used without reference to higher categorical technology, we refer the reader to Section 3.2. For direct and hands-on applications of this graphical calculus, we refer to Chapter 3 more generally.

In the following section, we give a more 'coordinate-independent' description of the 2-category of 2-vector spaces and introduce several of the concepts, ideas and

$$\begin{pmatrix} V_{11} & \cdots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{m1} & \cdots & V_{mn} \end{pmatrix}$$

(a) A 1-morphism $V : n \to m$

$$\begin{pmatrix} V_{11} \xrightarrow{\phi_{11}} V'_{11} & \dots & V_{1n} \xrightarrow{\phi_{1n}} V'_{1n} \\ \vdots & \ddots & \vdots \\ V_{m1} \xrightarrow{\phi_{m1}} V'_{m1} & \dots & V_{mn} \xrightarrow{\phi_{mn}} V'_{mn} \end{pmatrix}$$

(b) A 2-morphism $\phi : V \Rightarrow V'$

Figure 1: The 1- and 2-morphisms of the 2-category $\mathrm{Mat}(\mathrm{Vect}_k)$.

motivations relevant to our discussion of 3- and 4-vector spaces. Most of the material in this section is well established and expositional, an overview of various definitions of 2-vector spaces can be found in [BDSV15, App. A].

Semisimple categories

The category Vect_k of finite-dimensional k-vector spaces may be seen as a 'coordinate-independent' version of the category $\mathrm{Mat}(k)$ of natural numbers and matrices over k. Similarly, there is a 2-category $2\mathrm{Vect}_k$ of finite-dimensional 2-vector spaces which is equivalent to Kapranov and Voevodsky's 2-category $\mathrm{Mat}(\mathrm{Vect}_k)$. In the following section we define such finite-dimensional 2-vector spaces as k-linear categories which are *finite semisimple*. Here, we content ourselves with recalling the basic definitions and properties of semisimple categories — in the next section we closely re-examine these definitions with an eye towards further categorification, and clarify why semisimple k-linear categories deserve to be called 'finite-dimensional 2-vector spaces'.

Semisimplicity over commutative rings

Semisimplicity is a property of a k-linear category, or more generally, a property of a category enriched in the category of R-modules for R some commutative ring (and in particular — for $R = \mathbb{Z}$ — is a property of a category enriched in the category of abelian groups). For the sake of our later categorification, it will be useful to establish the basic definitions and properties in this more general setting and only later specialize to the case when $R = k$ is an algebraically closed field. We therefore let R be a commutative ring and refer to a category enriched in the category of R-modules as an R-linear category.

Definition I.2.1 (Zero object)**.** A *zero object* in a category $\mathcal{C}$ is an object that is both terminal and initial. A *pointed category* is a category with a zero object.

An object X in an R-linear category is a zero object if and only if its identity morphism is the zero element in $\mathrm{Hom}_{\mathcal{C}}(X, X)$, which in turn is equivalent to the R-module $\mathrm{Hom}_{\mathcal{C}}(X, X)$ being zero.

Definition I.2.2 (Direct sum in a linear category)**.** The *direct sum* (or *biproduct*) of two objects A_1 and A_2 in an R-linear category is an object $A_1 \oplus A_2$ with inclusion morphisms $i_j : A_j \to A_1 \oplus A_2$, and projection morphisms $p_j : A_1 \oplus A_2 \to A_j$, fulfilling the following conditions:

- $p_j \cdot i_j = \mathrm{id}_{A_j}$ for $j = 1, 2$;
- $p_1 \cdot i_2 = 0$ and $p_2 \cdot i_1 = 0$;
- $i_1 \cdot p_1 + i_2 \cdot p_2 = \mathrm{id}_{A_1 \oplus A_2}$.

An R-linear category is *additive* if it has a zero object and has (pairwise) direct sums.

Remark I.2.3 (Direct sum in a pointed category)*.* More generally, in a pointed category, two objects A_1 and A_2 have a direct sum if they have a categorical product $A_1 \times A_2$ and a categorical coproduct $A_1 \sqcup A_2$ such that the induced morphism

$$A_1 \sqcup A_2 \xrightarrow{\begin{pmatrix} \mathrm{id}_{A_1} & 0 \\ 0 & \mathrm{id}_{A_2} \end{pmatrix}} A_1 \times A_2$$

is an isomorphism.

Definition I.2.4 (Idempotent)**.** An *idempotent* in a category is a morphism $\gamma : A \to A$ such that $\gamma \circ \gamma = \gamma$. An idempotent *splits* if there are morphisms $i : B \to A$ and $r : A \to B$ such that $r \circ i = \mathrm{id}_B$ and $i \circ r = \gamma$. A category is *idempotent complete* if every idempotent splits.

Remark I.2.5 (Direct sums, zero objects and idempotent splittings are preserved by all linear functors)*.* In an R-linear category, direct sums, zero objects and idempotent splittings are 'equational' constructions, in that they may be defined in terms of the existence of certain morphisms satisfying certain equations. It follows that they are preserved by all R-linear functors. Recall that an absolute colimit is a colimit preserved by all R-linear functors. Direct sums, zero objects and idempotent splittings may be expressed as universal constructions, in particular as colimits, and are therefore absolute colimits. We discuss the significance of this observation in Section I.3.2.

Construction I.2.6 (Additive and idempotent completion). Any R-linear category $\mathcal{C}$ can be completed to an additive category $\mathcal{C}^{\oplus}$; here $\mathcal{C}^{\oplus}$ has as objects finite (possibly empty) lists of objects of $\mathcal{C}$, and as morphisms matrices of morphisms in $\mathcal{C}$ between the respective objects. Composition in $\mathcal{C}^{\oplus}$ is 'matrix multiplication' with sum and product replaced by sum and composition in $\mathcal{C}$.

Similarly, any category $\mathcal{C}$ can be completed to an idempotent complete category $\mathcal{C}^{\triangledown}$ whose objects are idempotents in $\mathcal{C}$ and whose morphisms $(e : A \to A) \to (e' : B \to B)$ are morphisms $f : A \to B$ such that $f \circ e = f = e' \circ f$. Composition is inherited from $\mathcal{C}$ and the identity on $(e : A \to A)$ is given by the morphism $e : A \to A$.

Example I.2.7 (Completion of the delooping of an algebra). Let A be an R-algebra and let $\mathrm{B}A$ be the one object R-linear category with endomorphism algebra A. Then, the additive and idempotent completion $(\mathrm{B}A^{\oplus})^{\triangledown}$ is the category of finitely generated projective A-modules.

Remark I.2.8 (Additive and idempotent completion is idempotent). If $\mathcal{C}$ is already additive and idempotent complete, then $(\mathcal{C}^{\oplus})^{\triangledown}$ is equivalent to $\mathcal{C}$; this is a consequence of the fact that direct sums, zero objects and idempotents are absolute colimits and that $(\mathcal{C}^{\oplus})^{\triangledown}$ is the free cocompletion under these colimits.

There are various equivalent definitions of semisimplicity. Recall that a *subobject* of an object X is an isomorphism class[7] of monomorphisms $Y \to X$, and that a non-zero object X of a pointed category is *simple* if its subobjects are all either zero objects or isomorphisms. Conventionally, an R-linear category is defined to be *semisimple* if it is abelian[8] and if every object is a finite direct sum of simple objects. A crucial property of semisimple categories is *Schur's lemma*: Every non-zero morphism between simple objects in a semisimple R-linear category is invertible. We now review two equivalent definitions of semisimplicity which do not explicitly impose abelianity and which will be better suited for our later categorification.

Proposition I.2.9 (Alternative characterizations of semisimplicity)**.** *An R-linear category is semisimple if and only if it is additive, idempotent complete and fulfills one of the following equivalent properties:*

a) every object is a finite direct sum of simple objects and the composite of any two non-zero morphisms between simple objects is again non-zero;

---

[7]Two monomorphisms $f : Y \to X$ and $g : Y' \to X$ are isomorphic if they are isomorphic in the over-category $\mathcal{C}/X$, that is, if there is an isomorphism $r : Y \to Y'$ such that $f = g \circ r$.

[8]An R-linear category is *abelian* if it has all finite limits and colimits and if for every morphism $f : A \to B$, the canonical morphism $\mathrm{coker}(\ker f) \to \ker(\mathrm{coker} f))$ is an isomorphism.

b) the endomorphism ring of every object is a semisimple ring[9]*.*

Proof. It is proven in [Jan92, Lem 2] that condition b) is equivalent to semisimplicity in the conventional abelian sense (the proof is given for the case that R is a field and that the endomorphism algebra of every object is moreover finite-dimensional, but generalizes without changes to the more general case considered here).

Moreover, every abelian semisimple category fulfills a) — by definition, every object is a finite direct sum of simple objects, and by Schur's lemma, the composite of nonzero morphisms between simple objects is again nonzero. To show that a) implies b) and hence that a) is also equivalent to conventional abelian semisimplicity, we first show that any nonzero morphism $f : X \to Y$ between simple objects in a category fulfilling a) is a monomorphism and hence, by simplicity of Y, an isomorphism. Indeed, let $x : Z \to X$ be a morphism such that $fx = 0$. Decomposing Z into a finite direct sum of simple objects with inclusions and projections $X_j \xrightarrow{i_j} Z \xrightarrow{p_j} X_j$ implies that $fxi_j = 0$ and hence, by assumption a), that $xi_j = 0$, and therefore $x = 0$. In particular, such categories fulfill a version of Schur's lemma — the endomorphism ring of any simple object is a division ring, and every morphism between nonisomorphic simple objects is zero. In particular, since every object is a finite direct sum of simple objects, it follows that the endomorphism ring of any object is a finite direct sum of matrix algebras over division rings and hence semisimple. □

Example I.2.10 (Modules of a semisimple algebra)*.* The canonical example of a semisimple R-linear category is the category of finitely generated modules of a semisimple algebra over R.

Remark I.2.11 (Semisimple categories as domainoids)*.* Recall that an algebra is called a domain if it has no zero divisors; the composition condition in Proposition I.2.9 a) can be understood as insisting that the category is a many-object version of a domain (a 'domainoid').

Warning I.2.12 (The domainoid condition is necessary)*.* In the literature, the 'domainoid' condition of Proposition I.2.9 a) is sometimes omitted and it is asserted that an additive, idempotent complete R-linear category in which every object decomposes as a finite direct sum of simple objects is automatically semisimple abelian. This is not correct; as a counterexample, suppose that $R = k$ is a field and consider the category fgProj(A) of finitely generated projective (fgp) modules of a finite-dimensional

[9] A *semisimple* ring is a ring A that decomposes into a finite direct sum of simple modules in the category of left (or equivalently right) A-modules. By the Artin-Wedderburn theorem, every semisimple ring is a finite product of matrix rings over divison rings.

k-algebra A with a unique simple module (such as the algebra $k[x]/(x^2)$). This category is k-linear, additive and idempotent complete (it is in fact precisely the additive and idempotent completion of the k-linear category $\mathrm{B}A$ with one object and endomorphism algebra A) but is in general not abelian and in particular not semisimple. Nevertheless, any object in $\mathrm{fgProj}(A)$ decomposes as a finite direct sum of simple objects. Indeed, since A is a finite-dimensional algebra with a unique simple module, it also has a unique indecomposable fgp module M, and every fgp module decomposes into a finite direct sum of M's. Even though M is not necessarily a simple module, we now show that M is a simple object in $\mathrm{fgProj}(A)$. Let $i : P \hookrightarrow M$ be a monomorphism between fgp modules and suppose that P is nonzero. Since P is a nonzero finite direct sum of M's, there is also a monomorphism $j : M \hookrightarrow P$ and hence a monomorphism $ij : M \to M$. Since every finitely generated module over a finite-dimensional k-algebra is in particular a finite-dimensional k-vector space and since every injective k-linear endomorphism on a finite-dimensional k-vector space is an isomorphism, it follows that the composite $ij : M \to M$ is an isomorphism in the category $\mathrm{fgProj}(A)$. Since i is a monomorphism, it follows that i is an isomorphism.

Semisimplicity over algebraically closed fields

From now on, we assume $R = k$ to be an algebraically closed field and restrict attention to finite semisimple categories.

Definition I.2.13 (Finite semisimple category)**.** A semisimple category over an algebraically closed field k is *finite* if every Hom-vector space $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is finite-dimensional and if there is a finite number of isomorphism classes of simple objects.

We are now ready to define the 2-category of 'finite-dimensional 2-vector spaces'.

Definition I.2.14 (The 2-category $2\mathrm{Vect}_k$)**.** The 2-category $2\mathrm{Vect}_k$ is the 2-category of finite semisimple k-linear categories, k-linear functors and natural transformations.

Remark I.2.15 (Every functor between semisimple categories is dualizable). It follows from our discussion in Section I.3.2 and Proposition I.3.13 that every k-linear functor between finite semisimple k-linear categories has a right and a left adjoint and in particular preserves limits and colimits.

Since every finite-dimensional division algebra over an algebraically closed field k is isomorphic to k itself, it is an immediate corollary of Schur's lemma that an object A in a finite semisimple k-linear category is simple if and only if its endomorphism algebra $\mathrm{End}_{\mathcal{C}}(A)$ is isomorphic to k, that is, if every endomorphism of A is proportional to the identity. This leads to the following alternative characterization of semisimplicity over algebraically closed fields [Müg03].

Proposition I.2.16 (A concrete characterization of semisimplicity)**.** *A k-linear category $\mathcal{C}$ is finite semisimple if and only if it is additive, idempotent complete and there is a finite set of objects $\{X_i\}_{i\in I}$ with $\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) \cong \delta_{i,j}k$ and such that the composition map*

$$\bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, B) \otimes \mathrm{Hom}_{\mathcal{C}}(A, X_i) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$$

is an isomorphism for each pair of objects A, B in $\mathcal{C}$.

Proof. Given a finite semisimple k-linear category $\mathcal{C}$, let $\{X_i\}_{i\in I}$ be a set of representatives of the simple objects of $\mathcal{C}$. Then, the conditions in Proposition I.2.16 are direct consequences of Schur's lemma and the fact that every object is a finite direct sum of simple objects. Conversely, given a k-linear category $\mathcal{C}$ fulfilling the conditions of Proposition I.2.16, then $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ decomposes into a finite direct sum of vectors $\eta_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, A) \otimes \mathrm{Hom}_{\mathcal{C}}(A, X_i)$. Composition

$$\mathrm{Hom}_{\mathcal{C}}(A, X_i) \otimes \mathrm{Hom}_{\mathcal{C}}(X_i, A) \to \mathrm{Hom}_{\mathcal{C}}(X_i, X_i) \cong k$$

induces a non-degenerate pairing with corresponding copairing $\eta_i : k \to \mathrm{Hom}_{\mathcal{C}}(A, X_i) \otimes \mathrm{Hom}_{\mathcal{C}}(X_i, A)$. In particular, $\mathrm{Hom}_{\mathcal{C}}(X_i, A)$ is a finite-dimensional vector space and the subalgebra $\mathrm{Hom}_{\mathcal{C}}(X_i, A) \otimes \mathrm{Hom}_{\mathcal{C}}(A, X_i)$ of $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is isomorphic to the endomorphism algebra $\mathrm{End}(\mathrm{Hom}_{\mathcal{C}}(X_i, A))$. Hence, $\mathrm{Hom}_{\mathcal{C}}(A, A) \cong \bigoplus_{i\in I} \mathrm{End}(\mathrm{Hom}_{\mathcal{C}}(X_i, A))$ is a finite-dimensional semisimple algebra. It follows from Proposition I.2.9a) that $\mathcal{C}$ is finite semisimple. □

Example I.2.17 (Examples of finite semisimple k-linear categories). The prototypical example of a finite semisimple k-linear category is the category of finite-dimensional k-vector spaces. More generally, the category of finite-dimensional representations of a finite-dimensional semisimple algebra is finite semisimple — and indeed, every finite semisimple category is of this form (see Proposition I.2.19).

From the perspective of Proposition I.2.16, an object A is simple if and only if it is isomorphic to one of the $\{X_i\}$. One may think of the objects $\{X_i\}$ as a chosen basis, 'orthonormal' with respect to the 'inner product' $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Vect}_k$. Indeed, Proposition I.2.16 shows that any semisimple k-linear category $\mathcal{C}$ with n isomorphism classes of simple objects is equivalent to the category Vect_k^n of n-tuples of vector spaces and n-tuples of linear maps. This finally allows us to relate finite semisimple categories to Kapranov and Voevodsky's 2-vector spaces.

Proposition I.2.18 (The equivalence between $2\mathrm{Vect}_k$ and $\mathrm{Mat}(\mathrm{Vect}_k)$)**.** *The* 2-*functor* $\mathrm{Mat}(\mathrm{Vect}_k) \to 2\mathrm{Vect}_k$ *which*

- *sends a natural number* n *to the finite semisimple category* Vect_k^n*;*
- *sends a* $m \times n$ *matrix of vector spaces* $\{V_{i,j}\}_{1\leq i\leq m, 1\leq j\leq n}$ *to the* k*-linear functor mapping an object* $(a_1, \ldots, a_n)$ *of* Vect_k^n *to the* Vect_k^m *object*

$$(\bigoplus_j V_{1,j} \otimes a_j, \ldots, \bigoplus_j V_{m,j} \otimes a_j);$$

- *sends a* $m \times n$ *matrix of linear maps* $\{V_{i,j} \xrightarrow{f_{i,j}} W_{i,j}\}$ *to the natural transformation* η *with components*

$$\eta_{(a_1,\ldots,a_n)} := (\bigoplus_j f_{1,j} \otimes \mathrm{id}_{a_j}, \ldots, \bigoplus_j f_{m,j} \otimes \mathrm{id}_{a_j})$$

is an equivalence.

Proof. Since every object is a finite direct sum of simple objects and since every morphism between simple objects is either zero or proportional to the identity, a k-linear functor between finite semisimple categories is completely determined by where it sends the simple objects. Similarly, given k-linear functors $F, G : \mathcal{C} \to \mathcal{D}$ between finite semisimple categories, a natural transformation $\eta : F \Rightarrow G$ is completely determined by its coefficients $\{\eta_{X_i} \in \mathrm{Hom}_{\mathcal{D}}(F(X_i), G(X_i))\}_i$ at the simple objects X_i of $\mathcal{C}$. Conversely, any family of morphisms $\{\eta_i : F(X_i) \to G(X_i)\}_i$ in $\mathcal{D}$, indexed by the simple objects X_i of $\mathcal{C}$, extends to a natural transformation $F \Rightarrow G$. This proves the proposition. □

Defining the (Deligne) *tensor product* $\mathcal{C} \boxtimes \mathcal{D}$ of finite semisimple k-linear categories $\mathcal{C}$ and $\mathcal{D}$ as the idempotent and direct sum completion $\big((\mathcal{C} \otimes \mathcal{D})^{\oplus}\big)^{\triangledown}$ of the k-linear category $\mathcal{C} \otimes \mathcal{D}$ with set of objects $\mathrm{ob}\, \mathcal{C} \times \mathrm{ob}\, \mathcal{D}$ and morphism spaces

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((c, d), (c', d')) := \mathrm{Hom}_{\mathcal{C}}(c, c') \otimes \mathrm{Hom}_{\mathcal{D}}(d, d')$$

equips $2\mathrm{Vect}_k$ with a symmetric monoidal structure for which the 2-functor $\mathrm{Mat}(\mathrm{Vect}_k) \to 2\mathrm{Vect}_k$ is a symmetric monoidal equivalence.

Semisimple categories as modules of semisimple algebras

Besides the *linear algebraic* ($\mathrm{Mat}(\mathrm{Vect}_k)$) and *categorical* ($2\mathrm{Vect}_k$) perspective on 2-vector spaces, we will now turn to a third *algebraic* perspective. Indeed, every finite semisimple category is the category of finite-dimensional modules of a finite-dimensional semisimple algebra. More generally, let $\mathrm{SSAlg}(\mathrm{Vect}_k)$ be the symmetric monoidal 2-category of semisimple finite-dimensional algebras, finite-dimensional bimodules and maps of bimodules.

Proposition I.2.19 (The equivalence between $2\mathrm{Vect}_k$ and $\mathrm{SSAlg}(\mathrm{Vect}_k)$)**.** *The functor* $\mathrm{Mod} : \mathrm{SSAlg}(\mathrm{Vect}_k) \to 2\mathrm{Vect}_k$ *which*

- *sends a finite-dimensional semisimple algebra to its category of finite-dimensional (left) modules;*
- *sends a finite-dimensional bimodule* ${}_BM_A$ *to the induced k-linear functor*

$$ {}_BM \otimes_A - : \mathrm{Mod}(A) \to \mathrm{Mod}(B); $$

- *sends a map of bimodules* $f : {}_BM_A \to {}_BN_A$ *to the induced natural transformation*

$$ {}_BM \otimes_A - \xrightarrow{f \otimes_A -} {}_BN \otimes_A - $$

is a symmetric monoidal equivalence.

Proof-Sketch. First note that every finite semisimple category is equivalent to Vect_k^n and hence is equivalent to the category of finite-dimensional modules of the finite-dimensional semisimple algebra $k \oplus \cdots \oplus k$. By Remark I.2.15, every k-linear functor from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(B)$ is a left adjoint and hence, by the Eilenberg-Watts theorem, is represented by a bimodule ${}_BM_A$. $\square$

I.3 Towards higher vector spaces

In the following section, we re-examine the definitions of Section I.2 with an eye towards categorifications and elaborate why the 2-category $2\mathrm{Vect}_k$ of finite semisimple k-linear categories is a good candidate for a 2-category of 'finite-dimensional 2-vector spaces'. This section is mostly expositional, and we only sketch definitions and proofs. In particular, its content is not strictly necessary for the mathematical development of later sections, but it conveys some of the key motivations and ideas reappearing throughout this thesis.

In this section, we denote the category of (possibly infinite-dimensional) k-vector spaces and linear maps by VECT_k.

I.3.1 Infinite-dimensional 2-vector spaces

A vector space is a set, equipped with the structure of an abelian group and an action of the base field k. Correspondingly, a 2-vector space should be a category with an appropriate notion of 'addition' for objects and morphisms and an action of k. This leads to two inherently different flavours of 2-vector space, in which k acts either on objects and morphisms, or only on morphisms of the 2-vector space. In other words, a 2-vector space could either be a category *internal*[10] to VECT_k with a vector space of objects and a vector space of morphisms, or it could be a category *enriched*[11] in VECT_k with a set of objects and vector spaces of morphisms. The internal perspective results in the notion of 2-vector space defined in [BC04]. By the Dold-Kan correspondence, the 2-category of such 2-vector spaces is equivalent to the 2-category of chain complexes of length 2 and is in particular a $(2,1)$-category in which every 2-morphism is invertible. For the purpose of defining 2-dimensional (and later higher dimensional) topological field theories, we will therefore focus on the enriched perspective and consider 2-vector spaces to be (amongst other things) categories enriched in VECT_k.

Profunctors as infinite-dimensional matrices

Since every vector space admits a basis, one can define a matrix calculus for arbitrary vector spaces. Explicitly, there is a category $\mathrm{MAT}(k)$ whose objects are sets and whose morphisms $X \to Y$ are $Y \times X$*-matrices*, that is, functions $M : Y \times X \to k$ such that for every $x \in X$, $M(y,x)$ is zero for all but finitely many $y \in Y$. The composite and Kronecker product of such infinite-dimensional matrices make $\mathrm{MAT}(k)$ into a symmetric monoidal category. The functor $\mathrm{MAT}(k) \to \mathrm{VECT}_k$ sending a set X to the vector space $k^X := \mathrm{Func}(X,k)$ of functions from X to k, and a $Y \times X$-matrix to the associated linear map $k^X \to k^Y$ is an equivalence.

[10] A category $\mathcal{C}$ *internal* to a category $\mathcal{A}$ with finite limits consist of $\mathcal{A}$-objects ob $\mathcal{C}$, mor $\mathcal{C}$, source and target $\mathcal{A}$-morphisms $s,t : \mathrm{mor}\ \mathcal{C} \to \mathrm{ob}\ \mathcal{C}$, an identity-assigning $\mathcal{A}$-morphism $e : \mathrm{ob}\ \mathcal{C} \to \mathrm{mor}\ \mathcal{C}$ and a composition $\mathcal{A}$-morphism $c : \mathrm{mor}\ \mathcal{C} \times_{\mathrm{ob}\ \mathcal{C}} \mathrm{mor}\ \mathcal{C} \to \mathrm{mor}\ \mathcal{C}$ fulfilling the obvious equations.

[11] A category $\mathcal{C}$ *enriched* in a monoidal category $\mathcal{V}$ consists of a set of objects ob $\mathcal{C}$ and for each pair of objects $A, B \in \mathrm{ob}\ \mathcal{C}$, a Hom-object $\mathrm{Hom}_{\mathcal{C}}(A,B)$ in $\mathcal{V}$ together with for each object $A \in \mathrm{ob}\ \mathcal{C}$ a $\mathcal{V}$-morphism $I \to \mathrm{Hom}_{\mathcal{C}}(A,A)$ and for each triple of objects $A, B, C \in \mathrm{ob}\ \mathcal{C}$ a composition $\mathcal{V}$-morphism $\mathrm{Hom}_{\mathcal{C}}(B,C) \otimes \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$ fulfilling the obvious equations — see [Kel82].

A natural categorification of $\mathrm{MAT}(k)$ is the symmetric monoidal 2-category Prof_k whose objects are VECT_k-enriched categories, whose 1-morphisms are (VECT_k-enriched) profunctors [Bén00] and whose 2-morphisms are natural transformations between profunctors. Recall that a *profunctor* $\mathcal{C} \nrightarrow \mathcal{D}$ (also known as *bimodule*, or *distributor*) is a VECT_k-enriched functor $\mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{VECT}_k$[12], and a natural transformation between profunctors $F, G : \mathcal{C} \nrightarrow \mathcal{D}$ is a natural transformations between the associated functors $\mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{VECT}_k$. The composite of profunctors $G : \mathcal{A} \nrightarrow \mathcal{B}$ and $F : \mathcal{B} \nrightarrow \mathcal{C}$ is defined analogously to the composite of matrices with the sum replaced by a coend (see [Lor15] for an introductory account of coends) in the cocomplete category VECT_k:

$$F \circ G(c, a) = \int^{b \in \mathcal{B}} F(c, b) \otimes G(b, a).$$

Remark I.3.1 (Cocomplete k-linear categories as 2-vector spaces)*.* Alternatively, 2-vector spaces are sometimes defined as cocomplete VECT_k-enriched category [BCJF15] — here the cocompleteness models 'linearity' at the level of objects. Whereas Prof_k naturally categorifies the category of matrices $\mathrm{MAT}(k)$, one may think of the category CCCat_k of cocomplete VECT_k-categories, cocontinuous functors and natural transformations as a natural categorification of VECT_k. And indeed, a natural categorification of the equivalence $\mathrm{MAT}(k) \to \mathrm{VECT}_k$ sending a set X to the vector space of functions $\mathrm{Func}(X, k)$ is the fully faithful 2-functor $\mathrm{Prof}_k \to \mathrm{CCCat}_k$ sending a VECT_k-enriched category $\mathcal{C}$ to the k-linear category of k-linear functors $\widehat{\mathcal{C}} := \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{VECT}_k)$ and sending a profunctor $\mathcal{C} \nrightarrow \mathcal{D}$ to the induced cocontinuous functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$. However, note that this 2-functor $\mathrm{Prof}_k \to \mathrm{CCCat}_k$ is not an equivalence. Whereas essential surjectivity of the functor $\mathrm{MAT}(k) \to \mathrm{VECT}_k$ arises from the fact that every k-module is free, it is not the case that every cocomplete VECT_k-category is of the form $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathrm{VECT}_k)$. Restricting to Prof_k as a reasonable 2-category of 2-vector spaces may be understood as restricting to the 'free' VECT_k-modules.

Example I.3.2 (Algebras as 2-vector spaces)*.* Every k-algebra A gives rise to a VECT_k-enriched category $\mathrm{B}A$ with one object and endomorphism algebra A. Conversely, every one-object VECT_k-enriched category is of this form. A profunctor between such one-object VECT_k-categories $\mathrm{B}A \to \mathrm{B}C$ is a $C - A$ bimodule. The 2-category $\mathrm{Alg}(\mathrm{VECT}_k)$ of k-algebras, bimodules and bimodule maps is therefore the full sub-2-category of Prof_k on the one-object VECT_k-enriched categories. In this sense, 2-linear

[12] The VECT_k-enriched tensor product $\mathcal{A} \otimes \mathcal{B}$ of VECT_k-enriched categories is defined as the VECT_k-enriched category whose object set is $\mathrm{ob}\, \mathcal{A} \times \mathrm{ob}\, \mathcal{B}$ and whose morphism vector spaces are the tensor products of the morphism spaces between the factors.

algebra subsumes classical algebra, and profunctors may be understood as 'horizontal' or 'many object' categorifications of bimodules.

The symmetric monoidal 2-category Prof_k can be understood as a first step in a sequence of categorifications of vector spaces; indeed, for $n = 0, 1, 2, 3, \ldots$ one may define the symmetric monoidal n-category of n-vector spaces as the '0-category' k ('of 0-vector spaces'), as the 1-category VECT_k ('of 1-vector spaces'), as the 2-category Prof_k ('of 2-vector spaces'), as the 3-category $2\mathrm{Prof}_k$ ('of 3-vector spaces') of Prof_k-enriched 2-categories, Prof_k-enriched 2-profunctors, pseudonatural transformations and modifications, and so on. The 'even higher Morita categories' of [JFS17] may be understood as a rigorous one-object/pointed variant of these profunctor-enriched higher profunctor categories.

I.3.2 Finite-dimensional 2-vector spaces

The category of finite-dimensional vector spaces Vect_k can be characterized categorically as the full subcategory of VECT_k on all *dualizable* vector spaces. Similarly, one may define 'finite-dimensional 2-vector spaces' to be *fully dualizable* objects in an appropriate symmetric monoidal 2-category of all 2-vector spaces.

Remark I.3.3 (Other characterizations of finite-dimensionality). There are other categorical characterizations singling out finite-dimensional vector spaces among all vector spaces. Our characterization via dualizability uses the *monoidal* structure of VECT_k. Alternative characterizations focus on the *categorical* structure of VECT_k. For example, the finite-dimensional vector spaces are the *compact* objects of VECT_k, that is, those objects V of VECT_k for which $\mathrm{Hom}_{\mathrm{VECT}_k}(V, -) : \mathrm{VECT}_k \to \mathrm{Set}$ preserves filtered colimits[13]. Given its relation to the cobordism hypothesis and its relevance to quantum information, we will focus on dualizability as our finiteness condition.

In the following, we recall that every fully dualizable object of Prof_k is equivalent to a finite semisimple k-linear category and that the sub-2-category $\mathrm{Prof}_k^{\mathrm{fd}}$ is equivalent to $2\mathrm{Vect}_k$. This observation is originally due to Tillmann [Til98], a version of which was reproduced in [BDSV15]. In the following, we give a further alternative proof working in the greater generality of $\mathcal{V}$-enriched categories, where $\mathcal{V}$ is an arbitrary nice symmetric monoidal category, and only specializing to $\mathcal{V} = \mathrm{VECT}_k$ at the

[13]Depending on the categorical context, there are various relations between compact and dualizable objects. For example, in a symmetric monoidal category $\mathcal{C}$ with compact tensor unit I, every dualizable object is compact: If V has a dual V^*, then $V^* \otimes - : \mathcal{C} \to \mathcal{C}$ has right adjoint $V \otimes -$ and hence preserves small colimits. By compactness of I, $\mathrm{Hom}_{\mathcal{C}}(V, -) \simeq \mathrm{Hom}_{\mathcal{C}}(I, V^* \otimes -) : \mathcal{C} \to \mathrm{Set}$ preserves filtered colimits.

last moment. This generality clarifies which behaviour of finite semisimple categories we should expect to re-appear in a categorified setting.

Dualizability

We recall the following standard terminology. For 1-morphisms $f : A \to B$ and $g : B \to A$ in a 2-category, we write $f \dashv g$, and say g is *right adjoint* to f or equivalently f is *left adjoint* to g, if there are 2-morphisms $\mathrm{ev} : f \circ g \Rightarrow \mathrm{id}_B,\ \mathrm{coev} : \mathrm{id}_A \Rightarrow g \circ f$ such that $(\mathrm{ev} \circ \mathrm{id}_f) \cdot (\mathrm{id}_f \circ \mathrm{coev}) = \mathrm{id}_f$ and $(\mathrm{id}_g \circ \mathrm{ev}) \cdot (\mathrm{coev} \circ \mathrm{id}_g) = \mathrm{id}_g$ (here, $\circ$ denotes horizontal, and $\cdot$ denotes vertical composition of 2-morphisms). In the following, we will often refer to these equations as the *cusp equations*. As a special case, an object X^* in a monoidal category $\mathcal{C}$ is *right dual* to an object X if it is a right adjoint in the 2-category $\mathrm{B}\mathcal{C}$ with one object and endomorphism category $\mathcal{C}$.

More generally, in an n-category[14] (or (∞, n)-category) $\mathcal{C}$ we say that a k-morphism (for $1 \leq k \leq n-1$) $\alpha : f \Rightarrow g$ between $(k-1)$-morphisms $f, g : A \to B$ (or objects f, g if $k = 1$) has a right adjoint $\beta : g \Rightarrow f$ if β is a right adjoint of α in the homotopy 2-category[15] of the $(n-k+1)$-category $\mathrm{Hom}_{\mathcal{C}}(A, B)$ (or in $\mathcal{C}$ if $k = 1$).

We follow [Lur09] and define a symmetric monoidal n-category to be *fully dualizable* if every object and every k-morphism for $1 \leq k \leq n-1$ in $\mathcal{C}$ has both a right and a left adjoint. For $\mathcal{C}$ a symmetric monoidal n-category, we denote the maximal fully dualizable subcategory by $\mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$. In the following, we think of $\mathcal{C}^{\mathrm{fd}}$ as the subcategory of 'finite' objects and 'finite' morphisms. We say that an object (or morphism) of $\mathcal{C}$ is *fully dualizable* if it is in (the essential image of) $\mathcal{C}^{\mathrm{fd}}$ (see [Lur09, Sec 2.3] for more details).

The following is a characterization of $(n-1)$-morphisms in $\mathcal{C}^{\mathrm{fd}}$.

Proposition I.3.4 ($(n-1)$-morphisms are in $\mathcal{C}^{\mathrm{fd}}$ iff they are left adjoint)**.** *Let $\mathcal{C}$ be a symmetric monoidal n-category and let f and g be $(n-2)$-morphisms in $\mathcal{C}^{\mathrm{fd}}$. Then, a $(n-1)$-morphism $\alpha : f \Rightarrow g$ is in $\mathcal{C}^{\mathrm{fd}}$ if and only if it is left adjoint.*

This is proven in Appendix A.

Remark I.3.5 (A characterization of $\mathcal{C}^{\mathrm{fd}}$). Proposition I.3.4 is a special case of a more general characterization of $\mathcal{C}^{\mathrm{fd}}$: For $0 \leq k < n-1$, we say that a k-morphism f (or object, for $k = 0$) is *fully left adjoint* if it has a right adjoint f^* for which

[14] We use the term 'n-category' informally and model independently; all categorical calculations employed here happen in dimensions $n = 2, 3$, all higher categorical statements are direct, model independent, consequences.

[15] The homotopy 2-category of an n-category $\mathcal{C}$ is the 2-category with the same objects and 1-morphisms of $\mathcal{C}$ and equivalence classes of 2-morphisms.

the evaluation and coevaluation $(k+1)$-morphisms are fully left adjoint, and that an $(n-1)$-morphism is fully left adjoint if it is left adjoint. For example, an object X in $\mathcal{C}$ is fully left adjoint if it has a right dual X^* such that evaluation and coevaluation have right adjoints, witnessed again by evaluation and coevaluations with right adjoints, and so on (cf. [Ara, Def 4.1.15]).

Following similar arguments as in the proof of Proposition I.3.4, it can be shown that an object X in a symmetric monoidal n-category is fully dualizable if it is fully left adjoint (a proof of this $k = 0$ case can be found in [Ara, Thm 4.1.19 and Cor 4.1.20]) and that for any $1 \leq k \leq (n-1)$ a k-morphism $\alpha : f \Rightarrow g$ between fully dualizable $(k-1)$-morphisms f and g is fully dualizable if and only if it is fully left adjoint.

Cauchy completeness and fully dualizable 2-vector spaces

Recall that a finite semisimple category is in particular additive and idempotent complete — this may be understood as a 'completeness condition'; it is not strictly necessary but it can be assumed without loss of generality and greatly simplifies the behaviour of semisimple categories and the morphisms between them.

Remark I.3.6 (Non-complete semisimple categories). Indeed, Turaev [Tur16] and Barrett-Westbury [BW96] use definitions of semisimplicity involving categories that are neither additive nor idempotent complete. Similarly, given a semisimple algebra A, the delooping $\mathrm{B}A$ may well deserve the name 'semisimple category'. Most generally, one may call a k-linear category $\mathcal{C}$ 'semisimple' if its additive and idempotent completion $(\mathcal{C}^{\oplus})^{\triangledown}$ is semisimple in the sense of Proposition I.2.9.

Additive and idempotent completeness are part of the more general phenomena of Cauchy completeness [Law73], a concept which plays a crucial role in the study of enriched profunctor categories.

To highlight the role of Cauchy completeness and its interaction with dualizability, we will establish some of the following results in the category of $\mathcal{V}$-enriched categories and $\mathcal{V}$-profunctors, where $\mathcal{V}$ is an arbitrary closed symmetric monoidal complete and cocomplete category. This has the advantage that results holding in this generality will fairly straight-forwardly categorify. And indeed, many of the propositions below will re-appear in a categorified form in Section 1.2.

Cauchy completeness in enriched profunctor categories

Let $\mathcal{V}$ be a closed symmetric monoidal complete and cocomplete category and let $\mathrm{Prof}_{\mathcal{V}}$ be the 2-category of $\mathcal{V}$-enriched categories, $\mathcal{V}$-profunctors and natural transformations. The tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{V}$-enriched categories $\mathcal{A}$, $\mathcal{B}$ is the $\mathcal{V}$-enriched category with objects $\mathrm{ob}\, \mathcal{A} \times \mathrm{ob}\, \mathcal{B}$ and hom-objects

$$\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}((a,b),(a',b')) = \mathrm{Hom}_{\mathcal{A}}(a,a') \otimes \mathrm{Hom}_{\mathcal{B}}(b,b').$$

This endows $\mathrm{Prof}_{\mathcal{V}}$ with the structure of a symmetric monoidal 2-category with tensor unit $\mathrm{B}I$, the $\mathcal{V}$-category with one object and endomorphism object the monoidal unit I of $\mathcal{V}$.

Every $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ gives rise to two profunctors $F_* : \mathcal{A} \nrightarrow \mathcal{B}$ and $F^* : \mathcal{B} \nrightarrow \mathcal{A}$ defined as

$$F_*(-,-) := \mathrm{Hom}_{\mathcal{B}}(-, F-) \qquad\qquad F^*(-,-) := \mathrm{Hom}_{\mathcal{B}}(F-,-).$$

As 1-morphisms in $\mathrm{Prof}_{\mathcal{V}}$, the profunctor F_* is left adjoint to the profunctor F^*. In particular, every object c of $\mathcal{C}$ gives rise to a left adjoint profunctor $c_* : \mathrm{B}I \nrightarrow \mathcal{C}$.

A $\mathcal{V}$-category is *Cauchy complete* if every left adjoint profunctor $\mathrm{B}I \nrightarrow \mathcal{C}$ is representable (that is, is isomorphic to c_* for some object c of $\mathcal{C}$). In fact, one can show that if $\mathcal{C}$ is Cauchy complete, then every left adjoint profunctor $\mathcal{B} \nrightarrow \mathcal{C}$ is representable. We define the *Cauchy completion* of $\mathcal{C}$ to be the category $\widehat{\mathcal{C}} := \mathrm{Prof}_{\mathcal{V}}(\mathrm{B}I, \mathcal{C})^L$ of left adjoint profunctors $\mathrm{B}I \nrightarrow \mathcal{C}$.

Example I.3.7 (The Cauchy completion of $\mathrm{B}I$ is $\mathcal{V}^{\mathrm{fd}}$). The monoidal category $\mathrm{Prof}_{\mathcal{V}}(\mathrm{B}I, \mathrm{B}I)$ is equivalent to $\mathcal{V}$. In particular, the Cauchy completion of the $\mathcal{V}$-category $\mathrm{B}I$ is the category $\mathcal{V}^{\mathrm{fd}}$ of dualizable objects in $\mathcal{V}$.

Since representable profunctors $\mathrm{B}I \nrightarrow \mathcal{C}$ are left adjoint, there is a fully faithful functor $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$. If $\mathcal{C}$ is already Cauchy complete, and hence every left adjoint profunctor $\mathrm{B}I \nrightarrow \mathcal{C}$ is representable, this inclusion $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ is an equivalence of $\mathcal{V}$-categories. In general, $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ is not an equivalence of $\mathcal{V}$-categories but nevertheless an equivalence in the 2-category $\mathrm{Prof}_{\mathcal{V}}$ (a 'pro-equivalence' or 'Morita equivalence'). Therefore, every $\mathcal{V}$-category is pro-equivalent to its Cauchy completion and the 2-category $\mathrm{Prof}_{\mathcal{V}}$ is equivalent to the full sub-2-category of Cauchy complete $\mathcal{V}$-categories.

Remark I.3.8 (Cauchy completeness and absolute colimits). Recall that an *absolute colimit* in a $\mathcal{V}$-category is a (weighted) colimit that is preserved by all $\mathcal{V}$-functors. It is shown in [Str83] that a $\mathcal{V}$-category is Cauchy complete if and only if it has all absolute colimits. In particular, the Cauchy completion of a $\mathcal{V}$-category is the completion under all absolute colimits.

Example I.3.9 (Cauchy completion of Mod(R)-enriched categories). If $\mathcal{V} = \mathrm{Mod}(R)$ is the category of modules over a commutative ring, then it can be shown (see for example [BDSV15, App A]) that all absolute colimits are generated by idempotent splittings and direct sums. In particular, a R-linear category is Cauchy complete if it is additive and idempotents split. The Cauchy completion of an R-linear category is precisely the additive and idempotent completion described in Construction I.2.6. In particular, the category of dualizable modules $\mathrm{Mod}(R)^{\mathrm{fd}}$ is the Cauchy completion of $\mathrm{B}R$ and hence the category of finitely generated, projective modules of R. If $R = k$ is a field, then $\mathrm{Vect}_k \cong \mathrm{VECT}_k^{\mathrm{fd}}$ is indeed the Cauchy completion of $\mathrm{B}k$.

Dualizability in enriched profunctor categories

Every $\mathcal{V}$-category $\mathcal{C}$ is 1-dualizable in $\mathrm{Prof}_{\mathcal{V}}$; its dual is the opposite $\mathcal{V}$-category $\mathcal{C}^{\mathrm{op}}$ with evaluation and coevaluation profunctor

$$\mathrm{ev}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \nrightarrow \mathrm{B}I \qquad \mathrm{coev}_{\mathcal{C}} : \mathrm{B}I \nrightarrow \mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}$$

both given by the functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$. The cusp equations are a direct consequence of the 'co-Yoneda lemma', the property that for functors $K : \mathcal{C} \to \mathcal{V}$ and $H : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ there are the following natural isomorphisms:

$$K \cong \int^{c \in \mathcal{C}} Kc \otimes \mathrm{Hom}_{\mathcal{C}}(-, c) \qquad H \cong \int^{c \in \mathcal{C}} Hc \otimes \mathrm{Hom}_{\mathcal{C}}(c, -)$$

The study of 2-dualizable $\mathcal{V}$-categories is much simplified if we play off the interaction between Cauchy completeness and dualizability. For example, the Hom-objects in a 2-dualizable $\mathcal{V}$-category are dualizable objects of $\mathcal{V}$.

Proposition I.3.10 (A $\mathcal{V}$-category is 1.5-dualizable iff it is enriched in $\mathcal{V}^{\mathrm{fd}}$)**.** *Let $\mathcal{C}$ be a $\mathcal{V}$-category. Then $\mathrm{ev}_{\mathcal{C}}$ has a right adjoint if and only if $\mathcal{C}$ is enriched in the full subcategory $\mathcal{V}^{\mathrm{fd}}$ of $\mathcal{V}$.*

Proof. Using that every left adjoint profunctor into a Cauchy complete category is representable, and that the Cauchy completion of $\mathrm{B}I$ is $\mathcal{V}^{\mathrm{fd}}$, it follows that $\mathrm{ev}_{\mathcal{C}}$ is left adjoint if and only if it is represented by a functor $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}^{\mathrm{fd}}$ which translates into $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$ factoring through $\mathcal{V}^{\mathrm{fd}}$. □

In particular, for every 2-dualizable $\mathcal{V}$-category there is an 'absolute Yoneda embedding' $\mathcal{C} \hookrightarrow \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}^{\mathrm{fd}})$ mapping an object c of $\mathcal{C}$ to the $\mathcal{V}^{\mathrm{fd}}$-valued presheaf $\mathrm{Hom}_{\mathcal{C}}(-, c)$ (cf. [BCJF15]).

Proposition I.3.11 (The absolute Yoneda embedding of 2-dualizable $\mathcal{V}$-categories is an equivalence)**.** *Let $\mathcal{C}$ be a 2-dualizable Cauchy complete $\mathcal{V}$-category. Then, the 'absolute Yoneda embedding' $\mathcal{C} \hookrightarrow \mathrm{Func}_{\mathcal{V}}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}^{\mathrm{fd}})$ is an equivalence.*

Proof. Since $\mathcal{C}$ is Cauchy complete, the embedding $\mathcal{C} \hookrightarrow \mathrm{Prof}_{\mathcal{V}}(\mathrm{B}I, \mathcal{C})^L$ into the category of left adjoint profunctors $\mathrm{B}I \nrightarrow \mathcal{C}$ is an equivalence. Moreover, since $\mathcal{V}^{\mathrm{fd}}$ is the Cauchy completion of $\mathrm{B}I$, it follows that the category $\mathrm{Prof}_{\mathcal{V}}(\mathcal{C}^{\mathrm{op}}, \mathrm{B}I)^L$ of left adjoint profunctors $\mathcal{C}^{\mathrm{op}} \nrightarrow \mathrm{B}I$ is equivalent to the $\mathcal{V}$-functor category $\mathrm{Func}_{\mathcal{V}}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}^{\mathrm{fd}})$. In these terms, the 'absolute Yoneda embedding' becomes the functor $\mathrm{Prof}_{\mathcal{V}}(\mathrm{B}I, \mathcal{C})^L \to \mathrm{Prof}_{\mathcal{V}}(\mathcal{C}^{\mathrm{op}}, \mathrm{B}I)^L$ given by

$$(F : \mathrm{B}I \nrightarrow \mathcal{C}) \mapsto (\mathrm{ev}_{\mathcal{C}} \circ (\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \otimes F) : \mathcal{C}^{\mathrm{op}} \nrightarrow \mathrm{B}I).$$

(For reasons of readability, we omit all unitor coherence equivalences.) This functor lands in the subcategory of left adjoint profunctors $\mathcal{C}^{\mathrm{op}} \nrightarrow \mathrm{B}I$ since $\mathrm{ev}_{\mathcal{C}}$ is left adjoint. Moreover, it is an equivalence with inverse functor $\mathrm{Prof}_{\mathcal{V}}(\mathcal{C}^{\mathrm{op}}, \mathrm{B}I)^L \to \mathrm{Prof}_{\mathcal{V}}(\mathrm{B}I, \mathcal{C})^L$ given by

$$(G : \mathcal{C}^{\mathrm{op}} \nrightarrow \mathrm{B}I) \mapsto ((G \otimes \mathrm{id}_{\mathcal{C}^{\mathrm{op}}}) \circ \mathrm{coev}_{\mathcal{C}} : \mathrm{B}I \nrightarrow \mathcal{C}).$$

This functor lands in the subcategory of left adjoint profunctors $\mathrm{B}I \nrightarrow \mathcal{C}$ since $\mathrm{coev}_{\mathcal{C}}$ is a left adjoint. □

As a direct consequence, we obtain the following useful characterization of representable profunctors into 2-dualizable Cauchy complete $\mathcal{V}$-categories.

Corollary I.3.12 (Profunctors between 2-dualizable Cauchy complete $\mathcal{V}$-categories representable iff $\mathcal{V}^{\mathrm{fd}}$-valued)**.** *Let $\mathcal{D}$ be a 2-dualizable Cauchy complete $\mathcal{V}$-category. Then, a profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ is representable if and only if it is valued in $\mathcal{V}^{\mathrm{fd}}$.*

Proof. Recall that every 2-dualizable $\mathcal{V}$-category is enriched in $\mathcal{V}^{\mathrm{fd}}$. Therefore, every representable profunctor into $\mathcal{D}$ is valued in $\mathcal{V}^{\mathrm{fd}}$. Conversely, the data of a profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ is a functor $\overline{P} : \mathcal{C} \to \mathrm{Func}_{\mathcal{V}}(\mathcal{D}^{\mathrm{op}}, \mathcal{V})$. If P is valued in $\mathcal{V}^{\mathrm{fd}}$, then this functor factors through $\mathrm{Func}_{\mathcal{V}}(\mathcal{D}^{\mathrm{op}}, \mathcal{V}^{\mathrm{fd}})$. But by Proposition I.3.11, the embedding $\mathcal{D} \hookrightarrow \mathrm{Func}_{\mathcal{V}}(\mathcal{D}^{\mathrm{op}}, \mathcal{V}^{\mathrm{fd}})$ is an equivalence. Therefore, $\overline{P}$ factors as a functor $\mathcal{C} \to \mathcal{D}$ followed by the Yoneda embedding $\mathcal{D} \hookrightarrow \mathrm{Func}_{\mathcal{V}}(\mathcal{D}^{\mathrm{op}}, \mathcal{V})$ — equivalently, P is representable. □

Lastly, we make the following observation on morphisms between 2-dualizable $\mathcal{V}$-categories.

Proposition I.3.13 (Functors between 2-dualizable Cauchy complete categories have right and left adjoints)**.** *Every $\mathcal{V}$-functor between Cauchy complete 2-dualizable $\mathcal{V}$-categories has right and left adjoint $\mathcal{V}$-functors.*

Proof. A $\mathcal{V}$-functor $f : \mathcal{C} \to \mathcal{D}$ gives rise to a representable profunctor F and hence to a right adjoint profunctor $F^* : \mathcal{D} \nrightarrow \mathcal{C}$. This profunctor F^* has left adjoint F and hence, by Proposition I.3.4, a right adjoint $F^{**} : \mathcal{C} \nrightarrow \mathcal{D}$. Thus, by the definition of Cauchy completeness, F^* itself is representable giving rise to a $\mathcal{V}$-functor $f^* : \mathcal{D} \to \mathcal{C}$ which is right adjoint to f. □

Dualizability in Prof_k

Returning to the case $\mathcal{V} = \mathrm{VECT}_k$ for k an algebraically closed field, we note that Proposition I.3.10, I.3.11 and Corollary I.3.13 suffice to completely characterize 2-dualizable k-linear categories.

Proposition I.3.14 (2-dualizability and Cauchy completeness is equivalent to finite semisimplicity)**.** *A k-linear category $\mathcal{C}$ is Cauchy complete and 2-dualizable if and only if it is finite semisimple.*

Proof. A direct computation shows that every finite semisimple k-linear category is 2-dualizable. Conversely, it follows from Proposition I.3.10 that every 2-dualizable VECT_k-category is enriched in Vect_k, that is, has finite-dimensional Hom-spaces. Moreover, by Proposition I.3.11, $\mathcal{C}$ is equivalent to the category of k-linear functors $\mathrm{Func}_k(\mathcal{C}^{\mathrm{op}}, \mathrm{Vect}_k)$ and is therefore abelian. Since both $\mathcal{C}$ and Vect_k are 2-dualizable and Cauchy complete, it follows from Proposition I.3.13 that every functor $F : \mathcal{C} \to \mathrm{Vect}_k$ has a right adjoint functor $\mathrm{Vect}_k \to \mathcal{C}$. In particular, for every object X of $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Vect}_k$ is left adjoint and hence preserves finite colimits. Therefore, every object of $\mathcal{C}$ is projective. It is shown in [BDSV15, Prop A.31] that an abelian category with finite-dimensional Hom-spaces over an algebraically closed field in which every object is projective is semisimple. We now show that $\mathcal{C}$ has a finite number of simple objects. Indeed, let J be a set of isomorphism classes of simple objects. Note that a k-linear functor out of a semisimple category is completely determined by its action on simple objects. In particular, there is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Vect}_k$ which maps every simple object isomorphic to an object in J to k, and all other simple objects to 0. By assumption, this functor F is representable. Hence, there exists an object Y in $\mathcal{C}$ such that $F \cong \mathrm{Hom}_{\mathcal{C}}(-, Y)$. But since every object in $\mathcal{C}$ is a finite direct sum of simple objects, such an object Y can only exist if the set J is finite. □

In conclusion, the fully dualizable subcategory $\mathrm{Prof}_k^{\mathrm{fd}} \to \mathrm{Prof}_k$ is (up to equivalence of subcategories of Prof_k) equivalent to $2\mathrm{Vect}_k$.

Proposition I.3.15 ($2\mathrm{Vect}_k$ is equivalent to Prof_k)**.** *Up to equivalence of subcategories,* $\mathrm{Prof}_k^{\mathrm{fd}}$ *is the symmetric monoidal* 2*-category of finite semisimple k-linear categories, k-linear functors and natural transformations.*

Proof. As seen above, every object of Prof_k is equivalent to a Cauchy complete object, and a Cauchy complete object is fully dualizable if and only if it is finite semisimple. By Proposition I.3.4, a 1-morphism between fully dualizable objects is in $\mathrm{Prof}_k^{\mathrm{fd}}$ if and only if it has a right adjoint, or equivalently if it is representable. □

Note that the symmetric monoidal 2-category $\mathrm{SSAlg}(\mathrm{Vect}_k)$ of semisimple finite-dimensional k-algebras is equivalent to $2\mathrm{Vect}_k$ (via the symmetric monoidal 2-functor from Proposition I.2.19) as a subcategory of Prof_k, and hence is another model for $\mathrm{Prof}_k^{\mathrm{fd}}$.

However, $2\mathrm{Vect}_k$ has a clear advantage over $\mathrm{SSAlg}(\mathrm{Vect}_k)$: By Proposition I.3.4, the 1-morphisms in $\mathrm{Prof}_k^{\mathrm{fd}}$ between Cauchy complete fully dualizable categories are functors rather than profunctors. In particular, the 1-morphisms between finite semisimple (and in particular Cauchy complete) categories are functors, whereas the 1-morphisms in the equivalent subcategory SSAlg are bimodules. This is important for our later categorification — whereas (weak) monoid objects in $\mathrm{Prof}_k^{\mathrm{fd}}$ are in general 'pro-monoidal' k-linear categories (that is, k-linear categories with a tensor product profunctor[16] $\mathcal{C} \otimes \mathcal{C} \nrightarrow \mathcal{C}$), (weak) monoid objects on Cauchy complete categories are ordinary k-linear monoidal categories and therefore much easier to study and manipulate.

[16] For example, pro-monoidal k-linear categories with a single object have appeared before under the name *sesquialgebra* [TWZ07]; a sesquialgebra is a monoid object in the category of k-algebras, bimodules and bimodule maps.

Chapter 1

Higher linear algebra

In this chaper, based on the first two sections of [DR18], we define semisimple 2*-categories, fusion* 2*-categories and spherical* 2*-categories, show that every semisimple* 2*-category is the* 2*-category of finite semisimple module categories of a multifusion category, and give examples.*

1.1 Introduction

One of the early successes of quantum topology was Turaev and Viro's construction of a 3-manifold invariant based on the representation theory of quantum $\mathfrak{sl}_2$ [TV92], and Barrett and Westbury's generalization of this construction to an invariant based on any spherical fusion category [BW96]. These invariants are defined by a 'state sum', a weighted average of numbers associated to fusion-categorical labelings of a triangulated manifold. From a more recent cobordism-hypothesis perspective, associated to a fusion category there is a local 3-dimensional field theory [DSPS17b], and the classical Turaev–Viro–Barrett–Westbury invariant is obtained by restricting to closed 3-manifolds. As the cobordism-hypothesis is non-constructive and invariants produced from it are not in general directly computable, explicit state sum constructions of invariants remain informative and useful, both mathematically and for their role in physical lattice field theories [LW05] and consequent relevance for condensed matter physics and topological quantum computation [KKR10].

In contrast to the situation in dimension 3, and despite a wealth of important field-theoretically-inspired 4-manifold invariants [Don83, Wit94, OS06, KM07], constructions of true 4-dimensional topological field theory invariants have been sparse and sporadic. The earliest was the Crane–Yetter 4-manifold invariant based on the modular data of the representations of quantum $\mathfrak{sl}_2$ [CY93], and its Crane–Yetter–Kauffman generalization using the data of any semisimple ribbon category [CKY97].

Around the same time, given the data of a finite 2-group, Yetter defined a state sum (in any dimension in fact) generalizing the Dijkgraaf–Witten invariant associated to a finite group [Yet93]; this was later generalized by Faria Martins–Porter to include a twisting cocycle [FMP07]. Mackaay attempted to systematize the data needed for a 4-dimensional state sum in a framework of certain monoidal 2-categories with trivial endomorphism categories, but the resulting notion did not encompass either the Crane–Yetter–Kauffman invariants or the Yetter–Dijkgraaf–Witten invariants and appears to only accommodate a twisted version of classical Dijkgraaf–Witten theory [Mac99]. More recently, given the data of a crossed-braided spherical fusion category, Cui constructed a state sum invariant of 4-manifolds that subsumes both the Crane–Yetter–Kauffman invariant and the Yetter–Dijkgraaf–Witten invariant, but does not incorporate either the twisted Yetter–Dijkgraaf–Witten case or hypothetical other instances of the Mackaay invariant [Cui16].

The quantum topology community has long expected that all these constructions should be expressible in a unified framework that associates a 4-dimensional field theory to some sort of 'spherical fusion 2-category' (analogous to the Barrett–Westbury framework for 3-dimensional theories from spherical fusion 1-categories), but the appropriate notion of fusion 2-category and of sphericality has remained unclear. In his recent survey article, *Beyond Anyons*, Wang notes, "One problem is to formulate a higher category theory that underlies all these theories, and study their application in [3+1]-dimensional topological phases of matter" [Wan18]. In this chapter, we completely address the relevant higher category theory by introducing a general purpose notion of fusion 2-category, based on a new notion of semisimple 2-category, and providing an appropriate corresponding sphericality condition. We define, given the data of a spherical fusion 2-category, a piecewise-linear 4-manifold invariant that specializes (for appropriate choices of the fusion 2-category) to all the aforementioned invariants, and therefore provides a unified framework for 4-dimensional semisimple topological field theory.

Semisimple 2-categories

We restrict attention to k-linear categories and 2-categories, where k is an algebraically closed field of characteristic zero.

For a monoidal linear 1-category to produce a full-fledged 3-dimensional topological field theory, it must be fully-dualizable in some 3-category of monoidal linear 1-categories. A convenient such 3-category is the 3-category of finite tensor categories; a finite tensor category is a category equivalent to the category of finite-dimensional modules over a finite-dimensional algebra, equipped with a monoidal structure such that every object has left and right duals. Any fully-dualizable finite tensor category must be semisimple [DSPS17b]; its underlying linear 1-category is therefore the category of modules for a finite-dimensional semisimple algebra. It therefore stands to reason that in building a categorical framework for 4-dimensional topological field theory, we should look for a notion of monoidal semisimple 2-category, and that we might expect the underlying semisimple linear 2-category to be the 2-category of modules for a finite semisimple tensor category, i.e. a "multifusion category".

Definition 1. *A* semisimple 2-category *is a locally semisimple 2-category, admitting adjoints for 1-morphisms, that is additive and idempotent complete.*

This is Definition 1.2.51 in the main text. Here 'locally semisimple' means that the Hom categories are semisimple linear categories, and *idempotent complete* is shorthand for the property that every separable monad admits a separable splitting (see Section 1.2.3 and Appendix B for extensive discussion of this condition).[1,2,3] The definition of semisimple 2-category does not explicitly demand the existence of any sort of additive decomposition of objects; nevertheless the local semisimplicity and the idempotent completeness conditions combine to ensure that, as one might hope given the name, in a semisimple 2-category every object decomposes as a finite direct sum of simple objects. A semisimple 2-category is called *finite* if it is locally finite semisimple and it has finitely many equivalence classes of simple objects. Finite semisimple 1-categories are a categorification of finite-dimensional vector spaces, and so are often referred to as (finite-dimensional) '2-vector spaces'. Similarly, finite semisimple 2-categories are a categorification of finite semisimple 1-categories, and so may be thought of as (finite-dimensional) '3-vector spaces'.

The above definition of semisimple 2-category does indeed have the desired relation to modules for multifusion categories.

[1] A separable splitting of a separable monad $E : B \to B$ in a 2-category $\mathcal{C}$ is an adjunction $\iota \vdash \rho$ with right invertible counit, together with an isomorphism of algebras $E \cong \iota \circ \rho$. The condition that a separable monad in a locally idempotent complete 2-category admits a separable splitting is equivalent to the condition that it admits a universal left module, that is an 'Eilenberg–Moore object', and also equivalent to the condition that it admits a universal right module, that is a 'Kleisli object'.

[2] Morrison and Walker have sketched an elegant theory of completeness for n-categories. We speculate that the notion of completeness we describe for 2-categories is, informally speaking, related to their notion of completeness in the same way that framed 2-dimensional field theory is related to oriented 2-dimensional field theory.

[3] In the context of a modular tensor category representing excitations of a (2+1)-dimensional topological phase of matter, the splitting of a commutative separable algebra can be thought of as 'anyon condensation' [Kon14].

Theorem 2. *The 2-category of finite semisimple module categories of a multifusion category is a finite semisimple 2-category.*

Theorem 3. *Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.*

These appear as Theorems 1.2.58 and 1.2.59.

Since finite semisimple 2-categories are exactly the 2-categories of modules for multifusion categories, one might wonder what utility finite semisimple 2-categories provide over the existing theory of multifusion categories. The crucial advantage becomes apparent when we add a monoidal structure to these 2-categories. In general, an additional monoidal structure on a multifusion category C would have to be encoded, somewhat intractably, as a $(\mathrm{C} \boxtimes \mathrm{C})$–C-bimodule together with further associativity structures and conditions. By contrast, a monoidal structure on a semisimple 2-category $\mathcal{C}$ will be describable *functorially*, that is simply as an ordinary 2-functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. A priori, a bimodule between tensor categories induces a 2-profunctor between the associated 2-categories of modules. (A finite semisimple 2-profunctor $\mathcal{C} \nrightarrow \mathcal{D}$ is a bilinear 2-functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to 2\mathrm{Vect}$, where 2Vect is the 2-category of '2-vector spaces', that is finite semisimple 1-categories.) However, it turns out that, thanks to the idempotent completeness of semisimple 2-categories, any 2-profunctor between semisimple 2-categories is a 2-functor.

Theorem 4. *Every finite semisimple 2-profunctor between finite semisimple 2-categories is equivalent to a 2-functor.*

This result appears as Corollary 1.2.62.

As an elementary example of a semisimple 2-category, consider the 2-category $\mathrm{Mod}(\mathrm{Vect}(\mathbb{Z}_2))$ of finite semisimple module categories for the category $\mathrm{Vect}(\mathbb{Z}_2)$ of $\mathbb{Z}_2$-graded vector spaces. This 2-category has two simple objects, namely the modules Vect and $\mathrm{Vect}(\mathbb{Z}_2)$; both these objects have endomorphism categories $\mathrm{Vect}(\mathbb{Z}_2)$, and the Hom category either direction between the two objects is Vect. The 2-category may therefore be drawn as follows:

This and other examples are described in Section 1.2.4. Note well that, as illustrated here and quite unlike the situation for semisimple 1-categories, in a semisimple 2-category there can be nontrivial morphisms between inequivalent simple objects.

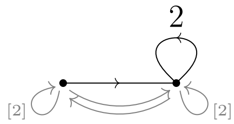
Fusion 2-categories

Because transformations between semisimple 2-categories can be encoded functorially, a monoidal structure on a semisimple 2-category can be encoded as an ordinary '2-functorial' monoidal 2-category.

Definition 5. *A* fusion 2-category *is a finite semisimple monoidal 2-category that has left and right duals for objects and a simple monoidal unit.*

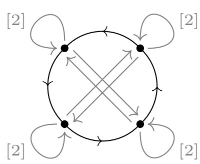
This appears as Definition 1.3.6. Examples of fusion 2-categories include 2-representations of a 2-group, 2-group-graded 2-vector spaces, modules for a braided fusion category, semisimple completions of crossed-braided fusion categories, and twisted versions thereof—see Section 1.3.1.

As a simple example, consider the fusion 2-category of modules of the symmetric fusion category $\mathrm{Vect}(\mathbb{Z}_2)$. There are two simple objects, the identity I (namely the module category $\mathrm{Vect}(\mathbb{Z}_2)$) and an object X (namely the module category Vect), with nontrivial fusion rule $X \square X \simeq X \boxplus X$. We may therefore depict this fusion 2-category as follows:



Here the black directed edges represent multiplication by the object X and the label indicates the multiplicity. The gray edges record the morphism categories: an unlabeled edge indicates a rank 1 category, that is Vect, and the label [2] indicates a rank 2 category, that is $\mathrm{Vect} \boxplus \mathrm{Vect}$. (In this case those rank 2 endomorphism fusion categories are $\mathrm{Vect}(\mathbb{Z}_2)$.) This sort of fusion graph, where an object has no fusion product containing an identity factor, is a completely new phenomenon in fusion 2-categories—in a fusion 1-category, the product of an object and its dual always has an identity summand, but in a fusion 2-category, this need not happen thanks to the existence of nontrivial nonequivalence morphisms between simple objects.

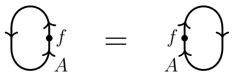
As another example, the fusion 2-category obtained as the semisimple completion of a $\mathbb{Z}_4$-crossed-braided structure on the Ising fusion category has the following structure:



The fusion structure of the simple objects is just the cyclic group $\mathbb{Z}_4$; the black edges denote multiplication by a generating object. Again the gray edges record the rank of the morphism categories. (In this case, the rank 2 endomorphism fusion categories are all $\mathrm{Vect}(\mathbb{Z}_2)$.) Note, again quite unlike what can happen in the context of fusion 1-categories, that there are only two connected components of simples in this fusion 2-category, despite the underlying order four fusion group of simple objects.

We expect that fusion 2-categories are fully-dualizable objects of an appropriate 4-category of tensor 2-categories, and therefore provide framed 4-dimensional local field theories, but to obtain oriented field theories and therefore oriented 4-manifold invariants, we need additional structure and properties on the fusion 2-categories. Recall that a *planar pivotal 2-category* is a 2-category with a functorial involutive coherent choice of adjoint for each 1-morphism. A *monoidal planar pivotal 2-category* is a planar pivotal 2-category with a compatible monoidal structure. A *pivotal 2-category* is a monoidal planar pivotal 2-category with a compatible involutive coherent choice of dual for each object. These definitions are given in detail in Section 1.3.2. (Note that what we call a 'pivotal 2-category' is presumably equivalent to what Barrett–Meusburger–Schaumann call a 'spatial Gray monoid' [BMS12].)

A pivotal 2-category will still not provide an oriented 4-dimensional field theory, just as a pivotal 1-category does not provide an oriented 3-dimensional field theory—what is needed is a sphericality condition. Recall that a pivotal 1-category is called 'spherical' when the left and right 'circular' traces of any 1-endomorphism $f : A \to A$ agree:



Analogously, for a 2-endomorphism $\alpha : 1_A \Rightarrow 1_A$ in a pivotal 2-category, there is a 'front' and a 'back 2-spherical trace' construction, which may be depicted graphically as follows:

Definition 6. *A* spherical 2-category *is a pivotal 2-category such that the front and back 2-spherical traces agree.*

This appears as Definition 1.3.42 in the main text. Examples of spherical fusion 2-categories include 2-representations of a 2-group, 2-group-graded 2-vector spaces, modules for a ribbon fusion category, and the semisimple completion of a crossed-braided spherical fusion category.[4] Spherical fusion 2-categories provide the desired data for constructing a state sum invariant of 4-manifolds.

Notation and conventions

Except where otherwise noted, we assume the field k to be algebraically closed, and starting from Section 1.2, to be of characteristic zero. Let Vect_k denote the 1-category of finite-dimensional k-vector spaces and linear maps. In the following, a *k-linear 1-category* is a Vect_k-enriched 1-category and a *k-linear 1-functor* is a Vect_k-enriched functor.

By a 2-category, we mean a weak 2-category, though throughout we will suppress unitors and associators from our notation. We define a *linear 2-category* to be a Vect_k-enriched 2-category, that is, a 2-category $\mathcal{C}$ whose 2-morphism sets are k-vector spaces such that horizontal and vertical composition of 2-morphisms are k-bilinear operations. A *linear 2-functor* will be a 2-functor that is locally k-linear.

Similar to [DSPS17b, Sec 2.1], we will distinguish between the *geometric* and the *functorial* direction of composition. If A, B and C are i-morphisms in an n-category, and $f : A \to B$ and $g : B \to C$ are $(i+1)$-morphisms, we denote their composition 'f followed by g' in functorial notation as $g \circ f$ and in geometric notation as $f \otimes_B g := g \circ f$. The object B is often left implicit and $f \otimes_B g$ is simply denoted by $f \otimes g$. For monoidal 2-categories, we will reserve the following symbols for the various composition conventions:

	object	1-morphism	2-morphism
functorial	$\square$	$\circ$	$\cdot$
geometric	$\boxtimes$	$\otimes$	$\times$

For uniformity, we will always use the functorial composition. Note that this is not the most naive categorification of the typical convention for tensor categories, which uses functorial composition for morphisms but geometric composition for objects.

We adopt the following notation:

- Given an object A in a 2-category, we denote its identity 1-morphism by $\mathbb{1}_A$; for a 1-morphism $f : A \to B$, we denote its identity 2-morphism by 1_f.

[4]Note that Mackaay [Mac99] used the term 'spherical fusion 2-category' to refer to what we might call 'circo-spherical endotrivial fusion 2-categories'—the 'sphericality' condition there is the equivalence of two categorical circular traces, not two 2-spherical traces, and the endomorphism fusion category of every indecomposable object is the trivial fusion category Vect. A 'circo-spherical endotrivial fusion 2-category' is a spherical fusion 2-category in our sense, but none of the aforementioned examples of spherical fusion 2-categories satisfy Mackaay's much more restrictive conditions.

- Given 1-morphisms $f, f' : A \to B$ and $g, g' : B \to C$ and 2-morphisms $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$, we often abbreviate the composites $1_g \circ \alpha$ and $\beta \circ 1_f$ by $g \circ \alpha$ and $\beta \circ f$, respectively.

- We write $A \simeq B$ for equivalent objects in a 2-category, and $f \cong g$ for isomorphic 1-morphisms.

- Recall that for 1-morphisms $f : A \to B$ and $g : B \to A$ in a 2-category, we write $f \dashv g$, and say g is right adjoint to f or equivalently f is left adjoint to g, if there are 2-morphisms $\epsilon : f \circ g \Rightarrow 1_B$, $\eta : 1_A \Rightarrow g \circ f$ such that $(\epsilon \circ 1_f) \cdot (1_f \circ \eta) = 1_f$ and $(1_g \circ \epsilon) \cdot (\eta \circ 1_g) = 1_g$. In the following, we will often refer to these equations as the *cusp equations*.

Outline

Section 1.2 concerns linear 2-categories; Section 1.2.1 discusses zero objects and direct sums in 2-categories and defines additive 2-categories. Section 1.2.2 defines presemisimple 2-categories, analyzes the direct sum decomposition of objects in such 2-categories, and defines the dimension of a presemisimple 2-category. Section 1.2.3 motivates categorified notions of idempotent and idempotent splitting, and defines idempotent complete 2-categories and an idempotent completion operation on 2-categories. Appendix B is a companion to Section 1.2.3, providing further technical details and many proofs concerning idempotent completion that are omitted from the main text. Section 1.2.4 defines semisimple 2-categories, proves the crucial result that semisimple 2-categories are exactly 2-categories of modules of multifusion categories, and shows that 2-profunctors between semisimple 2-categories are 2-functors; it then gives a variety of explicit constructions and examples of semisimple 2-categories.

Section 1.3 investigates monoidal structures on linear 2-categories. Section 1.3.1 gives a precise definition of a monoidal 2-category, defines prefusion and fusion 2-categories, and describes the graphical calculus of surfaces with defects that encodes 2-morphisms in a fusion 2-category; it then gives an extensive list of constructions and examples of fusion 2-categories. Section 1.3.2 recalls the notion and graphical calculus of planar pivotal 2-categories, and describes further the notion and graphical calculus for pivotal 2-categories; then it defines 2-spherical traces in pivotal 2-categories. Section 1.3.3 uses this notion of 2-spherical trace to define the notion of spherical 2-categories, mentions examples of spherical 2-categories, and then discusses dimensions of objects and 1-morphisms in spherical 2-categories.

1.2 On 3-vector spaces

We introduce a notion of *finite semisimple* 2*-category* — modelled after the phenomena observed in Section I.3 — as a potential candidate definition for 'finite-dimensional 3-vector spaces', categorifying the definitions and results of Section I.2.

From now on, and except where otherwise noted, we assume the field k to be algebraically closed and of characteristic zero.

1.2.1 Additive 2-categories

Zero objects. Recall that a *zero object* in a linear 1-category is an object 0 such that $\mathrm{Hom}(0,0)$ is the zero vector space. Note that an object whose identity morphism is the zero vector is necessarily a zero object. A zero object is unique up to equivalence, and is preserved by all functors. A 1-category with a zero object is called *pointed.* A *zero* 1*-morphism* in a linear 2-category is a 1-morphism $0_{A,B} : A \to B$ that is a zero object in the linear 1-category $\mathrm{Hom}(A, B)$. A linear 2-category is *locally pointed* if all its 1-morphism categories are pointed, that is have zero objects.

Definition 1.2.1 (Zero object in a 2-category)**.** A *zero object* in a locally pointed linear 2-category $\mathcal{C}$ is an object 0 such that $\mathrm{Hom}_{\mathcal{C}}(0,0)$ is the terminal 1-category.

Note that an object 0 in a locally pointed linear 2-category is a zero object if and only if its identity 1-morphism $\mathbb{1}_0 : 0 \to 0$ is a zero 1-morphism. Also, observe that a zero object in a locally pointed linear 2-category is uniquely determined up to equivalence, and is preserved by all linear 2-functors. We say that a linear 2-category is *pointed* if it is locally pointed and has a zero object.

Direct sums. Recall that a linear 1-category is called *additive* if it has a zero object and (pairwise) direct sums (see Definition I.2.2). A *locally additive* 2-category is a linear 2-category whose Hom-categories $\mathrm{Hom}(A, B)$ are additive for all objects A and B.

Definition 1.2.2 (Direct sum in a 2-category)**.** A *direct sum* of two objects A_1 and A_2 in a locally additive 2-category $\mathcal{C}$ is an object $A_1 \boxplus A_2$ together with inclusion and projection 1-morphisms $\iota_i : A_i \to A_1 \boxplus A_2$ and $\rho_i : A_1 \boxplus A_2 \to A_i$ for $i = 1, 2$, satisfying the following conditions:

- $\rho_i \circ \iota_i$ is isomorphic to $\mathbb{1}_{A_i}$ for $i = 1, 2$;

- $\rho_2 \circ \iota_1 \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2)$ and $\rho_1 \circ \iota_2 \in \mathrm{Hom}_{\mathcal{C}}(A_2, A_1)$ are zero objects;

- $1_{A_1 \boxplus A_2} \in \mathrm{Hom}_{\mathcal{C}}(A_1 \boxplus A_2, A_1 \boxplus A_2)$ is a direct sum of $\iota_1 \circ \rho_1$ and $\iota_2 \circ \rho_2$.

Observe that direct sums in a locally additive 2-category are uniquely determined up to equivalence and are preserved by all linear 2-functors.

Proposition 1.2.3 (Projection and inclusion are adjoint)**.** *Let $A_1 \boxplus A_2$ be a direct sum with inclusion and projection* 1*-morphisms $\iota_i : A_i \to A_1 \boxplus A_2$ and $\rho_i : A_1 \boxplus A_2 \to A_i$ for $i = 1, 2$. Then ρ_i is both a left and right adjoint of ι_i.*

Proof. The proof is analogous to the proof that every equivalence in a 2-category can be promoted to an adjoint equivalence. We show that ι_i is right adjoint to ρ_i; the proof for left-adjointness is similar. Let $\widetilde{\epsilon}_i : \rho_i \circ \iota_i \Rightarrow 1_{A_i}$, $\eta_i : 1_{A_1 \boxplus A_2} \cong \iota_1 \circ \rho_1 \oplus \iota_2 \circ \rho_2 \Rightarrow \iota_i \circ \rho_i$, and $\overline{\eta}_i : \iota_i \circ \rho_i \Rightarrow \iota_1 \circ \rho_1 \oplus \iota_2 \circ \rho_2 \cong 1_{A_1 \boxplus A_2}$ be the 2-morphisms provided by the definition of the direct sum. Define

$$\epsilon_i := \widetilde{\epsilon}_i \cdot (1_{\rho_i} \circ \overline{\eta}_i \circ 1_{\iota_i}) \cdot \left(1_{\rho_i \circ \iota_i} \circ \widetilde{\epsilon}_i^{-1}\right) : \rho_i \circ \iota_i \Rightarrow 1_{A_i}$$

Now observe that η_i and ϵ_i are the unit and counit of an adjunction $\rho_i \dashv \iota_i$. Checking that $(1_{\iota_i} \circ \epsilon_i) \cdot (\eta_i \circ 1_{\iota_i}) = 1_{\iota_i}$ is straightforward. The equation $(\epsilon_i \circ 1_{\rho_i}) \cdot (1_{\rho_i} \circ \eta_i) = 1_{\rho_i}$ follows (by the same calculation that shows that after appropriately modifying the 2-morphisms of an equivalence, one obtains an adjoint equivalence) using the fact that $\sum_j \overline{\eta}_j \cdot \eta_j = 1_{1_{A_1 \boxplus A_2}}$, hence that $1_{\rho_i} = 1_{\rho_i} \circ (\overline{\eta}_i \cdot \eta_i)$ and in particular that $1_{\rho_i} \circ \eta_i$ and $1_{\rho_i} \circ \overline{\eta}_i$ are inverse. □

Remark 1.2.4 (Direct sums and zero objects in 2-categories are preserved by all 2-functors). In a linear 2-category, direct sums and zero objects are 'equational' constructions, in that they may be defined in terms of the existence of certain morphisms satisfying certain equations. It follows that they are preserved by all linear 2-functors. Recall that an absolute (2-)colimit is a colimit preserved by all linear (2-)functors. Both direct sums and zero objects may be expressed as universal constructions, in particular as colimits, and are therefore absolute colimits.

Additivity and additive completion.

Definition 1.2.5 (Additive 2-category)**.** A linear 2-category is *additive* if it is locally additive, has a zero object, and has direct sums.

Construction 1.2.6 (Additive completion of a 2-category). Any locally additive linear 2-category $\mathcal{C}$ can be completed to an additive 2-category $\mathcal{C}^{\boxplus}$; here $\mathcal{C}^{\boxplus}$ has as objects finite (possibly empty) lists of objects of $\mathcal{C}$, as 1-morphisms matrices of 1-morphisms in $\mathcal{C}$, and as 2-morphisms matrices of 2-morphisms in $\mathcal{C}$. Horizontal composition of 1-morphisms in $\mathcal{C}^{\boxplus}$ is 'matrix multiplication' with sum and product replaced by direct sum and composition in $\mathcal{C}$.

Remark 1.2.7 (Additive completion is idempotent). If $\mathcal{C}$ is already additive, then $\mathcal{C}^{\boxplus}$ is equivalent to $\mathcal{C}$; this is a consequence of the fact that direct sums and zero objects are absolute colimits and that $\mathcal{C}^{\boxplus}$ is the free cocompletion under these colimits.

1.2.2 Presemisimple 2-categories

In the following, we develop a notion of 'presemisimple 2-category'. A presemisimple 2-category may be understood as a semisimple 2-category without the requirement of being 'categorified Cauchy complete' — although the definition simplifies considerably once we also impose the appropriate completeness conditions (see Definition 1.2.51) and every presemisimple 2-category can be completed to a semisimple 2-category, the notion of a presemisimple 2-category will nevertheless be useful for our state-sum construction in Chapter 2.

Presemisimple and semisimple 1-categories

Recall from Proposition I.2.9 that a semisimple 1-category may be defined as an additive and idempotent complete linear 1-category in which every object is a finite direct sum of simple objects and the composite of two non-zero morphisms between simple objects is again non-zero. Dropping the completeness conditions, we define a linear 1-category to be *presemisimple* if every object can be decomposed as a finite direct sum of simple objects, and the composition of any two nonzero morphisms between simple objects is nonzero.

Remark 1.2.8 (1-categories in which objects split into simples). Note that in a linear 1-category, asking merely that every object can be decomposed into a finite direct sum of simple objects achieves very little. For example, the category with one object whose endomorphism algebra is the algebra of 2-by-2 matrices satisfies this condition, even though it has nontrivial idempotents and the object will decompose upon idempotent completion. In fact, in any finite-dimensional algebra, a left-cancellative element is necessarily invertible; thus, in any category with one object and endomorphism algebra a finite-dimensional algebra, the object is simple, and so the category has the property that 'every object decomposes into a finite direct sum of simples'. (Also see Warning I.2.12.)

Note that that the additive and idempotent completion of a presemisimple 1-category is indeed semisimple: in this case the additive and idempotent completion does not produce any new simple objects, and so the category remains presemisimple, as required.

The definition of presemisimple 2-categories

We now discuss analogous 2-categorical notions. A 1-morphism $f : A \to B$ in a 2-category $\mathcal{C}$ is *fully faithful* if, for all objects X, the functor $\mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$ is fully faithful; a *subobject* of an object B of a 2-category is then an equivalence class of fully faithful 1-morphisms $A \to B$.

Definition 1.2.9 (Simple object in a 2-category)**.** A nonzero object in a linear 2-category is *simple* if its subobjects are either zero objects or equivalences.

A linear 2-category is called *locally semisimple* if all of its Hom categories are semisimple, and *locally finite semisimple* if all its Hom categories are finite semisimple.

Definition 1.2.10 (Decomposable object in a 2-category)**.** An object in a linear 2-category is *decomposable* if it is equivalent to a direct sum of nonzero objects, and it is *indecomposable* if it is nonzero and not decomposable.

For conciseness we will use the following terminology:

- A multifusion category is a finite semisimple linear monoidal 1-category whose objects have left and right duals.
- A fusion category is a multifusion category whose tensor unit is simple.
- An infusion category is a semisimple linear monoidal 1-category whose objects have left and right duals and whose tensor unit is simple.

Note that an infusion category may have infinitely many isomorphism classes of simple objects. We will see later (in Corollary 1.2.24) that infusion categories, though not quite 'categorical division algebras', function as 'categorical domains'.

We might expect to define a presemisimple 2-category to be a linear 2-category in which every object decomposes into simple objects and in which the composition of nonzero morphisms between simples is nonzero; in fact, we will find (see Proposition 1.2.23) that this composition property is implied merely asking the endomorphism categories of simples to be infusion categories, that is categorical domains.

Definition 1.2.11 (Presemisimple 2-category)**.** A *presemisimple* 2*-category* is a locally semisimple 2-category such that every 1-morphism admits a right adjoint and a left adjoint, such that every object is decomposable as a finite direct sum of simple objects, and such that the endomorphism category of every simple object is an infusion category.

We will see, from Corollary 1.2.19 below, that this definition is equivalent to the following somewhat more compact definition: *A presemisimple* 2*-category is a locally semisimple* 2*-category such that every* 1*-morphism admits a right adjoint and a left adjoint and such that every object is decomposable as a finite direct sum of objects with simple identity.*

Definition 1.2.12 (Finite presemisimple 2-category)**.** A presemisimple 2-category is *finite* if its Hom-categories are finite semisimple and if it has a finite number of equivalence classes of simple objects.

Remark 1.2.13 (2-categories in which objects split into simples)*.* As in the 1-categorical case, cf Remark 1.2.8, it is not particularly useful to consider linear (locally semisimple) 2-categories (with adjoints) merely such that every object decomposes as a finite sum of simples. In such a 2-category, the endomorphism categories of simple objects can be arbitrarily complicated and supposedly simple objects may decompose after an idempotent completion operation on the 2-category.

Remark 1.2.14 (Endomorphism categories are multifusion)*.* Observe that in a finite presemisimple 2-category, the endomorphism category of any object is a multifusion category.

Example 1.2.15 (The delooping of an infusion category)*.* Associated to an infusion category D, there is a presemisimple 2-category BD with a unique object $*$ and the endomorphism category $\mathrm{Hom}_{\mathrm{BD}}(*, *) = \mathrm{D}$.

Construction 1.2.16 (The unfolded finite presemisimple 2-category of a multifusion category)*.* More generally, let D be a multifusion category, and let $I \cong \bigoplus_{i\in\mathcal{I}} I_i$ be the simple decomposition of the tensor unit of D. Let $\mathrm{D}_{i,j}$ be the full additive subcategory of D containing the simple objects X that fulfill $I_j \otimes X \cong X \cong X \otimes I_i$. Recall that $\mathrm{D}_{i,i}$ is a fusion category, $\mathrm{D}_{i,j}$ is a $\mathrm{D}_{i,i}$-$\mathrm{D}_{j,j}$-bimodule category, and as a linear 1-category, $\mathrm{D} \cong \bigoplus_{i,j} \mathrm{D}_{i,j}$ [ENO05, Sec 2.4]. Associated to this multifusion category D, there is a finite presemisimple 2-category $\mathcal{D}$, the *unfolded* 2*-category* of D, with objects $i \in \mathcal{I}$ and 1-morphism categories $\mathrm{Hom}_{\mathcal{D}}(i, j) := \mathrm{D}_{i,j}$. (Regarding the process of 'folding' and 'unfolding' between tensor 1-categories and linear 2-categories, see Kuperberg [Kup03].)

Construction 1.2.17 (The folded multifusion category of a finite presemisimple 2-category). Given a finite presemisimple 2-category $\mathcal{D}$, one can conversely consider the associated *folded multifusion category* D; if I is a set of representative simple objects of $\mathcal{D}$, then the folded category is defined as $\mathrm{D} := \oplus_{(i,j)\in I\times I} \mathrm{Hom}_{\mathcal{D}}(i,j)$. Note well that folding and unfolding are not strictly inverse operations. For instance, a presemisimple 2-category $\mathcal{C}$ with only simple objects will have the same folding as its additive completion $\mathcal{C}^{\boxplus}$. (Note also that the unfolding of a multifusion category is never additive.) Nevertheless, we do expect that folding and unfolding produce inverse equivalences between the 3-category of multifusion categories and the 3-category of finite presemisimple 2-categories, where the 1-morphisms of multifusion categories are finite semisimple bimodule categories and the 1-morphisms between finite presemisimple 2-categories are finite semisimple 2-profunctors (also called '2-distributors', see Section 1.2.4). Thus we can consider giving a multifusion category as a method for providing the data of a finite presemisimple 2-category.

Decomposition in presemisimple 2-categories

Note that a presemisimple 2-category is not assumed to be additive, that is, it need not have a zero object or direct sums of objects, and it has no 1-morphism-level idempotent-completeness condition. Nevertheless, presemisimple 2-categories have a reasonably well behaved notion of decomposition of objects, as follows.

Simple objects and simple identities correspond.

Proposition 1.2.18 (Decomposition with simple identities implies simple objects and simple identities correspond)**.** *Let $\mathcal{C}$ be a locally semisimple 2-category such that every 1-morphism admits a right adjoint and a left adjoint and such that every object is decomposable as a finite direct sum of objects with simple identity. Then an object X is simple if and only if the identity 1-morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ is simple.*

Proof. A simple object can only decompose as itself (otherwise it would have a nontrivial subobject), and so by the decomposition assumption, it must have simple identity.

Now suppose X is an object with simple identity, and let $R : A \to X$ be a subobject, that is a fully faithful 1-morphism from a nonzero object A to X. Let $L : X \to A$ be a left adjoint of R, with unit $\eta : 1_X \Rightarrow R \circ L$ and counit $\epsilon : L \circ R \Rightarrow 1_A$. We will show that R is an equivalence (with inverse L), and thus X is simple. We do so by explicitly constructing inverses of the counit and unit of the adjunction.

Since by assumption $R : A \to X$ is fully faithful, the functor $R\circ - : \mathrm{Hom}_{\mathcal{C}}(A, A) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$ is full and faithful. In particular, by fullness, the composite $\eta \circ R : R \Rightarrow R \circ L \circ R$ is equal to $R \circ \delta$ for some 2-morphism $\delta : 1_A \Rightarrow L \circ R$. Postcomposing the equation $L \circ \eta \circ R = L \circ R \circ \delta$ with $\epsilon \circ L \circ R$ implies, using the cusp equation, that $\delta \cdot \epsilon = 1_{L\circ R}$. The functor $R \circ -$ sends both $\epsilon \cdot \delta$ and 1_{1_A} to $(R \circ \epsilon) \cdot (R \circ \delta) = (R \circ \epsilon) \cdot (\eta \circ R) = 1_R$, by the cusp equation. Faithfulness of $R \circ -$ implies that $\epsilon \cdot \delta = 1_{1_A}$.

Now the counit $\eta : 1_X \Rightarrow R \circ L$ is certainly nonzero, since otherwise by the cusp equation 1_L would be zero, implying L is zero and therefore R is zero, contradicting the fact that R is faithful and A is nonzero. By local semisimplicity and the simplicity of 1_X, there is a 2-morphism $\alpha : R \circ L \Rightarrow 1_X$ such that $\alpha \cdot \eta = 1_{1_X}$. By the fullness of R, there is a 2-morphism $r : 1_A \Rightarrow 1_A$ such that $R \circ r = (\alpha \circ R) \cdot (R \circ \delta) \in \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{C}}(A,X)}(R, R)$. Since ϵ and δ are inverse, precomposing this equation with $R \circ \epsilon$ gives $\alpha \circ R = (R \circ r) \cdot (R \circ \epsilon)$, and further precomposing with $\eta \circ R$ gives $R = (\alpha \cdot \eta) \circ R = R \circ r$. Faithfulness of R implies $r = 1_{1_A}$, so $R = (\alpha \circ R) \cdot (R \circ \delta)$. That last equation is (by 1-morphism precomposing with L and then postcomposing with η, respectively by 1-morphism precomposing with R and then precomposing with ϵ) equivalent to the equation $\eta \cdot \alpha = 1_{R\circ L}$. □

Corollary 1.2.19 (Decomposition with simple identities implies presemisimple)**.** *A locally semisimple 2-category is presemisimple if and only if every 1-morphism admits a right adjoint and a left adjoint and every object decomposes as a finite direct sum of objects with simple identity.*

Corollary 1.2.20 (Projections and inclusions are simple)**.** *Let $\boxplus X_i$ be a finite direct sum of simple objects in a presemisimple 2-category $\mathcal{C}$. The projection and inclusion 1-morphisms $\iota_i : X_i \leftrightarrows \boxplus X_i : \rho_i$ are necessarily simple 1-morphisms.*

Proof. By Proposition 1.2.3, we have $\mathrm{End}_{\mathcal{C}}(\iota_i) \cong \mathrm{Hom}_{\mathcal{C}}(1_{X_i}, \rho_i \circ \iota_i) \cong \mathrm{End}_{\mathcal{C}}(1_{X_i})$; hence ι_i is simple if and only if 1_{X_i} is, and 1_{X_i} in turn is simple if and only if X_i is. Since ι_i and ρ_i are adjoint, taking mates induces an isomorphism $\mathrm{End}_{\mathcal{C}}(\iota_i) \cong \mathrm{End}_{\mathcal{C}}(\rho_i)$, so ρ_i is also simple if and only if ι_i is. □

Simple objects and indecomposable objects correspond. It turns out that in a presemisimple 2-category, the notion of simple object and of indecomposable object coincide.

Proposition 1.2.21 (Simple if and only if indecomposable)**.** *An object in a presemisimple 2-category is simple if and only if it is indecomposable.*

Proof. Given a nontrivial decomposition of an object X as $A_1 \boxplus A_2$, the inclusion 1-morphism $A_1 \to X$ is fully faithful and therefore is a nontrivial subobject; thus X is not simple. Conversely, by the definition of presemisimplicity, any object is a sum of simple objects; for an indecomposable object, this sum can only have a single factor, and so the object itself is simple. □

Categorical domain Schur's lemma. We now show that in a presemisimple 2-category, the decomposition of an object into a sum of simple objects is unique. To show this we need the following 'categorical domain' version of Schur's lemma: though nonzero morphisms between simple objects in a presemisimple 2-category need not be equivalences (and therefore the endomorphism category of a simple object need not be a 'categorical division algebra'), nevertheless the composite of two nonzero morphisms between simple objects in a presemisimple 2-category cannot be zero (and therefore the endomorphism category of a simple object is a kind of 'categorical domain').

Definition 1.2.22 (Categorical domain)**.** A *categorical domain* is a semisimple monoidal 1-category with duals such that the tensor product of two nonzero objects is nonzero.

Proposition 1.2.23 (Categorical domain Schur's lemma)**.** *If $g : A \to B$ and $f : B \to C$ are nonzero* 1*-morphisms between simple objects in a presemisimple* 2*-category, then the composite $f \circ g$ is also nonzero.*

Proof. Let $g^* : B \to A$ be a right adjoint of g with counit $\epsilon : g \circ g^* \Rightarrow 1_B$. By assumption B is simple, and so by Corollary 1.2.19 and Proposition 1.2.18, the identity 1_B is simple. As a counit, ϵ must be nonzero, and as a nonzero morphism to a simple object in a semisimple category, it must have a section. If $f \circ g$ were zero, then $f \circ g \circ g^*$ would be zero and so the morphism $1_f \circ \epsilon$ would necessarily be zero. Precomposing with the section would imply that 1_f itself was zero, which in turn would force f to be zero. □

Corollary 1.2.24 (Infusion if and only if categorical domain)**.** *A semisimple monoidal* 1*-category with duals is infusion if and only if it is a categorical domain.*

Proof. By Proposition 1.2.23 and Proposition 1.2.18 applied to Example 1.2.15, an infusion category is a categorical domain. Conversely, if a semisimple monoidal 1-category with duals has two distinct simple subobjects of its tensor unit, then the product of those objects is zero [ENO05, Sec 2.4], preventing the category from being a domain. ☐

Note that if $g : A \to B$ and $f : B \to C$ are 1-morphisms in a linear 2-category and either one is a zero 1-morphism, then their composite $f \circ g$ is also a zero 1-morphism. Thus, morphisms between simple objects in a presemisimple 2-category satisfy a 'two out of three property', that if any two of f, g, and $f \circ g$ are nonzero, then so is the third; in this sense, though not necessarily equivalences, nonzero morphisms between simple objects in a presemisimple 2-category are a sort of 'very weak equivalences'.

Uniqueness of decomposition.

Proposition 1.2.25 (Decomposition into simples is unique in presemisimple 2-categories)**.** *The decomposition of any object in a presemisimple* 2*-category into a finite direct sum of simple objects is unique up to permutation and equivalence.*

Proof. Let $\boxplus_{i\in I} X_i$ and $\boxplus_{j\in J} X'_j$ be direct sum decompositions of an object X into simple objects, with inclusion and projection 1-morphisms $\iota_i : X_i \leftrightarrows X : \rho_i$ and $\iota'_j : X'_j \leftrightarrows X : \rho'_j$. Note that $\oplus_{j\in J}\, \rho_i \circ \iota'_j \circ \rho'_j \circ \iota_i \cong \rho_i \circ \iota_i \cong 1_{X_i}$. Since 1_{X_i} is simple, it follows that there exists a unique $f(i) \in J$ such that $\rho_i \circ \iota'_{f(i)} \circ \rho'_{f(i)} \circ \iota_i$ is nonzero (and in this case isomorphic to 1_{X_i}). By Proposition 1.2.23, a 1-morphism F between simple objects is zero if and only if $F^* \circ F$ is zero. Using Proposition 1.2.3, it follows that for every $i \in I$ there is a unique $f(i) \in J$ such that $\rho'_{f(i)} \circ \iota_i : X_i \to X'_{f(i)}$ is nonzero. The same argument applied to the decomposition $\oplus_{i\in I}\rho'_j \circ \iota_i \circ \rho_i \circ \iota'_j \cong 1_{X'_j}$ shows that for every $j \in J$ there is a unique $g(j) \in I$ such that $\rho'_j \circ \iota_{g(j)} : X_{g(j)} \to X'_j$ is nonzero. Thus $f : I \to J$ is a bijection with inverse $g : J \to I$ and $\rho'_{f(i)} \circ \iota_i$ is an equivalence since

$$1_{X_i} \cong \bigoplus_{j\in J} \rho_i \circ \iota'_j \circ \rho'_j \circ \iota_i \cong \rho_i \circ \iota'_{f(i)} \circ \rho'_{f(i)} \circ \iota_i$$
$$1_{X'_{f(i)}} \cong \bigoplus_{i'\in I} \rho'_{f(i)} \circ \iota_{i'} \circ \rho_{i'} \circ \iota'_{f(i)} \cong \rho'_{f(i)} \circ \iota_i \circ \rho_i \circ \iota'_{f(i)}.$$

Finally note that

$$\iota'_{f(i)} \circ \rho'_{f(i)} \circ \iota_i \cong \bigoplus_{j\in J} \iota'_j \circ \rho'_j \circ \iota_i \cong \iota_i$$
$$\rho'_{f(i)} \circ \iota_i \circ \rho_i \cong \bigoplus_{i'\in I} \rho'_{f(i)} \circ \iota_{i'} \circ \rho_{i'} \cong \rho'_{f(i)}.$$

Hence there is a bijection $f : I \to J$ and equivalences $e_i : X_i \to X'_{f(i)}$ such that $\iota'_{f(i)} \circ e_i \cong \iota_i$ and $e_i \circ \rho_i \cong \rho'_{f(i)}$, as required. ☐

Components of presemisimple 2-categories. A presemisimple 2-category may itself be 'decomposable' in the sense that its set of simple objects splits into two pieces, such that there are no nonzero morphisms between the simple objects in one piece and the simple objects in the other piece. Furthermore, each 'indecomposable' collection of simples will be completely connected in the sense that there is a nonzero morphism between any two simples in the collection; we will refer to such a completely connected collection of simples as a component of the 2-category.

Definition 1.2.26 (Components of presemisimple 2-categories)**.** Let $\mathcal{C}$ be a presemisimple 2-category. Two simple objects A and B in $\mathcal{C}$ are *in the same component* if there is a nonzero 1-morphism $A \to B$, that is if $\mathrm{Hom}_{\mathcal{C}}(A, B) \neq 0$. The *set of components* of $\mathcal{C}$, denoted $\pi_0\mathcal{C}$, is the quotient of the set of simples by the equivalence relation of being in the same component.

Note that being in the same component is indeed an equivalence relation: reflexivity is clear; symmetry follows from the fact that the right adjoint of a nonzero morphism is nonzero; and transitivity is precisely the content of the categorical domain Schur's lemma.

Given an indecomposable multifusion category (that is one that is not the direct sum of two nontrivial multifusion categories), the associated unfolded 2-category (see Construction 1.2.16) is connected (that is has a single component). More generally, the set of indecomposable factors of a multifusion category corresponds to the set of components of its unfolding.

Remark 1.2.27 (Components as indecomposable summands)*.* Though we will not need it, there is a natural notion of direct sum of linear 2-categories, and therefore of indecomposable linear 2-category. When a presemisimple 2-category $\mathcal{C}$ is in fact additive, we may think of its components as the summands in the finest direct sum decomposition of $\mathcal{C}$ into indecomposable linear 2-categories.

Dimensions of presemisimple 2-categories

In a finite semisimple 1-category, there are finitely many isomorphism classes of simple objects, the endomorphism algebra of any simple object is the base field, and there are no morphisms between non-isomorphic simples. There is therefore a natural invariant of such a category, namely the number of isomorphism classes of simple objects. This 'dimension' is of course a natural number. We now describe the analogous notion of dimension for finite presemisimple 2-categories. This notion is complicated by the fact that, in a presemisimple 2-category, the endomorphism fusion categories of simple objects are not necessarily trivial and there can be nontrivial morphisms between distinct simple objects. In particular, as a result, the dimension of a presemisimple 2-category will not necessarily be a natural number.

Recall the notion of the global dimension of a fusion category C [Müg03]: any simple object $x \in \mathrm{C}$ is isomorphic to its double dual x^{**}; given any isomorphism $a : x \to x^{**}$, one uses the counit of the duality $(x \dashv x^*)$ and the unit of the duality $(x^* \dashv x^{**})$ to form the quantum trace $\mathrm{Tr}(a) \in k$; the product of the quantum trace of a and the quantum trace of ${}^*(a^{-1})$ is independent of the choice of morphism a and is called the squared norm of the simple object x; the sum of the squared norms of a set of distinct simple objects is called the global dimension of the fusion category.

We describe the analogous notions for 1-morphisms in an appropriate 2-category.

Proposition 1.2.28 (Double adjunction is trivial)**.** *Any simple 1-morphism f in a finite presemisimple 2-category is isomorphic to its double right adjoint f^{**}.*

Proof. This is similar to the analogous result for fusion categories [ENO05, Prop 2.1]. The 1-morphism $f : A \to B$ is an object of the finite semisimple 1-category $\mathrm{Hom}(A, B)$. In any finite semisimple 1-category C, for any two objects $X, Y \in \mathrm{C}$, there is a noncanonical isomorphism $\mathrm{Hom}_{\mathrm{C}}(X, Y) \cong \mathrm{Hom}_{\mathrm{C}}(Y, X)$. Note that f^{**} is simple if and only if f is simple. We therefore have $\mathrm{Hom}(f, f^{**}) \cong \mathrm{Hom}(1_B, f^* \circ f^{**}) \cong \mathrm{Hom}(f^* \circ f^{**}, 1_B) \cong \mathrm{Hom}(f^{**}, f^{**}) \cong k$. Since both f and f^{**} are simple, the existence of a nontrivial morphism between them implies they are isomorphic. □

In a locally semisimple 2-category, every 2-endomorphism $\mu : g \Rightarrow g$ of a simple 1-morphism g is proportional to the identity 2-morphism; we denote the proportionality factor by $\langle \mu \rangle \in k$, that is $\mu = \langle \mu \rangle 1_g$.

Definition 1.2.29 (Squared norm of 1-morphism)**.** The *squared norm* of a simple 1-morphism $f : A \to B$ between simple objects in a finite presemisimple 2-category is the product

$$\|f\| := \Big\langle \epsilon_{f^*} \cdot (1_{f^*} \circ a) \cdot \eta_f \Big\rangle \quad \Big\langle \epsilon_f \cdot (a^{-1} \circ 1_{f^*}) \cdot \eta_{f^*} \Big\rangle \in k$$

where $a : f \Rightarrow f^{**}$ is an arbitrary 2-isomorphism, η_f and ϵ_f are the unit and counit of the adjunction $f \dashv f^*$, and η_{f^*} and ϵ_{f^*} are the unit and counit of the adjunction $f^* \dashv f^{**}$.

Here simplicity of the objects A and B is necessary to ensure that 1_A and 1_B are simple, so that we can extract scalars from the two 'quantum trace' endomorphisms in the formula for the squared norm. Simplicity of f ensures that the squared norm is independent of the choice of 2-morphism $a : f \Rightarrow f^{**}$, and hence only depends on the isomorphism class of f.

Definition 1.2.30 (Dimension of Hom category)**.** For simple objects A and B in a finite presemisimple 2-category $\mathcal{C}$, the *dimension* of the category $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is

$$\dim(\mathrm{Hom}_{\mathcal{C}}(A, B)) := \sum_{f:A\to B} \|f\|$$

where the sum is over isomorphism classes of simple 1-morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

This definition is analogous to that of the dimension of the 'off-diagonal subcategories' of a multifusion category [ENO05, Sec 2.4]. Note that for a simple object A, the dimension $\dim(\mathrm{Hom}_{\mathcal{C}}(A, A))$ is the usual global dimension of the fusion category $\mathrm{Hom}_{\mathcal{C}}(A, A)$; when over an algebraically closed field of characteristic zero, that dimension is always nonzero [ENO05, Thm 2.3].

Proposition 1.2.31 (Dimension is uniform within a component)**.** *Let* $\{A_i\}_{i\in I}$ *be the simple objects of a connected component of a finite presemisimple* 2*-category* $\mathcal{C}$*. Then the categories* $\mathrm{Hom}_{\mathcal{C}}(A_i, A_j)$ *all have the same dimension, for* $i, j \in I$*.*

Proof. The multifusion category $\bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{C}}(A_i, A_j)$ is indecomposable, because $\mathrm{Hom}_{\mathcal{C}}(A_i, A_j) \neq 0$ for simple objects in the same connected component. By [ENO05, Prop 2.17], it follows that the component categories $\mathrm{Hom}_{\mathcal{C}}(A_i, A_j)$ all have the same dimension. □

We now have a notion of the dimension of each component of a presemisimple 2-category (namely the dimension of any Hom category between simples in that component), and we are ready to assemble them into a notion of the dimension of the whole 2-category. Recall that for a 1-groupoid, the natural notion of size (the 'groupoid cardinality' [BHW10]) is the sum over components of the reciprocal of the size of the automorphism groups. The dimension for a presemisimple 2-category is analogous.

Definition 1.2.32 (Dimension of presemisimple 2-category)**.** The *dimension* of a finite presemisimple 2-category $\mathcal{C}$ is

$$\dim(\mathcal{C}) := \sum_{[x]\in\pi_0\mathcal{C}} \frac{1}{\dim(\mathrm{End}_{\mathcal{C}}(x))} \in k.$$

Here the sum is over components $[x]$ of $\mathcal{C}$, and x is any simple object in the component $[x]$.

Of course, the dimension of a finite presemisimple 2-category is only defined when the dimensions of all its endomorphism fusion categories are nonzero; this is ensured by our standing assumption that the base field is algebraically closed of characteristic zero.

As for fusion categories, when over an algebraically closed field of characteristic zero, the dimension of a presemisimple 2-category cannot vanish.

Proposition 1.2.33 (Dimension is nonzero)**.** *For $\mathcal{C}$ a finite presemisimple 2-category over an algebraically closed field of characteristic zero, the dimension $\dim(\mathcal{C})$ is nonzero.*

Proof. By [ENO05, Thm 2.3], a fusion category over $\mathbb{C}$ has positive real global dimension. Thus the dimension of a presemisimple 2-category over $\mathbb{C}$ is positive real, in particular nonzero. The result follows by noting that any finite presemisimple 2-category over an algebraic closed field k of characteristic zero can be defined over a subfield k' that is finitely generated over $\mathbb{Q}$ and which can therefore be embedded in $\mathbb{C}$. □

Remark 1.2.34 (Nonzero characteristic). We could proceed without a characteristic zero assumption, at the expense of restricting attention to *non-degenerate* finite presemisimple 2-categories $\mathcal{C}$, that is those for which the dimensions $\dim(\mathrm{End}_{\mathcal{C}}(x))$ are nonzero for all simple objects x and for which the overall dimension $\dim(\mathcal{C})$ is nonzero.

1.2.3 Idempotent complete 2-categories

A semisimple 1-category is a presemisimple 1-category that is also additive and idempotent complete. We described the notion of presemisimple 2-category and of additive 2-category; we now discuss idempotent completeness for 2-categories. Further details about idempotent completeness and idempotent completion, and a number of proofs, are given in Appendix B.

Categorified idempotents

Idempotents and split idempotents. In a 1-category, a *section-retraction pair* is a pair of 1-morphisms $i : A \to B$ and $r : B \to A$ such that $r \circ i = 1_A$. Associated to such a pair there is the 1-morphism $e := i \circ r : B \to B$, which is an *idempotent* (or 'projection'), meaning it is a 1-morphism $e : B \to B$ such that $e \circ e = e$. An arbitrary idempotent $e : B \to B$ is *splitable* (or more informally 'split') when there exists a section-retraction pair (i, r) such that $e = i \circ r$; a 'splitting' is a choice of such a pair. As before, a 1-category is idempotent complete if every idempotent splits. A 2-category $\mathcal{C}$ is *locally idempotent complete* if for all objects $A, B \in \mathcal{C}$, the 1-category $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is idempotent complete.

Idempotent monads and reflectively split idempotent monads. A natural categorification of the notion of section-retraction pair is the following: a *reflective subcategory* is a pair of functors $\iota : \mathcal{A} \to \mathcal{B}$ and $\rho : \mathcal{B} \to \mathcal{A}$ where ι is fully faithful and ρ is equipped with the structure of a left adjoint of ι. More generally, in a 2-category, a *reflective adjunction* is a fully faithful 1-morphism $\iota : A \to B$ together with a left adjoint $\rho : B \to A$. An adjunction $\iota \vdash \rho$ is reflective exactly when its counit is an isomorphism $\epsilon : \rho \circ \iota \xrightarrow{\cong} 1_A$; this condition on the counit is a strong categorification of the condition $r \circ i = 1_A$ on a section-retraction pair. Associated to a reflective adjunction $\iota \vdash \rho$ in a 2-category, there is the 1-morphism $E := \iota \circ \rho : B \to B$, which is an *idempotent monad* (a kind of 'categorified projection'), meaning it is a 1-morphism $E : B \to B$ equipped with a 2-isomorphism $m : E \circ E \to E$ (determined by the counit of the adjunction) and a 2-morphism $u : 1_B \to E$ (determined by the unit of the adjunction), such that $m \cdot (m \circ 1_E) = m \cdot (1_E \circ m)$ and $m \cdot (u \circ 1_E) = 1_E = m \cdot (1_E \circ u)$. An idempotent monad $E : B \to B$ is *reflectively splitable* (or more informally 'reflectively split') when there exists a reflective adjunction $\iota \vdash \rho$ and an isomorphism of monads $E \cong \iota \circ \rho$; a 'reflective splitting' is a choice of such an adjunction and isomorphism.

Monads and split monads. If we drop the fully faithful (equivalently counit isomorphism) condition in a reflective adjunction, we are left simply with 1-morphisms $\iota : A \to B$ and $\rho : B \to A$ forming an *adjunction* $\iota \vdash \rho$. The associated 1-morphism $E := \iota \circ \rho : B \to B$ is a *monad*, meaning it is a 1-morphism $E : B \to B$ equipped with a 2-morphism (not necessarily a 2-isomorphism) $m : E \circ E \to E$ and a 2-morphism $u : 1_B \to E$, satisfying the same equations as an idempotent monad. (Concisely, a monad E in a 2-category $\mathcal{C}$ is an algebra object in the endomorphism 1-category $\mathrm{Hom}_{\mathcal{C}}(B, B)$ of an object $B \in \mathcal{C}$.) An arbitrary monad $E : B \to B$ is *splitable* (or more informally 'split') when there exists an adjunction $\iota \vdash \rho$ and an isomorphism of monads $E \cong \iota \circ \rho$.

Categorified idempotent splitting

Uniqueness of splitting an idempotent. Given an idempotent $e : B \to B$ in a 1-category, if it admits a splitting, then there is a unique splitting. (That is, any two splittings $(i : A \to B, r : B \to A)$ and $(i' : A' \to B, r' : B \to A')$ are isomorphic by a unique isomorphism, namely the intertwiner $r' \circ i$.) Indeed, the splitting may be expressed either as a colimit or as a limit, as follows. Given an idempotent $e : B \to B$, consider the diagram $B \overset{e}{\underset{1}{\rightrightarrows}} B$. If it exists, the coequalizer $B \overset{e}{\underset{1}{\rightrightarrows}} B \xrightarrow{r} A$ provides a splitting of the idempotent (where the morphism $i : A \to B$ is determined by the universal property of the coequalizer). Similarly, if it exists, the equalizer $A \xrightarrow{i} B \overset{e}{\underset{1}{\rightrightarrows}} B$ provides a splitting of the idempotent (where the morphism $r : B \to A$ is then determined by the universal property of the equalizer). In particular, if either the coequalizer or equalizer exists, then the other does, and the coequalizing object is isomorphic to the equalizing object.

Uniqueness of reflectively splitting an idempotent monad. As idempotents in a 1-category have unique splittings (when they are split), so too idempotent monads in a locally idempotent complete 2-category, have unique reflective splittings (when they are reflectively split). However, an arbitrary monad in a 2-category, even if it admits a splitting, need not admit a unique splitting. We would like to restrict attention to a class of monads $E : B \to B$ for which the multiplication 2-morphism $m : E \circ E \to E$ need not be an isomorphism (by contrast with idempotent monads) but which nevertheless have a unique splitting property (as do idempotent monads).

Separable monads and separably split separable monads. The data of an idempotent monad in a 2-category $\mathcal{C}$ can be expressed as follows: it is a triple $(E : B \to B, m : E \circ E \to E, u : 1_B \to E)$ forming an algebra object in $\mathrm{Hom}_{\mathcal{C}}(B, B)$, such that there exists an E-E-bimodule map $c : E \to E \circ E$ that is a two-sided inverse to the multiplication $m : E \circ E \to E$. We can marginally weaken this notion of idempotent monad by only requiring there to exist a one-sided rather than two-sided inverse to the multiplication; this provides a version of categorified idempotent that is more lax than idempotent monad but stronger than arbitrary monad.

Definition 1.2.35 (Separable monad)**.** A monad $(E : B \to B, m : E \circ E \to E, u : 1_B \to E)$ in a 2-category is *separable* if there exists an E-E-bimodule map $c : E \to E \circ E$ that is a right inverse for the multiplication $m : E \circ E \to E$, that is such that $m \cdot c = 1_E$.[5]

Similarly, the data of a reflective splitting of an idempotent monad $E : B \to B$ can be expressed as follows: it is an adjunction $\iota \vdash \rho \equiv (\iota : A \to B, \rho : B \to A, \eta :$

[5]A separable monad in a 2-category with one object is a separable algebra object in the monoidal endomorphism category of that object. A classical separable algebra is a separable algebra object in the symmetric monoidal category of vector spaces.

$1_B \to \iota \circ \rho, \epsilon : \rho \circ \iota \to 1_A)$ such that there exists a 2-morphism $\phi : 1_A \to \rho \circ \iota$ that is a two-sided inverse to the counit $\epsilon : \rho \circ \iota \to 1_A$, together with an isomorphism of monads $E \cong \iota \circ \rho$. We can again marginally weaken the invertibility condition here by only requiring there to exist a one-sided inverse to the counit of the adjunction.

Definition 1.2.36 (Separable adjunction)**.** An adjunction $\iota \vdash \rho \equiv (\iota : A \to B, \rho : B \to A, \eta : 1_B \to \iota \circ \rho, \epsilon : \rho \circ \iota \to 1_A)$ in a 2-category is *separable* if there exists a 2-morphism $\phi : 1_A \to \rho \circ \iota$ that is a right inverse for the counit $\epsilon : \rho \circ \iota \to 1_A$, that is such that $\epsilon \cdot \phi = 1_A$.

The notion of separable adjunction is a categorification of section-retraction pair that is more lax than reflective adjunction but stronger than arbitrary adjunction; there is therefore a corresponding version of categorified idempotent splitting that is more lax than reflective splitting but stronger than arbitrary splitting.

Definition 1.2.37 (Separably split monad)**.** A separable monad $E : B \to B$ in a 2-category is *separably splitable* (or simply 'separably split') when there exists a separable adjunction $\iota \vdash \rho$ and an isomorphism of monads $E \cong \iota \circ \rho$; a *separable splitting* is a choice of such an adjunction and isomorphism.

Note that if a monad admits a separable splitting then the monad itself is necessarily separable.

Uniqueness of separable splittings of separable monads. Separable monads do indeed have unique separable splittings (when they are separably split), as desired:

Proposition 1.2.38 (Separable splittings are unique)**.** *A separable monad, in a locally idempotent complete 2-category, that admits a separable splitting, admits a unique up-to-equivalence separable splitting.*

This is an immediate corollary of Theorem B.1 in Appendix B, but we briefly and informally sketch the argument here. Recall how one sees the uniqueness of splittings for a 1-categorical idempotent $e : B \to B$: the colimit $\mathrm{colim}(B \underset{1}{\overset{e}{\rightrightarrows}} B)$ (necessarily unique) provides a splitting and the limit $\lim(B \underset{1}{\overset{e}{\rightrightarrows}} B)$ (necessarily unique) provides a splitting, and any splitting provides both a colimit and a limit; it follows that the splitting is unique and that the colimit and limit objects are isomorphic.

Observe that a 1-categorical idempotent $e : B \to B$ in a category C may be reexpressed (somewhat contortionistically) as a lax 2-semifunctor of 2-categories $* \xrightarrow{(B,e)} \mathrm{C}$, where we have reinterpreted C as a discrete 2-category; the 2-colimit, respectively 2-limit, of that lax functor is exactly the ordinary colimit, respectively limit, splitting of the idempotent as above. Now a monad $E : B \to B$ in a 2-category $\mathcal{C}$ is simply a lax 2-functor $* \xrightarrow{(B,E)} \mathcal{C}$, and we may consider the lax 2-colimit $\mathrm{colim}(* \xrightarrow{(B,E)} \mathcal{C})$ or lax 2-limit $\lim(* \xrightarrow{(B,E)} \mathcal{C})$; that 2-colimit, when it exists, is usually called a *Kleisli object* for the monad, and that 2-limit, when it exists, is usually called an *Eilenberg–Moore object* for the monad. Exactly as for an idempotent in the 1-categorical case, for a monad in the 2-categorical case, when the 2-colimit exists, it provides a splitting, and when the 2-limit exists, it provides a splitting.

The trouble is that a splitting need not provide a 2-colimit or a 2-limit; in particular, the 2-colimit and 2-limit objects need not be the same. However, provided we restrict attention to a separable monad in a locally idempotent complete 2-category, then (see Appendix B) either a 2-colimit or a 2-limit provides a separable splitting, and any separable splitting provides both a 2-colimit and a 2-limit; from this it follows that the separable splitting is unique and that the 2-colimit and 2-limit objects agree.

Remark 1.2.39 (Separable monads and separable splittings are preserved by all 2-functors). Recall from Remark 1.2.4 that direct sums and zero objects are preserved by all 2-functors and are therefore absolute 2-colimits. Similarly, both the separability of a monad and the existence of a separable splitting of a separable monad are 'equational' conditions, in that they are defined in terms of the existence of certain morphisms satisfying certain equations. Separable monads and their separable splittings are therefore preserved by all 2-functors. Since separable splittings of separable monads in a locally idempotent complete 2-category are 2-colimits, they are therefore absolute 2-colimits.

Categorified idempotent completeness

We have arrived at our proper context, a notion of 'lax' 2-categorical idempotent that admits unique splittings, and therefore a notion of 2-category in which all idempotents and categorical idempotents split.

Definition 1.2.40 (Idempotent complete 2-category)**.** A 2-category $\mathcal{C}$ is *idempotent complete* if it is locally idempotent complete and if every separable monad in $\mathcal{C}$ admits a separable splitting.

The above discussion shows that we can, as in this definition, sensibly ask that separable monads admit separable splittings. In practice, we will be interested in locally finite semisimple 2-categories, and in that restricted context, we can furthermore see that separable monads is the largest class of monads we would want to insist have splittings, as follows. As before, given a monad $E : B \to B$, if the lax 2-colimit $\operatorname{colim}(* \xrightarrow{(B,E)} \mathcal{C})$ exists, it provides a splitting of the monad and if the lax 2-limit $\lim(* \xrightarrow{(B,E)} \mathcal{C})$ exists, it also provides a splitting; indeed we may think of the 2-colimit as a 'universal right splitting' and similarly of the 2-limit as a 'universal left splitting'. We take it for granted that we want to consider splittings of monads that are universal, indeed preferably ones where the universal left and right splittings both exist and agree. Such a splitting must be a separable splitting of a separable monad.

Proposition 1.2.41 (Universally split monads are separable)**.** *Let $\mathcal{C}$ be a locally finite semisimple 2-category. If a monad in $\mathcal{C}$ admits a universal left splitting (that is, an Eilenberg–Moore object) or a universal right splitting (that is, a Kleisli object), then the monad is separable and admits a separable splitting.*

Proof. Suppose the monad $E : B \to B$ admits a universal left splitting, that is the lax 2-limit $A := \lim(* \xrightarrow{(B,E)} \mathcal{C})$ exists. By definition of a 2-limit and the notion of left module over a monad, the 2-limit A corepresents left E-module structures (see Appendix B). In particular, the category $\operatorname{Hom}_{\mathcal{C}}(B, A)$ is equivalent to the category $\operatorname{LMod}_E(B)$ of left E-module structures on the object B. Observe that this category $\operatorname{LMod}_E(B)$ of module structures is precisely the category $\operatorname{Mod}(E)$ of modules of E considered as an algebra object in the monoidal category $\operatorname{Hom}_{\mathcal{C}}(B, B)$. And this category $\operatorname{Mod}(E)$ of modules is of course a module category for $\operatorname{Hom}_{\mathcal{C}}(B, B)$. By local finite semisimplicity, the monoidal category $\operatorname{Hom}_{\mathcal{C}}(B, B)$ is finite semisimple and, because the base field is characteristic zero, it is a separable tensor category [DSPS17b, Cor 2.6.8]. The category $\operatorname{Hom}_{\mathcal{C}}(B, A)$ is also semisimple, therefore the module category $\operatorname{Mod}(E)$ is semisimple. By [DSPS17b, Prop 2.5.10], a semisimple module category over a separable tensor category is necessarily separable. By definition this means that the monad E is separable. By Theorem B.1, a separable monad with a univeral left splitting admits a separable splitting. The argument for right splittings is the same. □

Categorified idempotent completion

Construction 1.2.42 (Idempotent completion of a 1-category). A 1-category C can be completed to an idempotent complete 1-category $\mathrm{C}^{\triangledown}$, whose objects are idempotents in C and whose morphisms are bilodules. Here a 'left lodule' for an endomorphism $g : b \to b$ in a 1-category is a 1-morphism $f : a \to b$ such that $g \circ f = f$; similarly a 'right lodule' is a 1-morphism $h : b \to c$ such that $h \circ g = h$. A 'bilodule' from an endomorphism $e : b \to b$ to an endomorphism $e' : b' \to b'$ is a morphism $j : b \to b'$

that is a right lodule for e and a left lodule for e'. Note that if e and e' are split idempotents, the data of a bilodule is the same as the data of a morphism from the splitting object of e to the splitting object of e'.

If the category C is already idempotent complete, then the completion $\mathrm{C}^{\triangledown}$ is equivalent to C.

Construction 1.2.43 (Idempotent completion of a 2-category)*.* A locally idempotent complete 2-category $\mathcal{C}$ can be completed to an idempotent complete 2-category $\mathcal{C}^{\triangledown}$, whose objects are separable monads in $\mathcal{C}$, whose 1-morphisms are bimodules between those monads, and whose 2-morphisms are bimodule maps. See Appendix B.5 for a discussion of various properties of this idempotent completion construction.

Remark 1.2.44 (Idempotent completion is idempotent)*.* If the locally idempotent complete 2-category $\mathcal{C}$ is already idempotent complete, then the completion $\mathcal{C}^{\triangledown}$ is equivalent to $\mathcal{C}$; this is shown in Appendix B as Proposition B.9. This is a consequence of the fact that separable splittings of separable monads (in locally idempotent complete 2-categories) are absolute 2-colimits (see Remark 1.2.39), and the completion $\mathcal{C}^{\triangledown}$ is the free cocompletion under those colimits.

Remark 1.2.45 (Cauchy completion of a 1-category)*.* Recall that a linear 1-category is called 'Cauchy complete' if it has all absolute colimits. A linear 1-category is Cauchy complete if and only if it is additive and idempotent complete [BDSV15, Prop 2.11]. The Cauchy completion of a linear 1-category C is $(\mathrm{C}^{\oplus})^{\triangledown}$, where $\mathrm{C}^{\oplus}$ denotes the direct sum completion, and as above $\mathrm{C}^{\triangledown}$ denotes the idempotent completion.

Remark 1.2.46 (Cauchy completion of a 2-category)*.* A linear 2-category is 'Cauchy complete' if it has all absolute 2-colimits. We speculate that a linear locally Cauchy complete 2-category (or at least a locally finite semisimple 2-category) is Cauchy complete if and only if it is additive and idempotent complete, and we imagine that the Cauchy completion of $\mathcal{C}$ is given by $(\mathcal{C}^{\boxplus})^{\triangledown} \simeq (\mathcal{C}^{\triangledown})^{\boxplus}$.

A prototypical example of a locally idempotent complete 2-category that is not idempotent complete is the delooping BC of a multifusion category C; that is, BC is the 2-category with one object whose endomorphism category is C. We now show that the idempotent completion $(\mathrm{BC})^{\triangledown}$ of this 2-category is the 2-category Mod(C) of finite semisimple (right) module categories for the multifusion category. Note that the objects of the idempotent completion $(\mathrm{BC})^{\triangledown}$, separable monads in BC, are in this case just separable algebras in C.

Proposition 1.2.47 (The idempotent completion of the delooping of a multifusion category is the 2-category of modules)**.** *Let* C *be a multifusion category. The 2-functor* mod : $\mathrm{BC}^{\triangledown} \to \mathrm{Mod(C)}$*, taking a separable algebra in* C *to its category of left modules, is an equivalence.*

Proof. Every finite semisimple module category of a multifusion category (in characteristic zero) is the category of modules of a separable algebra [DSPS17b, Cor 2.6.9]. Thus the 2-functor mod is essentially surjective. Furthermore the category of internal bimodules between algebras is equivalent to the category of functors of module categories [EGNO15, Prop 7.11.1], and the 2-functor mod is therefore an equivalence on 1-morphism categories, as required. □

The delooping BC of a multifusion category includes into the 2-category Mod(C) of modules, by sending the unique object to the module category C_{C}.

Corollary 1.2.48 (Functors from the delooping of a multifusion category extend to modules)**.** *Let* C *be a multifusion category and let* $\mathcal{D}$ *be an idempotent complete 2-category. Every 2-functor* $\mathrm{BC} \to \mathcal{D}$ *extends uniquely (up to equivalence) to a 2-functor* $\mathrm{Mod(C)} \to \mathcal{D}$*.*

Proof. Observe that the composite $\mathrm{BC} \to (\mathrm{BC})^{\triangledown} \xrightarrow{\mathrm{mod}} \mathrm{Mod(C)}$ is the inclusion $\mathrm{BC} \to \mathrm{Mod(C)}$. In Appendix B, see especially Proposition B.14, we show that the idempotent completion is initial among idempotent complete targets; thus the functor $\mathrm{BC} \to \mathcal{D}$ extends to a functor $(\mathrm{BC})^{\triangledown} \to \mathcal{D}$. It follows that the composite $\mathrm{Mod(C)} \xrightarrow{\mathrm{mod}^{-1}} (\mathrm{BC})^{\triangledown} \to \mathcal{D}$ is the desired extension. □

Remark 1.2.49 (The idempotent completion of the delooping of a multifusion category is already additive)*.* Given the delooped 1-category BA of a finite-dimensional semisimple algebra A, to obtain the (additive) category of finite-dimensional modules Mod(A), one must both idempotent and additively complete BA. By contrast, the idempotent completion BC of the deloop of a multifusion category C already has direct sums and need not be further additively completed.

Direct sum decomposition in idempotent complete 2-categories

In a locally additive 2-category, a direct sum decomposition $X \simeq \boxplus_i X_i$ of an object X, with inclusion and projection 1-morphisms $\iota_i \colon X_i \leftrightarrows X : \rho_i$, induces by definition a direct sum decomposition of the identity 1-morphism $1_X \cong \bigoplus_i \iota_i \circ \rho_i \in \mathrm{Hom}(X, X)$. A crucial property of idempotent complete 2-categories is that, conversely, a direct sum decomposition of an identity 1-morphism induces a direct sum decomposition of the corresponding object.

Proposition 1.2.50 (Identity splitting implies object splitting)**.** *Let X be an object in an idempotent complete linear 2-category $\mathcal{C}$. If $1_X \cong \bigoplus_{i\in I} f_i$ is a finite decomposition of 1_X into nonzero 1-morphisms, then there is a finite decomposition $X \simeq \boxplus_{i\in I} X_i$ of X into nonzero objects with inclusions and projections $\iota_i \colon X_i \leftrightarrows X : \rho_i$ such that $f_i \cong \iota_i \circ \rho_i$.*

Proof. Let $r_i : f_i \Rightarrow 1_X$ and $s_i : 1_X \Rightarrow f_i$ be the inclusion and projection 2-morphisms exhibiting the direct sum decomposition $1_X \cong \bigoplus_{i\in I} f_i$. For each i, the following 2-morphisms form the multiplication m_i and unit u_i of a separable monad:

$$m_i := s_i \cdot (r_i \circ r_i) : f_i \circ f_i \Rightarrow f_i$$
$$u_i := s_i : 1_X \Rightarrow f_i$$

A separating section of m_i is given by $\Delta_i := (s_i \circ s_i) \cdot r_i$. Observe that $\Delta_i \cdot m_i = (s_i \circ s_i) \cdot (r_i \circ r_i) = 1_{f_i \circ f_i}$, and so Δ_i (and hence m_i) is an isomorphism.

Separably splitting each monad gives an object X_i and adjoint 1-morphisms $\iota_i : X_i \leftrightarrows X : \rho_i$ such that $\iota_i \circ \rho_i \cong f_i$. Because the monad is separably split, there is (by an argument given in the second part of the proof of Theorem B.1) a section $\delta_i : 1_{X_i} \Rightarrow \rho_i \circ \iota_i$ of the counit $\rho_i \circ \iota_i \Rightarrow 1_{X_i}$ such that $\Delta_i = \iota_i \circ \delta_i \circ \rho_i$. (Here we have omitted the isomorphism $\iota_i \circ \rho_i \cong f_i$ from the notation). Since Δ_i is an isomorphism, it follows that δ_i is also an isomorphism, and so $\rho_i \circ \iota_i \cong 1_{X_i}$.

By assumption the composite $s_i \cdot r_j = 0$ for $i \neq j$. Note that $s_i \cdot r_j = (\iota_i \circ \rho_i \circ r_j) \cdot (s_i \circ \iota_j \circ \rho_j)$. Precomposing with $(r_i \circ \iota_j \circ \rho_j)$ and postcomposing with $(\iota_i \circ \rho_i \circ s_j)$ gives $1_{\iota_i \circ \rho_i \circ \iota_j \circ \rho_j} = 0$. Left and right composing with ρ_i and ι_j gives $1_{\rho_i \circ \iota_j} = 0$, thus $\rho_i \circ \iota_j = 0$. Altogether then, these 1-morphisms exhibit a direct sum decomposition $X \simeq \boxplus_{i\in I} X_i$, as desired. □

1.2.4 Semisimple 2-categories

The definition of semisimple 2-categories

Recall that a presemisimple 1-category is a linear 1-category in which every object decomposes as a finite direct sum of simple objects, and in which the composition of any two nonzero morphisms between simples is nonzero. A semisimple 1-category is one that is moreover additive (all finite direct sums exist) and idempotent complete (all idempotents split). A presemisimple 2-category is a locally semisimple linear 2-category in which every 1-morphism has a left and a right adjoint, in which every object decomposes as a finite direct sum of simple objects, and in which the endomorphism category of every simple object is an infusion category. (As we have seen, this endomorphism condition implies that the composition of nonzero 1-morphisms between simples is nonzero.) We might expect to define a semisimple 2-category to be a presemisimple 2-category that is moreover additive (all finite direct sums exist) and idempotent complete (all separable monads separably split). But in fact we can sharpen that definition by dropping both the condition that objects decompose as finite direct sums and the condition that endomorphisms of simples are infusion—local semisimplicity and idempotent completeness will conspire to ensure the object-direct-sum-splitting and endomorphism infusion properties of the 2-category.

Definition 1.2.51 (Semisimple 2-category)**.** A *semisimple 2-category* is a locally semisimple 2-category, admitting adjoints for 1-morphisms, that is additive and idempotent complete.

Definition 1.2.52 (Finite semisimple 2-category)**.** A semisimple 2-category is *finite* if it is locally finite semisimple and it has finitely many equivalence classes of simple objects.

Remark 1.2.53 (All 2-functors preserve sums and idempotent splittings)*.* As in Remarks 1.2.7 and 1.2.44, direct sums and idempotent splittings are absolute constructions, that is they are preserved by all linear 2-functors. In particular, we need not restrict attention to a subclass of functors, but can consider all linear 2-functors as the natural morphisms of semisimple 2-categories.

Remark 1.2.54 (Completeness implies the splitting condition for 2-categories)*.* In the 1-categorical case, the characterization of semisimplicity in Proposition I.2.9b) implies that every object splits as a finite direct sum of simples. Similarly, in the 2-categorical case, because we have already assumed a splitting condition at the 1-morphism level via local semisimplicity, and because (by Proposition 1.2.50) in an idempotent complete 2-category splittings of identity 1-morphisms provide splittings of objects, it is again not necessary to impose a further object-level splitting condition in the definition of semisimple 2-category.

Proposition 1.2.55 (Semisimple implies presemisimple)**.** *A semisimple* 2*-category is presemisimple.*

Proof. By Corollary 1.2.19, we need only show that any object is a finite direct sum of objects with simple identity. For any object X, by local semisimplicity of the 2-category, there is a decomposition $1_X \cong \bigoplus_{i \in I} f_i$ into a finite direct sum of simple 1-morphisms f_i. By Proposition 1.2.50, there is a decomposition $X \simeq \boxplus_{i \in I} X_i$ with inclusions and projections $\iota_i \colon X_i \leftrightarrows X : \rho_i$ such that $f_i \cong \iota_i \circ \rho_i$ and $1_{X_i} \cong \rho_i \circ \iota_i$. If 1_{X_i} decomposed into a direct sum, then $f_i \cong \iota_i \circ \rho_i \cong \iota_i \circ 1_{X_i} \circ \rho_i$ would also decompose, contradicting the simplicity of f_i. □

By Proposition 1.2.25, it follows that objects in a semisimple 2-category decompose uniquely:

Corollary 1.2.56 (Decomposition into simples is unique in semisimple 2-categories)**.** *Every object in a semisimple 2-category is a finite direct sum of simple objects, and this decomposition is unique up to permutation and equivalence.*

Semisimple completion of presemisimple 2-categories

Given a presemisimple 2-category $\mathcal{C}$, combining the additive completion $(-)^{\oplus}$ from Construction 1.2.6 and the idempotent completion $(-)^{\triangledown}$ from Construction 1.2.43, we obtain a semisimple 2-category $(\mathcal{C}^{\oplus})^{\triangledown}$. The natural inclusion of presemisimple 2-categories $\mathcal{C} \to (\mathcal{C}^{\oplus})^{\triangledown}$ is of course not an equivalence, but we expect, at least when $\mathcal{C}$ is finite, that it is a 2-profunctor equivalence in the following sense. The natural notion of morphism between finite presemisimple 2-categories is not a 2-functor but a 2-profunctor (also called a 2-distributor): given finite presemisimple 2-categories $\mathcal{C}$ and $\mathcal{D}$, a *finite semisimple 2-profunctor* $M : \mathcal{C} \nrightarrow \mathcal{D}$ is a k-bilinear 2-functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to 2\mathrm{Vect}_k$ from the product to the 2-category $2\mathrm{Vect}_k$ of finite semisimple linear 1-categories. Finite presemisimple 2-categories thus form a 3-category with 1-morphisms the 2-profunctors, 2-morphisms the profunctor transformations, and 3-morphisms the profunctor modifications. A *2-profunctor equivalence* of finite presemisimple 2-categories is an equivalence in that 3-category. Though theoretically straightforward enough, in practice it is difficult to determine when finite presemisimple 2-categories are 2-profunctor equivalent—one is better off working with their corresponding (completed) semisimple 2-categories.

By contrast with finite presemisimple 2-categories, the natural notion of morphism between finite semisimple 2-categories is simply a 2-functor. Indeed, every finite semisimple 2-profunctor between finite semisimple 2-categories is in fact a 2-functor—this is established in Section 1.2.4. Altogether, finite semisimple 2-categories, with 2-functors, natural transformations, and modifications, form a 3-category. We expect the inclusion from finite semisimple to finite presemisimple 2-categories and the completion from finite presemisimple to finite semisimple 2-categories form an equivalence between the corresponding 3-categories.

Remark 1.2.57 (Dimension is invariant under completion)*.* The dimension of a presemisimple 2-category is invariant under semisimple completion. The fact that simple objects in a presemisimple 2-category can be detected by whether their identities are simple ensures that a simple object of a presemisimple 2-category remains simple during idempotent completion. It follows that the set of components (and the dimension of each component) is unchanged by idempotent completion of a presemisimple 2-category, and therefore the overall dimension is similarly unaffected. More generally, we expect the dimension of a presemisimple 2-category is invariant under 2-profunctor equivalence.

Semisimple 2-categories are module 2-categories of multifusion categories

Over a field of characteristic zero, in the 2-category of algebras, bimodules, and intertwiners, an algebra is fully dualizable if and only if it is finite-dimensional semisimple. The category of finite-dimensional modules of a finite-dimensional semisimple algebra is a finite semisimple 1-category. And in fact every finite semisimple 1-category is such a category of modules [BDSV15].

Analogously, over a field of characteristic zero, in the 3-category of finite tensor categories, bimodule categories, bimodule functors, and bimodule intertwiners, a finite tensor category is fully dualizable if and only if it is multifusion [DSPS17b]. The modules over a multifusion category is the prototypical semisimple 2-category; moreover, in fact every finite semisimple 2-category has this form. We now prove this correspondence between semisimple 2-categories and module 2-categories for multifusion categories, under our standing assumption that the base field is algebraically closed of characteristic zero.

Theorem 1.2.58 (The module 2-category of a multifusion category is semisimple)**.** *The 2-category of finite semisimple module categories of a multifusion category is a finite semisimple 2-category.*

Proof. Let C be a multifusion category and let Mod(C) denote the 2-category of finite semisimple right module categories of C. Given finite semisimple module categories $\mathrm{M_C}$ and $\mathrm{N_C}$ it is proven in [DSPS17b, Cor 2.5.6] that the category $\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{M}, \mathrm{N})$ is finite semisimple. By [DSPS17a, EO04] we know that Mod(C) is a 2-category in which every 1-morphism has a right and left adjoint.

The existence of a zero object and direct sums is immediate. The fact that separable monads split follows from the fact that a separable monad $p : \mathrm{N}_{\mathrm{C}} \to \mathrm{N}_{\mathrm{C}}$ in $\mathrm{Mod(C)}$ is a separable algebra in $\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})$ and as such gives rise to a finite semisimple right module category $p\text{−Mod}$ over $\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})$. The $\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})$-module functor $\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N}) \to \mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})$ corresponding to the object $p \in \mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})$ can then be split as the following composite of $\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})$-module functors:

$$\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N}) \xrightarrow{{}_p p \otimes -} p\text{−Mod} \xrightarrow{p \otimes_p -} \mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})$$

This splitting is separable; under the monoidal equivalence

$$p\text{−Mod−}p \to \mathrm{Hom}_{\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})}(p - \mathrm{Mod}, p - \mathrm{Mod})$$

the counit of the adjunction $p \otimes_p - \vdash_p p \otimes -$ corresponds to the multiplication $m : {}_p p \otimes p_p \Rightarrow p$ of p. Hence, the right inverse $\Delta : {}_p p_p \Rightarrow {}_p p \otimes p_p$ of m in $p\text{−Mod−}p$ gives rise to a right inverse of the counit in $\mathrm{Hom}_{\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{N},\mathrm{N})}(p\text{−Mod}, p\text{−Mod})$. Composing with $- \boxdot_{\mathrm{Hom}_{\mathrm{Mod(C)}}(N,N)} N$ and noting that $\mathrm{Hom}_{\mathrm{Mod(C)}}(N,N) \boxdot_{\mathrm{Hom}_{\mathrm{Mod(C)}}(N,N)} N \simeq N$, induces the required splitting of the C-module functor

$$p : \mathrm{N} \to p\text{−Mod} \boxdot_{\mathrm{Hom}_{\mathrm{C}}(\mathrm{N},\mathrm{N})} \mathrm{N} \to \mathrm{N}.$$

Finally, it is proven in [EGNO15, Cor 9.1.6] that every multifusion category admits only a finite number of equivalence classes of indecomposable module categories, and thus $\mathrm{Mod(C)}$ has only finitely many simple objects. □

Theorem 1.2.59 (A semisimple 2-category is modules for a multifusion category)**.** *Every finite semisimple 2-category is equivalent to the 2-category of finite semisimple module categories of a multifusion category.*

Proof. Let $\{X_i \mid i \in I\}$ denote a set of representatives of the equivalence classes of simple objects of the finite semisimple 2-category $\mathcal{C}$. Define the object $X := \boxplus_{i\in I} X_i$ and the multifusion category $\mathrm{C} := \mathrm{Hom}_{\mathcal{C}}(X,X)$, and let $\mathrm{Mod(C)}$ denote the 2-category of finite semisimple module categories of C. Observe that for any object $c \in \mathcal{C}$, the category $\mathrm{Hom}_{\mathcal{C}}(X,c)$ has a right C-module structure, and any 1-morphism $f : c \to d$ defines a module functor $\mathrm{Hom}_{\mathcal{C}}(X,f) := f \circ - : \mathrm{Hom}_{\mathcal{C}}(X,c) \to \mathrm{Hom}_{\mathcal{C}}(X,d)$. We will show that the 2-functor $\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathrm{Mod(C)}$ is an equivalence.

1. *Essential surjectivity on objects.* Let M be a finite semisimple C-module category. Following [DSPS17b, Cor 2.6.9], there is a separable algebra m in $\mathrm{C} =$

$\mathrm{Hom}_{\mathcal{C}}(X,X)$ such that $\mathrm{M} \simeq m\text{−Mod}$ as C-module categories. In particular, $m : X \to X$ is a separable monad in $\mathcal{C}$ and therefore admits an Eilenberg–Moore object X^m in $\mathcal{C}$ (see Appendix B.2). In particular, there is a left m-module $R : X^m \to X$ in $\mathcal{C}$ such that the induced functor $R \circ - : \mathrm{Hom}_{\mathcal{C}}(X, X^m) \to \mathrm{LMod}_m(X) = m\text{−Mod} \simeq \mathrm{M}$ is an equivalence. Since this functor is defined by left composition with a 1-morphism and the action of $\mathrm{C} = \mathrm{Hom}_{\mathcal{C}}(X,X)$ is by right composition, it inherits the structure of an equivalence of module categories.

2. *Essential surjectivity on* 1*-morphisms.* Let c and d be objects of $\mathcal{C}$. We show that the functor $\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathrm{Hom}_{\mathcal{C}}(c,d) \to \mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{Hom}_{\mathcal{C}}(X,c), \mathrm{Hom}_{\mathcal{C}}(X,d))$ is essentially surjective on objects. Since any linear 2-functor preserves direct sums, it suffices to prove this under the assumption that c and d are simple objects X_i and X_j. Let $\{\iota_i : X_i \leftrightarrows X : \rho_i\}_{i\in I}$ be a direct sum decomposition of $X \simeq \boxplus_{i\in I} X_i$ and fix isomorphisms $\lambda_i : 1_{X_i} \Rightarrow \rho_i \circ \iota_i$. Given a module functor $\Psi : \mathrm{Hom}_{\mathcal{C}}(X, X_i) \to \mathrm{Hom}_{\mathcal{C}}(X, X_j)$ with coherence isomorphism $\psi_{a,b} : \Psi(a \circ b) \Rightarrow \Psi(a) \circ b$ for $a \in \mathrm{Hom}_{\mathcal{C}}(X, X_i)$, $b \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, we define $h := \Psi(\rho_i) \circ \iota_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, X_j)$ and claim that $\mathrm{Hom}_{\mathcal{C}}(X,h)$ is naturally isomorphic to Ψ as a $\mathrm{Hom}_{\mathcal{C}}(X,X)$-module functor. Indeed, the following natural transformation provides such an isomorphism:

$$\left\{\eta_s : \Psi(s) \xRightarrow{\Psi(\lambda_i \circ s)} \Psi(\rho_i \circ \iota_i \circ s) \xRightarrow{\psi_{\rho_i, \iota_i s}} \Psi(\rho_i) \circ \iota_i \circ s = h \circ s\right\}_{s \in \mathrm{Hom}_{\mathcal{C}}(X, X_i)}$$

3. *Fully faithful on* 2*-morphisms.* Given 1-morphisms $f, g : X_i \to X_j$ we will now show that the map $\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathrm{Hom}_{\mathcal{C}}(f,g) \to \mathrm{Hom}_{\mathrm{Mod(C)}}(f \circ -, g \circ -)$ is an isomorphism. Indeed, a transformation between the module functors $f \circ -$ and $g \circ -$ is given by a natural transformation $\{\eta_s : f \circ s \Rightarrow g \circ s\}_{s \in \mathrm{Hom}_{\mathcal{C}}(X, X_i)}$ fulfilling $\eta_{s \circ r} = \eta_s \circ r$ for all $s \in \mathrm{Hom}_{\mathcal{C}}(X, X_i)$ and $r \in \mathrm{Hom}_{\mathcal{C}}(X,X)$. The map $\mathrm{Hom}_{\mathcal{C}}(X,-)$ is injective: if $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(f,g)$ fulfill that $\alpha \circ s = \beta \circ s$ for all $s \in \mathrm{Hom}_{\mathcal{C}}(X, X_i)$, then $\alpha \circ \rho_i = \beta \circ \rho_i$ and thus $\alpha = \beta$ by right invertibility of the 1-morphism ρ_i. It is also surjective: given a natural transformation of module functors η one can define the following 2-morphism $\alpha : f \Rightarrow g$ which fulfills $\alpha \circ s = \eta_s$:

$$\alpha : f \xRightarrow{f \circ \lambda_i} f \circ \rho_i \circ \iota_i \xRightarrow{\eta_{\rho_i} \circ \iota_i} g \circ \rho_i \circ \iota_i \xRightarrow{g \circ \lambda_i^{-1}} g$$ □

Remark 1.2.60 (Characterizing semisimple 2-categories over non-algebraically-closed fields). We expect that Theorems 1.2.58 and 1.2.59 hold as stated over a field of characteristic zero (not necessarily algebraically closed) provided there are only finitely many isomorphism classes of finite-dimensional division algebras over the field.

We expect taking the module 2-category gives an equivalence of 3-categories from multifusion categories (and their finite semisimple bimodules, bimodule functors, bimodule transformations) to semisimple 2-categories (and their 2-functors, transformations, modifications). Though semisimple 2-categories can be faithfully modeled by multifusion categories, semisimple 2-categories have a crucial theoretical advantage over multifusion categories: *while additional structure on multifusion categories must be encoded in a system of bimodule categories and relations between their relative tensor products, additional structure on semisimple 2-categories can be encoded functorially.* (See the next Section 1.2.4, which establishes that any finite semisimple 2-profunctor between semisimple categories is a 2-functor.) In particular, an additional monoidal operation on a multifusion category C would take the form of a $(\mathrm{C} \boxtimes \mathrm{C})$-C-bimodule category, whereas a monoidal structure on a semisimple 2-category $\mathcal{C}$ is simply a bilinear 2-functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

2-profunctors between semisimple 2-categories are 2-functors

Recall that a (finite semisimple) 2-profunctor $\mathcal{C} \nrightarrow \mathcal{D}$ between finite semisimple 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a k-bilinear 2-functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to 2\mathrm{Vect}_k$, where $2\mathrm{Vect}_k$ is the 2-category of finite semisimple 1-categories. Because every finite semisimple 2-category is locally finite semisimple, we may consider the 'absolute Yoneda embedding' as a 2-functor $y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Func}\,(\mathcal{C}^{\mathrm{op}}, 2\mathrm{Vect}_k)$, where Func denotes the 2-category of linear 2-functors.

The following result is a categorification of Proposition I.3.11.

Proposition 1.2.61 (The absolute Yoneda embedding of finite semisimple 2-categories is an equivalence)**.** *For $\mathcal{C}$ a finite semisimple 2-category, the absolute Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Func}\,(\mathcal{C}^{\mathrm{op}}, 2\mathrm{Vect}_k)$ is an equivalence.*

Proof. By the Yoneda lemma for 2-categories, the embedding $y_{\mathcal{C}}$ is an equivalence on 1-morphism categories; it therefore suffices to show $y_{\mathcal{C}}$ is essentially surjective, i.e. that any $2\mathrm{Vect}_k$-valued presheaf on $\mathcal{C}$ is representable. By Theorem 1.2.59, there is a multifusion category C such that $\mathcal{C} \simeq \mathrm{Mod}(\mathrm{C})$. Given a presheaf $P \in \mathrm{Func}(\mathrm{Mod}(\mathrm{C})^{\mathrm{op}}, 2\mathrm{Vect}_k)$, note that the finite semisimple 1-category $P(\mathrm{C}_{\mathrm{C}})$ is a right C-module by precomposition with the left action of C.

We claim that P is represented by this C-module $P(\mathrm{C}_{\mathrm{C}})$, that is there is an equivalence $P \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathrm{C})}\,(-, P(\mathrm{C}_{\mathrm{C}}))$. By Corollary 1.2.48, two 2-functors $\mathrm{Mod}(\mathrm{C})^{\mathrm{op}} \to 2\mathrm{Vect}_k$ are equivalent if their restrictions to the delooping $(\mathrm{BC})^{\mathrm{op}}$ are equivalent. Thus it suffices to show that $P(\mathrm{C}_{\mathrm{C}})$ is equivalent to $\mathrm{Hom}_{\mathrm{Mod}(\mathrm{C})}\,(\mathrm{C}_{\mathrm{C}}, P(\mathrm{C}_{\mathrm{C}}))$ as right

C-module categories. For any right C-module M_{C}, evaluation at the tensor unit $\mathrm{I} \in \mathrm{C}$ induces an equivalence $\mathrm{Hom}_{\mathrm{Mod(C)}}(\mathrm{C}_{\mathrm{C}}, M_{\mathrm{C}}) \to M_{\mathrm{C}}$, and so in particular does so for the module $P(\mathrm{C}_{\mathrm{C}})$, as required. $\square$

Analogously to Corollary I.3.12, we immediately obtain the following corollary of Proposition 1.2.61.

Corollary 1.2.62 (2-profunctors between semisimple 2-categories are 2-functors)**.** *Every finite semisimple 2-profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ between finite semisimple 2-categories is a 2-functor; that is, there is a 2-functor $F : \mathcal{C} \to \mathcal{D}$ such that $P(-,-) \simeq \mathrm{Hom}_{\mathcal{D}}(-, F-)$.*

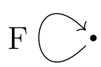
Proof. The 2-profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ corresponds to a 2-functor $\mathcal{C} \to \mathrm{Func}(\mathcal{D}^{\mathrm{op}}, 2\mathrm{Vect}_k)$. Postcomposing with the equivalence $\mathrm{Func}(\mathcal{D}^{\mathrm{op}}, 2\mathrm{Vect}_k) \simeq \mathcal{D}$ provided by Proposition 1.2.61 yields a 2-functor $\mathcal{C} \to \mathcal{D}$ representing P. $\square$

Examples of semisimple 2-categories

The canonical example of a semisimple 1-category is the 1-category Vect_k of finite-dimensional vector spaces, with linear functions as morphisms. Analogously, the canonical example of a semisimple 2-category is the 2-category of finite semisimple linear 1-categories, with linear functors and natural transformations as 1-morphisms and 2-morphisms; we denote this semisimple 2-category by $2\mathrm{Vect}_k$. Note that $2\mathrm{Vect}_k$ is the 2-category of finite semisimple module categories over the fusion category Vect_k [KV94, BDSV15].

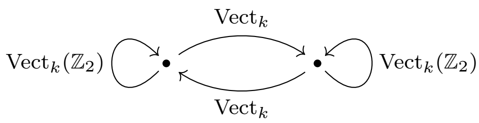
Module 2-categories of multifusion categories. As discussed in Section 1.2.4, the finite semisimple module categories of a multifusion 1-category form a finite semisimple 2-category, and any finite semisimple 2-category is of this form.

Example 1.2.63 (Fusion category with a unique indecomposable module category). Let F be a fusion category with a unique indecomposable module category, namely F itself, for instance the Fibonacci fusion category or the Ising fusion category. Then the semisimple 2-category $\mathrm{Mod(F)}$ of modules has a unique simple object F, which has (F-module) endomorphism fusion category again equivalent to F. The 2-category may therefore be schematically depicted as:



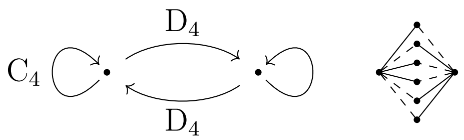
Note that in this special situation, the 2-category Mod(F) is equivalent to the additive completion of the delooped 2-category BF.

Example 1.2.64 (Module categories for $\mathrm{Vect}_k(\mathbb{Z}_2)$). Over an algebraically closed field of characteristic zero, the fusion category $\mathrm{Vect}_k(\mathbb{Z}_2)$ of $\mathbb{Z}_2$-graded vector spaces has, up to equivalence, two indecomposable module categories, namely Vect_k and $\mathrm{Vect}_k(\mathbb{Z}_2)$; the 2-category $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ thus has two corresponding simple objects. Both these simple objects have endomorphism categories $\mathrm{Vect}_k(\mathbb{Z}_2)$ (for the module Vect_k, though every endomorphism is trivial as a functor, there is a sign choice in the module functor structure), and $\mathrm{Hom}_{\mathrm{Vect}_k(\mathbb{Z}_2)}(\mathrm{Vect}_k, \mathrm{Vect}_k(\mathbb{Z}_2))$ and $\mathrm{Hom}_{\mathrm{Vect}_k(\mathbb{Z}_2)}(\mathrm{Vect}_k(\mathbb{Z}_2), \mathrm{Vect}_k)$ are both simply Vect_k. This 2-category may therefore be depicted as:



Note well, as visible here, how different the structure of semisimple 2-categories is from that of semisimple 1-categories: in a semisimple 2-category there can be nontrivial 1-morphisms between inequivalent simple objects.

Example 1.2.65 (Module categories for quantum $\mathfrak{sl}_2$ at level 4). Let C_4 denote the fusion category of representations of quantum $\mathfrak{sl}_2$ at level 4; this category has five simple objects, denoted $V_0, \ldots, V_4$. The indecomposable module categories of this fusion category (and more generally of quantum $\mathfrak{sl}_2$ at other levels) have been classified by Ocneanu [Ocn00] and Ostrik [Ost03a]: there is an indecomposable module category called A_5, coming from the action of C_4 on itself, and an indecomposable module category called D_4, whose corresponding algebra object is $V_0 \oplus V_4$. The module endomorphisms of A_5 is of course just C_4, and $\mathrm{Hom}_{\mathrm{C}_4}(\mathrm{A}_5, \mathrm{D}_4)$ and $\mathrm{Hom}_{\mathrm{C}_4}(\mathrm{D}_4, \mathrm{A}_5)$ are both simply D_4. The endomorphism fusion category $\mathrm{End}_{\mathrm{C}_4}(\mathrm{D}_4)$ of D_4, originally computed in [Ocn00] and described in more detail in [Got10], has eight simple objects and fusion graph that may be depicted (in the picture of the 2-category $\mathrm{Mod}(\mathrm{C}_4)$) as follows:



Here in $\mathrm{End}_{\mathrm{C}_4}(\mathrm{D}_4)$, the top node is the tensor unit, the left and right nodes are generators, and the solid and dashed lines give fusion with these generators, respectively.

Example 1.2.66 (Decomposing the module 2-category of a multi-component multifusion category). The 2-category of module categories of a multifusion category $\mathrm{M} = \mathrm{M}^1 \oplus \ldots \oplus \mathrm{M}^n$ with indecomposable components M^i is equivalent to the direct sum of 2-categories $\boxplus_i \mathrm{Mod}(\mathrm{M}^i)$. (See Remark 1.2.27.) Let $\mathrm{M}^i = \oplus_{j,k} \mathrm{M}^i_{jk}$ be the decomposition of M^i into fusion categories M^i_{jj} and their bimodule categories. By [EGNO15, Prop 7.17.5], for any j, there is an invertible bimodule (namely $\oplus_j \mathrm{M}^i_{jk}$) between the multifusion category M^i and the fusion category M^i_{jj}, and so, picking some index j_i for each i, we have $\mathrm{Mod}(\mathrm{M})$ is equivalent to $\boxplus_i \mathrm{Mod}(\mathrm{M}^i_{j_i j_i})$. (This presentation is convenient later when we consider monoidal structures on these semisimple 2-categories.)

Representations of groupoids. Many examples of semisimple 1-categories arise by taking a category of functors into Vect_k, and a natural family of domains for such functors are 1-groupoids.

Notation 1.2.67 (n-groupoids via homotopy groups). Recall that an n-groupoid is a homotopy n-type, that is a space whose only nontrivial homotopy groups occur in dimensions $0, 1, \ldots, n$. We will suppress k-invariant information from the notation and denote an n-groupoid simply by the tuple $(\pi_0, \pi_1, \ldots, \pi_n)$, where π_0 is a discrete set (namely the component set of the space), and π_i for $i > 0$ is a family of discrete groups indexed by π_0 (namely the family of the i-th homotopy groups of the various components of the space).

Alternatively, an n-groupoid may be viewed as an n-category all of whose morphisms are invertible. From that perspective, the k-invariant information is encoded in the weak structure data, that is the higher units, associators, and interchangers of the n-category. In the categorical view, the set π_0 is the set of equivalence classes of objects of the n-category, and the family π_i records, for each component of the n-category (i.e. element of π_0), the set of equivalence classes of automorphisms of the $(i-1)$-fold identity on an object in that component.

A 0-groupoid is therefore denoted simply π_0; a 1-groupoid with no nontrivial automorphisms is denoted $(\pi_0, *)$—here $*$ refers to the π_0-indexed family of groups that is trivial in every component; a connected 1-groupoid is denoted $(*, \pi_1)$; a 2-groupoid with no nontrivial 2-automorphisms is denoted $(\pi_0, \pi_1, *)$; a connected 2-groupoid is denoted $(*, \pi_1, \pi_2)$.

The notation permits the addition of monoidal structures. Recall that an n-group is by definition a connected n-groupoid. Thought of directly as an n-groupoid, this would be denoted $(*, \pi_1, \ldots, \pi_n)$, but we may instead consider its loop space as an $(n-1)$-groupoid with a (grouplike) monoidal structure; that grouplike monoidal $(n-1)$-groupoid will be denoted $(\pi_1, \ldots, \pi_n)$. Similarly, a (grouplike) k-fold monoidal n-group is, by definition, an $(n+k-1)$-groupoid $(\pi_0, \ldots, \pi_{n+k-1})$ such that π_i is trivial for $0 \leq i \leq k-1$. The k-th loop space of that $(n+k-1)$-groupoid is a k-fold monoidal $(n-k)$-groupoid denoted $(\pi_k, \ldots, \pi_n)$. For instance, a grouplike monoidal 0-groupoid (that is, a group) is denoted π_1; a grouplike 2-fold monoidal 0-groupoid (that is, an abelian group) is denoted π_2; a grouplike monoidal 1-groupoid (that is, the loop space of a 2-group) is denoted (π_1, π_2); a grouplike 2-fold monoidal 1-groupoid (that is, the double loop space of a 3-group) is denoted (π_2, π_3); a grouplike monoidal 2-groupoid (that is, the loop space of a 3-group) is denoted (π_1, π_2, π_3).

An n-groupoid is called finite when it has finitely many components and all the homotopy groups of every component are finite.

Recollection 1.2.68 (1-category of 1-representations of a 1-groupoid)*.* A representation of a finite 1-groupoid (π_0, π_1) is a 1-functor $(\pi_0, \pi_1) \to \mathrm{Vect}_k$. The linear 1-category $\mathrm{Rep}(\pi_0, \pi_1) := [(\pi_0, \pi_1), \mathrm{Vect}_k]$ of representations of a finite 1-groupoid (π_0, π_1) is a finite semisimple 1-category.

As a special case, we may think of a group π_1 as a connected 1-groupoid $(*, \pi_1)$, and consider the category of representations $\mathrm{Rep}(*, \pi_1) \equiv [(*, \pi_1), \mathrm{Vect}_k]$; the objects of this category are called simply "π_1-representations". As another special case, we may think of a set π_0 as a discrete 1-groupoid $(\pi_0, *)$, and consider the category of representations $\mathrm{Rep}(\pi_0, *) \equiv [(\pi_0, *), \mathrm{Vect}_k]$; the objects of this category are called "π_0-graded vector spaces". We summarize this situation in the following table:

Input	Notation		Definition	Name
groupoid (π_0, π_1)	$\mathrm{Rep}(\pi_0, \pi_1)$	$:=$	$[(\pi_0, \pi_1), \mathrm{Vect}_k]$	(π_0, π_1)-representation
group π_1	$\mathrm{Rep}(*, \pi_1)$	$:=$	$[(*, \pi_1), \mathrm{Vect}_k]$	π_1-representation
set π_0	$\mathrm{Rep}(\pi_0, *)$	$:=$	$[(\pi_0, *), \mathrm{Vect}_k]$	π_0-graded vector space

Given a group π_1, one may of course think of it as a discrete (grouplike monoidal) 1-groupoid $(\pi_1, *)$, and therefore consider the category $\mathrm{Vect}_k(\pi_1) := \mathrm{Rep}(\pi_1, *)$ of π_1-graded vector spaces. We defer attention to that case until later, when we are concerned with monoidal structures on these semisimple 1-categories. We then consider also more generally the category $\mathrm{Rep}(\pi_1, \pi_2)$ of representations of a 2-group (π_1, π_2).

2-representations of 2-groupoids. Many examples of semisimple 2-categories arise by taking a 2-category of 2-functors into $2\mathrm{Vect}_k$.

Example 1.2.69 (2-category of 2-representations of a 2-groupoid). For a finite 2-groupoid (π_0, π_1, π_2), a representation of (π_0, π_1, π_2) is a (weak) 2-functor $(\pi_0, \pi_1, \pi_2) \to 2\mathrm{Vect}_k$. The linear 2-category $2\mathrm{Rep}(\pi_0, \pi_1, \pi_2) := [(\pi_0, \pi_1, \pi_2), 2\mathrm{Vect}_k]$ of 2-representations of a finite 2-groupoid (π_0, π_1, π_2) is a finite semsimple 2-category; this can be seen as follows.

Given a finite 2-groupoid (π_0, π_1, π_2), let $k(\pi_0, \pi_1, \pi_2)$ denote its linearization, that is the linear 2-category with the same objects and 1-morphisms as the 2-groupoid, and with 2-morphism vector spaces freely generated by the 2-morphism sets of the 2-groupoid. Let $\widehat{k}(\pi_0, \pi_1, \pi_2)$ denote the finite presemisimple 2-category obtained by local additive and local idempotent completion of $k(\pi_0, \pi_1, \pi_2)$. Then let $\overline{k}(\pi_0, \pi_1, \pi_2)$ denote the multifusion category obtained by folding $\widehat{k}(\pi_0, \pi_1, \pi_2)$. Observe that the 2-category of 2-functors $[(\pi_0, \pi_1, \pi_2), 2\mathrm{Vect}_k]$ is equivalent to the 2-category of modules $\mathrm{Mod}(\overline{k}(\pi_0, \pi_1, \pi_2))$, and therefore that $2\mathrm{Rep}(\pi_0, \pi_1, \pi_2)$ is a finite semisimple 2-category, as desired.

There are a number of important special cases of this example of 2-representations of a 2-groupoid. We summarize them in the following table:

Input	Notation		Definition	Name
2-grpoid (π_0, π_1, π_2)	$2\mathrm{Rep}(\pi_0, \pi_1, \pi_2)$	$:=$	$[(\pi_0, \pi_1, \pi_2), 2\mathrm{Vect}_k]$	(π_0, π_1, π_2)-2-representation
2-group (π_1, π_2)	$2\mathrm{Rep}(*, \pi_1, \pi_2)$	$:=$	$[(*, \pi_1, \pi_2), 2\mathrm{Vect}_k]$	(π_1, π_2)-2-representation
1-grpoid (π_0, π_1)	$2\mathrm{Rep}(\pi_0, \pi_1, *)$	$:=$	$[(\pi_0, \pi_1, *), 2\mathrm{Vect}_k]$	(π_0, π_1)-graded 2-vector space
ab group π_2	$2\mathrm{Rep}(*, *, \pi_2)$	$:=$	$[(*, *, \pi_2), 2\mathrm{Vect}_k]$	π_2-2-representation
group π_1	$2\mathrm{Rep}(*, \pi_1, *)$	$:=$	$[(*, \pi_1, *), 2\mathrm{Vect}_k]$	π_1-2-vepresentation
set π_0	$2\mathrm{Rep}(\pi_0, *, *)$	$:=$	$[(\pi_0, *, *), 2\mathrm{Vect}_k]$	π_0-graded 2-vector space

Of course, given a group π_1, one may treat it as a discrete (grouplike monoidal) 2-groupoid $(\pi_1, *, *)$, and consider the 2-category $2\mathrm{Vect}_k(\pi_1) := 2\mathrm{Rep}(\pi_1, *, *)$ of π_1-graded 2-vector spaces. We defer discussion of that case to later attention to monoidal structures, where we also discuss more generally the 2-category $2\mathrm{Rep}(\pi_1, \pi_2, \pi_3)$ of 2-representations of a 3-group (π_1, π_2, π_3).

Example 1.2.70 (π_0-graded 2-vector spaces). For a finite set π_0, the 2-category $2\mathrm{Rep}(\pi_0, *, *)$ is the 2-category of π_0-graded finite semisimple 1-categories $\mathrm{C} = \bigoplus_{x \in \pi_0} \mathrm{C}_x$, with 1-morphisms the grading-preserving functors and 2-morphisms the natural transformations.

The simple objects of this 2-category are of the form $[x] := \bigoplus_{y \in \pi_0} \mathrm{C}_y$ with $\mathrm{C}_x = \mathrm{Vect}_k$ and $\mathrm{C}_y = 0$ for $y \neq x$. In this case, the 1-morphism category $\mathrm{Hom}_{2\mathrm{Rep}(\pi_0,*,*)}([x], [y])$ is Vect_k if $x = y$ and zero otherwise.

Example 1.2.71 (π_1-2-vepresentations are modules for graded vector spaces). For a finite group π_1, the 2-category $2\mathrm{Rep}(*, \pi_1, *)$ is the 2-category of finite semisimple 1-categories with a (weak) π_1-action, with 1-morphisms the intertwining functors and 2-morphisms the natural transformations. The structure of a π_1-action on a semisimple 1-category is equivalent to the structure of an action of the fusion 1-category $\mathrm{Vect}_k(\pi_1)$ of π_1-graded vector spaces (with the monoidal structure induced by the group structure on π_1). Thus the 2-category $2\mathrm{Rep}(*, \pi_1, *)$ is equivalent to the 2-category $\mathrm{Mod}(\mathrm{Vect}_k(\pi_1))$.

For instance, the 2-category $2\mathrm{Rep}(*, \mathbb{Z}_2, *)$ is equivalent to the 2-category $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ described in Example 1.2.64, with two simple objects having nonzero non-equivalence morphisms between them. More generally, the indecomposable module categories of $\mathrm{Vect}_k(\pi_1)$, hence the simple objects of $2\mathrm{Rep}(*, \pi_1, *)$, have been classified by Ostrik [Ost03a].

Example 1.2.72 (π_2-2-representations). For a finite abelian group π_2, the 2-category $2\mathrm{Rep}(*, *, \pi_2)$ is the 2-category of finite semisimple 1-categories C with a group homomorphism $\phi : \pi_2 \to \mathrm{Aut}(\mathrm{id} : \mathrm{C} \to \mathrm{C})$, with 1-morphisms the functors $F : \mathrm{C} \to \mathrm{C}'$ such that $F \circ \phi(g) = \phi'(g) \circ F$ for all $g \in \pi_2$ and 2-morphisms the natural transformations.

An object $(\mathrm{C}, \phi) \in 2\mathrm{Rep}(*, *, \pi_2)$ is simple if and only if $\mathrm{C} \simeq \mathrm{Vect}_k$. Hence the simple objects of $2\mathrm{Rep}(*, *, \pi_2)$ correspond to group homomorphisms $\phi : \pi_2 \to k^*$, and the 1-morphism category $\mathrm{Hom}_{2\mathrm{Rep}(*,*,\pi_2)}(\phi, \phi')$ is Vect_k if $\phi = \phi'$ and zero otherwise. Thus it happens that $2\mathrm{Rep}(*, *, \pi_2)$ is equivalent to $2\mathrm{Rep}(\mathrm{Hom}(\pi_2, k^*), *, *)$.

The equivalence classes of objects and Hom categories of the common generalization of this example and the previous example, namely $2\mathrm{Rep}(*, \pi_1, \pi_2)$, have been investigated by Elgueta [Elg07].

Remark 1.2.73 (Completion produces non-invertible simple 1-morphisms). Even though, by the discussion in Example 1.2.69, the 2-category $2\mathrm{Rep}(\pi_0, \pi_1, \pi_2)$ is the semisimple completion of the linearization $k(\pi_0, \pi_1, \pi_2)$ of the 2-groupoid (π_0, π_1, π_2), it can certainly happen that $2\mathrm{Rep}(\pi_0, \pi_1, \pi_2)$ has non-invertible simple 1-morphisms. Such morphisms are seen, for instance, in $2\mathrm{Rep}(*, \mathbb{Z}_2, *) \simeq \mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ from Remark 1.2.71 and Example 1.2.64.

Remark 1.2.74 (The dimension of groupoid-graded 2-vector spaces is the groupoid cardinality). For a finite set π_0, the semisimple 2-category $2\mathrm{Rep}(\pi_0, *, *)$ evidently has dimension the order of π_0. For a finite group π_1, by the preceding remark and Remark 1.2.57, we see that the dimension of the semisimple 2-category $2\mathrm{Rep}(*, \pi_1, *)$ is the dimension of the presemisimple 2-category $\mathrm{BVect}_k(\pi_1)$. As the global dimension of the fusion category $\mathrm{Vect}_k(\pi_1)$ is $|\pi_1|$, the dimension of $\mathrm{BVect}_k(\pi_1)$ is $1/|\pi_1|$. More generally, for a finite 1-groupoid (π_0, π_1), the dimension of the 2-category of (π_0, π_1)-graded 2-vector spaces $2\mathrm{Rep}(\pi_0, \pi_1, *)$ is the groupoid cardinality [BHW10] of (π_0, π_1), that is the sum over components of the reciprocal of the size of the automorphism groups.

1.3 On 4-vector spaces

A key advantage of the 2-category $2\mathrm{Vect}_k$ of finite semisimple categories over the equivalent 2-category $\mathrm{SSAlg}(\mathrm{Vect}_k)$ of finite-dimensional semisimple algebras is that the 1-morphisms in $2\mathrm{Vect}_k$ are functors whereas the 1-morphisms in $\mathrm{SSAlg}(\mathrm{Vect}_k)$ are bimodules — in particular, it is much easier to study (weak) monoid objects in $2\mathrm{Vect}_k$ (that is, monoidal finite semisimple categories) than it is to study (weak) monoid objects in $\mathrm{SSAlg}(\mathrm{Vect}_k)$ (that is, 'semisimple sesquialgebras' — see footnote 16), even though these concepts are equivalent. Similarly, even though every finite semisimple 2-category is the 2-category of finite semisimple module categories of a multifusion category, it follows from Corollary I.3.12 that every $2\mathrm{Vect}_k$-enriched 2-profunctor between finite semisimple 2-categories is representable. Hence, whereas the natural morphisms between multifusion categories are bimodule categories, the natural morphisms between finite semisimple 2-categories are 2-functors.

This allows us to take a first glimpse at '4-vector spaces', which we define as certain well-behaved monoid objects in the 3-category of finite semisimple 2-categories, or equivalently, as certain well behaved monoidal finite semisimple 2-categories. The fact that these 'fusion 2-categories' give rise to 4-manifold invariants (see Chapter 2) strongly suggests that fusion 2-categories are indeed fully dualizable objects in an appropriate 4-category.

1.3.1 Fusion 2-categories

The definition of fusion 2-categories

Monoidal structures on 2-categories. We will work with semistrict monoidal 2-categories, meaning that the underlying 2-category is strict and the monoidal structure is strictly unital and associative, though there may be a nontrivial interchange isomorphism between the two distinct ways of taking the monoidal product of two 1-morphisms. Such semistrict monoidal 2-categories first appeared in Gordon–Powers–Street [GPS95] and are often called "Gray monoids". A recent presentation of the notion occurs in Barrett–Meusburger–Schaumann (BMS) [BMS12]. Though structured somewhat differently, our notion of monoidal 2-category is equivalent to the BMS definition.

Definition 1.3.1 (Monoidal 2-category)**.** A *monoidal* 2*-category* consists of the following data:

D1. a strict 2-category $\mathcal{C}$;

D2. an "identity" object $\mathrm{I} \in \mathcal{C}$;

D3. strict "left and right tensor product" 2-functors

$$\begin{aligned} L_A &\equiv A \,\Box\, - : \mathcal{C} \to \mathcal{C} \\ R_A &\equiv - \,\Box\, A : \mathcal{C} \to \mathcal{C}, \end{aligned}$$

for each object $A \in \mathcal{C}$;

D4. an "interchange" 2-isomorphism

$$\phi_{f,g} : (f \,\Box\, B') \circ (A \,\Box\, g) \Rightarrow (A' \,\Box\, g) \circ (f \,\Box\, B)$$

for each pair of 1-morphisms $f : A \to A'$ and $g : B \to B'$;

subject to the following conditions:

C1. left and right multiplication agree: $L_A B = R_B A$ for objects $A, B \in \mathcal{C}$;

C2. the tensor product is strictly unital and associative:

$$\begin{aligned} L_{\mathrm{I}} &= \mathrm{id}_{\mathcal{C}} = R_{\mathrm{I}} \\ L_A L_B &= L_{A \Box B} \\ R_B R_A &= R_{A \Box B} \\ L_A R_B &= R_B L_A; \end{aligned}$$

C3. the interchanger respects identities:

$$\begin{aligned} \phi_{f,1_A} &= 1_{f \Box A} \\ \phi_{1_A,f} &= 1_{A \Box f} \end{aligned}$$

for object $A \in \mathcal{C}$ and 1-morphism $f : C \to D$;

C4. the interchanger respects composition:

$$\phi_{f'\circ f,g} = (\phi_{f',g} \circ (f \,\square\, B)) \cdot ((f' \,\square\, B') \circ \phi_{f,g})$$
$$\phi_{f,g'\circ g} = ((A' \,\square\, g') \circ \phi_{f,g}) \cdot (\phi_{f,g'} \circ (A \,\square\, g))$$

for $f : A \to A'$, $f' : A' \to A''$, $g : B \to B'$ and $g' : B' \to B''$;

C5. the interchanger is natural:

$$\phi_{f',g} \cdot ((\alpha \,\square\, B') \circ (A \,\square\, g)) = ((A' \,\square\, g) \circ (\alpha \,\square\, B)) \cdot \phi_{f,g}$$
$$\phi_{f,g'} \cdot ((f \,\square\, B') \circ (A \,\square\, \beta)) = ((A' \,\square\, \beta) \circ (f \,\square\, B)) \cdot \phi_{f,g}$$

for 1-morphisms $f, f' : A \to A', g, g' : B \to B'$ and 2-morphisms $\alpha : f \Rightarrow f'$, $\beta : g \Rightarrow g'$;

C6. the interchanger respects tensor product:

$$\phi_{A\square g,h} = A \,\square\, \phi_{g,h}$$
$$\phi_{f\square B,h} = \phi_{f,B\square h}$$
$$\phi_{f,g\square C} = \phi_{f,g} \,\square\, C$$

for $f : A \to A'$, $g : B \to B'$ and $h : C \to C'$.

Notation 1.3.2 (Horizontal composition of 1-morphisms). Though the tensor product of a monoidal 2-category does not provide a unique tensor of two 1-morphisms $f : A \to B$ and $g : C \to D$, as a matter of convenient notation, we use the symbol $f \,\square\, g$ to mean $(f \,\square\, D) \circ (A \,\square\, g)$. This convention is known as “nudging”. The ‘tensor’ $\alpha \square \beta$ of two 2-morphisms α and β is similarly used to mean the corresponding nudged composite.

Definition 1.3.3 (Linear monoidal 2-category)**.** A *linear monoidal* 2*-category* is a linear 2-category equipped with a monoidal structure such that, for all objects A, the functors $A \,\square\, -$ and $- \,\square\, A$ are linear.

Remark 1.3.4 (Strictification for monoidal 2-categories). Any weakly monoidal weak 2-category can be strictified to a monoidal 2-category of the flavor given in Definition 1.3.1; similarly any linear weakly monoidal weak 2-category can be strictified to a linear monoidal 2-category a la Definition 1.3.3. The feasibility of those strictifications follow from the usual stricitfication for tricategories [GPS95] and a corresponding Vect_k-enriched version. Because of this, we permit ourselves to work with semistrict monoidal 2-categories as described, even though most examples will arise in the first instance in a weaker form.

Duality in monoidal 2-categories.

Definition 1.3.5 (Duals in monoidal 2-categories)**.** In a monoidal 2-category, an object $A^\#$ is a *right dual* of an object A, equivalently A is a *left dual* of $A^\#$, if there exist counit and unit 1-morphisms $e : A \square A^\# \to \mathrm{I}$ and $i : \mathrm{I} \to A^\# \square A$ such that $(e \square A) \circ (A \square i) \cong 1_A$ and $1_{A^\#} \cong (A^\# \square e) \circ (i \square A^\#)$.

Definition 1.3.6 (Prefusion and fusion 2-categories)**.** A *prefusion 2-category* is a finite presemisimple monoidal 2-category that has left and right duals for objects and a simple monoidal unit. A *fusion 2-category* is a finite semisimple monoidal 2-category that has left and right duals for objects and a simple monoidal unit.

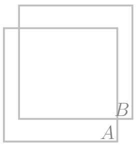
Being a (pre)fusion 2-category is a property of a linear monoidal 2-category, and this property is preserved under any linear 2-equivalence. Hence, every (pre)fusion linear *weakly* monoidal *weak* 2-category can be strictified to a (pre)fusion 2-category in the sense of Definition 1.3.6.

Remark 1.3.7 (State sum invariance under completion). The monoidal product in a prefusion or fusion 2-category is given by a 2-functor (as opposed to a 2-profunctor). It is not therefore the case that one can transport a prefusion structure on a presemisimple 2-category across an arbitrary 2-profunctor equivalence of presemisimple 2-categories (see Section 1.2.4). However, because every 2-profunctor between finite semisimple 2-categories is a 2-functor, it is the case that a prefusion structure on a presemisimple 2-category induces a fusion structure on the completed semisimple 2-category. (We might say that a prefusion 2-category is 'monoidally 2-profunctor equivalent' to its completion.) We expect the state sum will be invariant under this completion, that is the state sum invariant for a prefusion 2-category will be the same as the invariant for the associated (completed) fusion 2-category.

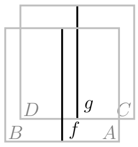
Remark 1.3.8 (State sum invariance under bimodule equivalence). A natural notion of equivalence between fusion 2-categories is asking for a monoidal 2-functor that is an equivalence of the underlying 2-categories; this is called a 'monoidal 2-functor equivalence'. (This is a categorification of the notion of monoidal functor equivalence between monoidal categories.) A coarser notion of equivalence between two prefusion or fusion 2-categories is asking for an invertible bimodule 2-category between them; this is called a 'bimodule equivalence'. (This notion is a categorification of the notion of bimodule or 'Morita' equivalence between monoidal categories.) The state sum of a fusion 2-category is of course invariant under monoidal 2-functor equivalence, but we speculate it is actually invariant under bimodule equivalence as well.

The graphical calculus of fusion 2-categories

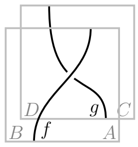
Calculus of monoidal structures. Recall from Section I.1 that a 2-category admits a graphical calculus of labeled 1-manifolds in the plane, so called 'string diagrams'. Monoidal 2-categories (and more generally semistrict 3-categories) admit a similar graphical calculus of 'surface diagrams' in 3-space [BMS12]:



The object $A \,\square\, B$

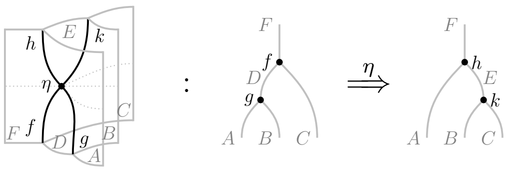


The 1-morphism $f \,\square\, g$



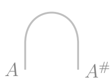
The interchanger $\phi_{f,g}$

The monoidal structure is depicted by layering surfaces behind one another, with the convention that tensor product occurs from back to front: that is, in a diagram for $A \,\square\, B$, the surface labeled A appears in front of the surface labeled B. Note that the tensor of 1-morphisms $f \,\square\, g$ is defined by Notation 1.3.2 and indeed in the diagram the morphism f appears slightly to the left of the morphism g. As drawn, the interchanger is depicted as a crossing of wires living in parallel planes. The following is a more complicated example of a surface diagram representing a 2-morphism in a monoidal 2-category:

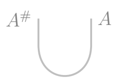


For clarity, here we have also explicitly depicted the source and target of the 2-morphism; that source and target appear in the surface diagram as the bottom and top horizontal slices, respectively. Note that we will often omit labels on regions, wires, and nodes, when it is clear from context what those labels should be.

Calculus of duality. In a monoidal 2-category in which every object has left and right duals, we can extend the graphical calculus of monoidal 2-categories by introducing the following diagrammatic notation for particular choices of counit and unit 1-morphisms of the object dualities:



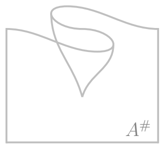
$e_A : A \,\square\, A^\# \to \mathrm{I}$



$i_A : \mathrm{I} \to A^\# \,\square\, A$

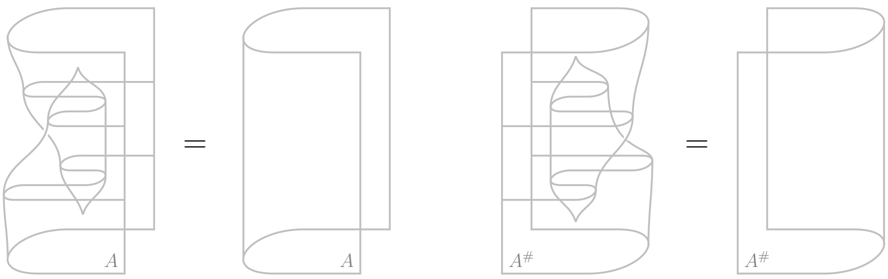
We may furthermore depict particular choices of the 2-isomorphisms $(e \square A) \circ (A \square i) \cong 1_A$ and $1_{A^\#} \cong (A^\# \square e) \circ (i \square A^\#)$ by cusps:

the cusp on A



the cusp on $A^\#$

These cusp 2-isomorphisms may be chosen to satisfy the swallowtail equations (though that fact will not be of any particular relevance, as later in our definition of pivotal 2-category we explicitly assume choices of cusps satisfying the swallowtail equations):



In these pictures, and henceforth, we use a tiny gap in a line specifically to indicate the presence of a categorical interchanger; crossings without a gap are merely coincidences of the depicted projection and do not signify a categorical operation.

Examples of fusion 2-categories

Most naturally occurring monoidal 2-categories are not semistrict, that is are not literally of the form described in Definition 1.3.1. In the following examples, whenever we say that something is a monoidal, respectively fusion, 2-category, what we mean is that it is a (fully weak) 3-category with one object which after (always possible [GPS95, Gur06]) strictification becomes a fusion or monoidal 2-category in the sense of Definition 1.3.1, respectively Definition 1.3.6.

Representations of groups, and group-graded vector spaces. We warm up by discussing examples of fusion 1-categories. Of course, the semisimple 1-category Vect_k itself is naturally a fusion 1-category using the tensor product of vector spaces. Recall the discussion of (monoidal) n-groupoids from Notation 1.2.67.

Recollection 1.3.9 (Group representations). For a finite group π_1, the semisimple 1-category $\mathrm{Rep}(*, \pi_1) := [(*, \pi_1), \mathrm{Vect}_k]$ of π_1-representations has a (symmetric) monoidal structure inherited from the monoidal structure on Vect_k, and with this structure, $\mathrm{Rep}(*, \pi_1)$ is a fusion 1-category. We will denote this fusion category by $\mathrm{Rep}(\pi_1)$.

More generally, for a finite 1-groupoid (π_0, π_1), the semisimple 1-category $\mathrm{Rep}(\pi_0, \pi_1) := [(\pi_0, \pi_1), \mathrm{Vect}_k]$ has a monoidal structure inherited from Vect_k, but note that this is a multifusion rather than fusion structure, that is the unit need not be simple. The notation $\mathrm{Rep}(\pi_0, \pi_1)$ will always refer either to the bare semisimple 1-category $[(\pi_0, \pi_1), \mathrm{Vect}_k]$ or to that category equipped with its symmetric multifusion structure.

Recollection 1.3.10 (Group-graded vector spaces). Given a finite group π_1 we may instead form the semisimple 1-category $\mathrm{Rep}(\pi_1, *) := [(\pi_1, *), \mathrm{Vect}_k]$ of π_1-graded vector spaces. This category has a monoidal structure induced by the group multiplication: the product is the (Day) convolution of functors $(\pi_1, *) \to \mathrm{Vect}_k$, or more concretely, the product of the functor $F(x) = \delta_{x,f} k$ and the functor $G(x) = \delta_{x,g} k$ is the functor $(F * G)(x) = \delta_{x,fg} k$. We will denote this fusion category, as before, by $\mathrm{Vect}_k(\pi_1)$.

More generally, for a finite 2-group (π_1, π_2), we could consider the semisimple 1-category $\mathrm{Rep}(\pi_1, \pi_2) := [(\pi_1, \pi_2), \mathrm{Vect}_k]$ with its convolution product; this is a monoidal semisimple 1-category, which will be denoted $\mathrm{Vect}_k(\pi_1, \pi_2)$, but in general it is multifusion rather than fusion.

Recollections 1.3.9 and 1.3.10 can be summarized in the following table:

Input	Notation		1-category	Monoidal structure
1-groupoid (π_0, π_1)	$\mathrm{Rep}(\pi_0, \pi_1)$	$:=$	$[(\pi_0, \pi_1), \mathrm{Vect}_k]$	symmetric from Vect_k (multifusion)
1-group π_1	$\mathrm{Rep}(\pi_1)$	$:=$	$[(*, \pi_1), \mathrm{Vect}_k]$	symmetric from Vect_k (fusion)
2-group (π_1, π_2)	$\mathrm{Vect}_k(\pi_1, \pi_2)$	$:=$	$[(\pi_1, \pi_2), \mathrm{Vect}_k]$	convolution product (multifusion)
1-group π_1	$\mathrm{Vect}_k(\pi_1)$	$:=$	$[(\pi_1, *), \mathrm{Vect}_k]$	convolution product (fusion)

Since every 2-group is in particular a 1-groupoid, the category $[(\pi_1, \pi_2), \mathrm{Vect}_k]$ has two distinct monoidal structures. (These structures are compatible in the sense that the symmetric monoidal structure together with a comonoidal structure associated to the convolution product give $[(\pi_1, \pi_2), \mathrm{Vect}_k]$ the structure of a Hopf 1-category.)

Recollection 1.3.11 (Twisted group-graded vector spaces). Given again a finite group π_1 and a 3-cocycle $w \in Z^3(\pi_1, k^*)$, we may twist the associator of the fusion category $\mathrm{Vect}_k(\pi_1)$ to obtain a new fusion category denoted $\mathrm{Vect}_k^w(\pi_1)$. Note here we may think of the cocycle either as a 'group-cohomology-style' cocycle for the ordinary group π_1 or as a topological cocycle on the space corresponding to the group, namely $\mathrm{B}\pi_1 = (*, \pi_1)$.

2-representations of 2-groups, and 2-group-graded 2-vector spaces. The semisimple 2-category $2\mathrm{Vect}_k$ is of course the canonical fusion 2-category, where the tensor of two finite semisimple linear 1-categories is the Deligne tensor product. This fusion 2-category has a unique equivalence class of simple objects, represented by the 1-category Vect_k.

Construction 1.3.12 (2-group 2-representations). Given a finite abelian group π_2, the semisimple 2-category $2\mathrm{Rep}(*, *, \pi_2) := [(*, *, \pi_2), 2\mathrm{Vect}_k]$ of π_2-2-representations inherits a (symmetric) monoidal structure from $2\mathrm{Vect}_k$ and with that structure is a fusion 2-category. We will denote this fusion 2-category by $2\mathrm{Rep}(\pi_2)$.

Similarly, given a finite 2-group (π_1, π_2), the semisimple 2-category $2\mathrm{Rep}(*, \pi_1, \pi_2) := [(*, \pi_1, \pi_2), 2\mathrm{Vect}_k]$ of (π_1, π_2)-2-representations inherits a monoidal structure from $2\mathrm{Vect}_k$, and this structure is again fusion. We will denote this fusion 2-category by $2\mathrm{Rep}(\pi_1, \pi_2)$.

More generally, given a finite 2-groupoid (π_0, π_1, π_2), the semisimple 2-category $2\mathrm{Rep}(\pi_0, \pi_1, \pi_2) := [(\pi_0, \pi_1, \pi_2), 2\mathrm{Vect}_k]$ has a monoidal structure, but it is in general multifusion rather than fusion, that is the unit need not be simple. The notation $2\mathrm{Rep}(\pi_0, \pi_1, \pi_2)$ will always refer either to the bare semisimple 2-category $[(\pi_0, \pi_1, \pi_2), 2\mathrm{Vect}_k]$ or to that 2-category equipped with its symmetric multifusion structure.

Construction 1.3.13 (2-group-graded 2-vector spaces). Given a finite group π_1, the semisimple 2-category $2\mathrm{Rep}(\pi_1, *, *) := [(\pi_1, *, *), 2\mathrm{Vect}_k]$ of π_1-graded 2-vector spaces has a monoidal structure induced by the group multiplication, namely the convolution product of 2-functors $(\pi_1, *, *) \to 2\mathrm{Vect}_k$, and is thereby a fusion 2-category. We will denote this fusion 2-category by $2\mathrm{Vect}_k(\pi_1)$.

Similarly, for a finite 2-group (π_1, π_2), the semisimple 2-category $2\mathrm{Rep}(\pi_1, \pi_2, *) := [(\pi_1, \pi_2, *), 2\mathrm{Vect}_k]$ of (π_1, π_2)-graded 2-vector spaces has a convolution product, and again is a fusion 2-category. We will of course denote this fusion 2-category by $2\mathrm{Vect}_k(\pi_1, \pi_2)$.

More generally, given a finite 3-group (π_1, π_2, π_3), the semisimple 2-category $2\mathrm{Rep}(\pi_1, \pi_2, \pi_3) := [(\pi_1, \pi_2, \pi_3), 2\mathrm{Vect}_k]$ of (π_1, π_2, π_3)-2-representations has a convolution monoidal structure; this monoidal semisimple 2-category will be denoted $2\mathrm{Vect}_k(\pi_1, \pi_2, \pi_3)$, but note it is in general multifusion rather than fusion.

Remark 1.3.14 (Bimodule equivalence of 2-representations and graded 2-vector spaces). The fusion 1-category $\mathrm{Rep}(\pi_1)$ of π_1-representations and the fusion 1-category $\mathrm{Vect}_k(\pi_1)$ of π_1-graded vector spaces are bimodule equivalent, and therefore lead to the same 3-manifold invariant. We suspect the 2-categorical situation is analogous, in that the fusion 2-category $\mathrm{Rep}(\pi_1, \pi_2)$ of (π_1, π_2)-2-representations is bimodule equivalent to the fusion 2-category $2\mathrm{Vect}_k(\pi_1, \pi_2)$ of (π_1, π_2)-graded 2-vector spaces, and that these fusion 2-categories therefore produce the same 4-manifold invariant.

Constructions 1.3.12 and 1.3.13 can be summarized in the following table:

Input	Notation		2-category	Monoidal structure
2-grpoid (π_0, π_1, π_2)	$2\mathrm{Rep}(\pi_0, \pi_1, \pi_2)$	$:=$	$[(\pi_0, \pi_1, \pi_2), 2\mathrm{Vect}_k]$	symm from $2\mathrm{Vect}_k$ (multifus)
2-group (π_1, π_2)	$2\mathrm{Rep}(\pi_1, \pi_2)$	$:=$	$[(*, \pi_1, \pi_2), 2\mathrm{Vect}_k]$	symm from $2\mathrm{Vect}_k$ (fusion)
ab 1-group (π_2)	$2\mathrm{Rep}(\pi_2)$	$:=$	$[(*, *, \pi_2), 2\mathrm{Vect}_k]$	symmfrom $2\mathrm{Vect}_k$ (fusion)
3-group (π_1, π_2, π_3)	$2\mathrm{Vect}_k(\pi_1, \pi_2, \pi_3)$	$:=$	$[(\pi_1, \pi_2, \pi_3), 2\mathrm{Vect}_k]$	conv product (mulitfus)
2-group (π_1, π_2)	$2\mathrm{Vect}_k(\pi_1, \pi_2)$	$:=$	$[(\pi_1, \pi_2, *), 2\mathrm{Vect}_k]$	conv product (fusion)
1-group (π_1)	$2\mathrm{Vect}_k(\pi_1)$	$:=$	$[(\pi_1, *, *), 2\mathrm{Vect}_k]$	conv product (fusion)

Since every 3-group is in particular a 2-groupoid, the 2-category $[(\pi_1, \pi_2, \pi_3), 2\mathrm{Vect}_k]$ has two distinct monoidal structures, one (symmetric) structure coming from the product on $2\mathrm{Vect}_k$ and one not-necessarily symmetric convolution structure coming from the 3-group itself. (We expect these structure to be compatible, in that the symmetric fusion structure together with a comonoidal structure associated to the convolution will form a Hopf 2-category.)

Remark 1.3.15 (The convolution product is the completion of group multiplication). Recall from Example 1.2.69 that the finite semisimple 2-category $[(\pi_0, \pi_1, \pi_2), 2\mathrm{Vect}_k]$ is the semisimple completion of the linearization $k(\pi_0, \pi_1, \pi_2)$ of the 2-groupoid (π_0, π_1, π_2). If the 2-groupoid is a 3-group (π_1, π_2, π_3), then $k(\pi_1, \pi_2, \pi_3)$, and hence its semisimple completion $[(\pi_1, \pi_2, \pi_3), 2\mathrm{Vect}_k]$, inherits a monoidal structure from the monoidal structure of (π_1, π_2, π_3) — the resulting monoidal structure is the convolution product on $2\mathrm{Vect}_k(\pi_1, \pi_2, \pi_3)$.

Construction 1.3.16 (Twisted 2-group-graded 2-vector spaces). Given a finite 2-group (π_1, π_2), and a 4-cocycle $\omega \in Z^4((\pi_1, \pi_2); k^*)$, one can form a fusion 2-category $2\mathrm{Vect}_k^\omega(\pi_1, \pi_2)$ of 'ω-twisted (π_1, π_2)-graded 2-vector spaces', as follows. (Here $Z^4((\pi_1, \pi_2); k^*)$ is the topological 4-cocycles, with k^* coefficients, of the space $(*, \pi_1, \pi_2)$ with only first and second homotopy groups.)

The 4-cocycle $\omega \in Z^4((\pi_1, \pi_2); k^*)$ provides the k-invariant for an extension of the 2-group (π_1, π_2) to a 3-group (π_1, π_2, k^*), with trivial action of π_1 on $\pi_3 = k^*$. Because the π_1-action on π_3 is trivial, we may think of this 3-group as a monoidal 2-category enriched in k^*-sets. (Here the enriching category of k^*-sets has tensor product $X \times_{k^*} Y := \mathrm{coeq}(X \times k^* \times Y \rightrightarrows X \times Y)$.) Base changing from k^* to k produces a k-linear monoidal 2-category denoted $(\pi_1, \pi_2, k^*)_k$. Precisely, this operation is base change along the functor from k^*-sets to (possibly infinite-dimensional) k-vector spaces, taking a k^*-set X to the k-vector space $k \otimes_{k(k^*)} k(X)$, where $k(X)$ is the free k-vector space on the set X. Note that the k-linear monoidal 2-category $(\pi_1, \pi_2, k^*)_k$ has finitely many equivalence classes of objects, all invertible, and finitely many isomorphism classes of 1-morphisms, all invertible, and that $\mathrm{Hom}_{(\pi_1,\pi_2,k^*)_k}(f,g)$ is k when f and g are isomorphic and 0 otherwise. In particular, $(\pi_1,\pi_2,k^*)_k$ is a locally presemisimple linear monoidal 2-category.

The fusion 2-category $2\mathrm{Vect}_k^\omega(\pi_1,\pi_2)$ is defined to be the semisimple completion of the local Cauchy completion of the monoidal 2-category $(\pi_1,\pi_2,k^*)_k$. That is, $2\mathrm{Vect}_k^\omega(\pi_1,\pi_2)$ is a completion of a twisted linearization of (π_1,π_2). When the twisting is trivial, the construction recovers $2\mathrm{Vect}_k(\pi_1,\pi_2)$ by the discussion in Example 1.2.69.

Remark 1.3.17 (Characterizing twisted 2-group-graded 2-vector spaces). A fusion 1-category C in which every simple object is invertible is necessarily equivalent to the fusion 1-category $\mathrm{Vect}_k^\omega(\pi_1)$ for some finite group π_1 and 3-cocycle $\omega \in Z^3(\pi_1;k^*)$; the group π_1 is determined as the group of isomorphism classes of simple objects of C. And of course every simple object of $\mathrm{Vect}_k^\omega(\pi_1)$ is invertible.

The situation for fusion 2-categories is more complicated. The fusion 2-category $2\mathrm{Vect}_k^\omega(\pi_1,\pi_2)$, for a finite 2-group (π_1,π_2) and 4-cocycle $\omega \in Z^4((\pi_1,\pi_2);k^*)$, can have non-invertible simple objects and non-invertible simple 1-morphisms. For instance, the fusion 2-category $2\mathrm{Vect}_k(*,\mathbb{Z}_2)$ has both non-invertible simple objects and 1-morphisms, as described below in Examples 1.3.20 and 1.3.21 (also see Example 1.2.64).

Even if a fusion 2-category has only invertible simple objects (so there is an obvious 'group of simple objects' π_1) and every simple 1-endomorphism of the unit object I is invertible (so there is an obvious 'group of simple 1-endomorphisms' π_2), it is still not necessarily the case that the fusion 2-category is equivalent to $2\mathrm{Vect}_k^\omega(\pi_1,\pi_2)$ for some (π_1,π_2) and ω. An example of such a fusion 2-category (that is not twisted 2-group graded 2-vector spaces) is provided by the completion of a $\mathbb{Z}_4$-crossed braided structure on the Ising category, see below Example 1.3.27.

Nevertheless, it is still possible to characterize twisted 2-group-graded 2-vector spaces as follows: a fusion 2-category $\mathcal{C}$ is monoidally equivalent to $2\mathrm{Vect}_k^\omega(\pi_1,\pi_2)$ for some finite 2-group (π_1,π_2) and 4-cocycle $\omega \in Z^4((\pi_1,\pi_2);k^*)$ if and only if

1. every component of $\mathcal{C}$ contains an invertible object;
2. every simple 1-morphism in $\mathrm{End}_{\mathcal{C}}(\mathrm{I})$ is invertible;
3. the group homomorphism $\mathcal{C}^\times \to \pi_0\mathcal{C}$, from the group of equivalence classes of invertible objects to the group of components, admits a section.

(Note that the existence of a group structure on the set of components depends on the existence of an invertible object in each component.) Here the groups π_1 and π_2 can be taken to be the group of components $\pi_0\mathcal{C}$ and the group of isomorphism classes of simple 1-morphisms in $\mathrm{End}_{\mathcal{C}}(\mathrm{I})$, respectively. The aforementioned $\mathbb{Z}_4$-crossed braided Ising category fails the third condition as in that case the group of invertible objects $\mathbb{Z}_4$ projects onto the group of components $\mathbb{Z}_2$.

Construction 1.3.18 (Abelian-group 2-vepresentations). Given an abelian group π_2, the semisimple 2-category of 'π_2-2-vepresentations' $2\mathrm{Rep}(*, \pi_2, *) := [(*, \pi_2, *), 2\mathrm{Vect}_k]$ has both a symmetric and a convolution structure, and both products are fusion, not merely multifusion. We therefore expect this construction provides in an appropriate sense 'fusion Hopf 2-categories'. We reserve the notation $2\mathrm{Vep}(\pi_2)$ for the semisimple 2-category $[(*, \pi_2, *), 2\mathrm{Vect}_k]$ equipped simultaneously with both monoidal structures; by contrast we would use $2\mathrm{Rep}(\pi_2, *)$ when thinking only of the symmetric fusion structure and $2\mathrm{Vect}_k(*, \pi_2)$ when thinking only of the convolution fusion structure.

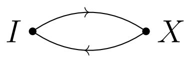
Braided fusion categories and their modules.

Construction 1.3.19 (Fusion 2-categories from braided fusion categories). The delooping of a braided fusion category C is a prefusion 2-category BC: the delooping of the underlying fusion category is a presemisimple 2-category, as in Example 1.2.15, and the braiding provides precisely the data of a monoidal structure on the delooping. By Proposition 1.2.47 the idempotent completion of the delooping BC is the semisimple 2-category of modules Mod(C). The monoidal structure on the delooping BC induces a monoidal structure on the completion, making Mod(C) a fusion 2-category. Module categories for C correspond to separable algebras in C, and the monoidal structure is the tensor product of algebras (which, note well, depends on the braiding of C). Directly as C-module categories, the monoidal product is given by the relative Deligne tensor over C, cf [BJS18].

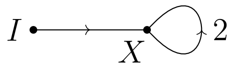
Example 1.3.20 (Fusion structures on $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$). Recall from Example 1.2.64 the structure of the semisimple 2-category $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$: there is the simple object $\mathrm{Vect}_k(\mathbb{Z}_2)$—which we will now abbreviate as I—and the simple object Vect_k—which we will now abbreviate as X—with morphism categories $\mathrm{Hom}(\mathrm{I},\mathrm{I}) \simeq \mathrm{Hom}(X,X) \simeq \mathrm{Vect}_k(\mathbb{Z}_2)$ and $\mathrm{Hom}(X,\mathrm{I}) \simeq \mathrm{Hom}(\mathrm{I},X) \simeq \mathrm{Vect}_k$.

There are two braidings on the fusion category $\mathrm{Vect}_k(\mathbb{Z}_2)$, namely the symmetric braiding and the super braiding, and thus two corresponding fusion structures on the 2-category $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$. We may calculate the fusion rules for each as follows. An object of $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ corresponds to a separable algebra in $\mathrm{Vect}_k(\mathbb{Z}_2)$. (This correspondence may either be seen as the classical Ostrik translation or as the fact that $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ is the idempotent—i.e. separable algebra—completion of the deloop $\mathrm{BVect}_k(\mathbb{Z}_2)$.) The object I corresponds to the trivial algebra $k \in \mathrm{Vect}_k(\mathbb{Z}_2)$ and the object X corresponds to the 'graded group algebra' $k(\mathbb{Z}_2) \in \mathrm{Vect}_k(\mathbb{Z}_2)$. The monoidal product of objects here is, as mentioned in Construction 1.3.19, the tensor product of algebras inside the braided fusion category. Thus I is evidently the identity in the fusion 2-category (for either braiding).

Now suppose the base field is $\mathbb{C}$ and the braiding is super. In this case the tensor product $\mathbb{C}(\mathbb{Z}_2) \otimes \mathbb{C}(\mathbb{Z}_2)$ is, by complex 'Bott periodicity', Morita equivalent to the trivial algebra $\mathbb{C}$. The corresponding fusion rule in the fusion 2-category is therefore $X \square X \simeq \mathrm{I}$. We may depict the fusion rules of this 2-category as follows, where the directed edge indicates multiplication by X:



For the symmetric braiding, the tensor product $\mathbb{C}(\mathbb{Z}_2) \otimes \mathbb{C}(\mathbb{Z}_2)$ is, by idempotent decomposing that algebra in the symmetric tensor category $\mathrm{Vect}_{\mathbb{C}}(\mathbb{Z}_2)$, isomorphic to the algebra $\mathbb{C}(\mathbb{Z}_2) \oplus \mathbb{C}(\mathbb{Z}_2)$. The corresponding fusion rule is therefore $X \square X \simeq X \boxplus X$. We may depict the fusion rules of this 2-category as follows, where again the directed edge is multiplication by X and the label indicates the multiplicity:



Note well that this sort of fusion graph, where an object has no fusion products containing identity factors, is completely new to fusion 2-categories: in a fusion 1-category, the product of an object and its dual has the identity as a summand, but in a fusion 2-category, because of the existence of nontrivial nonequivalence morphisms between simple objects, this need not be the case.

Example 1.3.21 (Fusion structures on $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ as twisted 2-vepresentations). Recall from Remark 1.2.71 that the semisimple 2-category $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ of module categories for $\mathrm{Vect}_k(\mathbb{Z}_2)$ is equivalent to the 2-category $2\mathrm{Rep}(*, \mathbb{Z}_2, *) := [(*, \mathbb{Z}_2, *), 2\mathrm{Vect}_k]$ of 2-vepresentations of $\mathbb{Z}_2$. The monoidal structure on $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ induced by the standard braiding (see Example 1.3.20) corresponds to the convolution product on $2\mathrm{Rep}(*, \mathbb{Z}_2, *)$, therefore gives the fusion 2-category $2\mathrm{Vect}_k(*, \mathbb{Z}_2)$ of $(*, \mathbb{Z}_2)$-graded 2-vector spaces (see Construction 1.3.13). The monoidal structure on $\mathrm{Mod}(\mathrm{Vect}_k(\mathbb{Z}_2))$ induced, by contrast, by the super braiding corresponds to the convolution product on $2\mathrm{Rep}(*, \mathbb{Z}_2, *)$ twisted by a nontrivial 4-cocycle $\omega \in Z^4((*, \mathbb{Z}_2); k^*)$, thus to the twisted 2-group-graded 2-category $2\mathrm{Vect}_k^\omega(*, \mathbb{Z}_2)$. Specifically, the super braiding fusion structure is obtained by twisting by the cocycle representing the order 2-element in $H^4((*, \mathbb{Z}_2); k^*) \cong \mathbb{Z}_4$. (Note more generally that for an abelian group π_2, the group $Z^4((*, \pi_2); k^*)$ twisting the fusion structure on $2\mathrm{Vect}_k(*, \pi_2) \simeq \mathrm{Mod}(\mathrm{Vect}_k(\pi_2))$ is the same as the group of 'abelian 3-cocycles' $Z^3_{\mathrm{ab}}(\pi_2, k^*)$ [EGNO15, Sec 8.4] that simultaneously twists the associator and the braiding of the fusion 1-category $\mathrm{Vect}_k(\pi_2)$.)

Remark 1.3.22 (One-component fusion 2-categories). Any fusion 2-category $\mathcal{C}$ with only one component is (monoidally 2-functor) equivalent to the 2-category $\mathrm{Mod}(\mathrm{C})$ of modules of a braided fusion category C; more specifically it is equivalent to the 2-category of modules of the braided fusion category $\mathrm{Hom}_{\mathcal{C}}(\mathrm{I}, \mathrm{I})$ of endomorphisms of the identity of $\mathcal{C}$. Indeed, any one-component finite semisimple 2-category is the 2-category of modules of any one of its fusion endocategories, so as a semisimple 2-category, the fusion 2-category $\mathcal{C}$ is equivalent to $\mathrm{Mod}(\mathrm{Hom}_{\mathcal{C}}(\mathrm{I}, \mathrm{I}))$. But using Corollary 1.2.48 (applied to functors $\mathrm{Mod}(\mathrm{C} \boxtimes \mathrm{C}) \to \mathrm{Mod}(\mathrm{C})$, with $\mathrm{C} = \mathrm{Hom}_{\mathcal{C}}(\mathrm{I}, \mathrm{I})$), the monoidal structure on $\mathrm{Mod}(\mathrm{Hom}_{\mathcal{C}}(\mathrm{I}, \mathrm{I}))$ is completely determined by the monoidal structure on $\mathrm{B}\,\mathrm{Hom}_{\mathcal{C}}(\mathrm{I}, \mathrm{I})$, that is by the braiding on $\mathrm{Hom}_{\mathcal{C}}(\mathrm{I}, \mathrm{I})$.

Crossed-braided fusion categories.

Construction 1.3.23 (From crossed-braided fusion categories to fusion 2-categories). Recall that a G-crossed-braided fusion category is a fusion category C, together with a G-grading $\mathrm{C} = \bigoplus_{g \in G} \mathrm{C}_g$, a G-action on C where $g \in G$ maps C_h to $\mathrm{C}_{ghg^{-1}}$, and a compatible crossed-braiding isomorphism $X \otimes Y \to g(Y) \otimes X$ whenever $X \in \mathrm{C}_g$. To any G-crossed-braided fusion category C, there is an associated monoidal 2-category $\mathcal{C}$, as follows [Cui16, Sec 6]. (To obtain a semistrict monoidal 2-category, for convenience we will start with a strict crossed-braided fusion category [DGNO10, Def 4.41].)

The objects of $\mathcal{C}$ are the elements $g \in G$, and these objects are all simple. The hom category $\mathrm{Hom}_{\mathcal{C}}(g, h)$ is the category $C_{hg^{-1}}$. The composition of 1-morphisms in $\mathcal{C}$ is given by the tensor product in C:

$$\begin{array}{c} \mathrm{Hom}_{\mathcal{C}}(g, h) \times \mathrm{Hom}_{\mathcal{C}}(f, g) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(f, h) \\ \mathrm{C}_{hg^{-1}} \times \mathrm{C}_{gf^{-1}} \xrightarrow{\otimes} \mathrm{C}_{hf^{-1}} \end{array}$$

The monoidal product $g \,\square\, h$ of objects $g, h \in \mathcal{C}$ is the object gh, and the monoidal unit is the identity element $e \in G$. The monoidal product of an object g on the left on 1- and 2-morphisms in $\mathcal{C}$ is given by the action of g in C:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(h,h') &\xrightarrow{g\Box -} \mathrm{Hom}_{\mathcal{C}}(g\Box h, g\Box h')\\ \mathrm{C}_{h'h^{-1}} &\xrightarrow{g(-)} \mathrm{C}_{gh'h^{-1}g^{-1}} = \mathrm{C}_{gh'(gh)^{-1}}\end{aligned}$$

By contrast, the monoidal product of an object g on the right in $\mathcal{C}$ is given by the identity operation in C:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(h,h') &\xrightarrow{-\Box g} \mathrm{Hom}_{\mathcal{C}}(h\Box g, h'\Box g)\\ \mathrm{C}_{h'h^{-1}} &\xrightarrow{\mathrm{id}} \mathrm{C}_{h'h^{-1}} = \mathrm{C}_{h'g(hg)^{-1}}\end{aligned}$$

Finally, the interchanger 2-isomorphism of $\mathcal{C}$ is given by the crossed-braiding of C; for 1-morphisms $X : g_1 \to g_2$, $Y : h_1 \to h_2$:

$$\begin{aligned}(X\Box h_2)\circ(g_1\Box Y) &\xRightarrow{\phi_{X,Y}} (g_2\Box Y)\circ(X\Box h_1)\\ X\otimes g_1(Y) &\xrightarrow{c_{X,g_1(Y)}} g_2(Y)\otimes X\end{aligned}$$

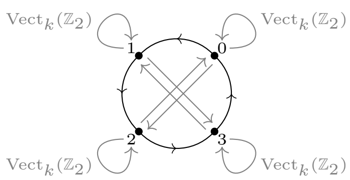
This construction defines a prefusion 2-category; the completion is therefore a fusion 2-category, as desired. If the grading is faithful, that is $\mathrm{C}_g \neq 0$ for all $g \in G$, then the resulting fusion 2-category has only one component, and is therefore equivalent (see Remark 1.3.22) to the completion of the deloop BC_e of the braided fusion category C_e.

Remark 1.3.24 (Inequivalent crossed-braided fusion categories giving equivalent fusion 2-categories)*.* Let C be an Ising fusion category, that is one with simple objects I, f, σ and fusion rules $f^2 \cong I, f\sigma \cong \sigma f \cong \sigma$, and $\sigma^2 \cong I \oplus f$. Equip this category with the $\mathbb{Z}_2$-grading $\mathrm{C}_0 = \mathrm{Vect}_k(\mathbb{Z}_2) = \langle I, f\rangle$ and $C_1 = \mathrm{Vect}_k = \langle\sigma\rangle$, and the trivial $\mathbb{Z}_2$-action. An Ising category admits four inequivalent braidings [DGNO10, App B], all of which restrict to the super braiding on $\mathrm{Vect}_k(\mathbb{Z}_2)$. Any of these braidings makes the given Ising category C into a $\mathbb{Z}_2$-crossed-braided fusion category, and there is thus an associated fusion 2-category. However, because the $\mathbb{Z}_2$-grading is faithful, each of these fusion 2-categories is equivalent to the completion of the deloop $\mathrm{BC}_0 = \mathrm{BVect}_k(\mathbb{Z}_2)$ of the super braided category $\mathrm{Vect}_k(\mathbb{Z}_2)$, and thus to the fusion 2-category of modules of the super braided category $\mathrm{Vect}_k(\mathbb{Z}_2)$.

Remark 1.3.25 (Invertible-object fusion 2-categories are not necessarily crossed-braided)*.* It is not the case that a fusion 2-category all of whose simple objects are invertible is equivalent to a fusion 2-category associated to a G-crossed-braided category. Given a fusion 2-category $\mathcal{C}$ with invertible simple objects, the equivalence classes of simple objects form a finite group G, and (picking representative simple objects g in each equivalence class $[g] \in G$) the semisimple 1-category $\mathrm{C} := \bigoplus_{g \in G} \mathrm{Hom}_{\mathcal{C}}(e, g)$ appears to want to be a G-crossed-braided category—but in general it is not possible to put an appropriate monoidal structure on that category. Specifically, choose equivalences $\psi(g,h) : g \,\Box\, h \to gh$ and 2-isomorphisms $\alpha(g_1, g_2, g_3) : \psi(g_1 g_2, g_3) \circ (\psi(g_1, g_2) \,\Box\, g_3) \Rightarrow \psi(g_1, g_2 g_3) \circ (g_1 \,\Box\, \psi(g_2, g_3))$; these isomorphisms α will satisfy the pentagon equations only up to some scalars $\omega(g_1, g_2, g_3, g_4) \in k^*$, and those scalars define a 4-cocycle $\omega \in Z^4(G; k^*)$. Only if that 4-cocycle is cohomologically trivial, is it possible to (adjust the choices of representing objects, equivalences, and 2-isomorphisms and then) define a multiplication giving C the structure of a G-crossed-braided fusion category. (Note that this process does not produce a unique crossed-braided category, but a collection of such categories all lifting the same fusion 2-category.) If that 4-cocycle is cohomologically nontrivial, then the fusion 2-category encodes a kind of twisting not presentable in the framework of crossed-braided categories.

Remark 1.3.26 (Endotrivial fusion 2-categories). A fusion 2-category is called *endotrivial* if the endomorphism fusion category of every indecomposable object is the trivial fusion category Vect_k. Endotrivial fusion 2-categories were called simply 'fusion 2-categories' in Mackaay [Mac99]. Note that of all the examples of fusion 2-categories described above, including those coming from 2-representations of 2-groups (Construction 1.3.12), 2-group-graded 2-vector spaces (Construction 1.3.13), braided fusion categories (Construction 1.3.19), and crossed-braided fusion categories (Construction 1.3.23), the only case that is endotrivial is the special case of 2-group-graded 2-vector spaces where the grading is in fact by a 1-group.

Example 1.3.27 ($\mathbb{Z}_4$-crossed braided Ising categories). For any Ising fusion category C, the $\mathbb{Z}_4$-grading $\mathrm{C}_0 = \mathrm{Vect}_k(\mathbb{Z}_2) = \langle I, f \rangle$, $\mathrm{C}_2 = \mathrm{Vect}_k = \langle \sigma \rangle$, and $\mathrm{C}_1 = \mathrm{C}_3 = 0$, together with the trivial $\mathbb{Z}_4$-action and any choice of braiding, gives C the structure of a $\mathbb{Z}_4$-crossed-braided fusion category. The associated fusion 2-category may be depicted as follows:



Here the nodes indicate the four simple objects. The gray lines show the structure of the underlying semisimple 2-category, where the unlabeled arrows denote the morphism space Vect_k. The directed black lines indicate the fusion structure of multiplication by the generating simple object 1.

Remark 1.3.28 (Fusion 2-categories must allow morphisms between inequivalent simple objects). By Theorem 1.2.59 and Proposition 1.2.47, every semisimple 2-category is the completion of the delooping of a multifusion category, and is therefore (see Constructions 1.2.16 and 1.2.17) 2-profunctor equivalent to the presemisimple unfolding of that multifusion category. That presemisimple 2-category has the attractive feature that the Hom categories between inequivalent simple objects are all trivial, and so it looks 'semisimple' in a more classical sense. For instance, the underlying semisimple 2-category in Example 1.3.27 is 2-profunctor equivalent to the presemisimple 2-category with two simple objects, each with endomorphism fusion category $\mathrm{Vect}_k(\mathbb{Z}_2)$, and no further morphisms. This is, however, a specious presentation: even if we are prepared to work up to 2-profunctor equivalence and even if we are prepared to work with presemisimple 2-categories, it is still *not possible* in general to give a monoidal fusion structure without allowing nontrivial morphisms between inequivalent simple objects. Example 1.3.27 illustrates this necessity: that fusion 2-category is not monoidally 2-profunctor equivalent to any prefusion 2-category with trivial Hom categories between inequivalent simple objects. Indeed the order four cyclic fusion group of the simple objects forces there to be at least two distinct simple objects in each connected component, in order to describe the monoidal structure.

1.3.2 Pivotal 2-categories

The definition and graphical calculus of planar pivotal 2-categories

Planar pivotal structures. A monoidal 1-category is called pivotal when it is equipped with a monoidal trivialization of the double dual functor; such a category has chosen isomorphisms from the double dual of each object to the object itself, in a way compatible with the tensor product of objects. This notion generalizes straightforwardly to 2-categories, by asking for a trivialization of the double adjoint of 1-morphisms, in a way compatible with composition. Such 2-categories are usually called 'pivotal', but because we will also be concerned with trivializing the double dual of objects in a monoidal 2-category and therefore will have a different need for the modifier 'pivotal', we will refer to 2-categories with a 1-morphism-level pivotal structure as 'planar pivotal'. For convenience, we will adopt a somewhat strictified definition as follows.

Definition 1.3.29 (Planar pivotal 2-category)**.** Let $\mathcal{C}$ be a strict 2-category in which every 1-morphism has a left and a right adjoint. A *planar pivotal structure* on $\mathcal{C}$ consists of the following data:

D1. a choice of right adjoint $f^* : B \to A$ for every 1-morphism $f : A \to B$,

D2. a choice of unit $\eta_f : 1_A \Rightarrow f^* \circ f$ and counit $\epsilon_f : f \circ f^* \Rightarrow 1_B$,

subject to the following conditions:

C1. the unit and counit satisfy the cusp equations:

$$(\epsilon_f \circ 1_f) \cdot (1_f \circ \eta_f) = 1_f$$
$$1_{f^*} = (1_{f^*} \circ \epsilon_f) \cdot (\eta_f \circ 1_{f^*})$$

C2. the choice of adjoint is functorial:

$$(1_A)^* = 1_A$$
$$(f \circ g)^* = g^* \circ f^*$$

C3. the choice of unit and counit is functorial:

$$\epsilon_{1_A} = 1_{1_A}$$
$$\eta_{1_A} = 1_{1_A}$$
$$\epsilon_{f \circ g} = \epsilon_f \cdot (f \circ \epsilon_g \circ f^*)$$
$$\eta_{f \circ g} = (g^* \circ \eta_f \circ g) \cdot \eta_g$$

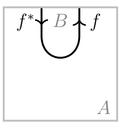
C4. the adjoint is involutive: $f^{**} = f$;

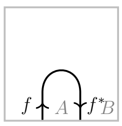
C5. right and left mates agree: for any 2-morphism $\alpha : f \Rightarrow g$ we have

$$\alpha^* := (1_{f^*} \circ \epsilon_g) \cdot (1_{f^*} \circ \alpha \circ 1_{g^*}) \cdot (\eta_f \circ 1_{g^*}) = (\epsilon_{g^*} \circ 1_{f^*}) \cdot (1_{g^*} \circ \alpha \circ 1_{f^*}) \cdot (1_{g^*} \circ \eta_{f^*}) =: {}^*\alpha$$

We refer to a 2-category with a planar pivotal structure simply as a *planar pivotal 2-category.*

Calculus of planar pivotal structures. Planar pivotal 2-categories admit a graphical calculus of oriented strings in the plane [Sel11]. The unit and counit of an adjunction $f \dashv f^*$ are depicted respectively as follows:

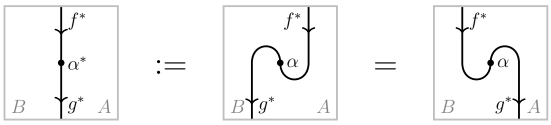




Here a wire labeled f with an upwards pointing orientation arrow refers to the morphism f, and the same wire with a downwards pointing orientation arrow refers to the morphism f^*; note that in order to avoid confusion later and contrary to typical convention, we will label this downwards pointing segment by f^*, the actual object associated to that wire segment, not by f, the object associated to the segment with reversed orientation.

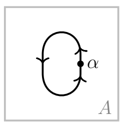
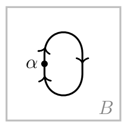
The cusp equations are similarly represented by the following pictures:

A 2-morphism $\alpha : f \Rightarrow g$ is depicted by a dot on a wire, with the lower half of the wire labeled f and the upper half labeled g. In this notation the mate 2-morphism $\alpha^* : g^* \Rightarrow f^*$ appears as follows:



Because taking the dual is involutive in a planar pivotal 2-category, we can form well-typed circular strings, known as ‘traces’.

Definition 1.3.30 (Planar trace)**.** In a planar pivotal 2-category, given a 1-morphism $f : A \to B$ and a 2-endomorphism $\alpha : f \Rightarrow f$, the *left planar trace* $\mathrm{tr}_L(\alpha) : 1_A \Rightarrow 1_A$ and *right planar trace* $\mathrm{tr}_R(\alpha) : 1_B \Rightarrow 1_B$ are

$\mathrm{tr}_L(\alpha) := \epsilon_{f^*} \cdot (1_{f^*} \circ \alpha) \cdot \eta_f =$  $\mathrm{tr}_R(\alpha) := \epsilon_f \cdot (\alpha \circ 1_{f^*}) \cdot \eta_{f^*} =$ 

Monoidal planar pivotal structures. If we add a monoidal structure to a planar pivotal 2-category, we insist that the monoidal structure respects the planar pivotal structure as follows.

Definition 1.3.31 (Monoidal planar pivotal 2-category)**.** A *monoidal planar pivotal 2-category* is a monoidal 2-category equipped with a planar pivotal structure such that

1. the adjoint of a tensor is the tensor of the adjoints:

$$(A \,\square\, f)^* = A \,\square\, f^*$$
$$(f \,\square\, A)^* = f^* \,\square\, A$$

2. the unit and counit for a tensor are the tensors of the units and counits:

$$\begin{aligned}\epsilon_{A\square f} &= A \square \epsilon_f \\ \eta_{A\square f} &= A \square \eta_f \\ \epsilon_{f\square A} &= \epsilon_f \square A \\ \eta_{f\square A} &= \eta_f \square A\end{aligned}$$

Here A is an object, $f : A \to B$ is a 1-morphism, and ϵ_f and η_f are the counit and unit of the adjunction between f and f^*.

The graphical calculus for planar pivotal 2-categories extends to one for monoidal planar pivotal 2-categories: 1-morphisms $f : A \to B$ are again represented by oriented strings, which are constrained to move within a single sheet of the surface diagram. If the monoidal category has duals, and therefore a graphical calculus where multiple sheets may connect via unit and counit morphisms depicted as 'cups' and 'caps', the 1-morphism strings may not intersect these cups and caps—such interaction requires further conditions, as described in the next section.

The definition and graphical calculus of pivotal 2-categories

Pivotal structures. In a 2-category with adjoints for 1-morphisms (or more simply a monoidal 1-category with duals for objects), a planar pivotal structure (respectively pivotal structure) is a choice of adjoint structure that is coherent (Definition 1.3.29-C1), functorial (C2,C3), and involutive (C4), such that left and right mates agree (C5).

Given a monoidal planar pivotal 2-category with duals for objects, we can insist that object duality itself is 'pivotal' and that the object-level duality interacts well with the 1-morphism-level duality. More specifically, below we will define a pivotal structure on a monoidal planar pivotal 2-category to be a choice of duality structure that is coherent (Definition 1.3.32-C1), compatible with tensor (C2,C3,C4), compatible with the existing adjoint structure (C5,C6), and involutive (C7), such that the right-over and left-under twist 2-morphisms between the left and right mates agree (C8).

Given a monoidal 2-category with chosen right dual $A^\#$ for every object A such that $A^{\#\#} = A$, along with chosen unit and counit 1-morphisms $i_A : \mathrm{I} \to A^\# \square A$ and

$e_A : A \,\Box\, A^\# \to \mathrm{I}$, and any 1-morphism $f : A \to B$, the right and left mate of f are defined by

$$\begin{aligned} f^\# &:= (A^\# \,\Box\, e_B) \circ (A^\# \,\Box\, f \,\Box\, B^\#) \circ (i_A \,\Box\, B^\#) : B^\# \to A^\# \\ {}^\# f &:= (e_{B^\#} \,\Box\, A^\#) \circ (B^\# \,\Box\, f \,\Box\, A^\#) \circ (B^\# \,\Box\, i_{A^\#}) : B^\# \to A^\#. \end{aligned}$$

Similarly for any 2-morphism $\alpha : g \Rightarrow h$ between 1-morphisms $g, h : A \to B$, the right and left (object-duality) mates of α are defined by

$$\begin{aligned} \alpha^\# &:= (A^\# \,\Box\, e_B) \circ (A^\# \,\Box\, \alpha \,\Box\, B^\#) \circ (i_A \,\Box\, B^\#) : g^\# \Rightarrow h^\# \\ {}^\# \alpha &:= (e_{B^\#} \,\Box\, A^\#) \circ (B^\# \,\Box\, \alpha \,\Box\, A^\#) \circ (B^\# \,\Box\, i_{A^\#}) : {}^\# g \Rightarrow {}^\# h. \end{aligned}$$

Definition 1.3.32 (Pivotal 2-category)**.** Let $\mathcal{C}$ be a monoidal planar pivotal 2-category in which every object has a left and a right dual. A *pivotal structure* on $\mathcal{C}$ consists of the following data:

D1. a choice of right dual $A^\#$ for every object A of $\mathcal{C}$;

D2. a choice of 1-morphisms (called *folds*) $i_A : \mathrm{I} \to A^\# \,\Box\, A$ and $e_A : A \,\Box\, A^\# \to \mathrm{I}$;

D3. a choice of invertible 2-morphisms (called *cusps*)

$$\begin{aligned} C_A &: (e_A \,\Box\, A) \circ (A \,\Box\, i_A) \Rightarrow 1_A \\ D_A &: 1_{A^\#} \Rightarrow \big(A^\# \,\Box\, e_A\big) \circ \big(i_A \,\Box\, A^\#\big) \end{aligned}$$

subject to the following conditions:

C1. the cusps satisfy the swallowtail equations:

$$\begin{aligned} \big[e_A \circ \big(C_A \,\Box\, A^\#\big)\big] \cdot \big[\phi_{e_A, e_A} \circ \big(A \,\Box\, i_A \,\Box\, A^\#\big)\big] \cdot [e_A \circ (A \,\Box\, D_A)] = 1_{e_A} \\ \big[\big(A^\# \circ C_A\big) \circ i_A\big] \cdot \big[\big(A^\# \,\Box\, e_A \,\Box\, A\big) \circ \phi_{i_A, i_A}\big] \cdot [(D_A \,\Box\, A) \circ i_A] = 1_{i_A} \end{aligned}$$

C2. the choice of dual is compatible with tensor:

$$\begin{aligned} \mathrm{I}^\# &= \mathrm{I} \\ (A \,\Box\, B)^\# &= B^\# \,\Box\, A^\# \end{aligned}$$

C3. the choice of folds is compatible with tensor:

$$\begin{aligned} i_I &= e_I = 1_I \\ i_{A \Box B} &= \big(B^\# \,\Box\, i_A \,\Box\, B\big) \circ i_B \\ e_{A \Box B} &= e_A \circ \big(A \,\Box\, e_B \,\Box\, A^\#\big) \end{aligned}$$

C4. the choice of cusps is compatible with tensor:

$$C_I = D_I = 1_{1_I}$$
$$C_{A\Box B} = [(C_A \Box B) \circ (A \Box C_B)] \cdot [(e_A \Box A \Box B) \circ (A \Box \phi_{e_B, i_A} \Box B) \circ (A \Box B \Box i_B)]$$
$$D_{A\Box B} = \left[\left(B^\# \Box A^\# \Box e_A\right) \circ \left(B^\# \Box \phi_{i_A, e_B} \Box A^\#\right) \circ \left(i_B \Box B^\# \Box A^\#\right)\right] \cdot \left[\left(B^\# \Box D_A\right) \circ \left(D_B \Box A^\#\right)\right]$$

C5. the folds intertwine duality of objects and adjunction of 1-morphisms: $(e_A)^* = i_{A^\#}$;

C6. the cusps intertwine duality of objects and mates of 2-morphisms: $(D_A)^* = C_{A^\#}$;

C7. the dual is involutive: $A^{\#\#} = A$;

C8. the right-over and left-under twists between the left and right mates agree: for any 1-morphism $f : A \to B$ we have

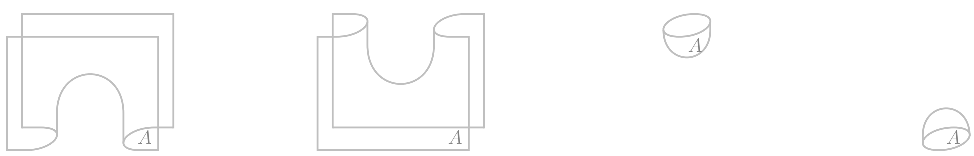
$$\theta_f = (\theta_{f^*})^* : {}^\# f \Rightarrow f^\#$$

where

$$\begin{aligned}
\theta_f := &\left[\left(A^\# \Box e_B\right) \circ \left(A^\# \Box f \Box B^\#\right) \circ \left(\epsilon_{{}^\# f} \Box A \Box B^\#\right) \circ \left(i_A \Box B^\#\right)\right] \\
&\cdot \left[\phi_{{}^\# f, e_B \circ (f \Box B^\#)} \circ \left((f^*)^\# \Box A \Box B^\#\right) \circ \left(i_A \Box B^\#\right)\right] \\
&\cdot \left[{}^\# f \circ \left(B^\# \Box e_B\right) \circ \left(B^\# \Box f \Box B^\#\right) \circ \left(B^\# \Box e_A \Box A \Box B^\#\right) \circ \left(\phi^{-1}_{(B^\# \Box f^*) \circ i_B, i_A} \Box B^\#\right)\right] \\
&\cdot \left[{}^\# f \circ \left(f \circ C_A^{-1} \circ f^*\right)^\#\right] \cdot \left[{}^\# f \circ \eta^\#_{f^*}\right] \cdot \left[{}^\# f \circ D_B\right]
\end{aligned}$$

For brevity, we refer to a monoidal planar pivotal 2-category with a pivotal structure as a *pivotal 2-category*. Though structured somewhat differently, we expect this definition is equivalent to the notion BMS call a 'spatial Gray monoid' [BMS12].

Calculus of pivotal structures. The graphical calculus for pivotal 2-categories is a calculus of surfaces with string defects, compare [BMS12]. By planar pivotality, the fold 1-morphisms e_A and i_A have adjoints $i_A^* = e_{A^\#}$ and $e_A^* = i_{A^\#}$. The units and counits of those adjunctions are referred to as a *crotch*, *saddle*, *birth of a circle*, and *death of a circle* and (extending the calculus of monoidal 2-categories with duals from Section 1.3.1) are depicted as follows:



$\epsilon_{i_{A^\#}} : i_{A^\#} \circ e_A \Rightarrow 1_{A \Box A^\#}$ $\eta_{e_A} : 1_{A\Box A^\#} \Rightarrow i_{A^\#} \circ e_A$ $\eta_{i_{A^\#}} : 1_{\mathrm{I}} \Rightarrow e_A \circ i_{A^\#}$ $\epsilon_{e_A} : e_A \circ i_{A^\#} \Rightarrow 1_{\mathrm{I}}$

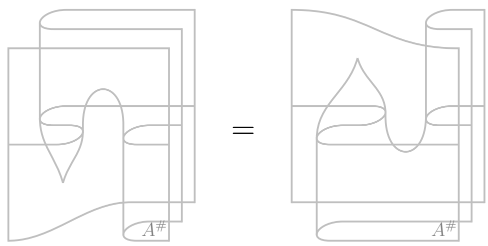
With this pictorial notation, one may draw a surface embedded in 3-dimensional space, with sheets labeled by objects, together with string defects on the surface labeled by 1-morphisms, and point defects on the strings labeled by 2-morphisms. It is reasonable to conjecture that the resulting 2-morphism of the pivotal 2-category is invariant under isotopy of the picture; BMS go some way towards establishing that result [BMS12]. Note that we will not need this full graphical calculus and none of our results depend on it—all equations we need will be explicitly established algebraically.

Warning 1.3.33 (Failure of invariance for the pivotal 2-category graphical calculus). Depending on one's perspective, either Definition 1.3.32 or its corresponding graphical calculus has a fairly serious and perhaps not evident drawback: the scalar value of a closed surface labeled by an object is not invariant under equivalence of objects. We do not know how to satisfactorily alter either the definition or the graphical calculus to eliminate this problem.

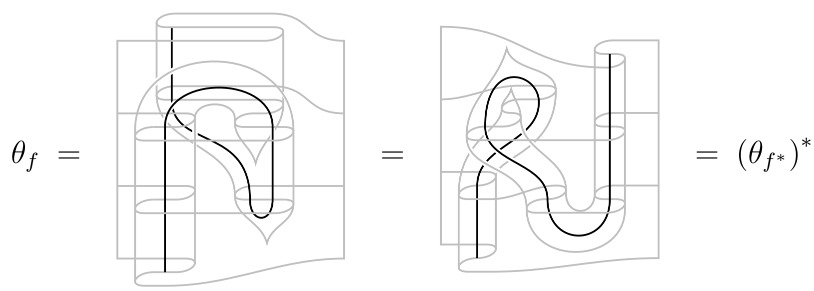
Remark 1.3.34 (Cusp flip condition). The condition C6, in Definition 1.3.32, that cusps intertwine duality of objects and mates of 2-morphisms is equivalent to the 'cusp flip equation':

$$\big[(A^{\#} \,\square\, e_A) \circ (\epsilon_{i_A} \,\square\, A^{\#})\big] \cdot \big[D_A \circ (e_{A^{\#}} \,\square\, A^{\#})\big] = \big[C_{A^{\#}} \circ (A^{\#} \,\square\, e_A)\big] \cdot \big[(e_{A^{\#}} \,\square\, A^{\#}) \circ (A^{\#} \,\square\, \eta_{e_A})\big]$$

This equation is depicted graphically as follows:



Remark 1.3.35 (Surface ribbon condition). The final condition C8 of Definition 1.3.32, that the right-over and left-under twists between mates agree, may be depicted graphically as follows:



The two surfaces may be thought of as the two simplest ways to take a surface with a line on it and deform the surface so that the normal vector to the line in the surface undergoes a full counterclockwise rotation.

Remark 1.3.36 (The loop category of a pivotal 2-category is ribbon). The surface ribbon condition drawn in Remark 1.3.35 is of course reminiscent of the ribbon condition that may be imposed on a braided pivotal category. Indeed, when the morphism f is an endomorphism of the identity, and the surface is therefore irrelevant, the surface ribbon condition reduces precisely to the ribbon condition:

$\theta_f =$  $=$  $= (\theta_{f^*})^*$

Thus, for a pivotal 2-category $\mathcal{C}$, the braided (planar) pivotal 1-category $\mathrm{End}_{\mathcal{C}}(\mathrm{I})$ is a ribbon category; cf [BMS12, Cor 4.9]. By the same comparison, the delooping of a ribbon category is a pivotal 2-category.

Spherical traces in pivotal 2-categories

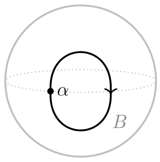
Recall that a pivotal monoidal 1-category is called 'spherical' when the left and right 'traces' of any 1-endomorphism agree. In our terminology, that situation is a planar pivotal 2-category $\mathcal{C}$ having just one object $*$, in which for any 1-morphism $f : * \to *$ and 2-morphism $\alpha : f \Rightarrow f$, the left planar trace $\mathrm{tr}_L(\alpha) : 1_* \Rightarrow 1_*$ is equal to the right planar trace $\mathrm{tr}_R(\alpha) : 1_* \Rightarrow 1_*$.

In a general planar pivotal 2-category, it does not make sense to ask that the left planar trace and right planar trace are equal, as when the 1-morphism $f : A \to B$ is not an endomorphism, the left planar trace $\mathrm{tr}_L(\alpha) : 1_A \Rightarrow 1_A$ and right planar trace $\mathrm{tr}_R(\alpha) : 1_B \Rightarrow 1_B$ are 2-endomorphisms of different objects. However, when the planar pivotal 2-category has a monoidal structure and is in fact pivotal, then we can compare the left and right planar traces by putting each one on a 2-sphere and then comparing the resulting 2-endomorphisms of the identity object.

Let $\mathcal{C}$ be a pivotal 2-category, and let $f : A \to B$ be a 1-morphism and $\alpha : f \Rightarrow f$ a 2-morphism in $\mathcal{C}$. As before $e_B : B \Box B^{\#} \to \mathrm{I}$ denotes the counit of the object duality; the composite $e_B \circ (\alpha \Box B^{\#})$ is an endomorphism of the 1-morphism $e_b \circ (f \Box B^{\#}) : A \Box B^{\#} \to \mathrm{I}$. The right planar trace of this composite is therefore a 2-endomorphism of the identity object,

$$\mathrm{tr}_R(e_B \circ (\alpha \Box B^{\#})) : 1_\mathrm{I} \Rightarrow 1_\mathrm{I}$$

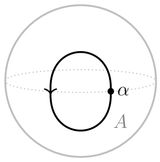
and has the graphical representation



Similarly, using the unit $i_{A^\#} : \mathrm{I} \to A \,\square\, A^\#$, we can form the composite $(\alpha \,\square\, A^\#) \circ i_{A^\#}$, which is an endomorphism of the 1-morphism $(f \,\square\, A^\#) \circ i_{A^\#} : \mathrm{I} \to B \,\square\, A^\#$. The left planar trace of this composite is therefore a 2-endomorphism of the identity object,

$$\mathrm{tr}_L((\alpha \,\square\, A^\#) \circ i_{A^\#}) : 1_\mathrm{I} \Rightarrow 1_\mathrm{I}$$

and has the graphical representation



As is evident from the graphical calculus (by pulling the wire around the back of the sphere), these left and right traces always agree in a pivotal 2-category.

Proposition 1.3.37 (Equality of left and right traces [BMS12, Lem 7.6])**.** *For an endomorphism $\alpha : f \Rightarrow f$ of a 1-morphism $f : A \to B$ in a pivotal 2-category, the following left and right traces agree:*

$$\mathrm{tr}_R(e_B \circ (\alpha \,\square\, B^\#)) = \mathrm{tr}_L((\alpha \,\square\, A^\#) \circ i_{A^\#})$$

Proof. The cusp is a 2-isomorphism $C_A : {}^\#(1_{A^\#}) \Rightarrow 1_A$. Given a 2-endomorphism $\beta : 1_A \Rightarrow 1_A$, define its *wrinkle dual* $\beta^+ : 1_{A^\#} \Rightarrow 1_{A^\#}$ as follows:

$$\beta^+ := C_{A^\#} \circ {}^\#\beta \circ C_{A^\#}^{-1}.$$

(Graphically the wrinkle dual of a 2-endomorphism is obtained by taking a surface, introducing a 'wrinkle', that is a cusp and its inverse, and then drawing the 2-endomorphism on 'the back side' of the wrinkle.) By imitating the usual graphical proof that ribbon categories are spherical, while keeping track of an underlying surface, one can show that the surface ribbon condition implies the following crucial 'circular wrinkle equation':

$$\mathrm{tr}_R(\alpha^\#) = \mathrm{tr}_L(\alpha)^+.$$

A graphical rendition of that argument (utilizing somewhat different conventions) appears in [BMS12, Fig 63]. The swallowtail equation directly implies the following relation between a 2-endomorphism $\beta : 1_A \Rightarrow 1_A$ and its wrinkle dual:

$$(A \,\square\, \beta^+) \circ i_{A^\#} = (\beta \,\square\, A^\#) \circ i_{A^\#}$$

The result now follows from a chain of equalities:

$$\begin{aligned}
\mathrm{tr}_R(e_B \circ (\alpha \,\square\, B^\#)) &= \mathrm{tr}_R(e_A \circ (A \,\square\, \alpha^\#)) \\
&= \mathrm{tr}_R(e_A \circ (A \,\square\, \mathrm{tr}_R(\alpha^\#)) \\
&= \mathrm{tr}_L((A \,\square\, \mathrm{tr}_R(\alpha^\#)) \circ i_{A^\#}) \\
&= \mathrm{tr}_L((A \,\square\, \mathrm{tr}_L(\alpha)^+) \circ i_{A^\#}) \\
&= \mathrm{tr}_L((\mathrm{tr}_L(\alpha) \,\square\, A^\#) \circ i_{A^\#}) \\
&= \mathrm{tr}_L((\alpha \,\square\, A^\#) \circ i_{A^\#})
\end{aligned}$$

Here the first equality follows from planar pivotality and the cusp-induced equation $\mathrm{tr}_R({}^\#(1_{A^\#})) = \mathrm{tr}_R(1_A)$. The third equality holds because $\mathrm{tr}_R(\beta) = \mathrm{tr}_L(\beta^*)$ for any 2-morphism β, and the $*$-mate of a 2-endomorphism of an identity 1-morphism is exactly the original 2-endomorphism. The fourth equality applies the circular wrinkle equation. The fifth is the aforementioned consequence of the swallowtail equation. □

Definition 1.3.38 (Back 2-spherical trace)**.** In a pivotal 2-category, the *back 2-spherical trace* of an endomorphism $\alpha : f \Rightarrow f$ of a 1-morphism $f : A \to B$ is

$$\mathrm{Tr}_B(\alpha) := \mathrm{tr}_R\left[e_B \circ (\alpha \,\square\, B^\#)\right] = \mathrm{tr}_L\left[(\alpha \,\square\, A^\#) \circ i_{A^\#}\right] : 1_\mathrm{I} \Rightarrow 1_\mathrm{I}.$$

Remark 1.3.39 (Circular trace equality is not sphericality). One might be tempted to think of the equivalence of the two traces in Definition 1.3.38 as a 'sphericality' condition on the pivotal 2-category, but that is not the situation. Indeed, the equivalence of those traces is a consequence purely of the pivotal structure on the 2-category, that is of the natural object-level involutive duality structure. There is, as described in the next section, a *further* sphericality condition that can be imposed on a pivotal 2-category, analogous to the sphericality condition that can be imposed on a pivotal 1-category.

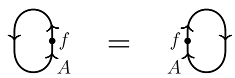
Remark 1.3.40 (Circular trace equality implies surface ribbon condition). That left and right traces agree, from Proposition 1.3.37, is one of the main properties of pivotal 2-categories we will use later in analyzing the state sum. We will not, in particular, directly use the surface ribbon condition C8 from the Definition 1.3.32

of pivotal structures. It may therefore seem like we have not used all the structure available in pivotal 2-categories, but that is not the case: following [BMS12, Lem 7.6 & Lem 7.15], it can be shown, for a presemisimple monoidal planar pivotal 2-category with a pivotal structure possibly not satisfying the surface ribbon condition, that the surface ribbon condition is actually equivalent to the condition that left and right traces agree. (This result is a generalization of the fact that a semisimple spherical braided pivotal 1-category is always ribbon [HPT16, Prop A.4].)

1.3.3 Spherical 2-categories

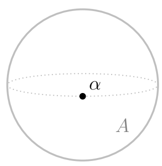
The definition of sphericality

Recall that a pivotal 1-category is called 'spherical' when the left and right 'circular' traces of a 1-endomorphism $f : A \to A$ agree:

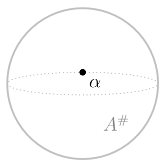


The terminology is motivated by the idea that those traces would agree if the corresponding string diagrams could move freely on a *2-sphere*. For emphasis then, we might say that the 'sphericality' condition on a pivotal 1-category is '2-sphericality'.

The analogous condition on a pivotal 2-category $\mathcal{C}$ arises as follows. Given an object A of $\mathcal{C}$ and a 2-endomorphism $\alpha : 1_A \Rightarrow 1_A$, we may form the back 2-spherical trace $\mathrm{Tr}_B(\alpha) : 1_\mathrm{I} \Rightarrow 1_\mathrm{I}$ from Definition 1.3.38. The corresponding graphical picture is



As the terminology suggests, there is another possible construction, namely the front 2-spherical trace:



Definition 1.3.41 (Front 2-spherical trace)**.** In a pivotal 2-category, the *front 2-spherical trace* of an endomorphism $\alpha : f \Rightarrow f$ of a 1-morphism $f : A \to B$ is

$$\mathrm{Tr}_F(\alpha) := \mathrm{tr}_R\left[e_{B^\#} \circ (B^\# \,\square\, \alpha)\right] = \mathrm{tr}_L\left[(A^\# \,\square\, \alpha) \circ i_A\right] : 1_\mathrm{I} \Rightarrow 1_\mathrm{I}.$$

In a pivotal 2-category, it is not necessarily the case that the front and back traces coincide. For instance, in the corresponding graphical surface calculus, the front trace $\mathrm{Tr}_F(1_{1_A})$ and back trace $\mathrm{Tr}_B(1_{1_A})$ correspond to A labeled spheres of opposite coorientation in $\mathbb{R}^3$.

Thus, the natural sphericality condition on a pivotal 2-category is that the front and back traces agree; this would be graphically sensible if, hypothetically, the corresponding surface diagrams could move freely in a *3-sphere*.[6] We might therefore say that the 'sphericality' condition on a pivotal 2-category is '3-sphericality'.

Definition 1.3.42 (Spherical 2-category)**.** A pivotal 2-category $\mathcal{C}$ is *spherical* if for every object A of $\mathcal{C}$ and every 2-endomorphism $\alpha : 1_A \Rightarrow 1_A$, the front and back traces of α agree:

$$\mathrm{Tr}_F(\alpha) = \mathrm{Tr}_B(\alpha)$$

Note that it is a consequence of the equality of front and back traces for 2-endomorphisms of the form $\alpha : 1_A \Rightarrow 1_A$ that the front and back traces in fact agree for any 2-morphism of the form $\alpha : f \Rightarrow f$. We will refer to a pivotal 2-category that is spherical simply as a 'spherical 2-category'. In a spherical 2-category, we will write $\mathrm{Tr}(\alpha)$ for the front or equivalently back 2-spherical trace of a 2-morphism $\alpha : f \Rightarrow f$ and will refer to it simply as the 'trace'.

Example 1.3.43 (2-group 2-representations)*.* For a finite 2-group (π_1, π_2), we expect the (symmetric) fusion 2-category $2\mathrm{Rep}(\pi_1, \pi_2) := [(*, \pi_1, \pi_2), 2\mathrm{Vect}_k]$ (see Construction 1.3.12) inherits a spherical structure from the spherical structure of $2\mathrm{Vect}_k$.

Example 1.3.44 (2-group-graded 2-vector spaces)*.* For a finite 2-group (π_1, π_2) and a 4-cocycle $\omega \in Z^4((\pi_1, \pi_2); k^*)$, the fusion 2-category $2\mathrm{Vect}_k^\omega(\pi_1, \pi_2)$ of ω-twisted (π_1, π_2)-graded 2-vector spaces (see Construction 1.3.16) should admit a canonical spherical structure as follows. We expect that every 3-group, seen as a monoidal 2-category, admits a canonical (up to an appropriate notion of equivalence) spherical structure. In particular, the 3-group (π_1, π_2, k^*) (determined by the twisting ω) has a spherical structure, which induces a spherical structure on the linearization $(\pi_1, \pi_2, k^*)_k$ and then in turn on its semisimple completion $2\mathrm{Vect}_k^\omega(\pi_1, \pi_2)$. More concretely, this spherical structure is obtained by taking a semi-strictification of the linear monoidal 2-category $(\pi_1, \pi_2, k^*)_k$ and using the group inverses of π_1 and π_2 to define the duals of objects and adjoints of 1-morphisms.

[6]Note that this idea of 'isotopy of surfaces on a 3-sphere' is merely a heuristic for understanding the definition of '3-sphericality'; even in the classical case of '2-spherical' 1-categories, it is not the case that isotopy on the sphere faithfully represents the categorical structure in question [Sel11, Sec 4.3].

Example 1.3.45 (Ribbon categories). By Remark 1.3.36, the delooping of a ribbon category is a pivotal 2-category. As there is only one object, the (self-dual) identity of the monoidal 2-category, this pivotal structure is evidently spherical.

Example 1.3.46 (Crossed-braided spherical categories). Construction 1.3.23 produces a fusion 2-category $\mathcal{C}$ from a crossed-braided fusion 1-category C. A spherical structure (in the ordinary 1-categorical sense) on the G-crossed-braided category C induces a pivotal structure (in the 2-categorical sense of Definition 1.3.32) on the corresponding fusion 2-category $\mathcal{C}$. (See Cui [Cui16, Sec 6] for a discussion of the duals of objects and adjoints of 1-morphisms in this case.) Note well that the 1-categorical sphericality condition on the braided 1-category C provides an equality of left and right traces, which corresponds to the equality of left and right traces (see Proposition 1.3.37) in a pivotal 2-category; that spherical condition a priori has nothing to do with 2-categorical sphericality. Nevertheless, in this particular case, because the G-action fixes the tensor unit of the 1-category C, it follows that the pivotal 2-category $\mathcal{C}$ associated to a G-crossed-braided fusion category is in fact always a spherical fusion 2-category.

Remark 1.3.47 (Mackaay's notion of sphericality). Mackaay [Mac99, Def 2.8] defined a notion of 'spherical monoidal 2-category'; a presemisimple monoidal 2-category that is Mackaay-spherical is spherical in our sense (Definition 1.3.42): conditions C1–C7 of Definition 1.3.32 follow from [Mac99, Def 2.3], condition C8 of Definition 1.3.32 follows using Remark 1.3.40 from [Mac99, Def 2.7 & Lem 2.12], and the sphericality condition of Definition 1.3.42 follows from [Mac99, Lem 2.13].

However, note that Mackaay's notion of 'spherical monoidal 2-category' is much more restrictive than our (Definition 1.3.42) notion of sphericality (i.e. 3-sphericality) and Mackaay's notion does not correspond to a graphical calculus of surface diagrams moving on a 3-sphere. Mackaay's definition insists on the existence of a 2-isomorphism between the two 'categorical traces' $e_{A^\#} \circ (A^\# \,\square\, f) \circ i_A$ and $e_A \circ (f \,\square\, A^\#) \circ i_{A^\#}$; that 2-isomorphism would be sensible if the corresponding surface diagrams lived in $S^2 \times [0, 1]$ rather than in S^3, but that is not always the case in relevant examples. For instance, as in Example 1.3.46, the monoidal 2-category associated to a crossed-braided category is spherical (in our 3-spherical sense) but does not satisfy Mackaay's $S^2 \times [0, 1]$-sphericality condition unless the G-action on the identity-graded component C_1 is trivial; see [Cui16, Sec 6].

Dimension in spherical prefusion 2-categories

In a prefusion 2-category $\mathcal{C}$, the monoidal unit I is simple, and hence the 1-morphism $\mathbb{1}_{\mathrm{I}}$ is simple. We therefore may and will identify $\mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathrm{I}})$ with the base field k. In a spherical prefusion 2-category $\mathcal{C}$, then, the trace $\mathrm{Tr}(\alpha)$ of any 2-morphism $\alpha : f \Rightarrow f$ may be canonically considered as an element of k.

Definition 1.3.48 (Dimensions of 1-morphisms and objects)**.** In a spherical prefusion 2-category, the *dimension* of a 1-morphism $f : A \to B$ is

$$\dim(f) := \mathrm{Tr}(1_f) \in k.$$

The *dimension* of an object A is

$$\dim(A) := \dim(\mathbb{1}_A) = \mathrm{Tr}(1_{\mathbb{1}_A}) \in k.$$

Graphically, the dimension of an object A is represented by an A-labeled sphere; the dimension of a 1-morphism f is represented by an A-labeled sphere with an f-labeled loop on it. By pivotality, the f-label may be placed either on the right or left of the loop, and by sphericality, the A-label and the loop may be placed on either the front or back of the sphere. Note that the conditions of planar pivotality ensure that the dimensions of isomorphic 1-morphisms are the same.

Note well that while the dimension of an object or 1-morphism in a prefusion 2-category $\mathcal{C}$ depends on the monoidal structure, the 'dimension of $\mathcal{C}$' itself refers to the dimension of the underlying presemisimple 2-category, see Definition 1.2.32, and therefore does not depend on the monoidal structure; this is rather in contrast with what one might expect from the fact that the dimension of a fusion 1-category certainly does depend on its monoidal structure.

Remark 1.3.49 (Pivotal adjoint equivalence preserves dimension). As mentioned in Warning 1.3.33, it is not in general the case that equivalent objects have the same dimension. Indeed, two objects will have the same dimension only when they are equivalent in a way compatible with the pivotal structure—we might call such an equivalence a 'pivotal adjoint equivalence'. In the constructions that follow, including in the formula for the state sum, we will use dimensions of objects, and so these constructions depend on choices of representative objects in each equivalence class of objects—however, we will show that the overall resulting state sum is independent of those choices.

We now show that in spherical prefusion 2-categories, the dimensions of simple objects and 1-morphisms do not vanish.

Lemma 1.3.50 (Planar trace is nonzero)**.** *For $f : A \to B$ a simple 1-morphism into a simple object B, in a planar pivotal presemisimple 2-category $\mathcal{C}$, the right planar trace* $\mathrm{tr}_R(1_f) \in \mathrm{End}_{\mathcal{C}}(1_B) = k$ *is nonzero.*

Proof. By definition, the trace in question is the composite of the unit $\eta_{f^*} : 1_B \Rightarrow f \circ f^*$ and the counit $\epsilon_f : f \circ f^* \Rightarrow 1_B$. Note that by adjunction $\mathrm{Hom}_{\mathcal{C}}(f \circ f^*, 1_B) \cong \mathrm{Hom}_{\mathcal{C}}(f, f) \cong k$ and similarly (or by semisimplicity) $\mathrm{Hom}_{\mathcal{C}}(1_B, f \circ f^*) \cong k$. Now both η_{f^*} and ϵ_f must be nonzero, as they are a unit and a counit. By Proposition 1.2.18, the identity 1_B is simple; there must therefore be nonzero maps $x : 1_B \Rightarrow f \circ f^*$ and $y : f \circ f^* \Rightarrow 1_B$ whose composite is the identity. Thus η_{f^*} and ϵ_f must be nonzero-proportional to x and y respectively, and by bilinearity of composition, it follows that the trace is nonzero-proportional to the identity, and therefore itself nonzero. □

Proposition 1.3.51 (Dimension of simple 1-morphism is nonzero)**.** *In a spherical prefusion* 2*-category* $\mathcal{C}$*, the dimension of a simple* 1*-morphism* $f : A \to B$ *is nonzero.*

Proof. By the duality between B and $B^\#$, we have $\mathrm{Hom}_{\mathcal{C}}(e_B \circ (f \Box B^\#), e_B \circ (f \Box B^\#)) \cong \mathrm{Hom}_{\mathcal{C}}(f, f) = k$. Thus $e_B \circ (f \Box B^\#) : A \Box B^\# \to \mathrm{I}$ is a simple 1-morphism to the (simple) identity. By Lemma 1.3.50, the right trace $\mathrm{tr}_R(1_{e_B \circ (f \Box B^\#)})$ is nonzero; but the dimension of f is $\mathrm{Tr}(1_f) = \mathrm{tr}_R(e_B \circ (1_f \Box B^\#)) = \mathrm{tr}_R(1_{e_B \circ (f \Box B^\#)})$. □

Corollary 1.3.52 (Dimension of simple object is nonzero)**.** *In a spherical prefusion* 2*-category* $\mathcal{C}$*, the dimension of a simple object A is nonzero.*

Proof. The dimension of the object is by definition the dimension of its identity, which is a simple 1-morphism. □

In addition to nonzero dimensions, the trace also provides a nondegenerate pairing on 2-morphism spaces.

Definition 1.3.53 (Pairing on Hom sets)**.** For 1-morphisms $f, g : A \to B$ in a spherical prefusion 2-category, the pairing $\langle \cdot, \cdot \rangle : \mathrm{Hom}_{\mathcal{C}}(f, g) \otimes \mathrm{Hom}_{\mathcal{C}}(g, f) \to k$ is given by

$$\langle \alpha, \beta \rangle := \mathrm{Tr}(\alpha \cdot \beta) = \mathrm{Tr}(\beta \cdot \alpha).$$

Proposition 1.3.54 (Pairing on Hom is nondegenerate)**.** *In a spherical prefusion* 2*-category, the pairing* $\langle \cdot, \cdot \rangle : \mathrm{Hom}_{\mathcal{C}}(f, g) \otimes \mathrm{Hom}_{\mathcal{C}}(g, f) \to k$ *is nondegenerate.*

Proof. Pick a collection $\{s_i\}$ of simple 1-morphisms $A \to B$, one in each isomorphism class. Without loss of generality we may assume that $f = \oplus_{i \in I} s_{k_i}$ and $g = \oplus_{j \in J} s_{l_j}$. Let $p_i : f \leftrightarrows s_{k_i} : \iota_i$ and $p'_j : g \leftrightarrows s_{l_j} : \iota'_j$ be the inclusion and projection 2-morphisms.

Suppose $\alpha : f \Rightarrow g$ is a 2-morphism such that $\langle \alpha, \beta \rangle = 0$ for all $\beta : g \Rightarrow f$. Every 2-morphism $g \Rightarrow f$ is a linear combination of the 2-morphisms $\iota_i \cdot p'_j$ for which $k_i = l_j$. Given i and j with $k_i = l_j$, by assumption and planar pivotality $0 = \langle \alpha, \iota_i \cdot p'_j \rangle = \mathrm{Tr}(\alpha \cdot \iota_i \cdot p'_j) = \mathrm{Tr}(p'_j \cdot \alpha \cdot \iota_i)$. Now $p'_j \cdot \alpha \cdot \iota_i : s_{k_i} \Rightarrow s_{l_j}$ is $\lambda_{i,j} 1_{s_k}$ for some scalar $\lambda_{i,j}$, where $k = k_i = l_j$. By bilinearity of composition, $\mathrm{Tr}(p'_j \cdot \alpha \cdot \iota_i) = \lambda_{i,j} \mathrm{Tr}(1_{s_k}) = \lambda_{i,j} \dim(s_k)$. As $\dim(s_k)$ is nonzero, the scalar $\lambda_{i,j}$ is forced to be zero for all i and j. By local semisimplicity, $p'_j \cdot \alpha \cdot \iota_i = 0$ for all i and j implies that α itself is zero. □

Chapter 2

A state-sum invariant for 4-manifolds

In this chaper, based on the last two sections of [DR18], we construct, given the data of a spherical prefusion 2-category, a state-sum invariant of oriented singular piecewise-linear 4-manifolds.

2.1 Introduction

A state sum for singular piecewise-linear 4-manifolds

Recall that a combinatorial n-sphere is a simplicial complex piecewise-linearly homeomorphic to the boundary of the standard $(n+1)$-simplex, and a combinatorial n-manifold is a simplicial complex such that the link of every vertex is a combinatorial $(n-1)$-sphere. Furthermore, a singular combinatorial n-manifold is a simplicial complex such that the link of every vertex is a combinatorial $(n-1)$-manifold. Given the data of a spherical fusion 1-category, Barrett and Westbury defined a state sum invariant of oriented combinatorial 3-manifolds. They prove invariance of their state sum by explicitly showing invariance under each 3-dimensional bistellar move. A bistellar move on a combinatorial n-manifold replaces a triangulation of a subcomplex isomorphic to a ball in $\partial\Delta^{n+1}$ by the complementary ball; these moves are known to generate piecewise-linear equivalence by a theorem of Pachner [Pac91]. Barrett and Westbury showed that in fact their invariant extends to singular combinatorial 3-manifolds by proving a generalization of the 3-dimensional case of Pachner's theorem: two singular combinatorial 3-manifolds are piecewise-linearly homeomorphic if and only if they are bistellar equivalent.

We will similarly show that our state sum is a piecewise-linear 4-manifold invariant by explicitly showing invariance under each 4-dimensional bistellar move. And, in fact, this invariant will also extend to singular combinatorial 4-manifolds because of the following result.

Theorem 7. *Two singular combinatorial 4-manifolds are piecewise-linearly homeomorphic if and only if they are bistellar equivalent.*

This appears as Theorem 2.2.6. The crucial fact that makes such a result possible is that the first nonshellable spheres do not appear until dimension 3.

We can now describe the state sum invariant itself, depending on a spherical fusion 2-category and a combinatorial 4-manifold. The sum is over labelings of the 1-simplices of the manifold by simple objects of the 2-category, and of 2-simplices of the manifold by simple 1-morphisms; the numbers being summed are, roughly speaking, the 10j symbols of the fusion 2-category corresponding to the given configuration of object and 1-morphism labels.

Definition 8. *Given an oriented singular combinatorial 4-manifold K and a spherical fusion 2-category $\mathcal{C}$, the* state sum *is the number*

$$Z_{\mathcal{C}}(K) := \sum_{\Gamma} \Big(\prod_{K_0} \dim(\mathcal{C})^{-1} \Big) \Big(\prod_{e \in K_1} \big(\dim(\Gamma(e)) \dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e))) n(\Gamma(e)) \big)^{-1} \Big) \Big(\prod_{s \in K_2} \dim(\Gamma(s)) \Big) Z(\Gamma).$$

A more precise version of this definition appears as Definition 2.2.18 in the main text. Here, K_i refers to the set of i-simplices of K, the sum is over the labelings Γ as described above, the dimension $\dim(\mathcal{C})$ is a kind of 'global dimension' of the underlying semisimple 2-category of $\mathcal{C}$, the dimensions $\dim(\Gamma(e))$ and $\dim(\Gamma(s))$ refer to appropriate scalar '2-spherical traces', $\dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))$ is the dimension of that fusion 1-category, $n(\Gamma(e))$ is the number of equivalence classes of simple objects in the connected component of the object $\Gamma(e)$, and finally $Z(\Gamma)$ is the aforementioned 10j symbol, which may be thought of as (derivable from) a piece of structural data of the fusion 2-category. This state sum definition is inspired by a combination of the state sums of Barrett–Westbury [BW96], Mackaay [Mac99], and Cui [Cui16].

Barrett and Westbury's state sum does not really need a full spherical fusion category as input—because the sum restricts attention to simple object labels, the nonsimple objects play essentially no role and therefore one need not insist that the input category have all direct sums. Our situation is somewhat analogous—because our state sum uses only simple object and simple 1-morphism labels, the necessary information is contained in the simples, and their tensor products, and the maps between such. In particular, we can drop the assumption that our fusion 2-category has direct sums, and even that it is idempotent complete. This leads to a notion of 'prefusion 2-category', a locally semisimple 2-category such that 1-morphisms have adjoints and every object decomposes as a direct sum of objects with simple identity, equipped with a monoidal structure for which objects have duals and the unit is simple. Our state sum works without modification for spherical prefusion 2-categories. This is convenient because various examples, for instance the deloop of a braided fusion category or a G-crossed-braided fusion category, are in the first instance prefusion 2-categories and must be additive and idempotent completed to obtain fusion 2-categories per se.

Theorem 9. *For an oriented singular piecewise-linear 4-manifold M and a spherical prefusion 2-category $\mathcal{C}$, the numerical state sum $Z_{\mathcal{C}}(K)$ is independent of the choice of triangulation K of M and therefore defines an oriented piecewise-linear invariant of the singular 4-manifold M.*

This Theorem appears, in a more precise form, as Theorem 2.2.19, and the proof, occupying all of Section 2.3, proceeds by direct combinatorial analysis of the effect of each bistellar move. As mentioned previously, for appropriate choices of the spherical prefusion 2-category, our invariant specializes to the Crane–Yetter–Kauffman invariant for ribbon categories, the (twisted) Yetter–Dijkgraaf–Witten invariant for finite 2-groups, the Mackaay invariant for endotrivial fusion 2-categories, and the Cui invariant for crossed-braided fusion categories.[1]

Note that 4-dimensional field theories associated to fusion 2-categories may not directly distinguish distinct smooth structures on closed 4-manifolds—for that one would presumably need nonsemisimple or derived algebraic structures, among other modifications. Nevertheless, as fusion 1-categories and particularly their classification have proven compelling quite apart from their associated closed 3-manifold invariants, we imagine fusion 2-categories and their classification will be a worthwhile subject in its own right.

Notation and conventions

In this chapter, we use the same notation and conventions as in Chapter 1.

[1]Bärenz and Barrett use the handle calculus to define a smooth 4-manifold invariant given the data of a pivotal functor from a spherical fusion 1-category to a ribbon fusion 1-category [BB18]. When the target ribbon category is modularizable, this invariant can be reformulated as a state sum invariant. We wonder whether there a fusion 2-category for which our state sum invariant gives the modularizable-target case of the Bärenz–Barrett invariant.

Outline

Section 2.2 defines the state sum invariant of singular piecewise-linear 4-manifolds. Section 2.2.1 recalls the relevant notions of combinatorial and singular combinatorial manifolds. Section 2.2.2 recalls bistellar transformations of combinatorial manifolds and the fundamental theorem that they generate piecewise-linear equivalence, and then proves that in fact bistellar moves generate piecewise-linear equivalence of singular combinatorial 4-manifolds. Section 2.2.3 motivates and discusses the technicalities of our state-sum labeling scheme, defines the 10j symbol and 10j action maps that form the core numerical information in the state sum, and finally gives the full state sum expression. Section 2.2.4 discusses how the state sum specializes, given specific choices of fusion 2-categories, to the Crane–Yetter–Kauffman, Yetter–Dijkgraaf–Witten, Mackaay, and Cui invariants.

Section 2.3 proves the invariance of the state sum. Section 2.3.1 shows that the sum is invariant of the labeling skeleton, that is of the choice of the particular representative simple objects and 1-morphisms used as labels; despite sounding innocuous, this invariance is nontrivial, in part because the dimensions of equivalent objects need not be the same, and the proof relies crucially on the local combinatorial manifold structure. Section 2.3.2 shows that the state sum is invariant under changing the chosen global order on the vertices, which was used to orient all the simplices and thus define the labeling scheme for the state sum; again despite sounding like a minor matter, the terms of the state sum change substantially under vertex reordering because object and 1-morphism labels are being replaced by composites of duals and adjoints, and proving invariance here uses much of the power of the pivotal 2-categorical context. Finally, Section 2.3.3 proves the state sum is invariant under changing the combinatorial structure of the 4-manifold, that is under the bistellar moves; it does so by defining a state sum for manifolds with boundary, in order to isolate the effect of a local bistellar move, and then explicitly computing each of the three 4-dimensional bistellar moves. This last bistellar proof uses an extensive collection of formulas relating the dimensions of objects and 1-morphisms in fusion 2-categories, formulas which are established in Appendix C.

2.2 A state-sum invariant of singular piecewise-linear 4-manifolds

Given a spherical prefusion 2-category $\mathcal{C}$ over an algebraically closed field k of characteristic zero, we define a k-valued invariant of closed oriented singular piecewise-linear 4-manifolds. We expect that a prefusion 2-category is a 4-dualizable object of an appropriate 4-category of linear monoidal 2-categories, and a spherical prefusion 2-category moreover has a canonical SO(4)-fixed point structure. There would therefore be a corresponding oriented local topological field theory, and we would expect our invariant restricts to the closed 4-manifold invariant of that field theory.

2.2.1 Combinatorial, piecewise-linear, and smooth manifolds

Simplicial complexes and piecewise-linear maps. Recall that a finite simplicial complex K is a finite set K_0 (of 'vertices') together with a collection of subsets of K_0 (the 'simplices' of K), such that a sub-subset of an element of the collection is again in the collection, and such that all singleton sets are in the collection. Each simplex σ of K determines a geometric simplex $|\sigma|$ in $\mathbb{R}^{K_0}$, namely the convex hull of the basis vectors corresponding to the vertices of that simplex; the standard geometric realization $|K|$ of a simplicial complex K is the union in $\mathbb{R}^{K_0}$ of the geometric simplices corresponding to the simplices of K. More generally, a geometric realization of the complex K in $\mathbb{R}^n$ is any subspace constructed as follows: choose an embedding of the vertices K_0 in $\mathbb{R}^n$ such that for any simplex σ of K, the embedded vertices of σ are linearly independent (and thus determine a corresponding geometric simplex as their convex hull), and such that for any two simplices σ and τ of K, the intersection of the corresponding geometric simplices in $\mathbb{R}^n$ is a face of each; the union of all the geometric simplices in $\mathbb{R}^n$ corresponding to simplices of K is the realization determined by the given vertex embedding. A subdivision of a simplicial complex K is a simplicial complex K' together with a geometric realization of K' that is, as a subspace of $\mathbb{R}^{K_0}$, the standard geometric realization $|K|$. A piecewise-linear map from a simplicial complex K to a simplicial complex L is a map $f : |K| \to |L|$ such that there exists a subdivision K' of K such that the map f is linear when restricted to each simplex of K'.

Combinatorial manifolds. A combinatorial n-ball is a simplicial complex piecewise-linearly homeomorphic to the standard n-simplex, and a combinatorial n-sphere is a simplicial complex piecewise-linearly homeomorphic to the boundary of the standard $(n+1)$-simplex.

Definition 2.2.1 (Combinatorial manifold)**.** A *combinatorial n-manifold* is a finite simplicial complex such that the link of every vertex is a combinatorial $(n-1)$-sphere.

A 'combinatorial n-manifold with boundary' is allowed to have vertices with links that are combinatorial $(n-1)$-balls; the boundary is the subcomplex of simplices whose vertices have links that are balls. Note that in a combinatorial n-manifold with boundary, the link of every k-simplex is necessarily a combinatorial $(n-k-1)$-sphere or a combinatorial $(n-k-1)$-ball.

Definition 2.2.2 (Singular combinatorial manifold)**.** A *singular combinatorial n-manifold* is a finite simplicial complex such that the link of every vertex is a combinatorial $(n-1)$-manifold.

A 'singular combinatorial n-manifold with boundary' is allowed to have vertices with links that are combinatorial $(n-1)$-manifolds with boundary; the boundary is the subcomplex of simplices whose vertices have links that have nonempty boundary. Note that in a singular combinatorial n-manifold with boundary, the link of every k-simplex, for $k \geq 1$, is necessarily a combinatorial $(n-k-1)$-sphere or a combinatorial $(n-k-1)$-ball.

An orientation on a singular combinatorial n-manifold with boundary is a choice of orientation of each n-simplex such that for every $(n-1)$-simplex not in the boundary, the orientations induced from the two adjacent n-simplices are opposite.

Piecewise-linear manifolds. A piecewise-linear (PL) manifold is a triple (M, K, ϕ) consisting of a topological manifold M, a finite simplicial complex K, and a homeomorphism $\phi : |K| \to M$ from the geometric realization of the complex to the manifold; a piecewise-linear map $(M, K, \phi) \to (M', K', \phi')$ is a map $M \to M'$ such that the induced map $|K| \to |K'|$ is piecewise-linear. Evidently, the category of PL manifolds and PL maps is equivalent to the category of combinatorial manifolds and PL maps. For convenience, then, we will work directly with combinatorial manifolds, and more generally with singular combinatorial manifolds. By a 'singular piecewise-linear manifold', or by merely 'singular manifold', we will mean a singular combinatorial manifold.

Piecewise-linear versus smooth 4-manifolds. In dimension 4 there is no functional difference between smooth and piecewise-linear structures on manifolds; thus our invariant of piecewise-linear 4-manifolds immediately provides a corresponding invariant of smooth 4-manifolds. More specifically, Whitehead [Whi40] proved that in any dimension, given a compact closed smooth manifold M, there is a combinatorial manifold K for which there exists a piecewise-differentiable homeomorphism from $|K|$ to M, and moreover such a combinatorial manifold is unique up to piecewise-linear homeomorphism. (A homeomorphism from a combinatorial manifold $|K|$ to a smooth manifold M is piecewise-differentiable if it is a smooth immersion when restricted to each simplex.) This association provides a canonical map from the diffeomorphism classes of smooth manifolds to the piecewise-linear homeomorphism classes of piecewise-linear manifolds. By work of Cerf [Cer68], Smale [Sma59], Munkres [Mun60], and Hirsch–Mazur [Hir63, HM74], this canonical map from smooth to piecewise-linear manifolds is a bijection in dimension 4.

2.2.2 Stellar and bistellar equivalence of singular combinatorial manifolds

Stellar subdivision and stellar equivalence of simplicial complexes. The stellar subdivision $S\Delta^k$ of the standard k-simplex Δ^k is obtained from the standard simplex by adding a new vertex in the interior and coning the boundary simplices to it; the resulting simplicial complex has $(k+1)$-many k-simplices and is concisely expressed as the join $\partial\Delta^k \star \{*\}$. Given a simplicial complex X and a k-simplex A of X, recall that the star of A can be expressed as the join $A \star \mathrm{lk}(A)$ of the simplex with its link $\mathrm{lk}(A)$. The stellar subdivision $S_A X$ of the complex X at A is obtained by replacing the star of A with the complex $SA \star \mathrm{lk}(A)$.

Two finite simplicial complexes are called stellar equivalent if they are related by a finite zigzag of stellar subdivisions. One of the first fundamental results of piecewise-linear topology is Alexander's theorem that stellar subdivision generates piecewise-linear equivalence.

Theorem 2.2.3 (Stellar equivalence of simplicial complexes [Ale30, New26, Lic99])**.** *Two finite simplicial complexes are piecewise-linear homeomorphic if and only if they are stellar equivalent.*

To produce a piecewise-linear invariant, whether of manifolds or otherwise, it therefore suffices to show that the invariant is unaffected by stellar subdivision. Unfortunately, even in a fixed dimension and in the context of manifolds, there are infinitely many distinct types of stellar subdivision, depending on the combinatorial structure of the link of the simplex being subdivided. Thus it is typically impractical to check invariance via stellar subdivision moves. Much more convenient is the finite collection of bistellar moves.

Bistellar moves and bistellar equivalence of combinatorial manifolds. Given a combinatorial n-manifold, one may obtain a piecewise-linearly homeomorphic combinatorial manifold by cone-subdividing an n-simplex: remove an n-simplex σ and replace it with the $(n+1)$ n-simplices produced by coning the boundary of σ. This top-dimensional stellar subdivision is also called a '$(1, n+1)$-bistellar move' and may be thought of as replacing a single simplex by the complement of a simplex in the standard combinatorial n-sphere $\partial\Delta^{n+1}$. More generally, for (p, q) with $p + q = n + 2$, let P denote the codimension zero combinatorial submanifold of $\partial\Delta^{n+1}$ with p n-simplices, and let Q denote the complementary codimension zero combinatorial submanifold with q n-simplices. Two combinatorial n-manifolds K and K' are related by a (p, q)-bistellar (or '(p, q)-Pachner') move if K' is obtained from K by removing a codimension zero combinatorial submanifold isomorphic to P and replacing it by one isomorphic to Q.

Two combinatorial manifolds are called bistellar equivalent if they are related by a finite series of bistellar moves. The fundamental result of combinatorial manifold theory is Pachner's theorem that bistellar moves generate piecewise-linear equivalence:

Theorem 2.2.4 (Bistellar equivalence of combinatorial manifolds [Pac91, Lic99])**.** *Two combinatorial n-manifolds are piecewise-linearly homeomorphic if and only if they are bistellar equivalent.*

To produce a piecewise-linear invariant of combinatorial manifolds, it therefore suffices to check that the invariant is unaffected by each of the finitely many bistellar moves.

Bistellar equivalence of singular combinatorial 4-manifolds. In a combinatorial 3-manifold, the link of a vertex is a 2-sphere; in a singular combinatorial 3-manifold, the link of a vertex is allowed to be any surface. Barrett and Westbury proved that Pachner's theorem extends to singular combinatorial 3-manifolds.

Theorem 2.2.5 (Bistellar equivalence of singular 3-manifolds [BW96])**.** *Two singular combinatorial 3-manifolds are piecewise-linearly homeomorphic if and only if they are bistellar equivalent.*

As the invariance of the Turaev–Viro–Barrett–Westbury state sum is established via invariance under bistellar moves, this result extended the state sum invariant to singular 3-manifolds.

In a singular combinatorial 4-manifold, the link of a vertex is allowed to be any combinatorial 3-manifold. We prove that Pachner's theorem extends to this case.

Theorem 2.2.6 (Bistellar equivalence of singular 4-manifolds)**.** *Two singular combinatorial 4-manifolds are piecewise-linearly homeomorphic if and only if they are bistellar equvialent.*

Proof. Of course bistellar equivalent singular combinatorial 4-manifolds are piecewise-linearly homeomorphic. Given two piecewise-linearly homeomorphic singular combinatorial 4-manifolds, they are stellar equivalent by Theorem 2.2.3. It suffices therefore to show that any stellar subdivision of a singular combinatorial 4-manifold M is a bistellar equivalence. Stellar subdivision at a 0-simplex is trivial, and stellar subdivision at a 4-simplex is itself a bistellar move. Let A be a k-simplex, with $1 \leq k \leq 3$; note that the link $\mathrm{lk}(A)$ is a combinatorial i-sphere, for $0 \leq i \leq 2$, and is therefore shellable. (The first nonshellable spheres arise in dimension 3 [Vin85].) Because the link $\mathrm{lk}(A)$ is shellable, the stellar subdivision $S_A M$ is bistellar equivalent to M by the proof of [Lic99, Cor 5.8]. (Note that Lickorish's proof of bistellar equivalence given shellable links applies in our context of singular manifolds.) □

Our state-sum will be invariant under bistellar moves, and though its invariance depends crucially on the assumption that the links of 1-, 2-, and 3-simplices are spheres, it will not require sphericality of vertex links. Thus, the state sum will give an invariant not only of piecewise-linear 4-manifolds but also of singular piecewise-linear 4-manifolds.

Convention (Manifolds may be singular)**.** *In the remainder, every time we refer to a 'combinatorial 4-manifold' we implicitly mean 'singular combinatorial 4-manifold'. The allowance of singularities will only be made explicit in certain key statements.*

2.2.3 States, associated states, the 10j action, and the partition function

Given a 4-dimensional topological field theory Z, the numerical invariant $Z(M)$ of a closed 4-manifold M may be thought of as the 'partition function' of the theory on that 'spacetime'. In the spirit of the path integral, we might imagine this invariant would be computed as an integral of an appropriate action, integrated over all possible physical histories in that spacetime and weighted by a normalization factor for each history. In the case of a topological field theory $Z_{\mathcal{C}}$ associated to a spherical prefusion 2-category $\mathcal{C}$, a possible 'physical history' of a combinatorial 4-manifold M is given by an assignment of an object of $\mathcal{C}$ to each 1-simplex of M, a compatible 1-morphism of $\mathcal{C}$ to each 2-simplex of M, and a compatible 2-morphism to each 3-simplex of M.

In practice, instead of considering all such assignments, we may reduce the path integral to a 'path sum', also called a 'state sum', by insisting that the labels be respectively by simple objects, simple 1-morphisms, and elements of chosen bases of the 2-morphism vector spaces. The 'action' term in the state sum will be given as a product of the 10j symbols that give the pentagonator structure data of the fusion operation of the fusion 2-category. The normalization factor will be an appropriate ratio of quantum dimensions of the simple objects and simple 1-morphisms in the given labeling. The state sum, finally, is the average, over all possible labelings of the manifold, weighted by these quantum dimension factors, of a product of the 10j symbols of the fusion 2-category. (Compare the Barrett–Westbury–Turaev–Viro state sum for combinatorial 3-manifolds based on a fusion 1-category, which is an average, weighted by quantum dimensions of objects, of products of the 6j symbols defining the associator data of the fusion category.)

States of a 4-manifold. Let $\mathcal{C}$ be a spherical prefusion 2-category and let K be an oriented (singular) combinatorial 4-manifold. As mentioned, roughly speaking a 'state' of the manifold would be a labeling of 1-simplices by simple objects, of 2-simplices by simple 1-morphisms, and of 3-simplices by 2-morphism basis elements. In fact, to streamline later proofs, we will proceed by only labeling 1-simples and 2-simplices, and implicitly sum over the 2-morphism bases as part of an associated state construction. (One may think of this 'partial state', with only 1-simplices and 2-simplices labeled, as corresponding to a state of the neighborhood of the 2-skeleton of the 4-manifold; the associated state construction will account for the effect of filling in the 3-simplices, while the action term will encode the process of filling in the 4-simplices.)

As the morphisms and 2-morphisms of the fusion 2-category $\mathcal{C}$ are of course directed, it is convenient to also have a consistent choice of direction on the simplices of the combinatorial 4-manifold K; this is achieved by choosing an ordering on the vertices of K.

Definition 2.2.7 (Ordered combinatorial manifold)**.** An *ordered oriented combinatorial 4-manifold* K^o is an oriented combinatorial 4-manifold K with a choice of total order o on its set of 0-simplices K_0.

Note that the 'global' order o on the vertices of K induces a 'local' order of the vertices of any n-simplex $\tau \in K_n$. For an order-preserving inclusion $[k] := \{0, 1, \ldots, k\} \hookrightarrow \{0, 1, \ldots, n\} =: [n]$, with image $\{j_0, \ldots, j_k\} \subset [n]$, there is a well-defined k-face of τ

determined by the vertices $\{j_0, \ldots, j_k\}$ of τ; that face will be denoted $\partial^o_{[j_0,\ldots,j_k]}\tau \in K_k$. These face maps give the ordered oriented combinatorial 4-manifold K^o the structure of a semisimplicial set $K^o : \Delta_+^{\mathrm{op}} \to \mathrm{Set}$ with $K^o([n]) = K_n$. (Here Δ_+ is the category of non-empty ordered finite sets with injective order-preserving maps.)

The total order o in an ordered oriented combinatorial manifold K^o is not required to respect the orientation in any particular way, and so there are a collection of signs governing how the order and the orientation interact: there is a function $\epsilon_o : K_4 \to \{+1, -1\}$ with $\epsilon_o(\mu) = +1$ if and only if the orientation of the 4-simplex μ agrees with the orientation induced by the vertex order o, and for every 4-simplex μ there is a function $\epsilon_o^\mu : \{\kappa \in K_3 \mid \kappa \subseteq \mu\} \to \{+1, -1\}$ with $\epsilon_o^\mu(\kappa) = +1$ if and only if the orientation of the face $\kappa \subseteq \mu$ induced from the orientation of μ agrees with the one induced from the total vertex order. (Recall that an orientation of an n-simplex is an even-permutation equivalence class of orderings of its vertices; we will denote the orientation associated to the vertex order $v_0, \ldots, v_n$ by $\langle v_0, \ldots, v_n\rangle$ and the opposite orientation by $-\langle v_0, \ldots, v_n\rangle$. The induced orientation of the face of an n-simplex opposite to the vertex v_i is $(-1)^i\langle v_0, \ldots, \widehat{v_i}, \ldots v_n\rangle$.)

As we will only be labeling the 1-simplices and 2-simplices of K, we will only be concerned with the associated 2-truncated semisimplicial set $K^o_{(2)} : \Delta^{\mathrm{op}}_{+,2} \to \mathrm{Set}$, which is the restriction of K^o to the full subcategory $\Delta^{\mathrm{op}}_{+,2}$ on the objects $\{[0], [1], [2]\}$. A spherical prefusion 2-category $\mathcal{C}$ also determines a 2-truncated semisimplicial set $\Delta\mathcal{C} : \Delta^{\mathrm{op}}_{+,2} \to \mathrm{Set}$ with $\Delta\mathcal{C}_0 = \{*\}$, $\Delta\mathcal{C}_1 = \{\text{simple objects of } \mathcal{C}\}$, and

$$\Delta\mathcal{C}_2 = \{(A, B, C, f) \mid A, B, C \in \Delta\mathcal{C}_1, f \text{ a simple 1-morphism in } \mathrm{Hom}(A \,\square\, B, C)\}.$$

Here the [01], [12], and [02] faces of (A, B, C, f) are respectively A, B, and C. (Note that this truncated semisimplicial set is not finite—we address that issue later by picking a skeleton of the prefusion 2-category—and that equivalent prefusion 2-categories may produce nonisomorphic associated semisimplicial sets.)

Definition 2.2.8 (State of a combinatorial manifold)**.** Given a spherical prefusion 2-category $\mathcal{C}$ and an ordered oriented combinatorial 4-manifold K^o, a $\mathcal{C}$*-state* of K^o is a natural transformation $\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}$.

Concretely, a $\mathcal{C}$-state is a function from the 1-simplices of K to simple objects of $\mathcal{C}$ and a function from the 2-simplices of K to simple 1-morphisms of $\mathcal{C}$ compatible with the face maps from 2- to 1-simplices. We will denote the set of $\mathcal{C}$-states of K^o by $[K^o, \Delta\mathcal{C}]$.

Given a $\mathcal{C}$-state $\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}$, the value of the state on a 1-simplex e or 2-simplex s is written simply as $\Gamma(e)$, respectively $\Gamma(s)$; it is convenient to have a succinct notation for the value of the state on the boundary simplices of 3- and 4-simplices.

Notation 2.2.9 (State labels of 1- and 2-simplices in 3- and 4-simplices). For a $\mathcal{C}$-state Γ, and $\tau \in K_p$ a p-simplex for p either 3 or 4:

$$[ij]^\tau_\Gamma := \Gamma\left(\partial^o_{[ij]}\tau\right) \quad \text{for } 0 \leq i < j \leq p$$
$$[ijk]^\tau_\Gamma := \Gamma\left(\partial^o_{[ijk]}\tau\right) \quad \text{for } 0 \leq i < j < k \leq p$$

Furthermore, in a 3-simplex the four 2-simplices naturally divide into two composable pairs; following Mackaay [Mac99], we use a compact notation for the composites of those pairs of 2-simplices.

Notation 2.2.10 (State labels of associated 3-simplices in 3- and 4-simplices). For a $\mathcal{C}$-state Γ, and $\tau \in K_p$ a p-simplex for p either 3 or 4:

$$[(ijk)l]^\tau_\Gamma := [ikl]^\tau_\Gamma \circ \left([ijk]^\tau_\Gamma \,\square\, 1_{[kl]^\tau_\Gamma}\right) \quad \text{for } 0 \leq i < j < k < l \leq p$$
$$[i(jkl)]^\tau_\Gamma := [ijl]^\tau_\Gamma \circ \left(1_{[ij]^\tau_\Gamma} \,\square\, [jkl]^\tau_\Gamma\right) \quad \text{for } 0 \leq i < j < k < l \leq p$$

Finally we have a shorthand for duals and adjoints of state labels:

Notation 2.2.11 (Duals and adjoints of state labels). For a $\mathcal{C}$-state Γ, and $\tau \in K_p$ a p-simplex for p either 3 or 4:

$$\overline{[ij]}^\tau_\Gamma := \left([ij]^\tau_\Gamma\right)^\# \quad \text{for } 0 \leq i < j \leq p$$
$$\overline{[ijk]}^\tau_\Gamma := \left([ijk]^\tau_\Gamma\right)^* \quad \text{for } 0 \leq i < j < k \leq p$$

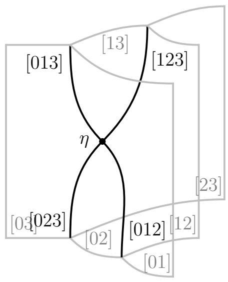
In all of these notations, we will often omit the super- and sub-scripts if the intended simplex and $\mathcal{C}$-labeling are clear.

The canonical associated state. As defined, a $\mathcal{C}$-state $\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}$ provides labels of the 2-simplices of K by 1-morphisms. Given a 3-simplex, there are vector spaces of associator-like 2-morphisms compatible with the given labels.

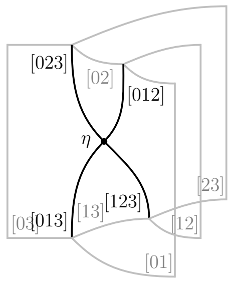
Definition 2.2.12 (Associator state spaces)**.** For a $\mathcal{C}$-state Γ on an ordered oriented combinatorial 4-manifold, the (positive and negative) *associator state spaces* of a 3-simplex $\kappa \in K_3$ are the vector spaces

$$V^+(\Gamma, \kappa) := \mathrm{Hom}_\mathcal{C}\left([(012)3]^\kappa_\Gamma, [0(123)]^\kappa_\Gamma\right)$$
$$V^-(\Gamma, \kappa) := \mathrm{Hom}_\mathcal{C}\left([0(123)]^\kappa_\Gamma, [(012)3]^\kappa_\Gamma\right)$$

Omitting the κ superscript and Γ subscript, vectors in these spaces are denoted graphically as follows:



$\eta \in V^+(\Gamma, \kappa)$



$\eta \in V^-(\Gamma, \kappa)$

If we were considering a state of the 4-manifold to include labelings of 3-simplices, those labels would be by elements of chosen bases for these associator state spaces. Instead we implicitly rather than explicitly sum over such bases by using a canonical copairing between the associator spaces, as follows. The pairing between $\mathrm{Hom}(f, g)$ and $\mathrm{Hom}(g, f)$ in a spherical prefusion 2-category, from Definition 1.3.53, gives, for any 3-simplex κ and labeling Γ, a pairing

$$\langle \cdot, \cdot \rangle_{\Gamma,\kappa} : V^-(\Gamma, \kappa) \otimes V^+(\Gamma, \kappa) \to k$$

By Proposition 1.3.54, this pairing is nondegenerate; there is therefore a canonically determined copairing

$$\cup_{\Gamma,\kappa} : k \to V^+(\Gamma, \kappa) \otimes V^-(\Gamma, \kappa)$$

providing, with the pairing, a duality between $V^+(\Gamma, \kappa)$ and $V^-(\Gamma, \kappa)$. We can tensor over all the 3-simplices of the manifold to obtain a 'global copairing':

$$\cup_\Gamma := \bigotimes_{\kappa \in K_3} \cup_{\Gamma,\kappa} : k \to \bigotimes_{\kappa \in K_3} \big(V^+(\Gamma, \kappa) \otimes V^-(\Gamma, \kappa)\big)$$

Definition 2.2.13 (Canonical associated state)**.** For a $\mathcal{C}$-state Γ on an ordered oriented combinatorial 4-manifold, the *canonical associated state* is

$$\cup_\Gamma(1) \in \bigotimes_{\kappa \in K_3} \big(V^+(\Gamma, \kappa) \otimes V^-(\Gamma, \kappa)\big)$$

Notice that this associated state may be thought of as a sum of 'complete states' of the manifold, where a complete state is a $\mathcal{C}$-state Γ labeling 1- and 2-simplices as before, and also a labeling of each 3-simplex κ by an element of a chosen basis of $V^+(\Gamma, \kappa)$.

The 10j symbol and the 10j action. Given a $\mathcal{C}$-state Γ on a 4-manifold K, a chosen 4-simplex $\mu \in K_4$, and for each 3-simplex $\kappa \subset \mu$ an 'associator vector' $v_\kappa \in V^{\epsilon_o^\mu(\kappa)}(\Gamma, \kappa)$, we can compose these five vectors (which are 2-morphisms in the prefusion 2-category) to obtain a 'pentagonator endomorphism'. The (numerical) 2-spherical trace of this pentagonator endomorphism is the '10j symbol' of that collection of associator vectors—these 10j symbols are the core linear-algebraic structure data of the prefusion 2-category.

Definition 2.2.14 (The 10j symbol)**.** For a $\mathcal{C}$-state Γ on an ordered oriented combinatorial 4-manifold K, and a chosen 4-simplex $\mu \in K_4$, the *10j symbol* is a map

$$z(\Gamma, \mu) : \bigotimes_{\kappa \in K_3, \kappa \subset \mu} V^{\epsilon_o^\mu(\kappa)}(\Gamma, \kappa) \to k$$

determined by the 2-spherical trace (Defintion 1.3.38) of the pentagonator endomorphism depicted in Figure 2.1.

(In the figure, the labels $[ijkl]$ denote elements of $V^+(\Gamma, \partial^o_{[ijkl]}\mu)$ and the labels $\overline{[ijkl]}$ denote elements of $V^-(\Gamma, \partial^o_{[ijkl]}\mu)$. As the monoidal unit I of the prefusion 2-category $\mathcal{C}$ is simple, and therefore the 1-morphism 1_I is also simple, we may and will identify the target $\mathrm{End}_\mathcal{C}(1_\mathrm{I})$ of the trace with the base field k.)

Explicitly, if the orientation of the 4-simplex agrees with the global order, i.e. $\epsilon_o(\mu) = +1$, then the map $z(\Gamma, \mu)$ has the form

$$z(\Gamma, \mu) : V^+(\Gamma, \partial^o_{[0123]}\mu) \otimes V^+(\Gamma, \partial^o_{[0134]}\mu) \otimes V^+(\Gamma, \partial^o_{[1234]}\mu) \otimes V^-(\Gamma, \partial^o_{[0124]}\mu) \otimes V^-(\Gamma, \partial^o_{[0234]}\mu) \to k$$

and is defined by the trace on the left side of the figure; if by contrast the orientation of the 4-simplex does not agree with the global order, i.e. $\epsilon_o(\mu) = -1$, then the map $z(\Gamma, \mu)$ has the form

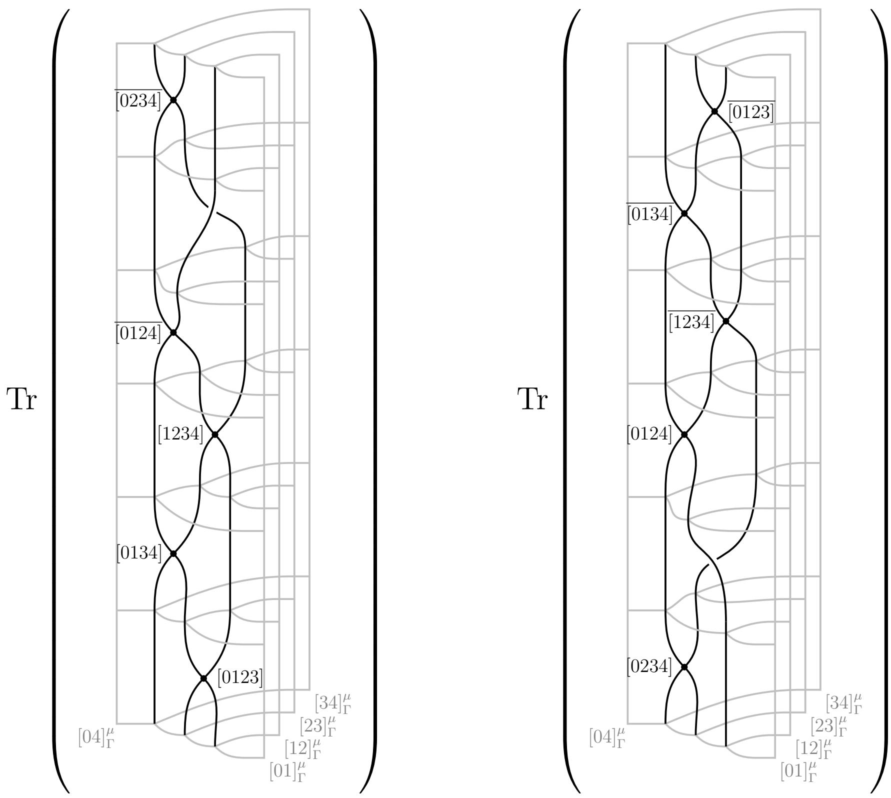
$$z(\Gamma, \mu) : V^+(\Gamma, \partial^o_{[0234]}\mu) \otimes V^+(\Gamma, \partial^o_{[0124]}\mu) \otimes V^-(\Gamma, \partial^o_{[1234]}\mu) \otimes V^-(\Gamma, \partial^o_{[0134]}\mu) \otimes V^-(\Gamma, \partial^o_{[0123]}\mu) \to k$$

and is defined by the trace on the right side of the figure.

Tensoring together the 10j symbols of all the 4-simplices of K, we obtain a global 10j symbol map, which we think of as playing the role of an action, applied to the state specified by Γ and a given collection of associator vectors.

Definition 2.2.15 (The 10j action)**.** For a $\mathcal{C}$-state Γ on an ordered oriented combinatorial 4-manifold K, the *10j action* is the map

$$z(\Gamma) := \Big(\bigotimes_{\mu \in K_4} z(\Gamma, \mu) \Big) : \bigotimes_{\kappa \in K_3} \big(V^+(\Gamma, \kappa) \otimes V^-(\Gamma, \kappa) \big) \to k$$



$z(\Gamma, \mu)$ if $\epsilon_o(\mu) = +1$ $z(\Gamma, \mu)$ if $\epsilon_o(\mu) = -1$

Figure 2.1: The definition of the 10j symbol $z(\Gamma, \mu)$.

In writing the domain of this linear map, we have used the fact that every 3-simplex $\kappa \in K_3$ appears as a face of precisely two 4-simplices $\mu_1, \mu_2 \in K_4$, and that $\epsilon_o^{\mu_1}(\kappa) = -\epsilon_o^{\mu_2}(\kappa)$. (We have also suppressed some swap maps reordering the factors in the tensor product in the domain.) In particular, we can and will consider the following numerical invariant of the state:

Definition 2.2.16 (The 10j action of the canonical associated state)**.** For a $\mathcal{C}$-state Γ on an ordered oriented combinatorial 4-manifold K, the *10j action of the canonical associated state* is the number

$$Z(\Gamma) := z(\Gamma) \circ \cup_\Gamma(1) \in k,$$

where $z(\Gamma)$ is the 10j action and $\cup_\Gamma(1)$ is the canonical associated state of the state Γ.

The partition function. We think of a combinatorial 4-manifold K being built up progressively as the sequence of its i-skeleta $K_{(i)}$:

$$\emptyset = K_{(-1)} \rightsquigarrow K_{(0)} \rightsquigarrow K_{(1)} \rightsquigarrow K_{(2)} \rightsquigarrow K_{(3)} \rightsquigarrow K_{(4)} = K$$

In keeping with the path-integral inspiration, the state sum invariant can be seen as a result of a corresponding sequence of progressive transformations:

-1. The initial empty state is represented by the unit of the base field:

$$K_{(-1)} \mapsto 1.$$

0. There are no possible labels of vertices of K, and so a state of the 0-skeleton is simply a scalar multiple of the collection of unlabeled vertices. Really, though, the vertices are labeled by the unique object in the 3-category that deloops the fusion 2-category, and in effect this amounts to labeling these vertices by the prefusion 2-category $\mathcal{C}$ itself. For some to-be-determined normalization factor $\phi(\mathcal{C}) \in k$, the canonical state of the 0-skeleton can therefore be seen as

$$K_{(0)} \mapsto \Big(\prod_{K_0} \phi(\mathcal{C})\Big)[K_0^{\mathcal{C}}].$$

Here $[K_0^{\mathcal{C}}]$ denotes the collection of vertices, each labeled by the prefusion 2-category $\mathcal{C}$.

1. A state of the 1-skeleton is a weighted sum of all possible labelings $\gamma : K^o_{(1)} \Rightarrow (\Delta\mathcal{C})_{(1)}$ of the 1-simplices $e \in K_1$ by simple objects $\gamma(e)$ of $\mathcal{C}$. Given a second to-be-determined scalar factor $\phi(\gamma(e)) \in k$ associated to each labeling object $\gamma(e)$, we may therefore think of the canonical state of the 1-skeleton as

$$K_{(1)} \mapsto \sum_{\gamma:K^o_{(1)} \Rightarrow (\Delta\mathcal{C})_{(1)}} \Big(\prod_{K_0} \phi(\mathcal{C})\Big)\Big(\prod_{e\in K_1} \phi(\gamma(e))\Big)[K_1^{\gamma}].$$

Here $[K_1^{\gamma}]$ denotes the 1-skeleton with the 1-simplices labeled according to the assignment γ.

2. Similarly a state of the 2-skeleton is a weighted sum of all possible labelings $\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}$ of the 1-simplices $e \in K_1$ by simple objects $\Gamma(e)$ and of the 2-simplices $s \in K_2$ by simple 1-morphisms $\Gamma(s)$. Given a third to-be-determined scalar factor $\phi(\Gamma(s)) \in k$ associated to each labeling 1-morphism $\Gamma(s)$, we imagine the canonical state of the 2-skeleton being

$$K_{(2)} \mapsto \sum_{\Gamma:K^o_{(2)} \Rightarrow \Delta\mathcal{C}} \Big(\prod_{K_0} \phi(\mathcal{C})\Big)\Big(\prod_{e\in K_1} \phi(\Gamma(e))\Big)\Big(\prod_{s\in K_2} \phi(\Gamma(s))\Big)[K_2^{\Gamma}].$$

Here $[K_2^{\Gamma}]$ denotes the 2-skeleton itself labeled according to Γ.

We have already discussed, in Definition 2.2.13, the 3-skeleton ‘canonical associated state’ coming from a state of the 2-skeleton, and in turn, in Definition 2.2.15, the 4-skeleton ‘10j action’ coming from a state of the 3-skeleton.

The three scalar factors for 0-, 1-, and 2-simplex labels are forced by asking the resulting state sum to be piecewise-linear homeomorphism invariant—more specifically by asking for the invariance of the state sum of a ball under the bistellar moves. The (3,3)-bistellar move (replacing a triangulation of a 4-ball with three 4-simplices by a different triangulation with three 4-simplices) involves summing over the 1-morphism labels $\Gamma(s)$ of 2-simplices s interior to the triangulations; this relation is satisfied if $\phi(\Gamma(s)) = \dim(\Gamma(s))$. The (2,4)-bistellar move (replacing a triangulation of a 4-ball with two 4-simplices by one with four 4-simplices) involves summing over the object labels $\Gamma(e)$ of a 1-simplex e interior to one of the triangulations; this relations is satisfied if $\phi(\Gamma(e)) = (\dim(\Gamma(e)) \dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))n(\Gamma(e)))^{-1}$. Here $\dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))$ denotes the global dimension of the fusion category $\mathrm{End}_{\mathcal{C}}(\Gamma(e))$, and $n(\Gamma(e))$ denotes the number of equivalence classes of simple objects in the connected component of $\Gamma(e)$. Finally, the (1,5)-bistellar move (replacing a triangulation of a 4-ball with one 4-simplex by one with five 4-simplices) involves ‘summing over’ the unique label $\mathcal{C}$ of an interior vertex of one of the triangulations; the resulting relation is satisfied if $\phi(\mathcal{C}) = \dim(\mathcal{C})^{-1}$. See Section 2.3.3 for a more detailed discussion of the 4-dimensional bistellar moves, and Lemmas 2.3.14, 2.3.15, and 2.3.16 for the statements of respectively the (3,3)-bistellar, (2,4)-bistellar, and (1,5)-bistellar relations that determine the scalar factors in the state sum.

These ingredients would piece together into a state sum expression for the whole 4-manifold, except that there are potentially infinitely many possible labelings $\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}$; we ensure that the sum is finite by restricting the labels to live in a skeletal subsemisimplicial set of $\Delta\mathcal{C}$, as follows.

Definition 2.2.17 (Simplicial skeleton for a prefusion 2-category)**.** A *simplicial skeleton* for a prefusion 2-category $\mathcal{C}$ is a subsemisimplicial set $\Delta\mathcal{C}^\omega \subseteq \Delta\mathcal{C}$ such that $(\Delta\mathcal{C}^\omega)_1$ contains precisely one object from each equivalence class of simple objects of $\mathcal{C}$, and the set of elements of $(\Delta\mathcal{C}^\omega)_2$ with faces $A, B, C \in (\Delta\mathcal{C}^\omega)_1$ contains precisely one 1-morphism from each isomorphism class of simple 1-morphisms of $\mathcal{C}$ from $A \square B$ to C.

Altogether, the 10j action of the associated state of the canonical skeletally-labeled state of the 2-skeleton gives our desired 4-manifold invariant.

Definition 2.2.18 (The state sum)**.** Given an oriented singular combinatorial 4-manifold K, with a chosen total order o on its vertices, and a spherical prefusion 2-category $\mathcal{C}$, with a chosen simplicial skeleton $\Delta\mathcal{C}^\omega$, the *state sum* is the number

$$Z_{\mathcal{C}}(K)_{o,\omega} := \sum_{\Gamma:K^o_{(2)}\Rightarrow\Delta\mathcal{C}^\omega}\Big(\prod_{K_0}\dim(\mathcal{C})^{-1}\Big)\Big(\prod_{e\in K_1}\big(\dim(\Gamma(e))\dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))n(\Gamma(e))\big)^{-1}\Big)\Big(\prod_{s\in K_2}\dim(\Gamma(s))\Big)Z(\Gamma).$$

Here the sum is over natural transformations $\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}^\omega$ that label the 1-simplices, respectively 2-simplices, of K by simple objects, respectively 1-morphisms, of the simplicial skeleton for $\mathcal{C}$. The dimension $\dim(\mathcal{C})$ is the dimension of the underlying presemisimple 2-category of $\mathcal{C}$, from Definition 1.2.32, the dimension $\dim(\Gamma(e))$ is the dimension of the object of the prefusion 2-category $\mathcal{C}$, from Definition 1.3.48, the dimension $\dim(\Gamma(s))$ is the dimension of the 1-morphism of the prefusion 2-category $\mathcal{C}$, also from Definition 1.3.48, the dimension $\dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))$ is the dimension of the fusion 1-category $\mathrm{End}_{\mathcal{C}}(\Gamma(e))$, the number $n(\Gamma(e)$ is the number of equivalence classes of simple objects in the connected component of the object $\Gamma(e)$, and the number $Z(\Gamma)$ is the 10j action of the canonical associated state of Γ, from Definition 2.2.16.

Theorem 2.2.19 (The state sum is an invariant of the manifold)**.** *Let M be an oriented singular piecewise-linear 4-manifold and let $\mathcal{C}$ be a spherical prefusion 2-category over an algebraically closed field of characteristic zero. The number $Z_{\mathcal{C}}(K)_{o,\omega}$ is independent of the choice of triangulation K of M, of the choice of order o on K, and of the choice of simplicial skeleton ω for $\mathcal{C}$, and therefore defines an oriented piecewise-linear invariant $Z_{\mathcal{C}}(M)$ of the singular 4-manifold M.*

The proof occupies all of Section 2.3.

Remark 2.2.20 (The state sum over other base fields). We expect Theorem 2.2.19 could be extended to arbitrary perfect fields; for fields that are not algebraically closed, one must insist that every endomorphism algebra of a simple 1-morphism in the spherical prefusion 2-category is the base field, and for fields of nonzero characteristic, one must insist the spherical prefusion 2-category be nondegenerate in the sense of Remark 1.2.34.

2.2.4 Special cases of the state sum invariant

The state sum invariant of Theorem 2.2.19 applies of course to any spherical prefusion 2-category. We now observe that this invariant simultaneously generalizes all previously known state-sum invariants of piecewise linear 4-manifolds, except possibly the dichromatic invariant for a pivotal functor with modularizable target.

Ribbon categories (the Crane–Yetter–Kauffman invariant)

The first state-sum invariant for 4-manifolds was constructed by Crane–Yetter [CY93] using the data of the modular category of representations of quantum $\mathfrak{sl}_2$; subsequently Crane–Yetter–Kauffman [CKY97] generalized this state sum to use the data of any semisimple ribbon category. This Crane–Yetter–Kauffman 4-manifold invariant for the semisimple ribbon category C agrees with our invariant for the unfolded spherical prefusion 2-category $\mathcal{C}$ associated to C, cf Constructions 1.2.16 and 1.3.19 and Example 1.3.45. This agreement can be seen directly by comparing the state sum constructions in that case, or as a corollary of the fact, discussed below, that our invariant generalizes the Cui invariant, which in turn generalizes the Crane–Yetter–Kauffman invariant.

Finite 2-groups (the Yetter–Dijkgraaf–Witten invariant)

Given the data of a finite 2-group, Yetter [Yet93] defined a 'finite 2-gauge theory' state-sum invariant of n-manifolds, generalizing the Dijkgraaf–Witten invariant associated to a finite group. Later, Faria Martins–Porter [FMP07] extended Yetter's construction to accommodate a twisting n-cocycle (thereby generalizing the twisted Dijkgraaf–Witten invariant). In the case of 4-manifolds, the (twisted) Yetter–Dijkgraaf–Witten invariant for the finite 2-group (π_1, π_2) with 4-cocycle ω agrees with our invariant for the spherical prefusion 2-category $2\mathrm{Vect}_k^\omega(\pi_1, \pi_2)$ of (twisted) 2-group-graded 2-vector spaces, cf Constructions 1.3.13 and 1.3.16 and Example 1.3.44. This agreement can be seen directly, or, again, by observing below that our invariant generalizes the Cui invariant which in turn generalizes the Yetter–Dijkgraaf–Witten invariant.

Endotrivial fusion 2-categories (the Mackaay invariant)

Recall that an endotrivial fusion 2-category is one where the endomorphism fusion category of every indecomposable object is the trivial fusion category Vect_k. Given the data of an endotrivial spherical fusion 2-category (cf Remark 1.3.47), Mackaay [Mac99] defined a state-sum invariant of 4-manifolds; in the endotrivial case, our state sum directly simplifies to Mackaay's formula [Mac99, Def 3.2]. (Note though, as in Remark 1.3.26, that we are not aware of any examples of endotrivial fusion 2-categories besides the 2-catgory $2\mathrm{Vect}_k^\omega(\pi_1)$ of twisted 1-group-graded 2-vector spaces, that is the twisted Dijkgraaf-Witten case. In particular, the examples coming from braided fusion categories, graded braided fusion categories, module tensor categories, 2-representations of 2-groups, and 2-group-graded 2-vector spaces, all have nontrivial endomorphism fusion categories.)

Crossed-braided fusion categories (the Cui invariant)

Given the data of a crossed-braided spherical fusion category, Cui [Cui16] defines a state-sum invariant of 4-manifolds, simultaneously generalizing the Crane–Yetter–Kauffman invariant and the Yetter–Dijkgraaf–Witten invariant. Recall from Construction 1.3.23 and Example 1.3.46 that a G-crossed-braided spherical fusion category can be interpreted as a spherical prefusion 2-category whose set of objects is the finite group G.

Cui's state-sum for a crossed-braided spherical fusion category C agrees with our state-sum for the corresponding spherical prefusion 2-category $\mathcal{C}$, as follows. (We will use the notation $\dim_{\mathrm{C}}$ to refer to dimensions of morphisms thought of in the spherical fusion category C and by contrast $\dim_{\mathcal{C}}$ for the dimensions of objects and morphisms in the spherical fusion 2-category $\mathcal{C}$.) The 10j action term $Z(\Gamma)$ in our state sum agrees with the expression associated to the 4-simplices in Cui [Cui16, Eq 22]. Observe that in the special case in question, the dimension $\dim_{\mathcal{C}}(g)$ of every simple object $g \in \mathcal{C}$ is 1, and therefore the dimension of any 1-morphism $f : g \to h$ is $\dim_{\mathcal{C}}(f) = \langle \mathrm{tr}_R(f) \rangle = \dim_{\mathrm{C}}(f)$. Hence, the 2-simplex normalization factor $\dim_{\mathcal{C}}(\Gamma(s))$ in our state sum agrees with the corresponding factor in Cui's formula.

Next, recall that the endomorphism category of any object $g \in \mathcal{C}$ is $\mathrm{End}_{\mathcal{C}}(g) = \mathrm{C}_e$, that is the identity-graded piece of the crossed-braided fusion 1-category C. More generally, the morphism category between objects $g, h \in \mathcal{C}$ is $\mathrm{Hom}_{\mathcal{C}}(g,h) = \mathrm{C}_{hg^{-1}}$, and so the number of objects in the component of g is $n(g) = |\{h \in G \,|\, \mathrm{C}_{hg^{-1}} \neq 0\}| = |\{h \in G \,|\, \mathrm{C}_h \neq 0\}|$. In particular, in this special case, neither the dimension $\dim_{\mathcal{C}}(g)$, nor the dimension $\dim(\mathrm{End}_{\mathcal{C}}(g))$, nor the number $n(g)$, depends on the object g. Hence, our 1-simplex normalization factor $\big(\prod_{e \in K_1} (\dim_{\mathcal{C}}(\Gamma(e)) \dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e))) n(\Gamma(e)))^{-1}\big)$ drastically simplifies to $\big(\dim(\mathrm{C}_e)\ |\{h \in G \,|\, \mathrm{C}_h \neq 0\}|\big)^{-|K_1|}$. Müger [Müg10] proves that in a crossed-braided fusion category C, the dimension of every nonzero graded piece is the same. Thus $\dim(\mathrm{C}) = |\{h \in G \,|\, \mathrm{C}_h \neq 0\}| \dim(\mathrm{C}_1)$ and the normalization factor in question is simply $(\dim(\mathrm{C}))^{-|K_1|}$, which indeed agrees with the 1-simplex factor in Cui's state sum.

Finally, observe that because there are $|\{h \in G \,|\, \mathrm{C}_h \neq 0\}|$ objects in each component of $\mathcal{C}$, the number of components of $\mathcal{C}$ is $|G|/|\{h \in G \,|\, \mathrm{C}_h \neq 0\}|$. The dimension of the whole prefusion 2-category $\mathcal{C}$ is therefore

$$\dim(\mathcal{C}) := \sum_{[x]\in\pi_0\mathcal{C}} \dim(\mathrm{End}_{\mathcal{C}}(x))^{-1} = |G||\{h \in G \mid \mathrm{C}_h \neq 0\}|^{-1} \dim(\mathrm{C}_e)^{-1} = |G| \dim(\mathrm{C})^{-1}.$$

In this case, our 0-simplex normalization factor is therefore $\left(|G| \dim(\mathrm{C})^{-1}\right)^{-|K_0|}$, which agrees with the corresponding factor in Cui's formula.

2.3 The state sum is a piecewise-linear homeomorphism invariant

As defined, the state sum expression $Z_{\mathcal{C}}(K)_{o,\omega}$ depends on the chosen labeling skeleton $\Delta\mathcal{C}^{\omega}$ of the semisimplicial labeling category $\Delta\mathcal{C}$, on the total order o on the vertices of the combinatorial 4-manifold K, and of course on the given combinatorial structure of K. We prove in turn that the numerical value of the state sum does not depend on each of these choices, and so, altogether, the state sum is an invariant of a piecewise-linear 4-manifold. (Recall from Section 2.2.2 that whenever we refer to a '4-manifold' we implicitly allow vertex singularities.)

2.3.1 The state sum is independent of the labeling skeleton

The state sum expression may be written as the sum over states

$$Z_{\mathcal{C}}(K)_{o,\omega} := \sum_{\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}^{\omega}} N(\Gamma)$$

where

$$N(\Gamma) := \Big(\prod_{K_0} \dim(\mathcal{C})^{-1}\Big)\Big(\prod_{e\in K_1} (\dim(\Gamma(e)) \dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))n(\Gamma(e)))^{-1}\Big)\Big(\prod_{s\in K_2} \dim(\Gamma(s))\Big) Z(\Gamma)$$

is the normalized 10j action.

Definition 2.3.1 (Equivalent states)**.** Given a spherical prefusion 2-category $\mathcal{C}$ and an ordered oriented combinatorial 4-manifold K^o, two $\mathcal{C}$-states $\Gamma, \Gamma' : K^o_{(2)} \Rightarrow \Delta\mathcal{C}$ are *equivalent* if for every edge $e \in K_1$, there are inverse equivalences

$$h_e : \Gamma(e) \rightleftarrows \Gamma'(e) : k_e$$

and for every 2-simplex $s \in K_2$, there are 2-isomorphisms

$$\Gamma(s) \cong k_{\partial^o_{02} s} \circ \Gamma'(s) \circ \left(h_{\partial^o_{01} s} \,\square\, h_{\partial^o_{12} s}\right).$$

Lemma 2.3.2 (Equivalent states have the same normalized 10j action)**.** *If the $\mathcal{C}$-states Γ and Γ' are equivalent, then their normalized 10j actions are the same: $N(\Gamma) = N(\Gamma')$.*

This lemma will be established below. Given two distinct labeling skeleta $\Delta\mathcal{C}^{\omega_1}$ and $\Delta\mathcal{C}^{\omega_2}$, for each object $A \in (\Delta\mathcal{C}^{\omega_1})_1$, choose inverse equivalences $h_A : A \rightleftarrows A' : k_A$ between A and the unique equivalent object $A' \in (\Delta\mathcal{C}^{\omega_2})_1$; similarly for each 1-morphism $(g : A \,\square\, B \to C) \in (\Delta\mathcal{C}^{\omega_1})_2$, choose an isomorphism $h_C \circ g \circ (k_A \,\square\, k_B) \cong g'$, where g' is the unique isomorphic 1-morphism $(g' : A' \,\square\, B' \to C') \in (\Delta\mathcal{C}^{\omega_2})_2$. Composing with these equivalences and isomorphisms provides a bijection $[K^o_{(2)}, \Delta\mathcal{C}^{\omega_1}] \cong [K^o_{(2)}, \Delta\mathcal{C}^{\omega_2}]$ between the set of $\mathcal{C}$-states with labels in the first skeleton and the set of $\mathcal{C}$-states with labels in the second skeleton; and of course this bijection takes each state to an equivalent state.

Corollary 2.3.3 (The state sum is invariant under change of labeling skeleton)**.** *Given a spherical prefusion 2-category $\mathcal{C}$, an oriented (singular) combinatorial 4-manifold K, a chosen total order o on its vertices, and any two simplicial skeleta $\Delta\mathcal{C}^{\omega_1}$ and $\Delta\mathcal{C}^{\omega_2}$ for $\mathcal{C}$, the corresponding state sums agree:*

$$Z_{\mathcal{C}}(K)_{o,\omega_1} = Z_{\mathcal{C}}(K)_{o,\omega_2}.$$

In light of this independence of the choice of simplicial skeleton, we will henceforth denote the state sum, associated to a spherical prefusion 2-category $\mathcal{C}$, an oriented combinatorial 4-manifold K, and a chosen total order o on vertices, by $Z_{\mathcal{C}}(K)_o$.

The 10j action is invariant under 1-morphism state changes

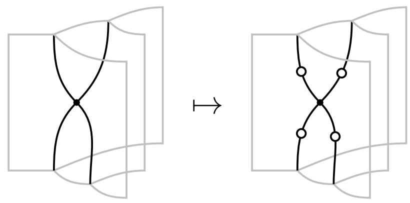
We begin by considering the situation where Γ and Γ' are equivalent $\mathcal{C}$-states, and in fact the object labels of the two states are equal, that is $\Gamma(e) = \Gamma'(e)$ for all 1-simplices $e \in K_1$.

Lemma 2.3.4 (Object-equal equivalent states have the same 10j action)**.** *If the $\mathcal{C}$-states Γ and Γ' are equivalent and moreover the equivalences $h_e : \Gamma(e) \rightleftarrows \Gamma'(e) : k_e$ are identities, then the corresponding 10j actions are equal: $Z(\Gamma) = Z(\Gamma')$.*

Proof. The two states differ only by a collection of 2-isomorphisms $\alpha_s : \Gamma(s) \Rightarrow \Gamma'(s)$, for $s \in K_2$. These chosen isomorphisms α_s induce, by pre- and post-composition, isomorphisms of the corresponding associator state spaces of every 3-simplex $\kappa \in K_3$:

$$l^+_\kappa : V^+(\Gamma, \kappa) \to V^+(\Gamma', \kappa)$$
$$l^-_\kappa : V^-(\Gamma, \kappa) \to V^-(\Gamma', \kappa)$$

The first of these isomorphisms, for instance, may be depicted as follows, where each white dot denotes either an α_s isomorphism or its inverse:



Recall that the pairing $\langle \cdot, \cdot \rangle_{\Gamma,\kappa} : V^-(\Gamma, \kappa) \otimes V^+(\Gamma, \kappa) \to k$ between the negative and positive associator state spaces is defined, see Definitions 1.3.53 and 1.3.38, as the 2-spherical trace of the composition. By the cyclicity of the planar trace (used in the definition of the 2-spherical trace), the various comparison isomorphisms α_s cancel out in the trace construction, and the associator state space isomorphisms l^+ and l^- therefore intertwine the pairings:

$$\langle \cdot, \cdot \rangle_{\Gamma,\kappa} = \langle \cdot, \cdot \rangle_{\Gamma',\kappa} \circ \left(l^-_\kappa \otimes l^+_\kappa\right) : V^-(\Gamma, \kappa) \otimes V^+(\Gamma, \kappa) \to k$$

It follows that the inverse isomorphisms intertwine the corresponding copairing:

$$\cup_{\Gamma,\kappa} = \left(l^+_\kappa \otimes l^-_\kappa\right)^{-1} \circ \cup_{\Gamma',\kappa} : k \to V^+(\Gamma, \kappa) \otimes V^-(\Gamma, \kappa)$$

Of course, these isomorphisms then intertwine the global copairings $\cup_\Gamma := \bigotimes_{\kappa \in K_3} \cup_{\Gamma,\kappa}$ and $\cup_{\Gamma'} := \bigotimes_{\kappa \in K_3} \cup_{\Gamma',\kappa}$.

Next, recall that the 10j action is defined, see Definition 2.2.15 and Figure 2.1, as a product of 2-spherical traces of pentagonator composites. Again by the cyclicity of the planar trace, the comparison isomorphisms α_s will cancel pairwise in this trace construction, and so the associator state space isomorphisms also intertwine the 10j action:

$$z(\Gamma) = z(\Gamma') \circ \Big(\bigotimes_{\kappa \in K_3} l^+_\kappa \otimes l^-_\kappa \Big)$$

Altogether then we find that the 10j actions for the states Γ and Γ' agree:

$$Z(\Gamma) := z(\Gamma) \circ \cup_\Gamma(1) = z(\Gamma') \circ \cup_\Gamma(1) =: Z(\Gamma') \qquad \square$$

Of course, none of the normalization factors in the normalized 10j action are affected by changing 1-morphism labels by isomorphisms, so the normalized 10j action is similarly unaffected by changes of 1-morphism labels.

The normalized 10j action is invariant under object state changes

Next we consider the situation where Γ and Γ' are equivalent $\mathcal{C}$-states, for which the 2-isomorphisms may be taken to be equalities and moreover for which the object inverse equivalences are given by a morphism and its chosen (planar pivotal) adjoint; that is, for every edge $e \in K_1$, there are inverse equivalences $h_e : \Gamma(e) \rightleftarrows \Gamma'(e) : h_e^*$ such that $\Gamma(s) = h^*_{\partial^\circ_{02}s} \circ \Gamma'(s) \circ \big(h_{\partial^\circ_{01}s} \,\Box\, h_{\partial^\circ_{12}s}\big)$.

Lemma 2.3.5 (Morphism-conjugate equivalent states have the same normalized 10j action)**.** *If the $\mathcal{C}$-states Γ and Γ' are equivalent, with inverse equivalences $h_e : \Gamma(e) \rightleftarrows \Gamma'(e) : h_e^*$ between object labels, and equalities $\Gamma(s) = h^*_{\partial^\circ_{02}s} \circ \Gamma'(s) \circ \big(h_{\partial^\circ_{01}s} \,\Box\, h_{\partial^\circ_{12}s}\big)$ relate the 1-morphism labels, then the corresponding normalized 10j actions are equal: $N(\Gamma) = N(\Gamma')$.*

The normalized 10j action may be considered to have five factors, the 0-simplex factor $\Big(\prod_{K_0} \dim(\mathcal{C})^{-1}\Big)$, the 1-simplex factor $\Big(\prod_{e\in K_1} (\dim(\Gamma(e)) \dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))n(\Gamma(e)))^{-1}\Big)$, the 2-simplex factor $\Big(\prod_{s\in K_2} \dim(\Gamma(s))\Big)$, the 3-simplex factor $\cup_\Gamma := \bigotimes_{\kappa\in K_3} \cup_{\Gamma,\kappa}$, and the 4-simplex factor $z(\Gamma) := \Big(\bigotimes_{\mu\in K_4} z(\Gamma,\mu)\Big)$. The 0-simplex factor is certainly unaffected by changing state labels. We address the other factors in turn.

The 1-simplex factor. The 1-simplex factor has three terms for each 1-simplex $e \in K_1$, the number $n(\Gamma(e))$ of equivalence classes of simple objects in the component, the dimension $\dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))$ of the endomorphism fusion category, and the dimension $\dim(\Gamma(e))$ of the object label itself. Given an equivalence of simple objects $h_e : \Gamma'(e) \simeq \Gamma(e)$, the two objects $\Gamma(e)$ and $\Gamma'(e)$ are in the same connected component and so the number of equivalence classes of simple objects in that component is evidently unchanged: $n(\Gamma(e)) = n(\Gamma'(e))$. Similarly, the endomorphism categories of $\Gamma'(e)$ and $\Gamma(e)$ are equivalent fusion 1-categories and therefore have the same dimension: $\dim(\mathrm{End}_{\mathcal{C}}(\Gamma'(e))) = \dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))$. The dimension of the simple object $\Gamma(e)$ itself is not, however, invariant under equivalence, rather it transforms according to the square of the planar trace of the chosen equivalence:

$$\begin{aligned}
\dim(\Gamma(e)) &:= \dim(1_{\Gamma(e)}) \\
&= \dim(h_e^* \circ h_e) && [1_{\Gamma(e)} \cong h_e^* \circ h_e] \\
&= \langle \mathrm{tr}_R(h_e)\rangle \dim(h_e^*) && [\Gamma'(e) \text{ simple}] \\
&= \langle \mathrm{tr}_R(h_e)\rangle \dim(h_e) && [\text{Prop. 1.3.37}] \\
&= \langle \mathrm{tr}_R(h_e)\rangle^2 \dim(1_{\Gamma'(e)}) && [\Gamma'(e) \text{ simple}] \\
&= \langle \mathrm{tr}_R(h_e)\rangle^2 \dim(\Gamma'(e))
\end{aligned}$$

As this planar trace recurs as a scalar transformation factor, it is worth having a compact notation for it:

$$\lambda_e := \langle \mathrm{tr}_R(h_e) \rangle$$

Recall that this trace of a simple 1-morphism between simple objects is nonzero by Lemma 1.3.50. Altogether, the 1-simplex factor transforms by the square inverse of that trace factor:

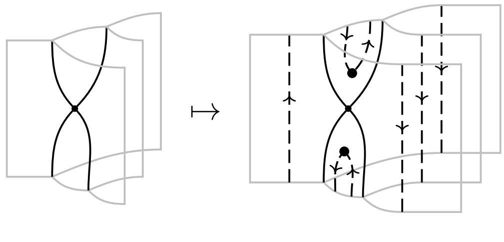
$$(\dim(\Gamma(e))\dim(\mathrm{End}_{\mathcal{C}}(\Gamma(e)))n(\Gamma(e)))^{-1} = \lambda_e^{-2}\,(\dim(\Gamma'(e))\dim(\mathrm{End}_{\mathcal{C}}(\Gamma'(e)))n(\Gamma'(e)))^{-1}.$$

The 2-simplex factor. The 2-simplex factor has just one term for each 2-simplex $s \in K_2$, namely the dimension $\dim(\Gamma(s))$ of the simple 1-morphism label. Given labels $\Gamma(s)$ and $\Gamma'(s)$ related by the equality $\Gamma(s) = h^*_{\partial^o_{02}s} \circ \Gamma'(s) \circ \left(h_{\partial^o_{01}s} \,\square\, h_{\partial^o_{12}s}\right)$, the corresponding dimensions are related by a trace factor for each edge of the 2-simplex:

$$\begin{aligned}
\dim(\Gamma(s)) &= \dim\left(h^*_{\partial^o_{[02]}s} \circ \Gamma'(s) \circ (h_{\partial^o_{[01]}s} \,\square\, h_{\partial^o_{[12]}s})\right) \\
&= \dim\left(h^*_{\partial^o_{[02]}s} \circ \Gamma'(s)\right)\ \lambda_{\partial^o_{[01]}s}\,\lambda_{\partial^o_{[12]}s} && [\Gamma'(\partial^o_{[01]}s), \Gamma'(\partial^o_{[12]}s) \text{ simple}] \\
&= \lambda_{\partial^o_{[01]}s}\,\lambda_{\partial^o_{[12]}s}\,\lambda_{\partial^o_{[02]}s}\ \dim(\Gamma'(s)) && [\text{Prop 1.3.37}]
\end{aligned}$$

The 3-simplex factor. The 3-simplex factor is a tensor over the 3-simplices $\kappa \in K_3$ of the copairing $\cup_{\Gamma,\kappa} : k \to V^+(\Gamma,\kappa) \otimes V^-(\Gamma,\kappa)$. To relate the copairing $\cup_{\Gamma,\kappa}$ and the copairing $\cup_{\Gamma',\kappa}$, we need to relate the corresponding associator state spaces $V^+(\Gamma,\kappa)$ and $V^+(\Gamma',\kappa)$, respectively $V^-(\Gamma,\kappa)$ and $V^-(\Gamma',\kappa)$. To that end, for each 1-simplex $e \in K_1$, choose a 2-isomorphism $\beta_e : h_e^* \circ h_e \Rightarrow 1_{\Gamma(e)}$, and define, for each 3-simplex $\kappa \in K_3$, an isomorphism of associator state spaces as follows:

$$r^+_\kappa : V^+(\Gamma,\kappa) \;\to\; V^+(\Gamma',\kappa)$$



Here, the dashed lines denote the equivalence $h_e : \Gamma(e) \to \Gamma'(e)$ or its adjoint, and the two black dots denote the 2-isomorphism $\beta_e : h_e^* \circ h_e \cong 1_{\Gamma(e)}$ and its inverse. The isomorphism $r^-_\kappa : V^-(\Gamma,\kappa) \to V^-(\Gamma',\kappa)$ is defined analogously.

Recall that the pairing $\langle \cdot, \cdot \rangle_{\Gamma,\kappa} : V^-(\Gamma,\kappa) \otimes V^+(\Gamma,\kappa) \to k$ is given by composing and taking a spherical trace. Precomposing this pairing with the isomorphisms of associator state spaces, we see that the pairings for the labelings Γ and Γ' are related by a scalar factor for each 1-simplex in the 3-simplex in question:

$$\langle \cdot, \cdot \rangle_{\Gamma,\kappa} = \left(\lambda_{\partial^o_{[01]}\kappa}\lambda_{\partial^o_{[12]}\kappa}\lambda_{\partial^o_{[23]}\kappa}\lambda_{\partial^o_{[03]}\kappa}\right) \langle \cdot, \cdot \rangle_{\Gamma',\kappa} \circ \left(r^-_\kappa \otimes r^+_\kappa\right).$$

It follows immediately that the corresponding copairings are related by the inverse scalar factors:

$$\cup_{\Gamma,\kappa} = \left(\lambda_{\partial^o_{[01]}\kappa}\lambda_{\partial^o_{[12]}\kappa}\lambda_{\partial^o_{[23]}\kappa}\lambda_{\partial^o_{[03]}\kappa}\right)^{-1} \left(r^+_\kappa \otimes r^-_\kappa\right)^{-1} \circ \cup_{\Gamma',\kappa}.$$

The 4-simplex factor. The 4-simplex factor is a tensor over the 4-simplices $\mu \in K_4$ of 10j symbols such as

$$z(\Gamma,\mu) : V^+(\Gamma,\partial^o_{[0123]}\mu) \otimes V^+(\Gamma,\partial^o_{[0134]}\mu) \otimes V^+(\Gamma,\partial^o_{[1234]}\mu) \otimes V^-(\Gamma,\partial^o_{[0124]}\mu) \otimes V^-(\Gamma,\partial^o_{[0234]}\mu) \to k$$

or

$$z(\Gamma,\mu) : V^+(\Gamma,\partial^o_{[0234]}\mu) \otimes V^+(\Gamma,\partial^o_{[0124]}\mu) \otimes V^-(\Gamma,\partial^o_{[1234]}\mu) \otimes V^-(\Gamma,\partial^o_{[0134]}\mu) \otimes V^-(\Gamma,\partial^o_{[0123]}\mu) \to k$$

depending on orientation. Recall that this 10j symbol $z(\Gamma,\mu)$ is defined as the spherical trace of one of the pentagonator composites depicted in Figure 2.1. To compare the 10j symbol $z(\Gamma,\mu)$ with the 10j symbol $z(\Gamma',\mu)$ for the alternative labeling Γ', we precompose with an appropriate tensor product of the isomorphisms r^+_κ and r^-_κ of associator state spaces. In the resulting spherical trace expression, after cancelling various β_e isomorphisms, exactly five planar trace scalar factors remain, one for each edge of the 4-simplex:

$$z(\Gamma,\mu) = \left(\lambda_{\partial^o_{[01]}\mu}\lambda_{\partial^o_{[12]}\mu}\lambda_{\partial^o_{[23]}\mu}\lambda_{\partial^o_{[34]}}\lambda_{\partial^o_{[04]}\mu}\right) z(\Gamma',\mu) \circ \left(\bigotimes_{\kappa\in K_3,\kappa\subseteq\mu} r_\kappa^{\epsilon^\mu_o(\kappa)}\right).$$

Spherical links cause factor cancellation. Combining the scalar factors calculated above, we see that the normalized 10j actions for the states Γ and Γ' are related by $N(\Gamma) = \gamma N(\Gamma')$ where

$$\gamma = \left(\prod_{e\in K_1}\lambda_e^{-2}\right)\cdot\left(\prod_{s\in K_2}\lambda_{\partial^o_{[01]}s}\lambda_{\partial^o_{[12]}s}\lambda_{\partial^o_{[02]}s}\right)\cdot$$
$$\left(\prod_{\kappa\in K_3}\lambda_{\partial^o_{[01]}\kappa}\lambda_{\partial^o_{[12]}\kappa}\lambda_{\partial^o_{[23]}\kappa}\lambda_{\partial^o_{[03]}\kappa}\right)^{-1}\cdot\left(\prod_{\mu\in K_4}\lambda_{\partial^o_{[01]}\mu}\lambda_{\partial^o_{[12]}\mu}\lambda_{\partial^o_{[23]}\mu}\lambda_{\partial^o_{[34]}\mu}\lambda_{\partial^o_{[04]}\mu}\right)$$

We can now observe that the terms of this scalar factor cancel precisely when the link of every k-simplex, for $k \geq 1$, is a combinatorial sphere.

Proof of Lemma 2.3.5. As a convenient compact notation, when $\tau \in K_p$ is a p-simplex, for p equal to 2 or 3, define $\lambda_\tau := \lambda_{\partial^o_{[0,p]}\tau}$, that is λ_τ is the trace factor associated to the maximal edge of the simplex τ. By direct combinatorial rearrangement, we may rewrite the products appearing in the factor γ as follows:

For $s \in K_2$: $$\lambda_{\partial^o_{[01]}s}\lambda_{\partial^o_{[12]}s}\lambda_{\partial^o_{[02]}s} = \Big(\prod_{e\in K_1, e\subseteq s} \lambda_e\Big)$$

For $\kappa \in K_3$: $$\lambda_{\partial^o_{[01]}\kappa}\lambda_{\partial^o_{[12]}\kappa}\lambda_{\partial^o_{[23]}\kappa}\lambda_{\partial^o_{[03]}\kappa} = \Big(\prod_{e\in K_1, e\subseteq \kappa} \lambda_e\Big)\Big(\prod_{s\in K_2, s\subseteq \kappa} \lambda_s\Big)^{-1}(\lambda_\kappa^2)$$

For $\mu \in K_4$: $$\lambda_{\partial^o_{[01]}\mu}\lambda_{\partial^o_{[12]}\mu}\lambda_{\partial^o_{[23]}\mu}\lambda_{\partial^o_{[34]}\mu}\lambda_{\partial^o_{[04]}\mu} = \Big(\prod_{e\in K_1, e\subseteq \mu} \lambda_e\Big)\Big(\prod_{s\in K_2, e\subseteq \mu} \lambda_s\Big)^{-1}\Big(\prod_{\kappa\in K_3, \kappa\subseteq \mu} \lambda_\kappa\Big)$$

Collecting terms we have

$$\gamma = \Big(\prod_{e\in K_1} \lambda_e^{\phi_e}\Big)\Big(\prod_{s\in K_2} \lambda_s^{\phi_s}\Big)\Big(\prod_{\kappa\in K_3} \lambda_\kappa^{\phi_\kappa}\Big)$$

where

$$\begin{aligned}
\phi_e &= -2 + |\{s\in K_2 \mid e\subseteq s\}| - |\{\kappa\in K_3 \mid e\subseteq \kappa\}| + |\{\mu\in K_4 \mid e\subseteq \mu\}| &&= -2+\chi(\mathrm{lk}(e))\\
\phi_s &= |\{\kappa\in K_3 \mid s\subseteq \kappa\}| - |\{\mu\in K_4 \mid s\subseteq \mu\}| &&= \chi(\mathrm{lk}(s))\\
\phi_\kappa &= -2 + |\{\mu\in K_4 \mid \kappa\subseteq \mu\}| &&= -2+\chi(\mathrm{lk}(\kappa))
\end{aligned}$$

Here $\mathrm{lk}(\tau)$ denotes the link of the simplex τ and $\chi(\mathrm{lk}(\tau))$ denotes the Euler characteristic of that link. Because by assumption K is a closed *singular* combinatorial 4-manifold, the link of every 1-, 2-, and 3-simplex is piecewise-linearly homeomorphic to a combinatorial sphere, and therefore has Euler characteristic 2 or 0 depending on parity; the exponents in the transformation factor γ vanish accordingly. □

Note crucially that the preceding proof did not use the Euler characteristic of the link of a vertex of the triangulation, and therefore applies to singular (that is vertex-singular) combinatorial 4-manifolds.

Equivalences of states factor into 1-morphism-only and object-only equivalences

We can wrap up the proof of independence of the choice of simplicial skeleton by factoring any equivalence of states into one that changes only the 1-morphism labels and one that appropriately changes only the object labels.

Proof of Lemma 2.3.2. Observe that if two $\mathcal{C}$-states Γ and Γ' are equivalent, then (by composing with the 2-isomorphism between the chosen inverse equivalence k_e and the adjoint inverse h_e^*) they are equivalent by a collection of adjoint inverse equivalences

$h_e : \Gamma(e) \rightleftarrows \Gamma'(e) : h_e^*$ and isomorphisms $\Gamma(s) \cong h^*_{\partial^o_{02}s} \circ \Gamma'(s) \circ \big(h_{\partial^o_{01}s} \,\square\, h_{\partial^o_{12}s}\big)$. There is then a $\mathcal{C}$-state Γ'' with $\Gamma''(e) = \Gamma(e)$ and $\Gamma''(s) = h^*_{\partial^o_{02}s} \circ \Gamma'(s) \circ \big(h_{\partial^o_{01}s} \,\square\, h_{\partial^o_{12}s}\big)$. As Γ and Γ'' are equivalent with identity 1-equivalences, and Γ'' and Γ' are equivalent with adjoint inverse equivalences and an equality of appropriate 2-simplex labels, the result follows from Lemma 2.3.4 and Lemma 2.3.5. □

2.3.2 The state sum is independent of the vertex ordering

Recall that in the state sum for an ordered oriented combinatorial 4-manifold K^o, the sum is over states $\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}^\omega$, that is labelings of the 1-simplices and 2-simplices of the semisimplicial 2-skeleton $K^o_{(2)}$ by elements of the skeletal labeling semisimplicial set $\Delta\mathcal{C}^\omega$. We will show that for any two choices of global vertex ordering, there is a bijection of the two sets of (skeletal) states and that corresponding states have the same normalized 10j action.

Lemma 2.3.6 (Reordering yields a 10j-preserving bijection of skeletal states). *If o and o' are global vertex orderings of the combinatorial 4-manifold K, then there is a bijection $\tau : [K^o_{(2)}, \Delta\mathcal{C}^\omega] \cong [K^{o'}_{(2)}, \Delta\mathcal{C}^\omega]$ of skeletal states for the two orderings, such that the bijection preserves the normalized 10j action: $N(\tau(\Gamma)) = N(\Gamma)$.*

Corollary 2.3.7 (The state sum is invariant under vertex reordering). *Given a spherical prefusion 2-category, an oriented combinatorial 4-manifold K, and any two orderings o and o' on the vertices of K, the corresponding state sums agree:*

$$Z_{\mathcal{C}}(K)_o = Z_{\mathcal{C}}(K)_{o'}.$$

Accordingly, we will henceforth denote the state sum simply by $Z_{\mathcal{C}}(K)$.

Transposition of states and associator states

Of course it suffices to prove Lemma 2.3.6 when the two orderings are related by a single transposition of the order of two order-adjacent vertices. Let σ denote the permutation corresponding to such a transposition, and let σ_τ denote the permutation of $0, \ldots, p$ induced by the restriction of σ to the p-simplex τ. The only relevant difference between the orderings o and o' are the face maps of simplices, which control the structure of the corresponding states; these face maps are related by $\partial^{o'}_{[i_1 \cdots i_r]}\tau = \partial^o_{[\sigma_\tau(i_1) \cdots \sigma_\tau(i_r)]}\tau$, where $0 \leq i_1 < \cdots < i_r \leq p$. Given a state for the order o, we define a corresponding state for a transposed order o', and then define an isomorphism of the corresponding associator state spaces.

State transposition. As before, let o and o' be vertex orderings related by a single adjacent transposition σ. Define a state transposition map

$$[K^o_{(2)}, \Delta\mathcal{C}] \to [K^{o'}_{(2)}, \Delta\mathcal{C}]$$
$$\Gamma \mapsto \Gamma^\sigma$$

where for $e \in K_1$, the transposed state label is

$$\Gamma^\sigma(e) = \begin{cases} \Gamma(e) & \text{if } \sigma_e = \text{id} \\ \Gamma(e)^\# & \text{if } \sigma_e = (01) \end{cases}$$

and for $s \in K_2$, the transposed state label $\Gamma^\sigma(s)$ is shown in Figure 2.2, for each relevant restriction of the transposition σ to the 2-simplex s. In that figure, as in Notation 2.2.9 and Notation 2.2.11, for instance $\overline{[01]}_\Gamma$ is shorthand for $\Gamma(\partial^o_{[01]} s)^\#$ and $[12]_{\Gamma^\sigma}$ is shorthand for $\Gamma^\sigma(\partial^{o'}_{[12]} s)$. Henceforth, when we write $[ij]$ without a subscript, it refers implicitly to $[ij]_\Gamma$. (Note that the dual of a simple object in a prefusion 2-category is again simple: an object is simple if and only if its identity is a simple 1-morphism, and taking mates induces an isomorphism between the endomorphisms of the identity of an object and the endomorphisms of the identity of its dual.)

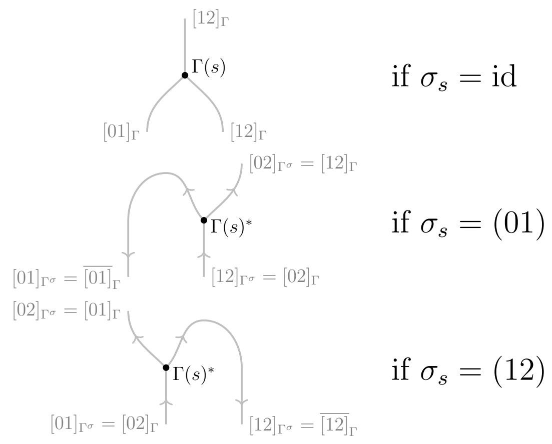


Figure 2.2: The transposed state label $\Gamma^\sigma(s)$, for each transposition σ_s of the 2-simplex s.

Associator state transposition. Recall from Definition 2.2.12 that for a state Γ and a 3-simplex $\kappa \in K_3$, the positive associator state space is $V^+(\Gamma,\kappa) := \text{Hom}_\mathcal{C}\left([(012)3]^\kappa_\Gamma, [0(123)]^\kappa_\Gamma\right)$, where the 1-morphisms $[(012)3]^\kappa_\Gamma$ and $[0(123)]^\kappa_\Gamma$ are each composites of two 2-simplex labels, as in Notation 2.2.10. The negative associator state space is similar, taking Hom in the opposite direction. Representative elements of these associator state spaces are depicted just after Definition 2.2.12.

For a vertex transposition σ and the corresponding transposed state Γ^σ, the associator state spaces are Hom-spaces between pairwise composites of the transposed state labels $\Gamma^\sigma(s)$ defined above. For instance, the negative associator state space $V^-(\Gamma^\sigma, \kappa)$ is shown in Figure 2.3, for the various restrictions of the transposition σ to the particular 3-simplex κ.

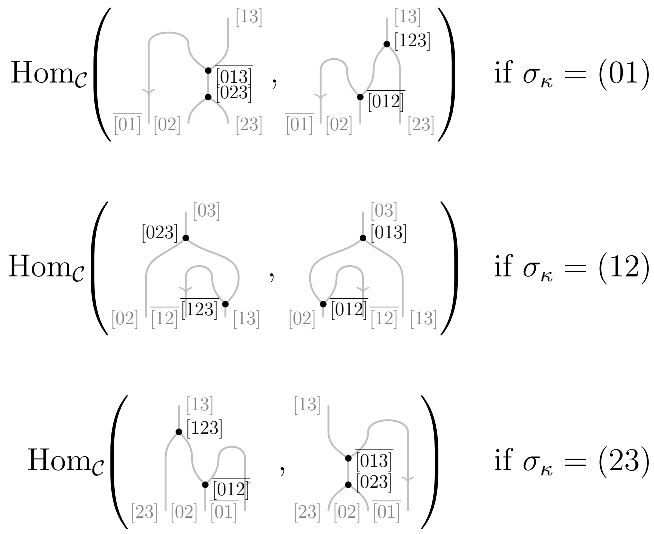


Figure 2.3: The negative associator state space $V^-(\Gamma^\sigma, \kappa)$, for the transpositions σ_κ of the 3-simplex κ.

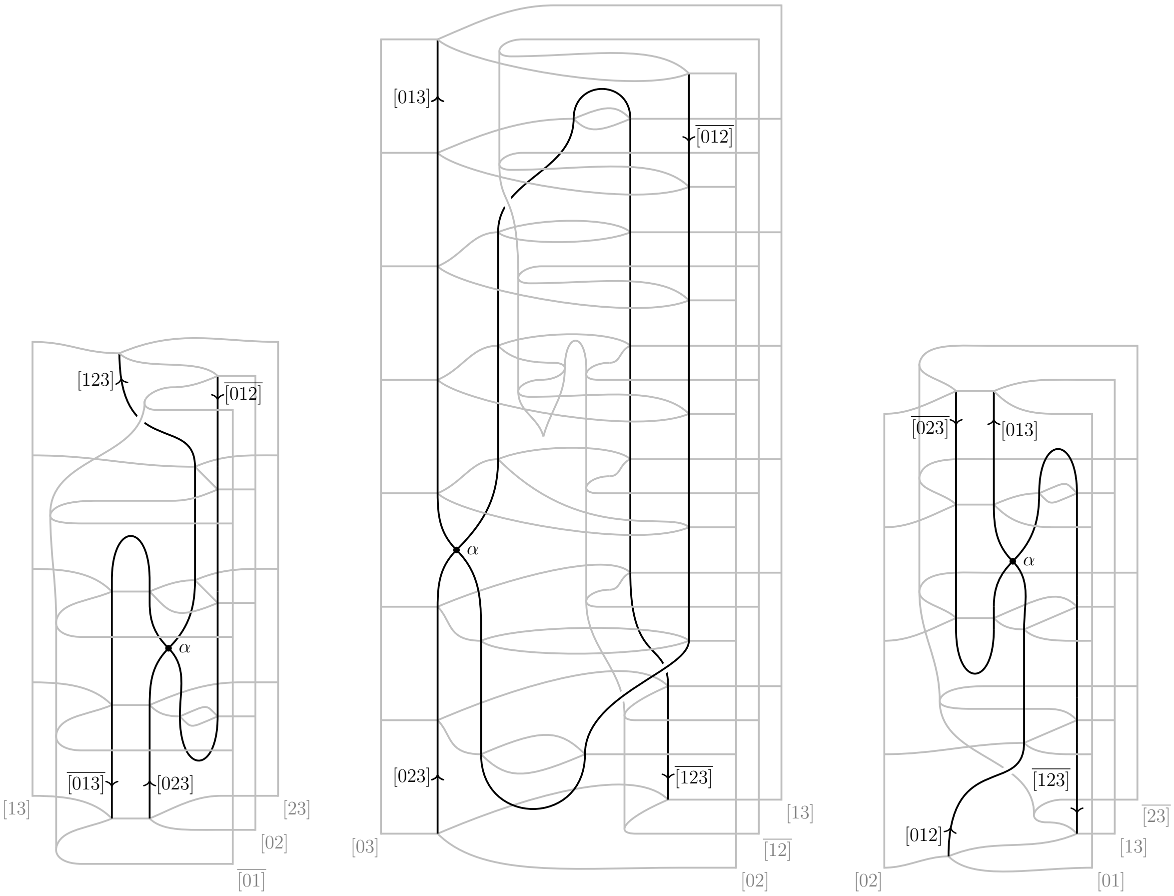
We now define isomorphisms of associator state spaces

$$\Phi^\pm_{\kappa,\sigma_\kappa} : V^\pm(\Gamma, \kappa) \to V^{\pm\mathrm{sgn}(\sigma_\kappa)}(\Gamma^\sigma, \kappa)$$

from the spaces for labeling Γ to the spaces for transposed labeling Γ^σ. Here $\mathrm{sgn}(\sigma_\kappa)$ denotes the sign of the permutation σ_κ. When $\sigma_\kappa = \mathrm{id}$, we set $\Phi^\pm_{\kappa,\mathrm{id}} := \mathrm{id}_{V^\pm(\Gamma,\kappa)}$. When σ restricts nontrivially to the 3-simplex κ, the isomorphism $\Phi^+_{\kappa,\sigma_\kappa}$ takes an associator state $\alpha \in V^+(\Gamma, \kappa)$ to one of the three composites $\Phi^+_{\kappa,\sigma_\kappa}(\alpha) \in V^-(\Gamma^\sigma, \kappa)$ depicted in Figure 2.4, according to the particular transposition σ_κ. The negative isomorphism $\Phi^-_{\kappa,\sigma_\kappa}$ is defined analogously by the vertical reflections of the diagrams in that figure.

State transposition preserves the normalized 10j action

We now show that the transposition of a state has the same normalized 10j action as the original state.



$\Phi^+_{\kappa,(01)}(\alpha)$ $\Phi^+_{\kappa,(12)}(\alpha)$ $\Phi^+_{\kappa,(23)}(\alpha)$

Figure 2.4: The definition of the transposed associator state $\Phi^+_{\kappa,\sigma_\kappa}(\alpha) \in V^-(\Gamma^\sigma, \kappa)$ for $\alpha \in V^+(\Gamma, \kappa)$.

Lemma 2.3.8 (State transposition preserves the normalized 10j action)**.** *Let o and o' be global vertex orderings of the oriented combinatorial 4-manifold K, related by an adjacent vertex transposition σ. For any state Γ of K with ordering o and transposed state Γ^σ of K with ordering o', the normalized 10j actions agree: $N(\Gamma^\sigma) = N(\Gamma)$.*

As the normalized 10j action is a normalization factor times the composite of the 10j symbol with the copairing of associator state spaces, it suffices to check that each of these three elements is appropriately preserved by state transposition.

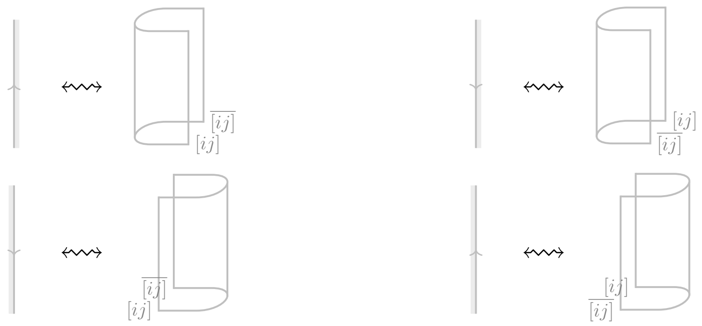
The copairing intertwines the associator state transposition.

Lemma 2.3.9 (The copairing intertwines the associator state transposition)**.** *For Γ a state of the 4-manifold K with ordering o, a 3-simplex $\kappa \in K_3$, and an adjacent* *vertex transposition* σ*, the isomorphisms of associator state spaces* $\Phi^{\pm}_{\kappa,\sigma_\kappa} : V^{\pm}(\Gamma, \kappa) \to V^{\pm \mathrm{sgn}(\sigma_\kappa)}(\Gamma^\sigma, \kappa)$ *intertwine the copairings* $\cup_{\Gamma,\kappa} : k \to V^+(\Gamma, \kappa) \otimes V^-(\Gamma, \kappa)$ *and* $\cup_{\Gamma^\sigma,\kappa} : k \to V^+(\Gamma^\sigma, \kappa) \otimes V^-(\Gamma^\sigma, \kappa)$ *in the sense that:*

$$sw^{\sigma_\kappa} \circ (\Phi^+_{\kappa,\sigma_\kappa} \otimes \Phi^-_{\kappa,\sigma_\kappa}) \circ \cup_{\Gamma,\kappa} = \cup_{\Gamma^\sigma,\kappa}.$$

Here sw^{σ_κ} *is trivial when* σ_κ *is trivial and is the swap of the two tensor factors when* σ_κ *is nontrivial.*

To establish this relation we will use a more compact graphical notation where we omit explicitly drawing the horizontal slices (that show the objects), instead recording only the nontrivial 1-morphisms as black wires (when they are elements of the state labeling) and gray wires (when they are fold 1-morphisms). For an object $[ij]$ (that is $\Gamma(\partial^o_{[ij]}\kappa)$), we graphically distinguish the four fold types $e_{[ij]}$, $i_{\overline{[ij]}} = e^*_{[ij]}$, $e_{\overline{[ij]}} = i^*_{[ij]}$, and $i_{[ij]}$ by an orientation arrow and a thin corona indicating the side of the fold with two sheets, as follows:



Proof of Lemma 2.3.9. This relation of the copairings of course follows from the corresponding relation for the pairing, namely

$$\langle \cdot, \cdot \rangle_{\Gamma,\kappa} \circ \left(\Phi^-_{\kappa,\sigma_\kappa} \otimes \Phi^+_{\kappa,\sigma_\kappa}\right)^{-1} \circ \mathrm{sw}^{\sigma_\kappa} = \langle \cdot, \cdot \rangle_{\Gamma^\sigma,\kappa}$$

Explicitly, we must show that for each of the three nontrivial transpositions $\sigma_\kappa = (01)$, (12), or (23), and for $\alpha \in V^-(\Gamma, \kappa), \beta \in V^+(\Gamma, \kappa)$, we have $\left\langle \Phi^-_{\kappa,\sigma_\kappa}(\beta), \Phi^+_{\kappa,\sigma_\kappa}(\alpha) \right\rangle_{\Gamma^\sigma,\kappa} = \langle \alpha, \beta \rangle_{\Gamma,\kappa}$.

For the transposition $\sigma_\kappa = (01)$, the proof appears as the first equation in Figure 2.5. There, 'isotopy' refers to moves allowed by Definition 1.3.1. Note that in this isotopy step the two grey circles interchange which one is on the outside. In the step using Proposition 1.3.37, note that the proposition is applied to the endomorphism obtained from the composite of α and β with just the right-hand wire traced out; in particular, this is an endomorphism of a 1-morphism from a tensor of two objects to a single object, so the left trace is enclosed by two grey fold circles while the right trace is enclosed by only one grey fold circle. For the transpositions $\sigma_\kappa = (12)$ and $\sigma_\kappa = (23)$, the proof appears as the second and third equations, respectively, in Figure 2.5. $\square$

The 10j symbol intertwines the associator state transposition.

Lemma 2.3.10 (The 10j symbol intertwines the associator state transposition)**.** *For Γ a state of the 4-manifold K with ordering o, a 4-simplex $\mu \in K_4$, and an adjacent vertex transposition σ, the isomorphisms $\Phi^{\pm}_{\kappa,\sigma_\kappa}$ of associator state spaces intertwine the 10j symbols in the sense that*

$$z(\Gamma^\sigma, \mu) \circ \left(\bigotimes_{\kappa \in K_3, \kappa \subseteq \mu} \Phi_{\kappa,\sigma_\kappa}^{\epsilon^\mu_{o'}(\kappa)\ \mathrm{sgn}(\sigma_\kappa)} \right) = z(\Gamma, \mu).$$

Proof. Note that the two sides of the equation do have the same domain, because $\epsilon^\mu_{o'}(\kappa)\ \mathrm{sgn}(\sigma_\kappa) = \epsilon^\mu_o(\kappa)$. Assume σ_μ is nontrivial and $\epsilon_{o'}(\mu) = +1$ (thus $\epsilon_o(\mu) = -1$); the case $\epsilon_{o'}(\mu) = -1$ is analogous. Abbreviate $\sigma_{ijkl} := \sigma_{\partial^{o'}_{[ijkl]}\mu} \in S_4$ for $0 \le i < j < k < l \le 4$. We need to show

$$z(\Gamma^\sigma, \mu) \circ \left(\Phi^{\mathrm{sgn}(\sigma_{0123})}_{\partial^{o'}_{[0123]}\mu, \sigma_{0123}} \otimes \Phi^{\mathrm{sgn}(\sigma_{0134})}_{\partial^{o'}_{[0134]}\mu, \sigma_{0134}} \otimes \Phi^{\mathrm{sgn}(\sigma_{1234})}_{\partial^{o'}_{[1234]}\mu, \sigma_{1234}} \otimes \Phi^{-\mathrm{sgn}(\sigma_{0124})}_{\partial^{o'}_{[0124]}\mu, \sigma_{0124}} \otimes \Phi^{-\mathrm{sgn}(\sigma_{0234})}_{\partial^{o'}_{[0234]}\mu, \sigma_{0234}} \right) \sim z(\Gamma, \mu)$$

where by "$\sim$" we mean they are equal up to some unspecified swap of the domain factors, depending on the particular permutation σ_μ in question.

Suppose $\sigma_\mu = (01) \in S_5$. The permutations $\sigma_{ijkl} \in S_4$ of the boundary faces are

$$\sigma_{0123} = (01) \qquad \sigma_{0134} = (01) \qquad \sigma_{1234} = \mathrm{id} \qquad \sigma_{0124} = (01) \qquad \sigma_{0234} = \mathrm{id}$$

The desired equation, with the appropriate swap accounted for, is

$$z(\Gamma^\sigma, \mu) \left(\Phi^-_{\partial^o_{[0123]}\mu,(01)}(\alpha), \Phi^-_{\partial^o_{[0134]}\mu,(01)}(\beta), \Phi^+_{\partial^o_{[0234]}\mu,\mathrm{id}}(\gamma), \Phi^+_{\partial^o_{[0124]}\mu,(01)}(\delta), \Phi^-_{\partial^o_{[1234]}\mu,\mathrm{id}}(\epsilon) \right) = z(\Gamma, \mu)(\gamma, \delta, \epsilon, \beta, \alpha)$$

Here $\alpha \in V^-(\Gamma, \partial^o_{[0123]}\mu)$, $\beta \in V^-(\Gamma, \partial^o_{[0134]}\mu)$, $\gamma \in V^+(\Gamma, \partial^o_{[0234]}\mu)$, $\delta \in V^+(\Gamma, \partial^o_{[0124]}\mu)$, and $\epsilon \in V^-(\Gamma, \partial^o_{[1234]}\mu)$. Note that since $\epsilon_{o'}(\mu) = +1$ and $\epsilon_o(\mu) = -1$, the expressions $z(\Gamma^\sigma, \mu)$ and $z(\Gamma, \mu)$ are defined by respectively the left and right traces in Figure 2.1. The equation is checked as in Figure 2.6.

Next, when the permutation is $\sigma_\mu = (12) \in S_5$, the permutations of the faces are

$$\sigma_{0123} = (12) \qquad \sigma_{0134} = \mathrm{id} \qquad \sigma_{1234} = (01) \qquad \sigma_{0124} = (12) \qquad \sigma_{0234} = \mathrm{id}$$

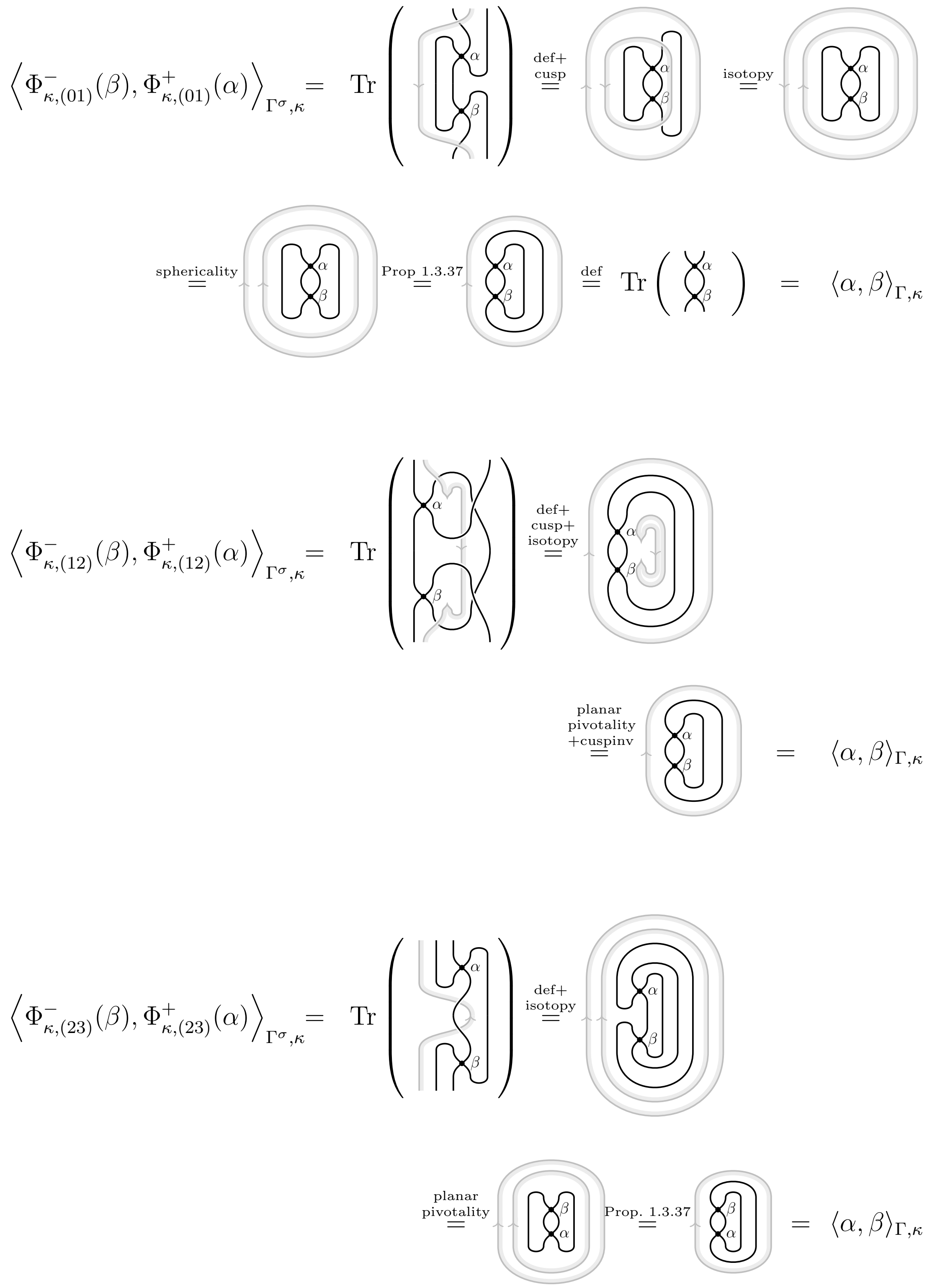


Figure 2.5: Relating the pairing of associator state spaces and the pairings of their transpositions.

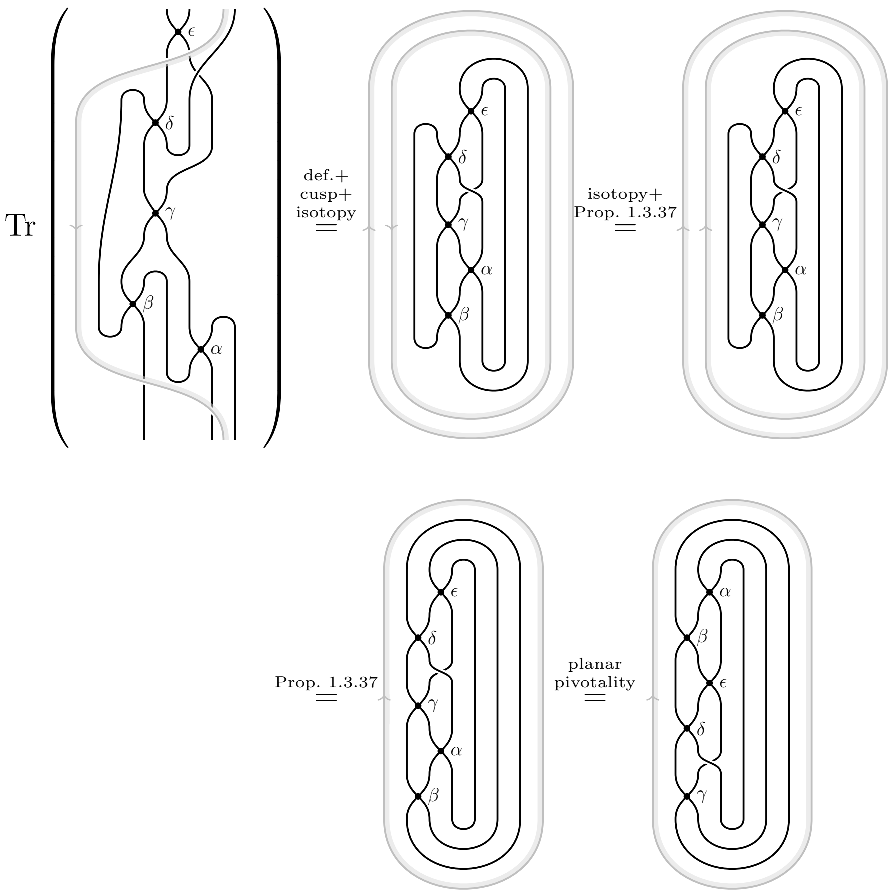


Figure 2.6: Relating the 10j symbol of a state and its (01) *transposition.*

The desired equation is

$$z(\Gamma^\sigma, \mu)\left(\Phi^-_{\partial^o_{[0123]}\mu,(12)}(\alpha), \Phi^+_{\partial^o_{[0234]}\mu,\mathrm{id}}(\beta), \Phi^-_{\partial^o_{[1234]}\mu,(01)}(\gamma), \Phi^+_{\partial^o_{[0124]}\mu,(12)}(\delta), \Phi^-_{\partial^o_{[0134]}\mu,\mathrm{id}}(\epsilon)\right)$$
$$= z(\Gamma, \mu)\,(\beta, \delta, \gamma, \epsilon, \alpha)$$

where $\alpha \in V^-(\Gamma, \partial^o_{[0123]}\mu)$, $\beta \in V^+(\Gamma, \partial^o_{[0234]}\mu)$, $\gamma \in V^-(\Gamma, \partial^o_{[1234]}\mu)$, $\delta \in V^+(\Gamma, \partial^o_{[0124]}\mu)$, and $\epsilon \in V^-(\Gamma, \partial^o_{[0134]}\mu)$. This is established as in Figure 2.7.

The cases $\sigma_\mu = (23)$ and $\sigma_\mu = (34)$ are analogous. □

The 10j normalization factors are preserved by state transposition.

Proof of Lemma 2.3.8. Composing Lemma 2.3.9 and Lemma 2.3.10, we see that the unnormalized 10j action is invariant under transposition: $Z(\Gamma^\sigma) = Z(\Gamma)$. To check that the normalized 10j action is similarly invariant, it remains only to see that the normalization factors are unaffected by transposition.

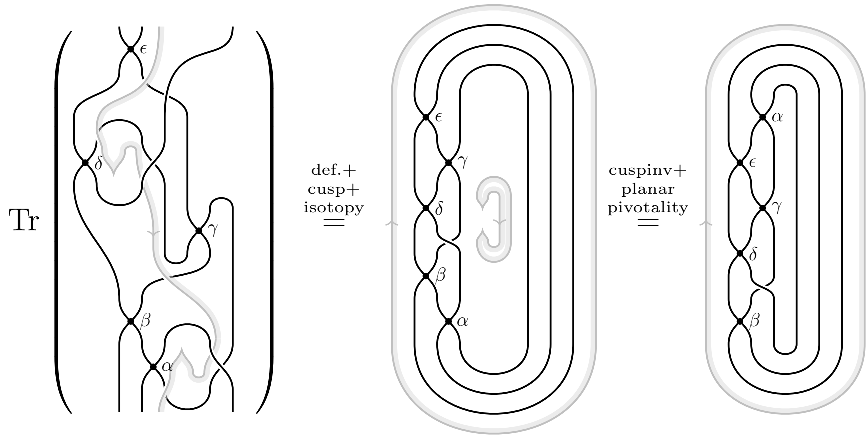


Figure 2.7: Relating the 10j symbol of a state and its (12) *transposition.*

For a 1-simplex $e \in K_1$, the transposed labeling $\Gamma^\sigma(e)$ is either $\Gamma(e)$ or $\Gamma(e)^\#$. By sphericality, therefore, we have $\dim(\Gamma^\sigma(e)) = \dim(\Gamma(e))$. Next, observe that for any object $A \in \mathcal{C}$, dualizing $(-)^\#$ defines an equivalence from the multifusion category $\mathrm{End}_\mathcal{C}(A)$ to $\mathrm{End}_\mathcal{C}(A^\#)^{\mathrm{mp}}$, where $(-)^{\mathrm{mp}}$ denotes the opposite monoidal product. As the global dimension of a multifusion category is the same as the global dimension of its monoidal opposite, it follows that $\dim(\mathrm{End}_\mathcal{C}(\Gamma^\sigma(e))) = \dim(\mathrm{End}_\mathcal{C}(\Gamma(e)))$ for any 1-simplex e. Evidently two simple objects are in the same component if and only if their duals are in the same component, so also the number of simple objects is unchanged by transposition: $n(\Gamma^\sigma(e)) = n(\Gamma(e))$.

Finally, note that for any 2-simplex $s \in K_2$, from Proposition 1.3.37 and sphericality, it follows that $\dim(\Gamma^\sigma(s)) = \dim(\Gamma(s))$. □

Transposition is a bijection of skeletal states

We have seen that give a state Γ of a given vertex-ordered 4-manifold, there is a corresponding state Γ^σ for the manifold with a permuted vertex order, and that Γ and Γ^σ have the same normalized 10j action. Equipped with that fact, we can now establish Lemma 2.3.6, that there is a bijection between skeletal states of a manifold and of the reordered manifold, such that the normalized 10j action is preserved, and therefore complete the proof of Corollary 2.3.7, that the state sum is invariant under vertex reorderings.

Proof of Lemma 2.3.6. Recall that o and o' are global vertex orders related by a permutation σ, and $\Delta\mathcal{C}^\omega$ is a simplicial skeleton for $\mathcal{C}$. For every simple object A

of $\mathcal{C}$, let A_0 denote the unique simple object of $\Delta\mathcal{C}^\omega$ equivalent to A; choose inverse equivalences $h_A : A \leftrightarrows A_0 : k_A$ such that when $A = A_0$, the equivalences h_A and k_A are identities. There is a natural transformation $X : \Delta\mathcal{C} \Rightarrow \Delta\mathcal{C}^\omega$ taking an object A to A_0 and taking a simple 1-morphism $f : A \Box B \to C$ to the unique 1-morphism isomorphic to $h_C \circ f \circ (k_A \Box k_B)$. This transformation induces a map $X_* : [K^o, \Delta\mathcal{C}] \to [K^o, \Delta\mathcal{C}^\omega]$, and note that by Lemma 2.3.2, this map preserves the normalized 10j action. The composite $X_*((-)^\sigma) : [K^o, \Delta\mathcal{C}^\omega] \to [K^{o'}, \Delta\mathcal{C}^\omega]$ taking a state Γ to $X_*(\Gamma^\sigma)$ therefore also preserves the normalized 10j action. It suffices to see that this composite is a bijection.

Define a map $Y_\sigma : [K^{o'}, \Delta\mathcal{C}^\omega] \to [K^{o'}, \Delta\mathcal{C}]$ on 1-simplices $e \in K_1$ by

$$Y_\sigma(\Gamma)(e) = \begin{cases} \Gamma(e) & \text{if } \sigma_e = \mathrm{id} \\ ((\Gamma(e)^\#)_0)^\# & \text{if } \sigma_e = (01) \end{cases}$$

and on 2-simplices $s \in K_2$ by

$$Y_\sigma(\Gamma)(s) = \begin{cases} \Gamma(s) & \text{if } \sigma_s = \mathrm{id} \\ \Gamma(s) \circ (h_{((\partial_{01}\Gamma(s)^\#)_0)^\#} \Box 1_{\partial_{12}\Gamma(s)}) & \text{if } \sigma_s = (01) \\ \Gamma(s) \circ (1_{\partial_{01}\Gamma(s)} \Box h_{((\partial_{12}\Gamma(s)^\#)_0)^\#}) & \text{if } \sigma_s = (12) \end{cases}$$

This map is well defined on 2-simplices because $(((A^\#)_0)^\#)_0 = A$ for any object A in the skeleton $\Delta\mathcal{C}^\omega$. Finally observe that the composite $X_*((Y_\sigma(-))^\sigma) : [K^{o'}, \Delta\mathcal{C}^\omega] \to [K^o, \Delta\mathcal{C}^\omega]$ is an inverse to $X_*((-)^\sigma) : [K^o, \Delta\mathcal{C}^\omega] \to [K^{o'}, \Delta\mathcal{C}^\omega]$, as required. $\Box$

2.3.3 The state sum is independent of the combinatorial structure

Lastly, we show that the state sum $Z_\mathcal{C}(K)$ is independent of the combinatorial structure of K.

Recall that, by Theorem 2.2.6, two singular combinatorial 4-manifolds are piecewise-linearly homeomorphic if and only if they are bistellar equivalent, that is if and only if there is a finite series of bistellar moves transforming one into the other. Hence, Theorem 2.2.19 is a direct consequence of the following lemma, which we prove in this section.

Lemma 2.3.11 (The state sum is invariant under bistellar equivalence)**.** *Let K and K' be bistellar equivalent oriented (singular) combinatorial 4-manifolds. Then, the corresponding state sums agree:*

$$Z_\mathcal{C}(K) = Z_\mathcal{C}(K')$$

To simplify notation, we will use the following abbreviations for simple objects X and 1-morphisms f in the spherical prefusion 2-category $\mathcal{C}$:

$$\begin{aligned} \mathrm{d}(X) &:= \dim(X)\ \dim\left(\mathrm{End}_{\mathcal{C}}(X)\right)\ n(X) \\ \mathrm{d}(f) &:= \dim(f) \end{aligned}$$

The state sum for combinatorial manifolds with boundary

Invariance under bistellar moves is most easily established if we extend the definition of $Z_{\mathcal{C}}$ to combinatorial manifolds with boundary. Let T be a closed oriented combinatorial 3-manifold, and let o be a total order on the vertices of T. Let $\epsilon_o : T_3 \to \{+1, -1\}$ be such that $\epsilon_o(\kappa) = +1$ if and only if the orientation of the 3-simplex $\kappa \in T_3$ coincides with the one induced from the order o_κ. To such a combinatorial 3-manifold we assign the following vector space:

$$W_{\mathcal{C}}(T, o, \Delta\mathcal{C}^\omega) := \bigoplus_{\Gamma : T^o_{(2)} \Rightarrow \Delta\mathcal{C}^\omega} \bigotimes_{\kappa \in T_3} V^{\epsilon_o(\kappa)}(\Gamma, \kappa)$$

Remark 2.3.12 (The 3-manifold vector space is not an invariant). The vector space $W_{\mathcal{C}}(T, o, \Delta\mathcal{C}^\omega)$ depends on the combinatorial structure of T and is not invariant under piecewise linear homeomorphisms. In particular, the hypothesized topological field theory extending the state sum $Z_{\mathcal{C}}$ will assign a certain subspace of $W_{\mathcal{C}}(T, o, \Delta\mathcal{C}^\omega)$ to a 3-manifold with triangulation T.

Denote by $\overline{T}$ the combinatorial manifold T with the opposite orientation. We define a nondegenerate pairing $\langle \cdot, \cdot \rangle_T : W_{\mathcal{C}}\left(\overline{T}, o, \Delta\mathcal{C}^\omega\right) \otimes W_{\mathcal{C}}\left(T, o, \Delta\mathcal{C}^\omega\right) \to k$ as follows:

$$\langle \cdot, \cdot \rangle_T := \dim(\mathcal{C})^{-|T_0|} \bigoplus_{\Gamma : T^o_{(2)} \Rightarrow \Delta\mathcal{C}^\omega} \left(\prod_{e \in T_1} \mathrm{d}(\Gamma(e)) \right)^{-1} \left(\prod_{s \in T_2} \mathrm{d}(\Gamma(s)) \right) \bigotimes_{\kappa \in T_3} \langle \cdot, \cdot \rangle_{\Gamma, \kappa}$$

For an oriented combinatorial 4-manifold K with boundary, and with a total order o on its vertices o, we define a linear map $Z_{\mathcal{C}}\left(K, o, \Delta\mathcal{C}^\omega\right) : k \to W_{\mathcal{C}}\left(\overline{\partial K}, o|_{\partial K}, \Delta\mathcal{C}^\omega\right)$ as follows:

$$\dim\left(\mathcal{C}\right)^{-|\mathrm{int}(K)_0|} \bigoplus_{\Sigma : \partial K^o_{(2)} \Rightarrow \Delta\mathcal{C}^\omega} \sum_{\substack{\Gamma : K^o_{(2)} \Rightarrow \Delta\mathcal{C}^\omega \\ \Gamma|_{\partial K} = \Sigma}} \left(\prod_{e \in \mathrm{int}(K)_1} \mathrm{d}(\Gamma(e)) \right)^{-1}$$
$$\left(\prod_{s \in \mathrm{int}(K)_2} \mathrm{d}(\Gamma(s)) \right) \left(\bigotimes_{\mu \in K_4} z(\Gamma, \mu) \right) \circ \left(\bigotimes_{\kappa \in K_3} \cup_{\Gamma, \kappa} \right)$$

Here, $z(\Gamma,\mu)$ and $\cup_{\Gamma,\kappa}$ are defined as before, and $i(K)_r$ denotes the finite set of r-simplices in the interior of K, that is $\mathrm{int}(K)_r := \{\tau \in K_r \mid \tau \notin \partial K_r\}$. The composition $\circ$ is over all vector spaces appearing both in the domain of $\left(\bigotimes_{\mu\in K_4} z(\Gamma,\mu)\right)$ and in the codomain of $\left(\bigotimes_{\kappa\in K_3} \cup_{\Gamma,\kappa}\right)$. Therefore, the codomain of $Z_{\mathcal{C}}(K, o, \Delta\mathcal{C}^\omega)$ is the vector space

$$\bigoplus_{\Sigma:\partial K^o_{(2)} \Rightarrow \Delta\mathcal{C}^\omega} \quad \bigotimes_{\kappa\in K_3 \text{ s.t.}\exists!\mu\in K_4 \text{ with } \kappa\subseteq\mu} V^{-\epsilon^\mu_o(\kappa)}(\Sigma,\kappa)$$

which indeed agrees with $W_{\mathcal{C}}(\overline{\partial K}, o|_{\partial K}, \Delta\mathcal{C}^\omega)$.

Lemma 2.3.13 (The state sum is the pairing of with-boundary state sums)**.** *Let K and K' be oriented combinatorial 4-manifolds with boundary and let $f : \partial K \to \partial K'$ be an orientation reversing simplicial isomorphism. Let o and o' be total orders on K_0 and K'_0 such that the simplicial isomorphism $f : \partial K \to \partial K'$ preserves the induced orders. Then,*

$$Z_{\mathcal{C}}(K \cup_f K') = \langle Z_{\mathcal{C}}\left(K, o, \Delta\mathcal{C}^\omega\right), Z_{\mathcal{C}}\left(K', o', \Delta\mathcal{C}^\omega\right)\rangle_{\partial K},$$

where we used f to identify the vector spaces $W_{\mathcal{C}}(\overline{\partial K'}, o'|_{\partial K'}, \Delta\mathcal{C}^\omega)$ and $W_{\mathcal{C}}(\partial K, o|_{\partial K}, \Delta\mathcal{C}^\omega)$.

Proof. This is a direct consequence of the definition of $Z_{\mathcal{C}}\left(K, o, \Delta\mathcal{C}^\omega\right)$ for combinatorial manifolds with boundary. □

The 4-dimensional bistellar moves

Recall that a bistellar move replaces a combinatorial n-manifold K by a combinatorial manifold obtained from replacing a codimension zero submanifold of K simplicially isomorphic to a subcomplex $I \subseteq \partial\Delta^{n+1}$ with the complementary subcomplex $J \subseteq \partial\Delta^{n+1}$. To show invariance of $Z_{\mathcal{C}}$ under such a move, it suffices by Lemma 2.3.13 to prove that the following linear maps $k \to W_{\mathcal{C}}\left(\partial I, o|_{\partial I}, \Delta\mathcal{C}^\omega\right)$ are equal:

$$Z_{\mathcal{C}}\left(I, o|_I, \Delta\mathcal{C}^\omega\right) = Z_{\mathcal{C}}\left(J, o|_J, \Delta\mathcal{C}^\omega\right)$$

Here, o is some fixed order on the vertices of Δ^{n+1}, which induces an orientation on Δ^{n+1}, and thus an orientation on the boundary $\partial\Delta^{n+1}$. The subcomplex $I \subseteq \partial\Delta^{n+1}$ carries the orientation induced by this orientation on $\partial\Delta^{n+1}$ and $J \subseteq \partial\Delta^{n+1}$ carries the opposite of the orientation induced by $\partial\Delta^{n+1}$.

For the three relevant 4-dimensional bistellar moves, we pick an order o on Δ^5 such that I_4 and J_4 contain the following 4-simplices, where we have labeled the vertices of Δ^5 according to the order o by $0 \ldots 5$ and indicated the orientation of each 4-simplex relative to the one induced from the order by a sign (these signs agree with the signs $\epsilon_{o|_I}(\mu)$ or $\epsilon_{o|_J}(\mu)$, respectively, introduced in Section 2.2.3):

	I_4	J_4
$(3,3)$-move	$\langle 01235\rangle$ $\langle 01345\rangle$ $\langle 12345\rangle$	$\langle 01234\rangle$ $\langle 01245\rangle$ $\langle 02345\rangle$
$(2,4)$-move	$\langle 01235\rangle$ $\langle 01345\rangle$	$\langle 02345\rangle$ $\langle 01245\rangle$ $\langle 02345\rangle$ $-\langle 12345\rangle$
$(1,5)$-move	$\langle 01235\rangle$	$\langle 02345\rangle$ $\langle 01245\rangle$ $\langle 02345\rangle$ $-\langle 12345\rangle$ $-\langle 01345\rangle$

In this notation, the relative sign $\epsilon^\mu(\kappa)$ for a 4-simplex $\mu = \lambda\langle v_0 \cdots v_4\rangle$ with $\lambda = \pm 1$, and a 3-simplex $\kappa = \langle v_0 \cdots \widehat{v}_i \cdots v_4\rangle$ is $\epsilon^\mu(\kappa) = \lambda(-1)^i$, for both ordered oriented complexes I and J.

The simplices in the interior of I and J are listed in Figures 2.8, 2.9 and 2.10.

	$\mathrm{int}(I)_k$	$\mathrm{int}(J)_k$
$k=4$	$\langle 01235\rangle$ $\langle 01345\rangle$ $\langle 12345\rangle$	$\langle 01234\rangle$ $\langle 01245\rangle$ $\langle 02345\rangle$
$k=3$	$\langle 0135\rangle$ $\langle 1235\rangle$ $\langle 1345\rangle$	$\langle 0124\rangle$ $\langle 0234\rangle$ $\langle 0245\rangle$
$k=2$	$\langle 135\rangle$	$\langle 024\rangle$

Figure 2.8: k-simplices in the interior of I and J for the $(3,3)$-bistellar move.

	$\mathrm{int}(I)_k$	$\mathrm{int}(J)_k$
$k=4$	$\langle 01235\rangle$ $\langle 01345\rangle$	$\langle 01234\rangle$ $\langle 01245\rangle$ $\langle 02345\rangle$ $\langle 12345\rangle$
$k=3$	$\langle 0135\rangle$	$\langle 0124\rangle$ $\langle 0234\rangle$ $\langle 0245\rangle$ $\langle 2345\rangle$ $\langle 1245\rangle$ $\langle 1234\rangle$
$k=2$		$\langle 024\rangle$ $\langle 245\rangle$ $\langle 234\rangle$ $\langle 124\rangle$
$k=1$		$\langle 24\rangle$

Figure 2.9: k-simplices in the interior of I and J for the $(2,4)$-bistellar move.

	$\mathrm{int}(I)_k$	$\mathrm{int}(J)_k$
$k=4$	$\langle 01235\rangle$	$\langle 01234\rangle$ $\langle 01245\rangle$ $\langle 02345\rangle$ $\langle 01345\rangle$ $\langle 12345\rangle$
$k=3$		$\langle 0124\rangle$ $\langle 0234\rangle$ $\langle 0245\rangle$ $\langle 2345\rangle$ $\langle 1245\rangle$ $\langle 1234\rangle$ $\langle 0134\rangle$ $\langle 0145\rangle$ $\langle 1345\rangle$ $\langle 0345\rangle$
$k=2$		$\langle 024\rangle$ $\langle 245\rangle$ $\langle 234\rangle$ $\langle 124\rangle$ $\langle 034\rangle$ $\langle 014\rangle$ $\langle 134\rangle$ $\langle 045\rangle$ $\langle 145\rangle$ $\langle 345\rangle$
$k=1$		$\langle 24\rangle$ $\langle 04\rangle$ $\langle 14\rangle$ $\langle 34\rangle$ $\langle 45\rangle$
$k=0$		$\langle 4\rangle$

Figure 2.10: k-simplices in the interior of I and J for the $(1,5)$-bistellar move.

From now on, we omit the choice of order from the notation and explicitly work with the simplices in Figures 2.8, 2.9 and 2.10 and the sign conventions outlined above. Given a $\mathcal{C}$-state $\Delta^5_{(2)} \Rightarrow \Delta\mathcal{C}$, we denote the simple object assigned to a 1-simplex $\langle ij\rangle$ by $[ij]$ and the simple 1-morphism assigned to a 2-simplex $\langle ijk\rangle$ by

$[ijk]$. Contrary to our previous notation, we henceforth almost always let the state be implicit and omit it from the notation. For a 3-simplex $\langle ijkl\rangle \in \Delta^5_3$, we recall Notation 2.2.10 and reintroduce the vector spaces from Definition 2.2.12:

$$V^+(ijkl) := \mathrm{Hom}_{\mathcal{C}}\left([(ijk)l],[i(jkl)]\right) \qquad V^-(ijkl) := \mathrm{Hom}_{\mathcal{C}}\left([i(jkl)],[(ijk)l]\right)$$

For a 4-simplex $\langle ijklm\rangle \in \Delta^5_4$, we recall the linear maps defined in Figures 2.1a and 2.1b (with again the state left implicit):

$$z_+(ijklm) : V^+(ijkl) \otimes V^+(ijlm) \otimes V^+(jklm) \otimes V^-(ijkm) \otimes V^-(iklm) \to k$$
$$z_-(ijklm) : V^+(iklm) \otimes V^+(ijkm) \otimes V^-(jklm) \otimes V^-(ijlm) \otimes V^-(ijkl) \to k$$

Precomposing $z_\pm(ijklm)$ with the appropriate maps $\cup_{abcd} : k \to V^+(abcd)\otimes V^-(abcd)$ (determined by the nondegenerate pairings $\langle\cdot,\cdot\rangle_{abcd} : V^-(abcd)\otimes V^+(abcd) \to k$), leads to linear maps of the following type for every 4-simplex $\langle ijklm\rangle \in \Delta^5_4$:

$$Z_+(ijklm) : V^+(ijkl) \otimes V^+(ijlm) \otimes V^+(jklm) \to V^+(iklm) \otimes V^+(ijkm)$$
$$Z_-(ijklm) : V^+(iklm) \otimes V^+(ijkm) \to V^+(ijkl) \otimes V^+(ijlm) \otimes V^+(jklm)$$

Using these maps, invariance under the various bistellar moves can then explicitly be reexpressed as the following lemmas.

In the following, all expressions are already postcomposed with appropriate pairings $\langle\cdot,\cdot\rangle$ to undo superfluous $\cup$-maps appearing on either side of the equations. To unclutter notation, the concatenation $A_1 \cdots A_n$ of linear maps $A_1, \ldots, A_n$ denotes composition over all vector spaces appearing both in the domain of some A_i and the codomain of some A_j with $j > i$. *Here, and in the following proofs, we will use this concatenation notation, and we will therefore omit all swap maps, all tensor product symbols, and all tensor products with identities.*

Lemma 2.3.14 (Invariance under the $(3,3)$-bistellar move)**.** *Let I and J be as in Figure 2.8. Then, the following holds for every $\mathcal{C}$-state $\partial I_{(2)} \Rightarrow \Delta\mathcal{C}^\omega$ of ∂I:*

$$\sum_{[135]} \mathrm{d}([135])\, Z_+(01235) Z_+(01345) Z_+(12345) = \sum_{[024]} \mathrm{d}([024])\, Z_+(02345) Z_+(01245) Z_+(01234)$$

The sums are over simple 1*-morphisms* $\Delta\mathcal{C}^\omega_2 \ni [ijk] : [ij] \,\Box\, [jk] \to [ik]$.

Lemma 2.3.15 (Invariance under the $(2,4)$-bistellar move)**.** *Let I and J be as in Figure 2.9. Then, the following holds for every $\mathcal{C}$-state $\partial I_{(2)} \Rightarrow \Delta\mathcal{C}^\omega$ of ∂I:*

$$Z_+(01235)Z_+(01345)$$
$$= \sum_{\substack{[24],[024],[245],\\ [234],[124]}} \frac{\mathrm{d}([024])\,\mathrm{d}([245])\,\mathrm{d}([234])\,\mathrm{d}([124])}{\mathrm{d}([24])} Z_+(02345)Z_+(01245)Z_+(01234)Z_-(12345)$$

The sums are over simple objects $[24] \in \Delta\mathcal{C}^\omega_1$*, and simple* 1*-morphisms* $\Delta\mathcal{C}^\omega_2 \ni [ijk] : [ij] \,\Box\, [jk] \to [ik]$*.*

Lemma 2.3.16 (Invariance under the $(1,5)$-bistellar move)**.** *Let I and J be as in Figure 2.10. Then, the following holds for every $\mathcal{C}$-state $\partial I_{(2)} \Rightarrow \Delta\mathcal{C}^\omega$ of ∂I:*

$$Z_+(01235) = \dim(\mathcal{C})^{-1} \sum_{\substack{[ij],0\leq i<j\leq 5 \\ i=4 \text{ or } j=4}} \sum_{\substack{[ijk],0\leq i<j<k\leq 5 \\ j=4 \text{ or } k=4}} \left(\prod_{\substack{[ij],i<j \\ i=4 \text{ or } j=4}} \mathrm{d}([ij]) \right)^{-1} \left(\prod_{\substack{[ijk],i<j<k \\ j=4 \text{ or } k=4}} \mathrm{d}([ijk]) \right)$$
$$\mathrm{Tr}_{V^+(0345)} \Big(Z_+(02345) Z_+(01245) Z_+(01234) Z_-(12345) Z_-(01345) \Big)$$

The sum is over simple objects $[ij] \in \Delta\mathcal{C}^\omega_1$ *and simple* 1*-morphisms* $\Delta\mathcal{C}^\omega_2 \ni [ijk] : [ij] \,\Box\, [jk] \to [ik]$*, and the (partial) trace is over the vector space* $V^+(0345)$*.*

The range of the sums and the appearance of the trace in the preceeding lemmas follow directly from comparing the explicit expressions for $Z_\mathcal{C}(I, o|_I, \Delta\mathcal{C}^\omega)$ and $Z_\mathcal{C}(J, o|_J, \Delta\mathcal{C}^\omega)$, where we consider I and J as oriented combinatorial manifolds with common boundary ∂I. Note that the expressions for $Z_\mathcal{C}(J, o|_J, \Delta\mathcal{C}^\omega)$ in Section 2.3.3 are written in terms of the maps $z_\pm$ instead of the maps $Z_\pm$, which are precomposed with $\cup$-maps. For example, the trace in Lemma 2.3.16 arises explicitly from the fact that both vector spaces $V^+(0345)$ and $V^-(0345)$ appear in the codomain of $\bigotimes_{\kappa\in J_3} \cup_{\Gamma,\kappa}$ and the domain of $\bigotimes_{\mu\in J_4} z(\Gamma,\mu)$ (explicitly, in the domain of $z_-(01345)$ and $z_+(02345)$, respectively) and are hence composed over, but the vector space $V^-(0345)$ in the domain of $z_+(02345)$ is transformed into the vector space $V^+(0345)$ in the codomain of $Z_+(02345)$. The formula in Lemma 2.3.16 therefore involves a trace over the vector space $V^+(0345)$ in the domain of $Z_-(01345)$ and the codomain of $Z_+(02345)$.

Pentagonator compositions in prefusion 2-categories

The '10j symbol' linear map $z(\Gamma,\mu)$ defined in Figure 2.1 extracts the matrix coefficients of the pentagonator of the monoidal 2-category $\mathcal{C}$. The corresponding 'partially dualized 10j symbol' maps $Z_\pm(ijklm)$ (defined in Section 2.3.3) can be thought of as first 'composing associators' along one side of the pentagon, and then 'uncomposing' along the other side of the pentagon. More precisely, we will express $Z_+(ijklm)$ as the composite of a linear map $\mathrm{LComp}_+(ijklm)$ (that composes associators along the long side of the pentagon), followed by the dual of a linear map $\mathrm{SComp}_-(ijklm)$ (that composes associators along the short side of the pentagon); similarly $Z_-(ijklm)$

will be a composite of a linear map $\mathrm{SComp}_+(ijklm)$ and the dual of a linear map $\mathrm{LComp}_-(ijklm)$. We will show that the dual of $\mathrm{SComp}_-(ijklm)$ is proportional to the inverse of $\mathrm{SComp}_+(ijklm)$ (this inverse may be though of as 'uncomposing along the short side of the pentagon') and that the dual of $\mathrm{LComp}_-(ijklm)$ is proportional to a section of $\mathrm{LComp}_+(ijklm)$ (this section is thus 'uncomposing along the long side of the pentagon'). This detailed decomposition of the 10j symbols will be crucial in proving invariance under the bistellar moves in Section 2.3.3.

Composing along the pentagon. For a 4-simplex $\langle ijklm\rangle \in \Delta^5_4$, we extend Notation 2.2.10 as follows:

$$[((ijk)l)m] := [ilm] \circ \left([ikl] \,\square\, 1_{[lm]}\right) \circ \left([ijk] \,\square\, 1_{[kl]} \,\square\, 1_{[lm]}\right)$$

The alternative parenthesizations $[(i(jkl))m]$, $[i((jkl)m)]$, $[i(j(klm))]$ are defined analogously. We define the following vector spaces:

$$\begin{aligned} V^+(ijklm) &:= \mathrm{Hom}_{\mathcal{C}}\left([((ijk)l)m], [i(j(klm))]\right) \\ V^-(ijklm) &:= \mathrm{Hom}_{\mathcal{C}}\left([i(j(klm))], [((ijk)l)m]\right) \end{aligned}$$

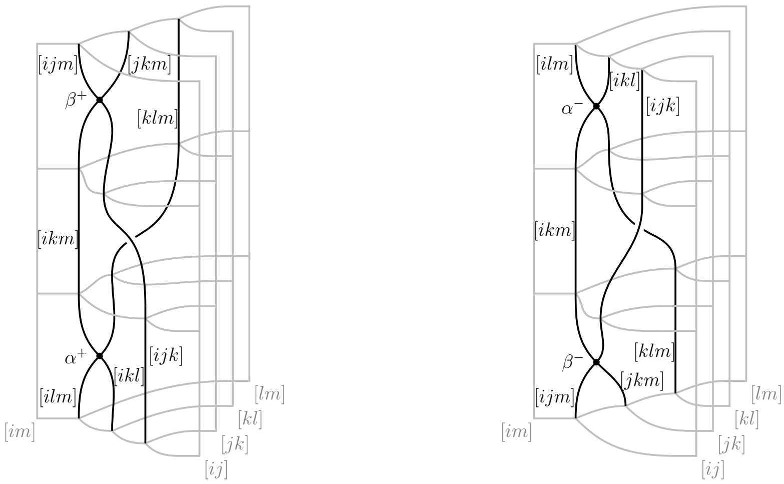
As mentioned, the linear maps $Z_\pm(ijklm)$ can be considered as first 'composing associators' along one side of the pentagon, factoring through the vector spaces $V^+(ijklm)$ or $V^-(ijklm)$ respectively, and then 'uncomposing' along the other side of the pentagon. In Figures 2.11 and 2.12 we therefore define linear maps

$$\begin{aligned} \mathrm{SComp}_\pm(ijklm) &: \bigoplus_{[ikm]} V^\pm(iklm) \otimes V^\pm(ijkm) \to V^\pm(ijklm) \\ \mathrm{LComp}_\pm(ijklm) &: \bigoplus_{\substack{[jl],[ijl] \\ [jkl],[jlm]}} V^\pm(ijkl) \otimes V^\pm(ijlm) \otimes V^\pm(jklm) \to V^\pm(ijklm) \end{aligned}$$

for 4-simplices $\langle ijklm\rangle \in \Delta^5_4$, expressing the composition of associators along the short (SComp) or long side (LComp) of the pentagon.

We denote the duals of these linear maps with respect to the non-degenerate pairing between V^+ and V^- by

$$\begin{aligned} \mathrm{SComp}^\vee_\pm(ijklm) &: V^\mp(ijklm) \to \bigoplus_{[ikm]} V^\mp(iklm) \otimes V^\mp(ijkm) \\ \mathrm{LComp}^\vee_\pm(ijklm) &: V^\mp(ijklm) \to \bigoplus_{\substack{[jl],[ijl] \\ [jkl],[jlm]}} V^\mp(ijkl) \otimes V^\mp(ijlm) \otimes V^\mp(jklm). \end{aligned}$$



$\mathrm{SComp}_+(ijklm)(\alpha^+, \beta^+)$ $\qquad$ $\mathrm{SComp}_-(ijklm)(\alpha^-, \beta^-)$

Figure 2.11: $\mathrm{SComp}_\pm(ijklm)$ *for* $\alpha^\pm \in V^\pm(iklm)$ *and* $\beta^\pm \in V^\pm(ijkm)$.

By definition, $Z_+(ijklm)$ and $Z_-(ijklm)$ are the direct sum coefficients of the following linear maps, respectively:

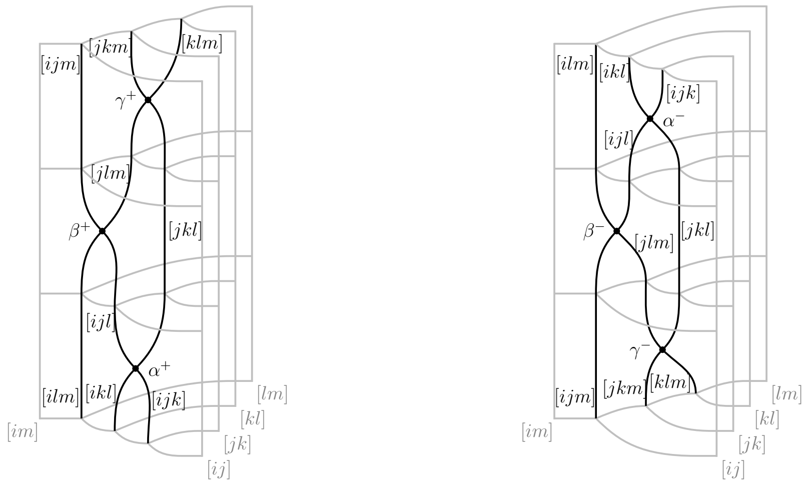
$$\mathrm{SComp}_-^\vee(ijklm) \circ \mathrm{LComp}_+(ijklm) \qquad \mathrm{LComp}_-^\vee(ijklm) \circ \mathrm{SComp}_+(ijklm)$$

Invertibility of the shortside pentagon composite. The aforementioned heuristic, that the map Z_+ is given by composing associators along one side of the pentagon and then 'uncomposing' along the other side, of course only makes sense if one of the composition maps LComp or SComp is indeed invertible. This is the content of the following lemma.

In the following, we will often write expressions such as $\lambda_i F$, where $\lambda_i \in k$, $i \in I$, I is a finite set, F is a linear map with (co)domain $\bigoplus_{i\in I} W_i$ and $\{W_i\}_{i\in I}$ is a family of vector spaces indexed by I. This is shorthand notation for the composite of F with the diagonal map $\bigoplus_{i\in I} \lambda_i \mathrm{id}_{W_i} : \bigoplus_{i\in I} W_i \to \bigoplus_{i\in I} W_i$.

Lemma 2.3.17 (The shortside pentagon composite is invertible)**.** *The function* $\mathrm{SComp}_+(ijklm)$ *is invertible with inverse* $\mathrm{d}([ikm])\,\mathrm{SComp}_-^\vee(ijklm)$.

Proof. We first prove that $\mathrm{SComp}_+(ijklm)$ is invertible. Consider the factorization of $\mathrm{SComp}_+(ijklm)$ in Figure 2.13. The top vertical map and the horizontal map are evidently isomorphisms, the bottom vertical map is an isomorphism since $\mathcal{C}$ is locally semisimple. Hence, $\mathrm{SComp}_+(ijklm)$ is invertible.



$\mathrm{LComp}_+(ijklm)(\alpha^+, \beta^+, \gamma^+)$ $\mathrm{LComp}_-(ijklm)(\alpha^-, \beta^-, \gamma^-)$

Figure 2.12: $\mathrm{LComp}_\pm(ijklm)$ *for* $\alpha^\pm \in V^\pm(ijkl)$*,* $\beta^\pm \in V^\pm(ijlm)$*, and* $\gamma^\pm \in V^\pm(jklm)$*.*

To show that $\mathrm{d}([ikm])\,\mathrm{SComp}_-^\vee(ijklm)$ is an inverse, it therefore suffices to prove that

$$\mathrm{d}([ikm])\,\mathrm{SComp}_-^\vee(ijklm) \circ \mathrm{SComp}_+(ijklm) = \mathrm{id}.$$

This follows from the calculation in Figure 2.14, for α, β, γ and δ of appropriate type, such that $\alpha \otimes \beta$ and $\gamma \otimes \delta$ are in a direct summand labeled by a simple object $[ikm]$ and $[ikm]'$, respectively. □

Right invertibility of the longside pentagon composite. While SComp_+ is invertible, allowing us to express $\widehat{Z}_+^\oplus$ as a 'composition around the pentagon', by contrast LComp_+ is not in general invertible. This situation is in contrast to the situation with endotrivial fusion categories, in which both SComp_+ and LComp_+ are invertible [Mac99]. However, we will show below that LComp_+ has a section, or a right inverse, and that $\widehat{Z}_-^\oplus$ can indeed be expressed as a composite of SComp_+ and this section; hence $\widehat{Z}_-^\oplus$ is itself a section of $\widehat{Z}_+^\oplus$. This suffices later to verify invariance under bistellar moves.

Lemma 2.3.18 (The longside pentagon composite is right invertible)**.** *The linear map* $\mathrm{LComp}_+(ijklm)$ *fulfills the following equation:*

$$\mathrm{LComp}_+(ijklm) \circ \left(\frac{\mathrm{d}([ijl])\,\mathrm{d}([jkl])\,\mathrm{d}([jlm])}{\mathrm{d}([jl])} \mathrm{LComp}_-^\vee(ijklm) \right) = \mathrm{id}$$

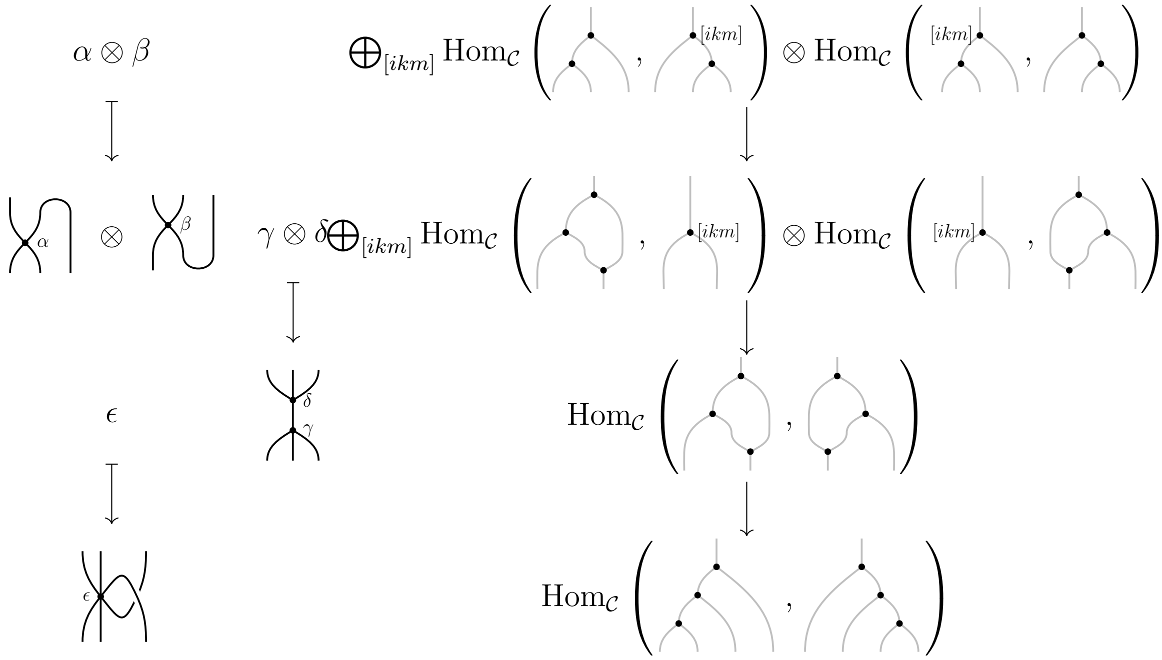


Figure 2.13: Factoring $\mathrm{SComp}_+(ijklm)$*.*

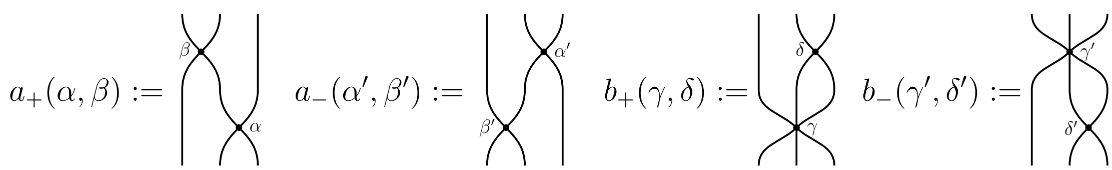
Proof. In this proof, we use the following notation:

$$V^+(i(jkl)m) := \mathrm{Hom}_{\mathcal{C}}([((ijk)l)m], [i((jkl)m)])$$
$$V^-(i(jkl)m) := \mathrm{Hom}_{\mathcal{C}}([i((jkl)m)], [((ijk)l)m])$$

We define linear maps

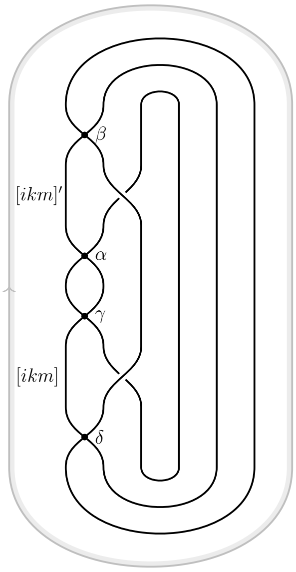
$$a_\pm : \bigoplus_{[ijl]} V^\pm(ijkl) \otimes V^\pm(ijlm) \to V^\pm(i(jkl)m)$$
$$b_\pm : \bigoplus_{[jl],[jkl],[jlm]} V^\pm(i(jkl)m) \otimes V^\pm(jklm) \to V^\pm(ijklm)$$

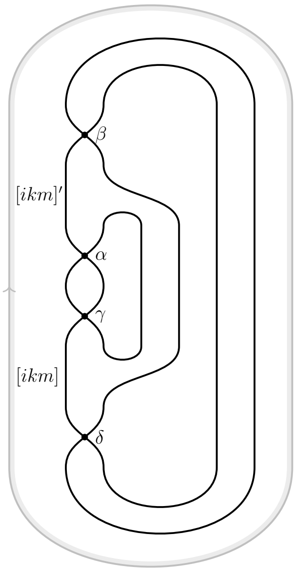
as follows, for $\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta$ and δ' of appropriate type:



Analogously to the proof of Lemma 2.3.17, it follows that a_+ is invertible with inverse $\mathrm{d}([ijl])\, a_-^\vee$.

$$\mathrm{d}([ikm]) \left\langle \mathrm{SComp}_{+}(ijklm)(\alpha,\beta), \mathrm{SComp}_{-}(ijklm)(\gamma,\delta) \right\rangle$$

$= \mathrm{d}([ikm])$ 

$= \mathrm{d}([ikm])$ 

$$\overset{\substack{\text{simplicity of} \\ [ikm],\,[ikm]'}}{=} \delta_{[ikm],[ikm]'} \; \mathrm{Tr}\,(\alpha \cdot \gamma) \;\; \mathrm{Tr}\,(\beta \cdot \delta) = \langle \alpha \otimes \beta, \gamma \otimes \delta \rangle$$

Figure 2.14: Pairing the positive and negative shortside pentagon composites.

The maps $a_{\pm}$ and $b_{\pm}$ give rise to a factorization of $\mathrm{LComp}_{\pm}(ijklm)$ as follows:

$$\mathrm{LComp}_{\pm}(ijklm) = b_{\pm} \circ \left(\bigoplus_{[jl],[jkl],[jlm]} a_{\pm} \otimes \mathrm{id}_{V^{\pm}(jklm)} \right)$$

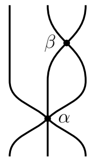
In particular, abbreviating the scalar $\lambda = \frac{\mathrm{d}([jkl])\mathrm{d}([jlm])}{\mathrm{d}([jl])}$, and omitting direct sum and tensor products with identities, the following holds:

$$\begin{aligned} \mathrm{LComp}_{+}(ijklm) \circ \left(\frac{\mathrm{d}([ijl])\,\mathrm{d}([jkl])\,\mathrm{d}([jlm])}{\mathrm{d}([jl]))} \mathrm{LComp}_{-}^{\vee}(ijklm) \right) \\ = b_{+} \circ a_{+} \circ \left(\mathrm{d}([ijl])\, a_{-}^{\vee} \right) \circ \left(\lambda b_{-}^{\vee} \right) = b_{+} \circ \left(\lambda b_{-}^{\vee} \right) \end{aligned}$$

To prove Lemma 2.3.18, it therefore suffices to show that $b_{+} \circ \left(\lambda b_{-}^{\vee} \right) = \mathrm{id}$, or equivalently that the following map is the identity:

$$\sum_{\alpha} \lambda \, \left\langle b_{-}(\widehat{\alpha}), - \right\rangle b_{+}(\alpha) : V^{+}(ijklm) \to V^{+}(ijklm)$$

Here, the sum is over a basis $\{\alpha\}$ of $\bigoplus_{[jl],[jkl],[jlm]} V^{+}(i(jkl)m) \otimes V^{+}(jklm)$ with corresponding dual basis $\{\widehat{\alpha}\}$ of $\bigoplus_{[jl],[jkl],[jlm]} V^{-}(i(jkl)m) \otimes V^{-}(jklm)$ with respect to the pairing $\langle\cdot,\cdot\rangle$. To prove this, we introduce the following auxilliary map:

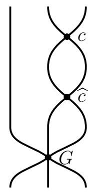
$$r : V^+(ijklm) \otimes V^-(jklm) \to V^+(i(jkl)m) \qquad r(\alpha,\beta) :=$$ 

It follows from definition that for $\widehat{F} \in V^-(i(jkl)m)$, $\widehat{c} \in V^-(jklm)$ and $G \in V^+(ijklm)$,

$$\left\langle b_-(\widehat{F},\widehat{c}), G\right\rangle = \left\langle \widehat{F}, r(G,\widehat{c})\right\rangle .$$

Choosing bases $\{F\}$ of $V^+(i(jkl)m)$ and $\{c\}$ of $V^+(jklm)$ with corresponding dual bases $\{\widehat{F}\}$ and $\{\widehat{c}\}$ induces a basis $\{\alpha\}$ of $\bigoplus_{[jl],[jkl],[jlm]} V^+(i(jkl)m) \otimes V^+(jklm)$ with which the above expression, evaluated at $G \in V^+(ijklm)$ becomes the following:

$$\sum_\alpha \lambda \left\langle b_-(\widehat{\alpha}), G\right\rangle b_+(\alpha) = \sum_{\substack{[jl],[jkl],[jlm]\\F,c}} \lambda \left\langle b_-(\widehat{F},\widehat{c}), G\right\rangle b_+(F,c) = \sum_{\substack{[jl],[jkl],[jlm]\\F,c}} \lambda \left\langle \widehat{F}, r(G,\widehat{c})\right\rangle b_+(F,c)$$

$$= \sum_{[jl],[jkl],[jlm],c} \lambda\, b_+\left(r(G,\widehat{c}),c\right) = \sum_{[jl],[jkl],[jlm],c} \frac{\mathrm{d}([jkl])\,\mathrm{d}([jlm])}{\mathrm{d}([jl])}$$ 

By Corollary C.7, this equals G. □

The summed normalized 10j symbols and their factorization. For a $\mathcal{C}$-state $\Delta^5_{(2)} \Rightarrow \Delta\mathcal{C}^\omega$ and a 4-simplex $\langle ijklm\rangle \in \Delta^5_4$, we define the following normalized linear maps:

$$\widehat{Z}_+(ijklm) := \mathrm{d}([ikm])\, Z_+(ijklm) \quad \widehat{Z}_-(ijklm) := \frac{\mathrm{d}([ijl])\,\mathrm{d}([jkl])\,\mathrm{d}([jlm])}{\mathrm{d}([jl])} Z_-(ijklm)$$

Taking the direct sum over simple objects $[jl] \in \Delta\mathcal{C}^\omega_1$ and compatible simple 1-morphisms $[ijl],[jkl],[jlm]$ and $[ikm]$ in $\Delta\mathcal{C}^\omega_2$ gives rise to the following linear maps:

$$\widehat{Z}^\oplus_+(ijklm) := \bigoplus_{\substack{[jl],[ijl],\\ [jkl],[jlm]}} \bigoplus_{[ikm]} \widehat{Z}_+(ijklm) :$$

$$\bigoplus_{\substack{[jl],[ijl],\\ [jkl],[jlm]}} V^+(ijkl) \otimes V^+(ijlm) \otimes V^+(jklm) \to \bigoplus_{[ikm]} V^+(iklm) \otimes V^+(ijkm)$$

$$\widehat{Z}^\oplus_-(ijklm) := \bigoplus_{[ikm]} \bigoplus_{\substack{[jl],[ijl],\\ [jkl],[jlm]}} \widehat{Z}_-(ijklm) :$$

$$\bigoplus_{[ikm]} V^+(iklm) \otimes V^+(ijkm) \to \bigoplus_{\substack{[jl],[ijl],\\ [jkl],[jlm]}} V^+(ijkl) \otimes V^+(ijlm) \otimes V^+(jklm)$$

Corollary 2.3.19 (The summed normalized 10j symbol factors around the pentagon)**.** *The direct sum map* $\widehat{Z}^{\oplus}_{+}(ijklm)$ *factors as follows through* $V^{+}(ijklm)$*:*

$$\widehat{Z}^{\oplus}_{+}(ijklm) = \big(\mathrm{SComp}_{+}(ijklm)\big)^{-1} \circ \mathrm{LComp}_{+}(ijklm)$$

Proof. By definition, $Z_{+}(ijklm)$ is a direct sum coefficient of the composite $\mathrm{SComp}^{\vee}_{-}(ijklm) \circ$ $\mathrm{LComp}_{+}(ijklm)$. The corollary therefore follows from Lemma 2.3.17 and the definition of $\widehat{Z}^{\oplus}_{+}(ijklm)$ as a normalized direct sum. □

Corollary 2.3.20 (The negative and positive summed normalized 10j symbols form a section-retraction pair)**.** $\widehat{Z}^{\oplus}_{-}(ijklm)$ *is a section of* $\widehat{Z}^{\oplus}_{+}(ijklm)$*:*

$$\widehat{Z}^{\oplus}_{+}(ijklm) \circ \widehat{Z}^{\oplus}_{-}(ijklm) = \mathrm{id}$$

Proof. Since $Z_{-}(ijklm)$ is a direct sum coefficient of $\mathrm{LComp}^{\vee}_{-}(ijklm) \circ \mathrm{SComp}_{+}(ijklm)$, it follows that the normalized direct sum map $\widehat{Z}^{\oplus}_{-}$ can be written as follows:

$$\widehat{Z}^{\oplus}_{-}(ijklm) = \left(\frac{\mathrm{d}([ijl])\,\mathrm{d}([jkl])\,\mathrm{d}([jlm])}{\mathrm{d}([jl])} \mathrm{LComp}^{\vee}_{-}(ijklm) \right) \circ \mathrm{SComp}_{+}(ijklm).$$

The corollary therefore follows from Corollary 2.3.19 and Lemma 2.3.18. □

Invariance under the bistellar moves

In the following, we prove Lemma 2.3.14, Lemma 2.3.15, and Lemma 2.3.16, showing invariance of the state sum under bistellar moves.

Invariance under the $(3,3)$*-bistellar move.* We start with the $(3,3)$-bistellar move and prove Lemma 2.3.14.

Proof of Lemma 2.3.14. Expressed in terms of the normalized $\widehat{Z}_{+}$ and $\widehat{Z}_{-}$, the equation in Lemma 2.3.14 reads as follows:

$$\sum_{[135]} \widehat{Z}_{+}(01235)\widehat{Z}_{+}(01345)\widehat{Z}_{+}(12345) = \sum_{[024]} \widehat{Z}_{+}(02345)\widehat{Z}_{+}(01245)\widehat{Z}_{+}(01234)$$

Rewritten in terms of the direct sums $\widehat{Z}^{\oplus}_{+}$ and $\widehat{Z}^{\oplus}_{-}$, and omitting all direct sum symbols, this becomes the following equation:

$$\widehat{Z}^{\oplus}_{+}(01235)\widehat{Z}^{\oplus}_{+}(01345)\widehat{Z}^{\oplus}_{+}(12345) = \widehat{Z}^{\oplus}_{+}(02345)\widehat{Z}^{\oplus}_{+}(01245)\widehat{Z}^{\oplus}_{+}(01234) \qquad \text{(B3-3)}$$

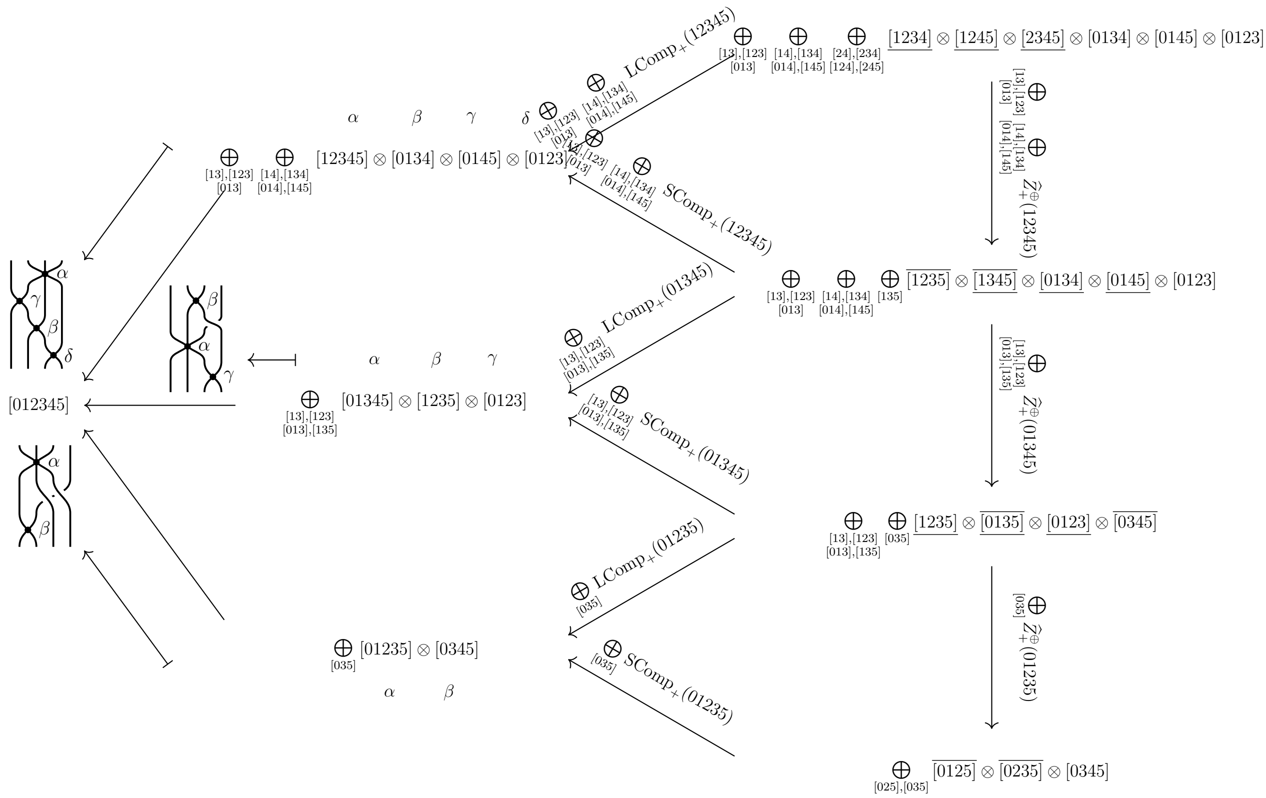


Figure 2.15: The left hand side of the equation in the proof of Lemma 2.3.14.

between linear maps

$$\bigoplus_{\substack{[13],[123]\\ [013]}} \bigoplus_{\substack{[14],[134]\\ [014],[145]}} \bigoplus_{\substack{[24],[234]\\ [124],[245]}} V^+(1234) \otimes V^+(1245) \otimes V^+(2345) \otimes V^+(0134) \otimes V^+(0145) \otimes V^+(0123)$$

$$\longrightarrow \bigoplus_{[025],[035]} V^+(0125) \otimes V^+(0235) \otimes V^+(0345).$$

The full expression, including direct sum symbols, for the left-hand side and right-hand side of equation (B3-3) are given as the composite of the rightmost column of vertical arrows in Figure 2.15 and Figure 2.16, respectively. In these figures, the vector spaces $V^+(ijkl)$ are abbreviated as $[ijkl]$ and the parts of the domain (codomain) on which the involved linear maps act non-trivially are underlined (overlined).

Consider the diagram in Figure 2.15. Each of the triangles in the right column commutes by Corollary 2.3.19. The squares in the left column can be shown to commute by explicitly comparing the graphical expressions of the involved maps. A similar analysis shows that the diagram in Figure 2.16 is commutative. It follows

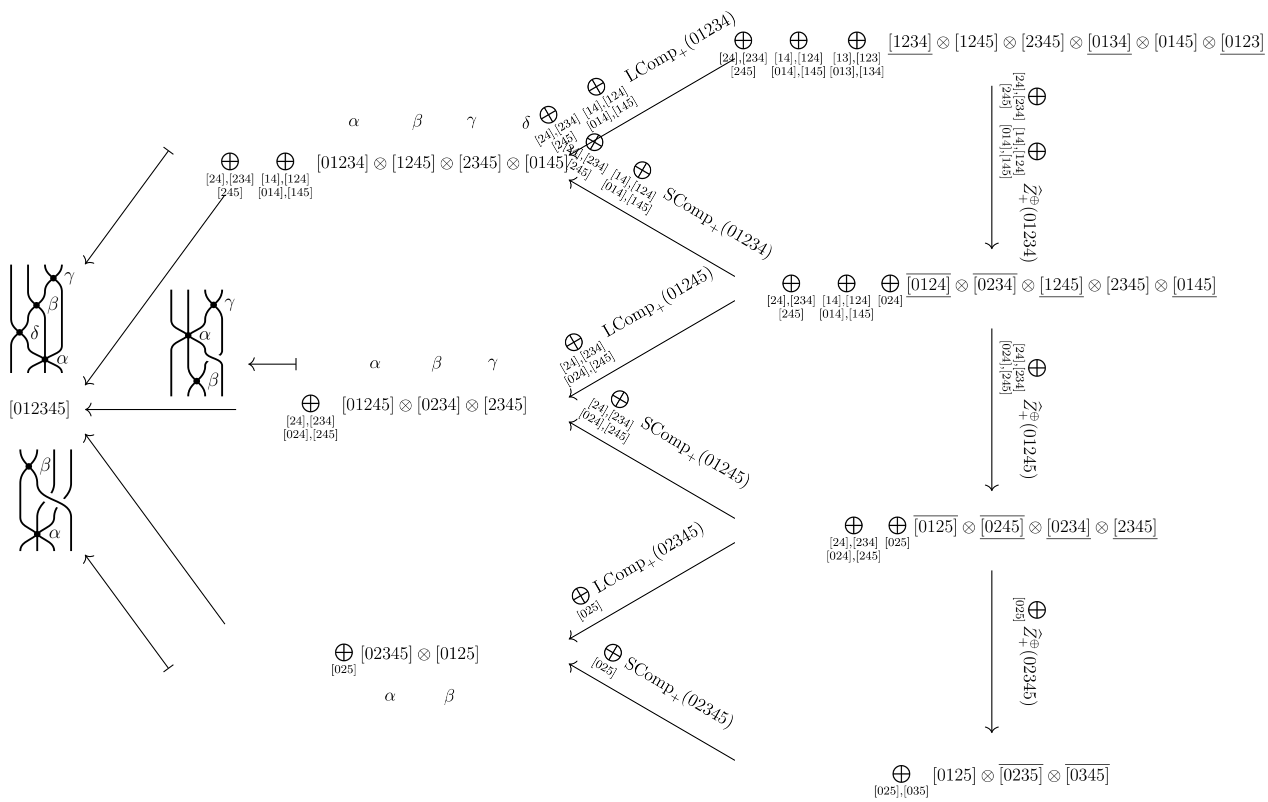


Figure 2.16: The right hand side of the equation in the proof of Lemma 2.3.14.

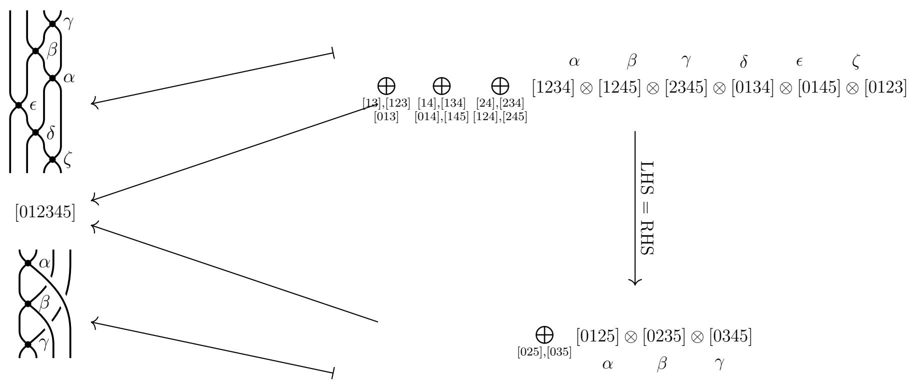


Figure 2.17: Comparing the left and right hand side of the equation in the proof of Lemma 2.3.14.

from an explicit comparison of the graphical expressions of the involved maps that the composites from the top-right to the left in Figures 2.15 and 2.16 coincide. Similarly, the composites from the bottom-right to the left coincide. Hence, these composites fit into a commutative diagram, depicted in Figure 2.17, in which the vertical map can either be the composite of the rightmost column of maps in Figure 2.15 or in Figure 2.16. The bottom map of Figure 2.17 is a composite of invertible linear maps (which again follows from local semisimplicity, analogously to the proof of Lemma 2.3.17) and is therefore invertible. Thus, the composites of the rightmost column of maps in Figure 2.15 and 2.16 coincide, proving Lemma 2.3.14. □

Remark 2.3.21 ((3, 3)-bistellar move as nonabelian 4-cocycle relation)*.* The (3, 3)-bistellar move essentially encodes the higher associativity equation fulfilled by the pentagonator of a monoidal 2-category. Accordingly, the proof of Lemma 2.3.14 proceeds by proving that two large graphical expressions are isotopic, making crucial use of the Gray monoid axioms (see Definition 1.3.1) which we use here to model monoidal 2-categories.

Invariance under the (2, 4)*-bistellar move.* Next, we prove Lemma 2.3.15, showing invariance of the state sum under the (2, 4)-bistellar move.

Proof of Lemma 2.3.15. Expressed in terms of the normalized maps $\widehat{Z}_+$ and $\widehat{Z}_-$, the equation in Lemma 2.3.15 becomes

$$\widehat{Z}_+(01235)\widehat{Z}_+(01345) = \sum_{\substack{[24],[024],[245],\\ [234],[124]}} \widehat{Z}_+(02345)\widehat{Z}_+(01245)\widehat{Z}_+(01234)\widehat{Z}_-(12345).$$

Rewritten in terms of the direct sum maps $\widehat{Z}_+^{\oplus}$ and $\widehat{Z}_-^{\oplus}$, and again omitting all direct sum symbols, this becomes the following equation

$$\widehat{Z}_+^{\oplus}(01235)\widehat{Z}_+^{\oplus}(01345) = \widehat{Z}_+^{\oplus}(02345)\widehat{Z}_+^{\oplus}(01245)\widehat{Z}_+^{\oplus}(01234)\widehat{Z}_-^{\oplus}(12345) \tag{B2-4}$$

between linear maps

$$\bigoplus_{\substack{[14],[134]\\ [014],[145]}} \bigoplus_{\substack{[13],[123]\\ [013],[135]}} V^+(0134) \otimes \otimes V^+(0145) \otimes V^+(1345) \otimes V^+(0123) \otimes V^+(1235)$$
$$\longrightarrow \bigoplus_{[035],[025]} V^+(0345) \otimes V^+(0235) \otimes V^+(0125).$$

Using equation (B3-3), this is equivalent to the following equation:

$$\widehat{Z}_+^{\oplus}(01235)\widehat{Z}_+^{\oplus}(01345) = \widehat{Z}_+^{\oplus}(01235)\widehat{Z}_+^{\oplus}(01345)\widehat{Z}_+^{\oplus}(12345)\widehat{Z}_-^{\oplus}(12345)$$

The lemma is therefore a direct consequence of Corollary 2.3.20. □

Invariance under the $(1,5)$*-bistellar move.* Lastly, we prove Lemma 2.3.16, showing invariance of the state sum under the $(1,5)$-bistellar move.

Proof of Lemma 2.3.16. Expressed in terms of the normalized $\widehat{Z}_+$ and $\widehat{Z}_-$, the equation in Lemma 2.3.16 becomes the following:

$$\widehat{Z}_+(01235) = \dim(\mathcal{C})^{-1} \sum_{\substack{[ij],0\le i<j\le 5\\ i=4 \text{ or } j=4}} \sum_{\substack{[ijk],0\le i<j<k\le 5\\ j=4 \text{ or } k=4}} \frac{\mathrm{d}([034])\,\mathrm{d}([045])\,\mathrm{d}([345])}{\mathrm{d}([035])\,\mathrm{d}([04])\,\mathrm{d}([34])\,\mathrm{d}([45])}$$
$$\mathrm{Tr}_{V^+(0345)}\left(\widehat{Z}_+(02345)\widehat{Z}_+(01245)\widehat{Z}_+(01234)\widehat{Z}_-(12345)\widehat{Z}_-(01345)\right)$$

Rewritten in terms of the direct sum maps $\widehat{Z}_+^{\oplus}$ and $\widehat{Z}_-^{\oplus}$, again omitting direct sum symbols, this becomes the equation

$$\widehat{Z}_+^{\oplus}(01235) = \dim(\mathcal{C})^{-1} \sum_{[04],[34],[45]} \sum_{[034],[045],[345]} \frac{\mathrm{d}([034])\,\mathrm{d}([045])\,\mathrm{d}([345])}{\mathrm{d}([035])\,\mathrm{d}([04])\,\mathrm{d}([34])\,\mathrm{d}([45])}$$
$$\mathrm{Tr}_{V^+(0345)}\left(\widehat{Z}_+^{\oplus}(02345)\widehat{Z}_+^{\oplus}(01245)\widehat{Z}_+^{\oplus}(01234)\widehat{Z}_-^{\oplus}(12345)\widehat{Z}_-^{\oplus}(01345)\right)$$

between linear maps

$$\bigoplus_{\substack{[13],[123]\\ [013],[135]}} \bigoplus_{[035]} V^+(0123)\otimes V^+(0135)\otimes V^+(1235) \longrightarrow \bigoplus_{[025],[035]} V^+(0235)\otimes V^+(0125).$$

Using equation (B2-4) and Corollary 2.3.20, this can be simplified as follows:

$$\widehat{Z}_+^{\oplus}(01235) = \dim(\mathcal{C})^{-1} \sum_{[04],[34],[45]} \sum_{[034],[045],[345]} \frac{\mathrm{d}([034])\,\mathrm{d}([045])\,\mathrm{d}([345])}{\mathrm{d}([035])\,\mathrm{d}([04])\,\mathrm{d}([34])\,\mathrm{d}([45])} \mathrm{Tr}_{V^+(0345)}\left(\widehat{Z}_+^{\oplus}(01235)\right)$$

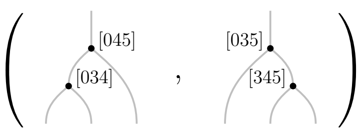
Since the 3-simplex $\langle 0345\rangle$ is not in the boundary of the 4-simplex $\langle 01235\rangle$, it follows that

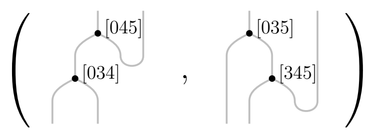
$$\mathrm{Tr}_{V^+(0345)}\left(\widehat{Z}_+^{\oplus}(01235)\right) = \dim(V^+(0345))\widehat{Z}_+^{\oplus}(01235).$$

Hence, to prove Lemma 2.3.16, it suffice to prove the following:

$$\dim(\mathcal{C}) = \sum_{[04],[34],[45]} \sum_{[034],[045],[345]} \frac{\mathrm{d}([034])\,\mathrm{d}([045])\,\mathrm{d}([345])}{\mathrm{d}([035])\,\mathrm{d}([04])\,\mathrm{d}([34])\,\mathrm{d}([45])} \dim(V_+(0345))$$

Observing that

$V^+(0345) = \mathrm{Hom}_{\mathcal{C}}$ 

$\cong \mathrm{Hom}_{\mathcal{C}}$ 

,

it therefore follows from Corollary C.6, that

$$\sum_{[04],[034],[045]} \dim(V_+(0345)) \frac{\mathrm{d}([034])\,\mathrm{d}([045])}{\mathrm{d}([04])} = \mathrm{d}([0(345)])\,.$$

Since [35] is a simple object, it follows from Proposition C.1 that

$$\mathrm{d}([0(345)]) = \frac{\mathrm{d}([345])\,\mathrm{d}([035])}{\dim([35])}.$$

The desired equation therefore simplifies to

$$\dim(\mathcal{C}) = \sum_{[34],[45],[345]} \frac{\mathrm{d}([345])^2}{\dim([35])\mathrm{d}([34])\,\mathrm{d}([45])}.$$

This equality is proven in Corollary C.5. □

These lemmas can now be assembled into a proof of Lemma 2.3.11, proving that the state sum is invariant under bistellar moves, and hence an invariant of singular piecewise linear 4-manifolds.

Proof of Lemma 2.3.11. Combining Lemmas 2.3.16, 2.3.15 and 2.3.14 proves Lemma 2.3.11. □

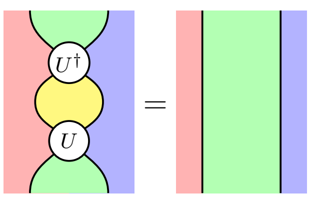
Chapter 3

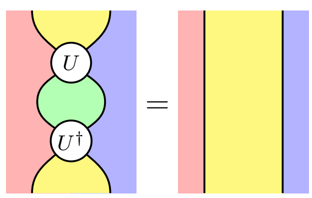
Biunitary constructions in quantum information

In this chaper, based on [RV19b], we present an infinite number of construction schemes involving unitary error bases, Hadamard matrices, quantum Latin squares and controlled families, many of which have not previously been described. Our results rely on biunitary connections, algebraic objects which play a central role in the theory of planar algebras. They have an attractive graphical calculus which allows simple correctness proofs for the constructions we present. We apply these techniques to construct a unitary error basis that cannot be built using any previously known method.

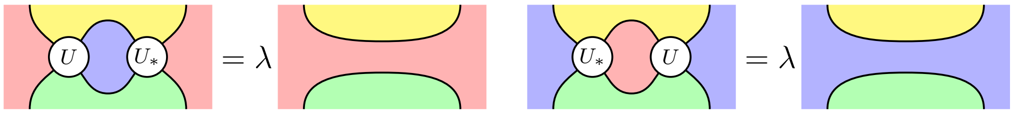
3.1 Introduction

Biunitary connections (or simply *biunitaries*) were introduced by Ocneanu [Ocn89] in 1989 as a central tool in the study and classification of subfactors. Here, we use an approach to biunitaries developed by Jones and others [Jon99, JMS13, MP14] within the theory of planar algebras, which studies the linear representation theory of algebraic structures in the plane. We can describe a biunitary informally as a 2-morphism in a 2-category with two inputs and two outputs, drawn below and above the vertex respectively, which is *vertically unitary* (3.1), and which is *horizontally unitary* up to a scalar factor λ (3.2):





(3.1)



(3.2)

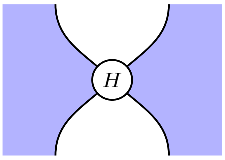
Working in the 2-category of 2-Hilbert spaces, diagrams of this sort represent simple linear algebra data: regions are labeled by indexing sets, and wires and vertices are labelled by indexed families of finite-dimensional Hilbert spaces and linear maps, respectively. Blank regions correspond to the trivial indexing set. In concrete terms, a biunitary therefore comprises a family of linear maps satisfying some algebraic properties.

The *type* of a biunitary is the shading pattern which surrounds the vertex. We show in Section 3.2 that a variety of structures in quantum information theory correspond exactly to biunitaries of particular types. Some important examples are given in Figure 3.1.[1] For example, complex Hadamard matrices and unitary error bases provide the mathematical foundation for an extremely rich variety of quantum computational phenomena, amongst them the study of mutually unbiased bases, quantum key distribution, quantum teleportation, dense coding and quantum error correction [DEBŻ10, Wer01, Sho96, KR03, Kni96b]. Nevertheless their general structure is notoriously difficult to understand; in dimension n, Hadamard matrices have only been classified up to $n = 5$ [Szö11, TŻ06], and the general structure of unitary error bases is virtually unknown for $n > 2$. Quantum Latin squares have been introduced much more recently [MV16, BN07, Mus17], generalizing classical Latin squares which have a wide range of applications in classical and quantum information [Sha49, MW15, BN17].

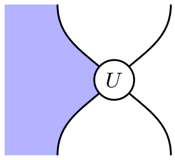
In the lower-right image of Figure 3.1, we see that the notation is 3-dimensional (that is, we use the monoidal structure of the 2-category 2Hilb), with the blue sheet lying beneath the yellow sheet; the colours do not convey mathematical information, but rather make the geometry easier to understand. Rotations by a quarter-turn, and reflections about the horizontal or vertical axes, preserve the given interpretations in terms of quantum structures.

Some of these characterizations are already known: Complex Hadamard matrices were characterized by Jones as biunitaries with alternating shaded and unshaded regions [Jon99], and unitary error bases were characterized by Vicary as biunitaries with one shaded and three unshaded regions [Vic12b, Vic12a]. Here we show that quantum Latin squares can be characterized as biunitaries with two adjacent shaded

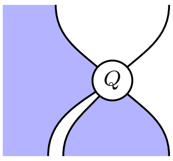
[1]Note that some of the inputs or outputs of the biunitary may in general be composite wires. For example, in Figure 3.1(c) the first input is composite, and in Figure 3.1(d) the first input and second output are composite.



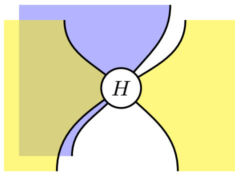
(a) Hadamard matrix (Had)[2]



(b) Unitary error basis (UEB)[3]



(c) Quantum Latin square (QLS)[4]



(d) Controlled family of Hadamards (Had*)

Figure 3.1: Biunitary characterizations of quantum structures.

regions and two adjacent unshaded regions. We also show that controlled families can be described by adding an additional shaded region in a certain way; in Figure 3.1, we illustrate one application of this idea to give a biunitary characterization of a controlled family of Hadamard matrices.

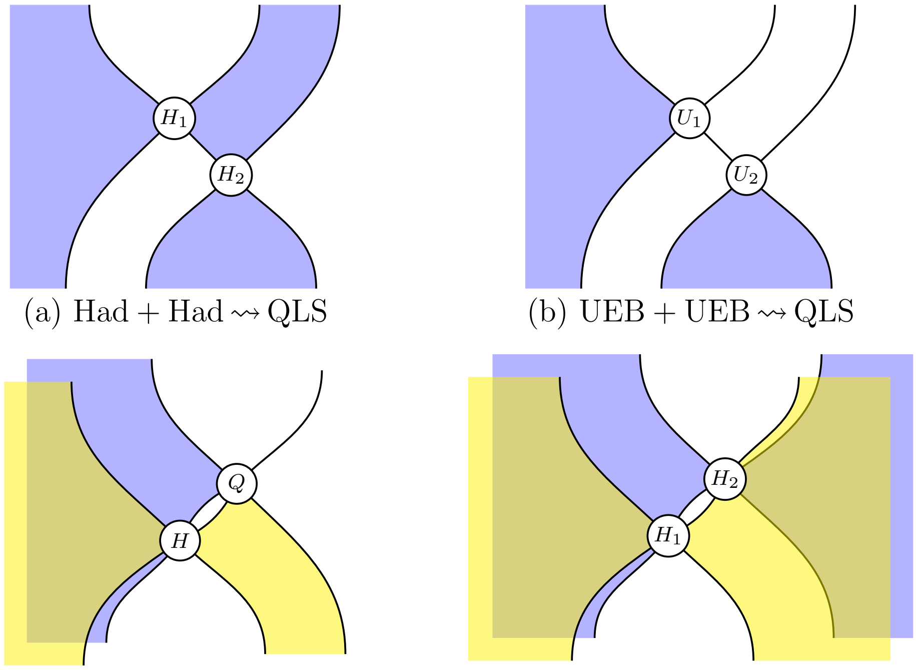
Composing biunitaries

Our main results in this chapter are based on the simple fact that the *diagonal* composite of two biunitaries is again biunitary. We show in Section 3.3 that, given the description of quantum combinatorial structures in terms of biunitaries as summarized above, one can immediately write down a large number of schemes for the construction of certain quantum structures from others. We give some examples in Figure 3.2; note that the biunitaries are connected diagonally in each case, as required.

[2] A (complex) *Hadamard matrix* is a square complex matrix with entries of modulus 1, which is proportional to a unitary matrix. Fundamental structures in quantum information, they are central in the theories of mutually unbiased bases, quantum key distribution, and other phenomena [DEBŻ10].

[3] A *unitary error basis* is a basis of unitary operators on a finite-dimensional Hilbert space, orthogonal with respect to the trace inner product. They provide the basic data for quantum teleportation, dense coding and error correction procedures [Wer01, Kni96b, Sho96].

[4] A *quantum Latin square* [MV16] is a square grid of vectors in a finite-dimensional Hilbert space, such that every row and every column is an orthonormal basis. They are quantum generalizations of classical Latin squares.



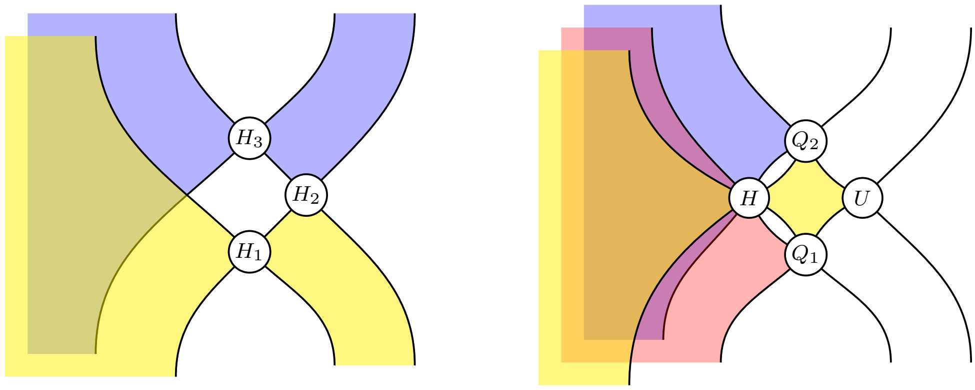
(a) Had + Had $\rightsquigarrow$ QLS (b) UEB + UEB $\rightsquigarrow$ QLS

(c) Had* + QLS $\rightsquigarrow$ UEB (d) Had* + Had* $\rightsquigarrow$ Had

Figure 3.2: Some biunitary composites of arity 2.

We explain each of these constructions briefly. Figure 3.2(a) gives a way to combine two Hadamard matrices to produce a quantum Latin square, generalizing a known construction.[5] (Note that the wires terminating near the upper-right of Figure 3.2(a) are interpreted as a *single* composite wire for the purpose of identifying it as having the basic quantum Latin square type of Figure 3.1, a method that we use repeatedly, and motivate formally with bracketings in Theorem 3.3.2.) Figure 3.2(b) describes a procedure for combining two unitary error bases to yield a quantum Latin square, a construction we believe to be new. Figure 3.2(c) combines a controlled family of Hadamard matrices and a quantum Latin square to give a unitary error basis, recovering the quantum shift-and-multiply construction [MV16, Def 18]. In Figure 3.2(d), two families of Hadamard matrices are combined to produce a single Hadamard matrix, recovering a construction of Hosoya and Suzuki [HS03, Sec 1] which generalizes a construction of Diţă [Diţ04, Sec 4]. These constructions can of course be iterated; for example, combining the constructions of Figures 3.2(a) and 3.2(c) gives a way to combine a controlled family of Hadamard matrices and two further Hadamard matrices to produce a single unitary error basis, again a new construction.

[5] When both Hadamard matrices are the same, this agrees with a known construction of a quantum Latin square from a single Hadamard matrix [MV16, Def 10].



(a) Had + Had + Had ⇝ UEB (b) Had** + QLS + QLS + UEB ⇝ UEB

Figure 3.3: Some biunitary composites of arities 3 and 4.

In all these cases, correctness of the construction follows immediately from the type-theoretic structure (that is, the shading pattern) of the diagram, relying only on diagonality of the composition; no further details need to be checked. Our approach therefore offers advantages even for those constructions that are already known, since the traditional proofs of correctness are nontrivial. To emphasize this point we compare our graphical techniques to traditional methods, in which constructions are defined using tensor notation. For example, the construction of Figure 3.2(c) would traditionally be written as follows [MV16, Def 18], where $U_{ab,c,d}$ is the (c,d)th matrix entry of the (a,b)th element of the unitary error basis, $Q_{b,d,c}$ is the coefficent of $|c\rangle$ in the (b,d)th position of the quantum Latin square, and $H^b_{a,d}$ is the (a,d)th coefficient of the bth Hadamard matrix:

$$U_{ab,c,d} := H^b_{a,d}\, Q_{b,d,c} \tag{3.3}$$

It is not trivial to write down correct expressions of this form, and to show that this indeed defines a unitary error basis requires a calculation of several lines [MV16, Thm 20] that invokes the distinct algebraic properties of the tensors $Q_{b,d,c}$ and $H^b_{a,d}$. In contrast, in our new approach, it would be easy to discover this construction by considering all ways the basic components can be diagonally composed; correctness is immediate, and all algebraic properties are subsumed by the single concept of biunitarity. Nonetheless, expression (3.3) can be immediately read off from the form of the biunitary composite.

Higher-arity constructions can also be described, such as those given in Figure 3.3. Both of these we believe to be new. In Figure 3.3(a), arising as a consequence of the constructions of Figures 3.2(a) and 3.2(c), three Hadamard matrices combine to produce a unitary error basis, an elegant construction which we believe to be new.[6] In Figure 3.3(b), which does not arise as a consequence of lower-arity constructions, we combine a double-controlled family of Hadamard matrices (H), two quantum Latin squares (Q_1, Q_2) and a unitary error basis (U) to produce a new unitary error basis. While the first example is simple and elegant, the second example is indicative of the more complex constructions this technique can produce. Further complex examples are given in Figures 3.8, 3.9 and 3.10.

For unitary error bases, we illustrate all constructions of arities 2 and 3 that arise from our methods, and we give examples of constructions of arities 4 and 8. Furthermore, in Section 3.3.5 we show that our methods give rise to an infinite family of logically independent constructions, none of which factor through any simpler construction between Hadamard matrices, unitary error bases, quantum Latin squares and controlled families thereof.

In Section 3.4 we consider the problem of equivalence from our new perspective. Hadamard matrices, unitary error bases, quantum Latin squares and controlled families all come with standard notions of equivalence. We give a new generic definition of equivalence for biunitaries, broader than the one used traditionally in the planar algebra literature, and show that it recovers precisely these usual notions of equivalence for each of the quantum structures we consider.

Finally, in Section 3.5 we use the 4-fold composite of Figure 3.3(b) to produce a unitary error basis on an 8-dimensional Hilbert space. We show that it cannot be produced by the two known UEB construction methods—algebraic, and quantum shift-and-multiply—even up to equivalence. This is a proof of principle that the biunitary methods we propose can give rise to genuinely new quantum structures.

Significance

By unifying quantum structures as special cases of the single notion of biunitary, and providing simple type-theoretical tools to understand the intricate interplay between them, we unify several already-known and seemingly-unrelated constructions [MV16, Wer01, HS03, Diţ04, BN07], uncover an infinite number of new constructions, and produce novel, concrete examples. These new tools may lead to further progress in questions of classification and applications of Hadamard matrices, unitary error bases and quantum Latin squares, and perhaps move us closer to full classification results for these important structures.

[6] When all three Hadamard matrices are the same, this agrees with a known construction of a unitary error basis from a single Hadamard matrix [MV16, Def 33] which we believe to be folklore.

As well as producing a number of new constructions, our methods encompass and unify the following constructions from the literature.

- The *Hadamard method* [MV16, Def 33], believed to be folklore, which produces a unitary error basis from a single Hadamard matrix. In expression (3.23) we give a biunitary presentation of a new generalization, in which three Hadamard matrices produce a unitary error basis.
- The method given in [BN07, Def 2.3] and [MV16, Def 10], which produces a quantum Latin square from a single Hadamard matrix. In Figure 3.6(a) we give a biunitary presentation of a new generalization, in which two Hadamard matrices produce a quantum Latin square.
- Werner's *shift-and-multiply construction* [Wer01] which produces a unitary error basis from a family of Hadamard matrices and a Latin square. This is a special case of the quantum shift-and-multiply construction discussed below.
- The *quantum shift-and-multiply construction* due to Musto and Vicary [MV16, Def 18] which produces a unitary error basis from a family of Hadamard matrices and a quantum Latin square. We give a biunitary description in Figure 3.7(b).
- *Diţă's construction* [Diţ04, Sec 4], which produces a Hadamard matrix from a Hadamard matrix and a family of Hadamard matrices. This method sees wide use in the literature [Diţ04, TŻ06, MRS07, Nic10], and is a special case of Hosoya's and Suzuki's construction, described below. We give a biunitary description in Figure 3.6(d).
- *Hosoya's and Suzuki's construction* [HS03, Sec 1], which produces a Hadamard matrix from two families of Hadamard matrices. We give a biunitary description in Figure 3.6(c).

There are many known constructions which are beyond our methods. For unitary error bases, we do not know a biunitary characterization of Knill's algebraic construction [Kni96a]. For Hadamard matrices, an analogue of Knill's construction are the Fourier matrices arising from finite abelian groups. Other examples include Petrescu's construction of continuous families of Hadamard matrices in prime dimension [Pet97], Wocjan's and Beth's construction [WB04] and its generalization by Musto [Mus17], or several other less-general constructions which only work in specific dimensions [Haa96, Szö12, Szö11, MRS07]. In all of these cases, the methods are not purely compositional; they make use of some additional group-theoretic or algebraic structure which is currently out of reach of the biunitary approach.

Notation and conventions

In contrast to the notation used in Chapter 2, we will from now on denote the n-element set $\{1, \ldots, n\}$ by $[n]$. The letters $a, b, d, e, f, g, h, i, j, k, r, s$ are used to denote indices, the letters n, m, p, q are used to denote dimensions. We use the following shorthand notations to refer to sets of quantum structures:

- UEB_n is the set of n-dimensional unitary error bases;
- QLS_n is the set of n-dimensional quantum Latin squares;
- Had_n is the set of n-dimensional Hadamard matrices;
- For $\mathrm{X} \in \{\mathrm{UEB}_n, \mathrm{QLS}_n, \mathrm{Had}_n\}$, $\mathrm{X}^{p_1,\ldots,p_k}$ is the set of lists of quantum structures of type X controlled by indices in $[p_1]$, $[p_2]$, $\ldots, [p_k]$.

For example, $\mathrm{UEB}^{n,p}_{n^2m}$ is the set of lists of n^2m-dimensional unitary error bases, controlled by indices taking values in $[n]$ and $[p]$.

Outline

Section 3.2 introduces quantum structures and their characterization in terms of biunitaries; Section 3.2.1 defines the 2-category 2Hilb, gives an elementary description of its graphical calculus, and defines biunitarity. Section 3.2.2 characterizes certain quantum structures — Hadamard matrices, unitary error bases, quantum Latin squares, and controlled families thereof — in terms of biunitaries of certain type.

Section 3.3 concerns the composite of biunitaries; in Section 3.3.1, the diagonal composite of biunitaries is defined and proven to be biunitary. In Section 3.3.2, 3.3.3 and 3.3.4, various new constructions of unitary error bases arising as diagonal composites of biunitaries are discussed. In Section 3.3.5 it is shown that this method leads to infinitely many conceptually distinct constructions.

Section 3.4 discusses a notion of equivalence of biunitaries which generalizes the usual notions of equivalence of Hadamard matrices and unitary error bases and investigates how this notion of equivalence interacts with our constructions.

In Section 3.5, we use biunitary composition to build a new 8-dimensional unitary error basis which we prove cannot be obtained from any of the previously known construction techniques. The 64 matrices comprising this unitary error basis are explicitly listed in Appendix E.

3.2 Biunitarity

In Section 3.2.1 we introduce our formalism, and give the definition of biunitarity. In Section 3.2.2 we recall the biunitary characterizations of Hadamard matrices and unitary error bases, and give new biunitary characterizations of quantum Latin squares and controlled families.

3.2.1 Mathematical foundations

In this chapter, we use the graphical calculus of monoidal 2-categories (sketched in Section I.1 and introduced in Section 1.3.1) applied to the 2-category 2Hilb of finite-dimensional 2-Hilbert spaces [Bae97]. This 2-category may be understood as a 'Hilbert', '$*$' or 'dagger' version of Kapranov and Voevodsky's 2-category of 2-vector spaces (see Section I.2). In particular, it can be described as follows:

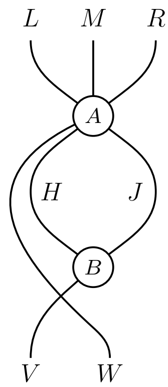
- objects are natural numbers $n, m, \ldots$;
- 1-morphisms $n \to m$ are $m \times n$-matrices of finite-dimensional Hilbert spaces;
- 2-morphisms are matrices of linear maps.

Composition of 1-morphisms is given by 'matrix multiplication' of matrices of Hilbert spaces, with addition and multiplication of complex numbers replaced by direct sum and tensor product, respectively. Composition of 2-morphisms is given by componentwise composition of linear maps. The 2-category has a monoidal structure, acting on objects as multiplication, and on 1- and 2-morphisms as the Kronecker product of matrices of Hilbert spaces and linear maps, respectively; this is represented graphically by 'layering' one diagram above another.

Elementary description.

To help the reader understand these concepts, we also give a direct account of the formalism in elementary terms, that can be used without reference to the higher categorical technology. In Figure 3.4 we indicate how to translate between the categorical language presented above and the more elementary language used here.

Recall that in the ordinary graphical calculus of the monoidal category Hilb, wires represent finite-dimensional Hilbert spaces and vertices represent linear maps between them, with wiring diagrams representing composite linear maps. For example, given linear maps $A : W \otimes H \otimes J \to L \otimes M \otimes R$ and $B : V \to H \otimes J$, we can describe a composite linear map $V \otimes W \to L \otimes M \otimes R$ graphically as follows:



The graphical calculus of 2Hilb may be understood as a generalization of this calculus that involves regions, as well as wires and vertices. In this elementary perspective, shaded regions are labeled by *finite sets*, indexed by a parameter; we write $i{:}n$ to indicate that the parameter i varies over the set $[n]$.[7] Wires and vertices now represent *families* of Hilbert spaces and linear maps respectively, indexed by the parameters of all adjoining regions. A composite surface diagram represents a family of composite linear maps, indexed by the parameters of all open regions, with closed

[7]For simplicity we will often omit these labels.

$$\phi : \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \Rightarrow \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \end{pmatrix} \circ \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}$$

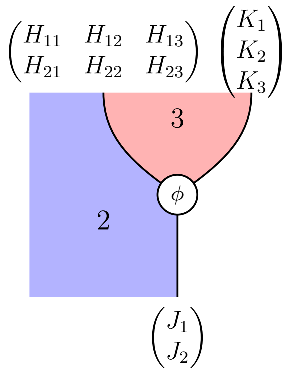
(a) A 2-morphism ϕ

$$\begin{pmatrix} J_1 \xrightarrow{\phi_1} (H_{11} \otimes K_1) \oplus (H_{12} \otimes K_2) \oplus (H_{13} \otimes K_3) \\ J_2 \xrightarrow{\phi_2} (H_{21} \otimes K_1) \oplus (H_{22} \otimes K_2) \oplus (H_{23} \otimes K_3) \end{pmatrix}$$

(b) The 2-morphism ϕ as a matrix of linear maps

$$\phi_{i,j} : J_i \to H_{i,j} \otimes K_j \quad \text{for } i \in [2] \text{ and } j \in [3]$$

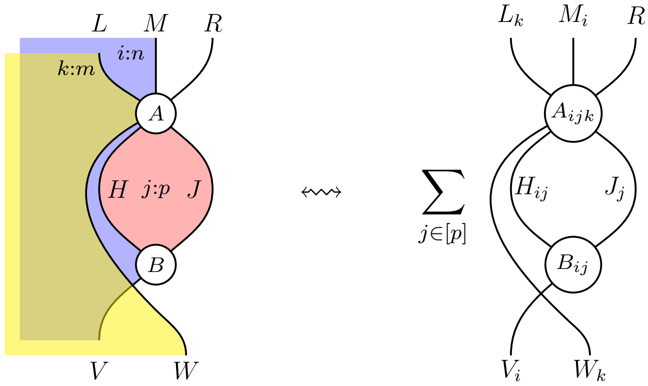
(c) The 2-morphism ϕ as a family of linear maps indexed by its adjacent regions



(d) Graphical representation of the 2-morphism ϕ

Figure 3.4: Translating between equivalent expressions for 2-morphisms.

regions being summed over. This is illustrated by the following example:



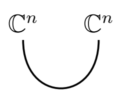
The diagram on the left represents an entire family of composite linear maps. The maps which comprise this family are given by the right-hand diagrams for different values of k and i, which index the open regions. The closed region labeled $j : p$ is summed over.

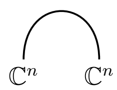
In some situations, particularly when dealing with equations of shaded diagrams with different connectivity, we may need to have multiple parameters $i : n$, $i' : n$ labelling the same region. In this case, we need an auxiliary rule that says the corresponding linear map is zero when $i \neq i'$.

Given this interpretation of diagrams D as families of linear maps D_i, we define two diagrams D, D' to be equal when all the corresponding linear maps D_i, D'_i are equal, and the scalar product λD as the family of linear maps λD_i.

Duality.

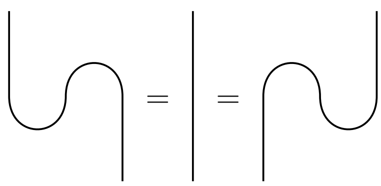
We define the linear maps $\eta : \mathbb{C} \to \mathbb{C}^n \otimes \mathbb{C}^n$ and $\epsilon : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}$ as follows:





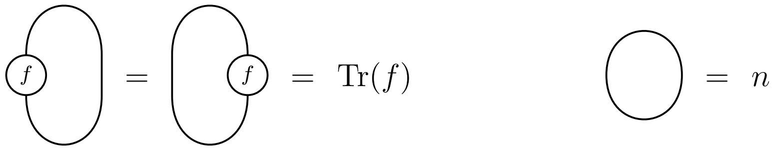
$$\eta :\ 1 \mapsto \sum_i |i\rangle \otimes |i\rangle \qquad\qquad \epsilon :\ |i\rangle \otimes |j\rangle \mapsto \delta_{ij} \tag{3.4}$$

These linear maps are the evaluation and coevaluation morphisms of a duality; the following equations can be demonstrated:



(3.5)

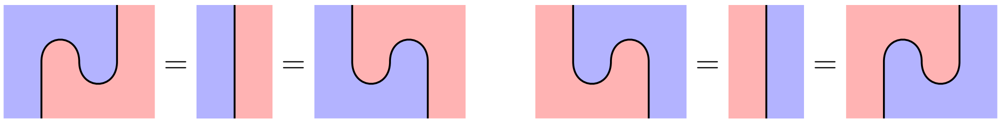
It can easily be verified that for a linear map $f : \mathbb{C}^n \to \mathbb{C}^n$, we have the following:



(3.6)

Since wires in our framework correspond to indexed families of Hilbert spaces, and assuming for simplicity that all Hilbert spaces are chosen to be of the form $\mathbb{C}^n$ for some $n \in \mathbb{N}$, we can introduce the following notation for families of linear maps of the form (3.4):

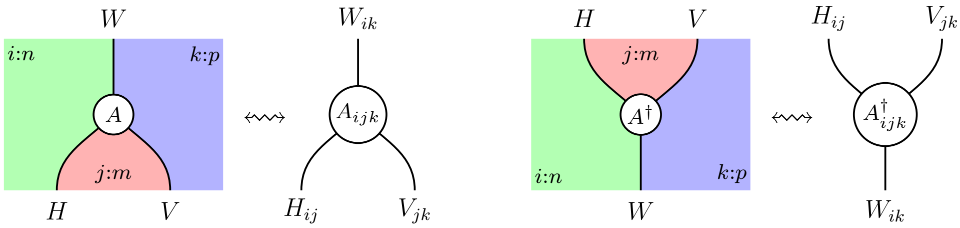
Then the following hold as a direct consequence of equations (3.5):



(3.7)

Dagger structure.

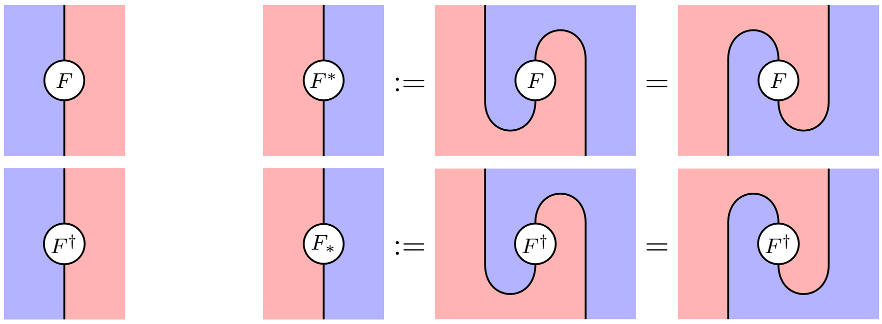
The 2-category 2Hilb is a dagger-2-category [HK16]. Expressed in our elementary terms, this means that given a family of linear maps, its *adjoint* (or *dagger*) is the family consisting of the adjoints of the linear maps:



Graphically, we can think of the adjoint as a reflection about a horizontal axis. This is justified, since the following holds:

(3.8)

In total, every vertex appears in four variants:



(3.9)

The equations on the right-hand sides can be shown to follow from the definitions (3.4).

A dagger structure gives rise to a general notion of unitarity.

Definition 3.2.1. A vertex U is *unitary* when it satisfies the following equations:

Standard boundaries.

In this chapter, we only make use of a restricted portion of this calculus: wires which bound only one shaded region always correspond to the 1-dimensional Hilbert space $\mathbb{C}$ for any value of the controlling parameter. (Wires that do not bound regions may correspond to Hilbert spaces of any finite dimension.) In particular, since they are 1-dimensional, the Hilbert spaces arising from standard boundaries are not depicted in the corresponding family of tensor diagrams:

(3.10)

This means that once the parameter $i : n$ for the region is given, no further labelling is needed for the wire itself.

The following properties may be verified for these standard boundaries:

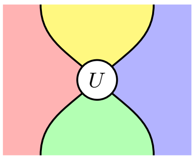
(3.11)

Many definitions and results of this chapter hold more generally, but the main application to constructions of Hadamard matrices, unitary error bases and quantum Latin squares in Sections 3.3.2–3.3.5 only involve this restricted calculus.

Biunitaries.

Having defined our graphical calculus, we now define biunitarity.

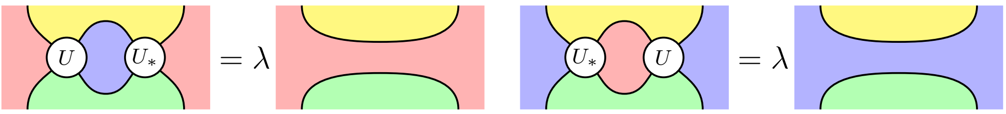
Definition 3.2.2. A *biunitary* is a vertex (or 2-morphism)



(3.12)

which is unitary (3.13), and which also satisfies the following *horizontal unitarity* equations (3.14) for some scalar λ:

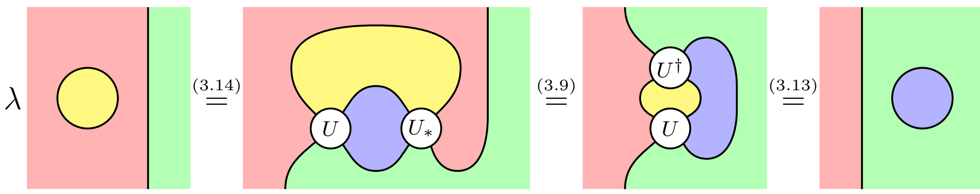
(3.13)



(3.14)

Note that biunitarity depends on a chosen partition of the input and output wires into two parts. Such a partition may not be unique; in particular, every unitary vertex U is biunitary with respect to the following partitions:

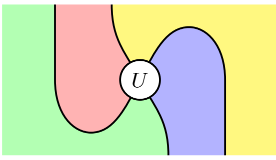
The scalar λ is uniquely determined and can be recovered as a consequence of equations (3.13) and (3.14):

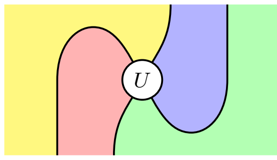


(3.15)

In particular, λ is real and positive. We will usually use the following equivalent formulation of biunitarity.

Definition 3.2.3. The *clockwise* and *anticlockwise quarter rotation* of a vertex U of type (3.12) is given by the following composites, respectively:





Proposition 3.2.4. *Given a vertex U of type* (3.12)*, the following are equivalent:*

1. *U is biunitary;*

2. *U is unitary, and its clockwise quarter rotation is proportional to a unitary;*

3. *U is unitary, and its anticlockwise quarter rotation is proportional to a unitary.*

Furthermore, in cases 2 and 3, the proportionality factor is unique up to a phase and given by a square root of λ.

Proof. The proposition follows straightforwardly from deformations of (3.14). □

Corollary 3.2.5. *Given a biunitary, arbitrary quarter rotations, or reflections about horizontal or vertical axes, are again proportional to biunitaries.*

In particular, as soon as we have characterized specific quantum structures in terms of biunitaries of certain types, we know that rotated and reflected versions of this type also correspond to this quantum structure, possibly after multiplication by a scalar.

3.2.2 Characterizing quantum structures

In this section we recall the biunitary characterizations of Hadamard matrices and unitary error bases, and give new characterizations of quantum Latin squares and controlled families. These results are summarized in Figure 3.5.

Except for Sections 3.3.1, 3.4 and the discussion of controlled families and interchangers in Section 3.2.2, all wires in the following diagrams are either standard boundaries (3.10) or Hilbert spaces that do not bound any region.

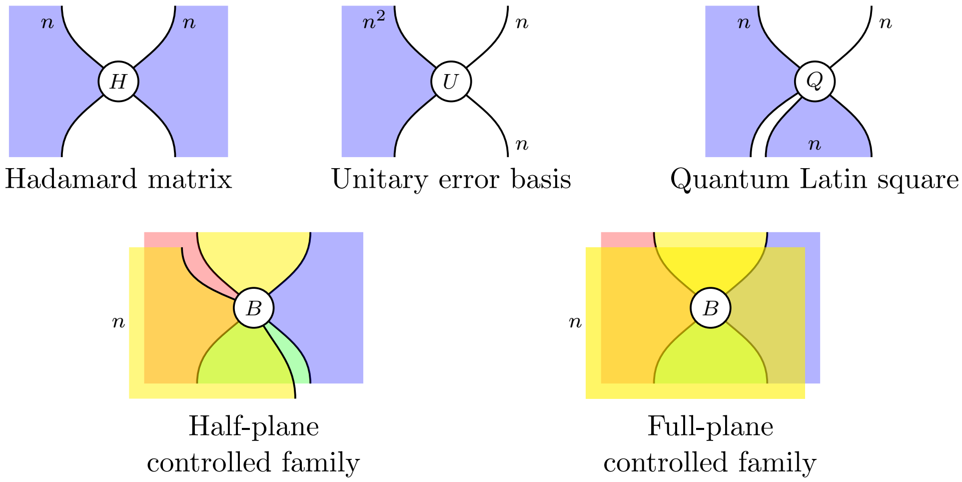


Figure 3.5: Quantum structures and their associated biunitary types.

Dimensional constraints.

For a linear map $f : H \to J$ to be unitary imposes certain algebraic constraints on the dimensions of H and J; namely, $\dim(H) = \dim(J)$. For a vertex of type (3.12) to be biunitary similarly induces certain constraints on the allowed labels for the surrounding regions and wires.

In all cases, these constraints are easily identified and solved for. For example, consider the following vertex U and its clockwise quarter rotation:

Here, n, m and p denote the dimensions of the corresponding region or wire, respectively. For the first of these to be unitary requires that $n = m$, while for the second to be proportional to a unitary requires $p = nm$. By Proposition 3.2.4, for U to be biunitary, we therefore require $(n, m, p) = (n, n, n^2)$, and the space of allowed types is parameterized by a single natural number. In a similar way, for the rest of this section, we will always label biunitaries by their allowed dimensions.

Hadamard matrices.

Hadamard matrices were identified by Jones to be characterized in terms of biunitarity [Jon99]. Complex Hadamard matrices play an important role in mathematical physics and quantum information theory [DEBŻ10]; in particular, they encode the data of a basis which is unbiased with respect to the computational basis.

Definition 3.2.6. A *Hadamard matrix* is a matrix $H \in \mathrm{Mat}_n(\mathbb{C})$ with the following properties, for $i, j \in [n]$:

$$H_{i,j}\overline{H}_{i,j} = 1 \tag{3.16}$$

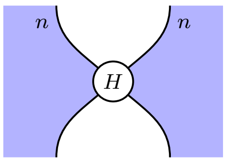
$$\textstyle\sum_k H_{i,k}\overline{H}_{j,k} = \delta_{i,j} n \tag{3.17}$$

$$\textstyle\sum_k \overline{H}_{k,i} H_{k,j} = \delta_{i,j} n \tag{3.18}$$

Properties (3.17) and(3.18) are equivalent, but we include them both for completeness.

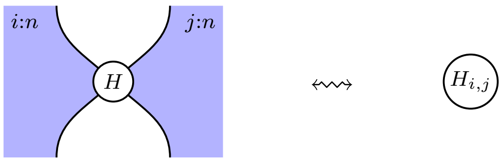
The biunitary characterization of Hadamard matrices is due to Jones in the setting of the spin model planar algebra, which our mathematical setup generalizes. It was shown in [Vic12a, Thm 4.5] that this characterization is equivalent to that of Coecke and Duncan in terms of interacting Frobenius algebras [CD08].

Proposition 3.2.7 ([Jon99, Sec 2.11])**.** *Hadamard matrices of dimension n correspond to biunitaries of the following type:*

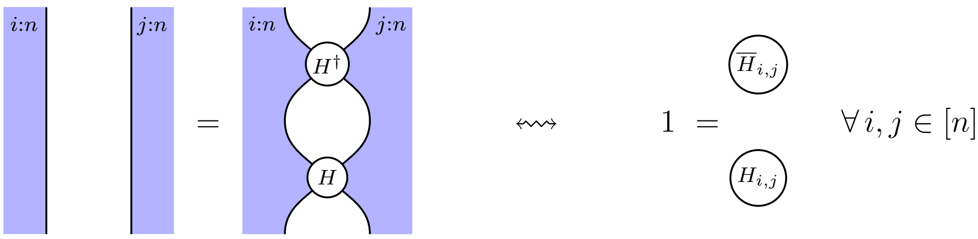


(3.19)

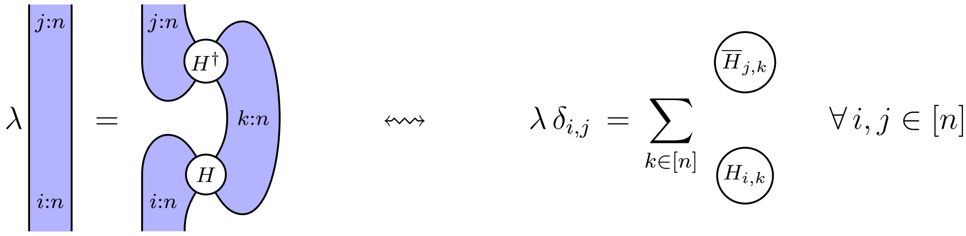
Proof. A vertex of type (3.19) represents a family of scalars $H_{i,j}$ controlled by $i, j \in [n]$:



The first vertical unitarity equation corresponds to the following equality of controlled families:

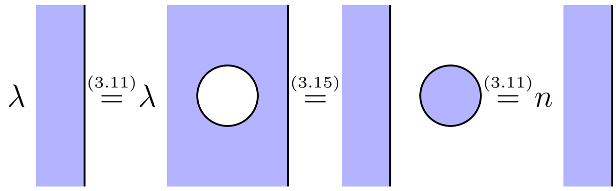


This means that $H_{i,j}\overline{H}_{i,j} = 1$ for all $i, j \in [n]$ which recovers condition (3.16). The other vertical composite gives the same condition. For horizontal unitarity, we consider the following equation:



In other words, $\sum_k H_{i,k}\overline{H}_{j,k} = \lambda\,\delta_{i,j}$ for all $i,j \in [n]$. Together with (3.16) this implies that $\lambda = n$ and recovers condition (3.17). Similarly, condition (3.18) is satisfied just when the other horizontal unitarity composite is satisfied. □

Following the argument (3.15), the scalar $\lambda = n$ could have been recovered as follows:



The same holds for unitary error bases and quantum Latin squares below.

Unitary error bases.

Originally introduced by Knill [Kni96b], unitary error bases are ubiquitous in modern quantum information theory. They lie at the heart of quantum error correcting codes [Sho96] and procedures such as superdense coding and quantum teleportation [Wer01].

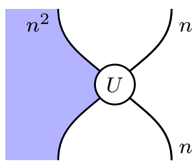
Definition 3.2.8 ([Kni96b])**.** A *unitary error basis* (UEB) on an n-dimensional Hilbert space H is a collection of unitary matrices $\{U_a \in \mathrm{U}(H) \,|\, a \in [n^2]\}$, satisfying the following orthogonality property, for $a, b \in [n^2]$:

$$\mathrm{Tr}(U_a^\dagger U_b) = n\,\delta_{a,b}$$

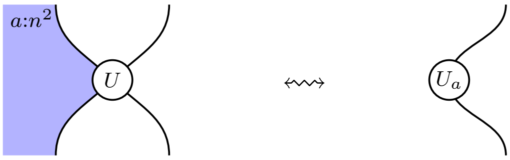
That is, a UEB is an orthogonal basis of the space $\mathrm{End}(H)$ consisting entirely of unitary matrices.

We denote the (i,j)th matrix element of the matrix U_a by $U_{a,i,j} = (U_a)_{i,j} = \langle i|\, U_a \,|j\rangle$.

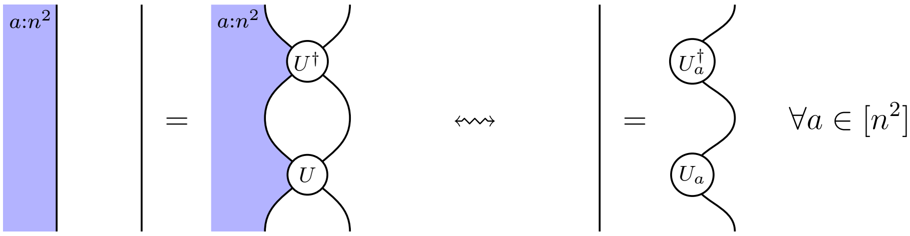
Proposition 3.2.9 ([Vic12a, Thm 4.2])**.** *Unitary error bases on an n-dimensional Hilbert space correspond to biunitaries of the following type:*



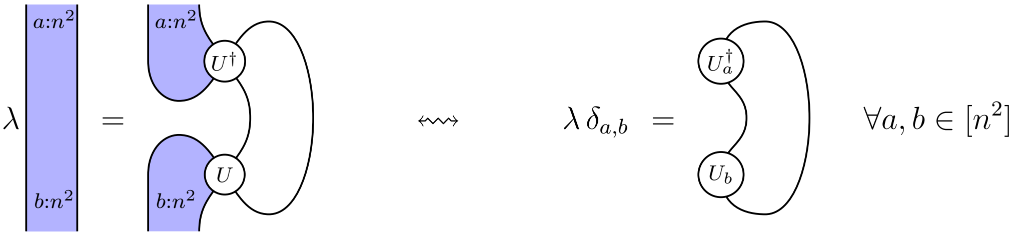
Proof. A vertex of the above type represents a family of linear maps U_a controlled by $a \in [n^2]$:



The first vertical unitarity equation corresponds to the following equality between controlled families:



Together with the other vertical composite[8], this implies that the linear maps U_a are unitary for all $a \in [n^2]$. For horizontal unitarity, we consider the following equation:



By equation (3.6), this means that $\mathrm{Tr}(U_a^\dagger U_b) = \lambda\,\delta_{a,b}$ for all $a, b \in [n^2]$. Since all matrices U_a are unitary, it follows that $\lambda = n$. Together with the other horizontal unitarity condition, this implies that the matrices $\frac{1}{\sqrt{n}} U_a$ form an orthonormal basis of $\mathrm{End}(H)$. □

Quantum Latin squares.

Quantum Latin squares were introduced by Musto and Vicary [MV16] as generalizations of classical Latin squares, with applications to the construction of unitary error bases. Related constructions were also introduced independently by Banica and Nicoară [BN07].

Definition 3.2.10 ([MV16, Def 1])**.** A *quantum Latin square* (QLS) on an n-dimensional Hilbert space H is a square grid of vectors $\{|Q_{a,b}\rangle \in H \mid a, b \in [n]\}$ such that

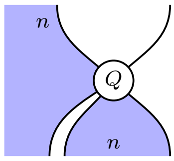
[8] Once we have fixed the dimensional constraints as described at the beginning of subsection 3.2.2, the two conditions on vertical composition (or horizontal composition, respectively) become equivalent. Strictly speaking, we therefore do not need to verify the 'other vertical composite'.

each row $\{|Q_{a,b}\rangle \mid b \in [n]\}$ and each column $\{|Q_{a,b}\rangle \mid a \in [n]\}$ form an orthonormal basis of H; for $a, b, c \in [n]$:

$$\langle Q_{a,b}|Q_{a,c}\rangle = \delta_{b,c} \qquad\qquad \langle Q_{a,c}|Q_{b,c}\rangle = \delta_{a,b}$$

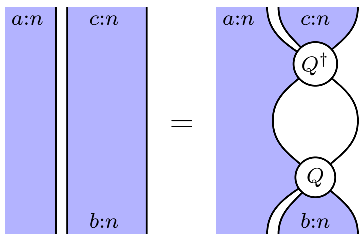
We denote the ith entry of the vector $|Q_{a,b}\rangle$ by $Q_{a,b,i} = \langle i|Q_{a,b}\rangle$.

Proposition 3.2.11. *Quantum Latin squares on an n-dimensional Hilbert space correspond to biunitaries of the following type:*

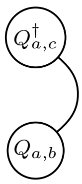


Proof. A vertex of the above type represents a family of vectors $|Q_{a,b}\rangle$ controlled by $a, b \in [n]$:

The first vertical unitarity equation corresponds to the following equality between controlled families:

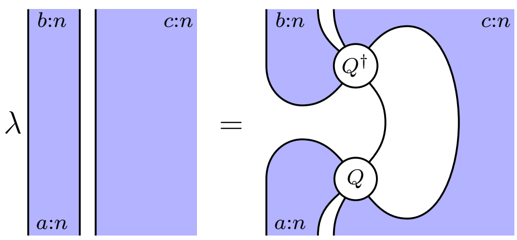


$\rightsquigarrow \qquad \delta_{b,c} \; = $

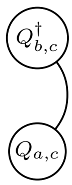


$\forall a, b, c \in [n]$

This means that $\langle Q_{a,c}|Q_{a,b}\rangle = \delta_{b,c}$ for all $a, b, c \in [n]$. Together with the other vertical composite this is equivalent to the fact that the rows $\{|Q_{a,b}\rangle \mid b \in [n]\}$ form orthonormal bases. For horizontal unitarity, we consider the following equation:



$\rightsquigarrow \qquad \lambda\, \delta_{a,b} \; = $



$\forall a, b, c \in [n]$

This means that $\langle Q_{b,c}|Q_{a,c}\rangle = \lambda\, \delta_{a,b}$ for all $a, b, c \in [n]$. Since all vectors $|Q_{a,b}\rangle$ are normalized, it follows that $\lambda = 1$. Together with the other horizontal unitarity condition this is equivalent to the fact that the columns $\{|Q_{a,b}\rangle \mid a \in [n]\}$ are orthonormal bases. □

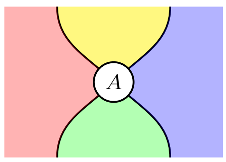
Controlled families.

In quantum information, we often want to describe lists of structures, parameterized by a given index. A standard name for such a list is a controlled family.

Definition 3.2.12. For a given quantum structure X, an *n-controlled family* is an ordered list of n instances of X.

In index notation, we reserve superscript for controlling indices. For example, a controlled family of Hadamard matrices would be written as $H^c_{a,b}$, where c iterates through the controlled family and a and b are the actual indices of the Hadamard matrix H^c.

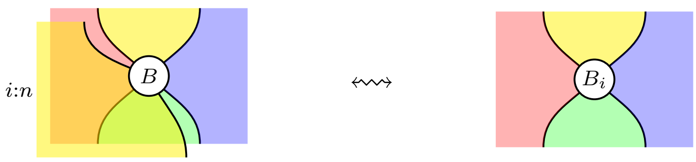
Proposition 3.2.13. *An n-controlled family of biunitaries of type*



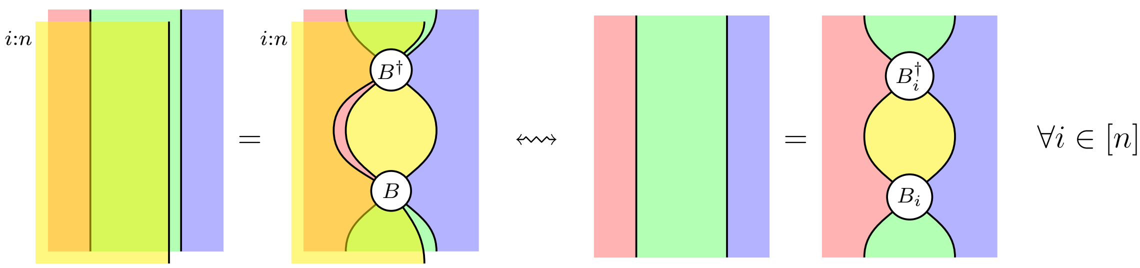
(3.20)

corresponds to a biunitary of the same type with an additional half-plane or full-plane sheet of dimension n attached:

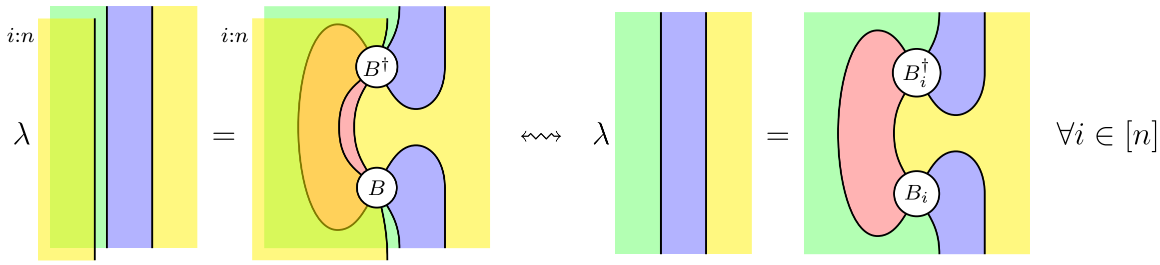
Proof. A half-plane biunitary B corresponds to a family of vertices of type (3.20) controlled by an index $i \in [n]$:



The first unitarity equation amounts to the following equation of controlled families:

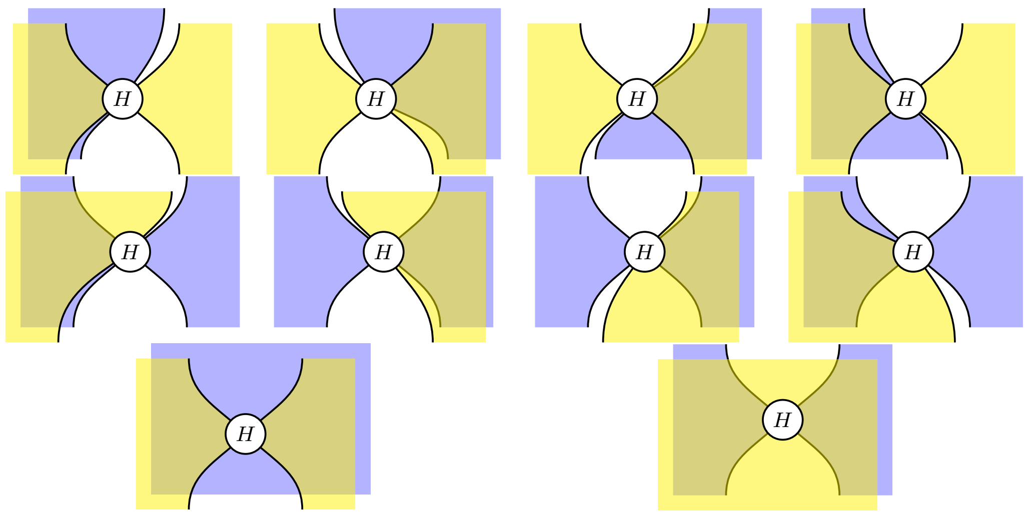


Together with the second vertical unitarity equation, this implies that the vertices B_i are unitary for each $i \in [n]$. For horizontal unitarity, we consider the following equation:



This means that the vertices B_i satisfy the first horizontal unitarity equation for each $i \in [n]$. In a similar way, the second horizontal unitarity equation for B corresponds to the second horizontal unitarity equation for the vertices B_i.
It follows that the half-plane control type corresponds to an indexed family of the underlying biunitary type. The proof for the full-plane control type is similar. □

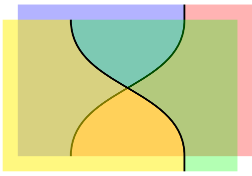
By Lemma 3.2.5, we could have put the half-plane controlling sheet in one of 4 different orientations. Furthermore, it makes no difference if the controlling sheet goes in front or behind. We therefore have 8 different half-plane controls and 2 different full plane controls, which we illustrate here for the case of Hadamard biunitaries:



In our pseudo-3d graphical notation, it can be hard to see if a rear sheet is actually connected to a vertex. In our diagrams, we will use the convention that all sheets drawn beneath a vertex are connected to it.

Interchangers.

The vertex representing the crossing of wires at different depths is called an *interchanger*:



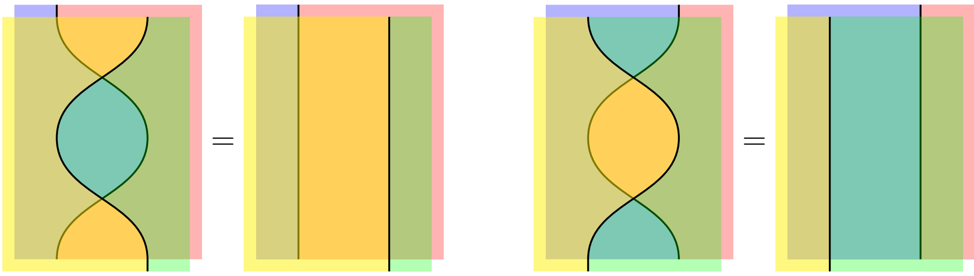
(3.21)

This is given canonically for all index values as the swap map $H \otimes J \to J \otimes H$.

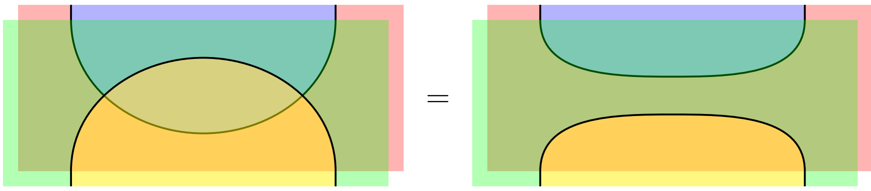
We now show that interchangers are biunitary, with scalar $\lambda = 1$.

Proposition 3.2.14. *The interchanger* (3.21) *is biunitary.*

Proof. Interchangers are unitary, as witnessed by the following equations:



They are also horizontally unitary, and thus biunitary, as witnessed by the following:



The other horizontal unitarity equation follows similarly. □

3.3 Biunitary composition

The results of this section are all corollaries of the following simple idea.

Theorem 3.3.1. *Arbitrary finite diagonal composites of biunitaries are again biunitary.*

Since we have established in Section 3.2 that biunitaries of various types correspond to different quantum structures, Theorem 3.3.1 suggests the possibility of building new quantum structures from existing ones by diagonal composition. In Section 3.3.1, we demonstrate that binary diagonal composites of biunitaries are again biunitary. We then consider the problem of diagonally composing the biunitaries corresponding to Hadamard matrices, quantum Latin squares, unitary error bases and controlled families to produce other such structures, investigating binary composites in Section 3.3.2, ternary composites in Section 3.3.3, and higher composites in Section 3.3.4. In Section 3.3.5, we argue that our methods gives rise to an infinite number of genuinely distinct constructions.

A *planar tiling* [DP93] is a partition of a rectangle by a finite number of rectangles, and gives the correct structure to describe the possible forms of an arbitrary finite diagonal composite of biunitaries.[9] This notation closely resembles Ocneanu's original paragroup notation for biunitaries and their composition [Ocn89]. The following are examples of planar tilings, which we always draw in a diagonal fashion to better match the biunitary pictures:

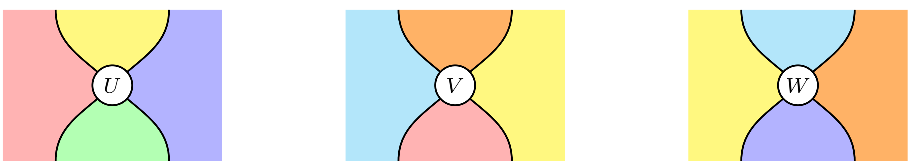
(3.22)

For the more complicated biunitary composites in Figure 3.8, we give the corresponding planar tiling to make the structure clear.

3.3.1 Diagonal composition

It is straightforward to see that the diagonal composite of two biunitaries is again biunitary.

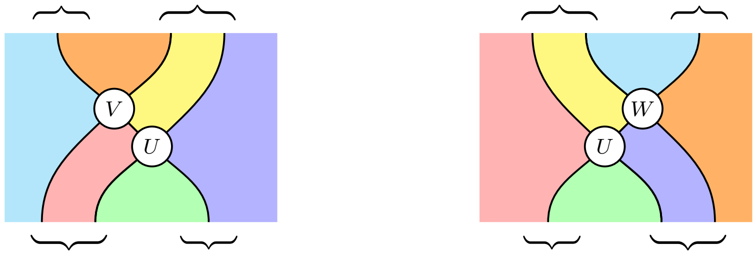
Theorem 3.3.2. *Let U, V and W be biunitaries of the following types:*



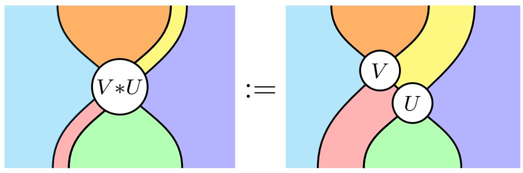
Then the following diagonal composites are biunitary, with respect to the indicated

[9] In particular, this implies that biunitaries can be organized as a double category [DP93].

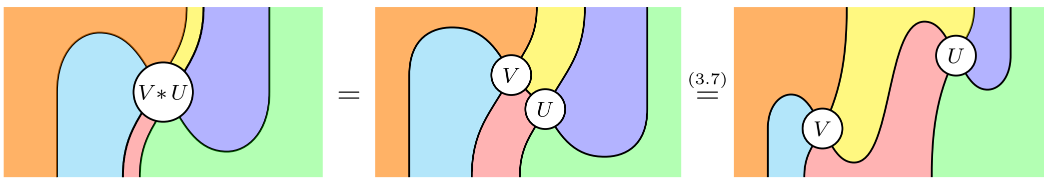
partitions of the input and output wires:



Proof. We will prove that



is biunitary; the proof for the other composite is completely analogous. The composite $V * U$ is vertically unitary, since it is the vertical composite of two unitary vertices. For horizontal unitarity, consider the anticlockwise rotation of $V * U$:



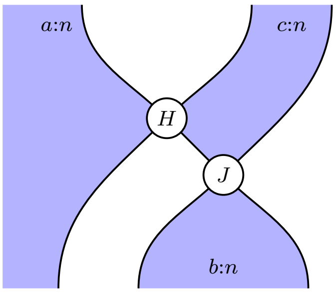
This is unitary up to a scalar, since by Proposition 3.2.4 it is the vertical composite of two vertices which are unitary up to a scalar. By Proposition 3.2.4, we conclude that $V * U$ is biunitary. □

Except for the pinwheel composite[10] [DP93], which can be handled separately, this shows that Theorem 3.3.1 holds.

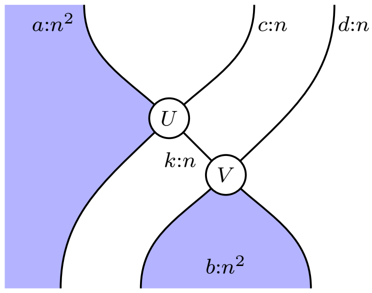
3.3.2 Binary composites

We give a number of quantum constructions listed in Figure 3.6 and Figure 3.7, each involving the diagonal composite of two biunitaries. Correctness of all these constructions follows as corollaries from Theorem 3.3.2, and the results of Section 3.2.2 as summarized in Figure 3.5.

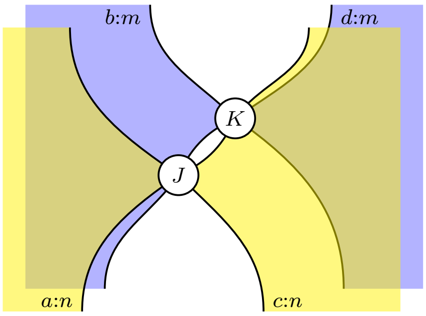
[10]The *pinwheel composite* is a way to compose five 2-morphisms in a double category, in a way which cannot be described in terms of repeated binary composites.



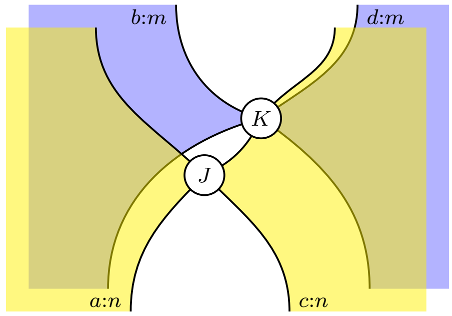
(a) $\mathrm{Had}_n + \mathrm{Had}_n \rightsquigarrow \mathrm{QLS}_n$



(b) $\mathrm{UEB}_n + \mathrm{UEB}_n \rightsquigarrow \mathrm{QLS}_{n^2}$



(c) $\mathrm{Had}_n^m + \mathrm{Had}_m^n \rightsquigarrow \mathrm{Had}_{nm}$ (Hosoya-Suzuki)



(d) $\mathrm{Had}_n + \mathrm{Had}_m^n \rightsquigarrow \mathrm{Had}_{nm}$ (Diţă)

Figure 3.6: Binary constructions of quantum Latin squares and Hadamard matrices.

Quantum Latin squares.

We begin by presenting two quantum Latin square constructions. The following construction produces a quantum Latin square from two Hadamard matrices, generalizing [BN07, Def 2.3] and [MV16, Def 10].

Corollary 3.3.3 (Had_n+$\mathrm{Had}_n \rightsquigarrow \mathrm{QLS}_n$)**.** *The construction of Figure 3.6(a) produces an n-dimensional quantum Latin square*

$$Q_{a,b,c} = \frac{1}{\sqrt{n}} H_{a,c} \, J_{c,b}$$

from the following data, with $a, b, c \in [n]$:

- *$H_{a,c}$ and $J_{c,b} \in \mathrm{Had}_n$, n-dimensional Hadamard matrices.*

The factor $\frac{1}{\sqrt{n}}$ arises as described in Proposition 3.2.4, since the biunitary J is of rotated Hadamard type. Such a biunitary is a *unitary* matrix; given an ordinary Hadamard matrix, we need to rescale it by a factor of $\frac{1}{\sqrt{n}}$ to obtain such a unitary.

The next construction, which we believe to be new, produces a quantum Latin square from two unitary error bases.

Corollary 3.3.4 ($\mathrm{UEB}_n + \mathrm{UEB}_n \rightsquigarrow \mathrm{QLS}_{n^2}$)**.** *The construction of Figure 3.6(b) produces an n^2-dimensional quantum Latin square*

$$Q_{a,b,cd} = \frac{1}{\sqrt{n}} \sum_{k\in[n]} U_{a,c,k}\, V_{b,k,d}$$

from the following data, with $a, b \in [n^2]$ and $c, d \in [n]$:

- *$U_{a,c,k}$ and $V_{b,k,d} \in \mathrm{UEB}_n$, n-dimensional unitary error bases.*

As with Corollary 3.3.3, the factor $\frac{1}{\sqrt{n}}$ arises since the biunitary V is of rotated UEB type.

Note that we concatenate indices corresponding to tensor products of Hilbert spaces or products of indexing sets; for example, for a QLS on a Hilbert space $V \otimes W$, the coefficient of the basis vector $|i, j\rangle = |i\rangle \otimes |j\rangle$ in the (a, b)th position of the quantum Latin square will be written as $Q_{a,b,ij}$. Similarly, if the indexing set of a UEB is the product of two sets $[n] \times [m]$ we denote its (a, b)th element by U_{ab} with coefficients $U_{ab,i,j}$.

Hadamard matrices.

The following construction produces a single Hadamard matrix from two controlled families.

Corollary 3.3.5 ([HS03], $\mathrm{Had}_n^m + \mathrm{Had}_m^n \rightsquigarrow \mathrm{Had}_{nm}$)**.** *The construction of Figure 3.6(c) produces an nm-dimensional Hadamard matrix*

$$H_{ab,cd} = J^b_{a,c}\, K^c_{b,d}$$

from the following data, with $a, c \in [n]$ and $b, d \in [m]$:

- *$J^b_{a,c} \in \mathrm{Had}_n^m$, an m-controlled family of n-dimensional Hadamard matrices;*
- *$K^c_{b,d} \in \mathrm{Had}_m^n$, an n-controlled family of m-dimensional Hadamard matrices.*

This construction was introduced in 2003 by Hosoya and Suzuki [HS03] under the name *generalized tensor product.* Originally, they defined their tensor product as a block matrix $(J^1, \ldots, J^m) \otimes (K^1, \ldots, K^n)$ with (i, j)th block given by

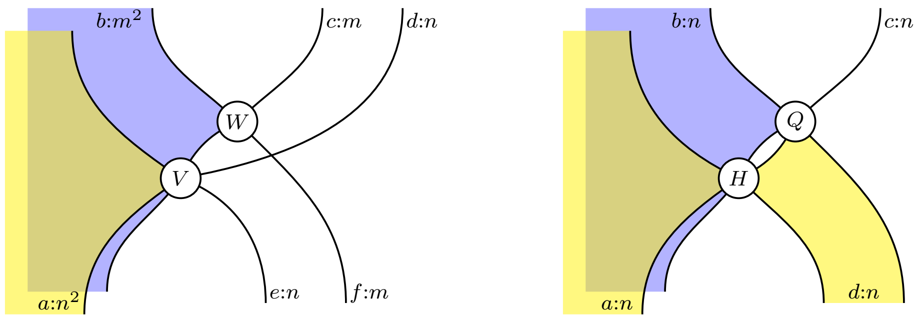
$$\mathrm{diag}\left(J^1_{i,j}, \ldots, J^m_{i,j}\right) K^j.$$

This coincides with the construction of Corollary 3.3.5.

A better known special case of this construction, due to Diţă [Diţ04], is a central tool in the study and classification of Hadamard matrices; we give it explicitly in Figure 3.6(d). Diţă's construction uses an n-dimensional Hadamard matrix J and an n-controlled family of m-dimensional Hadamard matrices $K^1, \ldots, K^n$ to obtain the Hadamard matrix $J \otimes (K^1, \ldots, K^n)$. The difference is that J is a single Hadamard matrix in Diţă's construction, rather than a controlled family of Hadamard matrices.

Unitary error bases.

We now turn our attention to unitary error bases. By a manual combinatorial check, it can be verified that the constructions in Figure 3.7 are the only possible binary constructions of UEBs using only Hadamard matrices, UEBs or QLSs and controlled families thereof.



(a) $\mathrm{UEB}_n^{m^2} + \mathrm{UEB}_m \rightsquigarrow \mathrm{UEB}_{nm}$ (b) $\mathrm{Had}_n^n + \mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_n$

Figure 3.7: Binary biunitary constructions of unitary error bases.

The following construction can be seen as the UEB analog of Diţă's construction given in Figure 3.6(d).

Corollary 3.3.6 ($\mathrm{UEB}_n^{m^2} + \mathrm{UEB}_m \rightsquigarrow \mathrm{UEB}_{nm}$)**.** *The construction of Figure 3.7(a) produces an nm-dimensional unitary error basis*

$$U_{ab,cd,ef} = V^b_{a,d,e} W_{b,c,f}$$

from the following data, with $a \in [n^2]$*,* $b \in [m^2]$*,* $c, f \in [m]$ *and* $d, e \in [n]$*:*

- $V^b_{a,d,e} \in \mathrm{UEB}_n^{m^2}$*, an* m^2*-controlled family of* n*-dimensional unitary error bases;*
- $W_{b,c,f} \in \mathrm{UEB}_m$*, an* m*-dimensional unitary error basis.*

In Figure 3.7(a), we have used biunitarity of the interchanger as established in Proposition 3.2.14.

It is also possible to compose biunitaries of different types to obtain unitary error bases, as shown by the following biunitary characterization of an existing construction, the *quantum shift-and-multiply* method, which simultaneously generalizes the shift-and-multiply method [Wer01] and the Hadamard method [MV16, Def 33].

Corollary 3.3.7 ([MV16], $\mathrm{Had}_n^n+\mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_n$)**.** *The construction of Figure 3.7(b) produces an n-dimensional unitary error basis*

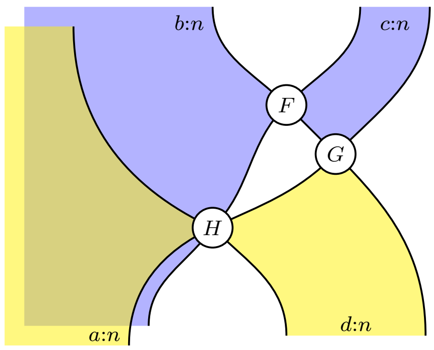
$$U_{ab,c,d} = H^b_{a,d}\, Q_{b,d,c}$$

from the following data, with $a, b, c, d \in [n]$:

- $H^b_{a,d} \in \mathrm{Had}_n^n$*, an n-controlled family of n-dimensional Hadamard matrices;*
- $Q_{b,d,c} \in \mathrm{QLS}_n$*, an n-dimensional quantum Latin square.*

3.3.3 Ternary constructions

We can easily obtain higher arity constructions by iterating some of the binary constructions of Figure 3.6 and Figure 3.7. For example, combining the constructions of Figure 3.6(a) and Figure 3.7(b) yields the following unitary error basis construction:



(3.23)

In index notation this corresponds to the expression

$$U_{ab,c,d} = \frac{1}{\sqrt{n}}\, H^b_{a,d}\, F_{b,c}\, G_{c,d}$$

built from the following data, with $a, b, c, d \in [n]$:

- $H^b_{a,d} \in \mathrm{Had}_n^n$, an n-controlled family of n-dimensional Hadamard matrices;
- $F_{b,c}$ and $G_{c,d} \in \mathrm{Had}_n$, n-dimensional Hadamard matrices.

This generalizes the Hadamard method [MV16, Def 33]. By definition, this construction factors through the quantum shift-and-multiply method of Figure 3.7(b).

More interestingly, there are ternary constructions that do not arise by iterating binary constructions involving our basic quantum structures. In this subsection, we list all ternary biunitary constructions of unitary error bases from Hadamard matrices, unitary error bases, quantum Latin squares and controlled families thereof, which do not factor through the constructions of Figure 3.7. We summarize them in Figure 3.8. Up to equivalence as defined in Section 3.4, we assert that this list is complete, although we do not prove completeness in a formal way. To our knowledge, all constructions in this section are new. As before, all these results are corollaries of Theorem 3.3.2, and the results of Section 3.2.2 as summarized in Figure 3.5. To improve readability, we indicate the form of the compositions by corresponding tiling diagrams.

The constructions of Figure 3.8(a) and Figure 3.8(b) can be seen as slight alterations of constructions that factor through the constructions of Figure 3.7, while the other constructions in Figure 3.8 do not seem to have binary analogues.

Corollary 3.3.8 ($\mathrm{Had}_n^{m^2,n} + \mathrm{UEB}_m^{n,n} + \mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_{nm}$)**.** *The construction of Figure 3.8(a) produces an nm-dimensional UEB*

$$U_{abc,de,fg} = H_{a,f}^{b,c}\, V_{b,e,g}^{c,f}\, Q_{c,f,d}$$

from the following data, with $a, c, d, f \in [n]$*,* $b \in [m^2]$ *and* $e, g \in [m]$*:*

- $H_{a,f}^{b,c} \in \mathrm{Had}_n^{m^2,n}$*, an* (m^2, n)*-controlled family of n-dimensional Hadamard matrices;*
- $V_{b,e,g}^{c,f} \in \mathrm{UEB}_m^{n,n}$*, an* (n, n)*-controlled family of m-dimensional unitary error bases;*
- $Q_{c,f,d} \in \mathrm{QLS}_n$*, an n-dimensional quantum Latin square.*

If the UEB V were not controlled, the construction would be the tensor product of V with the quantum shift-and-multiply UEB obtained as in Figure 3.7(b).

The following construction is also related to one of the binary constructions.

Corollary 3.3.9 ($\mathrm{Had}_{nm}^{n,m} + \mathrm{QLS}_n^{m,m} + \mathrm{QLS}_m \rightsquigarrow \mathrm{UEB}_{nm}$)**.** *The construction of Figure 3.8(b) produces an nm-dimensional UEB*

$$U_{abc,de,fg} = H_{a,eg}^{b,c}\, P_{e,b,f}^{c,g}\, Q_{c,g,d}$$

from the following data, with $a \in [nm]$ $b, e, f \in [n]$ *and* $c, d, g \in [m]$*:*

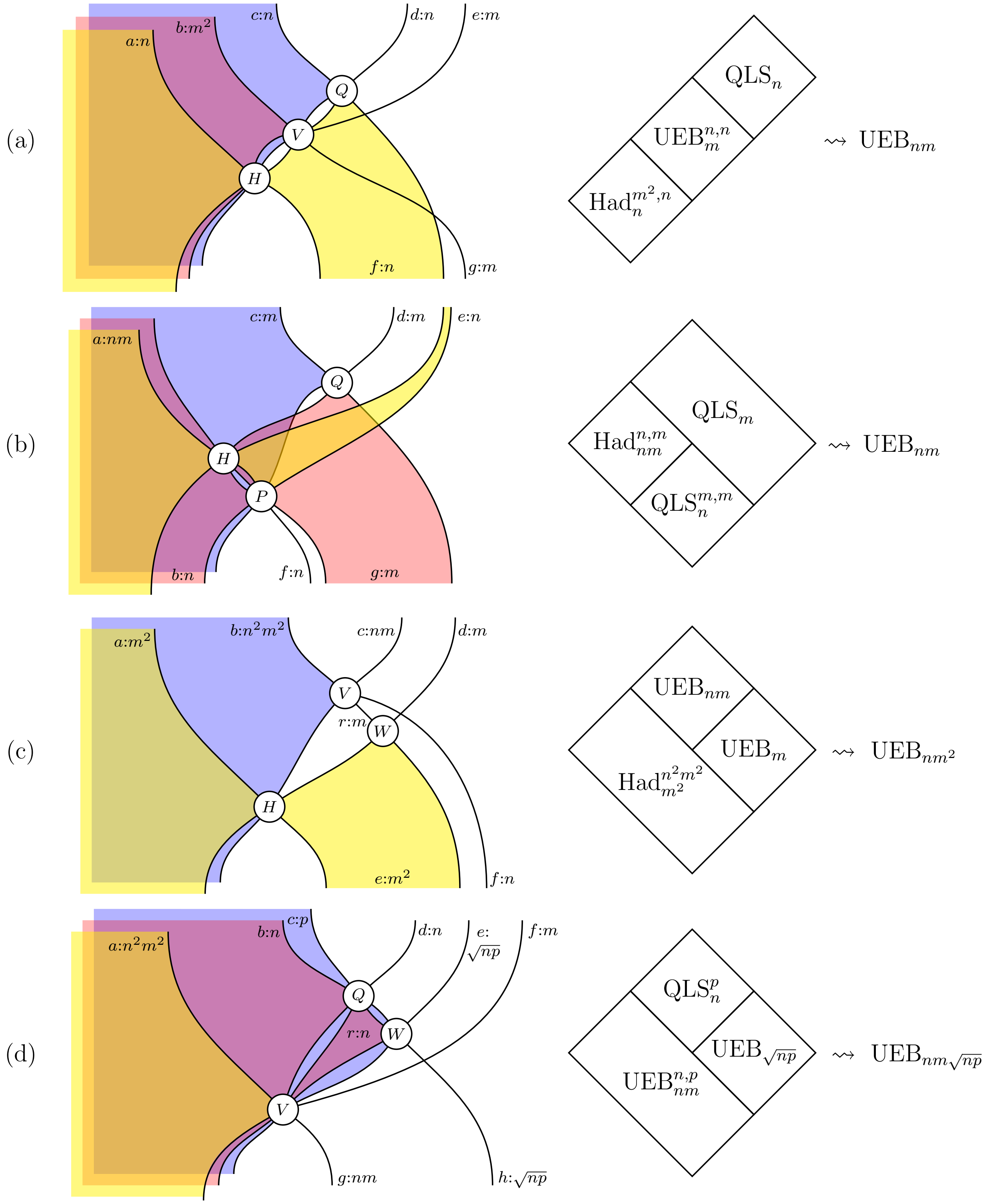


Figure 3.8: An overview of all ternary unitary error basis constructions.

- $H^{b,c}_{a,eg} \in \mathrm{Had}^{n,m}_{nm}$, *an* (n,m)*-controlled family of* nm*-dimensional Hadamard matrices;*

- $P^{c,g}_{e,b,f} \in \mathrm{QLS}^{m,m}_{n}$, *an* (m,m)*-controlled family of* n*-dimensional quantum Latin squares;*

- $Q_{c,g,d} \in \mathrm{QLS}_m$, *an* m*-dimensional quantum Latin square.*

In fact, taking the partial transpose of the resulting UEB (that is, bending the d wire down and the g wire up) leads to the quantum shift-and-multiply UEB generated from the Hadamard matrices H and a quantum Latin square obtained from the controlled tensor product (the QLS analogue of Diţă's construction) of P and Q. This relationship is surprising since taking the partial transpose does not in general preserve biunitarity.

The following is geometrically the simplest of our ternary constructions. It involves a closed wire, so the index expression includes a sum.

Corollary 3.3.10 ($\mathrm{Had}^{n^2m^2}_{m^2} + \mathrm{UEB}_{nm} + \mathrm{UEB}_m \rightsquigarrow \mathrm{UEB}_{nm^2}$). *The construction of Figure 3.8(c) produces an* nm^2*-dimensional UEB*

$$U_{ab,cd,ef} = \sum_{r \in [m]} H^b_{a,e}\, V_{b,c,rf}\, W_{e,r,d}$$

from the following data, with $a, e \in [m^2]$, $b \in [n^2m^2]$, $c \in [nm]$, $d \in [m]$, *and* $f \in [n]$*:*

- $H^b_{a,e} \in \mathrm{Had}^{n^2m^2}_{m^2}$, *an* n^2m^2*-controlled family of* m^2*-dimensional Hadamard matrices;*

- $V_{b,c,rf} \in \mathrm{UEB}_{nm}$, *an* nm*-dimensional unitary error basis;*

- $W_{e,r,d} \in \mathrm{UEB}_m$, *an* m*-dimensional unitary error basis.*

Our final ternary construction is the first to involve a sum over a closed region, which again gives rise to a summation.

Corollary 3.3.11 ($\mathrm{UEB}^{n,p}_{nm} + \mathrm{QLS}^p_n + \mathrm{UEB}_{\sqrt{np}} \rightsquigarrow \mathrm{UEB}_{nm\sqrt{np}}$). *For* $n, m, p \in \mathbb{N}$ *such that* $\sqrt{np} \in \mathbb{N}$*, the construction of Figure 3.8(d) produces an* $nm\sqrt{np}$*-dimensional UEB*

$$U_{abc,def,gh} := \sum_{r \in [n]} V^{b,c}_{a,rf,g}\, Q^c_{b,r,d}\, W_{rc,e,h}$$

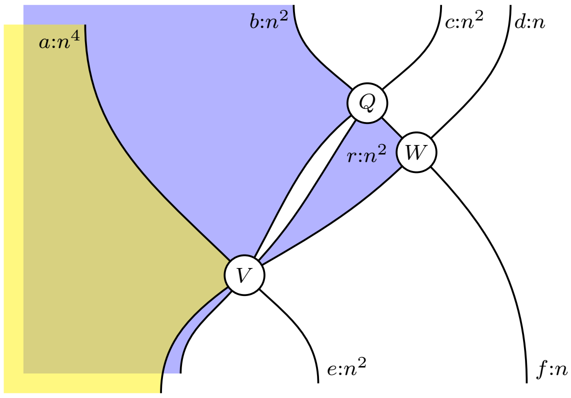
from the following data, with $a \in [n^2m^2]$, $b, d \in [n], c \in [p]$, $e, h \in [\sqrt{np}]$, $f \in [m]$, *and* $g \in [nm]$*:*

- $V^{b,c}_{a,rf,g} \in \mathrm{UEB}^{n,p}_{nm}$, *an* (n,p)*-controlled family of* nm*-dimensional unitary error bases;*

- $Q^{c}_{b,r,d} \in \mathrm{QLS}^{p}_{n}$, *an* p*-controlled family of* n*-dimensional quantum Latin squares;*

- $W_{rc,e,h} \in \mathrm{UEB}_{\sqrt{np}}$, *an* $\sqrt{np}$*-dimensional unitary error basis.*

A particularly simple case of this final construction is the following, which plays a role in our argument in Section 3.3.5 that our methods give rise to infinitely many distinct constructions:



This produces an n^3-dimensional UEB

$$U_{ab,cd,ef} := \sum_{r\in[n^2]} V^{b}_{a,r,e}\, Q_{b,r,c}\, W_{r,d,f}$$

from the following data, with $a \in [n^4]; b, c, e \in [n^2]$ and $d, f \in [n]$:

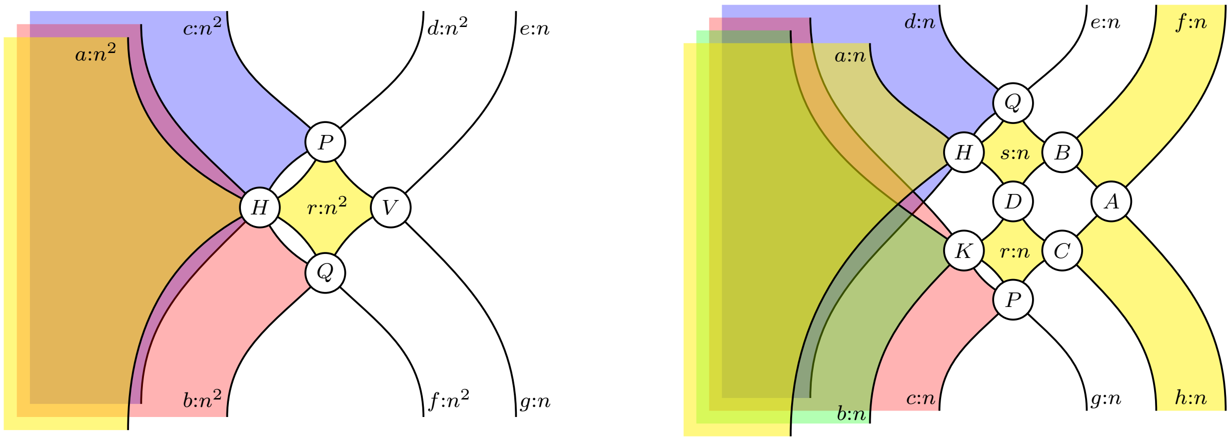
- $V^{b}_{a,r,e} \in \mathrm{UEB}^{n^2}_{n^2}$, an n^2-controlled family of n^2-dimensional unitary error bases;

- $Q_{b,r,c} \in \mathrm{QLS}_{n^2}$, an n^2-dimensional quantum Latin squares;

- $W_{r,d,f} \in \mathrm{UEB}_{n}$, an n-dimensional unitary error basis.

3.3.4 Higher constructions

Interesting biunitary composites exist for higher arity, and are easy to discover by *ad hoc* experimentation. We illustrate two examples which seem particularly elegant, and which, to our knowledge, are new constructions. Once again, these results are all corollaries of Theorem 3.3.1, and the results of Section 3.2.2 as summarized in Figure 3.5.



(a) $\mathrm{Had}_{n^2}^{n^2,n^2} + 2{\times}\mathrm{QLS}_{n^2} + \mathrm{UEB}_n \rightsquigarrow \mathrm{UEB}_{n^3}$ (b) $4{\times}\mathrm{Had}_n + 2{\times}\mathrm{Had}_n^n + 2{\times}\mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_{n^2}$

Figure 3.9: Some higher-arity constructions of unitary error bases.

Corollary 3.3.12 ($\mathrm{Had}_{n^2}^{n^2,n^2} + 2{\times}\mathrm{QLS}_{n^2} + \mathrm{UEB}_n \rightsquigarrow \mathrm{UEB}_{n^3}$)**.** *The construction in Figure 3.9(a) produces an n^3-dimensional UEB*

$$U_{abc,de,fg} = \sum_{r\in[n^2]} H_{a,r}^{b,c}\, P_{c,r,d}\, Q_{r,b,f}\, V_{r,e,g} \tag{3.24}$$

from the following data, with $a, b, c, d, f \in [n^2]$ and $e, g \in [n]$:

- $H_{a,r}^{b,c} \in \mathrm{Had}_{n^2}^{n^2,n^2}$*, an (n^2, n^2)-controlled family of n^2-dimensional Hadamard matrices;*
- $P_{c,r,d}, Q_{r,b,f} \in \mathrm{QLS}_{n^2}$*, n^2-dimensional quantum Latin squares;*
- $V_{r,e,g} \in \mathrm{UEB}_n$*, an n-dimensional unitary error bases.*

In Section 3.5, we will use this construction to produce a new unitary error basis that cannot be obtained by the most general previously known methods.

We now turn to the 8-ary construction of Figure 3.9(b).

Corollary 3.3.13 ($4{\times}\mathrm{Had}_n + 2{\times}\mathrm{Had}_n^n + 2{\times}\mathrm{QLS}_n \rightsquigarrow \mathrm{UEB}_{n^2}$)**.** *The construction in Figure 3.9(b) produces an n^2-dimensional UEB*

$$U_{abcd,ef,gh} = \frac{1}{n} \sum_{r,s\in[n]} A_{f,h}\, B_{s,f}\, C_{r,h}\, D_{s,r}\, H_{a,s}^{d}\, K_{b,r}^{c}\, Q_{d,s,e}\, P_{r,c,g}$$

from the following data, with $a, b, c, d, e, f, g, h \in [n]$:

- $A_{f,h}, B_{s,f}, C_{r,h}, D_{s,r} \in \mathrm{Had}_n$*, n-dimensional Hadamard matrices;*

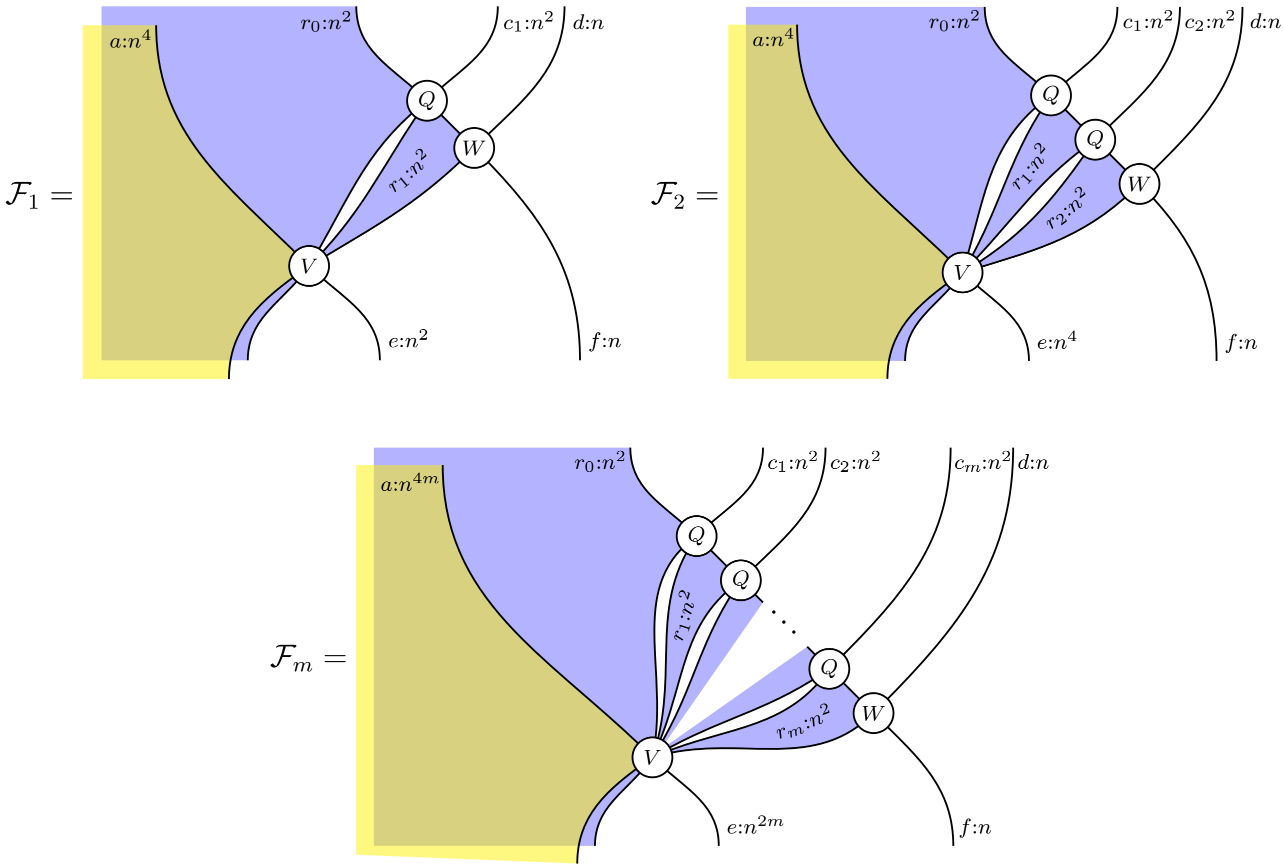


Figure 3.10: An infinite sequence of unitary error basis constructions.

- $H^d_{a,s}, K^c_{b,r} \in \mathrm{Had}^n_n$, *n-controlled families of n-dimensional Hadamard matrices;*
- $Q_{d,s,e}, P_{r,c,g} \in \mathrm{QLS}_n$, *n-dimensional quantum Latin squares.*

The factor $\frac{1}{n}$ arises from using two rotated Hadamard matrices A and D; see the discussion after Corollary 3.3.3.

3.3.5 An infinity of constructions

We have seen several examples of unitary error basis constructions which do not factor through compositions of lower arity only involving Hadamard matrices, unitary error bases, quantum Latin squares and controlled families thereof. We now argue that such constructions can be found for all arities, and hence that our methods lead to infinitely many conceptually distinct constructions.

Consider the sequence of unitary error basis constructions presented in Figure 3.10.

The construction $\mathcal{F}_m$ produces an n^{2m+1}-dimensional UEB

$$U_{ar_0,c_1\cdots c_m d,ef} := \sum_{r_1\in[n^2]} \cdots \sum_{r_m\in[n^2]} V^{r_0}_{a,r_1\cdots r_m,e} \left(\prod_{i\in[m]} Q_{r_{i-1},r_i,c_i} \right) W_{r_m,d,f}$$

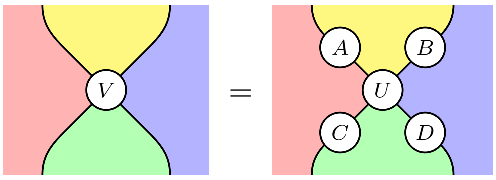
from the following data, with $a \in [n^{4m}]$, $e \in [n^{2m}]$, $r_i, c_i \in [n^2]$ and $d, f \in [n]$:

- $V^{r_0}_{a,r_1,\cdots,r_m,e} \in \mathrm{UEB}^{n^2}_{n^{2m}}$, an n^2-controlled family of n^{2m}-dimensional unitary error bases;
- $Q_{r_{i-1},r_i,c_i} \in \mathrm{QLS}_{n^2}$, an n^2-dimensional quantum Latin square;
- $W_{r_m,d,f} \in \mathrm{UEB}_n$, an n-dimensional unitary error basis.

By inspection, for each $m > 1$, the construction $\mathcal{F}_m$ does not factor through any simpler construction between Hadamards, unitary error bases, quantum Latin squares or controlled families thereof.

3.4 Equivalence of biunitaries

In the planar algebra literature, two biunitaries V and U are said to be *gauge equivalent* if there are unitaries A, B, C and D such that the following holds [Jon99, Def 2.11.10]:



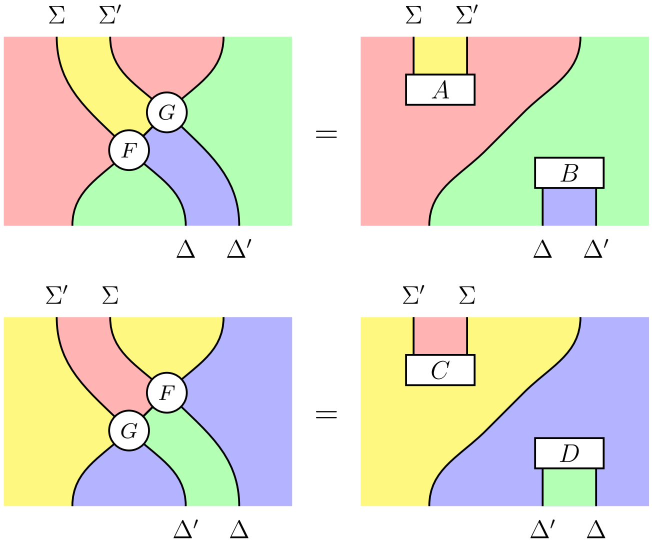
(3.25)

However, this notion of equivalence does not coincide with the usual notions of equivalence of Hadamard matrices, unitary error bases and quantum Latin squares [TŻ06, KR03, MV16]. For example, n-dimensional Hadamard matrices H and H' are said to be equivalent if there are scalars $\lambda_a, \mu_b \in U(1)$ and permutations $\sigma, \tau \in S_n$ such that $H'_{a,b} = \lambda_a \lambda_b H_{\sigma(a),\tau(b)}$ [TŻ06, Def 2.2]. Applying condition (3.25) to Hadamard matrices (3.19) accounts for the scalars λ_a and μ_b but not for the permutations σ and τ.

To remedy this, we suggest that two biunitaries should be considered equivalent if each can be obtained from the other by composition with biunitaries. (Note that (3.25) arises from biunitary composition of U with the unitaries A, B, C, D.) In Section 3.4.1, we make this precise. In Section 3.4.2, we verify that this gives the correct notion of equivalence for quantum structures, and investigate the consequences for some of the construction rules explored in Section 3.3.

3.4.1 Mathematical foundation

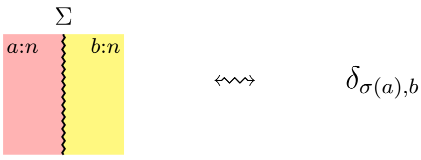
Definition 3.4.1. We say that a biunitary F is *minor reversible* if there exists a biunitary G and unitaries A, B, C, D such that the following hold:



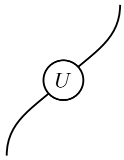
(Note that our usage of "unitary" is that of Definition 3.2.1.) That is, a biunitary is minor reversible if it is invertible with respect to biunitary composition along the minor diagonal direction ⁄. Similarly, we say that a biunitary is *major reversible* if it is invertible with respect to biunitary composition along the major diagonal direction ∖.

For example, a unitary U can be seen both as a minor reversible biunitary and a major reversible biunitary, respectively, depending on the chosen partition of the input and output wires:

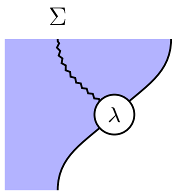
If F is minor reversible, then since A and C are invertible, it follows that the non-negative integer-valued matrix $\sigma := (\dim(\Sigma_{a,b}))_{a,b}$ is invertible with non-negative integer-valued inverse $\sigma' = (\dim(\Sigma'_{b,a}))_{b,a}$. In this case Σ defines a bijection on the label sets of the two adjacent regions, and we say that Σ is an *equivalence*, drawing it as follows:



It follows that a minor-reversible biunitary with no shaded region is simply a unitary



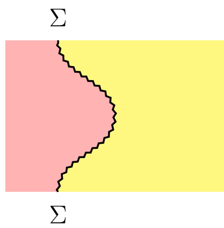
with Σ and Δ being the identity bijections between 1-element sets. A minor-reversible biunitary of the form



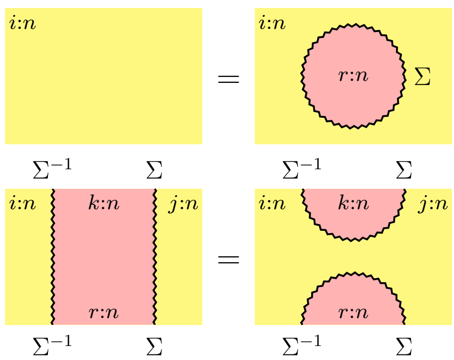
corresponds to a controlled family of scalars $\{\lambda_a \in U(1) \,|\, a \in \mathcal{A}\}$ with Σ acting as a permutation on the index set.

We make the following observation.

Proposition 3.4.2. *If Σ is an equivalence, then the following vertex is biunitary:*



Proof. Vertical unitarity is immediate, and horizontal unitarity follows from these calculations:



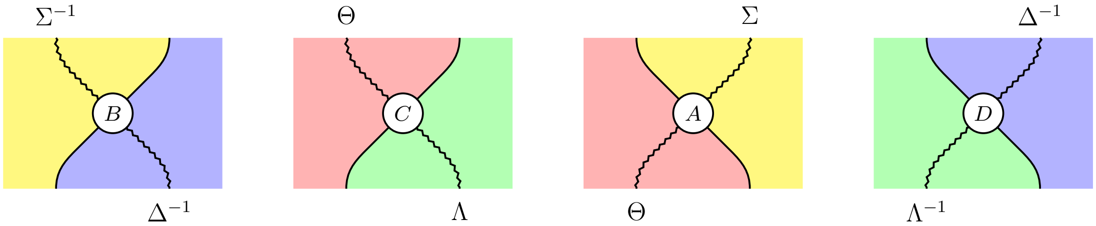
$$\rightsquigarrow \qquad 1 = \sum_{r \in [n]} \delta_{\sigma(r),i} \quad \forall i \in [n]$$

$$\rightsquigarrow \qquad \delta_{i,\sigma(k)}\delta_{k,r}\delta_{\sigma(k),j} = \delta_{i,j}\delta_{\sigma(k),j}\delta_{\sigma(r),j} \quad \forall i,j,k,r \in [n]$$

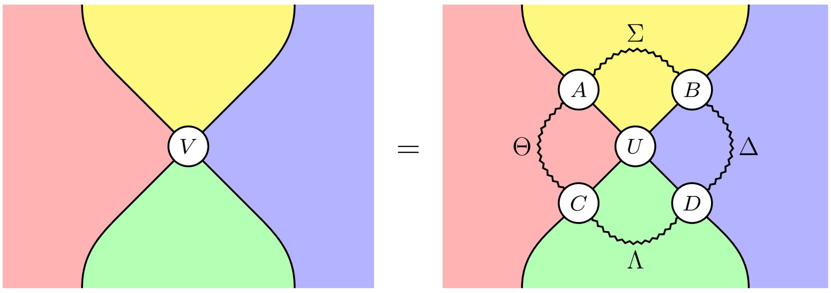
□

We are now ready to state the definition of equivalence of biunitaries, in which the unitaries in (3.25) are replaced by minor- and major-reversible biunitaries.

Definition 3.4.3. Two biunitaries U, V are *equivalent* if there exist minor-reversible biunitaries B, C and major-reversible biunitaries A, D



such that the following equation holds:



(3.26)

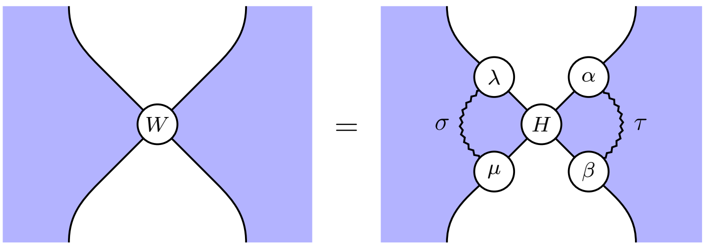
It is easy to check that this defines an equivalence relation on the set of biunitaries. Note that the right-hand side of (3.26) is a composite of 9 biunitaries, thanks to Proposition 3.4.2.

3.4.2 Equivalence for quantum structures

This leads to the following notions of equivalence of Hadamard matrices, unitary error bases and quantum Latin squares, agreeing with the respective notions proposed in the literature [TŻ06, KR03, MV16].

Hadamard matrices.

Two Hadamard matrices H and W are equivalent if the following equation holds:



Thus, H and W are equivalent if there are scalars $\lambda_a, \mu_a, \alpha_b$ and β_b and permutations $\sigma, \tau \in S_n$ such that

$$W_{a,b} = \lambda_a \mu_a H_{\sigma(a),\tau(b)} \alpha_b \beta_b.$$

Redefining $c_a := \lambda_a \mu_a$ and $d_b := \alpha_b \beta_b$, this becomes equivalent to the usual notion of equivalence of Hadamard matrices H and W: there are scalars $c_a, d_b \in U(1)$ and permutations $\sigma, \tau \in S_n$ such that

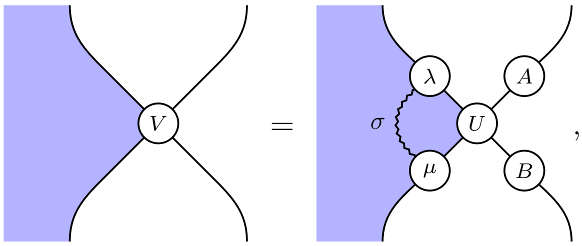
$$W_{a,b} = c_a d_b H_{\sigma(a),\tau(b)}.$$

Unitary error bases.

Two unitary error bases

$$\mathcal{U} = \left\{U_i \mid i \in [n^2]\right\} \qquad \mathcal{V} = \left\{V_i \mid i \in [n^2]\right\}$$

are equivalent if the following holds:

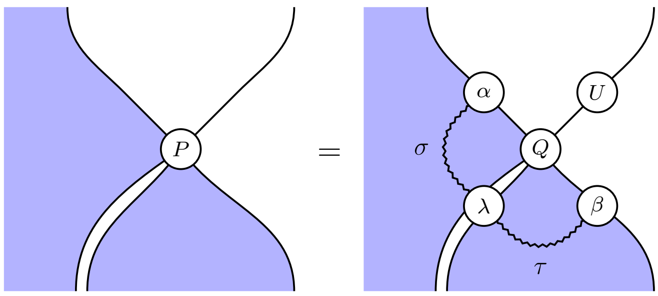


That is, they are equivalent if there are unitary matrices A, B, scalars $c_i \in U(1)$ and a permutation $\sigma \in S_{n^2}$ such that

$$V_i = c_i A U_{\sigma(i)} B.$$

Quantum Latin squares.

Two quantum Latin squares Q and P are equivalent if the following holds:

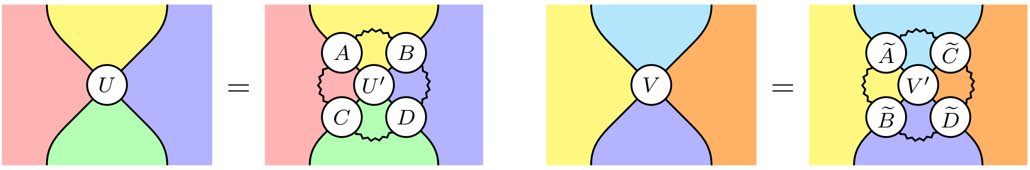


That is, they are equivalent if there is a unitary matrix U, scalars $c_{a,b} \in U(1)$ and permutations $\sigma, \tau \in S_n$ such that the following holds:

$$|P_{a,b}\rangle = c_{a,b} U \left|Q_{\sigma(a),\tau(b)}\right\rangle$$

Equivalence of controlled families of quantum structures can be defined in a similar way.

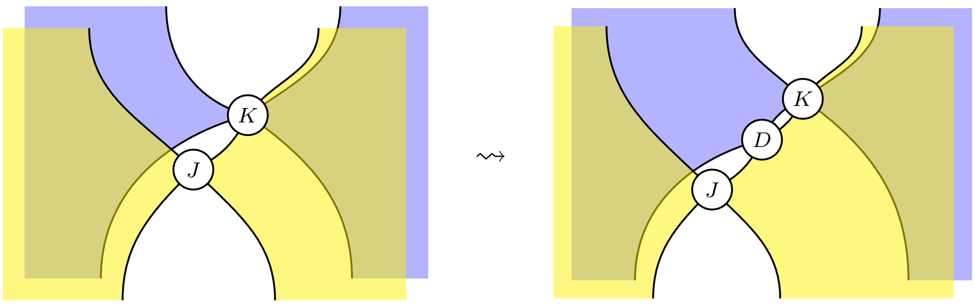
It is instructive to consider how the notion of equivalence interacts with composition of biunitaries. Consider two pairs of equivalent biunitaries of the following type:



Then in general, when B and $\widetilde{B}$ are not inverses with respect to composition along the minor diagonal, the following composites are not equivalent:

This behaviour was recognized by Werner [Wer01], who observed that it is possible to construct inequivalent shift-and-multiply unitary error bases even when all Hadamard matrices and Latin squares come from the same equivalence classes.

It is also exploited in Diţă's construction [Diţ04]. Consider the following equivalence transformation on the family of Hadamard matrices K in Figure 3.6(d):



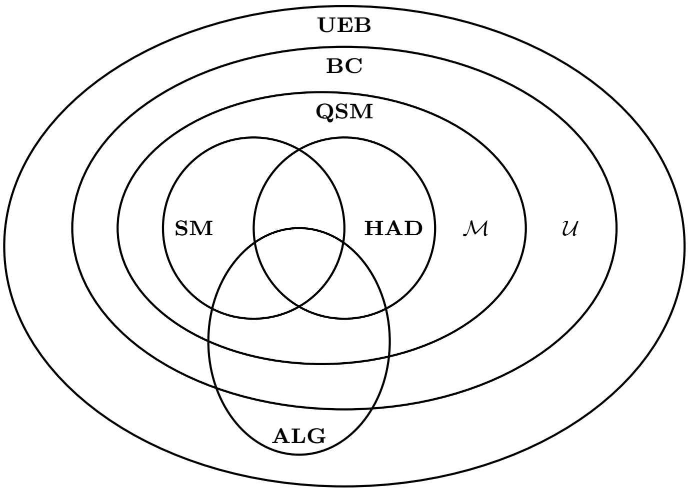
This allows us to introduce a controlled family of free scalars D in the resulting Hadamard matrix, a technique used to construct continuously-parameterized families of Hadamard matrices in [Diţ04, Sec 4]. This construction is one of the reasons why continuous families of Hadamard matrices are comparatively better understood in composite dimensions. In fact, it was conjectured by Popa [Pop83] that there are only finitely many inequivalent Hadamard matrices (and in particular no continuous families) in prime dimensions. This conjecture was disproven by Petrescu [Pet97] in 1997, who constructed several continuous families of Hadamard matrices in certain prime dimensions.

3.5 A new unitary error basis

Construction techniques for unitary error bases have been widely studied [Kni96a, KR03, MV16]. The methods proposed in the literature fall into the following two classes:

- Quantum shift-and-multiply (**QSM**) [MV16]. Requires a quantum Latin square and a family of Hadamard matrices. Generalizes the earlier shift-and-multiply (**SM)** [Wer01] and Hadamard (**HAD**) methods.
- Algebraic (**ALG**) [Kni96b]. Requires a finite group equipped with a projective representation satisfying certain requirements.

As shown in Corollary 3.3.7, the quantum shift-and-multiply method is a special case of our biunitary composition method (**BC**). We thus arrive at the following Venn diagram summarising all known constructions of unitary error bases, extending a Venn diagram in [MV16]:



In [MV16], a unitary error basis $\mathcal{M}$ was constructed which lies in **QSM**, but outside **SM**, **HAD** and **ALG**. In this section, we construct a unitary error basis $\mathcal{U}$ which lies in **BC**, but outside **QSM** and **ALG**. It follows that our biunitary composition techniques are able to produce genuinely new quantum structures.

In Section 3.5.1, we give the construction of $\mathcal{U}$. In Section 3.5.2 we show that it is not equivalent to a UEB arising from the algebraic construction, and in Section 3.5.3 we show it is not equivalent to one arising from the quantum shift-and-multiply construction. An accompanying *Mathematica* notebook is available at arXiv:1609.07775.

3.5.1 Constructing $\mathcal{U}$

We employ the construction of Figure 3.9(a) and Corollary 3.3.12 for $n = 2$, with the following definitions for the (constant family consisting of the) Hadamard matrix H, the quantum Latin squares P and Q, and the unitary error basis $\mathcal{V}$:

$$H := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & \text{-}1 & \text{-}i \\ 1 & \text{-}1 & 1 & \text{-}1 \\ 1 & \text{-}i & \text{-}1 & i \end{pmatrix}$$

$P :=$

$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$	$\lvert 4\rangle$
$\frac{1}{\sqrt{2}}(\lvert 2\rangle - \lvert 3\rangle)$	$\frac{1}{\sqrt{5}}(i\lvert 1\rangle + 2\lvert 4\rangle)$	$\frac{1}{\sqrt{5}}(2\lvert 1\rangle + i\lvert 4\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 2\rangle + \lvert 3\rangle)$
$\frac{1}{\sqrt{2}}(\lvert 2\rangle + \lvert 3\rangle)$	$\frac{1}{\sqrt{5}}(2\lvert 1\rangle + i\lvert 4\rangle)$	$\frac{1}{\sqrt{5}}(i\lvert 1\rangle + 2\lvert 4\rangle)$	$\frac{1}{\sqrt{2}}(\lvert 2\rangle - \lvert 3\rangle)$
$\lvert 4\rangle$	$\lvert 3\rangle$	$\lvert 2\rangle$	$\lvert 1\rangle$

$Q :=$

$\lvert 1\rangle$	$\lvert 4\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle$
$\lvert 4\rangle$	$\lvert 1\rangle$	$\lvert 3\rangle$	$\lvert 2\rangle$
$\lvert 3\rangle$	$\lvert 2\rangle$	$\lvert 1\rangle$	$\lvert 4\rangle$
$\lvert 2\rangle$	$\lvert 3\rangle$	$\lvert 4\rangle$	$\lvert 1\rangle$

$$\mathcal{V} := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \text{-}1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \text{-}1 & 0 \end{pmatrix} \right\}$$

The resulting unitary error basis $\widetilde{\mathcal{U}}$ is calculated according to formula (3.24). We then define an equivalent UEB $\mathcal{U}$ as follows:

$$\mathcal{U} = \left\{ U_{abc} := \widetilde{U}_{111}^{\dagger} \widetilde{U}_{abc} \,|\, a, b, c \in [4] \right\}$$

We choose $\mathcal{U}$ in this way to ensure that $U_{111} = \mathbb{1}$. An explicit list of the 64 8×8-matrices comprising the UEB $\mathcal{U}$ can be found in Appendix E, the commutativity structure of its elements is visualized in Figure 3.11.

3.5.2 Nice error bases

In this subsection we define nice error bases, and show that $\mathcal{U}$ is not equivalent to a nice error basis.

Definition 3.5.1 ([Kni96b]). A *nice error basis* is a unitary error basis $\mathcal{U} = \{U_i \mid i \in I\}$ with $\mathbb{1} \in \mathcal{U}$ that is (up to phases) closed under multiplication. In other words, for each $a, b \in I$, there exists a scalar $\omega(a, b) \in U(1)$ and an index $a * b \in I$, such that

$$U_a U_b = \omega(a, b) U_{a*b}.$$

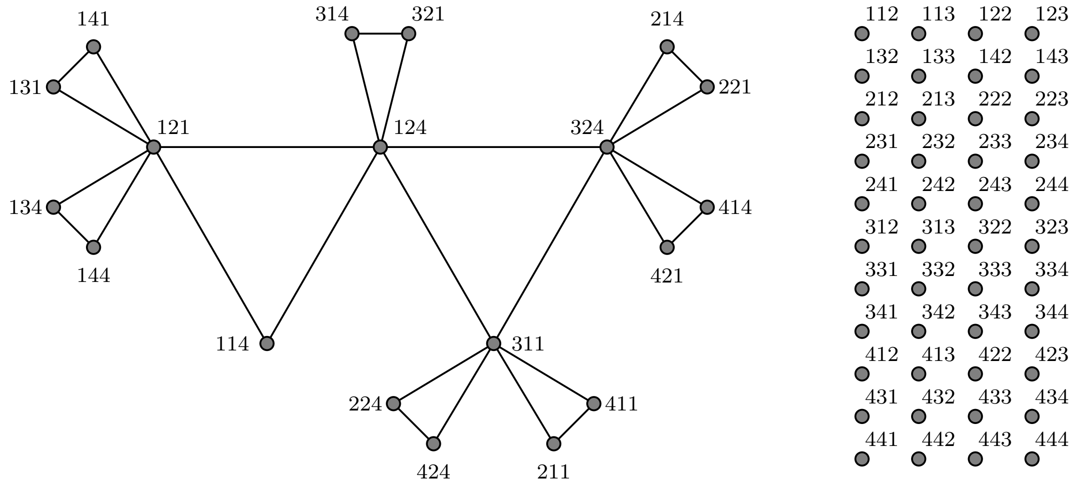


Figure 3.11: The graph with vertices given by elements of $\mathcal{U}$, and edges between commuting elements. The element $U_{111} = \mathbb{1}$ is omitted.

Nice unitary error bases correspond to certain projective representations of finite groups.

The following is a strong property of nice error bases.

Proposition 3.5.2 ([MV16, Prop 43]). *Let $\mathcal{V}$ be a unitary error basis containing the identity matrix, such that $\mathcal{V}$ is equivalent to a nice error basis. Then up to multiplication by a phase, $\mathcal{V}$ is closed under taking adjoints.*

It follows that our new unitary error basis $\mathcal{U}$ is not equivalent to a nice error basis.

Theorem 3.5.3. *The unitary error basis $\mathcal{U}$ is not equivalent to a nice unitary error basis.*

Proof. We note that $U_{112}^{\dagger}$ is not proportional to any matrix in $\mathcal{U}$, and hence by Proposition 3.5.2 the result follows. □

3.5.3 Quantum shift-and-multiply bases

Quantum shift-and-multiply UEBs were defined in [MV16], and the construction exactly matches the biunitary composite of Figure 3.7(b). In this subsection we demonstrate that all quantum shift-and-multiply UEBs have a particular commutativity property, and therefore demonstrate that our new UEB $\mathcal{U}$ does not arise in this way.

The following proposition gives a strong constraint on the structure of quantum shift-and-multiply bases.

Proposition 3.5.4. *Let $\mathcal{V}$ be an m-dimensional UEB which contains the identity matrix, such that $\mathcal{V}$ is equivalent to a quantum shift-and-multiply UEB. Then $\mathcal{V}$ contains m pairwise-commuting matrices.*

Proof. Quantum shift-and-multiply UEBs are of the form $V_{ab} = Q_a D_a^b$ for unitary matrices Q_a and unitary diagonal matrices D_a^b. Using the definition of equivalence of unitary error bases from Section 3.4.2, it follows that $\mathcal{V}$ is of the following form:

$$\mathcal{V} = \left\{ c_{ab} A Q_a D_a^b B \mid a, b \in [m] \right\}$$

Since $\mathbb{1} \in \mathcal{V}$, there are indices a_0, b_0 such that $c_{a_0 b_0} A Q_{a_0} D_{a_0}^{b_0} B = \mathbb{1}$. Defining the diagonal matrix $D := \left(c_{a_0 b_0} D_{a_0}^{b_0} \right)^\dagger$, this means that

$$A = B^\dagger D Q_{a_0}^\dagger$$

and hence that

$$\mathcal{V} = \left\{ c_{ab} B^\dagger D Q_{a_0}^\dagger Q_a D_a^b B \mid a, b \in [m] \right\}.$$

All matrices with $a = a_0$ pairwise commute, and there are m of these. □

The desired result follows.

Theorem 3.5.5. *The unitary error basis $\mathcal{U}$ is not equivalent to a quantum shift-and-multiply basis.*

Proof. The commutativity graph of $\mathcal{U}$ is shown in Figure 3.11. It is clear by inspection that every pairwise-commuting subset contains at most 4 elements (including the element $U_{111} = \mathbb{1}$, which is omitted from the graph.) The result then follows from Proposition 3.5.4. □

Chapter 4

The Morita theory of quantum graph isomorphisms

In this chaper, based on [MRV19], we classify instances of quantum pseudo-telepathy in the graph isomorphism game, exploiting a recently discovered connection between quantum information and the theory of quantum automorphism groups. Specifically, we show that graphs quantum isomorphic to a given graph are in bijective correspondence with Morita equivalence classes of certain Frobenius algebras in the category of finite-dimensional representations of the quantum automorphism algebra of that graph. We show that such a Frobenius algebra may be constructed from a central type subgroup of the classical automorphism group, whose action on the graph has coisotropic vertex stabilisers. In particular, if the original graph has no quantum symmetries, quantum isomorphic graphs are classified by such subgroups. We show that all quantum isomorphic graph pairs corresponding to a well-known family of binary constraint systems arise from this group-theoretical construction. We use our classification to show that, of the small order vertex-transitive graphs with no quantum symmetry, none is quantum isomorphic to a non-isomorphic graph. We show that this is in fact asymptotically almost surely true of all graphs.

4.1 Introduction

Quantum pseudo-telepathy [BBT05] is a well studied phenomenon in quantum information theory, where two non-communicating parties can use pre-shared entanglement to perform a task classically impossible without communication. Such tasks are usually formulated as games, where isolated players Alice and Bob are provided with inputs, and must return outputs satisfying some winning condition. One such game is the graph isomorphism game [AMR$^+$19], whose instances correspond to pairs of graphs Γ and Γ', and whose winning classical strategies are exactly graph isomorphisms $\Gamma \to \Gamma'$. Winning quantum strategies are called *quantum isomorphisms*. Quantum pseudo-telepathy is exhibited by graphs that are quantum but not classically isomorphic.

This work builds on two recent articles, in which Lupini, Mančinska and Roberson [LMR17] and the author, Musto and Verdon [MRV18] independently discovered a connection between these quantum isomorphisms and the *quantum automorphism groups* of graphs [Ban05a, BB09, BB07, BBC07, Bic03] studied in the framework of compact quantum groups [Wor98]. This connection has already proven to be fruitful, introducing new quantum information-inspired techniques to the study of quantum automorphism groups [Ban19, LMR17].

Here, we use this connection in the opposite direction, showing how results from the well developed theory of quantum automorphism groups have implications for the study of pseudo-telepathy. This may seem surprising, since pseudo-telepathy requires quantum isomorphisms between non-isomorphic graphs, not quantum automorphisms. However, we here show that the graphs quantum isomorphic to a given graph Γ can in fact be classified in terms of algebraic structures in the monoidal category $\mathrm{QAut}(\Gamma)$ of finite-dimensional representations of Banica's quantum automorphism Hopf C^*-algebra $A(\Gamma)$[1]. In other words, the quantum automorphism group of a graph, together with its action on the set of vertices of the graph, fully determines all graphs quantum isomorphic to it.

We further show that much information can be obtained just from the ordinary automorphism group of a graph. For example, if a graph has *no quantum symmetries* (see [BBC07]) it is possible to completely classify quantum isomorphic graphs in terms of certain subgroups of the ordinary automorphism group; as a consequence we show that no vertex-transitive graph of order ≤ 11 with no quantum symmetry [BB07, Sch18] is part of a pseudo-telepathic graph pair. Even if a graph does have quantum symmetries, it is still possible to construct quantum isomorphic graphs using only the structure of the ordinary automorphism group. In particular, we show that all pseudo-telepathic graph pairs arising from Lupini et al.'s version of Arkhipov's construction [Ark12, LMR17] — including the graph pairs corresponding to the well-known magic square [Mer90] and magic pentagram constraint systems— arise from certain $\mathbb{Z}_2^4$ or $\mathbb{Z}_2^6$ symmetries of one of the graphs.

[1]For a definition of this algebra, see [BB07, Def 2.1]. In Sections 4.2.2 and 4.2.3, we give an explicit description of the category $\mathrm{QAut}(\Gamma)$ which does not require knowledge of quantum automorphism groups.

Our classification results are more naturally expressed in terms of (finite) *quantum graphs*, originally introduced by Weaver [Wea15] and generalizing the noncommutative graphs of Duan, Severini and Winter [DSW13]. These quantum graphs generalize classical graphs, with a possibly non-commutative finite-dimensional C^*-algebra taking the role of the set of vertices. The notions of isomorphism and quantum isomorphism can both be generalized to the setting of quantum graphs [MRV18]; in particular, every quantum graph has a group of automorphisms $\mathrm{Aut}(\Gamma)$ and a category of quantum automorphisms $\mathrm{QAut}(\Gamma)$, which can again be understood as the category of finite-dimensional representations of a certain Hopf C^*-algebra. We are currently not aware of a direct application of quantum isomorphic quantum graphs in quantum information theory.[2] Nevertheless, our classification naturally includes quantum graphs, with the classification of quantum isomorphic classical graphs arising as a special case.

All results are derived in the 2-categorical framework of [MRV18].

The classification

For a quantum graph Γ, we classify quantum isomorphic quantum graphs Γ' in terms of *simple*[3] *dagger Frobenius monoids* in the representation categories $\mathrm{QAut}(\Gamma)$; these are dagger Frobenius monoids X (see Definition 4.2.2) in $\mathrm{QAut}(\Gamma)$ whose underlying algebra FX is simple, where $F : \mathrm{QAut}(\Gamma) \to \mathrm{Hilb}$ is the forgetful functor. In terms of the Hopf C^*-algebra $A(\Gamma)$ such a structure can equivalently be defined as a matrix algebra $\mathrm{Mat}_n(\mathbb{C})$ with normalized trace inner product $\langle A, B\rangle = \frac{1}{n}\mathrm{Tr}(A^\dagger B)$, equipped with a $*$-representation $\triangleright\colon A(\Gamma) \to \mathrm{End}(\mathrm{Mat}_n(\mathbb{C}))$ such that the following holds for all $x \in A(\Gamma)$ and $A, B \in \mathrm{Mat}_n(\mathbb{C})$:

$$\big(x_{(1)} \triangleright A\big)\,\big(x_{(2)} \triangleright B\big) = x \triangleright (AB) \qquad\qquad x \triangleright \mathbb{1}_n = \epsilon(x)\mathbb{1}_n \tag{4.1}$$

Here, we have used Sweedler's sumless notation for the comultiplication $\Delta(x) = x_{(1)} \otimes x_{(2)}$. We show that two such simple dagger Frobenius monoids produce isomorphic graphs if and only if they are *Morita equivalent*. Morita equivalence plays a central role in modern algebra and mathematical physics, in particular being used to classify module categories [Ost03a], rational conformal field theories [RFFS07] and gapped boundaries of two-dimensional gapped phases of matter [KK12].

[2] Although, see [Sta16] for a possible interpretation in terms of zero-error quantum communication.

[3] There exists a more general notion of simple Frobenius monoid in a semisimple monoidal category [KR08]; the simple Frobenius monoids appearing here are always simple in this broader sense.

Result 1 (Classification of quantum isomorphic quantum graphs — Corollary 4.3.7)**.** *For a quantum graph* Γ *there is a bijective correspondence between the following sets:*

- *Isomorphism classes of quantum graphs* Γ' *quantum isomorphic to* Γ*.*
- *Morita equivalence classes of simple dagger Frobenius monoids in* $\mathrm{QAut}(\Gamma)$*.*

We remark that this classification depends only on the quantum automorphism group of Γ, and not on its action on the (quantum) set of vertices.

For applications to pseudo-telepathy, we are of course interested in a classification of quantum isomorphic *classical* graphs.

Result 2 (Classification of quantum isomorphic classical graphs — Corollary 4.3.14)**.** *For a classical graph* Γ *there is a bijective correspondence between the following sets:*

- *Isomorphism classes of classical graphs* Γ' *quantum isomorphic to* Γ*.*
- *Morita equivalence classes of simple dagger Frobenius monoids in* $\mathrm{QAut}(\Gamma)$ *fulfilling a certain commutativity condition.*

In contrast to Result 1, the classification of quantum isomorphic classical graphs depends not only on the quantum automorphism group of Γ, but also on its action on the set of vertices.

Although some of the representation categories $\mathrm{QAut}(\Gamma)$ have been studied before [BC08, BB09], a general classification of Morita classes of simple dagger Frobenius monoids in all such categories seems unfeasible. We therefore focus on the *classical subcategory* of $\mathrm{QAut}(\Gamma)$; this is the full subcategory generated by the classical automorphisms[4] of Γ, and is equivalent to the category $\mathrm{Hilb}_{\mathrm{Aut}(\Gamma)}$ of $\mathrm{Aut}(\Gamma)$-graded Hilbert spaces. Using the well-known classification of Morita classes of Frobenius monoids in such categories [Ost03b], we can classify quantum isomorphic graphs in terms of central type subgroups of $\mathrm{Aut}(\Gamma)$. A *group of central type* [EGNO15, Def 7.12.21] (L, ψ) is a finite group L with a *non-degenerate* 2-cocycle $\psi : L \times L \to U(1)$; that is, a 2-cocycle such that the twisted group algebra $\mathbb{C}L^{\psi}$ is simple.

Result 3 (Quantum isomorphic quantum graphs from groups — Corollary 4.4.2)**.** *Every central type subgroup* (L, ψ) *of the automorphism group* $\mathrm{Aut}(\Gamma)$ *of a quantum graph* Γ *gives rise to a quantum graph* $\Gamma_{L,\psi}$ *and a quantum isomorphism* $\Gamma_{L,\psi} \to \Gamma$*. Moreover, if* Γ *has no quantum symmetries, this leads to a bijective correspondence between the following sets:*

[4] Equivalently, the classical subcategories can be understood as the categories of finite-dimensional representations of the commutative algebra of functions on $\mathrm{Aut}(\Gamma)$.

- *Isomorphism classes of quantum graphs* Γ' *quantum isomorphic to* Γ.
- *Central type subgroups* (L, ψ) *of* $\mathrm{Aut}(\Gamma)$ *up to the following equivalence relation:*

$$(L,\psi) \sim (L',\psi') \Leftrightarrow L' = gLg^{-1} \text{ and } \psi' \text{ is cohomologous to } \psi^g(x,y) := \psi(gxg^{-1}, gyg^{-1}) \text{ for some } g \in \mathrm{Aut}(\Gamma) \tag{4.2}$$

Classicality of the generated graph can also be expressed in group-theoretical terms. A nondegenerate 2-cocycle ψ of a group of central type L gives rise to a symplectic form[5] $\rho_\psi : L \times L \to U(1)$, where $\rho_\psi(a,b) := \psi(a,b)\overline{\psi}(aba^{-1}, a)$. In particular, a subset $S \subseteq L$ is said to be *coisotropic* if it contains its orthogonal complement $S^\perp$, defined as follows, where $Z_g = \{h \in L \mid hg = gh\}$ denotes the centralizer of $g \in L$:

$$S^\perp := \{g \in L \mid \rho_\psi(g,a) = 1 \ \forall a \in Z_g \cap S\}$$

For a subgroup $L \subseteq \mathrm{Aut}(\Gamma)$ and a vertex v of Γ we denote the corresponding stabilizer subgroup by $\mathrm{Stab}_L(v) := \{l \in L \mid l(v) = v\}$. We say that a central type subgroup (L, ψ) of $\mathrm{Aut}(\Gamma)$ *has coisotropic stabilizers* if the stabilizer subgroups $\mathrm{Stab}_L(v)$ are coisotropic for every vertex v of Γ.

Result 4 (Quantum isomorphic classical graphs from groups — Corollary 4.4.15)**.** *Every central type subgroup* (L, ψ) *of the automorphism group* $\mathrm{Aut}(\Gamma)$ *of a classical graph* Γ *with coisotropic stabilizers gives rise to a classical graph* $\Gamma_{L,\psi}$ *and a quantum isomorphisms* $\Gamma_{L,\psi} \to \Gamma$. *Moreover, if* Γ *has no quantum symmetries this leads to a bijective correspondence between the following sets:*

- *Isomorphism classes of classical graphs* Γ' *quantum isomorphic to* Γ.
- *Central type subgroups* (L, ψ) *of* $\mathrm{Aut}(\Gamma)$ *with coisotropic stabilizers up to the equivalence relation* (4.2).

Applications to pseudo-telepathy

We exhibit some first simple applications of this classification.

Application 1 (Corollary 4.5.6)**.** *The proportion of* n*-vertex graphs which admit a quantum isomorphism to a non-isomorphic graph goes to zero as* n *goes to infinity.*

In [BB07, Sch18] all vertex transitive graphs of order ≤ 11 without quantum symmetries are classified. The following is then a simple application of Result 4.

[5]See [BDGM14] for an introduction to symplectic forms on groups.

Application 2 (Theorem 4.5.8)**.** *None of the vertex transitive graphs of order* $\leq$ 11 *with no quantum symmetry admits a quantum isomorphism to a non-isomorphic graph.*

Conversely, we use Result 4 to construct graphs quantum isomorphic to a given graph. We will give an example of such a construction in the next paragraph. In fact, we show that all pseudo-telepathic graph pairs arising from Lupini et al.'s variant of Arkhipov's construction [LMR17, Ark12] are obtained by the central type subgroup construction of Result 4.

Application 3 (Theorem 4.5.14)**.** *All pseudo-telepathic graph pairs obtained from Arkhipov's construction [LMR17, Def 4.4 and Thm 4.5] arise from a central type subgroup of the automorphism group of one of the graphs, with coisotropic stabilizers. In particular, the central type subgroup can always taken to be isomorphic to either* $\mathbb{Z}_2^4$ *or* $\mathbb{Z}_2^6$*.*

Quantum isomorphisms from groups of central type

We now demonstrate how Result 4 — the construction of quantum isomorphisms between classical graphs from group-theoretical data — may be used in practise to produce pairs of graphs exhibiting pseudo-telepathy. Recall that the following input data are required:

1. A graph Γ;
2. A subgroup H of the automorphism group of Γ;
3. A nondegenerate 2-cocycle on H, such that the stabilizer subgroup $\mathrm{Stab}_H(v) \subset H$ is coisotropic for each vertex v of Γ.

We now describe a choice of such data which produces a pseudo-telepathic graph pair.

1. *The graph* Γ*.* The graph Γ is the *homogeneous BCS graph* introduced by Atserias et al. [AMR$^+$19, Fig 2] for the *binary magic square* (BMS). Explicitly, this graph is defined as follows. A binary magic square is a $3{\times}3$ matrix with entries drawn from $\{0,1\}$, such that each row and each column sum up to an even number. The following are examples:

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \qquad \begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix} \qquad \begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}$$

The definition of Γ is as follows.

- Vertices of Γ correspond to partial BMS; that is, binary magic squares in which only one row or column is filled. The following are examples:

$$\begin{pmatrix} 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}$$

  In total there are 24 distinct partial BMS, so the graph Γ has 24 vertices.

- We draw an edge between two vertices if the corresponding partial BMS are incompatible. For example, there is an edge between the vertices corresponding to the first and the last partial BMS above, but not between any other pair.

2. *The symmetry* $(\mathbb{Z}_2)^4$. Given a binary magic square, we can flip bits to obtain another binary magic square, so long as we preserve the parity of each row and each column. We denote such symmetries as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\begin{pmatrix} \times & \cdot & \times \\ \cdot & \cdot & \cdot \\ \times & \cdot & \times \end{pmatrix}} \begin{pmatrix} \neg a_{11} & a_{12} & \neg a_{13} \\ a_{21} & a_{22} & a_{23} \\ \neg a_{31} & a_{32} & \neg a_{33} \end{pmatrix}$$

These symmetries of binary magic squares induce symmetries of the graph Γ. They form a subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to $(\mathbb{Z}_2)^4$, and generated by the following transformations:

$$\underset{(1,0,0,0)}{\begin{pmatrix} \cdot & \times & \times \\ \cdot & \times & \times \\ \cdot & \cdot & \cdot \end{pmatrix}} \qquad \underset{(0,1,0,0)}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \times & \times & \cdot \\ \times & \times & \cdot \end{pmatrix}} \qquad \underset{(0,0,1,0)}{\begin{pmatrix} \times & \times & \cdot \\ \times & \times & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}} \qquad \underset{(0,0,0,1)}{\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \times & \times \\ \cdot & \times & \times \end{pmatrix}} \tag{4.3}$$

3. *A nondegenerate 2-cocycle on* $(\mathbb{Z}_2)^4$. It is well known that abelian groups of symmetric type — that is, groups of the form $A\times A$ for some abelian group A — admit nondegenerate 2-cocycles [BSZ01, Thm 5]. The Pauli matrices, which form a faithful projective representation[6] of $\mathbb{Z}_2^2$, give rise to such a 2-cocycle ψ_{P} on $\mathbb{Z}_2^2$:

$$P_{0,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad P_{1,0} = \sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad P_{0,1} = \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & \text{-}1 \end{pmatrix} \quad P_{1,1} = -i\sigma_Y = \begin{pmatrix} 0 & \text{-}1 \\ 1 & 0 \end{pmatrix}$$

[6] In quantum information theory, such faithful projective representations are known as *nice unitary error bases* [KR03]. See Definition 4.4.5.

$$P_{a_1,b_1}P_{a_2,b_2} = \psi_P((a_1,b_1),(a_2,b_2))P_{(a_1+a_2),(b_1+b_2)} \quad \forall\, a_1,a_2,b_1,b_2 \in \mathbb{Z}_2 \tag{4.4}$$

This induces a nondegenerate 2-cocycle ψ_{P^2} on $(\mathbb{Z}_2)^4$, corresponding to the projective representation consisting of pairwise tensor products of Pauli matrices:

$$U_{a,b,c,d} = P_{a,b} \otimes P_{c,d} \quad \forall\, a,b,c,d \in \mathbb{Z}_2 \tag{4.5}$$

We now verify that the stabilizer subgroups of the action of $(\mathbb{Z}_2)^4$ on Γ are coisotropic for the 2-cocycle ψ_{P^2} and its induced symplectic form ρ_{P^2}:

$$\rho_{\mathrm{P}^2}(a,b) = \psi_{\mathrm{P}^2}(a,b)\overline{\psi_{\mathrm{P}^2}(b,a)} \quad \forall\, a,b \in \mathbb{Z}_2^4 \tag{4.6}$$

The group $\mathbb{Z}_2^4$ can be understood as a four-dimensional vector space over the finite field $\mathbb{Z}_2$. From this perspective, order 2^k subgroups of $\mathbb{Z}_2^4$ correspond to k-dimensional subspaces and the symplectic form ρ_{P^2} is a symplectic form in the linear algebraic sense. Since all stabilizer subgroups are two-dimensional, they are coisotropic if and only if they are isotropic (and hence Lagrangian). A subgroup is isotropic if the restriction of the symplectic form (4.6) to this subgroup is trivial. By (4.4), the form ρ_{P^2} is trivial on two group elements of $\mathbb{Z}_2^4$ if the corresponding tensor products of Pauli matrices (4.5) commute. For example, let v be a vertex corresponding to a partial BMS in which only the first row is filled. Its stabilizer subgroup is generated by the group elements $(0,1,0,0)$ and $(0,0,0,1)$ (see (4.3)) with corresponding Pauli matrices $\sigma_Z \otimes \mathbb{1}_2$ and $\mathbb{1}_2 \otimes \sigma_Z$, which clearly commute. Similarly, the stabilizer subgroup of a middle column vertex is generated by the group elements $(1,0,1,0)$ and $(0,1,0,1)$ with corresponding commuting matrices $\sigma_X \otimes \sigma_X$ and $\sigma_Z \otimes \sigma_Z$.

A similar argument holds for all rows and columns, showing that all stabilizer subgroups are coisotropic.[7]

Our construction therefore produces a graph $\Gamma_{\mathbb{Z}_2^4,\psi_{\mathrm{P}^2}}$ that is quantum isomorphic to Γ. We show in Section 4.5.2 that this graph is isomorphic to the *inhomogenous BCS graph* for the binary magic square [AMR+19, Fig 1], which is known to be non-isomorphic to Γ. The two graphs therefore form a pseudo-telepathic graph pair.

[7]We note that the simultaneous assignment of $\mathbb{Z}_2^4$ group elements to symmetry transformations (4.3) and Pauli matrices (4.5) plays an important role in this argument. Other such assignments correspond to other, possibly non-cohomologous, nondegenerate 2-cocycles which might not have coisotropic stabilizers.

Notation and conventions

Dagger categories are defined in [Sel11]; recall that a *unitary* morphism in a dagger category is one whose †-adjoint is its inverse. Strict dagger 2-categories are defined in [HK16].[8]

We use the diagrammatic calculus for monoidal categories throughout; with the exception of Section 4.2.4, these diagrams will always represent morphisms in Hilb, the monoidal dagger category of finite-dimensional Hilbert spaces and linear maps.

'Frobenius algebra' and 'Frobenius monoid' are usually taken to be synonymous, but in this chapter we reserve the term 'Frobenius algebra' for Frobenius monoids in Hilb and use the term 'Frobenius monoid' to refer to Frobenius monoids in general monoidal categories, to aid the reader in distinguishing between the two cases.

All our definitions are adapted to the dagger (or $*$- or unitary) setting. In particular, when we say that two dagger Frobenius monoids in a dagger monoidal category are Morita equivalent we require that the corresponding invertible bimodules are compatible with the dagger structure (see Definition 4.2.33).

Whenever we say 'graph' or 'isomorphism' without the modifier 'quantum' we always refer to the ordinary notion (isomorphisms between quantum graphs are defined in Definition 4.2.15). Occasionally, to clearly distinguish between the two cases, we explicitly use the modifier 'classical' for isomorphisms between classical graphs and 'ordinary' for isomorphisms between quantum graphs.

All sets appearing in this chapter are finite, and all Hilbert spaces are finite-dimensional.

Outline

Section 4.2 recalls background material; Section 4.2.1 expresses Gelfand duality in terms of string diagrams, Section 4.2.2 and 4.2.3 introduce quantum graphs, quantum graph isomorphisms and the monoidal category of quantum graph automorphisms of a quantum graph, and Section 4.2.4 gives a brief introduction to Morita theory in monoidal categories.

In Section 4.3, we prove the general classification of quantum isomorphic graphs; Section 4.3.1 focuses on the classification of quantum isomorphic quantum graphs as in Result 1, Section 4.3.2 specializes this classification to classical graphs as in Result 2. Section 4.3.3 contains the proof of the main classification theorem. The

[8]Weak dagger 2-categories are the obvious generalization, with unitary associators and unitors.

classification theorem is based on a formal result about Morita equivalence in dagger 2-categories proven in Appendix D.

Section 4.4 restricts attention to the subcategory of classical automorphisms of a graph; Section 4.4.1 reexpresses the classification of Section 4.3 in terms of central type groups as in Result 3, Section 4.4.2 again specializes this classification to classical graphs resulting in the statement of Result 4.

Finally, in Section 4.5, we discuss several consequences of this classification. In Section 4.5.1, we investigate the central type subgroups of the automorphism groups of several explicit graphs leading to Applications 1 and 2. In Section 4.5.2, we show that all graph pairs arising from Arkhipov's pseudo-telepathy construction arise from a central type group resulting in Application 3.

4.2 Background

In this section, we recall various definitions and results; most of these are treated in greater detail in [MRV18].

4.2.1 String diagrams, Frobenius monoids and Gelfand duality

Most results in this chapter are derived using the graphical calculus of monoidal dagger categories (see Section I.1). Except for Appendix 4.2.4, we only use the graphical calculus of the compact closed [KL80, AC04] dagger category Hilb of finite-dimensional Hilbert spaces and linear maps.[9]

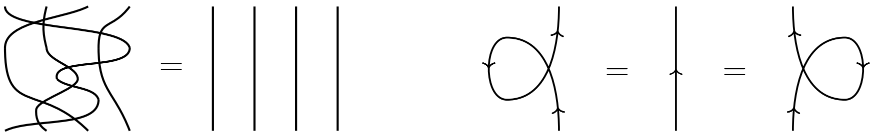
All finite-dimensional Hilbert spaces V have dual spaces $V^* = \mathrm{Hom}(V, \mathbb{C})$ represented in the graphical calculus as an oriented wire with the opposite orientation as V. Duality is characterized by the following evaluation and coevaluation linear maps:

$$f \otimes v \mapsto f(v) \qquad 1 \mapsto \mathbb{1}_V \qquad v \otimes f \mapsto f(v) \qquad 1 \mapsto \mathbb{1}_V \tag{4.7}$$

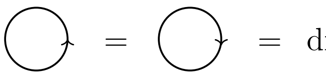
[9]Several of the definitions and theorems in Section 4.3 could be generalized to arbitrary idempotent complete compact closed dagger categories. However, Sections 4.3.2, 4.4 and 4.5 use the classification of special commutative dagger Frobenius algebras in Hilb and would need to be revised.

To define the second and fourth map, we have identified $V \otimes V^* \cong V^* \otimes V \cong \mathrm{End}(V)$. It may be verified that these maps indeed fulfill the cusp equations:

Together with the swap map $\sigma_{V,W} : v \otimes w \mapsto w \otimes v$, depicted as a crossing of wires, this leads to a very flexible topological calculus, allowing us to untangle arbitrary diagrams and straighten out any twists:



A closed circle evaluates to the dimension of the corresponding Hilbert space:



$$= \dim(H) \tag{4.8}$$

Frobenius monoids

We now recall the notion of a dagger Frobenius monoid in a monoidal dagger category.

Definition 4.2.1. A *monoid* in a monoidal category is an object M with multiplication and unit morphisms, depicted as follows:

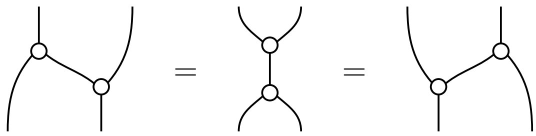
$m : M \otimes M \to M$ $\qquad\qquad$ $u : I \to M$

These morphisms satisfy the following associativity and unitality equations:

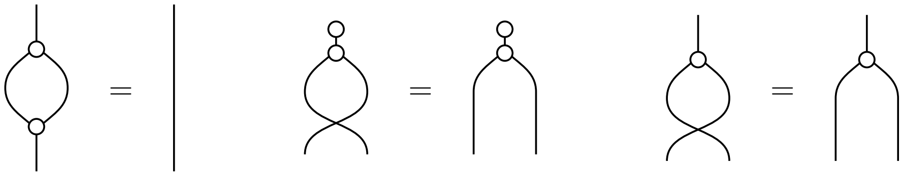
Analogously, a *comonoid* is an object C with a coassociative comultiplication $\delta : C \to C \otimes C$ and counit $\epsilon : C \to I$. The †-adjoint of a monoid in a monoidal dagger category is a comonoid.

Note that for the multiplication and unit morphisms of an monoid we simply draw white nodes rather than labeled boxes, for concision. Likewise, we draw the comultiplication and counit morphisms of the †-adjoint comonoid as white nodes. Despite having the same label in the diagram, they can be easily distinguished by their type.

Definition 4.2.2. A *dagger Frobenius monoid* in a monoidal dagger category is a monoid where the monoid and †-adjoint comonoid structures are related by the following Frobenius equations:



A dagger Frobenius monoid is *special* if equation (4.9a) holds. A dagger Frobenius algebra in Hilb is moreover *symmetric* or *commutative* if one of (4.9b) or (4.9c) holds[10].



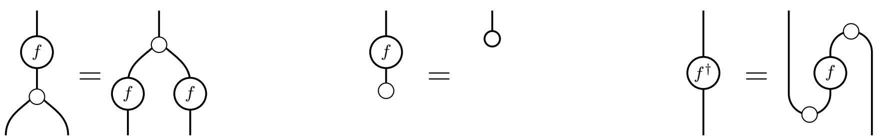
(4.9)

a) special b) symmetric c) commutative

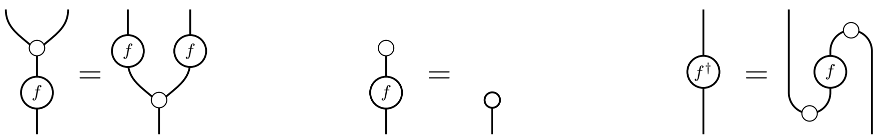
Dagger Frobenius monoids are closely related to dualities. In particular, it is a direct consequence of the Frobenius equation and unitality that the following cups and caps fulfil the cusp equations:

Finally, we define a notion of homomorphism between dagger Frobenius monoids.

Definition 4.2.3. A $*$-*homomorphism* $f : A \to B$ between dagger Frobenius monoids A and B is a morphism $f : A \to B$ satisfying the following equations:



A $*$-*cohomomorphism* $f : A \to B$ is a morphism $f : A \to B$ satisfying the following equations:



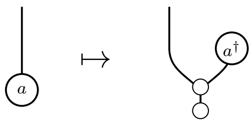
A $*$-*isomorphism* is a morphism which is both a $*$-homomorphism and a $*$-cohomomorphism.

[10]These definitions use the symmetric structure — or swap map — of the symmetric monoidal dagger category Hilb. Other categories appearing in this paper will in general not be braided.

We observe that the dagger of a $*$-homomorphism is a $*$-cohomomorphism, that every $*$-isomorphism is unitary, and that every unitary $*$-homomorphism between dagger Frobenius monoids is a $*$-isomorphism.

Recall that we refer to Frobenius monoids in Hilb as Frobenius algebras. A major reason for defining these structures is the fact that special symmetric dagger Frobenius algebras coincide with finite-dimensional C^*-algebras.

Theorem 4.2.4 ([Vic10, Thm 4.6 and 4.7]). *Every finite-dimensional C^*-algebra admits a unique inner product making it into a special symmetric dagger Frobenius algebra. Conversely, every special symmetric dagger Frobenius algebra A admits a unique norm such that the canonical involution, defined by its action on vectors $|a\rangle \in A$ as the following antihomomorphism, endows it with the structure of a C^*-algebra:*

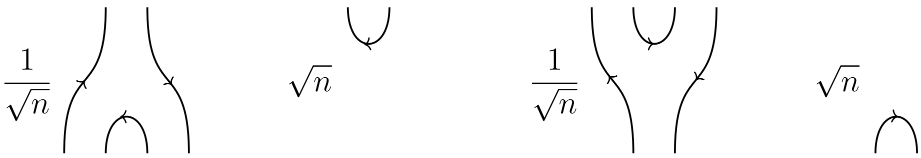


Moreover, the notions of $*$-homomorphism and $*$-isomorphism between special symmetric dagger Frobenius algebras coincide with the corresponding notions for finite-dimensional C^*-algebras.

One advantage of explicitly using special symmetric dagger Frobenius algebras instead of C^*-algebras is that Frobenius algebras already contain 'up-front' all emergent structures of finite-dimensional C^*-algebras, such as the comultiplication $\Delta = m^\dagger : H \to H \otimes H$; they are therefore more amenable to the purely compositional reasoning of the graphical calculus.

One important example of a special symmetric dagger Frobenius algebra is the endomorphism algebra of a Hilbert space.

Definition 4.2.5. The *endomorphism algebra* of a Hilbert space H is defined to be the following special symmetric dagger Frobenius algebra on $H \otimes H^*$ (where $n = \dim(H)$):



Remark 4.2.6. The normalization factors were chosen to make the endomorphism algebra special. This is not essential but simplifies some of our arguments. The endomorphism algebra of Definition 4.2.5 is $*$-isomorphic to the unique special symmetric dagger Frobenius algebra corresponding to the usual C^*-algebra structure on $\mathrm{End}(H)$

which is usually given with unnormalized multiplication and unit but normalized inner product $\langle A, B\rangle := \frac{1}{n}\mathrm{Tr}(A^\dagger B)$ to retain specialness. We prefer the normalization of Definition 4.2.5, since the normalized inner product does not arise as the canonical induced inner product on the tensor product Hilbert space $H \otimes H^*$.

Gelfand duality and Frobenius algebras

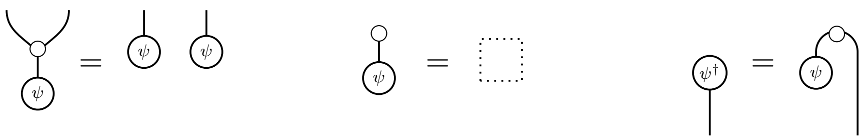
We now recall the graphical version of finite-dimensional Gelfand duality in the framework established by Coecke, Pavlović and Vicary [CPV13]. We first observe that every orthonormal basis of a Hilbert space H defines a special commutative dagger Frobenius algebra on H.

Example 4.2.7. Let $\{|i\rangle\}_{1\leq i\leq n}$ be an orthonormal basis of a Hilbert space H. Then the following multiplication and unit maps, together with their adjoints, form a special commutative dagger Frobenius algebra on H:

$$m : |i\rangle \otimes |j\rangle \mapsto \delta_{i,j}\,|i\rangle \qquad\qquad u : 1 \mapsto \sum_{i=1}^{n} |i\rangle$$

Conversely, every special commutative dagger Frobenius algebra A gives rise to an orthonormal basis of A; the basis vectors are given by the copyable elements of A, defined as follows.

Definition 4.2.8. A *copyable element* of a special commutative dagger Frobenius algebra A is a $*$-cohomomorphism $\psi : \mathbb{C} \to A$; that is, a vector $|\psi\rangle \in A$, such that the following hold:



Theorem 4.2.9 ([CPV13, Thm 5.1.])**.** *The copyable elements of a special commutative dagger Frobenius algebra A form an orthonormal basis of A for which the monoid is of the form given in Example 4.2.7.*

In other words, every special commutative dagger Frobenius algebra in Hilb is of the form of Example 4.2.7 for some orthonormal basis on a Hilbert space.

Given a special commutative dagger Frobenius algebra A, we denote its set of copyable elements by $\widehat{A}$. For such algebras A and B, it can easily be verified that every function $\widehat{A} \to \widehat{B}$ gives rise to a $*$-cohomomorphism between A and B and that conversely every $*$-cohomomorphism $A \to B$ comes from such a function $\widehat{A} \to \widehat{B}$. Therefore, Theorem 4.2.9 gives rise to the following Frobenius-algebraic version of finite-dimensional Gelfand duality.

Corollary 4.2.10 ([CPV13, Cor 7.2.]). *The category of special commutative dagger Frobenius algebras and $*$-cohomomorphisms in* Hilb *is equivalent to the category of finite sets and functions.*

Explicitly, this equivalence maps a special commutative dagger Frobenius algebra A to its set of copyable elements $\widehat{A}$ and a set X to the algebra associated to the orthonormal basis $\{|x\rangle \mid x \in X\}$ of the Hilbert space $\mathbb{C}^{|X|}$. Under this correspondence, we may therefore consider the category of finite sets as 'contained within Hilb' using the following identification.

Set	Hilb
sets of cardinality n	special comm. dagger Frobenius algebras of dim. n
elements of the set	copyable states of the Frobenius algebra
functions	$*$-cohomomorphisms
bijections	$*$-isomorphisms
the one element set $\{*\}$	the one-dimensional Frobenius algebra $\mathbb{C}$

Terminology 4.2.11. Throughout this chapter, we will take pairs of words in this table to be synonymous. In particular, we will denote a set and its corresponding commutative algebra by the same symbol. It will always be clear from context whether we refer to the set X or the algebra X.

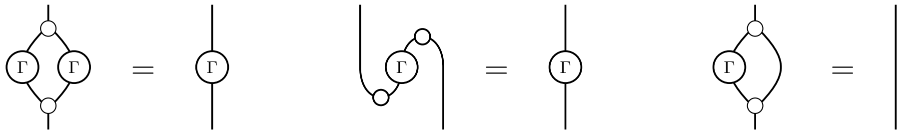
4.2.2 Quantum graphs and quantum graph isomorphisms

The fundamental idea of noncommutative topology is to generalize the correspondence between spaces and commutative algebras by considering noncommutative algebras in light of Gelfand duality.

Terminology 4.2.12. By analogy with Gelfand duality, we think of a special symmetric dagger Frobenius algebra as being associated to an imagined finite quantum set, just as a special commutative dagger Frobenius algebra is associated to a finite set. We follow Terminology 4.2.11 in denoting both the algebra and its associated imagined quantum set by the same symbol.

We can endow a quantum set with graph structure in the following way.

Definition 4.2.13. A *quantum graph* is a pair (V_Γ, Γ) of a special symmetric dagger Frobenius algebra V_Γ (the *quantum set of vertices*) and a self-adjoint linear map $\Gamma : V_\Gamma \to V_\Gamma$ (the *quantum adjacency matrix*) satisfying the following equations:



We will often omit the underlying algebra from the notation and denote quantum graphs (V_Γ, Γ) simply by Γ.

For a classical set V_Γ (that is, for a special commutative dagger Frobenius algebra), Definition 4.2.13 reduces to the definition of an adjacency matrix $\{\Gamma_{v,w}\}_{v,w\in V_\Gamma}$; from left to right, the conditions state that $\Gamma_{v,w}^2 = \Gamma_{v,w}$, that $\Gamma_{v,w} = \Gamma_{w,v}$, and that $\Gamma_{v,v} = 1$. Therefore, a quantum graph defined on a commutative algebra is precisely a graph in the usual sense.

Remark 4.2.14. Notions of quantum graph have been defined elsewhere. In [MRV18, Sec 7], we prove that:

- Our quantum graphs coincide with Weaver's finite-dimensional quantum graphs [Wea15], defined in terms of symmetric and reflexive quantum relations [KW12, Wea12].
- Our quantum graphs (Mat_n, Γ) on matrix algebras coincide with Duan, Severini and Winter's noncommutative graphs [DSW13].

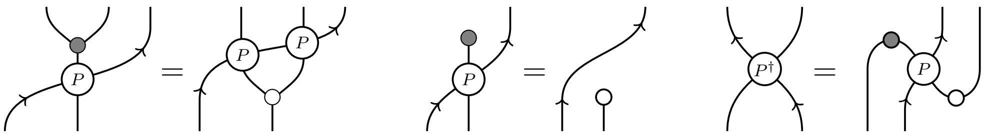
Definition 4.2.15. An (ordinary) *isomorphism* of quantum graphs Γ and Γ' is a $*$-isomorphism of the underlying Frobenius algebras $f : V_\Gamma \to V_{\Gamma'}$ intertwining the corresponding quantum adjacency matrices, i.e. such that $f\Gamma = \Gamma' f$. We denote the group of automorphisms of a quantum graph Γ by $\mathrm{Aut}(\Gamma)$.

For classical graphs, Definition 4.2.15 coincides with the usual notion of graph isomorphism. In particular, for a classical graph Γ, the group $\mathrm{Aut}(\Gamma)$ is the usual automorphism group.

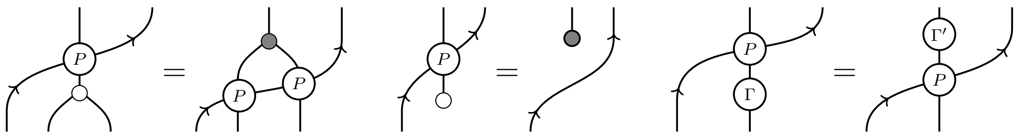
Quantum isomorphisms

We now come to the central definition of this work.

Definition 4.2.16. A *quantum isomorphism* between quantum graphs Γ and Γ' is a pair (H, P), where H is a Hilbert space and P is a linear map $P : H \otimes V_\Gamma \to V_{\Gamma'} \otimes H$ satisfying the following equations, where the algebras V_Γ and $V_{\Gamma'}$ are depicted as white and grey nodes respectively:



(4.10)



(4.11)

The *dimension* of a quantum isomorphism is defined as the dimension of the underlying Hilbert space H.

Notation 4.2.17. To clearly distinguish between the wires corresponding to the Hilbert space H and the wires corresponding to the algebras V_Γ and $V_{\Gamma'}$, we will always draw the Hilbert space wire with an orientation and leave the algebra wires unoriented.

Remark 4.2.18. There are classical and quantum isomorphisms between classical graphs, and ordinary (see Definition 4.2.15) and quantum isomorphisms between quantum graphs.

Remark 4.2.19. A one-dimensional quantum isomorphism between quantum graphs is an ordinary isomorphism (see Definition 4.2.15). In particular, a one-dimensional quantum isomorphism between classical graphs is a graph isomorphism.

A quantum isomorphism $(H, P) : \Gamma \to \Gamma'$ between classical graphs with adjacency matrices $\{\Gamma_{v,v'}\}_{v,v' \in V_\Gamma}$ and $\{\Gamma'_{w,w'}\}_{w,w' \in V_{\Gamma'}}$ can equivalently be expressed as a family of projectors $\{P_{v,w}\}_{v \in V_\Gamma, w \in V_{\Gamma'}}$ on H such that the following holds for all vertices $v, v_1, v_2 \in V_\Gamma$ and $w, w_1, w_2 \in V_{\Gamma'}$:

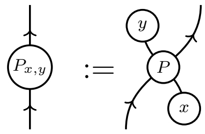
$$P_{v,w_1} P_{v,w_2} = \delta_{w_1,w_2} P_{v,w_1} \qquad \sum_{w \in V_{\Gamma'}} P_{v,w} = \mathbb{1}_H$$

$$P_{v_1,w} P_{v_2,w} = \delta_{v_1,v_2} P_{v_1,w} \qquad \sum_{v \in V_\Gamma} P_{v,w} = \mathbb{1}_H$$

$$\sum_{v' \in V_\Gamma} \Gamma_{v,v'} P_{v',w} = \sum_{w' \in V_{\Gamma'}} P_{v,w'} \Gamma'_{w',w}$$

We will refer to such families of projectors as *projective permutation matrices* [AMR+19]. Given a quantum isomorphism $(H, P) : \Gamma \to \Gamma'$ between classical graphs, the corresponding projective permutation matrix can be obtained as follows. A classical set

X corresponds to a special commutative dagger Frobenius algebra (Example 4.2.7); the elements of X form a basis of copyable elements of this algebra. Using this basis, the projectors $P_{x,y}$ can be obtained as follows:



(4.12)

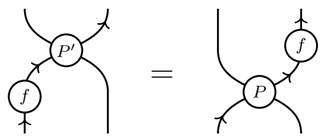
Like ordinary isomorphisms, quantum isomorphisms $(H, P) : \Gamma \to \Gamma'$ can only exist between quantum graphs with quantum vertex sets of equal dimension.

Proposition 4.2.20 ([MRV18, Prop 4.17])**.** *If there is a quantum isomorphism* $(H, P) : \Gamma \to \Gamma'$*, then* $\dim(V_\Gamma) = \dim(V_{\Gamma'})$*. In particular, quantum isomorphisms can only exist between classical graphs with an equal number of vertices.*

The 2-category QGraphIso

Quantum graphs and quantum isomorphisms can be organized into a 2-category. The 2-morphisms of this 2-category are defined as follows.

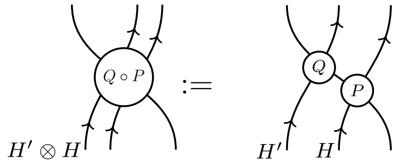
Definition 4.2.21. An *intertwiner* of quantum isomorphisms $(H, P) \to (H', P')$ is a linear map $f : H \to H'$ such that the following holds:



Definition 4.2.22 ([MRV18, Def 3.18 and Thm 3.20])**.** The dagger 2-category QGraphIso is defined as follows:

- **objects** are quantum graphs Γ, Γ', ...;
- **1-morphisms** $\Gamma \to \Gamma'$ are quantum isomorphisms $(H, P) : \Gamma \to \Gamma'$;
- **2-morphisms** $(H, P) \to (H', P')$ are intertwiners of quantum isomorphisms.

The composition of two quantum isomorphisms $(H, P) : A \to B$ and $(H', Q) : B \to C$ is a quantum isomorphism $(H' \otimes H, Q \circ P)$ defined as follows:

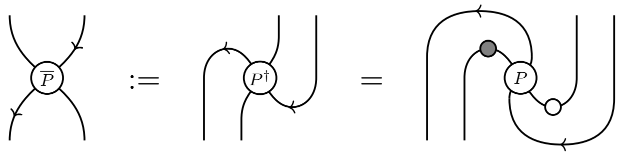


Vertical and horizontal composition of 2-morphisms is defined as the ordinary composition and tensor product of linear maps, respectively. The †-adjoint of a 2-morphism is defined as the Hilbert space adjoint of the underlying linear map.

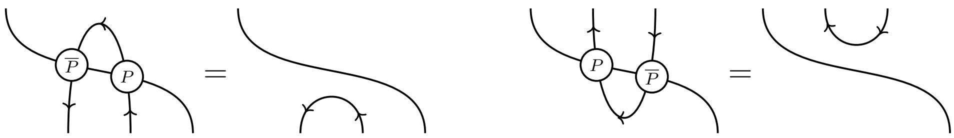
In [MRV18], we define a 2-category QGraph of quantum graphs and quantum *homomorphisms*. For the purpose of this work, it suffices to focus on quantum isomorphisms.

This 2-category QGraphIso has the advantage that every 1-morphism is dualizable.

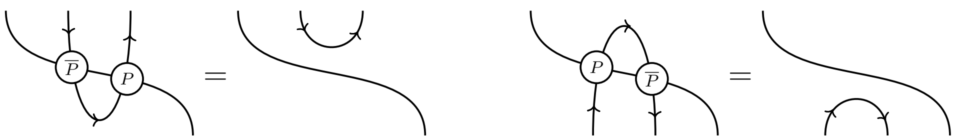
Theorem 4.2.23 ([MRV18, Thm 4.8])**.** *Every quantum isomorphism* $(H, P) : \Gamma \to \Gamma'$ *is dualizable in* QGraphIso*. In particular, this means that there is a quantum isomorphism* $(H^*, \overline{P}) : \Gamma' \to \Gamma$*, whose underlying linear map* $\overline{P} : H^* \otimes V_{\Gamma'} \to V_\Gamma \otimes H^*$ *is defined by equation* (4.13) *and fulfils equations* (4.14) *and* (4.15)*, expressing that the cups and caps* (4.7) *are intertwiners.*



(4.13)



(4.14)



(4.15)

In particular, the linear map $P : H \otimes V_\Gamma \to V_{\Gamma'} \otimes H$ *is unitary.*

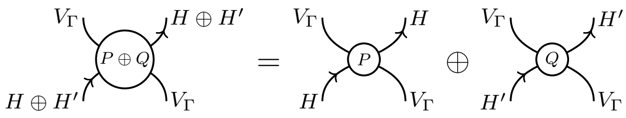
Proposition 4.2.24 ([MRV18, Prop 4.2])**.** *Equivalences in* QGraphIso *are ordinary isomorphisms as in Definition 4.2.15.*

4.2.3 The monoidal dagger category QAut(Γ)

For a quantum graph Γ, we write QAut(Γ) for the monoidal dagger category QGraphIso(Γ, Γ) of quantum automorphisms of Γ. For classical graphs Γ, the category QAut(Γ) (or rather the Hopf C^*-algebra for which it is the category of finite-dimensional representations) has been studied in the context of compact quantum groups [Ban05a, BB09, BB07, BBC07, Bic03, BC08].

Proposition 4.2.25 ([MRV18, Prop 5.19])**.** *Let* Γ *be a classical graph. The category* QAut(Γ) *is the category of finite-dimensional representations of Banica's quantum automorphism algebra* $A(\Gamma)$ *of the graph* Γ *(see e.g. [BB07, Def 2.1]).*

In particular, QAut(Γ) is *semisimple* (see [MRV18, Cor 6.21]). The *direct sum* of two quantum automorphisms $(H, P), (H', Q) : \Gamma \to \Gamma$ is defined as the direct sum of the underlying linear maps:



Conversely, a quantum isomorphism (H, P) is *simple* if it cannot be further decomposed into a non-trivial direct sum or equivalently, if it has no non-trivial interchangers, i.e. if $\mathrm{QGraphIso}((H, P), (H, P)) \cong \mathbb{C}$. Semisimplicity implies that every quantum isomorphism $\Gamma \to \Gamma$ is isomorphic to a direct sum of simple quantum isomorphisms. The decomposition is unique up to permutation of the summands.

Remark 4.2.26. By dimensional considerations, every ordinary isomorphism is a simple quantum isomorphism. However, in general not all simple quantum isomorphisms are ordinary isomorphisms.

Under composition of quantum isomorphisms, QAut(Γ) becomes a *monoidal* semisimple dagger category. In particular, since all quantum isomorphisms are dualizable, we obtain a monoidal semisimple dagger category with dualizable objects. For a finite number of simple objects such a structure is known as a *unitary fusion category*[11] (cf. Section 1.2.2 and [EGNO15]). In general, however, the number of simple objects of QAut(Γ) is not finite. In fact, the quantum automorphism category of most graphs, such as those of all complete graphs with four or more vertices, have an infinite number of simple objects (see for example equation (65) in [MRV18], giving a continuous family of simple quantum automorphisms of the complete graph on 4 vertices).

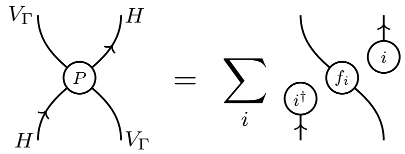
Definition 4.2.27. The *classical subcategory* of QAut(Γ) is the full semisimple monoidal subcategory of quantum automorphisms which are decomposable into a direct sum of ordinary automorphisms.

Remark 4.2.28. In [MRV18, Def 6.8, Rem 6.10], for classical graphs Γ, the objects of this classical subcategory were called *essentially classical* quantum automorphisms.

In other words, a quantum automorphism (H, P) in the classical subcategory is of the following form, where $\{|i\rangle\}$ is an orthonormal basis corresponding to the decomposition of the Hilbert space H into one-dimensional subspaces $H \cong \bigoplus_i \mathbb{C}\,|i\rangle$ and

[11]For fusion categories, it is additionally required that the monoidal unit is simple, which is straightforward to verify in our setting.

$f_i : \Gamma \to \Gamma$ are automorphisms:



(4.16)

We note that a quantum isomorphism between classical graphs is in the classical subcategory if and only if all projectors in its projective permutation matrix commute with each other [MRV18, Prop 6.9].

For a finite group G, let Hilb_G denote the category of finite-dimensional *G-graded Hilbert spaces*, whose objects are finite-dimensional Hilbert spaces H with a Hilbert space decomposition $H = \bigoplus_{g\in G} H_g$ and whose morphisms are grading-preserving linear maps. Hilb_G is a unitary fusion category[12]: the dagger functor is given by the Hilbert space adjoint of a graded linear map, and the monoidal product is defined by the following formula:

$$(H \otimes H')_g := \sum_{a,b\in G, ab=g} H_a \otimes H'_b$$

The simple objects of this fusion category are the one-dimensional G-graded Hilbert spaces. Thus, as a unitary fusion category, Hilb_G is generated by the elements of the group G — corresponding to isomorphism classes of one-dimensional G-graded Hilbert spaces — with tensor product induced by group multiplication.

Proposition 4.2.29. *Let Γ be a quantum graph. The classical subcategory of* $\mathrm{QAut}(\Gamma)$ *is equivalent to the unitary fusion category* $\mathrm{Hilb}_{\mathrm{Aut}(\Gamma)}$.

Proof. By Definition 4.2.27, the classical subcategory is a semisimple monoidal dagger category generated from the ordinary automorphisms of the quantum graph Γ. Thus, there is an obvious monoidal dagger equivalence taking such an automorphism $g \in \mathrm{Aut}(\Gamma)$ to the one-dimensional $\mathrm{Aut}(\Gamma)$-graded Hilbert space in grading g. □

In particular, there is a full inclusion $\mathrm{Hilb}_{\mathrm{Aut}(\Gamma)} \subseteq \mathrm{QAut}(\Gamma)$. In general the inclusion is strict; there will be simple quantum automorphisms which are *not* one-dimensional (see e.g. [MRV18, Exm 6.11]). However, for some graphs this is not the case.

Definition 4.2.30 ([BBC07]). A quantum graph Γ is said to have *no quantum symmetries* if every quantum automorphism is in the classical subcategory, i.e. if $\mathrm{QAut}(\Gamma) \cong \mathrm{Hilb}_{\mathrm{Aut}(\Gamma)}$; or equivalently, if all simple quantum automorphisms are 1-dimensional.

[12]The category Hilb_G is the dagger analogue of the fusion category Vect_G of G-graded vector spaces [EGNO15, Exm 2.3.6] in which every vector space is equipped with an inner product compatible with the grading.

4.2.4 A rapid introduction to Morita theory

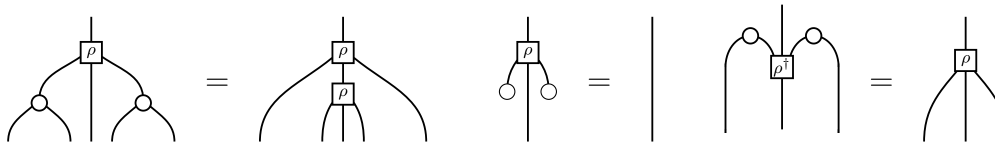
We now recall the theory of Morita equivalence in monoidal dagger categories. Similar expositions can be found in a variety of contexts [CR16, KR08, HVW14]. In the following, we focus on special dagger Frobenius monoids in monoidal dagger categories; however, most definitions and statements below have analogues for more general Frobenius monoids in monoidal categories.

Definition 4.2.31. A *dagger idempotent* in a dagger category is an endomorphism $p : A \to A$ such that $p^2 = p$ and $p^\dagger = p$. We say that a dagger idempotent *splits*, if there is an object V and a morphisms $i : V \to A$ such that $p = ii^\dagger$ and $i^\dagger i = 1_V$.

Example 4.2.32. A dagger idempotent in the dagger category Hilb of finite-dimensional Hilbert spaces and linear maps is an orthogonal projection. Dagger splitting expresses the projector as a map onto the image composed with its adjoint.

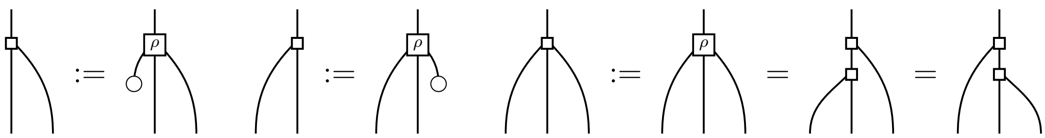
The splitting of an indempotent is unique up to a unitary isomorphism: Indeed, if (i, V) and (i', V') split the same idempotent, then $U = i^\dagger i' : V' \to V$ is unitary with $i' = iU$.

Definition 4.2.33. Let A and B be special dagger Frobenius monoids in a monoidal dagger category. An A–B-*dagger bimodule* is an object M together with an morphism $\rho : A \otimes M \otimes B \to M$ fulfilling the following equations:

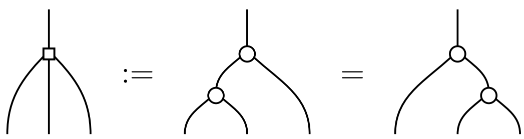


(4.17)

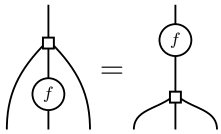
We usually denote an A–B-dagger bimodule M by ${}_AM_B$. For a dagger bimodule ${}_AM_B$, we introduce the following shorthand notation:



Every special dagger Frobenius monoid A gives rise to a trivial dagger bimodule ${}_AA_A$:

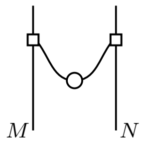


Definition 4.2.34. A *morphism of dagger bimodules* ${}_AM_B \to {}_AN_B$ is a morphism $f : M \to N$ that commutes with the action of the Frobenius monoids:



Two dagger bimodules are *isomorphic*, here written ${}_AM_B \cong {}_AN_B$, if there is a unitary morphism of dagger bimodules ${}_AM_B \to {}_AN_B$.

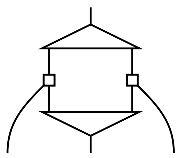
In a monoidal dagger category in which dagger idempotents split, we can compose dagger bimodules ${}_AM_B$ and ${}_BN_C$ to obtain an A–C-dagger bimodule ${}_AM \otimes_B N_C$ as follows. First note that the following endomorphism is a dagger idempotent:



The *relative tensor product* ${}_AM \otimes_B N_C$ is defined as the image of the splitting of this idempotent. We depict the morphism $i : M \otimes_B N \to M \otimes N$ as a downwards pointing triangle:

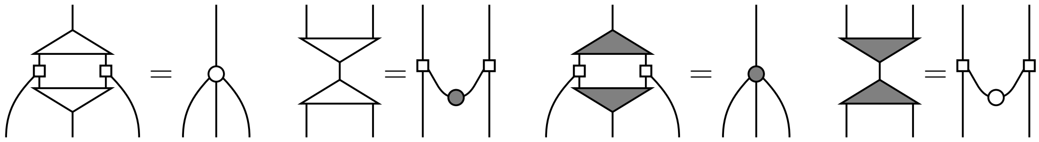
(4.18)

For dagger bimodules ${}_AM_B$ and ${}_BN_C$, the relative tensor product $M \otimes_B N$ is itself an A–C-dagger bimodule with the following action $A \otimes (M \otimes_B N) \otimes C \to M \otimes_B N$:



Definition 4.2.35. Two special dagger Frobenius monoids A and B are *Morita equivalent* if there are dagger bimodules ${}_AM_B$ and ${}_BN_A$ such that ${}_AM \otimes_B N_A \cong {}_AA_A$ and ${}_BN \otimes_A M_B \cong {}_BB_B$.

In other words, special dagger Frobenius monoids A (depicted as a white node) and B (depicted as a grey node) are Morita equivalent if there are dagger bimodules ${}_AM_B$ and ${}_BN_A$ and morphisms $i : A \to M \otimes N$ (depicted as a downwards-pointing white triangle) and $i' : B \to N \otimes M$ (depicted as a downwards-pointing grey triangle) such that the following equations hold:


(4.19)

It can easily be verified that $*$-isomorphic special dagger Frobenius monoids are Morita equivalent.

4.3 A classification of quantum isomorphic graphs

In this section we present our classification of quantum graphs Γ' quantum isomorphic to a given graph Γ in terms of algebraic structures in the monoidal category $\mathrm{QAut}(\Gamma)$.

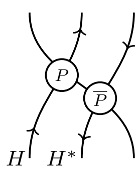
These results are based on the observation that dualizable 1-morphisms $\Gamma' \to \Gamma$ in a dagger 2-category give rise to dagger Frobenius monoids in the endomorphism category $\mathrm{Hom}(\Gamma, \Gamma)$. These Frobenius monoids can therefore be used to classify dualizable morphisms into Γ. This is a prominent technique employed, for example, in the theory of module categories [Ost03b, Ost03a, EGNO15] and the classification of subfactors [BKLR15].

In Section 4.3.1, we classify quantum graphs quantum isomorphic to a given quantum graph (Corollary 4.3.7). In Section 4.3.2, we restrict attention to classical graphs, and classify classical graphs quantum isomorphic to a given classical graph (Corollary 4.3.14).

4.3.1 Classifying quantum isomorphic quantum graphs

We first establish that dualizable 1-morphisms in QGraphIso into a quantum graph Γ give rise to Frobenius monoids in $\mathrm{QAut}(\Gamma)$.

Proposition 4.3.1. *A quantum isomorphism (H, P) between quantum graphs Γ' and Γ gives rise to a special dagger Frobenius monoid in* $\mathrm{QAut}(\Gamma)$*, whose underlying object is the composition $(H \otimes H^*, P \circ \overline{P})$, and whose underlying algebra is the endomorphism algebra of Definition 4.2.5:*



Proof. It is a standard fact in 2-category theory that the composition $P \circ \overline{P}$ of a 1-morphism with its dual in a dagger 2-category gives rise to a Frobenius monoid. In our case, all we need to show is that the structural morphisms of the endomorphism algebra are intertwiners for $P \circ \overline{P}$. This follows immediately from equations (4.14) and (4.15). □

The Frobenius monoids arising from dualizable 1-morphisms in Proposition 4.3.1 have an underlying endomorphism algebra. We abstract this property.

Definition 4.3.2. Let $\mathcal{C}$ be a monoidal dagger category with a faithful monoidal dagger functor $F : \mathcal{C} \to \text{Hilb}$. A *$F$-simple dagger Frobenius monoid* in $\mathcal{C}$ is a dagger Frobenius monoid A in $\mathcal{C}$ such that the underlying dagger Frobenius algebra FA in Hilb is $*$-isomorphic to an endomorphism algebra (Definition 4.2.5).

Every F-simple dagger Frobenius monoid A is special, since FA is special.

In the following, we will be concerned with F-simple dagger Frobenius monoids in QAut(Γ) where $F : \text{QAut}(\Gamma) \to \text{Hilb}$ is the evident forgetful functor.[13] From now on, we omit the functor F from the notation and refer to *simple dagger Frobenius monoids* in QAut(Γ).

Remark 4.3.3. Since QAut(Γ) is the category of finite-dimensional $*$-representations of the Hopf C^*-algebra $A(\Gamma)$, unpacking Definition 4.3.2 gives the definition (4.1) made in the introduction.

The main result of this section is that the converse of Proposition 4.3.1 is also true: simple dagger Frobenius monoids in QAut(Γ) give rise to quantum isomorphisms into Γ.

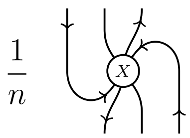
Theorem 4.3.4. *Let Γ be a quantum graph and let X be a simple dagger Frobenius monoid in* QAut(Γ)*. Then there exists a quantum graph Γ_X and a quantum isomorphism $(H, P) : \Gamma_X \to \Gamma$ such that X is $*$-isomorphic to $(H \otimes H^*, P \circ \overline{P})$.*

Proof. We will prove this in Section 4.3.3. □

Remark 4.3.5. From the perspective of category theory, the quantum graph Γ_X is both an Eilenberg-Moore and a Kleisli object [LS02] for the Frobenius monad X in the 2-category QGraphIso.

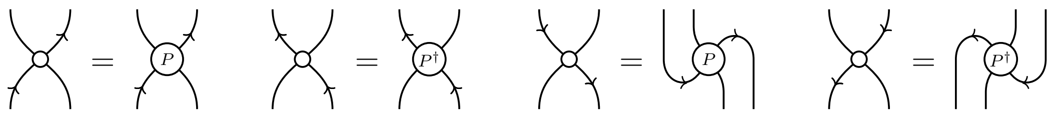
[13] The forgetful functor $\text{QAut}(\Gamma) \to \text{Hilb}$ takes a quantum isomorphism (H, P) to the Hilbert space H and an intertwiner to the underlying linear map; equivalently it is the forgetful functor of the finite-dimensional representation category $\text{QAut}(\Gamma) = \text{Rep}_{\text{fd}}(A(\Gamma))$. See [MRV18, Sec 3.3]

Although we postpone the details of the proof, we quickly sketch the reconstruction of the quantum graph Γ_X and the quantum isomorphism $\Gamma_X \to \Gamma$ from a simple dagger Frobenius monoid X in $\mathrm{QAut}(\Gamma)$. Note firstly that the Frobenius monoid X is a quantum isomorphism $(H \otimes H^*, X) : \Gamma \to \Gamma$ for which the endomorphism algebra is an intertwiner. It is then easy to check that the following linear map $x \in \mathrm{End}(H^* \otimes V_\Gamma \otimes H)$ is a dagger idempotent, i.e. is self-adjoint and fulfils $x^2 = x$:



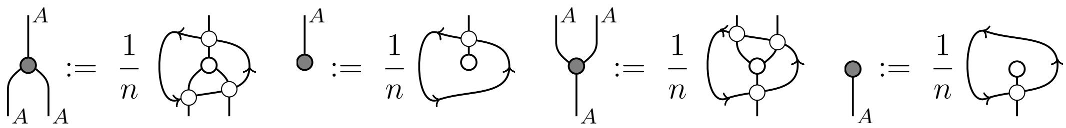
Here, $n = \dim(H)$ denotes the dimension of the Hilbert space H. Splitting this idempotent results in a new Hilbert space A and an isometry $i : A \to H^* \otimes V_\Gamma \otimes H$ which gives rise to (by bending wires) a map $P : H \otimes A \to V_\Gamma \otimes H$, so that X is of the form given in Proposition 4.3.1.

We now define the structure of a quantum graph on A. For this, we use the following shorthand notation:



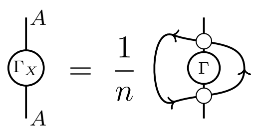
(4.20)

Using the algebra structure on V_Γ (depicted as a white node), we define an algebra structure on A (depicted as a grey node) as follows:



(4.21)

We will show in Section 4.3.3 that this makes A into a special symmetric dagger Frobenius algebra. The quantum graph Γ_X has vertex quantum set $V_{\Gamma_X} := A$ and quantum adjacency matrix $\Gamma_X : A \to A$, defined as follows:



We will prove in Section 4.3.3 that Γ_X is a quantum graph, and that P is a quantum isomorphism from Γ_X to Γ.

Remark 4.3.6. The algebra V_{Γ_X} is in general noncommutative, even if V_Γ is commutative. Quantum graphs therefore naturally emerge in Theorem 4.3.4, even if we restrict our attention to classical graphs Γ. For pseudo-telepathy, we are interested in classical graphs Γ_X, and therefore want V_{Γ_X} to be commutative; in Section 4.3.2, we give a necessary and sufficient condition on the Frobenius monoid X for this to be the case.

In summary, for every quantum isomorphism $\Gamma' \to \Gamma$ we get a simple dagger Frobenius monoid in QAut(Γ) (Proposition 4.3.1), and for every simple dagger Frobenius monoid in QAut(Γ) we get a quantum isomorphism $\Gamma' \to \Gamma$ (Theorem 4.3.4). With the right notion of equivalence (Definition 4.2.35) of simple dagger Frobenius monoids, this in fact gives us a classification of quantum graphs quantum isomorphic to Γ.

Corollary 4.3.7. *Let Γ be a quantum graph. The constructions of Proposition 4.3.1 and Theorem 4.3.4 induce a bijective correspondence between:*

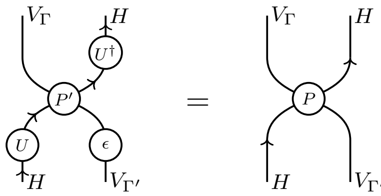
- *Isomorphism classes of quantum graphs Γ' such that there exists a quantum isomorphism $\Gamma' \to \Gamma$.*
- *Morita equivalence classes of simple dagger Frobenius monoids in* QAut(Γ).

Proof. This is a straightforward application of a general theorem (Theorem D.1), which holds in any dagger 2-category in which dagger idempotents split, and which is proved in Appendix D.

To apply this theorem, we note that dagger idempotents split in QGraphIso, as shown in [MRV18, Proof of Theorem 6.4]. The conditions of the theorem are therefore satisfied. The result follows immediately, since every 1-morphism in QGraphIso can be normalized to a special 1-morphism (see Appendix D) by multiplication with a scalar factor, dagger equivalences in QGraphIso are precisely ordinary isomorphisms (Proposition 4.2.24), and dagger Frobenius monoids in QAut(Γ) are split if and only if they are simple (Proposition 4.3.1 and Theorem 4.3.4). □

Remark 4.3.8. The classification in Corollary 4.3.7 only depends on the monoidal category QAut(Γ) and its fiber functor F : QAut(Γ) $\to$ Hilb. In the language of compact quantum groups, the classification of quantum graphs quantum isomorphic to a classical graph Γ depends only on the quantum automorphism group of Γ, and not on its action on the set of vertices V_Γ.

Remark 4.3.9. Corollary 4.3.7 provides a classification of all quantum graphs Γ' which are quantum isomorphic to a quantum graph Γ, but does not classify the explicit quantum isomorphisms between Γ' and Γ. Such a classification can in fact be achieved as follows. We take two quantum isomorphisms $(H,P) : \Gamma' \to \Gamma$ and $(H',P') : \Gamma'' \to \Gamma$ into Γ to be *equivalent* if there is an isomorphism of quantum graphs (Definition 4.2.15) $\epsilon : \Gamma' \to \Gamma''$ and a unitary map $U : H \to H'$ such that the following holds[14]:



(4.22)

It then follows from Remark D.4 that the constructions of Proposition 4.3.1 and Theorem 4.3.4 induce a bijection between the following sets:

- Quantum isomorphisms into Γ up to the equivalence relation (4.22).
- $*$-isomorphism classes of simple dagger Frobenius monoids in $\mathrm{QAut}(\Gamma)$.

In other words, $*$-isomorphism classes of simple dagger Frobenius monoids classify quantum isomorphisms into Γ up to dagger equivalence, while the coarser Morita equivalence classes only classify quantum graphs which are quantum isomorphic to Γ, without keeping track of the quantum isomorphism itself. For applications to pseudo-telepathy, we are mainly interested in this latter, coarser classification.

The following is a first, easy application of Corollary 4.3.7.

Corollary 4.3.10. *Let Γ be a quantum graph with trivial quantum automorphism group, that is,* $\mathrm{QAut}(\Gamma) \cong \mathrm{Hilb}$. *Then, every quantum graph that is quantum isomorphic to Γ is also isomorphic to Γ.*

Proof. The category $\mathrm{QAut}(\Gamma) \cong \mathrm{Hilb}$ has only one Morita equivalence class of simple dagger Frobenius monoids, corresponding to the graph Γ itself. □

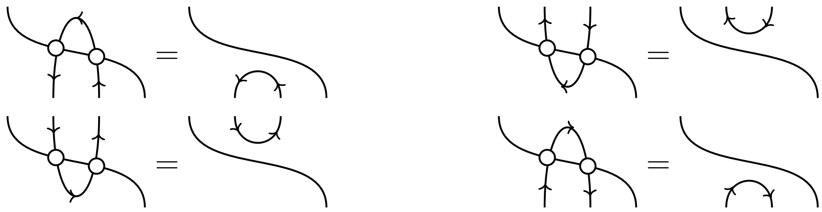
4.3.2 Classifying quantum isomorphic classical graphs

In Corollary 4.3.7, we classified quantum graphs Γ' quantum isomorphic to a quantum graph Γ in terms of Morita equivalence classes of simple dagger Frobenius monoids in $\mathrm{QAut}(\Gamma)$. However, as discussed in Remark 4.3.6, if Γ is a classical graph, then the quantum graph Γ_X corresponding to a simple dagger Frobenius monoid X in $\mathrm{QAut}(\Gamma)$ will in general not be classical.

[14]For classical graphs Γ, Γ' and Γ'', this translates into the following condition on projective permutation matrices. Two projective permutation matrices $\{P_{v',v}\}_{v' \in V_{\Gamma'}, v \in V_\Gamma}$ and $\{P'_{v'',v}\}_{v'' \in V_{\Gamma''}, v \in V_\Gamma}$ on Hilbert spaces H and H' are equivalent if there is a graph isomorphism $\epsilon : \Gamma' \to \Gamma''$ and a unitary $U : H \to H'$ such that for all $v \in V_\Gamma$ and $v' \in V_{\Gamma'}$ it holds that $P_{v',v} = U^\dagger P'_{\epsilon(v'),v} U$

In this section, we prove a necessary and sufficient condition for commutativity of the algebra V_{Γ_X}, and therefore classicality of the graph Γ_X. This results in a classification of classical graphs quantum isomorphic to a given classical graph Γ.

For a quantum isomorphism $(H, P) : \Gamma' \to \Gamma$, equations (4.14) and (4.15) are expressed in the shorthand notation (4.20) as follows:



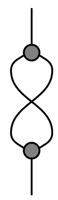
(4.23)

(4.24)

These equations look exactly like the second Reidemeister move from knot theory. Together with equations (4.10) and (4.11), this leads to a very flexible topological calculus, allowing us to move oriented Hilbert space wires almost freely through our diagrams, interconverting the algebra V_Γ (in the following depicted by white nodes) and the algebra $V_{\Gamma'}$ (depicted by grey nodes) when passing through the corresponding nodes.

We also recall the following piece of folklore [Bae].

Proposition 4.3.11. *Let A be a special symmetric dagger Frobenius algebra (depicted as a grey node). Then, the following endomorphism $P_{Z(A)} : A \to A$ is a projector onto the center of A:*

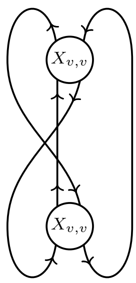


Proof. For an appropriately normalized matrix algebra (see e.g. Definition 4.2.5), Proposition 4.3.11 can easily be verified. Semisimplicity then extends this formula to general finite-dimensional C^*-algebras. $\square$

In particular, $\dim(Z(A)) = \mathrm{Tr}(P_{Z(A)})$, and A is commutative if and only if $\mathrm{Tr}(P_{Z(A)}) = \dim(A)$. We use this fact to derive our commutativity condition.

Theorem 4.3.12. *Let Γ be a classical graph, let $(H \otimes H^*, X)$ be a simple dagger Frobenius monoid in $\mathrm{QAut}(\Gamma)$ and let Γ_X be the associated quantum graph. Then, the dimension of the center of V_{Γ_X} can be expressed as follows, where $X_{v,v}$ are the* *diagonal components of the projective permutation matrix underlying X (see* (4.12)*):*

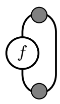
$$\dim(Z(V_{\Gamma_X})) \quad = \quad \frac{1}{\dim(H)} \sum_{v \in V_\Gamma} \tag{4.25}$$



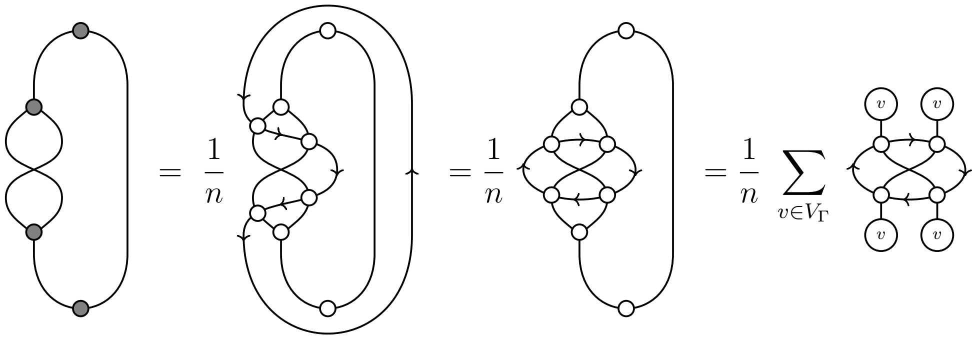
In particular, Γ_X *is classical if and only if* $\dim(Z(V_{\Gamma_X})) = |V_\Gamma|$.

Proof. Note that for a special symmetric dagger Frobenius algebra A (depicted as a grey node) and a linear map $f : A \to A$, the trace $\mathrm{Tr}(f)$ can be computed as follows:

$$\mathrm{Tr}(f) \quad =$$



Let $(H, P) : \Gamma_X \to \Gamma$ be a quantum isomorphism such that $P \circ \overline{P} = X$ (see Theorem 4.3.4). Using the shorthand notation (4.20) for P, the trace of the projector of Proposition 4.3.11 for the algebra V_{Γ_X} (depcited as a grey node) can be expressed as follows, where $n = \dim(H)$:



In the first equation, we have introduced a circle (4.8) to the right of the diagram and then enlarged this circle over parts of the diagram, converting grey V_{Γ_X}-nodes into white V_Γ-nodes in the process. In the last equation we used the expression from Example 4.2.7 for the commutative special dagger Frobenius algebra V_Γ.

Using $X = P \circ \overline{P}$, and untangling the above equation leads to the formula (4.25). In particular, Γ_X is classical if V_{Γ_X} is commutative, that is, if $\dim(Z(V_{\Gamma_X})) = \dim(V_{\Gamma_X})$. Since quantum isomorphisms preserve dimensions (Proposition 4.2.20), we have $\dim(V_{\Gamma_X}) = |V_\Gamma|$. Thus, Γ_X is classical if and only if $\dim(Z(V_{\Gamma_X})) = |V_\Gamma|$. $\square$

Remark 4.3.13. In contrast to the classification of quantum isomorphic quantum graphs (see Remark 4.3.8), the condition in Theorem 4.3.12 does not only depend on the abstract monoidal category with fiber functor $\mathrm{QAut}(\Gamma)$. In the language of compact quantum groups, the classification of classical graphs Γ' which are quantum isomorphic to a classical graph Γ depends both on the quantum automorphism group of Γ and its action on V_Γ.

We therefore obtain a classification of classical graphs which are quantum isomorphic to a classical graph Γ in terms of simple dagger Frobenius monoids in $\mathrm{QAut}(\Gamma)$.

Corollary 4.3.14. *Let Γ be a classical graph. Then, the construction of Proposition 4.3.1 induces a bijective correspondence between the following sets:*

- *Isomorphism classes of classical graphs Γ' such that there exists a quantum isomorphism $\Gamma' \to \Gamma$.*
- *Morita equivalence classes of simple dagger Frobenius monoids in $\mathrm{QAut}(\Gamma)$ for which the expression* (4.25) *evaluates to $|V_\Gamma|$.*

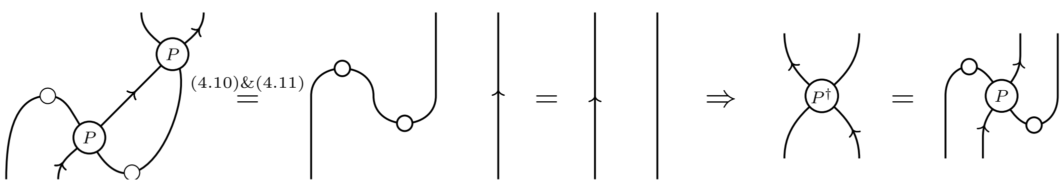
Proof. Corollary 4.3.14 follows from restricting the classification of quantum isomorphic quantum graphs (Corollary 4.3.7) to Morita equivalence classes of simple dagger Frobenius monoids fulfilling the conditions of Theorem 4.3.12 and their associated classical graphs. □

4.3.3 Proof of Theorem 4.3.4

In this section, we prove Theorem 4.3.4. We first introduce two technical propositions.

Proposition 4.3.15. *Let A and B be special symmetric dagger Frobenius algebras, and let $P : H \otimes A \to B \otimes H$ be a linear map fulfilling the first two equations of* (4.10) *and the first two equations of* (4.11)*. Then, P is unitary if and only if it also fulfils the last equation of* (4.10).

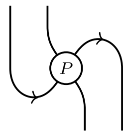
Proof. The 'if'-direction follows immediately from Theorem 4.2.23 (in the special case where the quantum adjacency matrices are identities). For the other direction, observe that if P is unitary, then the following implication holds:



□

We adopt the following terminology, originally introduced in [Ocn89, Jon99] and adapted to a categorical setting in [Vic12a, RV19b] (see Chapter 3):

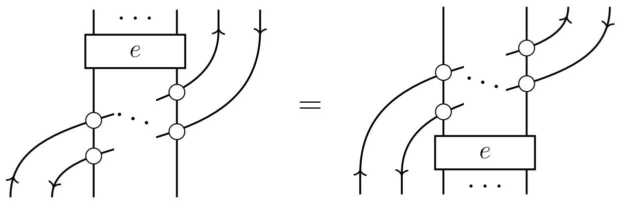
Definition 4.3.16. Let A, B and H be Hilbert spaces. A linear map $P : H \otimes A \to B \otimes H$ is *biunitary*, if it and the following 'quarter-rotation' are unitary:



From now on, we will use the shorthand (4.20) for P. It can straightforwardly be verified that a morphism $P : H \otimes A \to B \otimes H$ is biunitary if and only if the equations (4.23) and (4.24) hold.

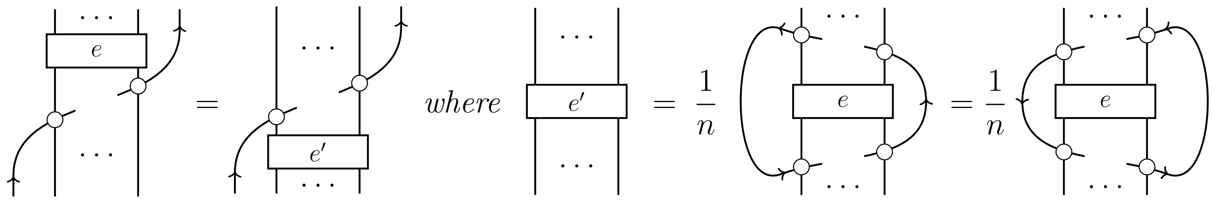
The following proposition allows linear maps that can pull through a double wire to 'jump' over a single wire, acquiring a surrounding bubble as they do so.

Proposition 4.3.17. *Let $S : H \otimes A \to B \otimes H$ be a biunitary linear map, written using the conventions above, and let $n = \dim(H)$. Let $e : B^{\otimes k} \to B^{\otimes r}$ be a linear map between tensor powers of A fulfilling the following:*



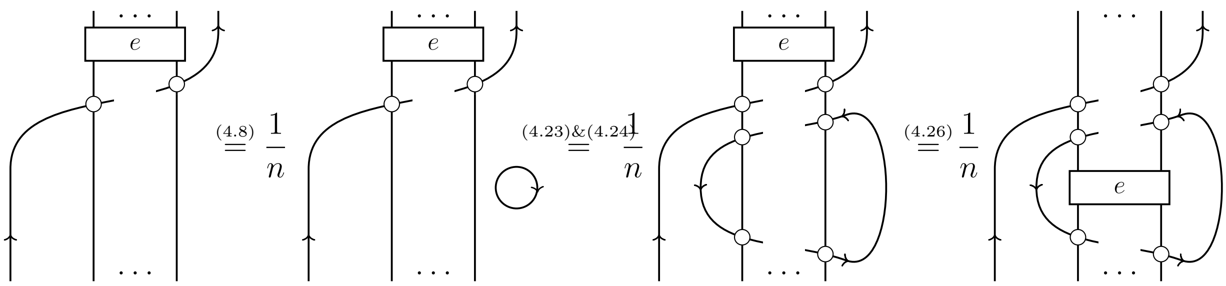
(4.26)

Then, the following holds:



Moreover, if $f : B^{\otimes l} \to B^{\otimes k}$ and $e : B^{\otimes k} \to B^{\otimes r}$ both fulfil (4.26), it follows that $(1_{B^{\otimes k}})' = 1_{A^{\otimes k}}$ and $(ef)' = e'f'$.

Proof.



The statements $(1_{B^{\otimes k}})' = 1_{A^{\otimes k}}$ and $(ef)' = e'f'$ are verified analogously. □

Remark 4.3.18. Proposition 4.3.17 is closely related to standard techniques in the setting of *planar algebras*. In particular, it is analogous to [Jon99, Prop 2.11.7 and Thm 2.11.8].

We now prove Theorem 4.3.4.

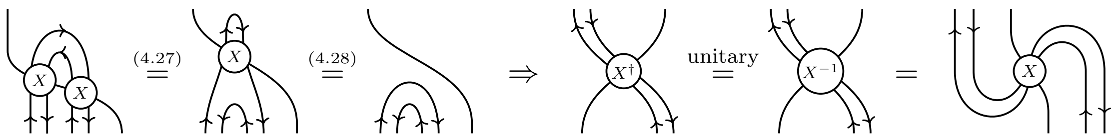
Theorem 4.3.4. *Let* Γ *be a quantum graph and let* X *be a simple dagger Frobenius monoid in* $\mathrm{QAut}(\Gamma)$*. Then there exist a quantum graph* Γ_X *and a quantum isomorphism* $(H, P) : \Gamma_X \to \Gamma$ *such that* X *is* $*$*-isomorphic to* $(H \otimes H^*, P \circ \overline{P})$.

Proof. A simple dagger Frobenius monoid in $\mathrm{QAut}(\Gamma)$ is $*$-isomorphic to a quantum isomorphism $(H \otimes H^*, X) : \Gamma \to \Gamma$, represented by a linear map $X : (H \otimes H^*) \otimes V_\Gamma \to V_\Gamma \otimes (H \otimes H^*)$, fulfilling:

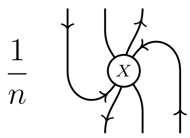

(4.27)


(4.28)

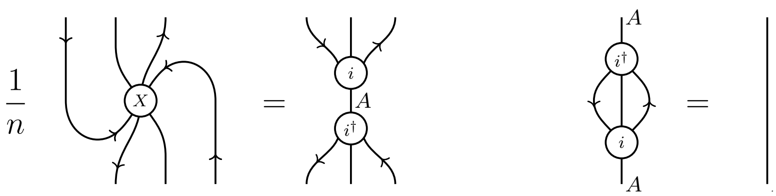
We first note that since X is a quantum isomorphism and therefore unitary (Theorem 4.2.23), the following holds:



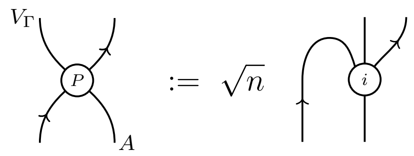
It then follows straightforwardly from that the following linear map $x \in \mathrm{End}(H^* \otimes V_\Gamma \otimes H)$ is a dagger idempotent, i.e. it is self-adjoint and fulfils $x^2 = x$:



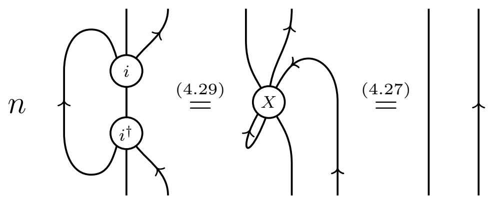
Splitting this idempotent (see Section 4.2.4) produces an isometry $i : A \to H^* \otimes V_\Gamma \otimes H$ from some Hilbert space A such that:


(4.29)

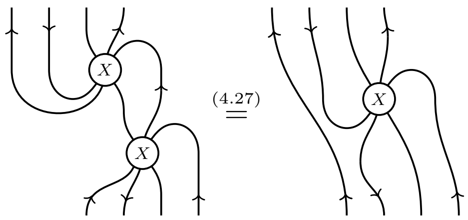
Note that this splitting is unique up to a unitary morphism. We now claim that the following linear map $P : H \otimes A \to V_\Gamma \otimes H$ is biunitary:



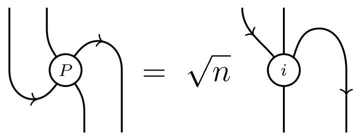
The unitarity equation $PP^\dagger = \mathbb{1}$ follows from:



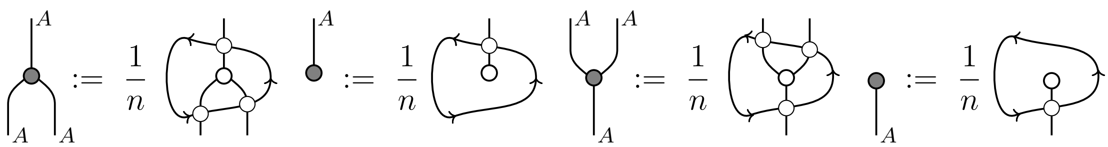
The other equation $P^\dagger P = \mathbb{1}$ follows from conjugating the three right-most wires of the following by i and using (4.29):



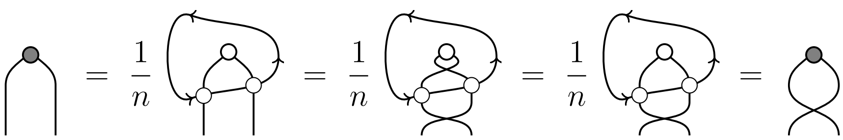
Unitarity of the quarter-rotation follows analogously:



From now on we will use the short-hand notation for P introduced in (4.20). Using the algebra on V_Γ, we define the following linear maps on A:

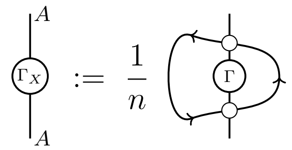


It follows from Proposition 4.3.17 that these structures form a special dagger Frobenius algebra. In fact, they form a special *symmetric* dagger Frobenius algebra, since we also have that



Here, the second equation is a direct consequence of the graphical calculus, moving the bottom right node all the way around the oriented loop to the left. The third equation is symmetry of the algebra on V_Γ.

We also define the following endomorphism on A, which is — again due to Proposition 4.3.17 — an adjacency matrix of a quantum graph.



It follows from Proposition 4.3.15 and Proposition 4.3.17 that P is a quantum graph isomorphism from Γ_X to Γ and it follows from (4.29) that $X = P \circ \overline{P}$. □

4.4 Frobenius monoids in classical subcategories

In Section 4.3, we classified quantum and classical graphs which are quantum isomorphic to a given quantum or classical graph Γ in terms of Morita equivalence classes of simple dagger Frobenius monoids in the monoidal category $\mathrm{QAut}(\Gamma)$. Although some of these categories $\mathrm{QAut}(\Gamma)$ have been studied in the framework of compact quantum groups [BC08, BB09], the general classification of Morita equivalence classes of Frobenius monoids in such categories seems unfeasible using current techniques.

We therefore focus on the much more tractable classical subcategories $\mathrm{Hilb}_{\mathrm{Aut}(\Gamma)} \subseteq \mathrm{QAut}(\Gamma)$ (see Proposition 4.2.29), where the Morita equivalence classes of Frobenius monoids are known [Ost03b]. Although these Frobenius monoids are in a sense 'classical', being sums of ordinary automorphisms, we will see in Section 4.5.2 (and have already seen in the introduction) that they can still give rise to quantum but not classically isomorphic graphs.

Moreover, if a quantum graph Γ has no quantum symmetries (Definition 4.2.30) — that is, if $\mathrm{QAut}(\Gamma) \cong \mathrm{Hilb}_{\mathrm{Aut}(\Gamma)}$ — we are able to completely classify quantum graphs quantum isomorphic to Γ in terms of straightforward group theoretical properties of the automorphism group of Γ.

4.4.1 Quantum isomorphic quantum graphs from groups

We state the Morita classification of special Frobenius monoids in the category of graded vector spaces.

Proposition 4.4.1 ([Ost03b, Exm 2.1])**.** *Let G be a finite group. Up to Morita equivalence, indecomposable,*[15] *symmetric,*[16] *special dagger Frobenius monoids in* Hilb_G *correspond to pairs (L, ψ) where L is a subgroup of G and $\psi : L \times L \to U(1)$ is a 2-cocycle up to the equivalence relation:*

$$(L,\psi) \sim (L',\psi') \Leftrightarrow L' = gLg^{-1} \text{ and } \psi' \text{ is cohomologous to} \\ \psi^g(x,y) := \psi(gxg^{-1}, gyg^{-1}) \text{ for some } g \in G \tag{4.30}$$

Proof. Morita equivalence classes of indecomposable, symmetric, special Frobenius monoids in the fusion category Vect_G of finite-dimensional G-graded vector spaces correspond to equivalence classes of semisimple indecomposable module categories over Vect_G whose classification in terms of pairs (L, ψ) up to the equivalence relation (4.30) can be found in the fusion category literature [Ost03b, Exm 2.1]. That this classification also applies to dagger Morita equivalence classes of indecomposable, symmetric, special dagger Frobenius monoids in Hilb_G can be seen as follows. Forgetting the Hilbert space structure, every such Frobenius monoid A in Hilb_G is Morita equivalent (in Vect_G) to a twisted group algebra $\mathbb{C}L^\psi$ (see (4.31)). This twisted group algebra can be endowed with an inner product making it into a special dagger Frobenius monoid in Hilb_G. Proposition 4.4.1 is proven once we show that we can promote the invertible bimodule (in Vect_G) between A and $\mathbb{C}L^\psi$ to an invertible dagger bimodule (Definition 4.2.33) in Hilb_G. More generally, we will show that if A and B are symmetric special dagger Frobenius monoids in Hilb_G and M is a G-graded A–B bimodule (in the category Vect_G), then M can be endowed with an inner product compatible with the grading (i.e. making M into a G-graded Hilbert space), giving it the structure of a dagger bimodule in the category Hilb_G.

For a symmetric, special dagger Frobenius monoid A in Hilb_G, we define an antilinear involution $(-)^* : A \to A$ as in Theorem 4.2.4. The fact that A is a dagger Frobenius monoid in Hilb_G implies the following, where $\langle \cdot, \cdot \rangle_A$ is the inner product on A:

$$\langle ab, c\rangle_A = \langle b, a^* c\rangle_A = \langle a, cb^*\rangle_A$$

Given a G-graded A–B bimodule M (in Vect_G), choose an arbitrary inner product $\langle \cdot, \cdot \rangle'$ on M which is compatible with the grading, and let $\{a_i\}_{i \in \mathcal{I}}$ and $\{b_j\}_{j \in \mathcal{J}}$ be G-homogeneous orthonormal bases of the G-graded Hilbert spaces underlying A and

[15] A Frobenius monoid is *indecomposable* if it is not a direct sum of non-trivial Frobenius monoids. We observe that all simple dagger Frobenius monoids are indecomposable.

[16] Frobenius monoids in pivotal (a.k.a. sovereign) categories are *symmetric* if [KR08, Eqn. (2.3)] holds. All simple dagger Frobenius monoids in Hilb_G (with the obvious pivotal structure) are symmetric.

B, containing the units of the respective algebras. We define the following new inner product on M which is also compatible with the grading:

$$\langle m_1, m_2\rangle := \sum_{i,j} \langle a_i m_1 b_j, a_i m_2 b_j\rangle'$$

To verify that M, equipped with this inner product, is a dagger bimodule, we need to verify the following equation (corresponding to the last equation of Definition 4.2.33) for all $a \in A$ and $b \in B$:

$$\langle a^* m_1 b^*, m_2\rangle = \langle m_1, a m_2 b\rangle$$

For simplicity, we will prove this equation for b being the unit of B. The general case is completely analogous.

$$\langle m_1, a m_2\rangle = \sum_{i,j} \langle a_i m_1 b_j, a_i a m_2 b_j\rangle' = \sum_{i,j} \left\langle a_i m_1 b_j, \sum_k \langle a_k, a_i a\rangle_A a_k m_2 b_j \right\rangle'$$

$$= \sum_{i,j,k} \overline{\langle a_i, a_k a^*\rangle_A} \, \langle a_i m_1 b_j, a_k m_2 b_j\rangle' = \sum_{j,k} \langle a_k a^* m_1 b_j, a_k m_2 b_j\rangle' = \langle a^* m_1, m_2\rangle$$

Here, the second and fourth equations use orthonormality of the basis $\{a_k\}_{k\in\mathcal{I}}$. □

The underlying algebra of the Frobenius monoid associated to (L, ψ) is the twisted group algebra $\mathbb{C}L^\psi$ defined on the Hilbert space $\mathbb{C}L$ with orthonormal basis given by the group elements and algebra structure defined as:

$$g \star_\psi h := \frac{1}{\sqrt{|L|}} \, \psi(g, h) gh \qquad\qquad e_\psi := \sqrt{|L|} \; \overline{\psi}(e, e) \; e \tag{4.31}$$

Here again, the normalization factors are chosen to make $\mathbb{C}L^\psi$ special (see Remark 4.2.6).

The Frobenius monoid (L, ψ) is simple in the sense of Definition 4.3.2, if the algebra $\mathbb{C}L^\psi$ is simple. Groups with 2-cocycles ψ such that $\mathbb{C}L^\psi$ is simple have a long history and are known as *groups of central type*, while the corresponding 2-cocycles are said to be *nondegenerate* (see [EGNO15, Def 7.12.21]). This leads to the following consequence of Corollary 4.3.7.

Corollary 4.4.2. *Let Γ be a quantum graph. Every subgroup of central type (L, ψ) of $\mathrm{Aut}(\Gamma)$ induces a quantum graph $\Gamma_{L,\psi}$ and a quantum isomorphism $\Gamma_{L,\psi} \to \Gamma$. Moreover, if Γ has no quantum symmetries, this gives rise to a bijective correspondence between the following sets:*

- *Isomorphism classes of quantum graphs* Γ' *such that there exists a quantum isomorphism* $\Gamma' \to \Gamma$.
- *Subgroups of central type* (L, ψ) *of* $\mathrm{Aut}(\Gamma)$ *up to the equivalence relation* (4.30).

Proof. The statement is a direct consequence of Proposition 4.4.1; the classification of Morita equivalence classes of simple dagger Frobenius monoids in the category of $\mathrm{Aut}(\Gamma)$-graded Hilbert spaces. □

Remark 4.4.3. Two Frobenius monoids in $\mathrm{Hilb}_{\mathrm{Aut}(\Gamma)} \subseteq \mathrm{QAut}(\Gamma)$ might be Morita equivalent in $\mathrm{QAut}(\Gamma)$ even if they are not in $\mathrm{Hilb}_{\mathrm{Aut}(\Gamma)}$. Therefore, the bijective correspondence of Corollary 4.4.2 holds only if Γ has no quantum symmetries, that is, if $\mathrm{QAut}(\Gamma) \cong \mathrm{Hilb}_{\mathrm{Aut}(\Gamma)}$.

This makes the classification of quantum isomorphic quantum graphs quite concrete, particularly if one of the graphs has no quantum symmetries.

Example 4.4.4. Let C_n be the cycle graph with $n \geq 5$ vertices. It is known [Ban05b, Lem 3.5] that C_n has no quantum symmetries. Therefore, quantum isomorphic quantum graphs Γ' are in correspondence with subgroups of central type of $\mathrm{Aut}(C_n) = D_n$. All subgroups of D_n are either cyclic or dihedral. For odd n, D_n has no subgroup of central type. For even n, the only such subgroups are the abelian groups $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, acting by 180-degree rotations and reflections on the cycle graph. Since there is only one nondegenerate second cohomology class of $\mathbb{Z}_2 \times \mathbb{Z}_2$, the equivalence relation (4.30) reduces to conjugacy of subgroups. If 4 does not divide n, there is only one conjugacy class of $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroups; if 4 divides n, there are two such conjugacy classes, depending on whether the line of reflection is through opposing edges or through opposing vertices. We therefore conclude the following:

- For odd n, C_n is only isomorphic to itself.
- For even n not divisible by 4, there is exactly one other quantum graph quantum isomorphic to C_n.
- For n divisible by 4, there are exactly two other quantum graphs quantum isomorphic to C_n.

We will show in Example 4.4.17 that none of these quantum graphs are classical graphs.

We now explicitly construct the simple dagger Frobenius monoid in $\mathrm{QAut}(\Gamma)$ corresponding to a subgroup of central type (L, ψ) of the automorphism group $\mathrm{Aut}(\Gamma)$ of a quantum graph Γ.

Every quantum isomorphism X in the classical subcategory $\mathrm{Hilb}_{\mathrm{Aut}(\Gamma)} \subseteq \mathrm{QAut}(\Gamma)$ (see Definition 4.2.27) is of the form (4.16) with some orthonormal basis $\{|l\rangle \mid l \in L\}$ of the underlying Hilbert space H and permutations $\{l \in L\}$ where $L \subseteq \mathrm{Aut}(\Gamma)$ is some subset of the automorphism group of Γ. If X is moreover a simple dagger Frobenius monoid, by Proposition 4.4.1 we can assume without loss of generality that L is a subgroup of central type of the automorphism group and that the basis $\{|l\rangle\}$ is determined by a *-isomorphism of algebras $\mathbb{C}L^{\psi} \cong \mathrm{End}(H)$. The data defining such an isomorphism is known in the quantum information community as a nice unitary error basis (see Chapter 3).

Definition 4.4.5 ([KR03])**.** A *nice unitary error basis* (nice UEB) for a group of central type (L, ψ) is a family of unitary endomorphisms $\{U_a \mid a \in L\}$ on some Hilbert space H with $|L| = \dim(H)^2$ and such that for all $a, b \in L$:

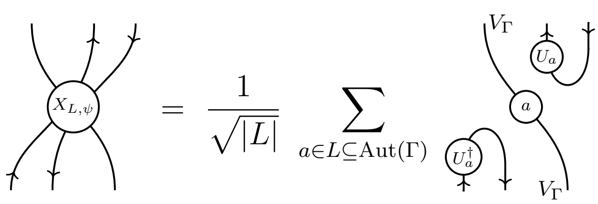
$$\mathrm{Tr}(U_a^{\dagger} U_b) = \dim(H)\ \delta_{a,b} \qquad\qquad U_a U_b = \psi(a,b) U_{ab} \tag{4.32}$$

The group L is called the *index group* of the nice UEB. From now on, and without loss of generality, we will always assume that $\psi(e,h) = 1 = \psi(h,e)$ and therefore that $U_e = \mathbb{1}_H$.

A nice UEB induces a *-isomorphism of algebras $\mathbb{C}L^{\psi} \to \mathrm{End}(H)$, $a \mapsto \sqrt{\dim(H)}^{-1} U_a$ (see Remark 4.2.6 for our normalization of the endomorphism algebra) and every $*$-isomorphism between $\mathbb{C}L^{\psi}$ and $\mathrm{End}(H)$ is of this form.

We summarize this discussion in the following proposition.

Proposition 4.4.6. *Let Γ be a quantum graph. Every simple dagger Frobenius monoid in $\mathrm{Hilb}_{\mathrm{Aut}(\Gamma)} \subseteq \mathrm{QAut}(\Gamma)$ is Morita equivalent to a simple dagger Frobenius monoid $(H \otimes H^*, X_{L,\psi})$ for some Hilbert space H, where the underlying linear map $X_{L,\psi} : (H \otimes H^*) \otimes V_\Gamma \to V_\Gamma \otimes (H \otimes H^*)$ is defined as follows:*



(4.33)

Here, (L, ψ) is a subgroup of central type of $\mathrm{Aut}(\Gamma)$ *and $\{U_a \mid a \in L\}$ is a corresponding nice UEB. The endomorphism $a : V_\Gamma \to V_\Gamma$ denotes the action of $a \in L \subseteq \mathrm{Aut}(\Gamma)$ on the quantum set of vertices V_Γ.*

Remark 4.4.7. Different nice UEBs for the same subgroup of central type (L, ψ) — that is, different $*$-isomorphisms $\mathbb{C}L^\psi \cong \mathrm{End}(H)$ — give rise to $*$-isomorphic, and in particular Morita equivalent, simple dagger Frobenius monoids $X_{L,\psi}$, and thus to isomorphic induced quantum graphs. Therefore, the particular choice of UEB does not play a role in the following classification.

Remark 4.4.8. The fact that $X_{L,\psi}$ is in the classical subcategory (Definition 4.2.27) does not mean that its splitting — the induced quantum isomorphism from some quantum graph $\Gamma_{L,\psi}$ to Γ — is an ordinary isomorphism. If this were not the case, we could never generate any non-isomorphic graph from Frobenius monoids in the classical subcategory. In fact, it is a direct consequence of Corollary 4.3.7 that the splitting is only an ordinary isomorphism if $X_{L,\psi}$ is Morita trivial in $\mathrm{QAut}(\Gamma)$; if Γ has no quantum symmetries, this only happens if L is trivial.

Remark 4.4.9. For a classical graph Γ, the underlying projective permutation matrix of the quantum isomorphism (4.33) is the following, for $v, w \in V_\Gamma$:

$$(X_{L,\psi})_{v,w} := \frac{1}{\sqrt{|L|}} \sum_{a \in L \subseteq \mathrm{Aut}(\Gamma)} \delta_{a(v),w} \; P_{U_a}$$

Here $P_{U_a} : \mathrm{End}(H) \to \mathrm{End}(H)$ denotes the projector on the one-dimensional subspace spanned by $U_a \in \mathrm{End}(H)$.

4.4.2 Quantum isomorphic classical graphs from groups

We now consider the conditions under which a central type subgroup of the automorphism group of a classical graph gives rise to a quantum isomorphic classical graph. In particular, we translate the classicality condition of Theorem 4.3.12 into a condition on subgroups of central type.

We first discuss some properties of nondegenerate 2-cocycles. We denote the *centralizer* of a group element $a \in L$ by $Z_a := \{b \in L \mid ab = ba\}$ and the *commutator* of two group elements $a, b \in L$ by $[a, b] := aba^{-1}b^{-1}$.

For a 2-cocycle $\psi : L \otimes L \to U(1)$, we define the following function:

$$\rho_\psi : L \otimes L \to U(1) \qquad\qquad \rho_\psi(a, b) := \psi(a, b)\overline{\psi}(aba^{-1}, a)$$

If L is abelian, it can be shown that ρ_ψ is an *alternating bicharacter* (that is, a homomorphism in both arguments such that $\rho_\psi(a, b) = \overline{\rho}_\psi(b, a)$). In the general setting, and for nondegenerate 2-cocycle ψ, the following still holds.

Proposition 4.4.10 ([EGNO15, Ex 7.12.22.v]). *Let (L, ψ) be a group of central type and let $x \in L$. Then $\rho_\psi(x, -)|_{Z_x} : Z_x \to U(1)$ is a multiplicative character of the centralizer Z_x and $\rho_\psi(x, -)|_{Z_x}$ is non-trivial for every $x \neq e_L$, that is:*

$$\rho_\psi(x, a) = 1 \quad \forall a \in Z_x \qquad \Rightarrow \qquad x = e_L \tag{4.34}$$

If (L, ψ) is a group of central type, we may therefore think of ρ_ψ as a nondegenerate alternating form on L. In particular, we borrow the following definitions and terminology from the theory of symplectic forms on groups [BDGM14].

Definition 4.4.11. Let (L, ψ) be a group of central type and let $S \subseteq L$ be a subset. The *orthogonal complement* $S^\perp$ of S is the following subset of L:

$$S^\perp := \{g \in L \mid \rho_\psi(g, a) = 1 \;\; \forall a \in Z_g \cap S\}$$

We say that a subset S is *coisotropic* if $S^\perp \subseteq S$.

Proposition 4.4.10 leads to the following observation.

Proposition 4.4.12. *For a group of central type (L, ψ) and a subgroup $H \subseteq L$ we define:*

$$\Phi_H^{L,\psi} \; := \sum_{\substack{a,b \in H \\ [a,b]=e}} \rho_\psi(a, b)$$

Then, $\Phi_H^{L,\psi} \in \mathbb{N}$ and $\Phi_H^{L,\psi} \leq |L|$ with equality if and only if H is coisotropic.

Proof. Using orthogonality of characters of the group $Z_a \cap H$, we calculate:

$$\Phi_H^{L,\psi} \;=\; \sum_{a \in H} \sum_{b \in Z_a \cap H} \rho_\psi(a, b) \;=\; \sum_{\substack{a \in H \\ \rho_\psi(a,b)=1 \;\forall b \in Z_a \cap H}} |Z_a \cap H|$$

Thus, $\Phi_H^{L,\psi}$ is a natural number. Again using orthogonality of characters and equation (4.34) we note the following:

$$\sum_{\substack{a \in L, b \in H \\ [a,b]=e}} \rho_\psi(a, b) \;=\; \sum_{b \in H} \sum_{a \in Z_b} \rho_\psi(a, b) \overset{(4.34)}{=} \sum_{b \in H} |Z_b| \, \delta_{b,e} \;=\; |L|$$

On the other hand, we find:

$$\sum_{\substack{a \in L, b \in H \\ [a,b]=e}} \rho_\psi(a, b) \;=\; \sum_{\substack{a,b \in H \\ [a,b]=e}} \rho_\psi(a, b) + \sum_{a \in L \setminus H} \sum_{b \in Z_a \cap H} \rho_\psi(a, b) \;=\; \Phi_H^{L,\psi} + \sum_{\substack{a \in L \setminus H \\ \rho_\psi(a,b)=1 \;\forall b \in Z_a \cap H}} |Z_a \cap H|$$

Therefore, we obtain the following formula for $\Phi_H^{L,\psi}$:

$$\Phi_H^{L,\psi} \;=\; |L| \;-\; \sum_{\substack{a\in L\setminus H \\ \rho_\psi(a,b)=1 \;\forall b\in Z_a\cap H}} |Z_a \cap H|$$

Proposition 4.4.12 is an immediate consequence. □

We now turn our attention back to graphs. For a vertex v of a classical graph Γ and a subgroup $L \subseteq \mathrm{Aut}(\Gamma)$, we denote the *stabilizer subgroup* of L by $\mathrm{Stab}_L(v) := \{h \in L \mid h(v) = v\}$.

Proposition 4.4.13. *Let Γ be a classical graph and let (L, ψ) be a subgroup of central type of $\mathrm{Aut}(\Gamma)$. Then, the dimension of the center of the algebra $V_{\Gamma_{L,\psi}}$ can be expressed as follows:*

$$\dim(Z(V_{\Gamma_{L,\psi}})) \;=\; \frac{1}{|L|} \sum_{v\in V_\Gamma} \Phi^{L,\psi}_{\mathrm{Stab}_L(v)}$$

Proof. Inserting the Frobenius algebra $X_{L,\psi}$ (4.33) into the expression (4.25) results in the following formula for the dimension of the center of the algebra $V_{\Gamma_{L,\psi}}$:

$$\dim(Z(V_{\Gamma_{L,\psi}})) \;=\; \frac{1}{|L|^{\frac{3}{2}}} \sum_{v\in V_\Gamma} \sum_{a,b\in \mathrm{Stab}_L(v)} \mathrm{Tr}(U_b U_a^\dagger U_b^\dagger U_a)$$

It is a direct consequence of (4.32) that the trace is only non-zero if $[a, b] = e$. In this case, $U_a U_b = \rho_\psi(a, b) U_b U_a$ and therefore $\mathrm{Tr}(U_b U_a^\dagger U_b^\dagger U_a) = \sqrt{|L|}\;\rho_\psi(a, b)$. This proves the theorem:

$$\dim(Z(V_{\Gamma_{L,\psi}})) \;=\; \frac{1}{|L|} \sum_{v\in V_\Gamma} \sum_{\substack{a,b\in \mathrm{Stab}_L(v) \\ [a,b]=e}} \rho_\psi(a, b) \;=\; \frac{1}{|L|} \sum_{v\in V_\Gamma} \Phi^{L,\psi}_{\mathrm{Stab}_L(v)}$$ □

Combining the formula of Proposition 4.4.13 with Proposition 4.4.12 leads to a necessary and sufficient condition for the quantum graph $\Gamma_{L,\psi}$ to be classical.

Theorem 4.4.14. *Let Γ be a classical graph and let (L, ψ) be a subgroup of central type of $\mathrm{Aut}(\Gamma)$. Then, $\Gamma_{L,\psi}$ is a classical graph if and only if all stabilizer subgroups are coisotropic; that is, for every vertex $v \in V_\Gamma$ the following holds:*

$$\mathrm{Stab}_L(v)^\perp := \{a \in L \mid \rho_\psi(a, b) = 1 \;\forall b \in Z_a \cap \mathrm{Stab}_L(v)\} \;\subseteq\; \mathrm{Stab}_L(v) \tag{4.35}$$

Proof. The graph $\Gamma_{L,\psi}$ is classical if $V_{\Gamma_{L,\psi}}$ is commutative, that is if $\dim(Z(V_{\Gamma_{L,\psi}})) = \dim(V_{\Gamma_{L,\psi}}) \overset{\text{Prop. 4.2.20}}{=} \dim(V_\Gamma) = |V_\Gamma|$. Using Proposition 4.4.13, $\Gamma_{L,\psi}$ is therefore classical if and only if the following holds:

$$\frac{1}{|L|} \sum_{v\in V_\Gamma} \Phi^{L,\psi}_{\mathrm{Stab}_L(v)} \;=\; |V_\Gamma| \tag{4.36}$$

It follows from Proposition 4.4.12 that $\Phi^{L,\psi}_{\mathrm{Stab}_L(v)} \leq |L|$. Thus, equation (4.36) holds if and only if $\Phi^{L,\psi}_{\mathrm{Stab}_L(v)} = |L|$ for every vertex $v \in V_\Gamma$ which in turn holds, again by Proposition 4.4.12, if and only if $\mathrm{Stab}_L(v)$ is coisotropic. □

We now summarize our results on quantum isomorphic classical graphs obtained from simple dagger Frobenius monoids in the classical subcategory.

Corollary 4.4.15. *Let Γ be a classical graph. Then, every subgroup of central type (L,ψ) of* $\mathrm{Aut}(\Gamma)$ *with coisotropic stabilizers induces a classical graph $\Gamma_{L,\psi}$ and a quantum isomorphism $\Gamma_{L,\psi} \to \Gamma$. Moreover, if Γ has no quantum symmetries, this gives rise to a bijective correspondence between the following sets:*

- *Isomorphism classes of classical graphs Γ' such that there exists a quantum isomorphism $\Gamma' \to \Gamma$.*
- *Subgroups of central type (L,ψ) of* $\mathrm{Aut}(\Gamma)$ *with coisotropic stabilizers up to the equivalence relation* (4.30).

Proof. Corollary 4.4.15 is a direct consequence of Theorem 4.4.14 and Corollary 4.3.14. □

We immediately make a simple observation based on the fact that trivial subgroups can never be coisotropic.

Proposition 4.4.16. *Let Γ be a classical graph, and let (L,ψ) be a non-trivial subgroup of central type of* $\mathrm{Aut}(\Gamma)$ *such that $\Gamma_{L,\psi}$ is a classical graph. Then, every vertex is stabilized by some non-trivial element of L, that is* $\mathrm{Stab}_L(v) \neq \{e\}$.

Proof. Note that $\{e\}^\perp = L$. Thus, if v is a vertex of Γ such that $\mathrm{Stab}_L(v) = \{e\}$, it follows from Theorem 4.4.14 that $L = \mathrm{Stab}_L(v)^\perp \overset{(4.35)}{\subseteq} \mathrm{Stab}_L(v) = \{e\}$, and thus that $L = \{e\}$ contradicting non-triviality of L. □

Example 4.4.17. Let C_n be the cycle graph with $n \geq 5$ vertices. We have seen in Example 4.4.4 that for even n there are either one or two quantum graphs Γ' which are quantum isomorphic to C_n, corresponding to conjugacy classes of central type subgroups $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset D_n$. These subgroups act by 180 degree rotation and reflection along some axis through either opposite vertices or opposite edges of C_n. In both cases, there are vertices with trivial stabilizer. It therefore follows from Proposition 4.4.16 that all quantum graphs quantum isomorphic to C_n are non-classical.

4.5 Quantum pseudo-telepathy

Quantum pseudo-telepathy is a well-studied phenomenon in quantum information theory, where two non-communicating parties can use pre-shared entanglement to perform a task classically impossible without communication [Bra03, BBT05, CHTW04]. Such tasks are usually formulated as games, where isolated players Alice and Bob are provided with inputs, and must return outputs satisfying some winning condition.

One such game is is the *graph isomorphism game* [AMR$^+$19], whose instances correspond to pairs of classical graphs Γ and Γ', and whose winning classical strategies are precisely graph isomorphisms $\Gamma' \to \Gamma$. Winning quantum strategies correspond to quantum isomorphisms.

Proposition 4.5.1 ([AMR$^+$19, Thm 5.4])**.** *Given classical graphs Γ and Γ', there is a winning quantum strategy for the graph isomorphism game if and only if there is a quantum isomorphism $(H, P) : \Gamma' \to \Gamma$.*

Therefore, two non-isomorphic graphs with a quantum isomorphism between them exhibit pseudo-telepathic behaviour.

Definition 4.5.2. A pair of non-isomorphic graphs (Γ, Γ') will be called *pseudo-telepathic* if there is a quantum isomorphism $\Gamma' \to \Gamma$.

We can therefore apply the results of Sections 4.3 and 4.4 to obtain the following classification of pseudo-telepathic graph pairs (Γ, Γ') in terms of structures in the monoidal category $\mathrm{QAut}(\Gamma)$.

Corollary 4.5.3. *Let Γ be a classical graph. There is a bijective correspondence between the following sets:*

- *Isomorphism classes of classical graphs Γ' such that (Γ, Γ') are pseudo-telepathic.*
- *Non-trivial Morita equivalence classes of simple dagger Frobenius monoids in* $\mathrm{QAut}(\Gamma)$ *for which the expression* (4.25) *evaluates to* $|V_\Gamma|$.

Proof. This is essentially the statement of Corollary 4.3.14 with the additional condition of non-triviality. Note that a simple dagger Frobenius monoid is Morita trivial if it is Morita equivalent to the monoidal unit I. On the other hand, under the correspondence of Corollary 4.3.14, the monoidal unit of $\mathrm{QAut}(\Gamma)$ corresponds to the isomorphism class of Γ itself. Excluding this trivial class leads to Corollary 4.5.3. $\square$

Similarly, we can translate the statement of Corollary 4.4.15 into a statement about pseudo-telepathic graph pairs.

Corollary 4.5.4. *Let Γ be a classical graph with no quantum symmetries. There is a bijective correspondence between the following sets:*

- *Isomorphism classes of classical graphs Γ' such that the pair (Γ, Γ') is pseudo-telepathic.*
- *Non-trivial subgroups of central type (L, ψ) of $\mathrm{Aut}(\Gamma)$ with coisotropic stabilizers up to the equivalence relation (4.30).*

4.5.1 Ruling out pseudo-telepathy

In this section, we demonstrate how Corollary 4.5.3 and Corollary 4.5.4 can be used to show that a graph Γ cannot exhibit pseudo-telepathy. We begin by showing that almost all graphs are not part of a pseudo-telepathic graph pair. We recall a result of Lupini et al. showing that almost all graphs have trivial quantum automorphism group.

Theorem 4.5.5 ([LMR17, Thm 3.14])**.** *Let G_n be the number of isomorphism classes of classical graphs with n vertices and let Q_n be the number of isomorphism classes of classical graphs with non-trivial quantum automorphism group. Then Q_n/G_n goes to zero as n goes to infinity.*

We combine this with our results to obtain the following corollary.

Corollary 4.5.6. *Let G_n be the number of isomorphism classes of classical graphs with n vertices and let PT_n be the number of isomorphism classes of classical graphs which are part of a pseudo-telepathic pair. Then PT_n/G_n goes to zero as n goes to infinity.*

Proof. If Γ has trivial quantum automorphism group, then it has no quantum symmetries and trival automorphism group $\mathrm{Aut}(\Gamma)$. There are therefore no non-trivial Morita equivalence classes of simple dagger Frobenius monoids in $\mathrm{QAut}(G)$; the result then follows from Corollary 4.5.4 and Theorem 4.5.5. □

We now consider various graphs known to have no quantum symmetries. We recall the following result.

Theorem 4.5.7 ([BB07, Sch18])**.** *The following is a complete list of all vertex-transitive graphs of order ≤ 11 with no quantum symmetries.*

Graph	Automorphism group
C_{11}, $C_{11}(2)$, $C_{11}(3)$	D_{11}
Petersen	S_5
C_{10}, $C_{10}(2)$, C_{10}^+, $\mathrm{Pr}(C_5)$	D_{10}
Torus	$S_3 \wr \mathbb{Z}_2$
C_9, $C_9(3)$	D_9
C_8, C_8^+	D_8
C_7	D_7
C_6	D_6
C_5	D_5
K_3	S_3
K_2	$\mathbb{Z}_2$

Here the graphs C_n, $C_n(m)$ and $C_{2n}^+ = C_{2n}(n)$ are circulant graphs; K_n are complete graphs; the Petersen graph is well-known; $\mathrm{Pr}(C_5)$ is the graph $C_5 \times K_2$; and Torus is the graph $K_3 \times K_3$, where $\times$ is the direct product; see [BB07] for more detail.

Theorem 4.5.8. *Vertex-transitive graphs of order* ≤ 11 *with no quantum symmetries cannot be part of a pseudo-telepathic graph pair.*

Proof. In this proof we make extensive use of the fact that the trivial subgroup of a group of central type cannot be coisotropic (see Proposition 4.4.16).

The automorphism groups of the complete graphs K_2 and K_3 have no nontrivial subgroups of central type, so by Corollary 4.5.4 cannot be part of a pseudo-telepathic graph pair.[17]

The circulant graphs all have dihedral automorphism group, which acts on them as on any cycle graph. As with the cycle graph (Examples 4.4.4 and 4.4.17), there are up to two conjugacy classes of nontrivial central type subgroups (all isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$), all of which have some trivial vertex stabilizers; so, by Corollary 4.5.4, they cannot be part of a pseudo-telepathic graph pair.

Similarly, $\mathrm{Pr}(C_5)$ has trivial vertex stabilizers under the action of the unique up-to-conjugacy central type subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$.

For the Petersen graph, all central type subgroups of S_5 are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$; there are two conjugacy classes of these subgroups. However, each of these conjugacy classes have vertices with trivial stabilizer.

For the torus graph, $S_3 \wr \mathbb{Z}_2$ has three conjugacy classes of central type subgroups, two isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and one isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. Again, it is straightforward

[17] In fact, it is well known that all quantum isomorphisms between graphs with fewer than four vertices are direct sums of classical isomorphisms [Wan98].

to check that all three conjugacy classes have vertices with trivial stabilizer; the corresponding quantum graphs are therefore non-classical. □

Remark 4.5.9. By Corollary 4.3.7, we also obtain a classification of quantum graphs which are quantum isomorphic to a classical graph with no quantum symmetries. We show how this works in the vertex-transitive case. The central type subgroups appearing in the proof of Theorem 4.5.8 are of the form $\mathbb{Z}_n \times \mathbb{Z}_n$ with $n = 2, 3$. There is only one cohomology class of nondegenerate 2-cocycles on $\mathbb{Z}_2 \times \mathbb{Z}_2$, so for those graphs with only $\mathbb{Z}_2 \times \mathbb{Z}_2$ central type subgroups, quantum isomorphic quantum graphs are in bijective correspondence with conjugacy classes of these subgroups. This implies that the circulant graphs of odd order have no quantum isomorphic quantum graph, $\mathrm{Pr}(C_5)$ and the circulant graphs of even order not divisible by 4 have one quantum isomorphic quantum graph, and the Petersen graph and the circulant graphs of order divisible by 4 have two quantum isomorphic quantum graphs.

We must be slightly more careful with the torus graph, since the central type subgroup $\mathbb{Z}_3 \times \mathbb{Z}_3$ has two cohomology classes $[\phi_1]$ and $[\phi_2]$ of nondegenerate 2-cocyles. It is straightforward to check that, for a subgroup $L \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ of $\mathrm{Aut}(\mathrm{Torus}) \cong S_3 \wr \mathbb{Z}_2$, the pairs $(L, [\phi_1])$ and $(L, [\phi_2])$ are equivalent under the relation (4.30). The torus graph therefore has three quantum isomorphic quantum graphs, corresponding to the two conjugacy classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroups, and the single conjugacy class of $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroups with either of the equivalent cohomology classes of 2-cocyles.

These quantum isomorphisms may have some interpretation in the theory of zero-error quantum communication [Sta16].

4.5.2 Binary constraint systems and Arkhipov's construction

In [Ark12], Arkhipov describes a construction of a non-local game from a connected non-planar graph Z and a specified vertex l^*, generalizing the famous magic square and magic pentagram games [Mer90]. In [LMR17, Def 4.4 and Thm 4.5], Lupini et al. translate this construction into a construction of a pseudo-telepathic graph pair $(X_0(Z), X(Z, l^*))$.

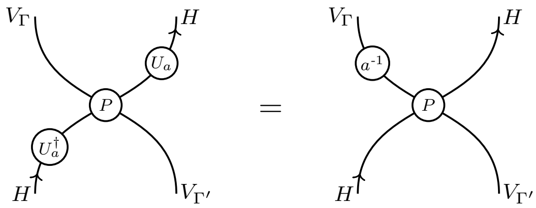
In this section, we show that the graph $X(Z, l^*)$ and the quantum isomorphism $X(Z, l^*) \to X_0(Z)$ always arise from subgroups of central type[18] of the automorphism group of the graph $X_0(Z)$, following the construction of Corollary 4.4.15. Moreover, these subgroups can always be taken to be isomorphic to either $\mathbb{Z}_2^4$ or $\mathbb{Z}_2^6$. The

[18]In particular, all these graph pairs correspond to Frobenius monoids in the classical subcategory of one of the graphs.

observations and constructions in this section generalize the binary magic square example from the introduction.

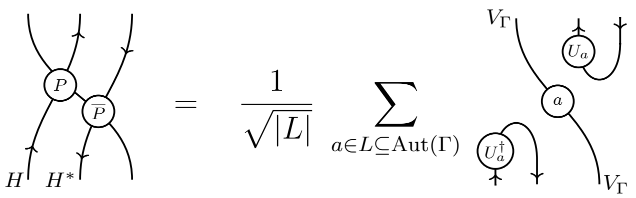
We first establish the following proposition, which allows us to recognize whether a graph Γ' which is quantum isomorphic to another graph Γ comes from a given central type subgroup of $\mathrm{Aut}(\Gamma)$.

Proposition 4.5.10. *Let Γ and Γ' be classical graphs, let (L,ψ) be a subgroup of central type of $\mathrm{Aut}(\Gamma)$ with coisotropic stabilizers and let $\{U_a \in U(H) | a \in L\}$ be a corresponding nice unitary error basis. Then Γ' is isomorphic to $\Gamma_{L,\psi}$ if and only if there exists a quantum isomorphism $(H,P) : \Gamma' \to \Gamma$ such that the following holds, for all $a \in L \subseteq \mathrm{Aut}(\Gamma)$:*



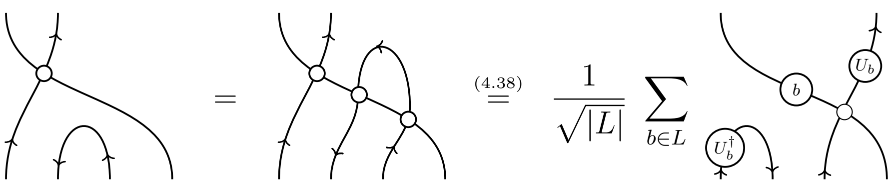
(4.37)

Proof. It follows from Proposition 4.3.1 that $\Gamma_{L,\psi}$ is isomorphic to Γ' if and only if there exists a quantum isomorphism $(H,P) : \Gamma' \to \Gamma$ such that $P \circ \overline{P} = X_{L,\psi}$:



(4.38)

Using the shorthand notation (4.20) for the quantum isomorphism P, and (4.24), this is equivalent to the following:



Contracting the first two bottom wires with U_a for $a \in L$ and using (4.32) completes the proof. □

In terms of the underlying projective permutation matrix P, condition (4.37) can be stated as follows, for all $a \in L \subseteq \mathrm{Aut}(\Gamma), v' \in V_{\Gamma'}$ and $v \in V_\Gamma$:

$$U_a P_{v',v} U_a^\dagger = P_{v',a(v)} \tag{4.39}$$

We give a brief summary of the construction of pseudo-telepathic graphs from binary constraint systems as developed in [AMR$^+$19, Sec 6].

Let $\mathcal{F}$ be a linear binary constraint system (see [AMR$^+$19, Sec 6.1 and 6.2]) with binary variables $x_1, \ldots, x_m \in \{+1, -1\}$ and constraints $C_1, \ldots, C_p$, where each C_l is an equation of the form $\prod_{x_i \in S_l} x_i = b_l$ for $S_l \subseteq \{x_1, \ldots, x_n\}$ and $b_l \in \{+1, -1\}$.[19] A *classical solution* is a solution of the constraint system with $x_i \in \{+1, -1\}$. A *quantum solution* is a solution for which the x_i are self-adjoint operators with eigenvalues ± 1 acting on some finite-dimensional Hilbert space H, and such that all operators appearing in the same constraint commute. A linear binary constraint system which admits a quantum but not a classical solution will be called *pseudo-telepathic*.

The *homogenisation* $\mathcal{F}_0$ of $\mathcal{F}$ is the constraint system in which we set the right hand side of every constraint equation to $+1$. For every linear binary constraint system $\mathcal{F}$, Atserias et al. construct a graph $\Gamma^{\mathcal{F}}$ whose vertices are pairs (C_l, f) of a constraint equation C_l of $\mathcal{F}$ together with a 'local' classical solution $f : S_l \to \{+1, -1\}$ of this equation, and with an edge between (C_l, f) and (C_k, g) if and only if the local solutions $f : S_l \to \{+1, -1\}$ and $g : S_k \to \{+1, -1\}$ are inconsistent on $S_l \cap S_k$. They show that a constraint system $\mathcal{F}$ has a classical solution if and only if the graphs $\Gamma^{\mathcal{F}_0}$ and $\Gamma^{\mathcal{F}}$ are isomorphic, and that if $\mathcal{F}$ has a quantum solution then these graphs are quantum isomorphic. (See [AMR$^+$19, Proof of Theorem 6.3] or the proof of Proposition 4.5.11 below for the construction of the quantum isomorphism arising from a quantum solution.)

We now show that all pseudo-telepathic graph pairs arising from a binary constraint system possessing a quantum solution satisfying a certain pair of conditions can be obtained from central type subgroups.

Proposition 4.5.11. *Let $\mathcal{F}$ be a linear binary constraint system and suppose that this system has a quantum solution $\{X_i \in \mathrm{End}(H)\}_{1 \le i \le m}$, acting on some Hilbert space H, with the following two properties:*

- *If $A \in \mathrm{End}(H)$ is such that $AX_i = X_i A$ for all $1 \le i \le m$, then $A \propto \mathbb{1}_H$.*
- *There is a group of central type (L, ψ) and a corresponding nice unitary error basis $\{U_a \in U(H) \mid a \in L\}$ such that the following holds for all $a \in L$ and $1 \le i \le m$:*

$$U_a^\dagger X_i U_a = p_i^a \; X_i \qquad \text{where } p_i^a \in \{+1, -1\} \tag{4.40}$$

Then, there is an embedding $L \hookrightarrow \mathrm{Aut}(\Gamma^{\mathcal{F}_0})$ and $\Gamma^{\mathcal{F}}$ is isomorphic to $\Gamma^{\mathcal{F}_0}_{L,\psi}$.

[19] Unlike [AMR$^+$19], we write our constraint systems in multiplicative form.

Proof. We first note that for each $a \in L$, $\{p_i^a\}_{1\leq i\leq m}$ forms a (global) classical solution of the homogenous constraint system $\mathcal{F}_0$ and thus gives rise to 'local' assignments which we denote by $[p^a]^l : S_l \to \{+1,-1\}$. This in turn gives rise to an automorphism p^a of $\Gamma^{\mathcal{F}_0}$, mapping a vertex (C_l, f) to the vertex $(C_l, [p^a]^l \cdot f)$ where $[p^a]^l \cdot f : S_l \to \{+1,-1\}$ denotes the pointwise multiplication of the assignments $[p^a]^l, f : S_l \to \{+1,-1\}$. This results in a group homomorphism $L \to \mathrm{Aut}(\Gamma^{\mathcal{F}_0})$, $a \mapsto p^a$, which is injective, since if $p^a = p^b$ it follows that $p_i^a = p_i^b$ for all $1 \leq i \leq m$ and thus that $U_a^\dagger X_i U_a = U_b^\dagger X_i U_b$, or equivalently that $U_a U_b^\dagger$ commutes with each x_i. Thus, by the first assumption, $U_a U_b^\dagger \propto \mathbb{1}_H$ and therefore $a = b$. Therefore, $a \mapsto p^a$ defines an embedding $L \hookrightarrow \mathrm{Aut}(\Gamma^{\mathcal{F}_0})$.

We now show that $\Gamma^{\mathcal{F}}$ is isomorphic to $\Gamma^{\mathcal{F}_0}_{L,\psi}$. In the proof of [AMR+19, Thm 6.3], from a quantum solution $\{X_i \in \mathrm{End}(H)\}_{1\leq i\leq m}$ a quantum isomorphism $(H, P) : \Gamma^{\mathcal{F}} \to \Gamma^{\mathcal{F}_0}$ is constructed as follows. Given a vertex (C_l, f) of $\Gamma^{\mathcal{F}}$, define the projector $Q_{(C_l,f)}$ on H as the projector onto the joint eigenspace of the commuting operators $\{X_i \mid x_i \in S_l\}$ with respective eigenvalues determined by $f : S_l \to \{+1,-1\}$. The quantum isomorphism $(H, P) : \Gamma^{\mathcal{F}} \to \Gamma^{\mathcal{F}_0}$ is then defined as the following projective permutation matrix, where $(C_k, f) \in V_{\Gamma^{\mathcal{F}}}$ and $(C_l, g) \in V_{\Gamma^{\mathcal{F}_0}}$:

$$P_{(C_k,f),(C_l,g)} := \delta_{k,l}\ Q_{(C_l,fg)}$$

If the given quantum solution fulfils the second condition of the theorem, then the projectors onto the joint eigenspaces fulfil the following equation:

$$U_a Q_{(C_l,g)} U_a^\dagger = Q_{(C_l,[p^a]^l\cdot g)}$$

Therefore, the following holds for the just defined projective permutation matrix P and for all $a \in L$ and vertices $v \in V_{\Gamma^{\mathcal{F}}}$ and $w \in V_{\Gamma^{\mathcal{F}_0}}$:

$$U_a P_{v,w} U_a^\dagger = P_{v,p^a(w)}$$

This is precisely condition (4.39). It thus follows from Proposition 4.5.10 that $\Gamma^{\mathcal{F}}$ and $\Gamma^{\mathcal{F}_0}_{L,\psi}$ are isomorphic. □

Remark 4.5.12. The first paragraph of the proof of Propositon 4.5.11 shows how automorphisms of the graph $\Gamma^{\mathcal{F}_0}$ arise from global classical solutions of the homogenous constraint system $\mathcal{F}_0$. This generalizes how the automorphism subgroup $\mathbb{Z}_2^4$ of the graph Γ in the example in the introduction arises from bit flip symmetries — or equivalently from global classical solutions of the binary magic square constraint system.

Remark 4.5.13. The global classical solutions (4.3) of the magic square constraint system discussed in the introduction arise as in equation (4.40) as the matrices of signs obtained from conjugating the entries of the following quantum solution of the inhomogenous[20] magic square constraint system by $U_{1,0,0,0} = \sigma_X \otimes \mathbb{1}_2$, $U_{0,1,0,0} = \sigma_Z \otimes \mathbb{1}_2$, $U_{0,0,1,0} = \mathbb{1}_2 \otimes \sigma_X$ and $U_{0,0,0,1} = \mathbb{1}_2 \otimes \sigma_Z$, respectively:

$$\begin{pmatrix} \mathbb{1}_2 \otimes \sigma_Z & \sigma_Z \otimes \sigma_Z & \sigma_Z \otimes \mathbb{1}_2 \\ \sigma_X \otimes \sigma_Z & \sigma_Y \otimes \sigma_Y & \sigma_Z \otimes \sigma_X \\ \sigma_X \otimes \mathbb{1}_2 & \sigma_X \otimes \sigma_X & \mathbb{1}_2 \otimes \sigma_X \end{pmatrix} \tag{4.41}$$

In particular, the inhomogenous magic square constraint system fulfils the conditions of Proposition 4.5.11 which leads to the proof of Theorem 4.5.14.

We now show that all pseudo-telepathic graph pairs generated from Lupini et al.'s translation of Arkhipov's construction arise from a central type subgroup of the automorphism group of one of the graphs. Recall that, in the introduction, we used tensor products of the Pauli UEB to define the 2-cocycles ψ_{P} on $\mathbb{Z}_2^2$ (4.4) and ψ_{P^2} on $\mathbb{Z}_2^4$ (4.5). We define the 2-cocycle ψ_{P^3} on $\mathbb{Z}_2^6$ analogously.

Theorem 4.5.14. *Let Z be a connected non-planar graph, let l^* be a specified vertex of Z and let $X_0(Z)$ and $X(Z, l^*)$ be the induced pseudo-telepathic graphs [LMR17, Def 4.4]. Then, there is a subgroup of central type (L, ψ) of $\mathrm{Aut}(X_0(Z))$,which is isomorphic to either $(\mathbb{Z}_2^4, \psi_{\mathrm{P}^2})$ or $(\mathbb{Z}_2^6, \psi_{\mathrm{P}^3})$ such that $X(Z, l^*)$ is isomorphic to the graph $X_0(Z)_{L,\psi}$.*

Proof. If Z is the bipartite complete graph $K_{3,3}$ or the complete graph K_5 with arbitrary specified vertex l^*, the associated pseudo-telepathic pair $(X_0(Z), X(Z, l^*))$ arise, respectively, from the magic square and magic pentagram binary constraint systems (see [Ark12]). In both cases, there are quantum solutions (see (4.41) and [Ark12, Fig II.2]) consisting of two-fold and three-fold tensor products of Pauli matrices, fulfilling the conditions of Proposition 4.5.11 for the subgroups $\mathbb{Z}_2^4$ and $\mathbb{Z}_2^6$ with 2-cocycle ψ_{P^2} and ψ_{P^3}, respectively, with the corresponding Pauli tensor product UEBs. Thus, $X(K_{3,3}, l^*)$ is isomorphic to $X_0(K_{3,3})_{\mathbb{Z}_2^4,\psi_{\mathrm{P}^2}}$, while $X(K_5, l^*)$ is isomorphic to $X_0(K_5)_{\mathbb{Z}_2^6,\psi_{\mathrm{P}^3}}$.

For a general connected non-planar graph Z, Arkhipov chooses a topological minor isomorphic either to $K_{3,3}$ or K_5 (such a topological minor exists due to the Pontryagin-Kuratowski theorem) and constructs a quantum solution which contains precisely the operators from the quantum solution to $K_{3,3}$ or K_5, respectively, reducing the

[20] In the inhomogenous magic square constraint system, all rows and columns multiply to 1 except for the middle column which multiplies to -1.

problem to either the magic square or magic pentagram. Thus, the obtained quantum solution again fulfils the conditions of Proposition 4.5.11 with $(L.\psi)$ isomorphic to either $(\mathbb{Z}_2^4, \psi_{\mathrm{P}^2})$ or $(\mathbb{Z}_2^6, \psi_{\mathrm{P}^3})$. □

Appendices

A An observation on dualizability

The following two basic categorical observations greatly simplify working with dualizability. The first observation is due to [Lur09, Rem 3.4.22] and is spelled out and proven in more detail in [DSPS17b, Lem 1.4.4].

Proposition A.1 (Left and right adjoint of 2-dualizable 1-morphism coincide [DSPS17b, Lem 1.4.4])**.** *Let $f : A \to B$ be a 1-morphism in a 3-category with right adjoint $f^* : B \to A$, witnessed by evaluation and coevaluation 2-morphisms $\mathrm{ev}_f : f \circ f^* \Rightarrow \mathrm{id}_B$ and $\mathrm{coev}_f : \mathrm{id}_A \Rightarrow f^* \circ f$ which themselves have right adjoints $\mathrm{ev}_f^*, \mathrm{coev}_f^*$. Then, f^* is a left adjoint of f witnessed by ev_f^* and coev_f^*.*

The second observation, which appears to be new, is a generalization of the fact that an object in a braided monoidal category has a right dual if and only if it has a left dual.

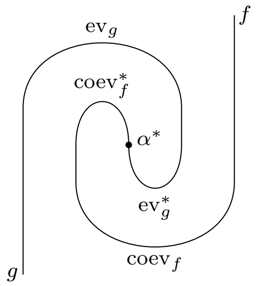
Proposition A.2 (2-morphism between 2-dualizable 1-morphism has left adjoint iff it has right adjoint)**.** *Let f and g be 1-morphisms in a 3-category with right adjoints f^* and g^* which are witnessed by left adjoint evaluation and coevaluation 2-morphisms. Then, a 2-morphism $\alpha : f \Rightarrow g$ has a right adjoint if and only if it has a left adjoint.*

Proof. We prove that if α has a right adjoint α^*, then it has a left adjoint. The converse follows from working in the 3-category with opposite 2-morphism direction $\mathcal{C}^{\mathrm{op}_2}$. (Indeed, in $\mathcal{C}^{\mathrm{op}_2}$, the converse corresponds to the statement that if f, g are 1-morphisms with left adjoints f^* and g^* which are witnessed by right adjoint evaluation and coevaluation maps, then a 2-morphism $\alpha : g \Rightarrow f$ with a right adjoint has a left adjoint. By the $\mathcal{C}^{\mathrm{op}_2}$ version of Proposition A.1, this assumption on f and g implies the original assumption of Proposition A.2.)

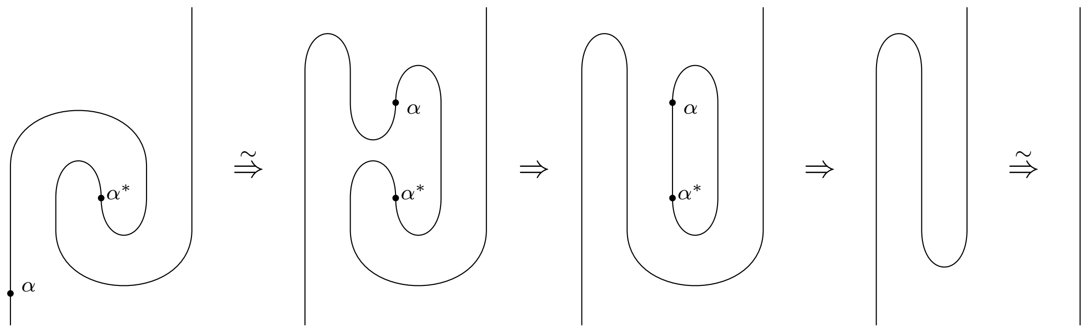
We work in the setting of Gray categories[21], using the 'movies-of-string-diagrams' calculus, detailled in [Bar14].

[21] A Gray category is a many-object version of a Gray monoid as in Definition 1.3.1, see [GPS95, BMS12] for more details.

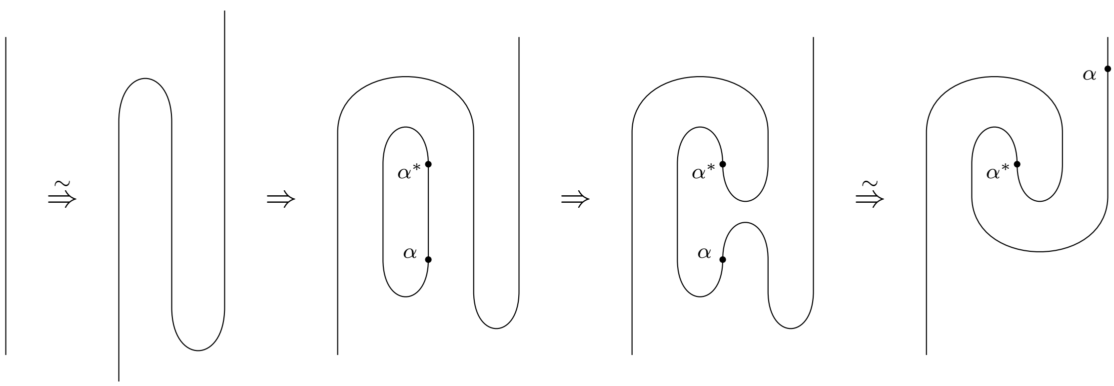
Denote the right adjoints of $f : A \to B$ and g by f^* and g^* with evaluation and coevaluation 2-morphisms $\mathrm{ev}_f : f \,\square\, f^* \to \mathrm{id}_A, \mathrm{coev}_f, \mathrm{ev}_g, \mathrm{coev}_g$ which themselves have right adjoints $\mathrm{ev}_f^*, \mathrm{coev}_f^*, \mathrm{ev}_g^*, \mathrm{coev}_g^*$. Let $\alpha^* : g \Rightarrow f$ denote the right adjoint of α with evaluation and coevaluation 3-morphisms $\mathrm{ev}_\alpha : \alpha \circ \alpha^* \Rrightarrow 1_g$ and $\mathrm{coev}_\alpha : 1_f \Rrightarrow \alpha^* \circ \alpha$. Then, it can be verified that the following 2-morphism $g \Rightarrow f$



is a left adjoint of α with evaluation 3-morphism



(here, the first and last 3-morphisms are isomorphisms composed from interchangers and cusp 3-isomorphisms, chosen to fulfill the swallowtail equations (see the discussion after Definition 1.3.1), the second 3-morphism is $\mathrm{ev}_{\mathrm{coev}_f}$ and the third 3-morphism is ev_α followed by $\mathrm{ev}_{\mathrm{ev}_g}$) and coevaluation 3-morphism



(here, the first and last 3-morphisms are isomorphisms composed from interchangers and cusp 3-isomorphisms, the second 3-morphism is $\mathrm{coev}_{\mathrm{coev}_f}$ followed by coev_α and the third 3-morphism is $\mathrm{coev}_{\mathrm{ev}_g}$). □

Example A.3 (Duals in braided monoidal categories). A braided monoidal category is a 3-category with one object $*$ and one 1-morphism id_*. Since id_* is invertible and hence fully dualizable, Proposition A.2 generalizes the well-known fact that an object in a braided monoidal category has a right dual if and only if it has a left dual.

Proposition I.3.4 is an immediate corollary.

B Separable monads and idempotent completion of 2-categories

B.1 Monads and their bimodules

Recall that a *monad* in a 2-category is a 1-morphism $P : A \to A$, together with 2-morphisms $P \circ P \overset{m}{\Rightarrow} P$ (the 'multiplication') and $1_A \overset{u}{\Rightarrow} P$ (the 'unit') fulfilling the following equations:

$$\left(P \circ P \circ P \xRightarrow{m \circ P} P \circ P \overset{m}{\Rightarrow} P\right) = \left(P \circ P \circ P \xRightarrow{P \circ m} P \circ P \overset{m}{\Rightarrow} P\right)$$

$$\left(P \xRightarrow{u \circ P} P \circ P \overset{m}{\Rightarrow} P\right) = \left(P \overset{1_P}{\Rightarrow} P\right) = \left(P \xRightarrow{P \circ u} P \circ P \overset{m}{\Rightarrow} P\right)$$

Given monads $(A \overset{P}{\to} A, P \circ P \xRightarrow{m_P} P, 1_A \xRightarrow{u_P} P)$ and $(B \overset{Q}{\to} B, Q \circ Q \xRightarrow{m_Q} Q, 1_B \xRightarrow{u_Q} Q)$, recall that a *$Q$–$P$-bimodule* ${}_Q M_P : A \to B$ is a 1-morphism $M : A \to B$ together with a 2-morphism $\rho : Q \circ M \circ P \Rightarrow M$ (the 'action') fulfilling the following equations:

$$\left(Q \circ Q \circ M \circ P \circ P \xRightarrow{m_Q \circ M \circ m_P} Q \circ M \circ P \overset{\rho}{\Rightarrow} M\right) = \left(Q \circ Q \circ M \circ P \circ P \xRightarrow{Q \circ \rho \circ P} Q \circ M \circ P \overset{\rho}{\Rightarrow} M\right)$$

$$\left(M \xRightarrow{u_Q \circ M \circ u_P} Q \circ M \circ P \overset{\rho}{\Rightarrow} M\right) = \left(M \overset{1_M}{\Rightarrow} M\right)$$

A *bimodule map* ${}_Q M_P \Rightarrow {}_Q N_P$ is a 2-morphism $f : M \Rightarrow N$ intertwining the action:

$$\left(Q \circ M \circ P \overset{\rho}{\Rightarrow} M \overset{f}{\Rightarrow} M\right) = \left(Q \circ M \circ P \xRightarrow{Q \circ f \circ P} Q \circ M \circ P \overset{\rho}{\Rightarrow} M\right)$$

B.2 Eilenberg–Moore and Kleisli morphisms and their splittings

Given a monad $P : A \to A$ in a 2-category $\mathcal{C}$, we write $\mathrm{LMod}_P(X)$ for the 1-category of left P-modules with domain X: its objects are pairs $(t : X \to A, \rho : P \circ t \Rightarrow t)$ of 1-morphisms t carrying a left module structure ρ of P, and its morphisms are 2-morphisms $t \Rightarrow t'$ in $\mathcal{C}$ intertwining the action of P. An *Eilenberg–Moore morphism* of P is a 1-morphism $R : A^P \to A$ together with a left P-module structure $\rho : P \circ R \Rightarrow R$, such that for every object X of $\mathcal{C}$, the induced functor

$$R \circ - : \mathrm{Hom}_{\mathcal{C}}(X, A^P) \to \mathrm{LMod}_P(X)$$

is an equivalence. In other words, an Eilenberg–Moore morphism is a universal left P-module. Analogously, a *Kleisli morphisms* $L : A \to A_P$ is a universal right P-module, that is a right P-module such that for every object X of $\mathcal{C}$, the induced functor

$$- \circ L : \mathrm{Hom}_{\mathcal{C}}(A_P, X) \to \mathrm{RMod}_P(X)$$

is an equivalence. The objects A^P and A_P are often known as Eilenberg–Moore and Kleisli objects, respectively.

A *splitting* of a monad $P : A \to A$ in a 2-category $\mathcal{C}$ is an adjunction $R \vdash L : A \to B$ together with an isomorphism of monads $\psi : R \circ L \Rightarrow P$. A splitting $(R \vdash L, \psi)$ is an *Eilenberg–Moore splitting* if $R : B \to A$ together with the action of P on R induced by ψ is an Eilenberg–Moore morphism of P. Note that any Eilenberg–Moore morphism $R : A^P \to A$ of a monad P admits a left adjoint $L : A \to A^P$ and a canonical isomorphism of monads $R \circ L \cong P$, hence gives rise to an Eilenberg–Moore splitting of P. Analogously, a *Kleisli splitting* of P is a splitting $(R \vdash L, \psi)$ such that $L : A \to B$ together with the induced P-action on L is a Kleisli morphism of P.

B.3 Separable monads and their splittings

Recall that a monad $P : A \to A$ is *separable* if there exists a P-P bimodule section $\Delta : P \Rightarrow P \circ P$ of the multiplication of P. A splitting $(R \vdash L, \psi)$ of a monad is *separable* if the counit $\epsilon : L \circ R \Rightarrow 1_B$ admits a section.

Theorem B.1 (Separable, Eilenberg–Moore, and Kleisli splittings are equivalent)**.** *Let $P : A \to A$ be a separable monad in a locally idempotent complete 2-category and let $(R \vdash L : A \to B, \psi)$ be a splitting of P. Then, the following are equivalent:*

1. *The splitting is separable.*
2. *The splitting is an Eilenberg–Moore splitting.*
3. *The splitting is a Kleisli splitting.*

Proof. We use the isomorphism ψ to identify P with $R \circ L$.

(1) $\Rightarrow$ (2) Suppose that $(R \vdash L : A \to B, \psi)$ is a separable splitting; that is suppose that the counit $\epsilon : L \circ R \Rightarrow 1_B$ is split by the section $\delta : 1_B \Rightarrow L \circ R$. For any object X of $\mathcal{C}$, we claim that the functor $R \circ - : \mathrm{Hom}_{\mathcal{C}}(X, B) \to \mathrm{LMod}_{R \circ L}(X)$ is an equivalence. To prove that $R \circ -$ is essentially surjective, we let $T : X \to A$ be a left $(R \circ L)$-module with action $\rho : R \circ L \circ T \Rightarrow T$. Precomposing $L \circ \rho$ with δ results in an idempotent 2-morphism $\rho' : L \circ T \Rightarrow L \circ T$. Splitting this idempotent yields a 1-morphism $S : X \to B$ and 2-morphisms $p : L \circ T \Rightarrow S$ and $i : S \Rightarrow L \circ T$ such that $p \cdot i = 1_S$ and $i \cdot p = \rho'$. Now observe that the left $(R \circ L)$-module $R \circ S$ is isomorphic to T, as follows. Using the adjunction between R and L, we turn p into a 2-morphism $p' : T \Rightarrow R \circ S$ and define the 2-morphism $i' : R \circ S \overset{R \circ i}{\Longrightarrow} R \circ L \circ T \overset{\rho}{\Rightarrow} T$. Direct

computation shows that i' and p' are inverse $R \circ L$-module maps. Full faithfulness of $R \circ -$ can be proven from the existence of $\delta : 1_B \Rightarrow L \circ R$ with $\epsilon \cdot \delta = 1_{1_B}$.

(2) ⇒ (1) Separability of P implies that there is an $(R \circ L)$–$(R \circ L)$-bimodule 2-morphism $\Delta : R \circ L \Rightarrow R \circ L \circ R \circ L$ that splits the multiplication of the monad $R \circ L$. Since R is an Eilenberg–Moore morphism of $R \circ L$ and Δ is a left $R \circ L$-module map, it follows that there is a 2-morphism $f : L \Rightarrow L \circ R \circ L$ such that $\Delta = R \circ f$. Using the adjunction between R and L, we obtain a 2-morphism $f' : R \Rightarrow R \circ L \circ R$. The fact that Δ is a right $(R \circ L)$-module, combined with the faithfulness of $R \circ -$, implies that f' is a left $(R \circ L)$-module map. Therefore, there is a 2-morphism $\delta : 1_B \Rightarrow L \circ R$ such that $f' = R \circ \delta$ and thus $\Delta = R \circ \delta \circ L$. The fact that Δ splits the multiplication of P then implies that δ splits the counit, i.e. $\epsilon \cdot \delta = 1_B$.

(1) ⇔ (3) Note that a splitting $(R \vdash L, \psi)$ is separable in $\mathcal{C}$ if and only if the splitting $(L \vdash R, \psi)$ is separable in $\mathcal{C}^{\mathrm{op}}$. (By $\mathcal{C}^{\mathrm{op}}$ we mean $\mathcal{C}$ with reversed direction of 1-morphisms, but not 2-morphisms.) Applying (1) ⇔ (2) to $\mathcal{C}^{\mathrm{op}}$ proves (1) ⇔ (3). □

In particular, Eilenberg–Moore objects and Kleisli objects of separable monads in locally idempotent complete 2-categories coincide.

Corollary B.2 (Eilenberg–Moore and Kleisli morphisms are adjoint)**.** *Given a separable monad P in a locally idempotent complete 2-category, a 1-morphism $R : A^P \to A$ is an Eilenberg–Moore morphism of P if and only if it has a left adjoint $L : A \to A^P$ that is a Kleisli morphism of P.*

Remark B.3 (Monadicity theorem for separable monads)*.* Recall that a 1-morphism $R : A \to B$ in a 2-category is called monadic if it has a left adjoint L and is an Eilenberg–Moore morphism for the induced monad $R \circ L$. Analogously, a 1-morphism R is separable monadic if it is monadic and if moreover the monad $R \circ L$ is separable. From this perspective, Theorem B.1 can be understood as a monadicity theorem for separable monads: a 1-morphism R is separable monadic if and only if it has a left adjoint such that the counit of the adjunction admits a section. A similar result specifically in the 2-category of categories appears in [Che15].

B.4 The relative composition of bimodules over a separable monad

Definition B.4 (Relative composition of modules)**.** Let $M_P : B \to C$ be a right module and let ${}_PN : A \to B$ be a left module of a separable monad $P : B \to B$ in a locally idempotent complete 2-category $\mathcal{C}$. We define their *relative composition*

$M \circ_P N : A \to C$ as the 1-morphism obtained from splitting the idempotent 2-morphism

$$M \circ N \xRightarrow{M \circ u \circ N} M \circ P \circ N \xRightarrow{M \circ \Delta \circ N} M \circ P \circ P \circ N \xRightarrow{\rho_M \circ \rho_N} M \circ N,$$

Here $u : 1_B \Rightarrow P$ is the unit of P, and $\rho_M : M \circ P \Rightarrow M$ and $\rho_N : P \circ N \Rightarrow N$ denote the action of P, and $\Delta : P \Rightarrow P \circ P$ is a P-P bimodule section of the multiplication of P.

Note that the resulting 1-morphism $M \circ_P N$ is, up to isomorphism, independent of the choice of section Δ. If ${}_QM_P$ and ${}_PN_R$ are bimodules, then the idempotent 2-morphism is a Q–R-bimodule map, inducing a Q–R-bimodule structure on the relative composition ${}_QM \circ_P N_R$.

We can reexpress the condition that a separable monad admits a separable splitting in terms of bimodule triviality as follows. We will say that a monad $P : A \to A$ is bimodule trivial (that is, trivial up to bimodule equivalence) if there are modules ${}_PM : B \to A$ and $N_P : A \to B$ such that ${}_PM \circ N_P \cong {}_PP_P$ as bimodules and $N \circ_P M \cong 1_B$.

Proposition B.5 (Separably split is bimodule trivial)**.** *In a locally idempotent complete* 2*-category, a separable monad* $P : A \to A$ *admits a separable splitting if and only if it is bimodule trivial.*

Proof. Let ${}_PM : B \to A$ and $N_P : A \to B$ be a bimodule trivialization of P. For any object X of $\mathcal{C}$, the functor ${}_PM \circ - : \mathrm{Hom}_{\mathcal{C}}(X, B) \to \mathrm{LMod}_P(X)$ is an equivalence with inverse $N \circ_P - : \mathrm{LMod}_P(X) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$. Hence, $M : B \to A$ is an Eilenberg–Moore morphism of P. The proposition follows from Theorem B.1 and the fact that every Eilenberg–Moore morphism gives rise to an Eilenberg–Moore splitting. □

B.5 Idempotent completion of a 2-category

A 2-category is *idempotent complete* if it is locally idempotent complete and if every separable monad admits a separable splitting. By Theorem B.1 this is equivalent to requiring that every separable monad admits an Eilenberg–Moore or Kleisli object.

Definition B.6 (Idempotent completion of a 2-category)**.** Let $\mathcal{C}$ be a locally idempotent complete 2-category. Its *idempotent completion* $\mathcal{C}^{\triangledown}$ is the 2-category whose objects are separable monads in $\mathcal{C}$, whose 1-morphisms are bimodules, and whose 2-morphisms are bimodule maps. The composition of 1-morphisms is the relative composition of bimodules. The identity 1-morphism on a separable monad P is the trivial bimodule ${}_P P_P$.

Remark B.7 (Well-definition of composition in the idempotent completion). As a splitting of an idempotent, the relative composite of bimodules is only defined up to isomorphism. As stated, the idempotent completion $\mathcal{C}^\triangledown$ is therefore only defined for locally idempotent complete 2-categories $\mathcal{C}$ with chosen splittings of their idempotent 2-morphisms. However, since splittings of an idempotent are unique up to a unique isomorphism, different choices of splittings give rise to equivalent completions $\mathcal{C}^\triangledown$, and (assuming the axiom of choice) we can always make such choices of splittings.

Proposition B.8 (The idempotent completion is idempotent complete)**.** *The idempotent completion $\mathcal{C}^\triangledown$ of a locally idempotent complete 2-category $\mathcal{C}$ is idempotent complete.*

Proof. The 2-category $\mathcal{C}^\triangledown$ is locally idempotent complete since any idempotent bimodule map $p : {}_A M_B \Rightarrow {}_A M_B$ splits in $\mathcal{C}$, resulting in a 1-morphism N and 2-morphisms $N \overset{r}{\Rightarrow} M \overset{s}{\Rightarrow} N$ such that $r \cdot s = p$ and $s \cdot r = 1_N$. The splitting 2-morphisms are themselves bimodule maps for the bimodule structure $A \circ N \circ B \overset{A\circ r\circ B}{\Longrightarrow} A \circ M \circ B \overset{\rho}{\Rightarrow} M \overset{s}{\Rightarrow} M$ on N (here $\rho : A \circ M \circ B \Rightarrow M$ denotes the A-B action on M) and hence split the idempotent p in $\mathcal{C}^\triangledown$.

We show that every separable monad in $\mathcal{C}^\triangledown$ admits a separable splitting. A separable monad in $\mathcal{C}^\triangledown$ is a monad $P : A \to A$ in $\mathcal{C}$ together with a bimodule ${}_P M_P$ and bimodule 2-morphisms $m : {}_P M \circ_P M_P \Rightarrow {}_P M_P$ and $u : {}_P P_P \Rightarrow {}_P M_P$ fulfilling the defining equations for a monad in $\mathcal{C}^\triangledown$, and such that m has an M–M-bimodule section $\Delta : M \circ_P M \Rightarrow M$. The 2-morphisms

$$\widehat{m} := M \circ M \Rightarrow M \circ_P M \overset{m}{\Rightarrow} M \qquad\qquad \widehat{u} := 1_A \Rightarrow P \overset{u}{\Rightarrow} M$$

make the 1-morphism $M : A \to A$ into a separable monad in $\mathcal{C}$. Here, the 2-morphism $1_A \Rightarrow P$ is the unit of the monad P and $M \circ M \Rightarrow M \circ_P M$ is the projection onto the image of the idempotent defining the composite $M \circ_P M$. By definition, the multiplication $\widehat{m}$ intertwines the left and right action of P on M, leading to bimodules ${}_P M_M$ and ${}_M M_P$. The bimodule morphisms

$$\eta := {}_P P_P \overset{u}{\Rightarrow} {}_P M_P \overset{\cong}{\Rightarrow} {}_P M \circ_M M_P \qquad\qquad \epsilon := {}_M M \circ_P M_M \overset{m}{\Rightarrow} {}_M M_M$$

constitute the unit and counit of an adjunction ${}_P M_M \vdash {}_M M_P$ in $\mathcal{C}^\triangledown$. By definition, the M-M bimodule section $\Delta : M \circ_P M \Rightarrow M$ is a right inverse of ϵ. The multiplication

$\widehat{m} : M \circ M \Rightarrow M$ induces an isomorphism of bimodules $\psi : {}_PM \circ_M M_P \Rightarrow {}_PM_P$. This isomorphism is compatible with the monad structure on ${}_PM \circ_M M_P$ induced from the adjunction ${}_PM_M \vdash {}_MM_P$ and defines an isomorphism of monads in $\mathcal{C}^\triangledown$. Hence, $({}_PM_M \vdash {}_MM_P, \psi)$ is a separable splitting of the separable monad ${}_PM_P$ in $\mathcal{C}^\Delta$. □

There is a fully faithful 2-functor $i_\mathcal{C} : \mathcal{C} \to \mathcal{C}^\triangledown$ that sends an object A of $\mathcal{C}$ to the trivial separable monad $1_A : A \to A$ and sends 1- and 2-morphisms of $\mathcal{C}$ to themselves seen as bimodules, or bimodule maps, for the respective trivial monad.

Proposition B.9 (A 2-category is idempotent complete when idempotent completion is an equivalence)**.** *A locally idempotent complete 2-category $\mathcal{C}$ is idempotent complete if and only if the 2-functor $i_\mathcal{C} : \mathcal{C} \to \mathcal{C}^\triangledown$ is an equivalence.*

Proof. Since $i_\mathcal{C}$ is fully faithful, it is an equivalence if and only if it is essentially surjective. Essential surjectivity of $i_\mathcal{C}$ is in turn equivalent to the requirement that any separable monad in $\mathcal{C}$ is split as a bimodule. By Proposition B.5, this is equivalent to every separable monad admitting a separable splitting. □

Corollary B.10 (Idempotent completion is an idempotent operation)**.** *For any locally idempotent complete 2-category, the 2-functor $i_{\mathcal{C}^\triangledown} : \mathcal{C}^\triangledown \to (\mathcal{C}^\triangledown)^\triangledown$ is an equivalence.*

This corollary is analogous to a result of Carqueville–Runkel [CR16], who show that replacing a 2-category by the 2-category of internal separable Frobenius algebras is an idempotent operation.

B.6 Idempotent completion of 2-functors

Any 2-functor $F : \mathcal{C} \to \mathcal{D}$ between locally idempotent complete 2-categories maps separable monads to separable monads, bimodules to bimodules, and bimodule maps to bimodule maps, hence gives rise to a 2-functor $F^\triangledown : \mathcal{C}^\triangledown \to \mathcal{D}^\triangledown$ between the completions.

Remark B.11 (Well-defiinition of idempotent completion of 2-functors). As in Remark B.7, $F^\triangledown$ is really defined for 2-functors $F : \mathcal{C} \to \mathcal{D}$ between 2-categories with chosen splittings of their idempotent 2-morphisms. Choosing different splittings leads to a 2-functor $F^{\triangledown'} : \mathcal{C}^{\triangledown'} \to \mathcal{D}^{\triangledown'}$ equivalent to $\mathcal{C}^{\triangledown'} \simeq \mathcal{C}^\triangledown \xrightarrow{F^\triangledown} \mathcal{D}^\triangledown \simeq \mathcal{D}^{\triangledown'}$, where $\mathcal{C}^{\triangledown'} \simeq \mathcal{C}^\triangledown$ and $\mathcal{D}^{\triangledown'} \simeq \mathcal{D}^\triangledown$ are the canonical 'splitting change' equivalences from Remark B.7.

More generally, we expect $(-)^\triangledown$ to be a 3-functor on the 3-category of locally idempotent 2-categories with chosen splittings, with 2-functors, natural transformations, and modifications between them. A choice of splittings for every locally idempotent complete 2-category provides an inverse to the forgetful 3-functor from the preceding 3-category to the 3-category of locally idempotent complete 2-categories, hence induces a 3-functor $(-)^{\triangledown}$ on the latter 3-category. Different choices of splittings lead to distinct, but equivalent, inverses to the forgetful 3-functor, and therefore to equivalent 3-functors $(-)^{\triangledown}$.

Henceforth, whenever we refer to $F^{\triangledown} : \mathcal{C}^{\triangledown} \to \mathcal{D}^{\triangledown}$, we have implicitly chosen a splitting of every idempotent 2-morphism in $\mathcal{C}$ and $\mathcal{D}$.

Remark B.12 (Idempotent completion is an idempotent 3-monad). Together with the transformation $i_C : \mathcal{C} \to \mathcal{C}^{\triangledown}$ and the equivalence $(\mathcal{C}^{\triangledown})^{\triangledown} \to \mathcal{C}^{\triangledown}$, we expect $(-)^{\triangledown}$ to be an idempotent 3-monad on the 3-category of locally idempotent complete 2-categories.

We establish several properties of $(-)^{\triangledown}$, all of them consequences of what it would mean to be a 3-monad on a 2-category.

Proposition B.13 (Invariance and functoriality of idempotent completion)**.** *Let $F, G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{D} \to \mathcal{E}$ be 2-functors between locally idempotent complete 2-categories. Then the following hold, where $\simeq$ denotes equivalence of 2-functors:*

1. *If $F \simeq G$, then $F^{\triangledown} \simeq G^{\triangledown}$.*
2. *$(H \circ G)^{\triangledown} \simeq H^{\triangledown} \circ G^{\triangledown}$.*
3. *$F^{\triangledown} \circ i_{\mathcal{C}} \simeq i_{\mathcal{D}} \circ F$.*
4. *$(i_{\mathcal{C}})^{\triangledown} \simeq i_{\mathcal{C}^{\triangledown}}$.*

Proof. The equivalences (2), (3) and (4) are direct consequences of the definition of $F^{\triangledown}$ and $i_{\mathcal{C}}$. We prove property (1). Let $\eta : F \Rightarrow G$ be a natural equivalence with component equivalences $\eta_A : F(A) \to G(A)$ for objects A in $\mathcal{C}$, and isomorphisms $\eta_f : \eta_B \circ F(f) \Rightarrow G(f) \circ \eta_A$ for 1-morphisms $f : A \to B$ in $\mathcal{C}$. For a separable monad $(A \xrightarrow{P} A, P \circ P \overset{m}{\Rightarrow} P, 1_A \overset{u}{\Rightarrow} P)$ in $\mathcal{C}$, we define — omitting all coherence isomorphisms of F and G — the $G(P)$–$F(P)$-bimodule $\eta^{\triangledown}_{(P,m,u)} := G(P) \circ \eta_A$ with action

$$G(P) \circ G(P) \circ \eta_A \circ F(P) \xrightarrow{G(m)\circ\eta_A\circ F(P)} G(P) \circ \eta_A \circ F(P) \xrightarrow{G(P)\circ\eta_P} G(P) \circ G(P) \circ \eta_A \xrightarrow{G(m)\circ\eta_A} G(P) \circ \eta_A.$$

This bimodule is invertible; an inverse is the bimodule $\eta_A^{-1} \circ G(P)$, with action as above, where $\eta^{-1} : G \Rightarrow F$ denotes an inverse of the natural equivalence η.

For a bimodule $(A \xrightarrow{M} B, Q \circ M \circ P \overset{\rho}{\Rightarrow} M)$ in $\mathcal{C}$, the $G(P)$–$F(P)$-bimodule morphisms

$$G(P) \circ \eta_B \circ F(M) \xRightarrow{\eta_P^{-1} \circ F(M)} \eta_B \circ F(P) \circ F(M) \xRightarrow{\eta_B \circ F(\rho)} \eta_B \circ F(M)$$
$$G(M) \circ G(P) \circ \eta_A \xRightarrow{G(\rho) \circ \eta_A} G(M) \circ \eta_A$$

induce bimodule isomorphisms

$$\eta^{\triangledown}_{(Q,m_Q,u_Q)} \circ_{F^{\triangledown}(Q,m_Q,u_Q)} F^{\triangledown}(M,\rho) \cong \eta_B \circ F(M)$$
$$G^{\triangledown}(M,\rho) \circ_{G^{\triangledown}(P,m_P,u_P)} \eta^{\triangledown}_{(P,m_P,u_P)} \cong G(M) \circ \eta_A.$$

Using these isomorphisms, we define the bimodule isomorphism $\eta^{\triangledown}_{(M,\rho)}$ as follows:

$$\eta^{\triangledown}_{(Q,m_Q,u_Q)} \circ_{F^{\triangledown}(Q,m_Q,u_Q)} F^{\triangledown}(M,\rho) \cong \eta_B \circ F(M) \overset{\eta_M}{\Longrightarrow} G(M) \circ \eta_A \cong G^{\triangledown}(M,\rho) \circ_{G^{\triangledown}(P,m_P,u_P)} \eta^{\triangledown}_{(P,m_P,u_P)}$$

It can be verified that the equivalences $\eta^{\triangledown}_{(P,m,u)}$ and the bimodule isomorphisms $\eta^{\triangledown}_{(M,\rho)}$ form a natural equivalence between $F^{\triangledown}$ and $G^{\triangledown}$. □

We now show that the 2-category $\mathcal{C}^{\triangledown}$ deserves the name 'idempotent completion': every 2-functor from the locally idempotent complete 2-category $\mathcal{C}$ into an idempotent complete 2-category $\mathcal{D}$ factors uniquely through $\mathcal{C}^{\triangledown}$.

Proposition B.14 (The idempotent completion is initial among idempotent complete targets)**.** *Let $\mathcal{C}$ be a locally idempotent complete 2-category and let $\mathcal{D}$ be an idempotent complete 2-category. Then, any 2-functor $F : \mathcal{C} \to \mathcal{D}$ uniquely extends to a 2-functor $\widehat{F} : \mathcal{C}^{\triangledown} \to \mathcal{D}$. That is, there exists a 2-functor $\widehat{F} : \mathcal{C}^{\triangledown} \to \mathcal{D}$ such that $\widehat{F} \circ i_{\mathcal{C}}$ is equivalent to F, and if $\widehat{F}, \widehat{F}' : \mathcal{C}^{\triangledown} \to \mathcal{D}$ are 2-functors such that $\widehat{F} \circ i_{\mathcal{C}} \simeq \widehat{F}' \circ i_{\mathcal{C}}$, then $\widehat{F}$ and $\widehat{F}'$ are equivalent.*

Proof. Since $\mathcal{D}$ is idempotent complete, the 2-functor $i_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}^{\triangledown}$ is an equivalence by Proposition B.9; we fix an inverse $i_{\mathcal{D}}^{-1} : \mathcal{D}^{\triangledown} \to \mathcal{D}$. Given a 2-functor $F : \mathcal{C} \to \mathcal{D}$, we define its extension

$$\widehat{F} := \mathcal{C}^{\triangledown} \xrightarrow{F^{\triangledown}} \mathcal{D}^{\triangledown} \xrightarrow{i_{\mathcal{D}}^{-1}} \mathcal{D}.$$

It follows from Proposition B.13(3) that $\widehat{F} \circ i_{\mathcal{C}} \simeq i_{\mathcal{D}}^{-1} \circ i_{\mathcal{D}} \circ F \simeq F$.

Given two extensions $\widehat{F}, \widehat{F}' : \mathcal{C}^{\triangledown} \to \mathcal{D}$ such that $\widehat{F} \circ i_{\mathcal{C}} \simeq \widehat{F}' \circ i_{\mathcal{C}}$, by Proposition B.13(2–4), we have

$$i_{\mathcal{D}}^{-1} \circ \left(\widehat{F} \circ i_{\mathcal{C}}\right)^{\triangledown} \simeq i_{\mathcal{D}}^{-1} \circ \widehat{F}^{\triangledown} \circ i_{\mathcal{C}}^{\triangledown} \simeq i_{\mathcal{D}}^{-1} \circ \widehat{F}^{\triangledown} \circ i_{\mathcal{C}^{\triangledown}} \simeq i_{\mathcal{D}}^{-1} \circ i_{\mathcal{D}} \circ \widehat{F} \simeq \widehat{F}.$$

It follows from Proposition B.13(1) that

$$\widehat{F} \simeq i_{\mathcal{D}}^{-1} \circ \left(\widehat{F} \circ i_{\mathcal{C}}\right)^{\triangledown} \simeq i_{\mathcal{D}}^{-1} \circ \left(\widehat{F}' \circ i_{\mathcal{C}}\right)^{\triangledown} \simeq \widehat{F}'.$$

□

C Dimension formulas in spherical prefusion 2-categories

We prove various formulas relating the dimensions of objects and 1-morphisms in spherical prefusion 2-categories.

Recall from Definition 1.2.30 that for simple objects A and B of a finite presemisimple 2-category $\mathcal{C}$, the dimension $\dim(\mathrm{Hom}_{\mathcal{C}}(A,B))$ is the sum over simple 1-morphisms from A to B of the squared norm of the 1-morphism. Similarly recall from Definition 1.2.32 that for a finite presemisimple 2-category $\mathcal{C}$, the dimension $\dim(\mathcal{C})$ is the sum over components of the reciprocal of the dimensions of the endomorphism categories. Finally recall from Definition 1.3.48 that for a 1-morphism f in a spherical prefusion 2-category, the dimension $\dim(f)$ is the (2-spherical) trace of the identity of f, and for an object A, the dimension $\dim(A)$ is the dimension of the identity of A.

For a simple object A of a spherical prefusion 2-category $\mathcal{C}$, we will denote by $n(A)$ the number of equivalence classes of simple objects in the component of A. We will also use the abbreviation $\mathrm{d}(A) := \dim(A)\dim(\mathrm{End}_{\mathcal{C}}(A))n(A)$.

Proposition C.1 (Dimension of 1-morphisms is relatively multiplicative)**.** *Let $g : A \to B$ and $f : B \to C$ be 1-morphisms in a spherical prefusion 2-category and assume that B is simple. Then*

$$\dim(f \circ g) = \langle \mathrm{tr}_R(1_g)\rangle \dim(f) = \langle \mathrm{tr}_L(1_f)\rangle \dim(g) = \frac{\dim(f)\dim(g)}{\dim(B)}$$

Proof. Simplicity of B implies that $\mathrm{tr}_R(1_g) = \langle \mathrm{tr}_R(1_g)\rangle 1_{1_B}$. Hence, $\dim(f \circ g) = \mathrm{Tr}(1_f \circ \mathrm{tr}_R(1_g)) = \langle \mathrm{tr}_R(1_g)\rangle \dim(f)$. The second equation follows analogously after an application of Proposition 1.3.37. The last equation follows since $\dim(f) = \langle \mathrm{tr}_L(1_f)\rangle \dim(B)$, $\dim(g) = \langle \mathrm{tr}_R(1_g)\rangle \dim(B)$, and $\dim(B)$ is nonzero. □

Lemma C.2 (Dimension of 1-morphisms is additive)**.** *Let $f : A \to B$ be a 1-morphism in a spherical prefusion 2-category. Then*

$$\sum_{h: A \to B} \dim\left(\mathrm{Hom}_{\mathcal{C}}(h,f)\right)\dim(h) = \dim(f),$$

where the sum is over representatives of the simple 1-morphisms $h : A \to B$.

Proof. Let $\{i_j : f_j \leftrightarrows f : p_j\}_{j\in J}$ be a direct sum decomposition of f into simple 1-morphisms. For $h : A \to B$ a simple 1-morphism, let $J_h \subseteq J$ be the subset of the

1-morphisms f_j isomorphic to h. Since $|J_h| = \dim(\mathrm{Hom}_{\mathcal{C}}(h, f))$, it follows that

$$\begin{aligned}\dim(f) = \mathrm{Tr}(1_f) &= \sum_{j\in J} \mathrm{Tr}(i_j \cdot p_j) = \sum_{j\in J} \mathrm{Tr}(p_j \cdot i_j) = \sum_{j\in J} \dim(f_j) \\ &= \sum_{h:A\to B} |J_h| \dim(h) = \sum_{h:A\to B} \dim\left(\mathrm{Hom}_{\mathcal{C}}(h, f)\right) \dim(h).\end{aligned}$$

In the second step we used that the trace is additive for 2-morphisms, that is $\mathrm{Tr}(\alpha + \beta) = \mathrm{Tr}(\alpha) + \mathrm{Tr}(\beta)$ for $\alpha, \beta : g \Rightarrow h$. □

Proposition C.3 (Dimension of a 1-morphism from its precompositions)**.** *Let $f : A \to B$ be a 1-morphism in a spherical prefusion 2-category $\mathcal{C}$. Then*

$$\sum_{C, C\xrightarrow{g}A} \frac{\dim(g)\dim(f\circ g)}{\mathrm{d}(C)} = \dim(f),$$

where the sum is over representatives of the simple objects C and simple 1-morphisms $g : C \to A$.

Proof. First, suppose that A is simple and fix a simple C. If A and C are in different components, then $\sum_{C\xrightarrow{g}A} \dim(g)\dim(f\circ g) = 0$. If A and C are in the same component, then

$$\begin{aligned}\sum_{C\xrightarrow{g}A} \dim(g)\dim(f\circ g) &= \sum_{C\xrightarrow{g}A} \langle \mathrm{tr}_L(1_g)\rangle\langle \mathrm{tr}_R(1_g)\rangle \quad \dim(C)\dim(f) \\ &= \dim\left(\mathrm{Hom}_{\mathcal{C}}(C, A)\right)\dim(C)\dim(f) \overset{\text{Prop. 1.2.31}}{=} \dim\left(\mathrm{End}_{\mathcal{C}}(C)\right)\dim(C)\dim(f).\end{aligned}$$

Summing over simples C gives the desired formula.

Now suppose that A is an arbitrary object and again fix a simple C. Let $\{\iota_i : A_i \leftrightarrows A : \rho_i\}_{i\in I}$ be a direct sum decomposition of A into simple objects $\{A_i\}_{i\in I}$. Observe that if $g : C \to A$ is a simple 1-morphism, then there is a unique $\alpha(g) \in I$ such that $\rho_{\alpha(g)} \circ g$ is nonzero — otherwise $g \cong \bigoplus_{i\in I} \iota_i \circ \rho_i \circ g$ would be a non-trivial direct sum decomposition of g. Similarly $\rho_{\alpha(g)} \circ g : C \to A_{\alpha(g)}$ itself must be a simple 1-morphism. Observe that the functions

$$\begin{array}{ccc} \{\text{iso classes of simples } C \to A\} & \leftrightarrow & \sqcup_{i\in I}\{\text{iso classes of simples } C \to A_i\} \\ g & \mapsto & \rho_{\alpha(g)} \circ g : C \to A_{\alpha(g)} \\ \iota_i \circ h & \leftarrow\!\shortmid & h : C \to A_i \end{array}$$

are inverse to each other. It therefore follows that

$$\begin{aligned}\sum_{C\xrightarrow{g}A} \dim(g)\dim(f\circ g) &= \sum_{i\in I}\sum_{C\xrightarrow{h}A_i} \dim(\iota_i\circ h)\dim(f\circ\iota_i\circ h) \\ &\overset{\text{Prop C.1}}{=} \sum_{i\in I}\sum_{C\xrightarrow{h}A_i} \langle \mathrm{tr}_L(1_{\iota_i})\rangle \dim(h)\dim(f\circ\iota_i\circ h).\end{aligned}$$

For a component $x \in \pi_0\mathcal{C}$, define the subset $I_x := \{i \in I \mid A_i \in x\}$. For a simple object C, let $[C] \in \pi_0\mathcal{C}$ denote the component of C. It follows from the previous calculation for simple A that

$$\sum_{i\in I} \sum_{C \xrightarrow{h} A_i} \langle \mathrm{tr}_L(1_{\iota_i})\rangle \dim(h) \dim(f\circ\iota_i\circ h) = \sum_{i\in I_{[C]}} \langle \mathrm{tr}_L(1_{\iota_i})\rangle \dim\left(\mathrm{End}_{\mathcal{C}}(C)\right) \dim(C) \dim(f\circ\iota_i).$$

By Proposition 1.2.3, ι_i is adjoint to ρ_i and hence $\mathrm{tr}_L(\iota_i) = \mathrm{tr}_R(\rho_i)$. Therefore,

$$\langle \mathrm{tr}_L(1_{\iota_i})\rangle \dim(f\circ\iota_i) = \dim(f\circ\iota_i\circ\rho_i).$$

We conclude that

$$\sum_{C \xrightarrow{g} A} \dim(g)\dim(f\circ g) = \dim\left(\mathrm{End}_{\mathcal{C}}(C)\right)\dim(C) \sum_{i\in I_{[C]}} \dim(f\circ\iota_i\circ\rho_i).$$

It follows that

$$\begin{aligned}\sum_{C, C \xrightarrow{g} A} \frac{\dim(g)\dim(f\circ g)}{\dim(C)\dim\left(\mathrm{End}_{\mathcal{C}}(C)\right) n(C)} &= \sum_{C} \frac{1}{n(C)} \sum_{i\in I_{[C]}} \dim(f\circ\iota_i\circ\rho_i) \\ &= \sum_{x\in\pi_0\mathcal{C}} \sum_{i\in I_x} \dim(f\circ\iota_i\circ\rho_i) = \sum_{i\in I} \dim(f\circ\iota_i\circ\rho_i) = \dim(f).\end{aligned}$$

In the last step, we have used that for 1-morphisms h and k, it holds that $\dim(h\oplus k) = \dim(h) + \dim(k)$. □

Corollary C.4 (Dimension of an object from incoming morphisms)**.** *Let A be an object in a spherical prefusion 2-category $\mathcal{C}$. Then*

$$\sum_{B, B \xrightarrow{f} A} \frac{\dim(f)^2}{\mathrm{d}(B)} = \dim(A),$$

where the sum is over representatives of the simple objects B and simple 1-morphisms $f : B \to A$.

Proof. This follows from Proposition C.3 for $f = 1_A$. □

Corollary C.5 (Dimension of a prefusion 2-category from its fusion rule morphisms)**.** *Let A be a simple object in a spherical prefusion 2-category $\mathcal{C}$. Then*

$$\sum_{B, C, f : B\Box C \longrightarrow A} \frac{\dim(f)^2}{\dim(A)\mathrm{d}(B)\mathrm{d}(C)} = \dim(\mathcal{C}),$$

where the sum is over representatives of the simple objects B and C and simple 1-morphisms f.

Proof. By sphericality and Corollary C.4, the left-hand side of this equation can be expressed as follows:

$$\sum_{B,C,B \xrightarrow{f} A \Box C^{\#}} \frac{\dim(f)^2}{\dim(A)\mathrm{d}(B)\mathrm{d}(C)} = \sum_C \frac{1}{\dim(A)\mathrm{d}(C)} \dim(A \Box C^{\#}) = \sum_C \frac{\dim(C)}{\mathrm{d}(C)}$$

Note that in the last equality we have used that the dimension of objects is multiplicative, that is $\dim(A \Box B) = \dim(A)\dim(B)$, and that the dimension of an object and its dual agree, that is $\dim(A^{\#}) = \dim(A)$. We furthermore have

$$\sum_C \frac{\dim(C)}{\mathrm{d}(C)} = \sum_C \frac{1}{\dim(\mathrm{End}_{\mathcal{C}}(C))n(C)} = \sum_{[x]\in\pi_0\mathcal{C}} \frac{1}{\dim(\mathrm{End}_{\mathcal{C}}(x))} = \dim(\mathcal{C}). \qquad \square$$

Corollary C.6 (Dimension of a 1-morphism from its factorizations)**.** *Let $f : A \to B$ be a 1-morphism in a spherical prefusion 2-category $\mathcal{C}$. Then*

$$\sum_{C,A \xrightarrow{h} C, C \xrightarrow{g} B} \dim\left(\mathrm{Hom}_{\mathcal{C}}(g \circ h, f)\right) \frac{\dim(g)\dim(h)}{\mathrm{d}(C)} = \dim(f),$$

where the sum is over representatives of the simple objects C and simple 1-morphisms g and h.

Proof. By pivotality, $\mathrm{Hom}_{\mathcal{C}}(g \circ h, f) \cong \mathrm{Hom}_{\mathcal{C}}(h, g^* \circ f)$. Lemma C.2 implies that, for fixed simple C,

$$\sum_{A \xrightarrow{h} C} \dim\left(\mathrm{Hom}_{\mathcal{C}}(h, g^* \circ f)\right) \; \dim(h) = \dim(g^* \circ f) = \dim(f^* \circ g).$$

The left-hand side of the desired formula becomes

$$\sum_{C,C \xrightarrow{g} B} \frac{\dim(g)\dim(f^* \circ g)}{\mathrm{d}(C)} \overset{\text{Prop C.3}}{=} \dim(f^*) = \dim(f). \qquad \square$$

Corollary C.7 (Total factorization of the identity on a 1-morphism)**.** *Let $A \xrightarrow{f} B$ be a 1-morphism in a spherical prefusion 2-category $\mathcal{C}$. Then*

$$\sum_{C,C \xrightarrow{g} B, A \xrightarrow{h} C} \frac{\dim(g)\dim(h)}{\mathrm{d}(C)} \sum_{\gamma \in \mathrm{Hom}_{\mathcal{C}}(g \circ h, f)} \gamma \cdot \widehat{\gamma} = 1_f$$

where the left sum is over representatives of the simple objects C and simple 1-morphisms g and h, the right sum is over a basis $\{\gamma\}$ of the vector space $\mathrm{Hom}_{\mathcal{C}}(g \circ h, f)$, and $\{\widehat{\gamma}\}$ denotes the dual basis of $\mathrm{Hom}_{\mathcal{C}}(f, g \circ h)$ with respect to the pairing $\langle \cdot, \cdot \rangle$ (see Definition 1.3.53).

Proof. First note that $\sum_\gamma \gamma \cdot \widehat{\gamma}$ is independent of the choice of basis. Let $\{s_i\}_{i\in I}$ be a set of representative simple 1-morphisms $s_i : A \to B$. For each $i \in I$, let $\{\alpha^i_j\}_{j\in J_i}$ be a basis of $\mathrm{Hom}_{\mathcal{C}}(s_i, f)$ and let $\{\beta^i_k\}_{k\in K_i}$ be a basis of $\mathrm{Hom}_{\mathcal{C}}(g \circ h, s_i)$. By local semisimplicity, the set

$$\mathcal{B} := \sqcup_{i\in I}\{\alpha^i_j \cdot \beta^i_k \in \mathrm{Hom}_{\mathcal{C}}(g \circ h, f) \mid j \in J_i, k \in K_i\}$$

forms a basis of $\mathrm{Hom}_{\mathcal{C}}(g \circ h, f)$. Let $\{\widehat{\alpha}^i_j \in \mathrm{Hom}_{\mathcal{C}}(f, s_i)\}_{j\in J_i}$ and $\{\widehat{\beta}^i_k \in \mathrm{Hom}_{\mathcal{C}}(s_i, g \circ h)\}_{k\in K_i}$ be dual bases to $\{\alpha^i_j\}_j$ and $\{\beta^i_k\}_k$ under the usual non-degenerate 'simple 1-morphism pairings':

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(f, s_i) \otimes \mathrm{Hom}_{\mathcal{C}}(s_i, f) &\to k & \qquad \mathrm{Hom}_{\mathcal{C}}(g \circ h, s_i) \otimes \mathrm{Hom}_{\mathcal{C}}(s_i, g \circ h) &\to k \\ \mu \otimes \nu &\mapsto \langle \mu \cdot \nu \rangle & \mu \otimes \nu &\mapsto \langle \mu \cdot \nu \rangle \end{aligned}$$

Direct calculation shows that the set

$$\sqcup_{i\in I}\left\{\frac{1}{\dim(s_i)}\ \widehat{\beta}^i_k \cdot \widehat{\alpha}^i_j\right\}_{j\in J_i, k\in K_i}$$

is a dual basis of the basis $\mathcal{B}$ with respect to the 'sphere-pairing' $\mathrm{Hom}_{\mathcal{C}}(g \circ h, f) \otimes \mathrm{Hom}_{\mathcal{C}}(f, g \circ h) \to k$ from Definition 1.3.53. With this choice of basis, the left-hand side of the desired equation becomes

$$\sum_{C, C \xrightarrow{g} B, A \xrightarrow{h} C} \frac{\dim(g)\dim(h)}{\mathrm{d}(C)} \sum_{i\in I} \frac{1}{\dim(s_i)} \sum_{j\in J_i, k\in K_i} \alpha^i_j \cdot \beta^i_k \cdot \widehat{\beta}^i_k \cdot \widehat{\alpha}^i_j.$$

Observing that $\sum_{k\in K_i} \beta^i_k \cdot \widehat{\beta}^i_k = \dim\left(\mathrm{Hom}_{\mathcal{C}}(s_i, g \circ h)\right)\ 1_{s_i}$, this expression becomes

$$\sum_{i\in I, j\in J_i} \frac{1}{\dim(s_i)} \alpha^i_j \cdot \widehat{\alpha}^i_j \left(\sum_{C, C \xrightarrow{g} B, A \xrightarrow{h} C} \frac{\dim(g)\dim(h)}{\mathrm{d}(C)} \dim\left(\mathrm{Hom}_{\mathcal{C}}(s_i, g \circ h)\right) \right).$$

By Corollary C.6 and the fact that for 1-morphism k, h in a locally semisimple 2-category $\mathrm{Hom}_{\mathcal{C}}(h, k) \cong \mathrm{Hom}_{\mathcal{C}}(k, h)$, the expression in parenthesis is $\dim(s_i)$. Hence, the expression in question reduces to

$$\sum_{i\in I, j\in J_i} \alpha^i_j \cdot \widehat{\alpha}^i_j = 1_f. \qquad \square$$

D Morita equivalence and dagger 2-categories

In this appendix, we prove the correspondence between equivalence classes of certain objects and Morita equivalence classes of certain Frobenius monads in dagger 2-categories in which dagger idempotents split. This is the main technical result needed for our classification of quantum isomorphic quantum graphs in Corollary 4.3.7.

The basic idea of this section can be summarized as follows. A dualizable 1-morphism $S : B \to A$ in a dagger 2-category $\mathbb{B}$ gives rise to a dagger Frobenius monoid $S \circ \overline{S}$ in $\mathbb{B}(A, A)$. It can be shown that two such Frobenius monoids are $*$-isomorphic if and only if the underlying 1-morphisms $S : B \to A$ and $S' : B' \to A$ are equivalent (in the sense that there is an equivalence $\epsilon : B \to B'$ such that $S' \circ \epsilon \cong S$). If we only consider Frobenius monoids up to Morita equivalence, then we cannot recover the 1-morphism S but we can still recover B, the source of S, up to equivalence. This is the content of Theorem D.1.

Here, we use the graphical calculus of 2-categories; objects are depicted as shaded regions, 1-morphisms $f : A \to B$ are depicted as wires bounded by A on the right and B on the left and 2-morphisms are depicted by vertices. Our diagrams should be read from bottom to top and from right to left to match the conventional right-to-left notation of function — and 1-morphism — composition. For an introduction to this calculus, see Section 1.3.1 or [Sel11, Mar14].

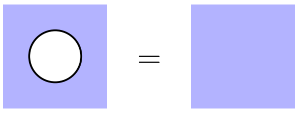
Recall that a 1-morphism $F : A \to B$ in a 2-category is an *equivalence* if there is a 1-morphism $G : B \to A$ such that $F \circ G \cong 1_B$ and $G \circ F \cong 1_A$. Similarly, we say that a 1-morphism in a dagger 2-category is a *dagger equivalence* if these 2-isomorphisms are also unitary.

In the following, we depict the objects A and B as white and blue regions, respectively. Recall that a 1-morphism $S : B \to A$ in a dagger 2-category has a *dual* $\overline{S} : A \to B$ if there are 2-morphisms $\epsilon : \overline{S} \circ S \Rightarrow 1_B$ and $\eta : 1_A \Rightarrow S \circ \overline{S}$, depicted as follows:

$\epsilon : \overline{S} \circ S \Rightarrow 1_B$ $\qquad$ $\eta : 1_A \Rightarrow S \circ \overline{S}$ $\qquad$ $\eta^\dagger : S \circ \overline{S} \Rightarrow 1_A$ $\qquad$ $\epsilon^\dagger : 1_B \Rightarrow \overline{S} \circ S$

These must satisfy the cusp equations:

We say that a 1-morphism $S : B \to A$ is *special* if it has a dual $\overline{S} : A \to B$ such that, in addition, the following holds:



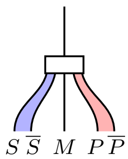
(42)

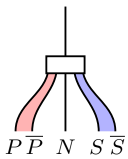
If $S : B \to A$ is a special 1-morphism, then $S \circ \overline{S}$ is a special dagger Frobenius monoid in $\mathbb{B}(A, A)$. Conversely, we say that a special dagger Frobenius monoid is *split* if it is $*$-isomorphic to $S \circ \overline{S}$ for some special 1-morphism.

We now state and prove the main technical result needed for the classification of quantum isomorphic graphs in Corollary 4.3.7. We believe the content and ideas of the following theorem to be well known; however, we could not find a similar statement and proof of appropriate generality in the literature.

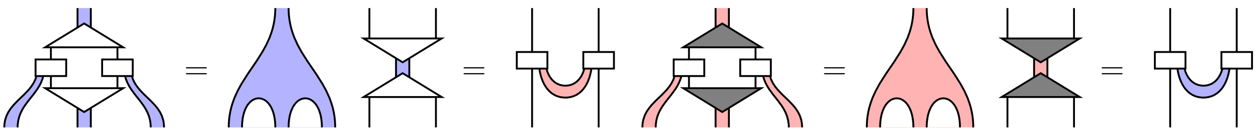
Theorem D.1. *Let $\mathbb{B}$ be a dagger 2-category in which all dagger idempotents split and let $S : X \to A$ and $P : Y \to A$ be special 1-morphisms. Then, the special dagger Frobenius monoids $S \circ \overline{S}$ and $P \circ \overline{P}$ are Morita equivalent if and only if X is dagger equivalent to Y.*

Proof. 1. Suppose that $S \circ \overline{S}$ and $P \circ \overline{P}$ are Morita equivalent. Then we claim that X (depicted as a blue region) and Y (depicted as a red region) are dagger equivalent. The object A is depicted as the white region and the invertible dagger bimodules ${}_{S\circ\overline{S}}M_{P\circ\overline{P}}$ and ${}_{P\circ\overline{P}}N_{S\circ\overline{S}}$ are depicted as follows:

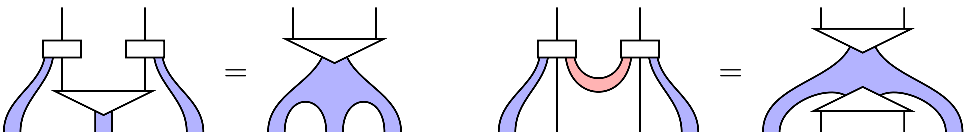




Following (4.19), we have additional 2-morphisms fulfilling the following equations:

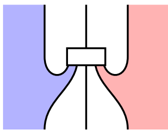


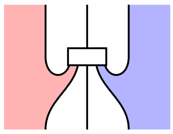
In particular, these equations imply the following:



(43)

Since ${}_{S\circ\overline{S}}M_{P\circ\overline{P}}$ and ${}_{P\circ\overline{P}}N_{S\circ\overline{S}}$ are dagger bimodules, it follows that the following two 2-morphisms are dagger idempotent:



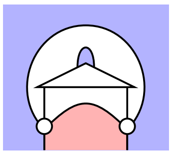


Splitting these idempotents produces 2-morphisms (here depicted as circular white nodes) fulfilling the following equations:

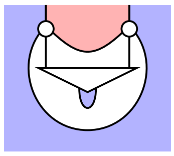
(44)

(45)

We claim that the resulting 1-morphisms $L : Y \to X$ and $R : X \to Y$ form a dagger equivalence. In fact, we claim that the following 2-morphisms are inverse to each other:

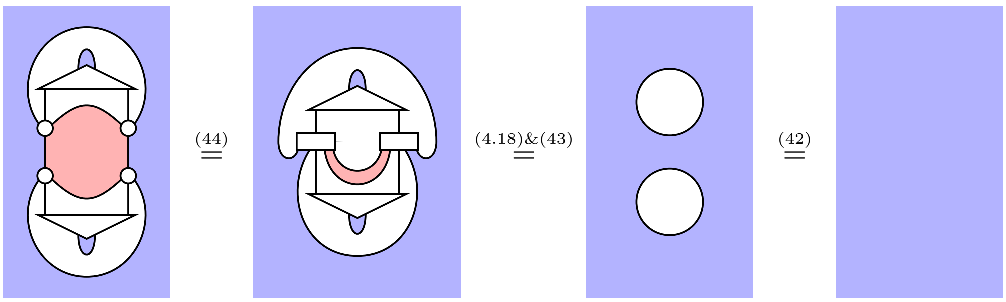


$\epsilon : L \circ R \Rightarrow 1_X$

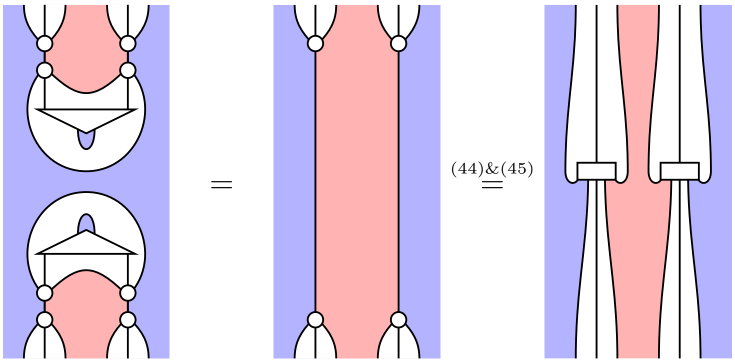


$\eta = \epsilon^\dagger : 1_X \Rightarrow L \circ R$

The equation $\epsilon\eta = 1_{1_X}$ can be proven as follows:

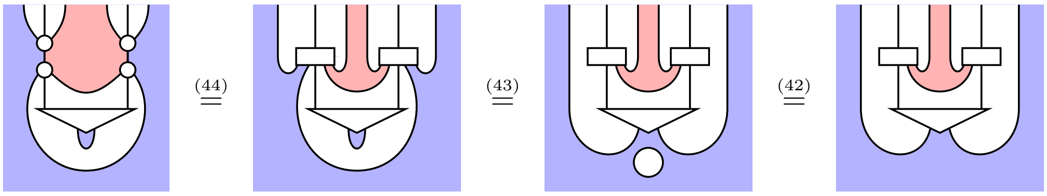


We note that it follows from (44) and (45) that the converse $\eta\epsilon = 1_{L\circ R}$ is equivalent to the following equation:

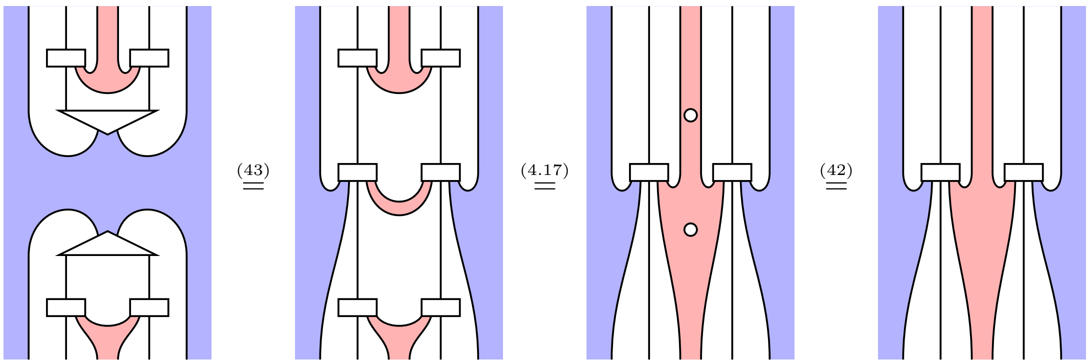


(46)

We can rewrite the involved 2-morphism as follows:



We can then prove equation (46) as follows:



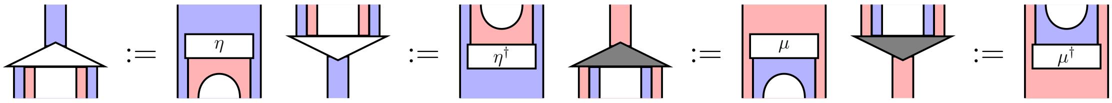
This concludes the proof that $L \circ R \cong 1_X$. The converse, $R \circ L \cong 1_Y$ can be proven analogously.

2. We claim that if $S : X \to A \to X$ and $P : Y \to A$ are special 1-morphisms such that X and Y are dagger equivalent, then $S \circ \overline{S}$ and $P \circ \overline{P}$ are Morita equivalent. Let $R : X \to Y$ and $L : Y \to X$ be an equivalence between X and Y and denote the corresponding 2-isomorphisms as follows:

$\eta : L \circ R \Rightarrow 1_X$ $\quad \eta^{-1} : 1_X \Rightarrow L \circ R$ $\quad \mu : R \circ L \Rightarrow 1_Y$ $\quad \mu^{-1} : 1_Y \Rightarrow R \circ L$

It can then be verified that the 1-morphisms $S \circ L \circ \overline{P}$ and $P \circ R \circ \overline{S}$ form bimodules

for which the following 2-morphisms establish Morita equivalence as in (4.19):



Corollary D.2. *Let $\mathbb{B}$ be a dagger 2-category in which dagger idempotents split and let A be an object of $\mathbb{B}$. The construction of a special dagger Frobenius monoid in $\mathbb{B}(A, A)$ from a special 1-morphism into A induces a bijection between the following sets:*

- *Dagger equivalence classes of objects X such that there exists a special 1-morphism $S : X \to A$.*
- *Morita equivalence classes of split special dagger Frobenius monoids $F \in \mathbb{B}(A, A)$.*

Proof. This function is well-defined by the *only if* condition of Theorem D.1, surjective by definition and injective by the *if* condition. □

Remark D.3. Statements similar to Theorem D.1 and Corollary D.2 hold for non-dagger special Frobenius monoids in non-dagger 2-categories.

Remark D.4. Corollary D.2 classifies objects X in $\mathbb{B}$ for which there exists some special 1-morphism $X \to A$. One may further obtain an explicit classification of these 1-morphisms as follows:

We say that two special 1-morphisms $S : X \to A$ and $P : Y \to A$ are *equivalent* if there exists a dagger equivalence $\epsilon : X \to Y$ and a unitary isomorphism $P \circ \epsilon \cong S$. It can then be shown that equivalence classes of special 1-morphisms into A are in one-to-one correspondence with split special dagger Frobenius monoids in $\mathbb{B}(A, A)$ *up to $*$-isomorphism.*

In other words, $*$-isomorphism classes of Frobenius monoids in $\mathbb{B}(A, A)$ classify 1-morphisms into A, while the coarser Morita equivalence classes just classify objects to which there exists some 1-morphism into A.

E The UEB $\mathcal{U}$

The following is the UEB $\mathcal{U}$ constructed in Section 3.5.

$$U_{111} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$U_{112} = \begin{pmatrix} 0&0&0&0&0&\frac{2}{\sqrt{5}}&\frac{i}{\sqrt{5}}&0\\ 0&0&0&0&\frac{2}{\sqrt{5}}&0&0&\frac{-i}{\sqrt{5}}\\ 0&0&0&0&\frac{-i}{\sqrt{5}}&0&0&\frac{2}{\sqrt{5}}\\ 0&0&0&0&0&\frac{i}{\sqrt{5}}&\frac{2}{\sqrt{5}}&0\\ 0&\frac{-1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0&0&0&0&0\\ \frac{-1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0&0\\ \frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0&0\\ 0&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0&0&0&0 \end{pmatrix}$$

$$U_{113} = \begin{pmatrix} 0&0&0&0&0&\frac{i}{\sqrt{5}}&\frac{2}{\sqrt{5}}&0\\ 0&0&0&0&\frac{i}{\sqrt{5}}&0&0&\frac{-2}{\sqrt{5}}\\ 0&0&0&0&\frac{-2}{\sqrt{5}}&0&0&\frac{i}{\sqrt{5}}\\ 0&0&0&0&0&\frac{2}{\sqrt{5}}&\frac{i}{\sqrt{5}}&0\\ 0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0&0&0&0\\ \frac{1}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}}&0&0&0&0\\ \frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0&0\\ 0&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0&0&0&0 \end{pmatrix}$$

$$U_{114} = \begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0 \end{pmatrix}$$

$$U_{121} = \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{pmatrix}$$

$$U_{122} = \begin{pmatrix} \frac{i}{\sqrt{5}}&0&0&\frac{2}{\sqrt{5}}&0&0&0&0\\ 0&\frac{-i}{\sqrt{5}}&\frac{2}{\sqrt{5}}&0&0&0&0&0\\ 0&\frac{2}{\sqrt{5}}&\frac{-i}{\sqrt{5}}&0&0&0&0&0\\ \frac{2}{\sqrt{5}}&0&0&\frac{i}{\sqrt{5}}&0&0&0&0\\ 0&0&0&0&\frac{-1}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}}\\ 0&0&0&0&0&\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0\\ 0&0&0&0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\ 0&0&0&0&\frac{1}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$U_{123} = \begin{pmatrix} \frac{2}{\sqrt{5}}&0&0&\frac{i}{\sqrt{5}}&0&0&0&0\\ 0&\frac{-2}{\sqrt{5}}&\frac{i}{\sqrt{5}}&0&0&0&0&0\\ 0&\frac{i}{\sqrt{5}}&\frac{-2}{\sqrt{5}}&0&0&0&0&0\\ \frac{i}{\sqrt{5}}&0&0&\frac{2}{\sqrt{5}}&0&0&0&0\\ 0&0&0&0&\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\ 0&0&0&0&0&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\ 0&0&0&0&0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\ 0&0&0&0&\frac{1}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$U_{124} = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix}$$

$$U_{131} = \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}$$

$$U_{132} = \begin{pmatrix} 0&\frac{2}{\sqrt{5}}&0&0&\frac{i}{\sqrt{5}}&0&0&0\\ \frac{2}{\sqrt{5}}&0&0&0&0&\frac{-i}{\sqrt{5}}&0&0\\ \frac{-i}{\sqrt{5}}&0&0&0&0&\frac{2}{\sqrt{5}}&0&0\\ 0&\frac{i}{\sqrt{5}}&0&0&\frac{2}{\sqrt{5}}&0&0&0\\ 0&0&0&\frac{-1}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}}&0\\ 0&0&\frac{-1}{\sqrt{2}}&0&0&0&0&\frac{1}{\sqrt{2}}\\ 0&0&\frac{1}{\sqrt{2}}&0&0&0&0&\frac{1}{\sqrt{2}}\\ 0&0&0&\frac{-1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0 \end{pmatrix}$$

$$U_{133} = \begin{pmatrix} 0&\frac{i}{\sqrt{5}}&0&0&\frac{2}{\sqrt{5}}&0&0&0\\ \frac{i}{\sqrt{5}}&0&0&0&0&\frac{-2}{\sqrt{5}}&0&0\\ \frac{-2}{\sqrt{5}}&0&0&0&0&\frac{i}{\sqrt{5}}&0&0\\ 0&\frac{2}{\sqrt{5}}&0&0&\frac{i}{\sqrt{5}}&0&0&0\\ 0&0&0&\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0\\ 0&0&\frac{1}{\sqrt{2}}&0&0&0&0&\frac{-1}{\sqrt{2}}\\ 0&0&\frac{1}{\sqrt{2}}&0&0&0&0&\frac{1}{\sqrt{2}}\\ 0&0&0&\frac{-1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0 \end{pmatrix}$$

$$U_{134} = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{pmatrix}$$

$$U_{141} = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}$$

$$U_{142} = \begin{pmatrix} 0&0&\frac{i}{\sqrt{5}}&0&0&0&0&\frac{2}{\sqrt{5}}\\ 0&0&0&\frac{-i}{\sqrt{5}}&0&0&\frac{2}{\sqrt{5}}&0\\ 0&0&0&\frac{2}{\sqrt{5}}&0&0&\frac{-i}{\sqrt{5}}&0\\ 0&0&\frac{2}{\sqrt{5}}&0&0&0&0&\frac{i}{\sqrt{5}}\\ \frac{-1}{\sqrt{2}}&0&0&0&0&\frac{-1}{\sqrt{2}}&0&0\\ 0&\frac{1}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}}&0&0&0\\ 0&\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0\\ \frac{1}{\sqrt{2}}&0&0&0&0&\frac{-1}{\sqrt{2}}&0&0 \end{pmatrix}$$

$$U_{143} = \begin{pmatrix} 0&0&\frac{2}{\sqrt{5}}&0&0&0&0&\frac{i}{\sqrt{5}}\\ 0&0&0&\frac{-2}{\sqrt{5}}&0&0&\frac{i}{\sqrt{5}}&0\\ 0&0&0&\frac{i}{\sqrt{5}}&0&0&\frac{-2}{\sqrt{5}}&0\\ 0&0&\frac{i}{\sqrt{5}}&0&0&0&0&\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{2}}&0&0&0&0&\frac{1}{\sqrt{2}}&0&0\\ 0&\frac{-1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0\\ 0&\frac{1}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0\\ \frac{1}{\sqrt{2}}&0&0&0&0&\frac{-1}{\sqrt{2}}&0&0 \end{pmatrix}$$

$$U_{144} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad U_{211} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{pmatrix} \quad U_{212} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & \frac{1}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{5}} & 0 & 0 & \frac{2i}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & \frac{2i}{\sqrt{5}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{213} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & \frac{2i}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & \frac{-2i}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} & 0 & 0 & \frac{-1}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{214} = \begin{pmatrix} 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad U_{221} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{222} = \begin{pmatrix} \frac{-1}{\sqrt{5}} & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2i}{\sqrt{5}} & \frac{i}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ \frac{2i}{\sqrt{5}} & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad U_{223} = \begin{pmatrix} \frac{2i}{\sqrt{5}} & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & \frac{-2i}{\sqrt{5}} & \frac{-i}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{5}} & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad U_{224} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{231} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \end{pmatrix} \quad U_{232} = \begin{pmatrix} 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & \frac{-1}{\sqrt{5}} & 0 & 0 & 0 \\ \frac{-2}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 & 0 \\ \frac{i}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{2i}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & \frac{2i}{\sqrt{5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 \end{pmatrix} \quad U_{233} = \begin{pmatrix} 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & \frac{2i}{\sqrt{5}} & 0 & 0 & 0 \\ \frac{-i}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{-2i}{\sqrt{5}} & 0 & 0 \\ \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & \frac{-1}{\sqrt{5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 \end{pmatrix}$$

$$U_{234} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{241} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{242} = \begin{pmatrix} 0 & 0 & \frac{-1}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & \frac{2i}{\sqrt{5}} & 0 & 0 & \frac{i}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{2i}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} \\ \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

$$U_{243} = \begin{pmatrix} 0 & 0 & \frac{2i}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} \\ 0 & 0 & 0 & \frac{-2i}{\sqrt{5}} & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{5}} & 0 & 0 & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} \\ \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \quad U_{244} = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad U_{311} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$U_{312} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} & \frac{-i}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} & 0 & 0 & \frac{i}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & \frac{-2}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{313} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{5}} & 0 & 0 & \frac{2}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & \frac{-i}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} & \frac{-i}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{314} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{321} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{322} = \begin{pmatrix} \frac{-i}{\sqrt{5}} & 0 & 0 & \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-2}{\sqrt{5}} & \frac{-i}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ \frac{-2}{\sqrt{5}} & 0 & 0 & \frac{i}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad U_{323} = \begin{pmatrix} \frac{-2}{\sqrt{5}} & 0 & 0 & \frac{i}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{i}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-i}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 \\ \frac{-i}{\sqrt{5}} & 0 & 0 & \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$U_{324} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{331} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad U_{332} = \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & 0 \\ \frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{5}} & 0 & 0 \\ \frac{-i}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{i}{\sqrt{5}} & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$U_{333} = \begin{pmatrix} 0 & \frac{i}{\sqrt{5}} & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 \\ \frac{i}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} & 0 & 0 \\ \frac{-2}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \quad U_{334} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{341} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{342} = \begin{pmatrix} 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} \\ 0 & 0 & 0 & \frac{i}{\sqrt{5}} & 0 & 0 & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \quad U_{343} = \begin{pmatrix} 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{5}} \\ 0 & 0 & 0 & \frac{2}{\sqrt{5}} & 0 & 0 & \frac{i}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \quad U_{344} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$U_{411} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix} \quad U_{412} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & 0 & 0 & \frac{-1}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{5}} & 0 & 0 & \frac{-2i}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & \frac{-2i}{\sqrt{5}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad U_{413} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & \frac{-2i}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{\sqrt{5}} & 0 & 0 & \frac{2i}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} & 0 & 0 & \frac{1}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & 0 & \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{414} = \begin{pmatrix} 0&0&0&i&0&0&0&0\\ 0&0&-i&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&i\\ 0&0&0&0&0&0&-i&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix} \quad U_{421} = \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&i&0&0&0\\ 0&0&0&0&0&i&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ -i&0&0&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0 \end{pmatrix} \quad U_{422} = \begin{pmatrix} \frac{1}{\sqrt{5}}&0&0&\frac{-2}{\sqrt{5}}&0&0&0&0\\ 0&\frac{-1}{\sqrt{5}}&\frac{-2}{\sqrt{5}}&0&0&0&0&0\\ 0&\frac{-2i}{\sqrt{5}}&\frac{i}{\sqrt{5}}&0&0&0&0&0\\ \frac{-2i}{\sqrt{5}}&0&0&\frac{-i}{\sqrt{5}}&0&0&0&0\\ 0&0&0&0&\frac{-i}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}}\\ 0&0&0&0&0&\frac{i}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0\\ 0&0&0&0&0&\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\ 0&0&0&0&\frac{i}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$U_{423} = \begin{pmatrix} \frac{-2i}{\sqrt{5}}&0&0&\frac{-i}{\sqrt{5}}&0&0&0&0\\ 0&\frac{2i}{\sqrt{5}}&\frac{-i}{\sqrt{5}}&0&0&0&0&0\\ 0&\frac{1}{\sqrt{5}}&\frac{2}{\sqrt{5}}&0&0&0&0&0\\ \frac{1}{\sqrt{5}}&0&0&\frac{-2}{\sqrt{5}}&0&0&0&0\\ 0&0&0&0&\frac{i}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}\\ 0&0&0&0&0&\frac{-i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\ 0&0&0&0&0&\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\ 0&0&0&0&\frac{i}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}} \end{pmatrix} \quad U_{424} = \begin{pmatrix} 0&0&0&0&0&i&0&0\\ 0&0&0&0&-i&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&i&0&0&0&0&0&0\\ -i&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix} \quad U_{431} = \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&i&0\\ 0&0&0&0&0&0&0&i\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&-i&0&0&0\\ 0&0&0&0&0&-i&0&0 \end{pmatrix}$$

$$U_{432} = \begin{pmatrix} 0&\frac{-2}{\sqrt{5}}&0&0&\frac{1}{\sqrt{5}}&0&0&0\\ \frac{-2}{\sqrt{5}}&0&0&0&0&\frac{-1}{\sqrt{5}}&0&0\\ \frac{i}{\sqrt{5}}&0&0&0&0&\frac{-2i}{\sqrt{5}}&0&0\\ 0&\frac{-i}{\sqrt{5}}&0&0&\frac{-2i}{\sqrt{5}}&0&0&0\\ 0&0&0&\frac{-1}{\sqrt{2}}&0&0&\frac{-i}{\sqrt{2}}&0\\ 0&0&\frac{-1}{\sqrt{2}}&0&0&0&0&\frac{i}{\sqrt{2}}\\ 0&0&\frac{1}{\sqrt{2}}&0&0&0&0&\frac{i}{\sqrt{2}}\\ 0&0&0&\frac{-1}{\sqrt{2}}&0&0&\frac{i}{\sqrt{2}}&0 \end{pmatrix} \quad U_{433} = \begin{pmatrix} 0&\frac{-i}{\sqrt{5}}&0&0&\frac{-2i}{\sqrt{5}}&0&0&0\\ \frac{-i}{\sqrt{5}}&0&0&0&0&\frac{2i}{\sqrt{5}}&0&0\\ \frac{2}{\sqrt{5}}&0&0&0&0&\frac{1}{\sqrt{5}}&0&0\\ 0&\frac{-2}{\sqrt{5}}&0&0&\frac{1}{\sqrt{5}}&0&0&0\\ 0&0&0&\frac{1}{\sqrt{2}}&0&0&\frac{i}{\sqrt{2}}&0\\ 0&0&\frac{1}{\sqrt{2}}&0&0&0&0&\frac{-i}{\sqrt{2}}\\ 0&0&\frac{1}{\sqrt{2}}&0&0&0&0&\frac{i}{\sqrt{2}}\\ 0&0&0&\frac{-1}{\sqrt{2}}&0&0&\frac{i}{\sqrt{2}}&0 \end{pmatrix} \quad U_{434} = \begin{pmatrix} 0&0&0&0&0&0&0&i\\ 0&0&0&0&0&0&-i&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&i&0&0\\ 0&0&0&0&-i&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}$$

$$U_{441} = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ i&0&0&0&0&0&0&0\\ 0&i&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&-i&0&0&0&0&0\\ 0&0&0&-i&0&0&0&0 \end{pmatrix} \quad U_{442} = \begin{pmatrix} 0&0&\frac{1}{\sqrt{5}}&0&0&0&0&\frac{-2}{\sqrt{5}}\\ 0&0&0&\frac{-1}{\sqrt{5}}&0&0&\frac{-2}{\sqrt{5}}&0\\ 0&0&0&\frac{-2i}{\sqrt{5}}&0&0&\frac{i}{\sqrt{5}}&0\\ 0&0&\frac{-2i}{\sqrt{5}}&0&0&0&0&\frac{-i}{\sqrt{5}}\\ \frac{-i}{\sqrt{2}}&0&0&0&0&\frac{-1}{\sqrt{2}}&0&0\\ 0&\frac{i}{\sqrt{2}}&0&0&\frac{-1}{\sqrt{2}}&0&0&0\\ 0&\frac{i}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0\\ \frac{i}{\sqrt{2}}&0&0&0&0&\frac{-1}{\sqrt{2}}&0&0 \end{pmatrix} \quad U_{443} = \begin{pmatrix} 0&0&\frac{-2i}{\sqrt{5}}&0&0&0&0&\frac{-i}{\sqrt{5}}\\ 0&0&0&\frac{2i}{\sqrt{5}}&0&0&\frac{-i}{\sqrt{5}}&0\\ 0&0&0&\frac{1}{\sqrt{5}}&0&0&\frac{2}{\sqrt{5}}&0\\ 0&0&\frac{1}{\sqrt{5}}&0&0&0&0&\frac{-2}{\sqrt{5}}\\ \frac{i}{\sqrt{2}}&0&0&0&0&\frac{1}{\sqrt{2}}&0&0\\ 0&\frac{-i}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0\\ 0&\frac{i}{\sqrt{2}}&0&0&\frac{1}{\sqrt{2}}&0&0&0\\ \frac{i}{\sqrt{2}}&0&0&0&0&\frac{-1}{\sqrt{2}}&0&0 \end{pmatrix}$$

$$U_{444} = \begin{pmatrix} 0&i&0&0&0&0&0&0\\ -i&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&i&0&0&0&0\\ 0&0&-i&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0 \end{pmatrix}$$

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